CHAPTER IV.

Specification of a General Form of Riemann's Integrals.

38. In the present chapter the problem of expressing the Riemann integrals is reduced to the determination of certain fundamental rational functions, called integral functions. The existence of these functions, and their principal properties, is obtained from the descriptive point of view natural to the Riemann theory.

It appears that these integral functions are intimately related to certain functions, the differential-coefficients of the integrals of the first kind, of which the ratios have been shewn (Chapter II. § 21) to be invariant for birational transformations of the surface. It will appear, further, in the next chapter, that when these integral functions are given, or, more precisely, when the equations which express their products, of pairs of them, in terms of themselves, are given, we can deduce a form of equation to represent the Riemann surface; thus these functions may be regarded as anterior to any special form of fundamental equation.

Conversely, when the surface is given by a particular form of fundamental equation, the calculation of the algebraic forms of the integral functions may be a problem of some length. A method by which it can be carried out is given in Chapter V. (§§ 72 ff.). Compare § 50 of the present chapter.

It is convenient to explain beforehand the nature of the difficulty from which the theory contained in §§ 38—44 of this chapter has arisen. Let the equation associated with a given Riemann surface be written

$$Ay^n+A_1y^{n-1}+\ldots+A_n=0,$$

wherein $A, A_1, \ldots, A_n$ are integral polynomials in x. An *integral* function is one whose poles all lie at the places $x=\infty$ of the surface; in this chapter the integral functions considered are all rational functions. If y be an integral function, the rational symmetric functions of the n values of y corresponding to any value of x, whose values, given by the equation, are $-A_1/A$, A_2/A, $-A_3/A$, etc., will not become infinite for any finite value of x, and will, therefore, be integral polynomials in x. Thus when y is an integral function, the polynomial A divides all the other polynomials A_1, $A_2, \ldots\ldots, A_n$. Conversely, when A divides these other polynomials, the form of the equation shews that y cannot become infinite for any finite value of x, and is therefore an integral function.

When y is not an integral function, we can always find an integral polynomial in x, say β, vanishing to such an order at each of the finite poles of y, that βy is an integral function. Then also, of course, $\beta^2 y^2, \beta^3 y^3, \ldots$ are integral functions: though it often happens that there is a polynomial β_2 of less order than β^2, such that $\beta_2 y^2$ is an integral function, and similarly an integral polynomial β_3 of less order than β^3, such that $\beta_3 y^3$ is an integral function; and similarly for higher powers of y.

In particular, if in the equation given we put $Ay = \eta$, the equation becomes

$$\eta^n + A_1 \eta^{n-1} + A_2 A \eta^{n-2} + \ldots + A_n A^{n-1} = 0,$$

and η is an integral function.

Suppose that y *is* an integral function. Then any rational integral polynomial in x and y is, clearly, also an integral function. But it does not follow, conversely, though it is sometimes true, that every integral rational function can be written as an integral polynomial in x and y. For instance on the surface associated with the equation

$$y^3 + By^2 x + Cyx^2 + Dx^3 - E(y^2 - x^2) = 0,$$

the three values of y at the places $x=0$ may be expressed by series of positive integral powers of x of the respective forms

$$y = x + \lambda x^2 + \ldots, \qquad y = -x + \mu x^2 + \ldots, \qquad y = E + \nu x + \ldots.$$

Thus, the rational function $(y^2 - Ey)/x$ is not infinite when $x=0$. Since y is an integral function, the function cannot be infinite for any other finite value of x. Hence $(y^2 - Ey)/x$ is an integral function. And it is not possible, with the help of the equation of the surface, to write the function as an integral polynomial in x and y. For such a polynomial could, by the equation of the surface, be reduced to the form of an integral polynomial in x and y of the second order in y; and, in order that such a polynomial should be equal to $(y^2 - Ey)/x$, the original equation would need to be reducible.

Ex. Find the rational relation connecting x with the function $\eta = (y^2 - Ey)/x$; and thus shew that η is an integral function.

39. We concern ourselves first of all with a method of expressing all rational functions whose poles are only at the places where x has the same finite value. For this value, say a, of x there may be several branch places: the most general case is when there are k places specified by such equations as

$$x - a = t_1^{w_1+1}, \ \ldots, \ x - a = t_k^{w_k+1}.$$

The orders of infinity, in these places, of the functions considered, will be specified by integral negative powers of $t_1, \ldots, t_k$ respectively. Let F be such a function. Let $\sigma+1$ be the least positive integer such that $(x-a)^{\sigma+1}F$ is finite at every place $x=a$. We call $\sigma+1$ the *dimension* of F. Let $f(x, y) = 0$ be the equation of the surface. In order that there may be any branch places at $x=a$, it is necessary that $\partial f/\partial y$ should be zero for this value of x. Since this is only true for a finite number of values of x, we shall suppose that the value of x considered is one for which there are no branch places.

We prove that there are rational functions $h_1, \dots, h_{n-1}$ infinite only at the n places $x=a$, such that every rational function whose infinities occur only at these n places can be expressed in the form

$$\left(\frac{1}{x-a}, 1\right)_\lambda + \left(\frac{1}{x-a}, 1\right)_{\lambda_1} h_1 + \dots + \left(\frac{1}{x-a}, 1\right)_{\lambda_{n-1}} h_{n-1} \dots\dots\dots \text{(A)},$$

in such a way that no term occurs in this expression which is of higher dimension than the function to be expressed: namely, if $\sigma+1$ be the dimension of the function to be expressed and σ_i+1 the dimension of h_i, the function can be expressed in such a way that no one of the integers

$$\lambda,\ \lambda_1+\sigma_1+1,\ \dots,\ \lambda_{n-1}+\sigma_{n-1}+1$$

is greater than $\sigma+1$. We may refer to this characteristic as the *condition of dimensions*. It is clear conversely that every expression of the form (A) will be a rational function infinite only for $x=a$.

Let the sheets of the surface at $x=a$ be considered in some definite order. A rational function which is infinite only at these n places may be denoted by a symbol $(R_1, R_2, \dots, R_n)$, where $R_1, R_2, \dots, R_n$ are the orders of infinity in the various sheets. We may call $R_1, R_2, \dots, R_n$ the *indices* of the function. Since the surface is unbranched at $x=a$, it is possible to find a certain polynomial in $\frac{1}{x-a}$, involving only positive integral powers of this quantity, the highest power being $\left(\frac{1}{x-a}\right)^{R_n}$, such that the function

$$(R_1, R_2, \dots, R_n) - \left(\frac{1}{x-a}, 1\right)_{R_n}, = (S_1, S_2, \dots, S_{n-1}, 0) \text{ say} \dots\dots\dots\text{(i)},$$

is not infinite in the nth sheet at $x=a$.

Consider then all rational functions, infinite only at $x=a$, of which the nth index is zero. It is in general possible to construct a rational function having prescribed values for the $(n-1)$ other indices, provided their sum be $p+1$. When this is not possible a function can be constructed* whose indices have a less sum than $p+1$, none of them being greater than the prescribed values. Starting with a set of indices $(p+1, 0, \dots, 0)$, consider how far the first index can be reduced by increasing the 2nd, 3rd, ..., $(n-1)$th indices. In constructing the successive functions with smaller first index, it will be necessary, in the most general case, to increase some of the 2nd, 3rd, ..., $(n-1)$th indices, and there will be a certain arbitrariness as to the way in which this shall be done. But if we consider only those functions of which the sum of the indices is less than $p+2$, there will be only a finite number

* The proof is given in the preceding Chapter, (§§ 24, 28).

possible for which the first index has a given value. There will therefore only be a finite number of functions of the kind considered*, for which the further condition is satisfied that *the first index is the least possible such that it is not less than any of the others.* Let this least value be r_1, and suppose there are k_1 functions satisfying this condition. Call them the reduced functions of the first class—and in general let any function whose nth index is zero be said to be of the first class when its first index is *greater or not less* than its other indices. In the same way reckon as functions of the second class all those (with nth index zero) whose second index is *greater than* the first index and *greater than or equal to* the following indices. Let the functions whose second index has the least value consistently with this condition be called the reduced functions of the second class; let their number be k_2 and their second index be r_2. In general, reckon to the ith class $(i < n)$ all those functions, with nth index zero, whose ith index is *greater than* the preceding indices and *not less than* the succeeding indices. Let there be k_i reduced functions of this class, with ith index equal to r_i. Clearly none of the integers $r_1, \ldots, r_{n-1}$ are zero.

Let now $$(s_1 \ldots s_{i-1}\, r_i\, s_{i+1} \ldots s_{n-1}\, 0),$$
where $$r_i > s_1, \ldots, r_i > s_{i-1}, r_i \geqq s_{i+1}, \ldots, r_i \geqq s_{n-1},$$

be *any* definite one of the k_i reduced functions of the ith class. Make a similar selection from the reduced functions of every class. And let

$$(S_1 \ldots S_{i-1}\, R_i\, S_{i+1} \ldots S_{n-1}\, 0)$$

be any function of the ith class other than a reduced function, so that

$$R_i > S_1, \ldots, R_i > S_{i-1}, R_i \geqq S_{i+1}, \ldots, R_i \geqq S_{n-1}.$$

Then by choice of a proper constant coefficient λ we can write

$$(S_1 \ldots S_{i-1}\, R_i\, S_{i+1} \ldots S_{n-1}\, 0) - \lambda\,(x-a)^{-(R_i - r_i)}\,(s_1 \ldots s_{i-1}\, r_i\, s_{i+1} \ldots s_{n-1}\, 0)$$

in the form

$$(T_1 \ldots T_{i-1}\, R_i'\, T_{i+1} \ldots T_{n-1},\, R_i - r_i) \ldots\ldots\ldots\ldots\ldots\text{(ii)},$$

where $R_i' < R_i$; T_1 may be as great as the greater of S_1, $R_i - (r_i - s_1)$, but is certainly less than R_i; and similarly $T_2, \ldots, T_{i-1}$ are certainly less than R_i; while T_{i+1} may be as great as the greater of S_{i+1}, $R_i - (r_i - s_{i+1})$, and is therefore not greater than R_i; and similarly $T_{i+2}, \ldots, T_{n-1}$ are certainly not greater than R_i.

* **Functions which have the same indices are here regarded as identical. Of course the general function with given indices may involve a certain number of arbitrary constants. By *the* function of given indices is here meant *any one* such, chosen at pleasure, which really becomes infinite in the specified way.**

Further, if $\left(\frac{1}{x-a}, 1\right)_{R_i - r_i}$ be a suitable polynomial of order $R_i - r_i$ in $(x-a)^{-1}$, we can write

$$(T_1 \dots T_{i-1}\, R_i' \dots T_{n-1},\, R_i - r_i) - \left(\frac{1}{x-a}, 1\right)_{R_i - r_i}$$
$$= (S'_1 \dots S'_{i-1}\, R''_i\, S'_{i+1} \dots S'_{n-1}\, 0) \dots\dots\dots\dots \text{(iii)},$$

where R''_i may be as great as the greater of R'_i, $R_i - r_i$, but is certainly less than R_i; S'_1 may be as great as the greater of T_1, $R_i - r_i$, but is certainly less than R_i; and similarly $S'_2, \dots, S'_{i-1}$ are certainly less than R_i; while S'_{i+1} may be as great as the greater of T_{i+1}, $R_i - r_i$, and is certainly not greater than R_i; and similarly $S'_{i+2}, \dots, S'_{n-1}$ are certainly not greater than R_i.

Hence there are two possibilities.

(1) Either $(S'_1 \dots S'_{i-1} R''_i S'_{i+1} \dots S'_{n-1}\, 0)$ is still of the ith class, namely, $\quad R''_i > S_1, \dots, R''_i > S'_{i-1}, R''_i \geqq S'_{i+1}, \dots, R''_i \geqq S'_{n-1}$,

and in this case the greatest value occurring among its indices (R''_i) is less than the greatest value occurring in the indices of $(S_1 \dots S_{i-1} R_i S_{i+1} \dots S_{n-1}\, 0)$.

(2) Or it is a function of another class, for which the greatest value occurring among its indices may be smaller than or as great as R_i (though not greater); but when this greatest value is R_i, it is not reached by any of the first i indices.

If then, using a term already employed, the greatest value occurring among the indices of any function $(R_1, \dots, R_n)$ be called the dimension of the function, we can group the possibilities differently and say, either $(S'_1 \dots S'_{i-1} R''_i S'_{i+1} \dots S'_{n-1}\, 0)$ is of lower dimension than

$$(S_1 \dots S_{i-1}\, R_i\, S_{i+1} \dots S_{n-1}\, 0),$$

or it is of the same dimension and then belongs to a more advanced class, that is, to an $(i+k)$th class where $k > 0$.

In the same way if $(t_1 \dots t_{i-1}\, r_i\, t_{i+1} \dots t_{n-1}\, 0)$ be any reduced function of the ith class other than $(s_1 \dots s_{i-1}\, r_i\, s_{i+1} \dots s_{n-1}\, 0)$, we can, by choice of a suitable constant coefficient μ, write

$$(t_1 \dots t_{i-1}\, r_i\, t_{i+1} \dots t_{n-1}\, 0) - \mu\, (s_1 \dots s_{i-1}\, r_i\, s_{i+1} \dots s_{n-1}\, 0)$$
$$= (t'_1 \dots t'_{i-1}\, r'_i\, t'_{i+1} \dots t'_{n-1}\, 0) \dots\dots\dots\dots\dots\dots \text{(iv)},$$

where $r'_i < r_i$, $t'_1 \dots t'_{i-1}$ may be respectively as great as the greater of the pairs $(t_1, s_1) \dots (t_{i-1}, s_{i-1})$ but are each certainly less than r_i, while similarly no one of $t'_{i+1}, \dots, t'_{n-1}$ is greater than r_i.

The function $(t'_1 \dots t'_{i-1}\, r'_i\, t'_{i+1} \dots t'_{n-1}\, 0)$ cannot be of the ith class, since no function of the ith class has its ith index less than r_i: and though the greatest value reached among its indices may be as great as r_i (and not greater), the number of indices reaching this value will be at least one less than for $(s_1 \dots s_{i-1} r_i s_{i+1} \dots s_{n-1} 0)$. Namely $(t'_1 \dots t'_{i-1} r'_i t'_{i+1} \dots t'_{n-1} 0)$ is certainly of more advanced class than $(s_1 \dots s_{i-1} r_i s_{i+1} \dots s_{n-1} 0)$, and not of higher dimension than this.

Denote now by $h_1, \dots, h_{n-1}$ the selected reduced functions of the 1st, 2nd, ..., $(n-1)$th classes. Then, having regard to the equations given by (ii), (iii), (iv), we can make the statement,

Any function $(S_1 \dots S_{i-1} R_i S_{i+1} \dots S_{n-1} 0)$ *can be expressed as a sum of* (1) *an integral polynomial in* $(x-a)^{-1}$, (2) *one of* $h_1, \dots, h_{n-1}$ *multiplied by such a polynomial,* (3) *a function* F *which is either of lower dimension than the function to be expressed or is of more advanced class.*

In particular when the function to be expressed is of the $(n-1)$th class the new function F will necessarily be of lower dimension than the function to be expressed.

Hence by continuing the process as far as may be needful, every function

$$f = (S_1 \dots S_{i-1} R_i S_{i+1} \dots S_{n-1} 0)$$

can be expressed in the form

$$\left(\frac{1}{x-a}, 1\right)_\lambda + \left(\frac{1}{x-a}, 1\right)_{\lambda_1} h_1 + \dots + \left(\frac{1}{x-a}, 1\right)_{\lambda_{n-1}} h_{n-1} + F_1, \quad\dots\dots(\text{v})$$

where F_1 is of lower dimension than f.

Applying this statement and recalling that there are lower limits to the dimensions of existent functions of the various classes, namely, those of the $k_1 + \dots + k_{n-1}$ reduced functions, and noticing that the reduction formula (v) can be applied to these reduced functions, we can, therefore, put every function $f = (S_1 \dots S_{i-1} R_i S_{i+1} \dots S_{n-1} 0)$ into a form

$$\left(\frac{1}{x-a}, 1\right)_\lambda + \left(\frac{1}{x-a}, 1\right)_{\lambda_1} h_1 + \dots + \left(\frac{1}{x-a}, 1\right)_{\lambda_{n-1}} h_{n-1}.$$

Now it is to be noticed that in the equations (ii), (iii), (iv), upon which this result is based, no terms are introduced which are of higher dimension than the function which it is desired to express: and that the same remark is applicable to equation (i).

Hence every function $(R_1, \dots, R_n)$ *can be written in the form* (A) *in such a way that the condition of dimensions is satisfied.*

40. In order to give an immediate example of the theory we may take the case of a surface of four sheets, and assume that the places $x = a$ are such that no rational function exists, infinite only there, whose aggregate order of infinity is less than $p+1$. In that case the specification of the reduced functions is an easy arithmetical problem. The reduced functions of the first class are $(m_1, m_2, m_3, 0)$, where m_1 is to be as small as possible without being smaller than m_2 or m_3: by the hypothesis we may take

$$m_1 + m_2 + m_3 = p + 1.$$

Those of the second class require m_2 as small as possible subject to

$$m_1 + m_2 + m_3 = p + 1,\ m_2 > m_1,\ m_2 \overline{\overline{>}} m_3:$$

those of the third class require m_3 greater than m_1 and m_2 but otherwise as small as possible subject to $m_1 + m_2 + m_3 = p + 1$. We therefore immediately obtain the reduced functions given in the 2nd, 3rd and 4th columns of the following table. The dimension of any function of the ith class being denoted by $\sigma_i + 1$, the values of σ_i are given in the fifth column, and the sum $\sigma_1 + \sigma_2 + \sigma_3$ in the sixth. The reason for the insertion of this value will appear in the next Article.

p	Reduced functions of the first class	Reduced functions of the second class	Reduced functions of the third class	$\sigma_1, \sigma_2, \sigma_3$	$\sigma_1+\sigma_2+\sigma_3$
$=3M-1$	$(M, M, M, 0)$	$(M-2, M+1, M+1, 0)$ $(M-1, M+1, M, 0)$ $(M, M+1, M-1, 0)$	$(M-1, M, M+1, 0)$	$M-1, M, M$	$3M-1$
$=3N-2$	$(N, N, N-1, 0)$ $(N, N-1, N, 0)$	$(N-1, N, N, 0)$	$(N-1, N-1, N+1, 0)$	$N-1, N-1, N$	$3N-2$
$=3P$	$(P+1, P, P, 0)$ $(P+1, P+1, P-1, 0)$ $(P+1, P-1, P+1, 0)$	$(P-1, P+1, P+1, 0)$ $(P, P+1, P, 0)$	$(P, P, P+1, 0)$	P, P, P	$3P$

Here the reduced functions of the various classes are written down in random order. Denoting those first written by h_1, h_2, h_3, we may exemplify the way in which the others are expressible by them in two cases.

(*a*) When $p = 3M - 1$, we have, μ being such a constant as in equation (iv) above (§ 39),

$$(M, M+1, M-1, 0) - \mu (M-2, M+1, M+1, 0) = \{M, M, M+1, 0\},$$

the right hand denoting a function whose orders of infinity in the various sheets are not higher than the indices given. If the order in the third sheet be less than $M + 1$, the right hand must be a function of the first class and therefore the order in the third sheet must be M. In that case, since a general function of aggregate order $p + 1$ contains two arbitrary constants, we have an expression of the form

$$(M, M+1, M-1, 0) = \mu h_2 + A h_1 + B,$$

for suitable values of the constants A, B.

If however there be no such reduction, we can choose a constant λ so that

$$\{M, M, M+1, 0\} - \lambda (M-1, M, M+1, 0) = \{M, M, M, 0\} = A' h_1 + B',$$

and thus obtain on the whole

$$(M, M+1, M-1, 0) = \mu h_2 + \lambda h_3 + A'h_1 + B',$$

for suitable values of the constants A', B'.

(*b*) When $p = 3P$ we obtain

$$\begin{aligned}(P+1, P+1, P-1, 0) &= \lambda h_1 + A\,(P, P+1, P, 0) + B \\ &= \lambda h_1 + A\,\{\mu h_2 + Ch_3 + D\} + B.\end{aligned}$$

Ex. 1. Shew for a surface of three sheets that we have the table

p	h_1, h_2	σ_1, σ_2	$\sigma_1+\sigma_2$
odd	$\left(\frac{p+1}{2}, \frac{p+1}{2}, 0\right)\left(\frac{p-1}{2}, \frac{p+3}{2}, 0\right)$	$\frac{p-1}{2}, \frac{p+1}{2}$	p
even	$\left(\frac{p+2}{2}, \frac{p}{2}, 0\right)\left(\frac{p}{2}, \frac{p+2}{2}, 0\right)$	$\frac{p}{2}, \frac{p}{2}$	p

Ex. 2. Shew, for a surface of n sheets, that if the places $x=a$ be such that it is impossible to construct a rational function, infinite only there, whose aggregate order of infinity is less than $p+1$, a set of reduced functions is given by

$$h_1 \ldots h_{r+1} = (k, \ldots k, k-1, \ldots, k-1, 0), (k-1, k, \ldots k, k-1, \ldots, k-1, 0) \ldots\ldots (k-1, \ldots, k-1, k, \ldots k, 0)$$
$$h_{r+2} \ldots h_{n-1} = (k-1, \ldots, k-1, k+1, k, \ldots k, 0)\ (k-1, \ldots, k-1, k, k+1, k, \ldots k, 0) \ldots\ldots$$
$$(k-1, \ldots, k-1, k, \ldots k, k+1, 0)$$

wherein $p+1 = (n-1)k - r$ $(r < n-1)$ and, in the first row, there are r numbers $k-1$ in each symbol, and, in the second row, there are $r+1$ numbers $k-1$ in each symbol. In each case $k, \ldots k$ denotes a set of numbers all equal to k and $k-1, \ldots, k-1$ denotes a set of numbers all equal to $k-1$.

The values of $\sigma_1, \ldots, \sigma_{r+1}$ are each $k-1$, those of $\sigma_{r+2}, \ldots, \sigma_{n-1}$ are each k. Hence

$$\sigma_1 + \ldots + \sigma_{r+1} + \sigma_{r+2} + \ldots + \sigma_{n-1} = (r+1)(k-1) + (n-r-2)k = (n-1)k - r - 1 = p.$$

Ex. 3. Shew that the resulting set of reduced functions is effectively independent of the order in which the sheets are supposed to be arranged at $x=a$.

41. For the case where rational functions exist, infinite only at the places $x=a$, whose aggregate order of infinity is less than $p+1$, the specification of their indices is a matter of greater complexity.

But we can at once prove that the property already exemplified and expressed by the equation $\sigma_1 + \ldots + \sigma_{n-1} = p$, or by the statement that the sum of the dimensions of the reduced functions is $p+n-1$, is true in all cases.

For consider a rational function which is infinite to the rth order in each sheet at $x=a$ and not elsewhere: if r be taken great enough, such a function necessarily exists and is an aggregate of $nr-p+1$ terms, one of these being an additive constant (Chapter III. § 37). By what has been proved, such a function can be expressed in the form

$$\left(\frac{1}{x-a}, 1\right)_\lambda + \left(\frac{1}{x-a}, 1\right)_{\lambda_1} h_1 + \ldots + \left(\frac{1}{x-a}, 1\right)_{\lambda_{n-1}} h_{n-1},$$

where the dimensions of the several terms, namely the numbers

$$\lambda,\ \lambda_1+\sigma_1+1,\ \dots,\ \lambda_{n-1}+\sigma_{n-1}+1,$$

are not greater than the dimension, r, of the function.

Conversely, the most general expression of this form in which $\lambda_1, \lambda_2, \dots, \lambda_{n-1}$ attain the upper limits prescribed by these conditions, is a function of the desired kind.*

But such general expression contains

$$(\lambda+1)+(\lambda_1+1)+\dots+(\lambda_{n-1}+1),$$

that is
$$(r+1)+(r-\sigma_1)+\dots+(r-\sigma_{n-1}),$$

or
$$nr-(\sigma_1+\dots+\sigma_{n-1})+1$$

arbitrary constants.

Since this must be equal to $nr-p+1$ the result enunciated is proved.

The result is of considerable interest—when the forms of the functions $h_1 \dots h_{n-1}$ are determined algebraically, we obtain the deficiency of the surface by finding the sum of the dimensions of $h_1 \dots h_{n-1}$. It is clear that a proof of the value of this sum can be obtained by considerations already adopted to prove Weierstrass's gap theorem. That theorem and the present result are in fact, here, both deduced from the same fact, namely, that the number of periods of a normal integral of the second kind is p.

42. Consider now the places $x=\infty$: let the character of the surface be specified by k equations

$$\frac{1}{x}=t_1^{w_1+1}, \dots, \frac{1}{x}=t_k^{w_k+1},$$

there being k branch places. A rational function g which is infinite only at these places will be called an integral function. If its orders of infinity at these places be respectively $r_1, r_2, \dots, r_k$ and $G\,[r_i/(w_i+1)]$ be the least positive integer greater than or equal to $r_i/(w_i+1)$, and $\rho+1$ denote the greatest of the k integers thus obtained, then it is clear that $\rho+1$ is the least positive integer such that $x^{-(\rho+1)}\,g$ is finite at every place $x=\infty$. We shall call $\rho+1$ the dimension of g.

Of such integral functions there are $n-1$ which we consider particularly, namely, using the notation of the previous paragraph, the functions

$$(x-a)^{\sigma_1+1}\,h_1,\ \dots\dots,\ (x-a)^{\sigma_{n-1}+1}\,h_{n-1},$$

which by the definitions of $\sigma_1, \dots\dots, \sigma_{n-1}$ are all finite at the places $x=a$, and are therefore infinite only for $x=\infty$. Denote $(x-a)^{\sigma_i+1}\,h_i$ by g_i. If h_i do not vanish at every place $x=\infty$, it is clear that the dimension of g_i is

* It is clear that this statement could not be made if any of the indices of the function to be expressed were less than the dimension of the function. For instance in the final equation of § 40 (a), unless μ, λ, A' be specially chosen, the right hand represents a function with its third index equal to $M+1$.

σ_i+1. If however h_i do so vanish, the dimension of g_i may conceivably be less than σ_i+1; denote it by ρ_i+1, so that $\rho_i \overline{\overline{<}} \sigma_i$. Then $x^{-(\rho_i+1)}\, g_i$, and therefore also $(x-a)^{-(\rho_i+1)}\, g_i, = (x-a)^{\sigma_i-\rho_i}\, h_i$, is finite at all places $x=\infty$: hence $(x-a)^{\sigma_i-\rho_i}\, h_i$ is a function which only becomes infinite at the places $x=a$. But, in the phraseology of § 39, it is clearly a function of the same class as h_i, it does not become infinite in the nth sheet at $x=a$, and is of less dimension than h_i if $\sigma_i > \rho_i$. That such a function should exist is contrary to the definition of h_i. Hence, in fact, $\sigma_i=\rho_i$. The reader will see that the same result is proved independently in the course of the present paragraph.

Let now F denote any integral function of dimension $\rho+1$. Then $x^{-(\rho+1)}\, F$ is finite at all places $x=\infty$: and therefore so also is $(x-a)^{-(\rho+1)}\, F$. This latter function is one of those which are infinite only at places $x=a$; if F do not vanish at all places $x=a$, the dimension $\sigma+1$ of $(x-a)^{-(\rho+1)}\, F$ will be $\rho+1$: in general we shall have $\sigma \overline{\overline{<}} \rho$.

By § 39 we can write

$$(x-a)^{-(\rho+1)}\, F = \left(\frac{1}{x-a}, 1\right)_\lambda + \left(\frac{1}{x-a}, 1\right)_{\lambda_1} h_1 + \ldots\ldots + \left(\frac{1}{x-a}, 1\right)_{\lambda_{n-1}} h_{n-1},$$

where
$$\sigma+1 \overline{\overline{>}} \lambda_i+\sigma_i+1,$$

and therefore, *a fortiori*,

$$\rho+1 \overline{\overline{>}} \lambda_i+\sigma_i+1 \overline{\overline{>}} \lambda_i+\rho_i+1.$$

Hence we can also write

$$F=(1, x-a)_\lambda\, (x-a)^{\rho-\lambda} + (1, x-a)_{\lambda_1}\, (x-a)^{\rho-\lambda_1-\sigma_1}\, g_1 + \ldots\ldots$$
$$+ (1, x-a)_{\lambda_{n-1}}\, (x-a)^{\rho-\lambda_{n-1}-\sigma_{n-1}}\, g_{n-1},$$

or say

$$F=(1, x)_\mu + (1, x)_{\mu_1}\, g_1 + \ldots\ldots + (1, x)_{\mu_{n-1}}\, g_{n-1}, \quad \ldots\ldots\ldots\ldots\text{(B)}$$

where
$$\mu_i+\rho_i+1=\rho-\sigma_i+\rho_i+1=\rho+1-(\sigma_i-\rho_i) \overline{\overline{<}} \rho+1,$$

namely, there is no term on the right whose dimension is greater than that of F (and each of $\mu, \mu_1, \ldots\ldots, \mu_{n-1}$ is a positive integer).

Hence the equation (B) is entirely analogous to the equation (A) obtained previously for the expression of functions which are infinite only at places $x=a$. The set $(1, g_1, \ldots\ldots, g_{n-1})$ will be called a *fundamental set* for the expression of rational integral functions*.

It can be proved precisely as in the previous Article that $\rho_1+\rho_2+\ldots\ldots+\rho_{n-1}=p$. For this purpose it is only necessary to consider a function

* The idea, derived from arithmetic, of making the integral functions the basis of the theory of all algebraic functions has been utilised by Dedekind and Weber, *Theor. d. alg. Funct. e. Veränd.* Crelle, t. 92. Kronecker, *U. die Discrim. alg. Fctnen.* Crelle, t. 91. Kronecker, *Grundzüge e. arith. Theor. d. algebr. Grössen*, Crelle, t. 92 (1882).

which is infinite at the places $x=\infty$ respectively to orders $r\,(w_1+1)$, ..., $r\,(w_k+1)$. And the equations $\Sigma\rho=\Sigma\sigma=p$, taken with $\sigma_i \overline{\overline{>}} \rho_i$, suffice to shew that $\sigma_i=\rho_i$. It can also be shewn that from the set $g_1 \ldots g_{n-1}$ we can conversely deduce a fundamental set $1, (x-b)^{-(\rho_i+1)}\, g_1, \ldots, (x-b)^{-(\rho_{n-1}-1)}\, g_{n-1}$ for the expression of functions infinite only at places $x=b$; these have the same dimensions as $1, g_1, \ldots, g_{n-1}$*.

43. Having thus established the existence of fundamental systems for integral rational functions, it is proper to refer to some characteristic properties of all such systems.

(*a*) If $G_1 \ldots G_{n-1}$ be any set of rational integral functions such that every rational integral function can be expressed in the form

$$(x,\,1)_\lambda + (x,\,1)_{\lambda_1}\, G_1 + \ldots\ldots + (x,\,1)_{\lambda_{n-1}}\, G_{n-1} \ldots\ldots\ldots\ldots\ldots \text{(C)},$$

there can exist no relations of the form

$$(x,\,1)_\mu + (x,\,1)_{\mu_1}\, G_1 + \ldots\ldots + (x,\,1)_{\mu_{n-1}}\, G_{n-1} = 0.$$

For if k such relations hold, independent of one another, k of the functions $G_1 \ldots G_{n-1}$ can be expressed linearly, with coefficients which are rational in x, in terms of the other $n-1-k$. Hence also $\beta_1 y, \beta_2 y^2, \ldots, \beta_{n-1-k}\, y^{n-1-k}$, $\beta_{n-k}\, y^{n-k}$, which are integral functions when $\beta_1, \ldots, \beta_{n-k}$ are proper polynomials in x, can be expressed linearly in terms of the $n-1-k$ linearly independent functions occurring among $G_1 \ldots G_{n-1}$, with coefficients which are rational in x. By elimination of these $n-1-k$ functions we therefore obtain an equation

$$A + A_1\, y + \ldots\ldots + A_{n-k}\, y^{n-k} = 0,$$

whose coefficients $A, A_1, \ldots\ldots, A_{n-k}$ are rational in x. Such an equation is inconsistent with the hypothesis that the fundamental equation of the surface is irreducible.

(*b*) Consider two places of the Riemann surface at which the independent variable, x, has the same value: suppose, first of all, that there are no branch places for this value of x. Let $\lambda, \lambda_1, \ldots\ldots, \lambda_{n-1}$ be constants. Then the linear function

$$\lambda + \lambda_1 G_1 + \ldots\ldots + \lambda_{n-1} G_{n-1}$$

cannot have the same value at these two places *for all values of* λ, $\lambda_1, \ldots\ldots, \lambda_{n-1}$.

For this would require that each of $G_1, \ldots\ldots, G_{n-1}$ has the same value at these two places. Denote these values by $a_1, \ldots\ldots, a_{n-1}$ respectively. We can choose coefficients $\mu_1, \ldots\ldots, \mu_{n-1}$ such that the function

$$\mu_1(G_1 - a_1) + \ldots\ldots + \mu_{n-1}(G_{n-1} - a_{n-1}),$$

* The dimension of an integral function is employed by Hensel, Crelle, t. 105, 109, 111; *Acta Math.* t. 18. The account here given is mainly suggested by Hensel's papers. For surfaces of three sheets see also Baur, *Math. Annal.* t. 43 and *Math. Annal.* t. 46.

which clearly vanishes at each of the two places in question, vanishes also at the other $n-2$ places arising for the same value of x. Denoting the value of x by c, it follows, since there are no branch places for $x=c$, that the function

$$[\mu_1(G_1-a_1)+\ldots\ldots+\mu_{n-1}(G_{n-1}-a_{n-1})]/(x-c)$$

is not infinite at any of the places $x=c$. It is therefore an integral rational function.

Now this is impossible. For then the function could be expressed in the form

$$(x, 1)_\lambda+(x, 1)_\mu G_1+\ldots\ldots+(x, 1)_\nu G_{n-1},$$

and it is contrary to what is proved under (a) that two expressions of these forms should be equal to one another.

Hence the hypothesis that the function

$$\lambda+\lambda_1 G_1+\ldots\ldots+\lambda_{n-1} G_{n-1}$$

can have the same value in each of two places at which x has the same value, is disproved.

If there be a branch place at $x=c$, at which two sheets wind, and no other branch place for this value of x, it can be proved in a similar way, that a linear function of the form

$$\lambda_1 G_1+\ldots\ldots+\lambda_{n-1} G_{n-1}$$

cannot vanish to the second order at the branch place, *for all values of* $\lambda_1,\ldots\ldots,\lambda_{n-1}$ namely, not all of $G_1,\ldots\ldots, G_{n-1}$ can vanish to the second order at the branch place. For then we could similarly find an integral function expressible in the form

$$(\mu_1 G_1+\ldots\ldots+\mu_{n-1} G_{n-1})/(x-c).$$

More generally, whatever be the order of the branch place considered, at $x=c$, and whatever other branch places may be present for $x=c$, it is always true that, *if all of* $G_1,\ldots\ldots, G_{n-1}$ *vanish at the same place* A *of the Riemann surface, they cannot all vanish at another place for which* x *has the same value; and if* A *be a branch place, they cannot all vanish at* A *to the second order.*

Ex. 1. Denoting the function

$$\lambda+\lambda_1 G_1+\ldots+\lambda_{n-1} G_{n-1}$$

by K, and its values in the n sheets for the same value of x by $K^{(1)}, K^{(2)},\ldots, K^{(n)}$, we have shewn that, for a particular value of x, we can always choose $\lambda, \lambda_1,\ldots, \lambda_{n-1}$, so that the equation $K^{(1)}=K^{(2)}$ is not verified. Prove, similarly, that we can always choose $\lambda, \lambda_1,\ldots, \lambda_{n-1}$ so that an equation of the form

$$m_1 K^{(1)}+m_2 K^{(2)}+\ldots+m_k K^{(k)}=0,$$

where $m_1,\ldots, m_{k-1}, m_k$ are given constants whose sum is zero, is not verified.

Ex. 2. Let $x=\gamma_1,\dots,\gamma_k$ be k distinct given values of x: then it is possible to choose coefficients λ, $\lambda_1,\dots$, μ, $\mu_1,\dots$, finite in number, such that the values of the function

$$(\lambda+\mu x+\nu x^2+\dots)+(\lambda_1+\mu_1 x+\nu_1 x^2+\dots)G_1+\dots+(\lambda_{n-1}+\mu_{n-1}x+\nu_{n-1}x^2+\dots)G_{n-1},$$

at the places $x=\gamma_1$, shall be all different, and also the values of the function, at the places $x=\gamma_2$, shall be all different, and, also, the values of the function, for each of the places $x=\gamma_3,\dots,\gamma_k$, shall be all different.

(*c*) If $1, H_1, H_2, \dots\dots, H_{n-1}$ be another fundamental set of integral functions, with the same property as $1, G_1, \dots\dots, G_{n-1}$, we shall have linear equations of the form

$$1 = 1$$

$$H_i = \alpha_i + \alpha_{i,1}\, G_1 + \dots\dots + \alpha_{i,\, n-1}\, G_{n-1} \quad\dots\dots\dots\dots\dots\dots \text{(D)},$$

where $\alpha_{i,\, j}$ is an integral polynomial in x.

Now in fact the determinant $|\, \alpha_{i,j} \,|$ is a constant $(i=1, 2, \dots, \overline{n-1}\,;$ $j=1, 2, \dots, \overline{n-1})$. For if $H_i^{(r)}$ denote the value of H_i, for a general value of x, in the rth sheet of the surface, we clearly have the identity

$$\begin{vmatrix} 1, & 1, & \dots\dots\dots\dots, & 1 \\ H_1^{(1)}, & H_1^{(2)}, & \dots\dots\dots, & H_1^{(n)} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ H_{n-1}^{(1)}, & H_{n-1}^{(2)}, & \dots\dots, & H_{n-1}^{(n)} \end{vmatrix} = \begin{vmatrix} 1, & 0, & \dots\dots\dots, & 0 \\ \alpha_1, & \alpha_{1,\,1}, & \dots\dots\dots, & \alpha_{1,\, n-1} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \alpha_{n-1}, & \alpha_{n-1,1}, & \dots\dots, & \alpha_{n-1,\, n-1} \end{vmatrix} \begin{vmatrix} 1, & 1, & \dots\dots\dots\dots, & 1 \\ G_1^{(1)}, & G_1^{(2)}, & \dots\dots\dots, & G_1^{(n)} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ G_{n-1}^{(1)}, & G_{n-1}^{(2)}, & \dots, & G_{n-1}^{(n)} \end{vmatrix}$$

If we form the square of this equation, the general term of the square of the left hand determinant, being of the form $H_i^{(1)}H_j^{(1)}+\dots\dots+H_i^{(n)}H_j^{(n)}$, will be a rational function of x which is infinite only for infinite values of x; it is therefore an integral polynomial in x. We shall therefore have a result which we write in the form

$$\Delta\,(1, H_1, \dots\dots, H_{n-1}) = \nabla^2 \,.\, \Delta\,(1, G_1, G_2, \dots\dots, G_{n-1}),$$

where ∇ is the determinant $|\, \alpha_{i,\, j}\,|$. $\Delta\,(1, H_1, \dots\dots, H_{n-1})$ may be called the discriminant of $1, H_1, \dots\dots, H_{n-1}$.

If β be such an integral polynomial in x that $\beta y, = \eta$, say, is an integral function, an equation of similar form exists when $1, \eta, \eta^2, \dots\dots, \eta^{n-1}$ are written instead of $1, H_1, \dots\dots, H_{n-1}$. Since then $\Delta\,(1, \eta, \eta^2, \dots\dots, \eta^{n-1})$ does not vanish for all values of x it follows that $\Delta\,(1, G_1, G_2, \dots\dots, G_{n-1})$ does not vanish for all values of x. (Cf. (*a*), of this Article.)

But because $1, H_1, H_2, \dots\dots, H_{n-1}$ are equally a set in terms of which all integral functions are similarly expressible, it follows that $\Delta\,(1, H_1, \dots\dots, H_{n-1})$ does not vanish for all values of x, and that

$$\Delta\,(1, G_1, \dots\dots, G_{n-1}) = \nabla_1^2\, \Delta\,(1, H_1, \dots\dots, H_{n-1}),$$

where ∇_1 is an integral function rationally expressible by x only.

Hence $\nabla^2 . \nabla_1^2 = 1$: *thus each of* ∇ *and* ∇_1 *is an absolute constant.*

Hence also the discriminants $\Delta (1, G_1, \ldots\ldots, G_{n-1})$ of all sets in terms of which integral functions are thus integrally expressible, are identical, save for a constant factor.

Let Δ denote their common value and $\eta_1, \ldots, \eta_n$ denote any n integral functions whatever; then if $\Delta (\eta_1, \eta_2, \ldots, \eta_n)$ denote the determinant which is the square of the determinant whose (s, r)th element is $\eta^{(r)}_s$, we can prove, as here, that there exists an equation of the form

$$\Delta (\eta_1, \eta_2, \ldots, \eta_n) = M^2 \Delta,$$

wherein M is an integral polynomial in x. The function $\Delta (\eta_1, \eta_2, \ldots, \eta_n)$ is called the discriminant of the set $\eta_1, \eta_2, \ldots, \eta_n$. Since this is divisible by Δ, it follows, if, for shortness, we speak of $1, H_1, \ldots, H_{n-1}$, equally with $\eta_1, \eta_2, \ldots, \eta_n$, as a set of n integral functions, that Δ *is the highest divisor common to the discriminants of all sets of n integral functions.*

(d) The sets $(1, G_1, \ldots\ldots, G_{n-1})$, $(1, H_1, \ldots\ldots, H_{n-1})$ are not supposed subject to the condition that, in the expression of an integral function in terms of them, no term shall occur of higher dimension than the function to be expressed. If $(1, g_1, \ldots\ldots, g_{n-1})$ be a fundamental system for which this condition is satisfied, the equation which expresses G_i in terms of $1, g_1, g_2, \ldots\ldots, g_{n-1}$ will not contain any of these latter which are of higher dimension than that of G_i. Let the sets $G_1, \ldots\ldots, G_{n-1}, g_1, \ldots\ldots, g_{n-1}$ be each arranged in the ascending order of their dimensions. Then the equations which express $G_1, G_2, \ldots\ldots, G_k$ in terms of $g_1, \ldots\ldots, g_{n-1}$ must contain at least k of the latter functions; for if they contained any less number it would be possible, by eliminating those of the latter functions which occur, to obtain an equation connecting $G_1, \ldots\ldots, G_k$ of the form

$$(x, 1)_\lambda + (x, 1)_{\lambda_1} G_1 + \ldots\ldots + (x, 1)_{\lambda_k} G_k = 0;$$

this is contrary to what is proved under (a).

Hence the dimension of g_k is not greater than the dimension of G_k: hence the sum of the dimensions of $G_1, G_2, \ldots\ldots, G_{n-1}$ is not less than the sum of the dimensions of $g_1, g_2, \ldots\ldots, g_{n-1}$. Hence, *the least value which is possible for the sum of the dimensions of a fundamental set* $(1, G_1, \ldots\ldots, G_{n-1})$ *is that which is the sum of the dimensions for the set* $(1, g_1, \ldots\ldots, g_{n-1})$, *namely, the least value is* $p + n - 1$.

We have given in the last Chapter a definition of p founded on Weierstrass's gap theorem: in the property that the sum of the dimensions of $g_1, \ldots, g_{n-1}$ is $p + n - 1$ we have, as already remarked, another definition, founded on the properties of integral rational functions.

Ex. 1. Prove that if $(1, g_1, \ldots, g_{n-1})$, $(1, h_1, \ldots, h_{n-1})$ be two fundamental sets both having the property that, in the expression of integral functions in terms of them, no terms occur of higher dimension than the function to be expressed, the dimensions of the individual functions of one set are the same as those of the individual functions of the other set, taken in proper order.

Ex. 2. Prove, for the surface

$$y^3 - by^2 + a_1 cy - a_1^2 a_2 = 0,$$

that the function

$$\eta = (y^2 - by + a_1 c)/a_1$$

satisfies the equation

$$\eta^3 - c\eta^2 + a_2 b\eta - a_2^2 a_1 = 0\ ;$$

and that

$$\Delta(1, y, \eta) = b^2c^2 + 18a_1a_2bc - 27a_1^2a_2^2 - 4a_1c^3 - 4a_2b^3,$$

$$\Delta(1, y, y^2) = a_1^2\,\Delta(1, y, \eta) \quad \Delta(1, \eta, \eta^2) = a_2^2\,\Delta(1, y, \eta) \quad \Delta(y, y^2, \eta) = a_1^2c^2\,\Delta(1, y, \eta).$$

In general $1, y, \eta$ are a fundamental set for integral functions, in this case.

44. Let now $(1, g_1, g_2, \ldots\ldots, g_{n-1})$ be any set of integral functions in terms of which any integral function can be expressed in the form

$$(x, 1)_\mu + (x, 1)_{\mu_1} g_1 + \ldots\ldots + (x, 1)_{\mu_{n-1}} g_{n-1},$$

and let the sum of the dimensions of $g_1, \ldots\ldots, g_{n-1}$ be $p + n - 1$.

There will exist integral polynomials in $x, \beta_1, \beta_2, \ldots\ldots, \beta_{n-1}$, such that $\beta_i y^i$ is an integral function: expressing this by $g_1, \ldots\ldots, g_{n-1}$ in the form above and solving for $g_1, \ldots\ldots, g_{n-1}$ we obtain* expressions of which the most general form is

$$g_i = \frac{\mu_{i,\,n-1}\, y^{n-1} + \ldots\ldots + \mu_{i,\,1}\, y + \mu_i}{D_i},$$

where $\mu_{i,\,n-1}, \ldots\ldots, \mu_{i,1}, \mu_i, D_i$ are integral polynomials in x. Denote this expression by $g_i(y, x)$.

Let the equation of the surface, arranged so as to be an integral polynomial in x and y, be written

$$f(y, x) = Q_0 y^n + Q_1 y^{n-1} + \ldots\ldots + Q_{n-1} y + Q_n = 0,$$

and let $\chi_i(y, x)$ denote the polynomial

$$Q_0 y^i + Q_1 y^{i-1} + \ldots\ldots + Q_{i-1} y + Q_i,$$

so that $\chi_0(y, x)$ is Q_0.

Let $\phi_0', \phi_1', \ldots\ldots, \phi'_{n-1}$ be quantities determined by equating powers of y in the *identity*

$$\begin{aligned}\phi_0' + \phi_1' \cdot g_1(y, x) + \phi_2' \cdot g_2(y, x) + \ldots\ldots + \phi'_{n-1} \cdot g_{n-1}(y, x) \\ = \chi_0 y^{n-1} + y^{n-2} \chi_1(y', x) + \ldots\ldots + y\, \chi_{n-2}(y', x) + \chi_{n-1}(y', x):\end{aligned}$$

* Since $g_1, \ldots, g_{n-1}$ are linearly independent.

in other words, if the equations expressing $1, y, y^2, \ldots\ldots, y^{n-1}$ in terms of $g_1(y, x), \ldots\ldots, g_{n-1}(y, x)$ be

$$\begin{aligned} 1 &= 1, \\ y &= a_1 + a_{1,1} g_1 + \ldots\ldots + a_{1,n-1} g_{n-1}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y^{n-1} &= a_{n-1} + a_{n-1,1} g_1 + \ldots\ldots + a_{n-1,n-1} g_{n-1}, \end{aligned}$$

where the coefficient $a_{i,j}$ is an integral polynomial in x divided by β_i, then

$$\begin{aligned} \phi_0' &= \chi_{n-1}(y', x) + a_1 \chi_{n-2}(y', x) + \ldots\ldots + a_{n-1}\chi_0 \\ \phi_1' &= a_{1,1} \chi_{n-2}(y', x) + \ldots\ldots + a_{n-1,1}\chi_0 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \phi'_{n-1} &= a_{1,n-1}\chi_{n-2}(y', x) + \ldots\ldots + a_{n-1,n-1}\chi_0. \end{aligned}$$

So that if we write

$$(1, y, y^2, \ldots\ldots, y^{n-1}) = \Omega\,(1, g, \ldots\ldots, g_{n-1}),$$

Ω being the matrix of the transformation, we have

$$(\phi_0', \phi_1', \ldots\ldots, \phi'_{n-1}) = \overline{\Omega}\,(\chi'_{n-1}, \chi'_{n-2}, \ldots\ldots \chi_1', \chi_0),$$

where $\chi_i' = \chi_i(y', x)$, and $\overline{\Omega}$ represents a transformation whose rows are the columns of Ω, its columns being the rows of Ω.

But if (Q) denote the substitution

$$\left| \begin{array}{llll} Q_{n-1}, & Q_{n-2}, & \ldots\ldots, Q_1, & Q_0 \\ Q_{n-2}, & Q_{n-3}, & \ldots\ldots, Q_0, & 0 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots \\ Q_1, & Q_0, & 0, & \ldots\ldots \\ Q_0, & 0, & \ldots\ldots & \end{array} \right|$$

we have

$$(\chi_{n-1}, \chi_{n-2}, \ldots\ldots, \chi_1, \chi_0) = (Q)\,(1, y, y^2, \ldots\ldots, y^{n-1}).$$

Hence, changing y' to y in ϕ_i' and writing therefore ϕ_i for ϕ_i', we may write

$$(\phi_0, \phi_1, \ldots\ldots, \phi_{n-1}) = \overline{\Omega}\,(Q)\,\Omega\,(1, g_1, g_2, \ldots\ldots, g_{n-1}) \;\ldots\ldots\ldots\ldots(\text{E}).$$

Either this, or the original definition, which is equivalent to

$$\begin{aligned} &\phi_0(y', x) + \phi_1(y', x)\, g_1(y, x) + \ldots\ldots + \phi_{n-1}(y', x)\, g_{n-1}(y, x) \\ &= \frac{f(y', x) - f(y, x)}{y' - y} \\ &= \chi_0 y'^{n-1} + y'^{n-2}\chi_1(y, x) + \ldots\ldots + y'\chi_{n-2}(y, x) + \chi_{n-1}(y, x) \\ &= \chi_0 y^{n-1} + y^{n-2}\chi_1(y', x) + \ldots\ldots + y\chi_{n-2}(y', x) + \chi_{n-1}(y', x) \ldots\ldots(\text{F}), \end{aligned}$$

may be used as the definition of the forms $\phi_0, \phi_1, \ldots\ldots, \phi_{n-1}$.

The latter form will now be further changed for the purposes of an immediate application: let $y_1, \ldots\ldots, y_n$ denote the values of y corresponding to any general value of x for which the values of y are distinct. Denote $\phi_i(y_r, x)$, $g_i(y_r, x)$, by $\phi_i^{(r)}$, $g_i^{(r)}$, etc.

Then putting in (F) in turn $y = y' = y_1$ and $y' = y_1$, $y = y_s$, we obtain

$$\phi_0^{(1)} + \phi_1^{(1)} g_1^{(1)} + \dots\dots + \phi_{n-1}^{(1)} g_{n-1}^{(1)} = \left(\frac{\partial f}{\partial y}\right)_{y_1} = f'(y_1) \text{ say,}$$

$$\phi_0^{(1)} + \phi_1^{(1)} g_1^{(s)} + \dots\dots + \phi_{n-1}^{(1)} g_{n-1}^{(s)} = 0, \qquad (s = 2, 3, \dots\dots, n).$$

Hence if, with arbitrary constant coefficients $c_0, c_1, \dots\dots, c_{n-1}$, we write

$$c_0\phi_0^{(1)} + c_1\phi_1^{(1)} + \dots\dots + c_{n-1}\,\phi_{n-1}^{(1)} = \phi^{(1)},$$

we have

$$\begin{vmatrix} c_0 & c_1 & \dots\dots & c_{n-1} & \phi^{(1)} \\ 1 & g_1^{(1)} & \dots\dots & g_{n-1}^{(1)} & f'(y_1) \\ \dots & \dots & \dots & \dots & \dots \\ 1 & g_1^{(r)} & \dots\dots & g_{n-1}^{(r)} & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & g_1^{(n)} & \dots\dots & g_{n-1}^{(n)} & 0 \end{vmatrix} = 0,$$

or

$$\frac{\phi^{(1)}}{f'(y_1)} \begin{vmatrix} 1 & g_1^{(1)} & \dots\dots & g_{n-1}^{(1)} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 1 & g_1^{(n)} & \dots\dots & g_{n-1}^{(n)} \end{vmatrix} = \begin{vmatrix} c_0 & c_1 & \dots\dots & c_{n-1} \\ 1 & g_1^{(2)} & \dots\dots & g_{n-1}^{(2)} \\ \dots & \dots & \dots & \dots \\ 1 & g_1^{(n)} & \dots\dots & g_{n-1}^{(n)} \end{vmatrix} \dots\dots\dots(G);$$

and we shall find this form very convenient: it clearly takes an indeterminate form for some values of x.

If we put all of $c_1, \dots\dots, c_{n-1}, = 0$ except c_r, and put $c_r = 1$, and multiply both sides of this equation by the determinant which occurs on the left hand, the right hand becomes

$$S_r + S_{r,1}\, g_1^{(1)} + \dots\dots + S_{r,n-1}\, g_{n-1}^{(1)},$$

where, if $s_{i,j} = g_i^{(1)} g_j^{(1)} + g_i^{(2)} g_j^{(2)} + \dots\dots + g_i^{(n)} g_j^{(n)}$, $S_{i,j}$ means the minor of $s_{i,j}$ in the determinant

$$\Delta(1, g_1, g_2, \dots\dots, g_{n-1}) = \begin{vmatrix} n & s_1 & s_2 & \dots\dots & s_{n-1} \\ s_1 & s_{1,1} & s_{1,2} & \dots\dots & s_{1,n-1} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ s_{n-1} & s_{n-1,1} & s_{n-1,2} & \dots\dots & s_{n-1,n-1} \end{vmatrix}$$

Since this is true for every sheet, we therefore have

$$\frac{\phi_r}{f'(y)} = \frac{S_r + S_{r,1}\, g_1 + \dots\dots + S_{r,n-1}\, g_{n-1}}{\Delta(1, g_1, \dots\dots, g_{n-1})}$$

$$= \frac{1}{\Delta}\frac{\partial\Delta}{\partial s_r} + \frac{1}{\Delta}\frac{\partial\Delta}{\partial s_{r,1}}\, g_1 + \dots\dots + \frac{1}{\Delta}\frac{\partial\Delta}{\partial s_{r,n-1}}\, g_{n-1} \dots\dots\dots\dots\dots (H),$$

and therefore, also

$$f'(y)\, g_r = s_r\, \phi_0 + s_{r,1}\, \phi_1 + \ldots\ldots + s_{r,\,n-1}\, \phi_{n-1} \ldots\ldots\ldots\ldots\ldots (\mathrm{H}').$$

The equation (H) has the remarkable property that it determines the functions $\dfrac{\phi_r}{f'(y)}$ from the functions g_i *with a knowledge of these latter only.*

But we can also express $g_1, \ldots\ldots, g_{n-1}$ so that they are determined from $\dfrac{\phi_0}{f'(y)}, \dfrac{\phi_1}{f'(y)}, \ldots\ldots, \dfrac{\phi_{n-1}}{f'(y)}$, *with a knowledge of these only.*

For let these latter be denoted by $\gamma_0, \gamma_1, \ldots\ldots, \gamma_{n-1}$: and, in analogy with the definition of $s_{r,i}$, let $\sigma_{r,i} = \sum\limits_{s=1}^{n} \gamma_r^{(s)}\, \gamma_i^{(s)}$.

Then from equation (H)

$$\sum_{s=1}^{n} \gamma_r^{(s)}\, g_i^{(s)} = \frac{1}{\Delta}\left[S_r s_i + S_{r,1}\, s_{i,1} + \ldots\ldots + S_{r,\,n-1}\, s_{i,\,n-1}\right]$$
$$= 0 \text{ or } 1 \text{ according as } i \neq r \text{ or } i = r.$$

Therefore, also, by equation (H),

$$\sigma_{r,i} = \sum_{s=1}^{n} \gamma_r^{(s)}\, \gamma_i^{(s)} = \frac{1}{\Delta}\left[S_r \sum_{s=1}^{n} \gamma_i^{(s)} + S_{r,1} \sum_{s=1}^{n} g_1^{(s)}\, \gamma_i^{(s)} + \ldots\ldots + S_{r,\,n-1} \sum_{s=1}^{n} g_{n-1}^{(s)}\, \gamma_i^{(s)}\right]$$
$$= \frac{1}{\Delta} S_{r,i},$$

so that equation (H) may be written

$$\gamma_r = \sigma_{r,0} + \sigma_{r,1}\, g_1 + \ldots\ldots + \sigma_{r,\,n-1}\, g_{n-1}.$$

If then $\Sigma_{r,i}$ denote the minor of $\sigma_{r,i}$ in the determinant of the quantities $\sigma_{r,i}$—which determinant we may call $\nabla(\gamma_0, \gamma_1, \ldots\ldots, \gamma_{n-1})$—we have, in analogy with (H),

$$g_r = \frac{1}{\nabla}\left(\Sigma_r\, \gamma_0 + \Sigma_{r,1}\, \gamma_1 + \ldots\ldots + \Sigma_{r,\,n-1}\, \gamma_{n-1}\right) \ldots\ldots\ldots\ldots\ldots (\mathrm{K})^*.$$

Of course $\nabla = \dfrac{1}{\Delta}$ and $\Sigma_{r,i} = \dfrac{1}{\Delta} s_{r,i}$, and equation (K) is the same as (H′).

Ex. 1. Verify that if the integral functions $g_1, \ldots, g_{n-1}$ have the forms

$$g_1(x, y) = \frac{\chi_1(x, y)}{D_1},\ g_2(x, y) = \frac{\chi_2(x, y)}{D_2},\ \ldots,\ g_{n-1}(x, y) = \frac{\chi_{n-1}(x, y)}{D_{n-1}},$$

wherein $D_1, \ldots, D_{n-1}$ are integral polynomials in x, then $\phi_0, \ldots, \phi_{n-1}$ are given by

$$\phi_0(x, y) = y^{n-1},\ \phi_1(x, y) = D_1 y^{n-2},\ \ldots,\ \phi_{n-1}(x, y) = D_{n-1}.$$

* The equations (H) and (K) are given by Hensel. In his papers they arise immediately from the method whereby the forms of $\gamma_1, \gamma_2, \ldots\ldots$ are found.

Ex. 2. Prove from the expressions here obtained that

$$\sum_{s=1}^{n} [\phi_i/f'(y)]_s = 0, \qquad (i=1, 2, \ldots, n-1),$$

and infer that $$\sum_{s=1}^{n} (dv/dx)_s = 0,$$

v being any integral of the first kind.

45. We are now in a position to express the Riemann integrals.

Let $P^{x,\,c}_{x_1,\,x_2}$ be a general integral of the third kind, infinite only at the places x_1, x_2. Writing, in the neighbourhood of x_1, $x - x_1 = t_1^{w_1+1}$, dP/dx will (§§ 14, 16) be infinite like

$$\frac{1}{(w_1+1)\,t_1^{w_1}}\frac{d}{dt_1}\left[\log t_1 + A + A_1 t_1 + A_2 t_1^2 + \ldots\ldots\right],$$

namely, like

$$\frac{1}{w_1+1}\left[\frac{1}{x-x_1} + \frac{A_1}{t_1^{w_1}} + \frac{2A_2}{t_1^{w_1-1}} + \ldots\ldots\right];$$

thus $(x - x_1)\dfrac{dP}{dx}$ is finite at the place x_1 and is there equal to $\dfrac{1}{w_1+1}$. Similarly $(x-x_2)\dfrac{dP}{dx}$ is finite at x_2 and there equal to $-\dfrac{1}{w_2+1}$.

Assume now, first of all, for the sake of simplicity, that at neither $x = x_1$ nor $x = x_2$ are there any branch places; let the finite branch places be at $x = a_1$, $x = a_2$, ...

At any one of these where, say, $x = a + t^{w+1}$, dP/dx is infinite like

$$\frac{1}{(w+1)\,t^w}\frac{d}{dt}[B + B_1 t + B_2 t^2 + \ldots],$$

and therefore $(x-a)\dfrac{dP}{dx}$ is zero to the first order at the place.

Hence, if $$\alpha = (x-a_1)(x-a_2)\ldots$$
be the integral polynomial which vanishes at all the finite branch places of the surface, and g be any integral function whatever, the function

$$K = \alpha \,.\, g \,.\, (x-x_1)(x-x_2)\frac{dP}{dx}$$

is a rational function which is finite for all finite values of x and vanishes at every finite branch place.

Therefore the sum of the values of K in the n sheets, for any value of x, being a symmetrical function of the values of K belonging to that value of x, is a rational function of x only, which is finite for finite values of x and is therefore an integral polynomial in x. Since it vanishes for all the values of

x which make the polynomial α zero, it is divisible by α, and may be written in the form αJ.

Let the polynomial J be written in the form

$$\lambda_1 (x - x_2) - \lambda_2 (x - x_1) + (x - x_1)(x - x_2) H,$$

wherein λ_1 and λ_2 are constants and H is an integral polynomial in x. This is uniquely possible. Let H be of degree $\mu - 1$ in x; denote it by $(x, 1)^{\mu-1}$.

Then, on the whole,

$$\left(g\frac{dP}{dx}\right)_1 + \dots + \left(g\frac{dP}{dx}\right)_n = \frac{\lambda_1}{x - x_1} - \frac{\lambda_2}{x - x_2} + (x, 1)^{\mu-1}.$$

Multiply this equation by $x - x_1$ and consider the case when $x = x_1$, there being by hypothesis no branch place at $x = x_1$. Thus we obtain the value of λ_1; namely, it is the value of g at the place x_1. This we denote by $g(x_1, y_1)$. Similarly λ_2 is $g(x_2, y_2)$. Further, at an infinite place where $x = t^{-(w+1)}$,

$$\frac{dP}{dx} = -\frac{t^{w+2}}{w+1}\frac{dP}{dt},$$

so that $x^2 dP/dx$ is finite at all places $x = \infty$. Hence if $\rho + 1$ be the dimension of the integral function g, and we write

$$\left(gx^{-\rho-1}\,.\,x^2\frac{dP}{dx}\right)_1 + \dots + \left(gx^{-\rho-1}\,.\,x^2\frac{dP}{dx}\right)_n$$
$$= \frac{g(x_1, y_1)}{x^{\rho-1}(x - x_1)} - \frac{g(x_2, y_2)}{x^{\rho-1}(x - x_2)} + \frac{(x, 1)^{\mu-1}}{x^{\rho-1}},$$

we can infer, since ρ cannot be negative, that *μ is at most equal to ρ.*

Hence, taking g in turn equal to $1, g_1, \dots, g_{n-1}$, the dimensions of these functions being denoted by $0, \tau_1 + 1, \dots, \tau_{n-1} + 1$, we have the equations

$$\left(\frac{dP}{dx}\right)_1 + \dots\dots\dots + \left(\frac{dP}{dx}\right)_n = \frac{1}{x - x_1} - \frac{1}{x - x_2},$$

$$g_1^{(1)}\left(\frac{dP}{dx}\right)_1 + \dots + g_1^{(n)}\left(\frac{dP}{dx}\right)_n = \frac{g_1(x_1, y_1)}{x - x_1} - \frac{g_1(x_2, y_2)}{x - x_2} + (x, 1)^{\tau'-1},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$g_{n-1}^{(1)}\left(\frac{dP}{dx}\right)_1 + \dots + g_{n-1}^{(n)}\left(\frac{dP}{dx}\right)_n = \frac{g_{n-1}(x_1, y_1)}{x - x_1} - \frac{g_{n-1}(x_2, y_2)}{x - x_2} + (x, 1)^{\tau'_{n-1} - 1},$$

where $\tau'_1, \dots, \tau'_{n-1}$ are *positive integers not greater than* $\tau_1, \dots, \tau_{n-1}$ *respectively.*

Let these equations be solved for $\left(\frac{dP}{dx}\right)_1$: then in accordance with equations (G) on page 63 we have, after removal of the suffix,

$$f'(y)\frac{dP}{dx}=(x,\,1)^{\tau'_1-1}\phi_1+(x,\,1)^{\tau'_2-1}\phi_2+\ldots+(x,\,1)^{\tau'_{n-1}-1}\phi_{n-1}$$
$$+\frac{\phi_0+\phi_1 g_1(x_1,\,y_1)+\ldots+\phi_{n-1}g_{n-1}(x_1,\,y_1)}{x-x_1}$$
$$-\frac{\phi_0+\phi_1 g_1(x_2,\,y_2)+\ldots+\phi_{n-1}g_{n-1}(x_2,\,y_2)}{x-x_2},$$

where ϕ_i stands for $\phi_i(x,\,y)$.

This, by the method of deduction, is the most general form which dP/dx can have; the coefficients in the polynomials $(x,\,1)^{\tau'_i-1}$ are in number, *at most*,

$$\tau_1+\tau_2+\ldots+\tau_{n-1},$$

or p; and no other element of the expression is undetermined. Now the most general form of dP/dx is known to be

$$\lambda_1\frac{dv_1}{dx}+\ldots+\lambda_p\frac{dv_p}{dx}+\left(\frac{dP}{dx}\right),$$

wherein $\left(\frac{dP}{dx}\right)$ is any special form of $\frac{dP}{dx}$ having the necessary character, and $\lambda_1,\,\ldots,\,\lambda_p$ are arbitrary constants. Hence, by comparison of these forms, we can infer the two results—

(i) The most general form of integral of the first kind is

$$\int^x\frac{dx}{f'(y)}\left[(x,\,1)^{\tau'_1-1}\phi_1(x,\,y)+\ldots+(x,\,1)^{\tau'_{n-1}-1}\phi_{n-1}(x,\,y)\right],$$

wherein $\tau'_i \overline{\gtrless}\, \tau_i$ and the coefficients in $(x,\,1)^{\tau'_i-1}$ are arbitrary:

(ii) A special and actual form of integral of the third kind logarithmically infinite at the two finite, ordinary, places $(x_1,\,y_1)$, $(x_2,\,y_2)$, namely like $\log\,[(x-x_1)/(x-x_2)]$, and elsewhere finite, is

$$\int^x\frac{dx}{f'(y)}\left[\frac{\phi_0(x,\,y)+\phi_1(x,\,y)\,g_1(x_1,\,y_1)+\ldots+\phi_{n-1}(x,\,y)\,g_{n-1}(x_1,\,y_1)}{x-x_1}\right.$$
$$\left.-\frac{\phi_0(x,\,y)+\phi_1(x,\,y)\,g_1(x_2,\,y_2)+\ldots+\phi_{n-1}(x,\,y)\,g_{n-1}(x_2,\,y_2)}{x-x_2}\right],$$

or

$$\int^x\frac{dx}{f'(y)}\int_{x_2}^{x_1}d\xi\,\frac{d}{d\xi}\left[\frac{\phi_0(x,\,y)+\phi_1(x,\,y)\,g_1(\xi,\,\eta)+\ldots+\phi_{n-1}(x,\,y)\,g_{n-1}(\xi,\,\eta)}{x-\xi}\right].$$

In the actual way in which we have arranged the algebraic proof of this result we have only considered values of the current variable x for which the n sheets of the surface are distinct: the reader may verify that the result is valid for all values of x, and can be deduced by means of the definitions of the forms $\phi_0,\,\ldots,\,\phi_{n-1}$, which have been given, other than the equation (G).

Ex. Apply the method to obtain the form of the general integral of the first kind only.

We shall find it convenient sometimes to use a single symbol for the expression

$$\frac{\phi_0(x, y) + \phi_1(x, y)\, g_1(\xi, \eta) + \ldots + \phi_{n-1}(x, y)\, g_{n-1}(\xi, \eta)}{(x-\xi) f'(y)},$$

and may denote it by (x, ξ). Then the result proved is that an elementary integral of the third kind is given by

$$P^{x, c}_{x_1, x_2} = \int_c^x dx\, [(x, x_1) - (x, x_2)].$$

This integral can be rendered normal, that is, chosen so that its periods at the p period loops of the first kind are zero, by the addition of a suitable linear aggregate of the p integrals of the first kind.

Now it can be shewn, as in Chapter II. § 19, that if $E^{x, c}_{\xi}$ denote an elementary integral of the second kind, the function of (x, y) given by the difference

$$D_\xi P^{x, c}_{\xi, x_2} - E^{x, c}_{\xi},$$

wherein D_ξ denotes a differentiation, is not infinite at (ξ, η). It follows from the form of $P^{x, c}_{\xi, x_2}$, here, that this function does not depend upon (x_2, y_2). Hence it is nowhere infinite, as a function of (x, y). Therefore, if not independent of (x, y), it is an aggregate of integrals of the first kind. Thus we infer that one form of an elementary integral of the second kind, which is once algebraically infinite at an *ordinary* place (ξ, η), like $-(x-\xi)^{-1}$, is given by

$$\int^x \frac{dx}{f'(y)} \frac{d}{d\xi} \left[\frac{\phi_0(x, y) + \phi_1(x, y)\, g_1(\xi, \eta) + \ldots + \phi_{n-1}(x, y)\, g_{n-1}(\xi, \eta)}{x - \xi}\right].$$

The direct deduction of the integral of the second kind when the infinity is at a branch place, which is given below, § 47, will furnish another proof of this result.

46. We proceed to obtain the form of an integral of the third kind when one or both of its infinities (x_1, y_1), (x_2, y_2) are at finite branch places; and when there may be other branch places for $x = x_1$ or $x = x_2$.

As before, let α be the integral polynomial vanishing at all the finite branch places. The function

$$g\alpha\,(x - x_1)(x - x_2)\, dP/dx$$

will vanish at all the places $x = x_1$: and though it may vanish at some of these to more than the first order, it will vanish at (x_1, y_1) only to as high order as $(x - x_1)$. Hence the sum of the values of this function in the several sheets for the same value of x is of the form αJ, where J is a polynomial in x which does not vanish, in general, for $x = x_1$ or $x = x_2$.

Hence as before (§ 45) we can write

$$\left(g\frac{dP}{dx}\right)_1 + \dots + \left(g\frac{dP}{dx}\right)_n = \frac{\lambda_1}{x-x_1} - \frac{\lambda_2}{x-x_2} + (x, 1)^{\mu-1}.$$

Multiply this equation by $x - x_1$ and consider the limiting form of the resulting equation as (x, y) approaches to (x_1, y_1): let $w+1$ be the number of sheets which wind at this place. Recalling that the limiting value of $(x-x_1)\,dP/dx$ is $1/(w+1)$, we see that $w+1$ terms of the left hand, corresponding to the $w+1$ sheets at the discontinuity of the integral, will take a form

$$\frac{1}{w+1}\left[1 + A_1 t\epsilon + 2A_2 t^2\epsilon^2 + \dots\right]\left[g(x_1, y_1) + Ct + Dt^2 + \dots\right],$$

where ϵ is a $(w+1)$th root of unity. The limit of this when $t=0$ is $g(x_1, y_1)/(w+1)$; the corresponding terms of the left will therefore have $g(x_1, y_1)$ as limit. The other terms of the left hand will vanish.

Hence $\lambda_1 = g(x_1, y_1)$, $\lambda_2 = g(x_2, y_2)$. The determination of the upper limit for μ and the rest of the deduction proceed exactly as before. Thus,

The expression already given for an integral of the third kind holds whether (x_1, y_1), (x_2, y_2) *be branch places or ordinary places.*

If we denote the form of integral of the third kind thus determined by $P^{x,\, c}_{x_1,\, x_2}$, the zero c being assigned arbitrarily, it follows, as in § 45, above, that an elementary integral of the second kind, which is infinite at a branch place x_1, is given by

$$\lim._{x_1'=x_1}\left[P^{x,\,c}_{x_1',\,x_2} - P^{x,\,c}_{x_1,\,x_2}\right] t_{x_1}^{-1} = \lim._{t_{x_1}=0}\left[\int_c^x dx\,\{(x, x_1') - (x, x_1)\}\right] t_{x_1}^{-1}$$
$$= \lim.\, P^{x,\,c}_{x_1',\,x_1} \,.\, t_{x_1}^{-1}\,.$$

Now if we write t for t_{x_1} and $x_1' = x_1 + t^{w+1}$, the coefficient of $dx/f'(y)$ in the integrand of the form here given for $P^{x,\,c}_{x_1',\,x_1}$ is

$$\frac{\phi_0 + \phi_1\,.\,(g_1 + tg_1' + \dots) + \dots + \phi_{n-1}\,.\,(g_{n-1} + tg'_{n-1} + \dots)}{x - x_1 - t^{w+1}}$$
$$- \frac{\phi_0 + \phi_1\,.\,g_1 + \dots + \phi_{n-1}\,.\,g_{n-1}}{x - x_1},$$

wherein $\phi_0, \dots, \phi_{n-1}$ are functions of x, y, and $g_1, \dots, g_{n-1}, g_1', g_2', \dots$ are written for $g_1(x_1, y_1), \dots, g_{n-1}(x_1, y_1), Dg_1(x_1, y_1), Dg_2(x_1, y_1), \dots$, respectively, D denoting a differentiation in regard to t. Hence the ultimate form is

$$t\,.\,\frac{\phi_1 g_1' + \dots + \phi_{n-1} g'_{n-1}}{x - x_1}\,.$$

That is, introducing ξ, η, instead of x_1, y_1, an elementary integral of the second kind, infinite at a finite branch place (ξ, η), is given by

$$\int^x \frac{dx}{f'(y)} \frac{\phi_1(x, y)\, g_1'(\xi, \eta) + \ldots + \phi_{n-1}(x, y)\, g'_{n-1}(\xi, \eta)}{x - \xi},$$

where $g_1'(\xi, \eta)$, ... are the differential coefficients in regard to the infinitesimal at the place. It has been shewn in (b) § 43 that these differential coefficients cannot be all zero.

Sufficient indications for forming the integrals when the infinities are at infinite places of the surface are given in the examples below (1, 2, 3, ...); in fact, by a linear transformation of the independent variable of the surface we are able to treat places at infinity as finite places.

Ex. 1. Shew that an integral of the third kind with infinities at (x_1, y_1), (x_2, y_2) can also be written in the form

$$\int \frac{dx}{f'(y)} \left[\frac{\lambda_1^{-1} \phi_0(x, y) + \Sigma \lambda_1^{\tau_r} \phi_r(x, y)\, g_r(x_1, y_1)}{x - x_1} - \frac{\lambda_2^{-1} \phi_0(x, y) + \Sigma \lambda_2^{\tau_r} \phi_r(x, y)\, g_r(x_2, y_2)}{x - x_2} \right],$$

wherein $\lambda_1 = (x-a)/(x_1 - a)$, $\lambda_2 = (x-a)/(x_2 - a)$, τ_{r+1} is the dimension of g_r, and a is any arbitrary finite quantity.

It can in fact be immediately verified that the difference between this form and that previously given is an integral of the first kind. Or the result may be obtained by considering the surface with an independent variable $\xi = (x-a)^{-1}$ and using the forms of § 39 of this chapter for the fundamental set for functions infinite only at places $x = a$. The corresponding forms of the functions ϕ are then obtainable by equations (H) § 44.

Ex. 2. Obtain, as in the previous and present Articles, corresponding forms for integrals of the second kind.

Ex. 3. Obtain the forms for integrals of the third and second kinds which have an infinity at a place $x = \infty$.

It is only necessary to find the limits of the results in Examples 1 and 2 as (x_1, y_1) approaches the prescribed place at infinity. It is clearly convenient to take $a = 0$.

Ex. 4. For a surface of the form

$$y^2 = x(x - a_1) \ldots\ldots (x - a_{2p+1}),$$

wherein $a_1, \ldots, a_{2p+1}$ are finite and different from zero and from each other, we may* take the fundamental set $(1, g_1)$ to be $(1, y)$, and so obtain $(\phi_0, \phi_1) = (y, 1)$. Assuming this, obtain the forms of all the integrals, for infinite and for finite positions of the infinities.

Ex. 5. In the case of Example 4 for which $p = 1$, the integral of Example 1, when a is taken 0, is

$$\tfrac{1}{2} \int \frac{dx}{y} \left[\frac{x_1}{x} \frac{y + x^2 x_1^{-2} y_1}{x - x_1} - \frac{x_2}{x} \frac{y + x^2 x_2^{-2} y_2}{x - x_2} \right].$$

Putting $x_1 = \infty$ and $y_1 = m x_1^2 + n x_1 + A + B x_1^{-1} + \ldots$, this takes the form

$$-\tfrac{1}{2} \int \frac{dx}{y} \left[\frac{y + m x^2}{x} + \frac{x_2 y}{(x - x_2)\, x} + \frac{x y_2}{x_2 (x - x_2)} \right]$$

or

$$-\tfrac{1}{2} \int \frac{dx}{y} \left[m x + \frac{y + y_2}{x - x_2} + \frac{y_2}{x_2} \right].$$

* Chap. V. § 56.

Prove that this integral is infinite at one place $x=\infty$ like $\log\left(\frac{1}{x}\right)$ and is otherwise infinite only at (x_2, y_2), namely like $-\log(x-x_2)$, if (x_2, y_2) be not a branch place.

Ex. 6. Prove in Example 5 that the limit of

$$\tfrac{1}{2}\int\frac{dx}{y}\left[\frac{x_1}{x}\frac{y+x^2x_1^{-2}y_1}{x-x_1}+\frac{y+mx^2}{x}\right]$$

as (x_1, y_1) approaches that place (∞, ∞) where $y=mx^2+nx+A+B/x+\ldots$, is

$$-\tfrac{1}{2}\int\frac{dx}{y}(y+mx^2+nx),$$

and that the expansion of this integral in the neighbourhood of this place is

$$-x-\frac{A}{2m}\frac{1}{x}+\ldots\ldots,$$

and that it is otherwise finite. It is therefore an integral of the second kind with this place as its infinity. The process by which the integral is obtained is an example of the method followed in the present and the last Articles, for obtaining an elementary integral of the second kind from an elementary integral of the third kind.

47. We give now a direct deduction of the integral of the second kind whose infinity is at a finite place (ξ, η): we suppose that $(w+1)$ sheets of the surface wind at this place, and find the integral which is there infinite like an expression of the form

$$\frac{A_1}{t}+\frac{A_2}{t^2}+\ldots+\frac{A_w}{t^w}+\frac{A_{w+1}}{x-\xi},$$

t being the infinitesimal at the place.

Firstly, let F be an integral which is infinite like the single term $(x-\xi)^{-1}$, so that in the neighbourhood of the infinity its expansion has a form

$$F=\frac{1}{x-\xi}+A+Bt+Ct^2+\ldots.$$

Forming as before the sum of the values of the functions $g.(x-\xi)^2\,dF/dx$ in the n sheets of the surface, g being any integral function, we obtain an expression

$$\sum_{i=1}^{i=n}\left[g\,(x-\xi)^2\frac{dF}{dx}\right]_i=\lambda+\mu\,(x-\xi)+(x-\xi)^2.(x, 1)^{\mu-1}.$$

Putting $x=\xi$ we infer, since all terms on the left except those belonging to the place (ξ, η) vanish, that

$$\lambda=-(w+1)\,g\,(\xi, \eta).$$

Differentiating, and then putting $x=\xi$, we obtain, from the terms on the left belonging to the infinity,

$$\mu\,\underline{|w+1}=\lim.\Sigma\left\{\frac{d^{w+1}g}{dt^{w+1}}.(x-\xi)^2\frac{dF}{dx}+g.\frac{d^{w+1}}{dt^{w+1}}\left[-1+(x-\xi)^2\frac{d}{dx}(A+Bt+\ldots)\right]\right\},$$

the summation extending to $(w+1)$ terms.

Now

$$\frac{d}{dx}\left[(x-\xi)^2\frac{d}{dx}(A+Bt+\ldots)\right]=\frac{1}{(w+1)^2t^w}\frac{d}{dt}[t^{w+2}(B+2Ct+\ldots)]$$

vanishes when t is zero: hence

$$\mu=-\frac{1}{\underline{|w}}D^{w+1}g(\xi,\eta).$$

Hence we can prove as before that, save for additive terms which are integrals of the first kind, the integral which is infinite like $(x-\xi)^{-1}$ is given by

$$F=-(w+1)\int\frac{dx}{f'(y)}\frac{\phi_0+\phi_1g_1(\xi,\eta)+\ldots+\phi_{n-1}g_{n-1}(\xi,\eta)}{(x-\xi)^2}$$
$$-\frac{1}{\underline{|w}}\int\frac{dx}{f'(y)}\frac{D^{w+1}[\phi_0+\phi_1g_1(\xi,\eta)+\ldots+\phi_{n-1}g_{n-1}(\xi,\eta)]}{x-\xi}.$$

This result is true whether (ξ,η) be a branch place or an ordinary place.

Consider now the integral, say E, which is infinite at (ξ,η) like t^{-m}, m being a positive integer less than $w+1$. At this place, therefore, $(x-\xi)\,dE/dx$ is infinite like $-\frac{m}{w+1}\cdot\frac{1}{t^m}$. If, as before, we consider the sum of the n values of the expression $\alpha\,.\,g\,.\,(x-\xi)\,dE/dx$, wherein g is any integral function and α is the integral polynomial before used, which vanishes at all the finite branch points of the surface, we shall obtain

$$\sum_{i=1}^{n}\left[g\,.\,(x-\xi)\frac{dE}{dx}\right]_i=\lambda+(x-\xi)\,(x,\,1)^{\mu-1}.$$

To find λ, let x approach to ξ. Then all the terms on the left, except those for the $w+1$ sheets which wind at the infinity of E, vanish: for such a non-vanishing term we have an expansion of the form

$$\left[g+tDg+\frac{t^2}{\underline{|2}}D^2g+\ldots\right]\left[-\frac{m}{w+1}\frac{1}{t^m}+A+Bt+Ct^2+\ldots\right],$$

where D denotes, as usual, a differentiation in regard to the infinitesimal of the surface at (ξ,η), and g is written for $g(\xi,\eta)$. The sum of these $w+1$ expansions is

$$-\left[\frac{1}{\underline{|m-1}}D^mg+\frac{m}{w+1}g\Sigma\frac{1}{t^m}+\frac{m}{w+1}g'.\Sigma\frac{1}{t^{m-1}}+\ldots+\frac{m}{w+1}\frac{1}{\underline{|m-1}}D^{m-1}g\,.\,\Sigma\frac{1}{t}\right]$$
$$+(w+1)Ag+(Ag'+Bg)\Sigma t+\ldots.$$

Now in fact every summation Σt^r, being a sum of terms of the form

$$t^r+\epsilon^r t^r+\ldots+\epsilon^{(w+1)r}t^r,$$

wherein ϵ is a primitive $(w+1)$th root of unity, will be zero unless r be a multiple of $w+1$. Thus the terms involving negative powers of t in the sum will vanish: those involving positive powers of t will vanish ultimately when $t=0$; and in fact A is zero, otherwise E would contain the logarithmic term $A\log(x-\xi)$ when (x, y) is near to (ξ, η). Hence on the whole

$$\lambda = -\frac{1}{\lfloor m-1} D^m g(\xi, \eta).$$

Then, proceeding as before, we obtain an expression of the integral in the form,

$$-\frac{1}{\lfloor m-1}\int^x \frac{dx}{f'(y)}\cdot\frac{1}{x-\xi} D^m [\phi_0(x, y) + \dots + \phi_{n-1}(x, y)\, g_{n-1}(\xi, \eta)].$$

Thus, denoting the expression

$$\phi_0(x, y) + \sum_1^{n-1} \phi_r(x, y)\, g_r(\xi, \eta)$$

by Φ, an integral which is infinite like an expression

$$\frac{A_1}{t} + \dots + \frac{A_w}{t^w} + \frac{A_{w+1}}{x-\xi},$$

is given by

$$-(w+1)A_{w+1}\int^x \frac{dx}{f'(y)}\frac{\Phi}{(x-\xi)^2}$$

$$-\int^x \frac{dx}{f'(y)}\cdot\frac{1}{x-\xi}\left[A_1 D + \frac{A_2}{\lfloor 1} D^2 + \frac{A_3}{\lfloor 2} D^3 + \dots + \frac{A_w}{\lfloor w-1} D^w + \frac{A_{w+1}}{\lfloor w} D^{w+1}\right]\Phi.$$

Of course the differentiations at the place (ξ, η) must be understood in the sense in which they arise in the work. If $\phi(\xi, \eta)$ be any function of ξ, η, $D\phi(\xi, \eta)$ means that we substitute in $\phi(x, y)$, for x, $\xi + t^{w+1}$, and for y, an expression of the form $\eta + P(t)$, that we then differentiate this function of t in regard to t, and afterwards regard t as evanescent.

Ex. 1. Obtain this result by repeated differentiation of the integral $P_{\xi, a}^{x, c}$.

Ex. 2. Obtain by the formula the integral which is infinite like $A/t + B/t^2$ in the neighbourhood of (0, 0), the surface being $y^2 = x(x, 1)_3$. Verify that the integral obtained actually has the property required.

48. The determinant $\Delta(1, g_1, \dots, g_{n-1})$, of which the general element is

$$s_{ij} = g_i^{(1)} g_j^{(1)} + \dots + g_i^{(n)} g_j^{(n)},$$

can be written in the form

$$\begin{vmatrix} n, & x^{-\tau_1-1}s_1, & \dots\dots, & x^{-\tau_{n-1}-1}s_{n-1} \\ x^{-\tau_1-1}s_1, & x^{-\tau_1-\tau_2-2}s_{1,2}, & \dots\dots, & x^{-\tau_1-\tau_{n-1}-2}s_{1,n-1} \\ \dots & \dots & \dots & \dots \\ x^{-\tau_{n-1}-1}s_{n-1}, & x^{-\tau_{n-1}-\tau_1-2}s_{n-1,1}, & \dots\dots, & x^{-2\tau_{n-1}-2}s_{n-1,n-1} \end{vmatrix} x^{2n-2+2p}.$$

In this form the determinant factor is finite at every place $x=\infty$: hence also $x^{-(2p-2+2n)}\Delta(1, g_1, \dots, g_{n-1})$ is finite (including zero) at infinity. Thus

$\Delta(1, g_1, \ldots, g_{n-1})$, which is an integral polynomial in x, is of not higher order than $2n-2+2p$ in x.

But when the sheets of the surface for $x=\infty$ are separate, it is not of less order; it is in fact easy to shew that if *for any value of x, $x=a$, there be several branch places, at which respectively w_1+1, w_2+1, ... sheets wind, then $\Delta(1, g_1, \ldots, g_{n-1})$ contains the factor $(x-a)^{w_1+w_2+\cdots}$.*

For, writing, in the neighbourhood of these places respectively,

$$x-a=t_1^{w_1+1},\ x-a=t_2^{w_2+1}, \ldots,$$

the determinant (§ 43)

$$\begin{vmatrix} 1, & g_1^{(1)}, & . & . & , & g_{n-1}^{(1)} \\ 1, & g_1^{(2)}, & . & . & , & g_{n-1}^{(2)} \\ . & . & . & . & & . \\ 1, & g_1^{(n)}, & . & . & , & g_{n-1}^{(n)} \end{vmatrix},$$

of which $\Delta(1, g_1, \ldots, g_{n-1})$ is the square, can, for values of x very near to $x=a$, be written in a form in which one row divides by t_1, another row by t_1^2, ..., another row by $t_1^{w_1}$, in which also another row divides by t_2, another row by t_2^2, ..., and another row by $t_2^{w_2}$, and so on.

Thus this determinant has the factor $t_1^{\frac{1}{2}w_1(w_1+1)}\, t_2^{\frac{1}{2}w_2(w_2+1)} \ldots$, and hence the square of this determinant has the factor $(x-a)^{w_1}(x-a)^{w_2} \ldots$.

Therefore, when there are no branch places at infinity, $\Delta(1, g_1, \ldots, g_{n-1})$ has at least an order Σw, $=2n+2p-2$ (§ 6).

In that case then $\Delta(1, g_1, \ldots, g_{n-1})$ *is exactly of order* $2n+2p-2$: *and, when all the branch places occur for different values of x, its zeros are the branch places of the surface, each entering to its appropriate order.*

When the surface is branched at infinity, choose a value $x=a$ where all the sheets are separate: and let $g_i=(x-a)^{r_i+1} h_i$. Then by putting $\xi=(x-a)^{-1}$ we can similarly prove that $\Delta(1, h_1, \ldots, h_{n-1})$ is an integral polynomial in ξ of precisely the order $2n+2p-2$. But it is immediately obvious that

$$\Delta(1, g_1, \ldots, g_{n-1})=(x-a)^{2n+2p-2}\, \Delta(1, h_1, \ldots, h_{n-1}).$$

Hence if the lowest power of ξ in $\Delta(1, h_1, \ldots, h_{n-1})$ be ξ^s, $\Delta(1, g_1, \ldots, g_{n-1})$ is an integral polynomial of order $2n+2p-2-s$. In this case the zeros of $\Delta(1, g_1, \ldots, g_{n-1})$, which arise for finite values of x, are the branch places, each occurring to its appropriate order, provided all the branch places occur for different values of x: and $\Delta(1, h_1, \ldots, h_{n-1})$ vanishes for $x=\infty$ to an order expressing the number of branch places there.

Ex. 1. For the surface $y^4=x^2(x-1)(x-a)$ there are two branch places at $x=0$, and a branch place at each of the places $x=1$, $x=a$, where all the sheets wind. Thus

$$2n+2p-2=w=2\,.\,1+3+3=8.$$

* Chap. II. § 21.

For this surface fundamental integral functions are given by $g_1=y$, $g_2=y^2/x$, $g_3=y^3/x$. With these values, prove that $\Delta(1, g_1, g_2, g_3)=-256x^2(x-1)^3(x-a)^3$, there being a factor x^2 corresponding to the superimposed branch places at $x=0$, while the other factors are of the same orders as the branch places corresponding to them.

Ex. 2. The surface $y^4=x^2(x-1)$ is similar to that in the last example, but there is a branch place at infinity at which the four sheets wind, so that, in the notation of this Article, $s=3$. As in the last example $2n+2p-2=8$, and $1, y, y^2/x, y^3/x$ are a fundamental system of integral functions. Prove that, now, $\Delta(1, g_1, g_2, g_3)$ is equal to $-256x^2(x-1)^3$, its order in x being $2n+2p-2-s=8-3=5$.

49. In accordance with the previous Chapter* the most general rational function having poles at $p+1$ independent places, is of the form $AF+B$, where F is a special function of this kind and A, B are arbitrary constants. The function will therefore become quite definite if we prescribe the coefficient of the infinite term at one of the $p+1$ poles—the so-called residue there—and also prescribe a zero of the function.

Limiting ourselves to the case where the $p+1$ poles are finite ordinary places of the surface, we proceed, now, to shew that the unique function thus determined can be completely expressed in terms of the functions introduced in this chapter. It will then be seen that we are in a position to express any rational function whatever.

If the general integral of the third kind here obtained with unassigned zero be denoted by $P^z_{x,a}$, the current variables being now (z, s), instead of (x, y), the infinities of the function being at x and a, the function

$$f'(s)\frac{dP^z_{x,a}}{dz}=\frac{\phi_0(z,s)+\phi_1(z,s)\,g_1(x,y)+\ldots\ldots+\phi_{n-1}(z,s)\,g_{n-1}(x,y)}{z-x}$$
$$-\frac{\phi_0(z,s)+\phi_1(z,s)\,g_1+\ldots\ldots+\phi_{n-1}(z,s)\,g_{n-1}}{z-a}$$
$$+\phi_1(z,s)\,(z,1)^{\tau_1-1}+\ldots\ldots+\phi_{n-1}(z,s)\,(z,1)^{\tau_{n-1}-1},$$

wherein $g_1, \ldots, g_{n-1}$ are written for the values of the functions $g_1(z,s), \ldots, g_{n-1}(z,s)$ at the place denoted by a, contains p disposeable coefficients, namely, those in the polynomials $(z,1)^{\tau_1-1}, \ldots\ldots, (z,1)^{\tau_{n-1}-1}$.

Let now $c_1, \ldots\ldots, c_p$ denote p finite, ordinary places of the surface, the values of z at these places being actually $c_1, \ldots, c_p$, which are so situated that the determinant

$$\Delta=\begin{vmatrix}\phi_1^{(1)}, \phi_1^{(1)}c_1, \ldots\ldots, \phi_1^{(1)}c_1^{\tau_1-1}, \ldots\ldots, \phi^{(1)}_{n-1}, \phi^{(1)}_{n-1}c_1, \ldots\ldots, \phi^{(1)}_{n-1}c_1^{\tau_{n-1}-1}\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ \phi_1^{(p)}, \phi_1^{(p)}c_p, \ldots\ldots, \phi_1^{(p)}c_p^{\tau_1-1}, \ldots\ldots, \phi^{(p)}_{n-1}, \phi^{(p)}_{n-1}c_p, \ldots\ldots, \phi^{(p)}_{n-1}c_p^{\tau_{n-1}-1}\end{vmatrix}$$

wherein $\phi_i^{(r)}$ is the value of $\phi_i(z,s)$ at the place c_r, does not vanish. That it is always possible to choose such p places is clear: for if $v_1, \ldots\ldots, v_p$ denote a

* Chap. III. § 37.

set of independent integrals of the first kind, the vanishing of Δ expresses the condition that a rational function of the form

$$f'(s)\left[\lambda_1 \frac{dv_1}{dz} + \ldots\ldots + \lambda_p \frac{dv_p}{dz}\right],$$

involving only $p-1$ disposeable ratios $\lambda_1 : \lambda_2 : \ldots\ldots : \lambda_p$, vanishes at each of the places $c_1, \ldots\ldots, c_p$.

Choose the p coefficients in the function $f'(s)\, dP/dz$, so that this function vanishes at $c_1, \ldots\ldots, c_p$: and denote the function dP/dz, with these coefficients, by $\psi(x, a; z, c_1, \ldots\ldots, c_p)$, so that $\Delta f'(s)\, \psi(x, a; z, c_1 \ldots\ldots c_p)$ is equal to the determinant

$$\begin{vmatrix} [z, x] - [z, a], & \phi_1(z, s), & z\phi_1(z, s), & \ldots, & z^{\tau_1 - 1}\phi_1(z, s), & \ldots, & z^{\tau_{n-1}-1}\phi_{n-1}(z, s) \\ [c_1, x] - [c_1, a], & \phi_1^{(1)}, & c_1\phi_1^{(1)}, & \ldots, & c_1^{\tau_1-1}\phi_1^{(1)}, & \ldots, & c_1^{\tau_{n-1}-1}\phi_{n-1}^{(1)} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ [c_p, x] - [c_p, a], & \phi_1^{(p)}, & c_p\phi_1^{(p)}, & \ldots, & c_p^{\tau_1-1}\phi_1^{(p)}, & \ldots, & c_p^{\tau_{n-1}-1}\phi_{n-1}^{(p)} \end{vmatrix},$$

where $[z, x]$ denotes the expression

$$\frac{\phi_0(z, s) + \phi_1(z, s)\, g_1(x, y) + \ldots + \phi_{n-1}(z, s)\, g_1(x, y)}{z - x}.$$

Suppose now that (z, s) is a finite place, not a branch place, such that none of the minors of the elements of the first row of this determinant vanish. Consider $\psi(x, a; z, c_1, \ldots\ldots, c_p)$ *as a function of* (x, y). It is clearly a rational function; *and is in fact rationally expressed in terms of all the quantities involved.* It is infinite at each of the places $z, c_1, c_2, \ldots\ldots, c_p$—and in fact as x approaches z, the limit of $(z - x)\, \psi(x, a; z, c_1, \ldots\ldots, c_p)$ is the same as that of

$$\frac{\phi_0(z, s) + \Sigma\, \phi_r(z, s)\, g_r(x, y)}{f'(s)},$$

namely, unity (§ 44, F): so that at $x = z$, ψ is infinite like $-(x - z)^{-1}$. And at $c_1, \ldots, c_p$ it is similarly seen to be infinite to the first order.

To obtain its behaviour when x is at infinity, we notice that, by the definition of the dimension of $g_i(x, y)$, the expression

$$\frac{g_i(x, y)}{z - x} + g_i(x, y)\left[\frac{1}{x} + \frac{z}{x^2} + \ldots + \frac{z^{\tau_i - 1}}{x^{\tau_i}}\right],$$

which is of the form

$$-x^{-(\tau_i + 1)}\, g_i(x, y)\left[z^{\tau_i} + \frac{z^{\tau_i + 1}}{x} + \frac{z^{\tau_i + 2}}{x^2} + \ldots\right]$$

is finite for infinite values of x. If then we add to the first column of the determinant which expresses the value of $\Delta f'(s)\, \psi(x, a; z, c_1, \ldots, c_p)$, the following multiples of the succeeding p columns

$$\frac{g_1(x, y)}{x^{\tau_1'}} - \frac{g_1(a, b)}{a^{\tau_1'}},\ \frac{g_2(x, y)}{x^{\tau_2'}} - \frac{g_2(a, b)}{a^{\tau_2'}},\ \ldots (\tau_1' = 1, 2, \ldots, \tau_1;\ \tau_2' = 1, 2, \ldots, \tau_2;\ \ldots),$$

the determinant will contain only quantities which remain finite for infinite values of x.

On the whole then, as the reader can now immediately see, we can summarise the result as follows.

$\psi(x, a; z, c_1, \ldots\ldots, c_p)$ is a rational function of x, having only $p+1$ poles, each of the first order, namely $z, c_1, \ldots\ldots, c_p$. It is infinite at z like $-(x-z)^{-1}$ and it vanishes at $x=a$.

It is immediately seen that if a function of x of the form

$$\lambda_1 \frac{dv_1}{dx} + \ldots\ldots + \lambda_p \frac{dv_p}{dx},$$

which is so chosen that it is zero at all of $c_1, \ldots, c_p$ except c_i and is unity at c_i, be denoted by $\omega_i(x)$, then $\psi(x, a; z, c_1 \ldots c_p)$ is infinite at c_i like $\dfrac{\omega_i(z)}{x-c_i}$.

Let now $R(x, y)$ be a rational function of (x, y) with poles at the finite ordinary places $z_1, z_2, \ldots, z_Q$: let its manner of infinity at z_i be the same as that of $-\lambda_i(x-z_i)^{-1}$. Then the function

$$R(x, y) - \lambda_1 \psi(x, a; z_1, c_1, \ldots, c_p) - \ldots - \lambda_Q \psi(x, a; z_Q, c_1, \ldots, c_p)$$

is a rational function of (x, y) which is only infinite at $c_1, \ldots, c_p$. Since however these latter places are independent*, no such function exists—nor does there exist a rational function infinite only in places falling *among* $c_1, \ldots, c_p$. Hence the function just formed is a constant; thus

$$R(x, y) = \lambda_1 \psi(x, a; z_1, c_1, \ldots, c_p) + \ldots + \lambda_Q \psi(x, a; z_Q, c_1, \ldots, c_p) + \lambda.$$

Conversely an expression such as that on the right hand here will represent a rational function having $z_1, \ldots, z_Q$ for poles, for *all* values of the coefficients $\lambda_1, \ldots, \lambda_Q, \lambda$, which satisfy the conditions necessary that this expression be finite at each of $c_1, \ldots, c_p$; these conditions are expressed by the p equations

$$\lambda_1 \omega_i(z_1) + \lambda_2 \omega_i(z_2) + \ldots + \lambda_Q \omega_i(z_Q) = 0,$$

where $i = 1, 2, \ldots, p$.

When these conditions are independent the function contains therefore

$$Q - p + 1$$

arbitrary constants—in accordance with the result previously enunciated (Chapter III. § 37). The excess arising when these conditions are not independent is immediately seen to be also expressible in the same way as before.

We thus obtain the Riemann-Roch Theorem for the case under consideration.

The function $\psi(x, a; z, c_1, \ldots, c_p)$ will sometimes be called Weierstrass's function. The modification in the expression of it which is necessary when

* In the sense employed Chapter III. § 23.

some of its poles are branch points, will appear in a subsequent utilization of the function (Chapter VII.*). The modification necessary when some of these poles are at infinity is to be obtained, conformably with § 39 of the present chapter by means of the transformation $x = (\xi - m)^{-1}$, whereby the place $x = \infty$ becomes a finite place $\xi = m$.

50. The theory contained in this Chapter can be developed in a different order, on an algebraical basis.

Let the equation of the surface be put into such a form as

$$y^n + y^{n-1} a_1 + \ldots + y a_{n-1} + a_n = 0,$$

wherein $a_1, \ldots, a_n$ are *integral* polynomials in x: so that y is an integral function of x.

By algebraical methods only it can be shewn that a set of integral functions $g_1, \ldots, g_{n-1}$ exists having the property that every integral function can be expressed by them in a form

$$(x, 1)_\lambda + (x, 1)_{\lambda_1} g_1 + \ldots + (x, 1)_{\lambda_{n-1}} g_{n-1},$$

in such a way that no term occurs in the expression which is of higher dimension than the function to be expressed; and that the sum of the dimensions of $g_1, \ldots, g_{n-1}$ is not less than $n-1$ but is less than that of any other set $(1, h_1, \ldots, h_{n-1})$, in terms of which all integral functions can be expressed in such a form as

$$[(x, 1)_\lambda + (x, 1)_{\lambda_1} h_1 + \ldots + (x, 1)_{\lambda_{n-1}} h_{n-1}]/(x, 1)_m.$$

If the sum of the dimensions of $g_1, \ldots, g_{n-1}$ be then written in the form $p + n - 1$, p is called the deficiency of the fundamental algebraic equation.

The expressions of the functions $g_1, g_2, \ldots, g_{n-1}$ being once obtained, and the forms $\phi_0, \phi_1, \ldots, \phi_{n-1}$ thence deduced as in this Chapter, the integrals of the first kind can be shewn, as in this Chapter or otherwise†, to have the form

$$\int \frac{dx}{f'(y)} [(x, 1)^{\tau'_1 - 1} \phi_1 + \ldots\ldots + (x, 1)^{\tau'_{n-1} - 1} \phi_{n-1}],$$

wherein $\tau'_1 \leqq \tau_1$, etc., $\tau_i + 1$ being the dimension of g_i. Thus the number of terms which enter is at most $\tau_1 + \ldots\ldots + \tau_{n-1}$ or p. But it can in fact be shewn algebraically that every one of these terms is an integral of the first kind, namely, that an integral of the form

$$\int \frac{dx}{f'(y)} x^r \phi_i \qquad (i = 1, 2, \ldots\ldots, n-1)$$

is everywhere finite‡ provided $0 \ngtr r \ngtr \tau_i - 1$.

* The reader may, with advantage, consult the early parts (e.g. §§ 122, 130) of that chapter at the present stage.

† Hensel, *Crelle*, 109.

‡ For this we may use the definition (G) or the definition (H) (§ 44). The reader may refer to Hensel, *Crelle*, 105, p. 336.

Then the forms of the integrals of the second and third kind will follow as in this Chapter: and an algebraic theory of the expression of rational functions of given poles can be built up on the lines indicated in the previous article (§ 49) of this Chapter. In this respect Chapter VII. may be regarded as a continuation of the present Chapter.

A method for realising the expressions of $g_1, \ldots, g_{n-1}$ for a given form of fundamental equation is explained in Chapter V. (§ 73).

For Kronecker's determination of a fundamental set of integral functions, for which however the sum of the dimensions is not necessarily so small as $p+n-1$, the reader may refer to the account given in Harkness and Morley, *Theory of Functions*, p. 262. It is one of the points of interest of the system here adopted that the method of obtaining them furnishes an algebraic determination of the deficiency of the surface.