CHAPTER XIX

ELLIPTIC FUNCTIONS AND INTEGRALS

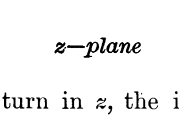
187. Legendre's integral I and its inversion. Consider

$$z = \int_0^w \frac{dw}{\sqrt{(1 - w^2)(1 - k^2w^2)}}, \qquad 0 < k < 1. \tag{I}$$

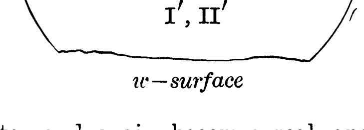
The Riemann surface for the integrand* has branch points at $w = \pm 1$ and $\pm 1/k$ and is of two sheets. Junction lines may be drawn between $+1, +1/k$ and $-1, -1/k$. For very large values of w, the radical $\sqrt{(1 - w^2)(1 - k^2w^2)}$ is approximately $\pm kw^2$ and hence there is no danger of confusing the values of the function. Across the junction lines the surface may be connected as indicated, so that in the neighborhood of $w = \pm 1$ and $w = \pm 1/k$ it looks like the surface for $\sqrt{w}$. Let $+1$ be the value of the integrand at $w = 0$ in the upper sheet. Further let

$$K = \int_0^1 \frac{dw}{\sqrt{(1 - w^2)(1 - k^2w^2)}}, \qquad iK' = \int_1^{\frac{1}{k}} \frac{dw}{\sqrt{(1 - w^2)(1 - k^2w^2)}}. \tag{1}$$

Let the changes of the integral be followed so as to map the surface on the z-plane. As w moves from o to a, the integral (I) increases by K, and z moves from O to A. As w continues straight on, z makes a right-angle turn and increases by pure imaginary increments to the total amount iK' when w reaches b. As w continues there is another right-angle turn in z, the integrand again becomes real, and z moves down to C. (That z reaches C follows from the facts that the



z—plane



w—surface

* The reader unfamiliar with Riemann surfaces (§ 184) may proceed at once to identify (I) and (2) by Ex. 9, p. 475 and may take (1) and other necessary statements for granted.

integral along an infinite quadrant is infinitesimal and that the direct integral from 0 to $i\infty$ would be pure imaginary like dw.) If w is allowed to continue, it is clear that the map of I will be a rectangle $2K$ by K' on the z-plane. The image of all four half planes of the surface is as indicated. The conclusion is reasonably apparent that w as the inverse function of z is doubly periodic with periods $4K$ and $2iK'$.

The periodicity may be examined more carefully by considering different possibilities for paths upon the surface. A path surrounding the pairs of branch points 1 and k^{-1} or -1 and $-k^{-1}$ will close on the surface, but as the integrand has opposite signs on opposite sides of the junction lines, the value of the integral is $2iK'$. A path surrounding -1, $+1$ will also close; the small circuit integrals about -1 or $+1$ vanish and the integral along the whole path, in view of the opposite values of the integrand along fa in I and II, is twice the integral from f to a or is $4K$. Any path which closes on the surface may be resolved into certain multiples of these paths. In addition to paths which close on the surface, paths which close in w may be considered. Such paths may be resolved into those already mentioned and paths running directly between 0 and w in the two sheets. All possible values of z for any w are therefore $4mK + 2niK' \pm z$. The function $w(z)$ has the periods $4K$ and $2iK'$, is an odd function of z as $w(-z) = w(z)$, and is of the second order. The details of the discussion of various paths is left to the reader.

Let $w = f(z)$. The function $f(z)$ vanishes, as may be seen by the map, at the two points $z = 0,\ 2K$ of the rectangle of periods, and at no other points. These zeros of w are simple, as $f'(z)$ does not vanish. The function is therefore of the second order. There are poles at $z = iK',\ 2K + iK'$, which must be simple poles. Finally $f(K) = 1$. The position of the zeros and poles determines the function except for a constant multiplier, and that will be fixed by $f(K) = 1$; the function is wholly determined. The function $f(z)$ may now be identified with $\operatorname{sn} z$ of § 177 and in particular with the special case for which K and K' are so related that the multiplier $g = 1$.

$$w = f(z) = \frac{\Theta(K)}{H(K)}\frac{H(z)}{\Theta(z)} = \operatorname{sn} z, \qquad z = u. \tag{2}$$

For the quotient of the theta functions has simple zeros at $0,\ 2K$, where the numerator vanishes, and simple poles at $iK',\ 2K + iK'$, where the denominator vanishes; the quotient is 1 at $z = K$; and the derivative of $\operatorname{sn} z$ at $z = 0$ is $g \operatorname{cn} 0 \operatorname{dn} 0 = g = 1$, whereas $f'(0) = 1$ is also 1.

The imposition of the condition $g = 1$ was seen to impose a relation between $K,\ K',\ k,\ k',\ q$ by virtue of which only one of the five remained independent. The definition of K and K' as definite integrals also makes them functions $K(k)$ and $K'(k)$ of k. But

$$
\begin{aligned}
iK'(k) = \int_1^{\frac{1}{k}} &\frac{dw}{\sqrt{(1-w^2)(1-k^2w^2)}} \\
&= i\int_0^1 \frac{dw_1}{\sqrt{(1-w_1^2)(1-k'^2w_1^2)}} = iK(k')
\end{aligned}
\tag{3}
$$

if $w = (1 - k'^2w_1^2)^{\frac{1}{2}}$ and $k^2 + k'^2 = 1$. Hence it appears that K may be computed from k' as K' from k. This is very useful in practice when k^2 is near 1 and k'^2 near 0. Thus let

$$
\begin{aligned}
&e^{-\pi\frac{K}{K'}} = q' = \frac{1}{2}\frac{1-\sqrt{k}}{1+\sqrt{k}} + \frac{2}{2^5}\left(\frac{1-\sqrt{k}}{1+\sqrt{k}}\right)^5 + \cdots, \qquad \log q \log q' = \pi^2, \\
&\sqrt{\frac{2K'}{\pi}} = \Theta_1(0, q') = 1 + 2q' + 2q'^4 + \cdots, \qquad K = -\frac{K'}{\pi}\log q';
\end{aligned}
\tag{4}
$$

and compare with (37) of p. 472. Now either k or k' is greater than 0.7, and hence either q or q' may be obtained to five places with only one term in its expansion and with a relative error of only about 0.01 per cent. Moreover either q or q' will be less than $1/20$ and hence a single term $1 + 2q$ or $1 + 2q'$ gives K or K' to four places.

188. As in the relation between the Riemann surface and the z-plane the whole real axis of z corresponds periodically to the part of the real axis of w between -1 and $+1$, the function $\operatorname{sn} x$, for real x, is real. The graph of $y = \operatorname{sn} x$ has roots at $x = 2mK$, maxima or minima alternately at $(2m+1)K$, inflections inclined at the angle 45° at the roots, and in general looks like $y = \sin(\pi x/2K)$. Examined more closely, $\operatorname{sn}\frac{1}{2}K = (1+k')^{-\frac{1}{2}} > 2^{-\frac{1}{2}} = \sin\frac{1}{4}\pi$; it is seen that the curve $\operatorname{sn} x$ has ordinates numerically greater than $\sin(\pi x/2K)$. As

$$
\operatorname{cn} x = \sqrt{1 - \operatorname{sn}^2 x}, \qquad \operatorname{dn} x = \sqrt{1 - k^2\operatorname{sn}^2 x}, \tag{5}
$$

the curves $y = \operatorname{cn} x$, $y = \operatorname{dn} x$, may readily be sketched in. It may be noted that as $\operatorname{sn}(x + K) \neq \operatorname{cn} x$, the curves for $\operatorname{sn} x$ and $\operatorname{cn} x$ cannot be superposed as in the case of the trigonometric functions.

The segment $0, iK'$ of the pure imaginary axis for z corresponds to the whole upper half of the pure imaginary axis for w. Hence $\operatorname{sn} ix$ with x real is pure imaginary and $-i\operatorname{sn} ix$ is real and positive for $0 \leqq x < K'$ and becomes infinite for $x = K'$. Hence $-i\operatorname{sn} ix$ looks in general like $\tan(\pi x/2K')$. By (5) it is seen that the curves for $y = \operatorname{cn} ix$, $y = \operatorname{dn} ix$ look much like $\sec(\pi x/2K')$ and that $\operatorname{cn} ix$ lies above $\operatorname{dn} ix$. These functions are real for pure imaginary values.

It was seen that when k and k' interchanged, K and K' also interchanged. It is therefore natural to look for a relation between the elliptic functions $\operatorname{sn}(z, k)$, $\operatorname{cn}(z, k)$, $\operatorname{dn}(z, k)$ formed with the modulus k

and the functions sn (z, k'), cn (z, k'), dn (z, k') formed with the complementary modulus k' It will be shown that

$$\operatorname{sn}(iz, k) = i\frac{\operatorname{sn}(z, k')}{\operatorname{cn}(z, k')}, \qquad \operatorname{sn}(z, k) = -i\frac{\operatorname{sn}(iz, k')}{\operatorname{cn}(iz, k')},$$

$$\operatorname{cn}(iz, k) = \frac{1}{\operatorname{cn}(z, k')}, \qquad \operatorname{cn}(z, k) = \frac{1}{\operatorname{cn}(iz, k')},$$

$$\operatorname{dn}(iz, k) = \frac{\operatorname{dn}(z, k')}{\operatorname{cn}(z, k')}, \qquad \operatorname{dn}(z, k) = \frac{\operatorname{dn}(iz, k')}{\operatorname{cn}(iz, k')}.$$

Consider sn (iz, k). This function is periodic with the periods $4K$ and $2iK'$ if iz be the variable, and hence with periods $4iK$ and $2K'$ if z be the variable. With z as variable it has zeros at 0, $2iK$, and poles at K', $2iK + K'$. These are precisely the positions of the zeros and poles of the quotient $H(z, q')/H_1(z, q')$, where the theta functions are constructed with q' instead of q. As this quotient and sn (iz, k) are of the second order and have the same periods,

$$\operatorname{sn}(iz, k) = C\frac{H(z, q')}{H_1(z, q')} = C_1\frac{\operatorname{sn}(z, k')}{\operatorname{cn}(z, k')}.$$

The constant C_1 may be determined as $C_1 = i$ by comparing the derivatives of the two sides at $z = 0$. The other five relations may be proved in the same way or by transformation.

The theta series converge with extreme rapidity if q is tolerably small, but if q is somewhat larger, they converge rather poorly. The relations just obtained allow the series with q to be replaced by series with q' and one of these quantities is surely less than $1/20$. In fact if $\nu = \pi x/2K$ and $\nu' = \pi x/2K'$, then

$$\begin{aligned}\operatorname{sn}(x, k) &= \frac{\sqrt[4]{q}}{\sqrt{k}}\,\frac{2\sin\nu - 2q^2\sin 3\nu + 2q^6\sin 5\nu - \cdots}{1 - 2q\cos 2\nu + 2q^4\cos 4\nu - 2q^9\cos 6\nu + \cdots}\\ &= \frac{1}{\sqrt{k}}\,\frac{\sinh\nu' - q'^2\sinh 3\nu' + q'^6\sinh 5\nu' - \cdots}{\cosh\nu' + q'^2\cosh 3\nu' + q'^6\cosh 5\nu' + \cdots}.\end{aligned} \tag{6}$$

The second series has the disadvantage that the hyperbolic functions increase rapidly, and hence if the convergence is to be as good as for the first series, the value of q' must be considerably less than that of q, that is, K' must be considerably less than K. This can readily be arranged for work to four or five places. For

$$q'^6 = e^{-6\pi\frac{K}{K'}}, \qquad \cosh 5\nu' = \tfrac{1}{2}\left(e^{\frac{5\pi x}{2K'}} + e^{-\frac{5\pi x}{2K'}}\right), \qquad 0 \leqq x \leqq K',$$

where owing to the periodicity of the functions it is never necessary to take $x > K'$. The term in q'^6 is therefore less than $\frac{1}{2}q'^{3\frac{1}{2}}$. If the term in q'^6 is to be equally negligible with that in q^6,

$$2q^6 = \tfrac{1}{2}q'^{\frac{7}{2}} \quad \text{with} \quad \log q \log q' = \pi^2,$$

from which q' is determined as about $q' = .02$ and q as about $q = .08$; the neglected term is about 0.0000005 and is barely enough to effect six-place work except through the multiplication of errors. The value of k corresponding to this critical value of q is about $k = 0.85$.

Another form of the integral under consideration is

$$F(\phi, k) = \int_0^\phi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^y \frac{dw}{\sqrt{1-w^2}\sqrt{1-k^2w^2}} = x, \tag{7}$$

$$\sin\phi = y = \operatorname{sn} x, \qquad \phi = \operatorname{am} x, \qquad \cos\phi = \sqrt{1-\operatorname{sn}^2 x} = \operatorname{cn} x,$$

$$\Delta\phi = \sqrt{1-k^2y^2} = \sqrt{1-k^2\sin^2\phi} = \operatorname{dn} x, \qquad k'^2 = 1-k^2,$$

$$x = \operatorname{sn}^{-1}(y, k) = \operatorname{cn}^{-1}(\sqrt{1-y^2}, k) = \operatorname{dn}^{-1}(\sqrt{1-k^2y^2}, k).$$

The angle ϕ is called the *amplitude* of x; the functions $\operatorname{sn} x$, $\operatorname{cn} x$, $\operatorname{dn} x$ are the *sine-amplitude, cosine-amplitude, delta-amplitude* of x. The half periods are then

$$\begin{aligned} K &= \int_0^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = F\left(\frac{1}{2}\pi, k\right), \\ K' &= \int_0^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt{1-k'^2\sin^2\theta}} = F\left(\frac{1}{2}\pi, k'\right), \end{aligned} \tag{8}$$

and are known as the *complete elliptic integrals of the first kind.*

189. The elliptic functions and integrals often arise in problems that call for a numerical answer. Here k^2 is given and the complete integral K or the value of the elliptic functions or of the elliptic integral $F(\phi, k)$ are desired for some assigned argument. The values of K and $F(\phi, k)$ in terms of $\sin^{-1}k$ are found in tables (B. O. Peirce, pp. 117–119), and may be obtained therefrom. The tables may be used by inversion to find the values of the function $\operatorname{sn} x$, $\operatorname{cn} x$, $\operatorname{dn} x$ when x is given; for $\operatorname{sn} x = \operatorname{sn} F(\phi, k) = \sin\phi$, and if $x = F$ is given, ϕ may be found in the table, and then $\operatorname{sn} x = \sin\phi$. It is, however, easy to compute the desired values directly, owing to the extreme rapidity of the convergence of the series. Thus

$$\sqrt{\frac{2K}{\pi}} = \Theta_1(0), \quad \sqrt{\frac{2Kk'}{\pi}} = \Theta(0), \quad \frac{1+\sqrt{k'}}{\sqrt{2\pi}}\sqrt{K} = \frac{1}{2}(\Theta_1(0)+\Theta(0)),$$

$$\begin{aligned} \sqrt{K} &= \frac{\sqrt{2\pi}}{1+\sqrt{k'}}(1+2q^4+\cdots) = \sqrt{-\frac{K'}{\pi}\log q'} \\ &= \frac{\sqrt{-2\log q'}}{1+\sqrt{k}}(1+2q'^4+\cdots). \end{aligned} \tag{9}$$

The elliptic functions are computed from (6) or analogous series. To compute the value of the elliptic integral $F(\phi, k)$, note that if

$$\cot\lambda = \frac{\operatorname{dn} x}{\sqrt{k'}} = \frac{1 + 2q\cos 2\nu + 2q^4\cos 4\nu + \cdots}{1 - 2q\cos 2\nu + 2q^4\cos 4\nu + \cdots}, \tag{10}$$

$$\tan\left(\frac{1}{4}\pi - \lambda\right) = \frac{\cot\lambda - 1}{\cot\lambda + 1} = 2q\frac{\cos 2\nu + q^8\cos 6\nu + \cdots}{1 + 2q^4\cos 4\nu + \cdots};$$

and $\tan(\frac{1}{4}\pi - \lambda) = 2q\cos 2\nu$ or $\tan(\frac{1}{4}\pi - \lambda) = \dfrac{2q\cos 2\nu}{1 + 2q^4\cos 4\nu}$ (10′)

are two approximate equations from which $\cos 2\nu$ may be obtained; the first neglects q^4 and is generally sufficient, but the second neglects only q^8. If k^2 is near 1, the proper approximations are

$$\cot\lambda = \frac{1}{\sqrt{k}}\frac{\operatorname{dn}(x, k)}{\operatorname{cn}(x, k)} = \frac{\operatorname{dn}(ix, k')}{\sqrt{k}} = \frac{1 + 2q'\cosh 2\nu' + \cdots}{1 - 2q'\cosh 2\nu' + \cdots}, \tag{11}$$

$$\tan(\tfrac{1}{4}\pi - \lambda) = 2q'\cosh 2\nu' \text{ or } \tan(\tfrac{1}{4}\pi - \lambda) = \frac{2q'\cosh 2\nu'}{1 + 2q'^4\cosh 4\nu'}. \tag{11′}$$

Here $q'^8\cosh 8\nu' < q'^4$ is neglected in the second, but $q'^4\cosh 4\nu' < q'^2$ in the first, which is not always sufficient for four-place work. Of course if ϕ with $\operatorname{sn} x = \sin\phi$ or if $y = \operatorname{sn} x$ is given, $\operatorname{dn} x = \sqrt{1 - k^2\operatorname{sn}^2 x}$ and $\operatorname{cn} x = \sqrt{1 - \operatorname{sn}^2 x}$ are readily computed.

As an example take $\displaystyle\int_0^\phi \frac{d\theta}{\sqrt{1 - \frac{1}{4}\sin^2\theta}}$ and find K, $\operatorname{sn}\frac{2}{3}K$, $F(\frac{1}{8}\pi, \frac{1}{2})$. As $k'^2 = \frac{3}{4}$ and $\sqrt{k'} > 0.9$, the first term of (37), p. 472, gives q accurately to five places. Compute in the form: ($\mathrm{Lg} = \log_{10}$)

$\mathrm{Lg}\, k'^2 = 9.87506$	$\mathrm{Lg}(1 - \sqrt{k'}) = 8.84136$	$\mathrm{Lg}\, 2\pi = 0.7982$
$\mathrm{Lg}\sqrt{k'} = 9.96876$	$\mathrm{Lg}(1 + \sqrt{k'}) = 0.28569$	$2\,\mathrm{Lg}(1 + \sqrt{k'}) = 0.5714$
$\sqrt{k'} = 9.93060$	$\mathrm{Lg}\, 2q = 8.55567$	$\mathrm{Lg}\, K = 0.2268$
$1 - \sqrt{k'} = 0.06940$	$2q = 0.03595$	$K = 1.686$
$1 + \sqrt{k'} = 1.93060$	$q = 0.01797$	Check with table.

$$\operatorname{sn}\frac{2}{3}K = 2\frac{\sqrt[4]{q}}{\sqrt{k}}\frac{\sin\frac{1}{3}\pi - q^2\sin\pi + \cdots}{1 - 2q\cos\frac{2}{3}\pi + \cdots} = 2\frac{\sqrt[4]{q}}{\sqrt{\frac{1}{2}}}\frac{\frac{1}{2}\sqrt{3}}{1 + q}.$$

$$\operatorname{sn}\frac{2}{3}K = \frac{\sqrt{6}\,\sqrt[4]{q}}{1.01797}$$

$\frac{1}{2}\mathrm{Lg}\, 6 = 0.38908$	$\mathrm{Lg}\operatorname{sn}\frac{2}{3}K = 9.9450$
$\frac{1}{4}\mathrm{Lg}\, q = 9.56366$	$\operatorname{sn}\frac{2}{3}K = 0.8810.$
$-\mathrm{Lg}\, 1.018 = 9.99226$	

$\phi = \frac{1}{8}\pi \qquad \Delta\phi = \operatorname{dn} x = \sqrt{1 - \frac{1}{4}\sin^2\frac{1}{8}\pi} = \sqrt{1 - \frac{1}{2}\sin\frac{1}{8}\pi}\,\sqrt{1 + \frac{1}{2}\sin\frac{1}{8}\pi}.$

$\frac{1}{2}\sin\frac{1}{8}\pi = 0.19134$	$\lambda = 43°\,28'\,28''$	$\text{Lg}\,42.20 = 1.6253$
$1-\frac{1}{2}\sin\frac{1}{8}\pi = 0.80866$	$\frac{1}{4}\pi - \lambda = 1°\,31'\,32''$	$\text{Lg}\,K = 0.2268$
$1+\frac{1}{2}\sin\frac{1}{8}\pi = 1.19134$	$\text{Lg}\tan = 8.42540$	$-\text{Lg}\,180 = 7.7447$
$\frac{1}{2}\text{Lg}(1-\frac{1}{2}\sin\frac{1}{8}\pi) = 9.95388$	$\text{Lg}\,2q = 8.55567$	$\text{Lg}\,x = 9.5968$
$\frac{1}{2}\text{Lg}(1+\frac{1}{2}\sin\frac{1}{8}\pi) = 0.03802$	$\text{Lg}\cos 2\nu = 9.86973$	$x = 0.3952$
$-\text{Lg}\sqrt{k'} = 0.03124$	$2\nu = 42°\,12'$	Check with table.
$\text{Lg}\cot\lambda = 0.02314$	$180x = K(42.20)$	

As a second example consider a pendulum of length a oscillating through an arc of 300°. Find the period, the time when the pendulum is horizontal, and its position after dropping for a third of the time required for the whole descent. Let $x^2+y^2 = 2ay$ be the equation of the path and $h = a\left(1+\frac{1}{2}\sqrt{3}\right)$ the greatest height. When $y = h$, the energy is wholly potential and equals mgh; and mgy is the general value of the potential energy. The kinetic energy is

$$\frac{m}{2}\left(\frac{dy}{dt}\right)^2 = \frac{\frac{1}{2}ma^2}{2ay-y^2}\left(\frac{dy}{dt}\right)^2 \quad\text{and}\quad \frac{\frac{1}{2}ma^2}{2ay-y^2}\left(\frac{dy}{dt}\right)^2 + mgy = mgh$$

is the equation of motion by the principle of energy. Hence

$$t = \int_0^y \frac{a\,dy}{\sqrt{2g}\sqrt{(h-y)(2ay-y^2)}} = \sqrt{\frac{a}{g}}\int_0^w \frac{dw}{\sqrt{(1-w^2)(1-k^2w^2)}},\quad w^2 = \frac{y}{h},\quad k^2 = \frac{h}{2a},$$

$$\sqrt{g/a}t = \text{sn}^{-1}(w, k), \qquad w = \text{sn}\left(\sqrt{g/a}t, k\right), \qquad y = h\,\text{sn}^2\left(\sqrt{g/a}t, k\right),$$

are the integrated results. The quarter period, from highest to lowest point, is $K\sqrt{a/g}$; the horizontal position is $y = a$, at which t is desired; and the position for $\sqrt{g/a}t = \frac{2}{3}K$ is the third thing required.

$$k^2 = 0.93301, \qquad 2q' = \frac{1-\sqrt{k}}{1+\sqrt{k}}, \qquad K = -\frac{K'}{\pi}\log q' = \frac{-2\,\text{Lg}\,q'}{M\left(1+\sqrt{k}\right)^2}.$$

$\text{Lg}\,k^2 = 9.96988$	$\text{Lg}(1-\sqrt{k}) = 8.23553$	$\text{Lg}\,2 = 0.3010$
$\text{Lg}\sqrt{k} = 9.99247$	$-\text{Lg}(1+\sqrt{k}) = 9.70272$	$\text{Lg}^2 q'^{-1} = 0.3734$
$\sqrt{k} = 0.98280$	$-\text{Lg}\,2 = 9.69897$	$-\text{Lg}\,M = 0.3622$
$1-\sqrt{k} = 0.01720$	$\text{Lg}\,q' = 7.63722$	$-2\,\text{Lg}(1+\sqrt{k}) = 9.4034$
$1+\sqrt{k} = 1.98280$	$q' = 0.00434$	$\text{Lg}\,K = 0.4420.$

Hence $K = 2.768$ and the complete periodic time is $4K\sqrt{a/g}$.

$$y = a, \qquad w^2 = \frac{a}{h}, \qquad \text{cn}\,w = \sqrt{1-a/h}, \qquad \text{dn}\,w = \sqrt{1-k^2a/h}.$$

$$\frac{1}{\sqrt{k}}\frac{\text{dn}\,w}{\text{cn}\,w} = \sqrt[4]{\frac{4}{3}k^2} = \cot\lambda, \qquad \tan\left(\frac{1}{4}\pi-\lambda\right) = 2q'\cosh 2\nu', \qquad 2\nu' = \frac{\pi K}{K'}\sqrt{\frac{g}{a}}\frac{t}{K}.$$

$\text{Lg}\,k^2 = 9.96988$	$\lambda = 43°\,26'\,12''$	$2\nu' = 1.813$
$\text{Lg}\,4 = 0.60206$	$\frac{1}{4}\pi-\lambda = 1°\,33'\,48''$	$\text{Lg}\,2\nu' = 0.2584$
$-\text{Lg}\,3 = 9.52288$	$\text{Lg}\tan = 8.43603$	$-\text{Lg}^2 q'^{-1} = 9.6266$
$\text{Lg}\cot^4\lambda = 0.09482$	$\text{Lg}\,2q' = 9.93825$	$\text{Lg}\,M = 9.6378$
$\text{Lg}\cot\lambda = 0.02370$	$\text{Lg}\cosh 2\nu' = 0.49778$	$\text{Lg}\sqrt{\frac{g}{a}}\frac{t}{K} = 9.5228.$

Hence the time for $y = a$ is $t = 0.3333\, K\sqrt{a/g} = \frac{1}{3}$ whole time of ascent.

$$y = h\,\mathrm{sn}^2\sqrt{\frac{g}{a}}\frac{2}{3}K\sqrt{\frac{a}{g}} = \frac{h}{k}\left(\frac{\sinh \pi K/3\,K' - q'^2 \sinh \pi K/K'}{\cosh \pi K/3\,K' + q'^2 \cosh \pi K/K'}\right)^2$$

$$= 2\,ak\left(\frac{q'^{-\frac{1}{3}} - q'^{\frac{1}{3}} - q'^2(q'^{-1} - q')}{q'^{-\frac{1}{3}} + q'^{\frac{1}{3}} + q'^2(q'^{-1} + q')}\right)^2 = 2\,ak\left(\frac{q'^{-\frac{1}{3}} - q'^{\frac{1}{3}} - q'}{q'^{-\frac{1}{3}} + q'^{\frac{1}{3}} + q'}\right)^2.$$

$$\tfrac{1}{3}\,\mathrm{Lg}\,q' = 9.21241 \qquad q'^{\frac{1}{3}} = 0.1631 \qquad y = 2\,ak\left(\frac{5.9645}{6.2993}\right)^2.$$
$$-\tfrac{1}{3}\,\mathrm{Lg}\,q' = 0.78759 \qquad q'^{-\frac{1}{3}} = 6.1319$$

This gives $y = 1.732\,a$, which is very near the top at $h = 1.866\,a$. In fact starting at 30° from the vertical the pendulum reaches 43° in a third and 90° in another third of the total time of descent. As $\mathrm{sn}\,\frac{1}{2}K$ is $(1 + k')^{-\frac{1}{2}}$ it is easy to calculate the position of the pendulum at half the total time of descent.

EXERCISES

1. Discuss these integrals by the method of mapping:

(α) $z = \int_0^w \frac{dw}{\sqrt{(a^2 - w^2)(b^2 - w^2)}}$, $\quad a > b > 0, \quad w = b\,\mathrm{sn}\,az, \quad k = \frac{b}{a}$,

(β) $z = \int_0^w \frac{dw}{\sqrt{w(1 - w)(1 - k^2w)}}$, $\quad w = \mathrm{sn}^2\left(\frac{1}{2}z, k\right), \quad z = 2\,\mathrm{sn}^{-1}(\sqrt{x}, k)$,

(γ) $z = \int_0^w \frac{dw}{\sqrt{(1 + w^2)(1 + k'^2w^2)}}$, $\quad w = \frac{\mathrm{sn}\,(z, k)}{\mathrm{cn}\,(z, k)} = \mathrm{tn}\,(z, k), \quad z = \mathrm{tn}^{-1}(w, k)$.

2. Establish these Maclaurin developments with the aid of § 177:

(α) $\mathrm{sn}\,z = z - (1 + k^2)\frac{z^3}{3\,!} + (1 + 14\,k^2 + k^4)\frac{z^5}{5\,!} - \cdots,$

(β) $\mathrm{cn}\,z = 1 - \frac{z^2}{2\,!} + (1 + 4\,k^2)\frac{z^4}{4\,!} - (1 + 44\,k^2 + 16\,k^4)\frac{z^6}{6\,!} + \cdots,$

(γ) $\mathrm{dn}\,z = 1 - k^2\frac{z^2}{2\,!} + k^2(4 + k^2)\frac{z^4}{4\,!} - k^2(16 + 44\,k^2 + k^4)\frac{z^6}{6\,!} + \cdots.$

3. Prove $\int_0^\phi \frac{d\phi}{\sqrt{1 - l^2\sin^2\phi}} = \frac{1}{l}\int_0^\psi \frac{d\psi}{\sqrt{1 - l^{-2}\sin^2\psi}}$, $\quad l > 1, \quad \sin^2\psi = l^2\sin^2\phi$.

4. Carry out the computations in these cases:

(α) $\int_0^\phi \frac{d\theta}{\sqrt{1 - 0.1\sin^2\theta}}$ to find K, $\quad \mathrm{sn}\,\frac{2}{3}K, \quad F\left(\frac{1}{8}\pi, \frac{1}{\sqrt{10}}\right)$,

(β) $\int_0^\phi \frac{d\theta}{\sqrt{1 - 0.9\sin^2\theta}}$ to find K, $\quad \mathrm{sn}\,\frac{1}{3}K, \quad F\left(\frac{1}{3}\pi, \frac{3}{\sqrt{10}}\right)$.

5. A pendulum oscillates through an angle of (α) 180°, (β) 90°, (γ) 340°. Find the periodic time, the position at $t = \frac{2}{3}K$, and the time at which the pendulum makes an angle of 30° with the vertical.

6. With the aid of Ex. 3 find the arc of the lemniscate $r^2 = 2a^2 \cos 2\phi$. Also the arc from $\phi = 0$ to $\phi = 30°$, and the middle point of the arc.

7. A bead moves around a vertical circle. The velocity at the top is to the velocity at the bottom as $1:n$. Express the solution in terms of elliptic functions.

8. In Ex. 7 compute the periodic time if $n = 2$, 3, or 10.

9. Neglecting gravity, solve the problem of the jumping rope. Take the x-axis horizontal through the ends of the rope, and the y-axis vertical through one end. Remember that "centrifugal force" varies as the distance from the axis of rotation. The first and second integrations give

$$dx = \frac{a^2 dy}{\sqrt{(b^2 - y^2)^2 - a^2}}, \qquad y = \sqrt{b^2 - a^2}\,\mathrm{sn}\left(\frac{\sqrt{b^2 + a^2}x}{a^2},\ \sqrt{\frac{b^2 - a^2}{b^2 + a^2}}\right).$$

10. Express $\int \frac{d\theta}{\sqrt{a - \cos\theta}}$, $a > 1$, in terms of elliptic functions.

11. A ladder stands on a smooth floor and rests at an angle of 30° against a smooth wall. Discuss the descent of the ladder after its release from this position. Find the time which elapses before the ladder leaves the wall.

12. A rod is placed in a smooth hemispherical bowl and reaches from the bottom of the bowl to the edge. Find the time of oscillation when the rod is released.

190. Legendre's Integrals II and III. The treatment of

$$\int_0^w \frac{\sqrt{1 - k^2w^2}}{\sqrt{1 - w^2}}\,dw = \int_0^w \frac{(1 - k^2w^2)\,dw}{\sqrt{(1 - w^2)(1 - k^2w^2)}} \tag{II}$$

by the method of conformal mapping to determine the function and its inverse does not give satisfactory results, for the map of the Riemann surface on the z-plane is not a simple region. But the integral may be treated by a change of variable and be reduced to the integral of an elliptic function. For with $w = \mathrm{sn}\,u$, $u = \mathrm{sn}^{-1} w$,

$$\begin{aligned}\int_0^w \frac{(1 - k^2w^2)\,dw}{\sqrt{(1 - w^2)(1 - k^2w^2)}} &= \int_0^u (1 - k^2 \mathrm{sn}^2 u)\,du \\ &= u - k^2 \int_0^u \mathrm{sn}^2 u du.\end{aligned} \tag{12}$$

The problem thus becomes that of integrating $\mathrm{sn}^2 u$. To effect the integration, $\mathrm{sn}^2 u$ will be expressed as a derivative.

The function $\mathrm{sn}^2 u$ is doubly periodic with periods $2K$, $2iK'$, and with a pole of the second order at $u = iK'$. But now

$$\Theta(u + 2K) = \Theta(u), \qquad \Theta(u + 2iK') = -q^{-1}e^{-\frac{i\pi}{K}u}\Theta(u)$$

$$\log\Theta(u + 2K) = \log\Theta(u),\ \log(\Theta + 2iK') = \log\Theta(u) - \frac{i\pi}{K}u - \log(-q).$$

It then appears that the second derivative of $\log \Theta(u)$ also has the periods $2K$, $2iK'$. Introduce the zeta function

$$Z(u) = \frac{d}{du}\log\Theta(u) = \frac{\Theta'(u)}{\Theta(u)}, \qquad Z'(u) = \frac{d}{du}\frac{\Theta'(u)}{\Theta(u)}. \tag{13}$$

The expansion of $\Theta'(u)$ shows that $\Theta'(u) = 0$ at $u = mK$. About $u = iK'$ the expansions of $Z'(u)$ and $\operatorname{sn}^2 u$ are

$$Z'(u) = -\frac{1}{(u - iK')^2} + a_0 + \cdots, \qquad \operatorname{sn}^2 u = \frac{1}{k^2}\frac{1}{(u - iK')^2} + b_0 + \cdots.$$

Hence $\qquad k^2 \operatorname{sn}^2 u = -Z'(u) + Z'(0), \qquad Z'(0) = \Theta''(0)/\Theta(0),$

and
$$k^2\int_0^u \operatorname{sn}^2 u\, du = -Z(u) + uZ'(0),$$

$$\int_0^u (1 - k^2\operatorname{sn}^2 u)\,du = u(1 - Z'(0)) + Z(u). \tag{14}$$

The derivation of the expansions of $Z'(u)$ and $\operatorname{sn}^2 u$ about $u = iK'$ are easy.

$$\Theta(u) = C\prod\left(1 - q^{2n+1}e^{\pm\frac{i\pi}{K}u}\right), \qquad \log\Theta(u) = \sum\log\left(1 - q^{2n+1}e^{\pm\frac{i\pi}{K}u}\right) + \log C$$

$$\log\Theta(u) = \log\left(1 - qe^{-\frac{i\pi}{K}u}\right) + \text{ function analytic near } u = iK'.$$

$$\frac{\Theta'(u)}{\Theta(u)} = \frac{i\pi q e^{-\frac{i\pi}{K}u}}{K\left(1 - qe^{-\frac{i\pi}{K}u}\right)} + \cdots = \frac{i\pi q}{K\left(e^{\frac{i\pi}{K}u} - q\right)} + \cdots,$$

$$f(u) = e^{\frac{i\pi}{K}u} = f(iK') + (u - iK')f'(iK') + \cdots = q + (u - iK')\frac{i\pi}{K}q + \cdots,$$

$$\frac{\Theta'(u)}{\Theta(u)} = \frac{+1}{u - iK'} + \cdots, \qquad \frac{d}{du}\frac{\Theta'(u)}{\Theta(u)} = \frac{-1}{(u - iK')^2} + \cdots.$$

$$\operatorname{sn}(u + iK') = \frac{1}{k}\frac{1}{\operatorname{sn} u}, \qquad \operatorname{sn}^2(u + iK') = \frac{1}{k^2}\frac{1}{\operatorname{sn}^2 u},$$

$$f(u) = \operatorname{sn} u = uf'(0) + \tfrac{1}{6}u^3 f'''(0) + \cdots = u + cu^3 + \cdots,$$

$$\operatorname{sn}^2(u + iK') = \frac{1}{k^2}\frac{1}{\operatorname{sn}^2 u} = \frac{1}{k^2}\left(\frac{1}{u} - cu + \cdots\right)^2 = \frac{1}{k^2}\left(\frac{1}{u^2} - 2c + \cdots\right),$$

$$\operatorname{sn}^2 u = \frac{1}{k^2}\left(\frac{1}{(u - iK')^2} - 2c + \cdots\right).$$

In a similar manner may be treated the integral

$$\int_0^w \frac{dw}{(w^2 - \alpha)\sqrt{(1 - w^2)(1 - k^2w^2)}} = \int_0^u \frac{du}{\operatorname{sn}^2 u - \alpha}. \tag{III}$$

Let a be so chosen that $\operatorname{sn}^2 a = \alpha$. The integral becomes

$$\int_0^u \frac{du}{\operatorname{sn}^2 u - \operatorname{sn}^2 a} = \frac{1}{2\operatorname{sn} a\operatorname{cn} a\operatorname{dn} a}\int\frac{2\operatorname{sn} a\operatorname{cn} a\operatorname{dn} a}{\operatorname{sn}^2 u - \operatorname{sn}^2 a}du. \tag{15}$$

The integrand is a function with periods $2K$, $2iK'$ and with simple poles at $u = \pm a$. To find the residues at these poles note

$$\lim_{u \doteq \pm a} \frac{u \mp a}{\operatorname{sn}^2 u - \operatorname{sn}^2 a} = \lim_{u \doteq \pm a} \frac{1}{2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u} = \frac{\pm 1}{2 \operatorname{sn} a \operatorname{cn} a \operatorname{dn} a}.$$

The coefficient of $(u \mp a)^{-1}$ in expanding about $\pm a$ is therefore ± 1. Such a function may be written down. In fact

$$\frac{2 \operatorname{sn} a \operatorname{cn} a \operatorname{dn} a}{\operatorname{sn}^2 u - \operatorname{sn}^2 a} = \frac{H'(u-a)}{H(u-a)} - \frac{H'(u+a)}{H(u+a)} + C$$
$$= \mathrm{Z}_1(u-a) - \mathrm{Z}_1(u+a) + C,$$

if $Z_1 = H'/H$. The verification is as above. To determine C let $u = 0$.

Then $\quad C = -\dfrac{2 \operatorname{cn} a \operatorname{dn} a}{\operatorname{sn} a} + 2\,\mathrm{Z}_1(a), \quad$ but $\quad \operatorname{sn} u = \dfrac{1}{\sqrt{k}} \dfrac{H(u)}{\Theta(u)},$.

and $\quad \dfrac{d}{du} \log \operatorname{sn} u = \dfrac{\operatorname{cn} u \operatorname{dn} u}{\operatorname{sn} u} = \mathrm{Z}_1(u) - \mathrm{Z}(u).$

Hence C reduces to $2\,\mathrm{Z}(a)$ and the integral is

$$\int_0^u \frac{du}{\operatorname{sn}^2 u - \operatorname{sn}^2 a} = \frac{1}{2 \operatorname{sn} a \operatorname{cn} a \operatorname{dn} a} \left[\log \frac{H(a-u)}{H(a+u)} + 2u\mathrm{Z}(a) \right]. \quad (16)$$

The integrals here treated by the substitution $w = \operatorname{sn} u$ and thus reduced to the integrals of elliptic functions are but special cases of the integration of any rational function $R(w, \sqrt{W})$ of w and the radical of the biquadratic $W = (1 - w^2)(1 - k^2w^2)$. The use of the substitution is analogous to the use of $w = \sin u$ in converting an integral of $R(w, \sqrt{1 - w^2})$ into an integral of trigonometric functions. Any rational function $R(w, \sqrt{W})$ may be written, by rationalization, as

$$R(w, \sqrt{W}) = \frac{R(w) + R(w)\sqrt{W}}{R(w) + R(w)\sqrt{W}} = \frac{R(w) + R(w)\sqrt{W}}{R(w)}$$
$$= R_1(w) + \frac{R(w)}{\sqrt{W}} = R_1(w) + \frac{w R_2(w^2) + R_3(w^2)}{\sqrt{W}}$$

where R means not always the same function. The integral of $R(w, \sqrt{W})$ is thus reduced to the integral of $R_1(w)$ which is a rational fraction, plus the integral of $wR_2(w^2)/\sqrt{W}$ which by the substitution $w^2 = u$ reduces to an integral of $R(u, \sqrt{(1-u)(1-k^2u)}$ and may be considered as belonging to elementary calculus, plus finally

$$\int \frac{R_3(w^2)}{\sqrt{W}} dw = \int R_3(\operatorname{sn}^2 u)\, du, \qquad w = \operatorname{sn} u.$$

By the method of partial fractions R_3 may be resolved and

$$\int \operatorname{sn}^{2n} u\, du \qquad n \gtrless 0, \qquad \int \frac{du}{(\operatorname{sn}^2 u - \alpha)^n} \qquad n > 0$$

are the types of integrals which must be evaluated to finish the integration of the given $R(w, \sqrt{W})$. An integration by parts (B. O. Peirce, No. 567) shows that for the first type n may be lowered if positive and raised if negative until the integral is expressed in terms of the integrals of $\text{sn}^2 x$ and $\text{sn}^0 x = 1$, of which the first is integrated above. The second type for any value of n may be obtained from the integral for $n = 1$ given above by differentiating with respect to α under the sign of integration. Hence the whole problem of the integration of $R(w, \sqrt{W})$ may be regarded as solved.

191. With the substitution $w = \sin\phi$, the integral II becomes

$$E(\phi, k) = \int_0^\phi \sqrt{1 - k^2\sin^2\theta}\,d\theta = \int_0^w \frac{\sqrt{1 - k^2w^2}}{\sqrt{1 - w^2}}\,dw \qquad (17)$$
$$= u(1 - \text{Z}'(0)) + \text{Z}(u), \qquad u = F(\phi, k).$$

In particular $E(\frac{1}{2}\pi, k)$ is called the complete integral of the second kind and is generally denoted by E. When $\phi = \frac{1}{2}\pi$, the integral $u = F(\phi, k)$ becomes the complete integral K. Then

$$E = K(1 - \text{Z}'(0)) + \text{Z}(K) = K(1 - \text{Z}'(0)), \qquad (18)$$

and

$$E(\phi, k) = EF(\phi, k)/K + \text{Z}(u). \qquad (19)$$

The problem of computing $E(\phi, k)$ thus reduces to that of computing K, E, $F(\phi, k) = u$, and $\text{Z}(u)$. The methods of obtaining K and $F(\phi, k)$ have been given. The series for $\text{Z}(u)$ converges rapidly. The value of E may be found by computing $K(1 - \text{Z}'(0))$.

For the convenience of logarithmic computation note that

$$\frac{K - E}{K} = \text{Z}'(0) = \frac{\Theta''(0)}{\Theta(0)} = \sqrt{\frac{\pi}{2Kk'}}\cdot\frac{2\pi^2}{K^2}(q - 4q^4 + 9q^9 - \cdots)$$

or

$$K - E = \tfrac{1}{2}\pi/\sqrt{k'}\cdot(2\pi/K)^{\frac{3}{2}}q(1 - 4q^3 + \cdots). \qquad (20)$$

Also

$$\text{Z}(u) = \frac{\Theta'(u)}{\Theta(u)} = \frac{2q\pi}{K}\,\frac{\sin 2\nu - 2q^3\sin 4\nu + \cdots}{1 - 2q\cos 2\nu + 2q^4\cos 4\nu - \cdots} \qquad (21)$$

where $\nu = \pi u/2K$. These series neglect only terms in q^8, which will barely affect the fifth place when $k \leqq \sin 82°$ or $k^2 \leqq 0.98$. The series as written therefore cover most of the cases arising in practice. For instance in the problem which gives the name to the elliptic functions and integrals, the problem of finding the arc of the ellipse $x = a\sin\phi$, $y = b\cos\phi$,

$$ds = \sqrt{a^2\cos^2\phi + b^2\sin^2\phi}\,d\phi = a\sqrt{1 - e^2\sin^2\phi}\,d\phi;$$

the eccentricity e may be as high as 0.99 without invalidating the approximate formulas. An example follows.

Let it be required to determine the length of the quadrant of an ellipse of eccentricity $e = 0.9$ and also the length of the portion over half the semiaxis major. Here the series in q' converge better than those in q, but as the proper expression to replace $\mathrm{Z}(u)$ has not been found, it will be more convenient to use the series in q and take an additional term or two. As $k = 0.9$, $k'^2 = 0.19$.

$\mathrm{Lg}\, k'^2 = 9.27875$	$\mathrm{Lg}(1-\sqrt{k'}) = 9.53120$	$5 \text{ diff.} = 6.55515$
$\mathrm{Lg}\, \sqrt{k'} = 9.81969$	$\mathrm{Lg}(1+\sqrt{k'}) = 0.22017$	$\mathrm{Lg}\, 16 = 1.20412$
$\sqrt{k'} = 0.66022$	$\text{diff.} = 9.31103$	$\mathrm{Lg}\text{ term } 2 = 5.35103$
$1-\sqrt{k'} = 0.33978$	$\mathrm{Lg}\, 2 = 0.30103$	$\text{term } 1 = 0.10233$
$1+\sqrt{k'} = 1.66022$	$\mathrm{Lg}\text{ term } 1 = 9.01000$	$\text{term } 2 = 0.00002$
		$q = 0.10235.$

$\mathrm{Lg}\, q = 9.0101$	$\mathrm{Lg}\, 2\pi = 0.7982$	$\mathrm{Lg}\, \frac{1}{2}\pi/\sqrt{k'} = 0.3764$
$3\, \mathrm{Lg}\, q = 7.0303$	$-2\, \mathrm{Lg}(1+\sqrt{k'}) = 9.5597$	$\frac{3}{2}\log 2\pi/K = 0.6603$
$4\, \mathrm{Lg}\, q = 6.0404$	$\mathrm{Lg}(1+2q^4) = 0.0001$	$\mathrm{Lg}\, q = 9.0101$
$q^3 = 0.0011$	$\mathrm{Lg}\, K = 0.3580$	$\mathrm{Lg}(1-4q^3) = 9.9981$
$q^4 = 0.0001$	$K = 2.280$	$\mathrm{Lg}(K-E) = 0.0449.$

Hence $K - E = 1.109$ and $E = 1.171$. The quadrant is $1.171\, a$. The point corresponding to $x = \frac{1}{2}a$ is given by $\phi = 30°$. Then $\mathrm{dn}\, F = \sqrt{1 - 0.2025}$.

$\mathrm{Lg}\, \mathrm{dn}\, F = 9.9509$	$\frac{1}{4}\pi - \lambda = 8° 31\frac{1}{2}'$	$\cos 2\nu = 0.7323$
$\mathrm{Lg}\, \sqrt{k'} = 9.8197$	$\mathrm{Lg}\tan = 9.1758$	Hence 4ν near $90°$
$\mathrm{Lg}\cot\lambda = 0.1312$	$\mathrm{Lg}\, 2q = 9.3111$	$1 + 2q^4\cos 4\nu = 1.0000$
$\lambda = 36° 28\frac{1}{2}'$	$\mathrm{Lg}\cos 2\nu = 9.8647$	$2\nu = 42° 55'.$

Now $180\, F = K(42.92)$. The computation for F, Z, $E(\frac{1}{6}\pi)$ is then

$\mathrm{Lg}\, K = 0.3580$	$\mathrm{Lg}\, 2\pi/K = 0.4402$	$\mathrm{Lg}\, E/K = 9.7106$
$\mathrm{Lg}\, 42.92 = 1.6326$	$\mathrm{Lg}\, q = 9.0101$	$\mathrm{Lg}\, F = 9.7353$
$-\mathrm{Lg}\, 180 = 7.7447$	$\mathrm{Lg}\sin 2\nu = 9.8331$	$EF/K = 0.2792$
$\mathrm{Lg}\, F = 9.7353$	$-\mathrm{Lg}(1 - 2q\cos 2\nu) = 0.0705$	$\mathrm{Z} = 0.2256$ *
$F = 0.5436$	$\mathrm{Lg}\, \mathrm{Z} = 9.3539$	$E(\frac{1}{6}\pi) = 0.5048.$

The value of Z marked * is corrected for the term $-2q^3\sin 4\nu$. The part of the quadrant over the first half of the axis is therefore $0.5048\, a$ and $0.666\, a$ over the second half. To insure complete four-figure accuracy in the result, five places should have been carried in the work, but the values here found check with the table except for one or two units in the last place.

EXERCISES

1. Prove the following relations for $\mathrm{Z}(u)$ and $\mathrm{Z}_1(u)$.

$$\mathrm{Z}(-u) = -\mathrm{Z}(u), \qquad \mathrm{Z}(u + 2K) = \mathrm{Z}(u), \qquad \mathrm{Z}(u + 2iK') = \mathrm{Z}(u) - i\pi/K.$$

If
$$\mathrm{Z}_1(u) = \frac{d}{du}\log H(u) = \frac{H'(u)}{H(u)}, \qquad \mathrm{Z}_1(u + iK') = \mathrm{Z}(u) - \frac{i\pi}{2K},$$

$$\frac{1}{\mathrm{sn}^2 u} = -\mathrm{Z}_1'(u) + \mathrm{Z}'(0), \qquad \int \frac{du}{\mathrm{sn}^2 u} = -\mathrm{Z}_1(u) + u\mathrm{Z}'(0),$$

$$\mathrm{Z}_1(u) - \mathrm{Z}(u) = \frac{d}{du}\log \mathrm{sn}\, u = \frac{\mathrm{cn}\, u\, \mathrm{dn}\, u}{\mathrm{sn}\, u}, \qquad \mathrm{Z}_1(0) = \infty.$$

2. An elliptic function with periods $2K$, $2iK'$ and simple poles at $a_1, a_2, \cdots, a_n$ with residues $c_1, c_2, \cdots, c_n$, $\Sigma c = 0$, may be written

$$f(u) = c_1 Z_1(u - a_1) + c_2 Z_1(u - a_2) + \cdots + c_n Z_1(u - a_n) + \text{const.}$$

3. $$\frac{k^2 \operatorname{sn} a \operatorname{cn} a \operatorname{dn} a \operatorname{sn}^2 u}{1 - k^2 \operatorname{sn}^2 a \operatorname{sn}^2 u} = \frac{1}{2} Z(u - a) - \frac{1}{2} Z(u + a) + Z'(a),$$

$$k^2 \operatorname{sn} a \operatorname{cn} a \operatorname{dn} a \int_0^u \frac{\operatorname{sn}^2 u \operatorname{dn}}{1 - k^2 \operatorname{sn}^2 a \operatorname{sn}^2 u} = \frac{1}{2} \log \frac{\Theta(a - u)}{\Theta(a + u)} + uZ'(a).$$

4. (α) $$\int \frac{\lambda du}{\operatorname{sn}^2 \sqrt{\lambda} u} = \lambda u Z'(0) - \sqrt{\lambda} Z(\sqrt{\lambda} u) - \sqrt{\lambda} \frac{\operatorname{cn} \sqrt{\lambda} u \operatorname{dn} \sqrt{\lambda} u}{\operatorname{sn} \sqrt{\lambda} u} + C$$

$$= \lambda u - \sqrt{\lambda} E(\phi = \sin^{-1} \operatorname{sn} \sqrt{\lambda} u) - \sqrt{\lambda} \frac{\operatorname{cn} \sqrt{\lambda} u \operatorname{dn} \sqrt{\lambda} u}{\operatorname{sn} \sqrt{\lambda} u} + C,$$

(β) $$\int \frac{k'^2 du}{\operatorname{dn}^2 u} = \int \operatorname{dn}^2 u du - k^2 \frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u} = E(\phi = \sin^{-1} \operatorname{sn} u) - k^2 \frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u},$$

(γ) $$\int \frac{\operatorname{cn}^2 u du}{\operatorname{sn}^2 u \operatorname{dn}^2 u} = u - 2E(\phi = \sin^{-1} \operatorname{sn} u) + \frac{\operatorname{cn} u}{\operatorname{sn} u \operatorname{dn} u}(1 - 2 \operatorname{dn}^2 u).$$

5. Find the length of the quadrant and of the portion of it cut off by the latus rectum in ellipses of eccentricity $e = 0.1, 0.5, 0.75, 0.95$.

6. If e is the eccentricity of the hyperbola $x^2/a^2 - y^2/b^2 = 1$, show that

$$s = \frac{b^2}{ae} \int_0^\phi \frac{\sec^2 \phi d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}, \qquad \text{where } \frac{ae}{b^2} y = \tan \phi, \qquad k = \frac{1}{e},$$

$$= \frac{b^2}{ae} F(\phi, k) - aeE(\phi, k) + ae \tan \phi \sqrt{1 - k^2 \sin^2 \phi}.$$

7. Find the arc of the hyperbola cut off by the latus rectum if $e = 1.2, 2, 3$.

8. Show that the length of the jumping rope (Ex. 9, p. 511) is

$$a(k'K/\sqrt{2} + \sqrt{2} E/k').$$

9. A flexible trough is filled with water. Find the expression of the shape of a cross section of the trough in terms of $F(\phi, k)$ and $E(\phi, k)$.

10. If an ellipsoid has the axes $a > b > c$, find the area of one octant.

$$\frac{1}{4}\pi c^2 + \frac{\pi ab}{4 \sin \phi}\left[\frac{c^2}{a^2} F(\phi, k) + \frac{a^2 - c^2}{a^2} E(\phi, k)\right], \qquad \cos \phi = \frac{c}{a}, \qquad k^2 = \frac{b^2 - c^2}{b^2 \sin^2 \phi}.$$

11. Compute the area of the ellipsoid with axes 3, 2, 1.

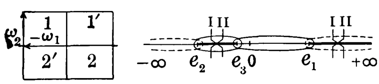
12. A hole of radius b is bored through a cylinder of radius $a > b$ centrally and perpendicularly to the axis. Find the volume cut out.

13. Find the area of a right elliptic cone, and compute the area if the altitude is 3 and the semiaxes of the base are $1\frac{1}{2}$ and 1.

192. Weierstrass's integral and its inversion. In studying the general theory of doubly periodic functions (§ 182), the two special functions $p(u)$, $p'(u)$ were constructed and discussed. It was seen that

$$\begin{aligned} z &= \int_\infty^w \frac{dw}{\sqrt{4w^3 - g_2 w - g_3}}, \qquad w = p(z), \qquad \infty = p(0), \\ &= \int_\infty^w \frac{dw}{\sqrt{4(w - e_1)(w - e_2)(w - e_3)}}, \qquad e_1 + e_2 + e_3 = 0, \end{aligned} \tag{22}$$

where the fixed limit ∞ has been added to the integral to make $w = \infty$ and $z = 0$ correspond and where the roots have been called e_1, e_2, e_3. Conversely this integral could be studied in detail by the method of mapping; but the method to be followed is to make only cursory use of the conformal map sufficient to give a hint as to how the function $p(z)$ may be expressed in terms of the functions sn z and cn z. The discussion will be restricted to the case which arises in practice, namely, when g_2 and g_3 are real quantities. There are two cases to consider, one when all three roots are real, the other when one is real and the other two are conjugate imaginary. The root e_1 will be taken as the largest real root, and e_2 as the smallest root if all three are real. Note that the sum of the three is zero.



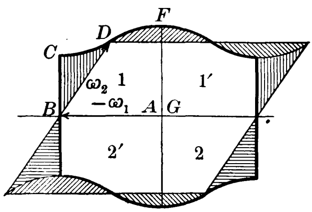
In the case of three real roots the Riemann surface may be drawn with junction lines e_2, e_3, and e_1, ∞. The details of the map may readily be filled in, but the observation is sufficient that there are only two essentially different paths closed on the surface, namely, about e_2, e_3 (which by deformation is equivalent to one about e_1, ∞) and about e_3, e_1 (which is equivalent to one about e_2, $-\infty$). The integral about e_2, e_3 is real and will be denoted by $2\omega_1$, that about e_3, e_1 is pure imaginary and will be denoted by $2\omega_2$. If the function $p(z)$ be constructed as in § 182 with $\omega = 2\omega_1$, $\omega' = 2\omega_2$ the function will have as always a double pole at $z = 0$. As the periods are real and pure imaginary, it is natural to try to express $p(z)$ in terms of sn z. As $p(z)$ depends on two constants g_2, g_3, whereas sn z depends on only the one k, the function $p(z)$ will be expressed in terms of sn $(\sqrt{\lambda} z, k)$, where the two constants λ, k are to be determined so as to fulfill the identity $p'^2 = 4p^3 - g_2 p - g_3$. In particular try

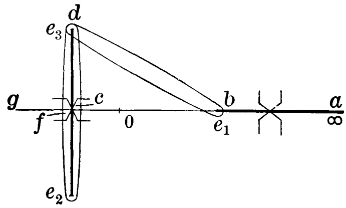
$$p(z) = A + \frac{\lambda}{\text{sn}^2(\sqrt{\lambda} z, k)}, \qquad A, \lambda, k \text{ constants.}$$

This form surely gives a double pole at $z = 0$ with the expansion $1/z^2$. The determination is relegated to the small text. The result is

$$p(z) = e_2 + \frac{e_1 - e_2}{\operatorname{sn}^2(\sqrt{\lambda}z,\, k)}, \qquad k^2 = \frac{e_3 - e_2}{e_1 - e_2} < 1,$$
$$\lambda = e_1 - e_2 > 0, \qquad \omega_1\sqrt{\lambda} = K, \qquad \omega_2\sqrt{\lambda} = iK'. \tag{23}$$

In the case of one real and two conjugate imaginary roots, the Riemann surface may be drawn in a similar manner. There are again two independent closed paths, one about e_2, e_3 and another about e_3, e_1. Let the integrals about these paths be respectively $2\,\omega_1$ and $2\,\omega_2$. That





$2\,\omega_1$ is real may be seen by deforming the path until it consists of a very distant portion along which the integral is infinitesimal and a path in and out along e_1, ∞, which gives a real value to the integral. As $2\,\omega_2$ is not known to be pure imaginary and may indeed be shown to be complex, it is natural to try to express $p(z)$ in terms of $\operatorname{cn} z$ of which one period is real and the other complex. Try

$$p(z) = A + \mu\,\frac{1 + \operatorname{cn}(2\sqrt{\mu}z,\, k)}{1 - \operatorname{cn}(2\sqrt{\mu}z,\, k)}.$$

This form surely gives a double pole at $z = 0$ with the expansion $1/z^2$. The determination is relegated to the small text. The result is

$$p(z) = e_1 + \mu\,\frac{1 + \operatorname{cn}(2\sqrt{\mu}z,\, k)}{1 - \operatorname{cn}(2\sqrt{\mu}z,\, k)}, \qquad k^2 = \frac{1}{2} - \frac{3\,e_1}{4\,\mu} < 1,$$
$$\mu^2 = (e_1 - e_2)(e_1 - e_3), \qquad \sqrt{\mu}\omega_1 = K, \qquad \sqrt{\mu}\omega_2 = \tfrac{1}{2}(K + iK'). \tag{23'}$$

To verify these determinations, substitute in $p'^2 = 4p^3 - g_2 p - g_3$.

$$p(z) = A + \frac{\lambda}{\operatorname{sn}^2(\sqrt{\lambda}z,\, k)}, \qquad p'(z) = -\frac{2\lambda^{\frac{3}{2}}}{\operatorname{sn}^3(\sqrt{\lambda}z,\, k)}\operatorname{cn}(\sqrt{\lambda}z,\, k)\operatorname{dn}(\sqrt{\lambda}z,\, k),$$

$$4\lambda^3\,\frac{(1 - \operatorname{sn}^2)(1 - k^2\operatorname{sn}^2)}{\operatorname{sn}^6} = 4\left(A^3 + \frac{3A^2\lambda}{\operatorname{sn}^2} + \frac{3A\lambda^2}{\operatorname{sn}^4} + \frac{\lambda^3}{\operatorname{sn}^6}\right) - g_2 A - \frac{g_2\lambda}{\operatorname{sn}^2} - g_3.$$

Equate coefficients of corresponding powers of sn^2. Hence the equations

$$4A^3 - g_2 A - g_3 = 0, \qquad 4\lambda^2 k^2 = 12A^2 - g_2\lambda, \qquad -\lambda(1 + k^2) = 3A.$$

The first shows that A is a root e. Let $A = e_2$. Note $-g_2 = e_1e_2 + e_1e_3 + e_2e_3$.

$$\lambda \cdot \lambda k^2 = 3e_2^2 + e_1e_2 + e_1e_3 + e_2e_3 = (e_1 - e_2)(e_3 - e_2),$$

$$\lambda + \lambda k^2 = -3e_2 = e_1 - e_2 + e_3 - e_2,$$

by virtue of the relation $e_1 + e_2 + e_3 = 0$. The solution is immediate as given.

To verify the second determination, the substitution is similar.

$$p(z) = A + \mu \frac{1 + \operatorname{cn} 2\sqrt{\mu}z}{1 - \operatorname{cn} 2\sqrt{\mu}z}, \qquad p'(z) = -\frac{4\mu^{\frac{3}{2}} \operatorname{sn} \operatorname{dn}}{(1 - \operatorname{cn})^2}.$$

$$[p'(z)]^2 = 16\mu^3 \frac{(1 + \operatorname{cn})(k'^2 + k^2 \operatorname{cn}^2)}{(1 - \operatorname{cn})^3} = 4\mu^3 [t^3 + 2(1 - 2k^2)t^2 + t]$$

where $t = (1 + \operatorname{cn})/(1 - \operatorname{cn})$. The identity $p'^2 = 4p^3 - g_2p - g_3$ is therefore

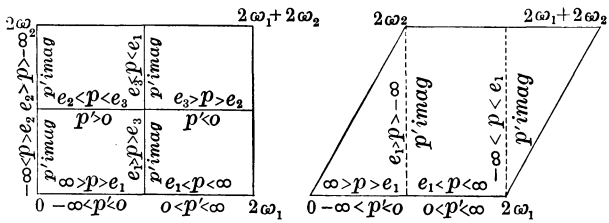
$$4\mu^3[t^3 + 2(1 - 2k^2)t^2 + t] = 4(A^3 + 3A^2\mu t + 3A\mu t^2 + \mu^3t^3) - g_2A - g_2\mu t - g_3.$$

$$4A^3 - g_2A - g_3 = 0, \qquad 4\mu^2 = 12A^2 - g_2, \qquad 2\mu(1 - 2k^2) = 3A.$$

Here let $A = e_1$. The solution then appears at once from the forms

$$\mu^2 = 3e_1^2 + e_1e_2 + e_1e_3 + e_2e_3 = (e_1 - e_2)(e_1 - e_3), \qquad \mu(1 - 2k^2) = 3A/2.$$

The expression of the function p in terms of the functions already studied permits the determination of the value of the function, and by inversion permits the solution of the equation $p(z) = c$. The function $p(z)$ may readily be expressed directly in terms of the theta series. In fact the periodic properties of the function and the corresponding properties of the quotients of theta series allow such a representation



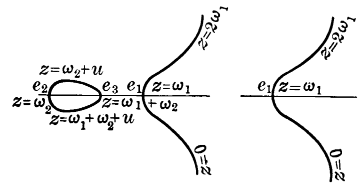
to be made from the work of § 175, provided the series be allowed complex values for q. But for practical purposes it is desirable to have the expression in terms of real quantities only, and this is the reason for a different expression in the two different cases here treated.*

The values of z for which $p(z)$ is real may be read off from (23) and (23′) or from the correspondence between the w-surface and the z-plane. They are indicated on the figures. The functions p and p' may be used to express parametrically the curve

$$y^2 = 4x^3 - g_2x - g_3 \quad \text{by} \quad y = p'(z), \qquad x = p(z).$$

* It is, however, *possible*, if desired, to transform the given cubic $4w^3 - g_2w - g_3$ with two complex roots into a similar cubic with all three roots real and thus avoid the duplicate forms. The transformation is not given here.

The figures indicate in the two cases the shape of the curves and the range of values of the parameter. As the function p is of the second order, the equation $p(z) = c$ has just two roots in the parallelogram, and as $p(z)$ is an even function, they will be of the form $z = a$ and $z = 2\omega_1 + 2\omega_2 - a$ and be symmetrically situated with respect to the center of the figure except in case a lies on the sides of the parallelogram so that $2\omega_1 + 2\omega_2 - a$ would lie on one of the excluded sides. The value of the odd function p' at these two points is equal and opposite. This corresponds precisely to the fact that to one value $x = c$ of x there are two equal and opposite values of y on the curve $y^2 = 4x^3 - g_2x - g_3$. Conversely to each point of the parallelogram corresponds one point of the curve and to points symmetrically situated with respect to the center correspond points of the curve symmetrically situated with respect to the x-axis. Unless z is such as to make both $p(z)$ and $p'(z)$ real, the point on the curve will be imaginary.



193. The curve $y^2 = 4x^3 - g_2x - g_3$ may be studied by means of the properties of doubly periodic functions. For instance

$$Ax + By + C = Ap'(z) + Bp(z) + C = 0$$

is the condition that the parameter z should be such that its representative point shall lie on the line $Ax + By + C = 0$. But the function $Ap'(z) + Bp(z) + C$ is doubly periodic with a pole of the third order; the function is therefore of the third order and there are just three points z_1, z_2, z_3 in the parallelogram for which the function vanishes. These values of z correspond to the three intersections of the line with the cubic curve. Now the roots of the doubly periodic function satisfy the relation

$$z_1 + z_2 + z_3 - 3 \times 0 = 2m_1\omega_1 + 2m_2\omega_2.$$

It may be observed that neither m_1 nor m_2 can be as great as 3. If conversely z_1, z_2, z_3 are three values of z which satisfy the relation $z_1 + z_2 + z_3 = 2m_1\omega_1 + 2m_2\omega_2$, the three corresponding points of the cubic will lie on a line. For if z_3' be the point in which a line through z_1, z_2 cuts the curve,

$$z_1 + z_2 + z_3' = 2m_1'\omega_1 + 2m_2'\omega_2, \qquad z_3 - z_3' = 2(m_1 - m_1')\omega_1 + 2(m_2 - m_2')\omega_2,$$

and hence z_3, z_3' are identical except for the addition of periods and must therefore be the same point on the parallelogram.

One application of this condition is to find the tangents to the curve from any point of the curve. Let z be the point from which and z' that to which the tangent is drawn. The condition then is $z + 2z' = 2m_1\omega_1 + 2m_2\omega_2$, and hence

$$z' = -\tfrac{1}{2}z, \qquad z' = -\tfrac{1}{2}z + \omega_1, \qquad z' = -\tfrac{1}{2}z + \omega_2, \qquad z' = -\tfrac{1}{2}z + \omega_1 + \omega_2$$

are the four different possibilities for z' corresponding to $m_1 = m_2 = 0$; $m_1 = 1$, $m_2 = 0$; $m_1 = 0$, $m_2 = 1$; $m_1 = 1$, $m_2 = 1$. To give other values to m_1 or m_2 would merely reproduce one of the four points except for the addition of complete periods. Hence there are four tangents to the curve from any point of the curve. The question of the reality of these tangents may readily be treated. Suppose z denotes a real point of the curve. If the point lies on the infinite portion, $0 < z < 2\omega_1$, and the first two points z' will also satisfy the conditions $0 < z' < 2\omega_1$ except for the possible addition of $2\omega_1$. Hence there are always two real tangents to the curve from any point of the infinite branch. In case the roots e_1, e_2, e_3 are all real, the last two points z' will correspond to real points of the oval portion and all four tangents are real; in the case of two imaginary roots these values of z' give imaginary points of the curve and there are only two real tangents. If the three roots are real and z corresponds to a point of the oval, z is of the form $\omega_2 + u$ and all four values of z' are complex,

$$-\tfrac{1}{2}\omega_2 - \tfrac{1}{2}u, \qquad -\tfrac{1}{2}\omega_2 - \tfrac{1}{2}u + \omega_1, \qquad +\tfrac{1}{2}\omega_2 - \tfrac{1}{2}u, \qquad +\tfrac{1}{2}\omega_2 - \tfrac{1}{2}u + \omega_1,$$

and none of the tangents can be real. The discussion is complete.

As an inflection point is a point at which a line may cut a curve in three coincident points, the condition $3z = 2m_1\omega_1 + 2m_2\omega_2$ holds for the parameter z of such points. The possible different combinations for z are nine:

$$\begin{array}{lll} z = 0 & \tfrac{2}{3}\omega_2 & \tfrac{4}{3}\omega_2 \\ \tfrac{2}{3}\omega_1 & \tfrac{2}{3}\omega_1 + \tfrac{2}{3}\omega_2 & \tfrac{2}{3}\omega_1 + \tfrac{4}{3}\omega_2 \\ \tfrac{4}{3}\omega_1 & \tfrac{4}{3}\omega_1 + \tfrac{2}{3}\omega_2 & \tfrac{4}{3}\omega_1 + \tfrac{4}{3}\omega_2. \end{array}$$

Of these nine inflections only the three in the first column are real. When any two inflections are given a third can be found so that $z_1 + z_2 + z_3$ is a complete period, and hence the inflections lie three by three on twelve lines.

If p and p' be substituted in $Ax^2 + Bxy + Cy^2 + Dx + Ey + F$, the result is a doubly periodic function of order 6 with a pole of the 6th order at the origin. The function then has 6 zeros in the parallelogram connected by the relation

$$z_1 + z_2 + z_3 + z_4 + z_5 + z_6 = 2m_1\omega_1 + 2m_2\omega_2,$$

and this is the condition which connects the parameters of the 6 points in which the cubic is cut by the conic $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. One application of interest is to the discussion of the conics which may be tangent to the cubic at three points z_1, z_2, z_3. The condition then reduces to $z_1 + z_2 + z_3 = m_1\omega_1 + m_2\omega_2$. If m_1, m_2 are 0 or any even numbers, this condition expresses the fact that the three points lie on a line and is therefore of little interest. The other possibilities, apart from the addition of complete periods, are

$$z_1 + z_2 + z_3 = \omega_1, \qquad z_1 + z_2 + z_3 = \omega_2, \qquad z_1 + z_2 + z_3 = \omega_1 + \omega_2.$$

In any of the three cases two points may be chosen at random on the cubic and the third point is then fixed. Hence there are three conics which are tangent to the cubic at any two assigned points and at some other point. Another application of interest is to the conics which have contact of the 5th order with the cubic. The condition is then $6z = 2m_1\omega_1 + 2m_2\omega_2$. As m_1, m_2 may have any of the 6 values from 0 to 5, there are 36 points on the cubic at which a conic may have contact of the 5th order. Among these points, however, are the nine inflections obtained by giving m_1, m_2 even values, and these are of little interest because the conic reduces to the inflectional tangent taken twice. There remain 27 points at which a conic may have contact of the 5th order with the cubic.

EXERCISES

1. The function $\zeta(z)$ is defined by the equation

$$-\zeta'(z) = p(z) \quad \text{or} \quad \zeta(z) = -\int p(z)\,dz = \frac{1}{z} - \frac{1}{3}c_1 z^3 + \cdots.$$

Show by Ex. 4, p. 516, that the value of ζ in the two cases is

$$\zeta(z) = -e_1 z + \sqrt{\lambda}\,E(\phi, k) + \sqrt{\lambda}\,\frac{\operatorname{cn}\sqrt{\lambda}z\operatorname{dn}\sqrt{\lambda}z}{\operatorname{sn}\sqrt{\lambda}z},$$

$$\zeta(z) = -(\mu + e_1)z + 2\sqrt{\mu}\,E(\phi, k) + \sqrt{\mu}\,\frac{\operatorname{cn}\sqrt{\mu}z}{\operatorname{sn}\sqrt{\mu}z\operatorname{dn}\sqrt{\mu}z}(2\operatorname{dn}^2\sqrt{\mu}z - 1),$$

where $\quad \lambda = e_1 - e_2, \quad k^2 = (e_3 - e_2)/(e_1 - e_2), \quad \phi = \sin^{-1}\operatorname{sn}\sqrt{\lambda}z,$

and $\quad \mu = \sqrt{(e_1 - e_2)(e_1 - e_3)}, \quad k^2 = \frac{1}{2} - 3e_1/4\mu, \quad \phi = \sin^{-1}\operatorname{sn}\sqrt{\mu}z.$

2. In case the three roots are real show that $p(z) - e_i$ is a square.

$$\sqrt{p(z) - e_1} = \sqrt{\lambda}\,\frac{\operatorname{cn}\sqrt{\lambda}z}{\operatorname{sn}\sqrt{\lambda}z}, \quad \sqrt{p(z) - e_2} = \frac{\sqrt{\lambda}}{\operatorname{sn}\sqrt{\lambda}z}, \quad \sqrt{p(z) - e_3} = \sqrt{\lambda}\,\frac{\operatorname{dn}\sqrt{\lambda}z}{\operatorname{sn}\sqrt{\lambda}z}.$$

What happens in case there is only one real root?

3. Let $p(z; g_2, g_3)$ denote the function p corresponding to the radical

$$\sqrt{4p^3 - g_2 p - g_3}.$$

Compute $p(\frac{1}{2}; 1, 0)$, $p(\frac{1}{4}; 0, \frac{1}{2})$, $p(\frac{3}{4}; 13, 6)$. Solve $p(z; 1, 0) = 2$, $p(z; 0, \frac{1}{2}) = 3$, $p(z; 13, 6) = 10$.

4. If 6 of the 9 points in which a cubic cuts $y^2 = 4x^3 - g_2 x - g_3$ are on a conic, the other three are in a straight line.

5. If a conic has contact of the second order with the cubic at two points, the points of contact lie on a line through one of the inflections.

6. How many of the points at which a conic may have contact of the 5th order with the cubic are real? Locate the points at least roughly.

7. If a conic cuts the cubic in four fixed and two variable points, the line joining the latter two passes through a fixed point of the cubic.

8. Consider the space curve $x = \operatorname{sn} t$, $y = \operatorname{cn} t$, $z = \operatorname{dn} t$. Show that to each point of the rectangle $4K$ by $4iK'$ corresponds one point of the curve and conversely. Show that the curve is the intersection of the cylinders $x^2 + y^2 = 1$ and $k^2x^2 + z^2 = 1$. Show that a plane cuts the curve in 4 points and determine the relation between the parameters of the points.

9. How many osculating planes may be drawn to the curve of Ex. 8 from any point on it? At how many points may a plane have contact of the 3d order with the curve and where are the points?

10. In case the roots are real show that $\zeta(z)$ has the form

$$\zeta(z) = \frac{\eta_1}{\omega_1}z + \sqrt{\lambda}\,\mathrm{Z}_1(\sqrt{\lambda}z), \qquad \eta_1 = \sqrt{\lambda}E - \frac{Ke_1}{\sqrt{\lambda}}.$$

Hence $$\log \sigma(z) = \int \zeta(z)\,dz = \frac{1}{2}\frac{\eta_1}{\omega_1}z^2 + \log H(\sqrt{\lambda}z) + C$$

or $$\sigma(z) = Ce^{\frac{1}{2}\frac{\eta_1}{\omega_1}z^2} H(\sqrt{\lambda}z).$$

11. By general methods like those of § 190 prove that

$$\frac{1}{p(z) - p(a)} = -\frac{1}{p'(a)}[\zeta(z+a) - \zeta(z-a) - 2\zeta(a)],$$

and $$\int \frac{dz}{p(z) - p(a)} = -\frac{1}{p'(a)}\log\frac{\sigma(z+a)}{\sigma(z-a)} + 2\frac{z\zeta(a)}{p'(a)}.$$

12. Let the functions θ be defined by these relations:

$$\theta(z) = H\left(\frac{Ku}{\omega_1}\right),\quad \theta_1(z) = H_1\left(\frac{Ku}{\omega_1}\right),\quad \theta_2(z) = \Theta\left(\frac{Ku}{\omega_1}\right),\quad \theta_3(z) = \Theta_1\left(\frac{Ku}{\omega_1}\right)$$

with $q = e^{\frac{\pi i \omega_2}{\omega_1}}$. Show that the θ-series converge if ω_1 is real and ω_2 is pure imaginary or complex with its imaginary part positive. Show more generally that the series converge if the angle from ω_1 to ω_2 is positive and less than 180°.

13. Let $$\sigma(z) = e^{\frac{\eta_1}{2\omega_1}z^2}\frac{\theta(z)}{\theta'(0)},\qquad \sigma_\alpha(z) = e^{\frac{\eta_1}{2\omega_1}z^2}\frac{\theta_\alpha(z)}{\theta_\alpha(0)}.$$

Prove $\sigma(z + 2\omega_1) = -e^{2\eta_1(z+\omega_1)}\sigma(z)$ and similar relations for $\sigma_\alpha(z)$.

14. Let $$2\eta_2 = \frac{2\eta_1\omega_2}{\omega_1} - \frac{\pi i}{\omega_1},\quad \text{or}\quad \eta_1\omega_2 - \eta_2\omega_1 = \frac{\pi i}{2}.$$

Prove $\sigma(z + 2\omega_2) = -e^{2\eta_2(z+\omega_2)}\sigma(z)$ and similar relations for $\sigma_\alpha(z)$.

15. Show that $\sigma(-z) = -\sigma(z)$ and develop $\sigma(z)$ as

$$\sigma(z) = z + \left[\frac{\eta_1}{2\omega_1} + \frac{1}{6}\frac{\theta'''(0)}{\theta'(0)}\right]z^3 + \cdots = z + 0\cdot z^3 + \cdots,\quad \text{if}\quad \eta_1 = -\frac{\omega_1}{3}\frac{\theta'''(0)}{\theta'(0)}.$$

16. With the determination of η_1 as in Ex. 15 prove that

$$\frac{d}{dz}\log\sigma(z) = \zeta(z),\qquad -\frac{d^2}{dz^2}\log\sigma(z) = -\zeta'(z) = p(z)$$

by showing that $p(z)$ as here defined is doubly periodic with periods $2\omega_1$, $2\omega_2$, with a pole $1/z^2$ of the second order at $z = 0$ and with no constant term in its development. State why this identifies $p(z)$ with the function of the text.