CHAPTER II.

THE FUNDAMENTAL FUNCTIONS ON A RIEMANN SURFACE.

12. IN the present chapter the theory of the fundamental functions is based upon certain *a priori* existence theorems*, originally given by Riemann. At least two other methods might be followed: in Chapters IV. and VI. sufficient indications are given to enable the reader to establish the theory independently upon purely algebraical considerations: from Chapter VI. it will be seen that still another basis is found in a preliminary theory of plane curves. In both these cases the ideas primarily involved are of a very elementary character. Nevertheless it appears that Riemann's descriptive theory is of more than equal power with any other; and that it offers a generality of conception to which no other theory can lay claim. It is therefore regarded as fundamental throughout the book.

It is assumed that the *Theory of Functions* of Forsyth will be accessible to readers of the present book; the aim in the present chapter has been to exclude all matter already contained there. References are given also to the treatise of Harkness and Morley*.

13. Let t be the infinitesimal† at any place of a Riemann surface: if it is a finite place, namely, a place at which the independent variable x is finite, the values of x for all points in the immediate neighbourhood of the place are expressible in the form $x = a + t^{w+1}$: if an infinite place, $x = t^{-(w+1)}$. There exists a function which save for certain additive moduli is one-valued on the whole surface and everywhere finite and continuous, save at the place in question, in the neighbourhood of which it can be expressed in the form

$$\frac{A}{t^r} + \frac{A_1}{t^{r-1}} + \dots + \frac{A_{r-1}}{t} + C + P(t).$$

* See for instance: Forsyth, *Theory of Functions of a Complex Variable*, 1893; Harkness and Morley, *Treatise on the Theory of Functions*, 1893; Schwarz, *Gesam. math. Abhandlungen*, 1890. The best of the early systematic expositions of many of the ideas involved is found in C. Neumann, *Vorlesungen über Riemann's Theorie*, 1884, which the reader is recommended to study. See also Picard, *Traité d'Analyse*, Tom. II. pp. 273, 42 and 77.

† For the notation see Chapter I. §§ 2, 3.

Herein, as throughout, $P(t)$ denotes a series of positive integral powers of t vanishing when $t = 0$, $C, A, \ldots, A_{r-1}$, are constants whose values can be arbitrarily assigned beforehand, and r is a positive integer whose value can be assigned beforehand.

We shall speak of all such functions as integrals of the second kind: but the name will be generally restricted to that *particular function whose behaviour near the place is that of

$$-\frac{1}{t} + C + P(t).$$

This function is not entirely unique. We suppose the surface dissected by $2p$ cuts†, which we shall call period loops; they subserve the purpose of rendering the function one-valued over the whole of the dissected surface. We impose the further condition that the periods of the function for transit across the p loops of the first kind‡ shall be zero; then the function is unique save for an additive constant. It can therefore be made to vanish at an arbitrary place. The special function§ so obtained whose infinity is that of $-\frac{1}{t}$ is then denoted by $\Gamma_a^{x, c}$, c denoting the place where the function vanishes and x the current place. When the infinity is an ordinary place, at which either $x = a$ or $x = \infty$, the function is infinite either like $-\frac{1}{x-a}$ or $-x$. The periods of $\Gamma_a^{x, c}$ for transit of the period loops of the second kind will be denoted by $\Omega_1, \ldots, \Omega_p$.

14. Let $(x_1 y_1)$, $(x_2 y_2)$ be any two places of the surface: and let the infinitesimals be respectively denoted by t_1, t_2, so that in the neighbourhood of these places we have the equations $x - x_1 = t_1^{w_1+1}$, $x - x_2 = t_2^{w_2+1}$. Let a cut be made between the places $(x_1 y_1)$, $(x_2 y_2)$. There exists a function, here denoted by $\Pi_{x_1, x_2}^{x, c}$, which (α) is one-valued over the whole dissected surface, (β) has p periods arising for transit of the period loops of the second kind and has no periods at the period loop of the first kind, (γ) is everywhere continuous and finite save near $(x_1 y_1)$ and $(x_2 y_2)$, where it is infinite respectively like $\log t_1$ and $-\log t_2$, and, (δ), vanishes when the current place denoted by x is the place denoted by c. This function is unique. If the cut between $(x_1 y_1)$, $(x_2 y_2)$ be not made, the function is only definite apart from an additive integral multiple of $2\pi i$, whose value depends on the

* This particular function is also called an *elementary* integral of the second kind.

† Those ordinarily called the a, b curves; see Forsyth, p. 354. Harkness and Morley, p. 242, etc.

‡ Those called the a cuts.

§ The fact that the function has no periods at the period loops of the first kind is generally denoted by calling the function a *normal* integral of the second kind.

path by which the variable is supposed to pass from c. It will be called* the integral of the third kind whose infinity is like that of $\log(t_1/t_2)$.

15. Beside these functions there exist also certain integrals of the first kind—in number p. They are everywhere continuous and finite and one-valued on the dissected surface. For transit of the period loops of the first kind, one of them, say v_i, has no periods except for transit of the i^{th} loop, a_i. This period is here taken to be 1. The periods of v_i for transit of the period loops of the second kind are here denoted by $\tau_{i1}, \ldots, \tau_{ip}$. We may therefore form the scheme of periods

	a_1	a_2	....	a_p	b_1	....	b_p
v_1	1	0		0	τ_{11}		τ_{1p}
v_2	0	1		0	τ_{21}		τ_{2p}
⋮							
v_p	0	0		1	τ_{p1}		τ_{pp}

Each of these functions v_i is unique when a zero is given. They will therefore be denoted by $v_1^{x,c}, \ldots, v_p^{x,c}$, the zero denoted by c being at our disposal.

The periods τ_{ij} have certain properties which will be referred to in their proper place: in particular $\tau_{ij} = \tau_{ji}$, so that they are certainly not equivalent to more than $\frac{1}{2}p(p+1)$ algebraically independent constants. As a fact, in accordance with the previous chapter, when $p > 1$ they are subject to $\frac{1}{2}p(p+1) - (3p-3) = \frac{1}{2}(p-2)(p-3)$ relations.

16. In regard to these enunciations, the reader will notice that the word period here used for that additive constant arising for transit of a period loop —namely, in consequence of a path leading from one edge of the period loop to the opposite edge—would be more properly ~~called~~ the period for circuit of this path than the period for transit of the loop.

The integrals here specified are more precisely called the *normal elementary* integrals of their kinds. The general integral of the first kind is a linear function of $v_1, \ldots, v_p$ with constant coefficients; its periods at the first p loops will not have the same simple forms as have those of $v_1 \ldots v_p$. The general integral of the third kind, infinite like $C\log(t_1/t_2)$, C being a constant, is obtained by adding a general integral of the first kind to $C\Pi_{x_1,x_2}^{x,c}$; similarly for the general integral of the second kind.

The function $\Pi_{x_1,x_2}^{x,c}$ has† the property expressed by the equation

$$\Pi_{x_1,x_2}^{x,c} = \Pi_{x,c}^{x_1,x_2}.$$

* More precisely, the *normal elementary* integral of the third kind.

† Forsyth, p. 453. Harkness and Morley, p. 445.

A more general integral of the third kind having the same property is

$$\Pi_{x_1, x_2}^{x, c} + \sum_{i,j}^{1..p} A_{ij} v_i^{x, c} v_j^{x_1, x_2},$$

wherein the arbitrary coefficients satisfy the equations $A_{ij} = A_{ji}$. The property is usually referred to as the theorem of the interchange of argument (x) and parameter (x_1).

The property allows the consideration of

$$\Pi_{x_1', x_2}^{x, c}$$

as *a function of* x_1' for fixed positions of x, c, x_2. In this regard a remark should be made:

For an ordinary position of x, the function

$$\Pi_{x_1', x_2}^{x, c} - \log(x_1' - x) = \Pi_{x, c}^{x_1', x_2} - \log(x_1' - x)$$

is a finite continuous function of x_1' when x_1' is in the neighbourhood of x. But if x_1 be a branch place where $w+1$ sheets wind, and x_1', x be two positions in its neighbourhood, the functions of x

$$\Pi_{x_1', x_2}^{x, c} - \log(x_1' - x), \quad \Pi_{x_1, x_2}^{x, c} - \frac{1}{w+1}\log(x_1 - x)$$

are respectively finite as x approaches x_1' and x_1, so that

$$\Pi_{x, c}^{x_1', x_2} - \log(x_1' - x)$$

is not a finite and continuous function *of* x_1' for positions of x_1' up to and including the branch place x_1.

In this case, let the neighbourhood of the branch place be conformally represented upon a simple plane closed area and let ξ_1, ξ_1', ξ be the representatives thereon of the places x_1, x_1', x. Then the correct statement is that

$$\Pi_{x, c}^{x_1', x_2} - \log(\xi_1' - \xi)$$

is a continuous function of x_1' or ξ_1' up to and including the branch place x_1. This is in fact the form in which the function $\Pi_{x, c}^{x_1', x_2}$ arises in the proof of its existence upon which our account is based*.

In a similar way the function

$$\Gamma_{x_1'}^{x, c},$$

regarded as a function of x_1', is such that

$$\Gamma_{x_1'}^{x, c} + \frac{1}{\xi - \xi_1'}$$

is a finite continuous function of ξ_1' in the immediate neighbourhood of x.

* The reader may consult Neumann, p. 220.

17. It may be desirable to give some simple examples of these integrals.

(*a*) For the surface represented by

$$y^2 = x(x-a_1)\ldots(x-a_{2p+1}),$$

wherein $a_1, \ldots, a_{2p+1}$ are all finite and different from zero and each other, consider the integral

$$\tfrac{1}{2}\int\frac{dx}{y}\left(\frac{y+\eta}{x-\xi}-\frac{y+\eta_1}{x-\xi_1}\right),$$

(ξ, η), (ξ_1, η_1) being places of the surface other than the branch places, which are

$$(0,\ 0), (a_1,\ 0), \ldots, (a_{2p+1},\ 0).$$

It is clearly infinite at these places respectively like $\log(x-\xi)$, $-\log(x-\xi_1)$.

It is not infinite at $(\xi, -\eta)$, $(\xi_1, -\eta_1)$; for $(y+\eta)/(x-\xi)$, $(y+\eta_1)/(x-\xi_1)$ are finite at these places respectively.

At a place $x=\infty$, where $x=t^{-1}$, $y=\epsilon t^{-p-1}(1+P_1(t))$, ϵ being ± 1, and $P_1(t)$ a series of positive integral powers of t vanishing for $t=0$, we have

$$\frac{y+\eta}{y}=1+\eta\epsilon t^{p+1}(1+P_1(t)),\ \frac{dx}{x-\xi}=-\frac{dt}{t}(1-\xi t)^{-1},$$

and the integral has the form

$$-\tfrac{1}{2}\int\frac{dt}{t}\,t\,[A+P_2(t)],$$

A being a constant. It is therefore finite.

At a place $y=0$, for instance where

$$x=a_1+t^2,\ y=Bt\,[1+P_3(t)],$$

B being a constant, the integral has the form

$$C\int dt\,[1+P_4(t)],$$

C being a constant, and is finite.

Thus it is an elementary integral of the third kind with infinities at (ξ, η), (ξ_1, η_1).

It may be similarly shewn that the integral

$$\tfrac{1}{2}\int\frac{dx}{x}\left(\frac{y}{x}-\frac{y+\eta_1}{x-\xi_1}\right)$$

is infinite at (ξ_1, η_1) like $-\log(x-\xi_1)$ and is not elsewhere infinite except at (0, 0).

Near (0, 0), we have $x=t^2$, $y=Dt\,[1+P_5(t^2)]$ and this integral is infinite like

$$\int\frac{dt}{t}=\log t.$$

It is therefore an elementary integral of the third kind with one infinity at the branch place (0, 0) and the other at (ξ_1, η_1).

Consider next the integral

$$\tfrac{1}{2}\int\frac{dx}{y}\,\frac{d}{d\xi}\left(\frac{y+\eta}{x-\xi}\right)=\tfrac{1}{2}\int\frac{dx}{y}\,\frac{y+\eta+(x-\xi)\,\eta'}{(x-\xi)^2},$$

where $\eta'=\dfrac{d\eta}{d\xi}$. It can easily be seen that it is not infinite save at (ξ, η). Writing for the neighbourhood of this place, which is supposed not to be a branch place,

$$y=\eta+(x-\xi)\,\eta'+\tfrac{1}{2}(x-\xi)^2\,\eta''+\ldots\ldots,$$

the integral becomes

$$\int \frac{dx}{(x-\xi)^2} \frac{\eta+(x-\xi)\eta'+\frac{1}{4}(x-\xi)^2\eta''+\dots}{\eta+(x-\xi)\eta'+\frac{1}{2}(x-\xi)^2\eta''+\dots},$$

which is equal to

$$\int \frac{dx}{(x-\xi)^2}\left[1-\frac{1}{4}\frac{\eta''}{\eta}(x-\xi)^2+\dots\right].$$

Thus the integral is there infinite like $-\frac{1}{x-\xi}$, and is thus an elementary integral of the second kind.

The elementary integral of the second kind for a branch place, say (0, 0), is a multiple of

$$\frac{1}{2}\int \frac{dx}{xy}.$$

In fact near $x=0$, writing $x=t^2$, $y=Dt[1+P(t^2)]$, this integral becomes

$$\frac{1}{D}\int \frac{dt}{t^2}[1+P(t^2)]^{-1}$$

or

$$\frac{1}{D}\int \frac{dt}{t^2}[1+Et^2+Ft^4+\dots]$$

which is equal to

$$\frac{1}{D}\left[-\frac{1}{t}+Et+\dots\right]$$

as desired.

The integral is clearly not infinite elsewhere.

Example 1. Verify that the integral last considered is the limit of

$$\frac{1}{2D}\int \frac{dx}{y}\left[\frac{y+\eta}{x-\xi}-\frac{y}{x}\right]$$

as the place (ξ, η) approaches indefinitely near to (0, 0).

Example 2. Shew that the general integral of the first kind for the surface is

$$\int \frac{dx}{y}(A_1+A_2x+\dots+A_{p-1}x^{p-1}).$$

(β) We have in the first chapter §§ 2, 3 spoken of a circumstance that can arise, that two sheets of the surface just touch at a point and have no further connexion, and we have said that we regard the points of the sheets as distinct places. Accordingly we may have an integral of the third kind which has its infinities at these two places, or an integral of the third kind having one of its infinities at one of these places. For example, on the surface

$$f(x, y)=(y-m_1x)(y-m_2x)+(x, y)_3+(x, y)_4=0$$

where $(x, y)_3$, $(x, y)_4$ are integral homogeneous polynomials of the degrees indicated by the suffixes, with quite general coefficients, and m_1, m_2 are finite constants, there are at $x=0$ two such places, at both of which $y=0$.

In this case

$$\int \frac{dx}{f'(y)},$$

where $f'(y)=\frac{\partial f}{\partial y}$, is a constant multiple of an integral of the third kind with infinities at these two places (0, 0); and

$$\int \frac{y-m_1x+Ax^2+Bxy+Cy^2}{Lx+My}\frac{dx}{f'(y)}$$

is a constant multiple of an integral of the third kind, provided A, B, C be so chosen that $y-m_1x+Ax^2+Bxy+Cy^2$ vanishes at one of the two places other than (0, 0) at which $Lx+My$ is zero. Its infinities are at (i) the uncompensated zero of $Lx+My$ which is not at (0, 0), (ii) the place (0, 0) at which the expression of y in terms of x is of the form

$$y=m_2x+Px^2+Qx^3+\ldots$$

In fact, at a branch place of the surface where $x=a+t^2$, $f'(y)$ is zero of the first order, and $dx=2t\,dt$; thus $\int\frac{dx}{f'(y)}$ is finite at the branch places. At each of the places (0, 0), $f'(y)$ is zero of the first order, $Lx+My$ is zero of the first order and $y-m_1x+Ax^2+Bxy+Cy^2$ is zero at these places to the first and second order respectively. These statements are easy to verify; they lead immediately to the proof that the integrals have the character enunciated.

The condition given for the choice of A, B, C will not determine them uniquely—the integral will be determined save for an additive term of the form

$$\int(Px+Qy)\frac{dx}{f'(y)},$$

where P, Q are undetermined constants. The reader may prove that this is a general integral of the first kind. The constants P, Q may be determined so that the integral of the third kind has no periods at the period loops of the first kind, whose number in this case is two. The reasons that suggest the general form written down will appear in the explanation of the geometrical theory.

(γ) The reader may verify that for the respective cases

$$y^6=(x-a)(x-b)^2(x-c)^3,$$
$$y^4=(x-a)(x-b)\ (x-c)^2,$$
$$y^6=(x-a)(x-b)\ (x-c)^4,$$
$$y^7=(x-a)(x-b)\ (x-c)^5,$$

the general integrals of the first kind are

$$\int\frac{dx}{y^5}(x-b)(x-c)^2,$$
$$\int\frac{dx}{y^3}(x-c),$$
$$\int\frac{dx}{y^5}(x-c)^2[Ay+B(x-c)],$$
$$\int\frac{dx}{y^6}(x-c)^2[Ay^2+By(x-c)+C(x-c)^2],$$

where A, B, C are arbitrary constants.

See an interesting dissertation "de Transformatione aequationis $y^n=R(x)$..." Eugen. Netto (Berlin, Gust. Schade, 1870).

(δ) *Ex.* Prove that if F denote any function everywhere one valued on the Riemann surface and expressible in the neighbourhood of every place in the form

$$\frac{A_1}{t}+\frac{A_2}{t^2}+\ldots+B+B_1t+B_2t^2+\ldots$$

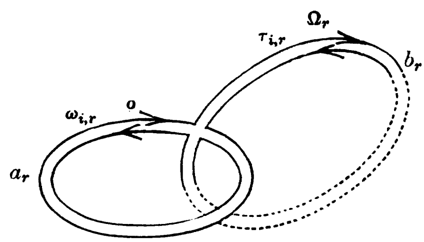
the sum of the coefficients of the logarithmic terms $\log t$ of the integral $\int^x F\,dx$, for all places where such a term occurs, is zero.

It is supposed that the number of places where negative powers of t occur in the expansion of F is finite, but it is not necessary that the number of negative powers be finite. The theorem may be obtained by contour integration of $\int F dx$, and clearly generalizes a property of the integral of the third kind.

18. The value of the integral* $\int \Gamma_a^{x,c}\, dv_i^{x,c}$ taken round the p closed curves formed by the two sides of the pairs of period loops $(a_1, b_1), \ldots, (a_p, b_p)$, in such a direction that the interior of the surface is always on the left hand, is equal to the value taken round the sole infinity, namely the place a, in a counter-clockwise direction. Round the pair a_r, b_r the value obtained is

$$\Omega_r \int dv_i^{x,c},$$

taken once positively in the direction of the arrow head round what in the figure is the outer side of b_r. This value is $\Omega_r(-\omega_{ir})$, where ω_{ir} denotes the period of v_i for transit of a_r, namely, from what in the figure is the inside of the oval a_r to the outside.



The relations indicated by the figure for the signs adopted for ω_{ir}, τ_{ir} and the periods of $\Gamma_a^{x,c}$ will be preserved throughout the book.

Since ω_{ir} is zero except when $r = i$, the sum of these p contour integrals is $-\omega_{i,i}\,\Omega_i$. Taken in a counter-clockwise direction, round the pole of $\Gamma_a^{x,c}$, where

$$\Gamma_a^{x,c} = -\frac{1}{t} + A + Bt + Ct^2 + \ldots,$$

the integral gives

$$\int \left[-\frac{1}{t} + A + Bt + Ct^2 + \ldots \right] \left[Dv_i^{a,c} + tD^2 v_i^{a,c} + \ldots \right] dt,$$

where D denotes $\dfrac{d}{dt}$. Hence, as $\omega_{i,i} = 1$,

$$\Omega_i = 2\pi i \left(Dv_i^{x,c} \right)_a.$$

* Cf. Forsyth, pp. 448, 451. Harkness and Morley, p. 439.

This is true whether a be a branch place or a place at infinity (for which, if not a branch place, $x = t^{-1}$) or an ordinary finite place. In the latter case

$$\Omega_i = 2\pi i \frac{d}{dx}\left(v_i^{x,c}\right).$$

Similarly the reader may prove that the periods of $\Pi_{x_1,x_2}^{x,c}$ are

$$0, \ldots\ldots 0,\ 2\pi i v_i^{x_1,x_2}, \ldots\ldots\ 2\pi i v_p^{x_1,x_2}.$$

In this case it is necessary to enclose x_1 and x_2 in a curve winding w_1+1 times at x_1, w_2+1 times at x_2, in order that this curve may be closed.

19. From these results we can shew that the integral of the second kind is derivable by differentiation from the integral of the third kind. Apart from the simplicity thus obtained, the fact is interesting because, as will appear, the analytical expression of an integral of the third kind is of the same general form whether its infinities be branch places or not; this is not the case for integrals of the second kind.

We can in fact prove the equation

$$D_{t_{x_1}} \Pi_{x_1,x_2}^{x,c} = \Gamma_{x_1}^{x,c},$$

namely, if, to take the most general case, x_1 be a winding place and x_1' a place in its neighbourhood such that $x_1' = x_1 + t_{x_1}^{w+1}$, the equation,

$$\lim_{t_{x_1}=0} \frac{1}{t_{x_1}}\left[\Pi_{x_1',x_2}^{x,c} - \Pi_{x_1,x_2}^{x,c}\right] = \Gamma_{x_1}^{x,c}.$$

For, let the neighbourhood of the branch place x_1 be conformally represented upon a simple closed area without branch place, by means of the infinitesimal of x, as explained in the previous chapter. Let ξ_1', ξ_1 be the representatives of the places x_1', x_1, and ξ the representative of a place x which is very near to x_1, but is so situate that we may regard x_1' as ultimately infinitely closer to x_1 than x is.

Then
$$x - x_1 = (\xi - \xi_1)^{w+1},$$
$$x - x_1' = (\xi - \xi_1')\,[C + P(\xi - \xi_1')],$$

where C does not vanish for $x_1' = x$,

and
$$\Pi_{x_1',x_2}^{x,c} = \log(x - x_1') + \Phi' = \log(\xi - \xi_1') + \phi',$$

where ϕ' is finite for the specified positions of the places *and remains finite when* ξ_1' *is taken infinitely near to* ξ_1 (§ 16).

Also
$$\Pi_{x_1,x_2}^{x,c} = \frac{1}{w+1}\log(x - x_1) + \phi = \log(\xi - \xi_1) + \phi,$$

where ϕ is also finite. Therefore

$$\Pi^{x, c}_{x_1', x_2} - \Pi^{x, c}_{x_1, x_2} = \log\left(1 - \frac{\xi_1' - \xi_1}{\xi - \xi_1}\right) + \phi' - \phi$$

$$= -(\xi_1' - \xi_1)\left[\frac{1}{\xi - \xi_1} + \tfrac{1}{2}\frac{\xi_1' - \xi_1}{(\xi - \xi_1)^2} + \dots\right] + \phi' - \phi,$$

and thus

$$\lim_{\xi_1' = \xi_1} \left[\frac{\Pi^{x, c}_{x_1', x_2} - \Pi^{x, c}_{x_1, x_2}}{\xi_1' - \xi_1}\right] = -\frac{1}{\xi - \xi_1} + \psi,$$

where ψ is finite.

Now as ξ_1' moves up to ξ_1, for a fixed position of ξ, we have

$$\xi_1' - \xi_1 = (x_1' - x_1)^{\frac{1}{w+1}} = t_{x_1},$$

and

$$\Gamma^{x, c}_{x_1} = \Gamma^{\xi, c}_{\xi_1} = -\frac{1}{\xi - \xi_1} + \vartheta,$$

where ϑ is finite.

Hence

$$D_{t_{x_1}} \Pi^{x, c}_{x_1, x_2} - \Gamma^{x, c}_{x_1}$$

is finite when x is near to x_1.

Moreover it does not depend on x_2. For from the equation

$$\Pi^{x, c}_{x_1, x_2} = \Pi^{x_1, x_2}_{x, c},$$

we may regard $\Pi^{x, c}_{x_1, x_2}$ as a function of x_1, which is determinate save for an additive constant by the specification of x and c only. This additive constant, which is determined by the condition that the function vanishes when $x_1 = x_2$, is the only part of the function which depends on x_2. It disappears in the differentiation.

Finally, by the determination of the periods previously given, it follows that

$$D_{t_{x_1}} \Pi^{x, c}_{x_1, x_2} - \Gamma^{x, c}_{x_1}$$

has no periods at the $2p$ period loops. Hence it is a constant, and therefore zero since it vanishes when $x = c$.

Corollary i.

Hence $$D_{t_x} \Gamma^{x, c}_{x_1} = D_{t_x} D_{t_{x_1}} \Pi^{x, c}_{x_1, x_2} = D_{t_{x_1}} D_{t_x} \Pi^{x_1, x_2}_{x, c} = D_{t_{x_1}} \Gamma^{x_1, c}_{x},$$

of which neither depends on the constant position c.

Corollary ii.

The functions

$$D_{t_{x_1}} \Gamma^{x, c}_{x_1},\ D^2_{t_{x_1}} \Gamma^{x, c}_{x_1},\ D^3_{t_{x_1}} \Gamma^{x, c}_{x_1},\ \dots$$

are respectively infinite like

$$-\frac{1}{t_{x_1}{}^2},\quad -\frac{\lfloor 2}{t_{x_1}{}^3},\quad -\frac{\lfloor 3}{t_{x_1}{}^4},\ \ldots.$$

We shall generally write $D_{x_1}, D^2_{x_1}, \ldots$ instead of $D_{t_{x_1}}, D^2_{t_{x_1}}, \ldots$. When x_1 is an ordinary place D_{x_1} will therefore mean $\frac{d}{dx_1}$, etc.

Corollary iii.

By means of the example (δ) of § 17 it can now be shewn that the infinite parts of the integral

$$\int F dx,$$

in which F is any uniform function of position on the undissected surface having only infinities of finite order, are those of a sum of terms consisting of proper constant multiples of integrals of the third kind and differential coefficients of these in regard to the parametric place.

20. One particular case of Cor. iii. of the last Article should be stated. A function which is everywhere one-valued on the undissected surface must be somewhere infinite. As in the case of uniform functions on a single infinite plane (which is the particular case of a Riemann surface for which the deficiency is zero), such functions can be divided into rational and transcendental, according as all their infinities are of finite order and of finite number or not. Transcendental functions which are uniform on the surface will be more particularly considered later. A rational uniform function can be expressed rationally in terms of x and y*. But since the function can be expressed in the neighbourhood of any of its poles in the form

$$C + \frac{A_1}{t} + \frac{A_2}{t^2} + \ldots + \frac{A_m}{t^m} + P(t),$$

we can, by subtracting from the function a series of terms of the form

$$-\left[A_1 \Gamma_a^{x, c} + A_2 D_a \Gamma_a^{x, c} + \ldots + \frac{A_m}{\lfloor m-1} D_a^{m-1}\, \Gamma_a^{x, c}\right],$$

obtain a function nowhere infinite on the surface and having no periods at the first p period loops. Such a function is a constant†. Hence F can also be expressed by means of normal integrals of the second kind only. Since F has no periods at the period loops of the second kind there are for all rational functions certain necessary relations among the coefficients $A_1, \ldots, A_m$. These are considered in the next Chapter.

* Forsyth, p. 369. Harkness and Morley, p. 262.
† Forsyth, p. 439.

21. Of all rational functions there are p whose importance justifies a special mention here; namely, the functions

$$\frac{dv_1}{dx}, \frac{dv_2}{dx}, \dots \frac{dv_p}{dx}.$$

In the first place, these cannot be all zero for any ordinary finite place a of the surface. For they are, save for a factor $2\pi i$, the periods of the normal integral $\Gamma_a^{x, c}$. If the periods of this integral were zero, it would be a rational uniform function of the first order; in that case the surface would be representable conformally upon another surface of one sheet*, $\xi = \Gamma_a^{x, c}$ being the new independent variable; and the transformation would be reversible (Chap. I. § 6). Hence the original surface would be of deficiency zero; in which case the only integral of the first kind is a constant. The functions are all infinite at a branch place a. But it can be shewn as here that the quantities to which they are there proportional, namely $D_a v_1, \dots, D_a v_p$, cannot be all zero. The functions are all zero at infinity, but similarly it can be shewn that the quantities, $Dv_1, \dots, Dv_p$, cannot be all zero there.

Thus p linearly independent linear aggregates of these quantities cannot all vanish at the same place. We remark, in connexion with this property, that surfaces exist of all deficiencies such that $p-1$ linearly independent linear aggregates of these quantities vanish in an infinite number of sets of *two* places. Such surfaces are however special, and their equation can be put† into the form

$$y^2 = (x, 1)_{2p+2}.$$

We have seen that the statement of the property requires modification at the branch places, and at infinity; this particularity is however due to the behaviour of the independent variable x. We shall therefore state the property by saying: there is no place at which all the differentials $dv_1, \dots, dv_p$ vanish. A similar phraseology will be adopted in similar cases. For instance, we shall say that each of $dv_1, dv_2, \dots, dv_p$ has‡ $2p-2$ zeros, some of which may occur at infinity.

In the next place, since any general integral of the first kind

$$\lambda_1 v_1^x + \dots + \lambda_p v_p^x$$

must necessarily be finite all over any other surface upon which the original surface is conformally and reversibly represented and therefore must be an integral of the first kind thereon, it follows that the rational function

$$\lambda_1 \frac{dv_1}{dx} + \dots + \lambda_p \frac{dv_p}{dx}$$

* I owe this argument to Prof. Klein.

† See below, Chap. V.

‡ See Forsyth, p. 461. Harkness and Morley, p. 450.

is necessarily transformed with the surface into

$$M\left(\lambda_1 \frac{dV_1}{d\xi} + \dots + \lambda_p \frac{dV_p}{d\xi}\right),$$

where $V_i = v_i$ is an integral of the first kind, not necessarily normal, on the new surface, ξ being the new independent variable, and $M = \frac{d\xi}{dx}$.

Thus, the ratios of the integrands of the first kind are transformed into ratios of integrands of the first kind; they may be said to be invariant for birational transformation.

This point may be made clearer by an example. The general integral of the first kind for the surface

$$y^2 = (x, 1)_8$$

can be shewn to be

$$\int \frac{dx}{y} (A + Bx + Cx^2),$$

A, B, C being arbitrary constants.

If then $\phi_1 : \phi_2 : \phi_3$ denote the ratios of any three linearly independent integrands of the first kind for this surface, we have

$$1 : x : x^2 = a_1\phi_1 + b_1\phi_2 + c_1\phi_3 : a_2\phi_1 + b_2\phi_2 + c_2\phi_3 : a_3\phi_1 + b_3\phi_2 + c_3\phi_3$$

for proper values of the constants $a_1, b_1, \dots, c_3$,

and hence

$$(a_1\phi_1 + b_1\phi_2 + c_1\phi_3)(a_3\phi_1 + b_3\phi_2 + c_3\phi_3) = (a_2\phi_1 + b_2\phi_2 + c_2\phi_3)^2.$$

Such a relation will therefore hold for all the surfaces into which the given one can be birationally transformed.

22. It must be remarked that the determination of the normal integrals here described depends upon the way in which the fundamental period loops are drawn. An integral of the first kind which is normal for one set of period loops will be a linear function of the integrals of the first kind which are normal for another set; and an integral of the second or third kind, which is normal for one set of period loops, will for another set differ from a normal integral by an additive linear function of integrals of the first kind.