CHAPTER XVIII

FUNCTIONS OF A COMPLEX VARIABLE

178. General theorems. The complex function $u(x, y) + iv(x, y)$, where $u(x, y)$ and $v(x, y)$ are single valued real functions continuous and differentiable partially with respect to x and y, has been defined as a function of the complex variable $z = x + iy$ when and only when the relations $u'_x = v'_y$ and $u'_y = -v'_x$ are satisfied (§ 73). In this case the function has a derivative with respect to z which is independent of the way in which Δz approaches the limit zero. Let $w = f(z)$ be a function of a complex variable. Owing to the existence of the derivative the function is necessarily continuous, that is, if ϵ is an arbitrarily small positive number, a number δ may be found so small that

$$|f(z) - f(z_0)| < \epsilon \quad \text{when} \quad |z - z_0| < \delta, \tag{1}$$

and moreover this relation holds uniformly for all points z_0 of the region over which the function is defined, provided the region includes its bounding curve (see Ex. 3, p. 92).

It is further assumed that the derivatives u'_x, u'_y, v'_x, v'_y are continuous and that therefore the derivative $f'(z)$ is continuous.* The function is then said to be an *analytic function* (§ 126). All the functions of a complex variable here to be dealt with are analytic in general, although they may be allowed to fail of being analytic at certain specified points called *singular points*. The adjective "analytic" may therefore usually be omitted. The equations

$$w = f(z) \quad \text{or} \quad u = u(x, y), \qquad v = v(x, y)$$

define a transformation of the xy-plane into the uv-plane, or, briefer, of the z-plane into the w-plane; to each point of the former corresponds one and only one point of the latter (§ 63). If the Jacobian

$$\begin{vmatrix} u'_x & u'_y \\ v'_x & v'_y \end{vmatrix} = (u'_x)^2 + (v'_x)^2 = |f'(z)|^2 \tag{2}$$

* It may be proved that, in the case of functions of a complex variable, the continuity of the derivative follows from its existence, but the proof will not be given here.

of the transformation does not vanish at a point z_0, the equations may be solved in the neighborhood of that point, and hence to each point of the second plane corresponds only one of the first:

$$x = x(u, v), \qquad y = y(u, v) \quad \text{or} \quad z = \phi(w).$$

Therefore it is seen that *if* $w = f(z)$ *is analytic in the neighborhood of* $z = z_0$, *and if the derivative* $f'(z_0)$ *does not vanish, the function may be solved as* $z = \phi(w)$, where ϕ is the inverse function of f, and is likewise analytic in the neighborhood of the point $w = w_0$. It may readily be shown that, as in the case of real functions, the derivatives $f'(z)$ and $\phi'(w)$ are reciprocals. Moreover, it may be seen that *the transformation is conformal*, that is, that the angle between any two curves is unchanged by the transformation (§ 63). For consider the increments

$$\Delta w = [f'(z_0) + \zeta]\,\Delta z = f'(z_0)\,[1 + \zeta/f'(z_0)]\,\Delta z. \qquad f'(z_0) \neq 0.$$

As Δz and Δw are the chords of the curves before and after transformation, the geometrical interpretation of the equation, apart from the infinitesimal ζ, is that the chords Δz are magnified in the ratio $|f'(z_0)|$ to 1 and turned through the angle of $f'(z_0)$ to obtain the chords Δw (§ 72). In the limit it follows that the tangents to the w-curves are inclined at an angle equal to the angle of the corresponding z-curves plus the angle of $f'(z_0)$. The angle between two curves is therefore unchanged.

The existence of an inverse function and of the geometric interpretation of the transformation as conformal both become illusory at points for which the derivative $f'(z)$ vanishes. Points where $f'(z) = 0$ are called *critical points* of the function (§ 183).

It has further been seen that the integral of a function which is analytic over any simply connected region is independent of the path and is zero around any closed path (§ 124); if the region be not simply connected but the function is analytic, the integral about any closed path which may be shrunk to nothing is zero and the integrals about any two closed paths which may be shrunk into each other are equal (§ 125). Furthermore Cauchy's result that the value

$$f(z) = \frac{1}{2\pi i}\int_{\bigcirc} \frac{f(t)}{t - z}\,dt \tag{3}$$

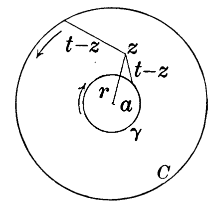
of a function, which is analytic upon and within a closed path, may be found by integration around the path has been derived (§ 126). By a transformation the Taylor development of the function has been found whether in the finite form with a remainder (§ 126) or as an infinite series (§ 167). It has also been seen that any infinite power series which converges is differentiable and hence defines an analytic function within its circle of convergence (§ 166).

It has also been shown that the sum, difference, product, and quotient of any two functions will be analytic for all points at which both functions are analytic, except at the points at which the denominator, in the case of a quotient, may vanish (Ex. 9, p. 163). The result is evidently extensible to the case of any rational function of any number of analytic functions.

From the possibility of development in series follows that *if two functions are analytic in the neighborhood of a point and have identical values upon any curve drawn through that point*, or even upon any set of points which approach that point as a limit, *then the functions are identically equal within their common circle of convergence and over all regions which can be reached by* (§ 169) *continuing the functions analytically.* The reason is that a set of points converging to a limiting point is all that is needed to prove that two power series are identical provided they have identical values over the set of points (Ex. 9, p. 439). This theorem is of great importance because it shows that if a function is defined for a dense set of real values, any one extension of the definition, which yields a fraction that is analytic for those values and for complex values in their vicinity, must be equivalent to any other such extension. It is also useful in discussing *the principle of permanence of form;* for if the two sides of an equation are identical for a set of values which possess a point of condensation, say, for all real rational values in a given interval, and if each side is an analytic function, then the equation must be true for all values which may be reached by analytic continuation.

For example, the equation $\sin x = \cos(\frac{1}{2}\pi - x)$ is known to hold for the values $0 \leqq x \leqq \frac{1}{2}\pi$. Moreover the functions $\sin z$ and $\cos z$ are analytic for all values of z whether the definition be given as in § 74 or whether the functions be considered as defined by their power series. Hence the equation must hold for all real or complex values of x. In like manner from the equation $e^x e^y = e^{x+y}$ which holds for real rational exponents, the equation $e^z e^w = e^{z+w}$ holding for all real and imaginary exponents may be deduced. For if y be given any rational value, the functions of x on each side of the sign are analytic for all values of x real or complex, as may be seen most easily by considering the exponential as defined by its power series. Hence the equation holds when x has any complex value. Next consider x as fixed at any desired complex value and let the two sides be considered as functions of y regarded as complex. It follows that the equation must hold for any value of y. The equation is therefore true for any value of z and w.

179. Suppose that a function is analytic in all points of a region except at some one point within the region, and let it be assumed that the function ceases to be analytic at that point because it ceases to be continuous. The discontinuity may be either finite or infinite. In case the discontinuity is finite let $|f(z)| < G$ in the neighborhood of the point $z = a$ of discontinuity. Cut the point out with a small circle and apply Cauchy's Integral to a ring surrounding the point. The integral is applicable because at all points on and within the ring the function is analytic. If the small circle be replaced by a smaller circle into which it may be shrunk, the value of the integral will not be changed.



$$f(z) = \frac{1}{2\pi i}\left[\int_C \frac{f(t)}{t-z}\,dt + \int_{\gamma_i} \frac{f(t)}{t-z}\,dt\right], \qquad i = 1, 2, \cdots.$$

Now the integral about γ_i which is constant can be made as small as desired by taking the circle small enough; for $|f(t)| < G$ and $|t - z| > |a - z| - r_i$, where r_i is the radius of the circle γ_i and hence the integral is less than $2\pi r_i G/[|z - a| - r_i]$. As the integral is constant, it must therefore be 0 and may be omitted. The remaining integral about C, however, defines a function which is analytic at $z = a$. Hence if $f(a)$ be chosen as defined by this integral instead of the original definition, the discontinuity disappears. *Finite discontinuities may therefore be considered as due to bad judgment in defining a function at some point;* and may therefore be disregarded.

In the case of infinite discontinuities, the function may either *become infinite for all methods of approach* to the point of discontinuity, or it may *become infinite for some methods of approach and remain finite for other methods.* In the first case the function is said to have a *pole* at the point $z = a$ of discontinuity; in the second case it is said to have an *essential singularity.* In the case of a pole consider the reciprocal function

$$F(z) = \frac{1}{f(z)}, \qquad z \neq a, \qquad F(a) = 0.$$

The function $F(z)$ is analytic at all points near $z = a$ and remains finite, in fact approaches 0, as z approaches a. As $F(a) = 0$, it is seen that $F(z)$ has no finite discontinuity at $z = a$ and is analytic also at $z = a$. Hence the Taylor expansion

$$F(z) = a_m(z - a)^m + a_{m+1}(z - a)^{m+1} + \cdots$$

is proper. If E denotes a function neither zero nor infinite at $z = a$, the following transformations may be made.

$$F(z) = (z-a)^m E_1(z), \qquad f(z) = (z-a)^{-m} E_2(z),$$

$$f(z) = \frac{C_{-m}}{(z-a)^m} + \frac{C_{-m+1}}{(z-a)^{m-1}} + \cdots + \frac{C_{-1}}{z-a}$$
$$+ C_0 + C_1(z-a) + C_2(z-a)^2 + \cdots.$$

In other words, a function which has a pole at $z = a$ may be written as the product of some power $(z-a)^{-m}$ by an E-function; and as the E-function may be expanded, the function may be expanded into a power series which contains a certain number of negative powers of $(z-a)$. *The order m of the highest negative power* is called *the order of the pole.* Compare Ex. 5, p. 449.

If the function $f(z)$ be integrated around a closed curve lying within the circle of convergence of the series $C_0 + C_1(z-a) + \cdots$, then

$$\int_{\bigcirc} f(z)\,dz = \int \frac{C_{-m}dz}{(z-a)^m} + \cdots + \int_{\bigcirc} \frac{C_{-1}dz}{z-a}$$
$$+ \int_{\bigcirc} [C_0 + C_1(z-a) + \cdots]dz = 2\pi i C_{-1},$$

or
$$\int_{\bigcirc} f(z)\,dz = 2\pi i C_{-1}; \tag{4}$$

for the first $m-1$ terms may be integrated and vanish, the term $C_{-1}/(z-a)$ leads to the logarithm $C_{-1}\log(z-a)$ which is multiple valued and takes on the increment $2\pi i C_{-1}$, and the last term vanishes because it is the integral of an analytic function. The total value of the integral of $f(z)$ about a small circuit surrounding a pole is therefore $2\pi i C_{-1}$. The value of the integral about any larger circuit within which the function is analytic except at $z = a$ and which may be shrunk into the small circuit, will also be the same quantity. The coefficient C_{-1} of the term $(z-a)^{-1}$ is called *the residue of the pole;* it cannot vanish if the pole is of the first order, but may if the pole is of higher order.

The discussion of the behavior of a function $f(z)$ when z becomes infinite may be carried on by making a transformation. Let

$$z' = \frac{1}{z}, \qquad z = \frac{1}{z'}, \qquad f(z) = f\left(\frac{1}{z'}\right) = F(z'). \tag{5}$$

To large values of z correspond small values of z'; if $f(z)$ is analytic for all large values of z, then $F(z')$ will be analytic for values of z' near the origin. At $z' = 0$ the function $F(z')$ may not be defined by (5); but if $F(z')$ remains finite for small values of z', a definition may be given so that it is analytic also at $z' = 0$. In this case $F(0)$ is said to be the value of $f(z)$ when z is infinite and the notation $f(\infty) = F(0)$ may be used. If $F(z')$ does not remain finite but has a pole at $z' = 0$, then $f(z)$ is said to have a pole of the same order at $z = \infty$; and if $F(z')$ has an essential singularity at $z' = 0$, then $f(z)$ is said to have an essential singularity at $z = \infty$. Clearly if $f(z)$ has a pole at $z = \infty$, the value of $f(z)$ must become indefinitely great no matter how z becomes infinite; but if $f(z)$ has an essential singularity at $z = \infty$, there will be some ways in which z may become infinite so that $f(z)$ remains finite, while there are other ways so that $f(z)$ becomes infinite.

Strictly speaking there is no point of the z-plane which corresponds to $z' = 0$. Nevertheless it is convenient to speak as if there were such a point, to call it *the point at infinity*, and to designate it as $z = \infty$. If then $F(z')$ is analytic for $z' = 0$ so that $f(z)$ may be said to be analytic at infinity, the expansions

$$F(z') = C_0 + C_1 z' + C_2 z'^2 + \cdots + C_n z'^n + \cdots =$$

$$f(z) = C_0 + \frac{C_1}{z} + \frac{C_2}{z^2} + \cdots + \frac{C_n}{z^n} + \cdots$$

are valid; the function $f(z)$ has been *expanded about the point at infinity into a descending power series in* z, and the series will converge for all points z *outside* a circle $|z| = R$. For a pole of order m at infinity

$$f(z) = C_{-m} z^m + C_{-m+1} z^{m-1} + \cdots + C_{-1} z + C_0 + \frac{C_1}{z} + \frac{C_2}{z^2} + \cdots.$$

Simply because it is convenient to introduce the concept of the point at infinity for the reason that in many ways the totality of large values for z does not differ from the totality of values in the neighborhood of a finite point, it should not be inferred that the point at infinity has all the properties of finite points.

EXERCISES

1. Discuss $\sin(x + y) = \sin x \cos y + \cos x \sin y$ for permanence of form.

2. If $f(z)$ has an essential singularity at $z = a$, show that $1/f(z)$ has an essential singularity at $z = a$. Hence infer that there is some method of approach to $z = a$ such that $f(z) \doteq 0$.

3. By treating $f(z) - c$ and $[f(z) - c]^{-1}$ show that at an essential singularity a function may be made to approach any assigned value c by a suitable method of approaching the singular point $z = a$.

4. Find the order of the poles of these functions at the origin:

(α) $\cot z$, (β) $\csc^2 z \log(1 - z)$, (γ) $z(\sin z - \tan z)^{-1}$.

5. Show that if $f(z)$ vanishes at $z = a$ once or n times, the quotient $f'(z)/f(z)$ has the residue 1 or n. Show that if $f(z)$ has a pole of the mth order at $z = a$, the quotient has the residue $-m$.

6. From Ex. 5 prove the important theorem that: If $f(z)$ is analytic and does not vanish upon a closed curve and has no singularities other than poles within the curve, then

$$\frac{1}{2\pi i}\int_{\bigcirc}\frac{f'(z)}{f(z)}dz = n_1 + n_2 + \cdots + n_k - m_1 - m_2 - \cdots - m_l = N - M,$$

where N is the total number of roots of $f(z) = 0$ within the curve and M is the sum of the orders of the poles.

7. Apply Ex. 6 to $1/P(z)$ to show that a polynomial $P(z)$ of the nth order has just n roots within a sufficiently large curve.

8. Prove that e^z cannot vanish for any finite value of z.

9. Consider the residue of $zf'(z)/f(z)$ at a pole or vanishing point of $f(z)$. In particular prove that if $f(z)$ is analytic and does not vanish upon a closed curve and has no singularities but poles within the curve, then

$$\frac{1}{2\pi i}\int_{\bigcirc}\frac{zf'(z)}{f(z)}dz = n_1a_1 + n_2a_2 + \cdots + n_ka_k - m_1b_1 - m_2b_2 - \cdots - m_lb_l,$$

where $a_1, a_2, \cdots, a_k$ and $n_1, n_2, \cdots, n_k$ are the positions and orders of the roots, and $b_1, b_2, \cdots, b_l$ and $m_1, m_2, \cdots, m_l$ of the poles of $f(z)$.

10. Prove that $\Theta_1(z)$, p. 469, has only one root within a rectangle $2K$ by $2iK'$.

11. State the behavior (analytic, pole, or essential singularity) at $z = \infty$ for:

(α) $z^2 + 2z$, (β) e^z, (γ) $z/(1 + z)$, (δ) $z/(z^3 + 1)$.

12. Show that if $f(z) = (z - \alpha)^k E(z)$ with $-1 < k < 0$, the integral of $f(z)$ about an infinitesimal contour surrounding $z = \alpha$ is infinitesimal. What analogous theorem holds for an infinite contour?

180. Characterization of some functions. The study of the limitations which are put upon a function when certain of its properties are known is important. For example, *a function which is analytic for all values of z including also $z = \infty$ is a constant.* To show this, note that as the function nowhere becomes infinite, $|f(z)| < G$. Consider the difference $f(z_0) - f(0)$ between the value at any point $z = z_0$ and at the origin. Take a circle concentric with $z = 0$ and of radius $R > |z_0|$. Then by Cauchy's Integral

$$f(z_0) - f(0) = \frac{1}{2\pi i}\left[\int_{\bigcirc}\frac{f(t)}{t - z_0}dt - \int_{\bigcirc}\frac{f(t)}{t - 0}dt\right] = \frac{z_0}{2\pi i}\int_{\bigcirc}\frac{f(t)dt}{t(t - z_0)},$$

or

$$|f(z_0) - f(0)| < \frac{|z_0|}{2\pi}\frac{2\pi RG}{R(R - |z_0|)} = \frac{G|z_0|}{R - |z_0|}.$$

By taking R large enough the difference, which is constant, may be made as small as desired and hence must be zero; hence $f(z) = f(0)$.

Any rational function $f(z) = P(z)/Q(z)$, where $P(z)$ and $Q(z)$ are polynomials in z and may be assumed to be devoid of common factors, can have as singularities merely poles. There will be a pole at each point at which the denominator vanishes; and if the degree of the numerator exceeds that of the denominator, there will be a pole at infinity of order equal to the difference of those degrees. Conversely it may be shown that *any function which has no other singularity than a pole of the mth order at infinity must be a polynomial of the mth order;* that *if the only singularities are a finite number of poles, whether at infinity or at other points, the function is a rational function;* and finally that *the knowledge of the zeros and poles with the multiplicity or order of each is sufficient to determine the function except for a constant multiplier.*

For, in the first place, if $f(z)$ is analytic except for a pole of the mth order at infinity, the function may be expanded as

$$f(z) = a_{-m}z^m + \cdots + a_{-1}z + a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots,$$

or

$$f(z) - [a_{-m}z^m + \cdots + a_{-1}z] = a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots.$$

The function on the right is analytic at infinity, and so must its equal on the left be. The function on the left is the difference of a function which is analytic for all finite values of z and a polynomial which is also analytic for finite values. Hence the function on the left or its equal on the right is analytic for all values of z including $z = \infty$, and is a constant, namely a_0. Hence

$$f(z) = a_0 + a_{-1}z + \cdots + a_{-m}z^m \quad \text{is a polynomial of order } m.$$

In the second place let $z_1, z_2, \cdots, z_k, \infty$ be poles of $f(z)$ of the respective orders $m_1, m_2, \cdots, m_k, m$. The function

$$\phi(z) = (z - z_1)^{m_1}(z - z_2)^{m_2} \cdots (z - z_k)^{m_k} f(z)$$

will then have no singularity but a pole of order $m_1 + m_2 + \cdots + m_k + m$ at infinity; it will therefore be a polynomial, and $f(z)$ is rational. As the numerator $\phi(z)$ of the fraction cannot vanish at $z_1, z_2, \cdots, z_k$, but must have $m_1 + m_2 + \cdots + m_k + m$ roots, the knowledge of these roots will determine the numerator $\phi(z)$ and hence $f(z)$ except for a constant multiplier. It should be noted that if $f(z)$ has not a pole at infinity but has a zero of order m, the above reasoning holds on changing m to $-m$.

When $f(z)$ has a pole at $z = a$ of the mth order, the expansion of $f(z)$ about the pole contains certain negative powers

$$P(z - a) = \frac{c_{-m}}{(z - a)^m} + \frac{c_{-m+1}}{(z - a)^{m-1}} + \cdots + \frac{c_{-1}}{z - a}$$

and the difference $f(z) - P(z - a)$ is analytic at $z = a$. The terms $P(z - a)$ are called *the principal part of the function $f(z)$ at the pole a.*

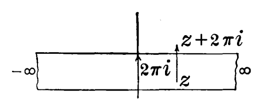
If the function has only a finite number of finite poles and the principal parts corresponding to each pole are known,

$$\phi(z) = f(z) - P_1(z - z_1) - P_2(z - z_2) - \cdots - P_k(z - z_k)$$

is a function which is everywhere analytic for finite values of z and behaves at $z = \infty$ just as $f(z)$ behaves there, since $P_1, P_2, \cdots, P_k$ all vanish at $z = \infty$. If $f(z)$ is analytic at $z = \infty$, then $\phi(z)$ is a constant; if $f(z)$ has a pole at $z = \infty$, then $\phi(z)$ is a polynomial in z and all of the polynomial except the constant term is the principal part of the pole at infinity. Hence *if a function has no singularities except a finite number of poles, and the principal parts at these poles are known, the function is determined except for an additive constant.*

From the above considerations it appears that if a function has no other singularities than a finite number of poles, the function is rational; and that, moreover, the function is determined in factored form, except for a constant multiplier, when the positions and orders of the finite poles and zeros are known; or is determined, except for an additive constant, in a development into partial fractions if the positions and principal parts of the poles are known. All single valued functions other than rational functions must therefore have either an infinite number of poles or some essential singularities.

181. The exponential function $e^z = e^x(\cos y + i \sin y)$ has no finite singularities and its singularity at infinity is necessarily essential. The function is periodic (§ 74) with the period $2\pi i$, and hence will take on all the different values which it can have, if z, instead of being allowed all values, is restricted to have its pure imaginary part y between two limits $y_0 \leqq y < y_0 + 2\pi$; that is, to consider the values of e^z it is merely necessary to consider the values in a strip of the z-plane parallel to the axis of reals and of breadth 2π (but lacking one edge). For convenience the strip may be taken immediately above the axis of reals. The function e^z becomes infinite as z moves out toward the right, and zero as z moves out toward the left in the strip. If $c = a + bi$ is any number other than 0, there is one and only one point in the strip at which $e^z = c$. For



$$e^x = \sqrt{a^2 + b^2} \quad \text{and} \quad \cos y + i \sin y = \frac{a}{\sqrt{a^2 + b^2}} + i \frac{b}{\sqrt{a^2 + b^2}}$$

have only one solution for x and only one for y if y be restricted to an interval 2π. All other points for which $e^z = c$ have the same value for x and some value $y \pm 2n\pi$ for y.

Any rational function of e^z, as

$$R(e^z) = C\,\frac{e^{nz} + a_1 e^{(n-1)z} + \cdots + a_{n-1}e^z + a_n}{e^{mz} + b_1 e^{(m-1)z} + \cdots + b_{m-1}e^z + b_m},$$

will also have the period $2\pi i$. When z moves off to the left in the strip, $R(e^z)$ will approach Ca_n/b_m if $b_m \neq 0$ and will become infinite if $b_m = 0$. When z moves off to the right, $R(e^z)$ must become infinite if $n > m$, approach C if $n = m$, and approach 0 if $n < m$. The denominator may be factored into terms of the form $(e^z - \alpha)^k$, and if the fraction is in its lowest terms each such factor will represent a pole of the kth order in the strip because $e^z - \alpha = 0$ has just one simple root in the strip. Conversely it may be shown that: *Any function $f(z)$ which has the period $2\pi i$, which further has no singularities but a finite number of poles in each strip, and which either becomes infinite or approaches a finite limit as z moves off to the right or to the left, must be $f(z) = R(e^z)$, a rational function of e^z.*

The proof of this theorem requires several steps. Let it first be assumed that $f(z)$ remains finite at the ends of the strip and has no poles. Then $f(z)$ is finite over all values of z, including $z = \infty$, and must be merely constant. Next let $f(z)$ remain finite at the ends of the strip but let it have poles at some points in the strip. It will be shown that a rational function $R(e^z)$ may be constructed such that $f(z) - R(e^z)$ remains finite all over the strip, including the portions at infinity, and that therefore $f(z) = R(e^z) + C$. For let the principal part of $f(z)$ at any pole $z = c$ be

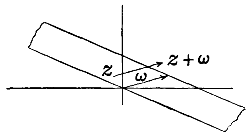
$$P(z-c) = \frac{c_{-k}}{(z-c)^k} + \frac{c_{-k+1}}{(z-c)^{k-1}} + \cdots + \frac{c_{-1}}{z-c}; \quad \text{then} \quad \frac{c_{-k}e^{kc}}{(e^z - e^c)^k} = \frac{c_{-k}}{(z-c)^k} + \cdots$$

is a rational function of e^z which remains finite at both ends of the strip and is such that the difference between it and $P(z-c)$ or $f(z)$ has a pole of not more than the $(k-1)$st order at $z = c$. By subtracting a number of such terms from $f(z)$ the pole at $z = c$ may be eliminated without introducing any new pole. Thus all the poles may be eliminated, and the result is proved.

Next consider the case where $f(z)$ becomes infinite at one or at both ends of the strip. If $f(z)$ happens to approach 0 at one end, consider $f(z) + C$, which cannot approach 0 at either end of the strip. Now if $f(z)$ or $f(z) + C$, as the case may be, had an infinite number of zeros in the strip, these zeros would be confined within finite limits and would have a point of condensation and the function would vanish identically. It must therefore be that the function has only a finite number of zeros; its reciprocal will therefore have only a finite number of poles in the strip and will remain finite at the ends of the strips. Hence the reciprocal and consequently the function itself is a rational function of e^z. The theorem is completely demonstrated.

If the relation $f(z + \omega) = f(z)$ is satisfied by a function, the function is said to have the period ω. The function $f(2\pi iz/\omega)$ will then have the period $2\pi i$. Hence it follows that *if $f(z)$ has the period ω, becomes infinite or remains finite at the ends of a strip of vector breadth*

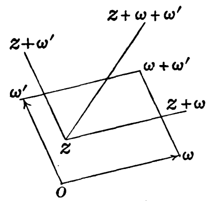
ω, *and has no singularities but a finite number of poles in the strip, the function is a rational function of* $e^{2\pi iz/\omega}$. In particular if the period is 2π, the function is rational in e^{iz}, as is the case with $\sin z$ and $\cos z$; and if the period is π, the function is rational in $e^{iz/2}$, as is $\tan z$. It thus appears that the single valued elementary functions, namely, rational functions, and rational functions of the exponential or trigonometric functions, have simple general properties which are characteristic of these classes of functions.



182. Suppose a function $f(z)$ has two independent periods so that

$$f(z+\omega)=f(z), \qquad f(z+\omega')=f(z).$$

The function then has the same value at z and at any point of the form $z+m\omega+n\omega'$, where m and n are positive or negative integers. The function takes on all the values of which it is capable in a parallelogram constructed on the vectors ω and ω'. Such a function is called *doubly periodic*. As the values of the function are the same on opposite sides of the parallelogram, only two sides and the one included vertex are supposed to belong to the figure. It has been seen that some doubly periodic functions exist (§ 177); but without reference to these special functions many important theorems concerning doubly periodic functions may be proved, subject to a subsequent demonstration that the functions do exist.



If a doubly periodic function has no singularities in the parallelogram, it must be constant; for the function will then have no singularities at all. *If two periodic functions have the same periods and have the same poles and zeros* (each to the same order) *in the parallelogram, the quotient of the functions is a constant; if they have the same poles and the same principal parts at the poles, their difference is a constant.* In these theorems (and all those following) it is assumed that the functions have no essential singularity in the parallelogram. The proof of the theorems is left to the reader. If $f(z)$ is doubly periodic, $f'(z)$ is also doubly periodic. The integral of a doubly periodic function taken around any parallelogram equal and parallel to the parallelogram of periods is zero; for the function repeats itself on opposite sides of the figure while the differential dz changes sign. Hence in particular

$$\int_{\square} f(z)\,dz=0, \qquad \int_{\square}\frac{f'(z)}{f(z)}\,dz=0, \qquad \int_{\square}\frac{f'(z)\,dz}{f(z)-C}=0.$$

The first integral shows that *the sum of the residues of the poles in the parallelogram is zero;* the second, that *the number of zeros is equal to the number of poles* provided multiplicities are taken into account; the third, that *the number of zeros of* $f(z) - C$ *is the same as the number of zeros or poles of* $f(z)$, because the poles of $f(z)$ and $f(z) - C$ are the same.

The common number m of poles of $f(z)$ or of zeros of $f(z)$ or of roots of $f(z) = C$ in any one parallelogram is called *the order of the doubly periodic function.* As the sum of the residues vanishes, it is impossible that there should be a single pole of the first order in the parallelogram. Hence there can be no functions of the first order and the simplest possible functions would be of the second order with the expansions

$$\frac{1}{(z-a)^2} + c_0 + c_1(z-c) + \cdots \quad \text{or} \quad \frac{1}{z-a_1} + c_0 + \cdots \quad \text{and} \quad \frac{-1}{z-a_2} + c'_0 + \cdots$$

in the neighborhood of a single pole at $z = a$ of the second order or of the two poles of the first order at $z = a_1$ and $z = a_2$. Let it be assumed that when the periods ω, ω' are given, a doubly periodic function $g(z, a)$ with these periods and with a double pole at $z = a$ exists, and similarly that $h(z, a_1, a_2)$ with simple poles at a_1 and a_2 exists.

Any doubly periodic function $f(z)$ *with the periods* ω, ω' *may be expressed as a polynomial in the functions* $g(z, a)$ *and* $h(z, a_1; a_2)$ *of the second order.* For in the first place if the function $f(z)$ has a pole of even order $2k$ at $z = a$, then $f(z) - C[g(z, a)]^k$, where C is properly chosen, will have a pole of order less than $2k$ at $z = a$ and will have no other poles than $f(z)$. Hence the order of $f(z) - C[g(z, a)]^k$ is less than that of $f(z)$. And if $f(z)$ has a pole of odd order $2k + 1$ at $z = a$, the function $f(z) - C[g(z, a)]^k h(z, a, b)$, with the proper choice of C, will have a pole of order $2k$ or less at $z = a$ and will gain a simple pole at $z = b$. Thus although $f - Cg^k h$ will generally not be of lower order than f, it will have a complex pole of odd order split into a pole of even order and a pole of the first order; the order of the former may be reduced as before and pairs of the latter may be removed. By repeated applications of the process a function may be obtained which has no poles and must be constant. The theorem is therefore proved.

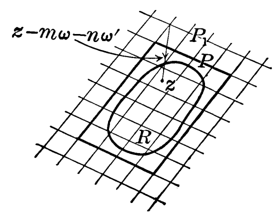
With the aid of series it is possible to write down some doubly periodic functions. In particular consider the series

$$p(z) = \frac{1}{z^2} + \sum{}' \left[\frac{1}{(z - m\omega - n\omega')^2} - \frac{1}{(m\omega + n\omega')^2} \right]$$

$$\text{and} \qquad p'(z) = -2 \sum \frac{1}{(z - m\omega - n\omega')^3}, \qquad (6)$$

where the second Σ denotes summation extended over all values of m, n, whether positive or negative or zero, and Σ' denotes summation extended over all these values except the pair $m = n = 0$. As the summations extend over all possible values for m, n, the series constructed for $z + \omega$ and for $z + \omega'$ must have the same terms as those for z, the only difference being a different arrangement of the terms. If, therefore, the series are absolutely convergent so that the order of the terms is immaterial, the functions must have the periods ω, ω'.

Consider first the convergence of the series $p'(z)$. For $z = m\omega + n\omega'$, that is, at the vertices of the net of parallelograms one term of the series becomes infinite and the series cannot converge. But if z be restricted to a finite region R about $z = 0$, there will be only a finite number of terms which can become infinite. Let a parallelogram P large enough to surround the region be drawn, and consider only the vertices which lie outside this parallelogram. For convenience of computation let the points $z = m\omega + n\omega'$ outside P be considered as arranged on successive parallelograms P_1, P_2, $\cdots$, P_k, $\cdots$. If the number of vertices on P be ν, the number on P_1 is $\nu + 8$ and on P_k is $\nu + 8k$. The shortest vector $z - m\omega - n\omega'$ from z to any vertex of P_1 is longer than a, where a is the least altitude of the parallelogram of periods. The total contribution of P_1 to $p'(z)$ is therefore less than $(\nu + 8)a^{-3}$ and the value contributed by all the vertices on successive parallelograms will be less than



$$S = \frac{\nu + 8}{a^3} + \frac{\nu + 8 \cdot 2}{(2\,a)^3} + \frac{\nu + 8 \cdot 3}{(3\,a)^3} + \cdots + \frac{\nu + 8 \cdot k}{(ka)^3} + \cdots.$$

This series of positive terms converges. Hence the infinite series for $p'(z)$, when the first terms corresponding to the vertices within P_1 are disregarded, converges absolutely and even uniformly so that it represents an analytic function. The whole series for $p'(z)$ therefore represents a doubly periodic function of the *third order* analytic everywhere except at the vertices of the parallelograms where it has a pole of the third order. As the part of the series $p'(z)$ contributed by vertices outside P is uniformly convergent, it may be integrated from 0 to z to give the corresponding terms in $p(z)$ which will also be absolutely convergent because the terms, grouped as for $p'(z)$, will be less than the terms of lS where l is the length of the path of integration from 0 to z. The other terms of $p'(z)$, thus far disregarded, may be integrated at sight to obtain the corresponding terms of $p(z)$. Hence $p'(z)$ is really the derivative of $p(z)$; and as $p(z)$ converges absolutely except for the vertices of the parallelograms, it is clearly doubly periodic of the *second order* with the periods ω, ω', for the same reason that $p'(z)$ is periodic.

It has therefore been shown that doubly periodic functions exist, and hence the theorems deduced for such functions are valid. Some further important theorems are indicated among the exercises. They lead to the inference that any doubly periodic function which has the periods ω, ω' and has no other singularities than poles may be expressed as a rational function of $p(z)$ and $p'(z)$, or as an irrational function of $p(z)$ alone, the only irrationalities being square roots. Thus by employing only the general methods of the theory of functions of a complex variable an entirely new category of functions has been characterized and its essential properties have been proved.

EXERCISES

1. Find the principal parts at $z = 0$ for the functions of Ex. 4, p. 481.

2. Prove by Ex. 6, p. 482, that $e^z - c = 0$ has only one root in the strip.

3. How does $e^{(e^z)}$ behave as z becomes infinite in the strip?

4. If the values $R(e^z)$ approaches when z becomes infinite in the strip are called exceptional values, show that $R(e^z)$ takes on every value other than the exceptional values k times in the strip, k being the greater of the two numbers n, m.

5. Show by Ex. 9, p. 482, that in any parallelogram of periods the sum of the positions of the roots less the sum of the positions of the poles of a doubly periodic function is $m\omega + n\omega'$, where m and n are integers.

6. Show that the terms of $p'(z)$ may be associated in such a way as to prove that $p'(-z) = -p'(z)$, and hence infer that the expansions are

$$p'(z) = -2z^{-3} + 2c_1z + 4c_2z^3 + \cdots, \qquad \text{only odd powers,}$$

and

$$p(z) = z^{-2} + c_1z^2 + c_2z^4 + \cdots, \qquad \text{only even powers.}$$

7. Examine the series (6) for $p'(z)$ to show that $p'(\frac{1}{2}\omega) = p'(\frac{1}{2}\omega') = p'(\frac{1}{2}\omega + \frac{1}{2}\omega') = 0$. Why can $p'(z)$ not vanish for any other points in the parallelogram?

8. Let $p(\frac{1}{2}\omega) = e$, $p(\frac{1}{2}\omega') = e'$, $p(\frac{1}{2}\omega + \frac{1}{2}\omega') = e''$. Prove the identity of the doubly periodic functions $[p'(z)]^2$ and $4[p(z) - e][p(z) - e'][p(z) - e'']$.

9. By examining the series defining $p(z)$ show that any two points $z = a$ and $z = a'$ such that $p(a) = p(a')$ are symmetrically situated in the parallelogram with respect to the center $z = \frac{1}{2}(\omega + \omega')$. How could this be inferred from Ex. 5?

10. With the notations $g(z, a)$ and $h(z, a_1, a_2)$ of the text show:

$$(\alpha)\ \frac{p'(z) + p'(a)}{p(z) - p(a)} = 2h(z, 0, a), \qquad \frac{p'(z) + p'(a)}{p(z) - p(a)} = -2h(z, a, 0),$$

$$(\beta)\ \frac{p'(z) + p'(a_2)}{p(z) - p(a_2)} - \frac{p'(z) + p'(a_1)}{p(z) - p(a_1)} = 2h(z, a_1, a_2),$$

$$(\gamma)\ \frac{1}{4}\left[\frac{p'(z) + p'(a)}{p(z) - p(a)}\right]^2 - p(z) = g(z, a) = p(z - a) + \text{const.},$$

$$(\delta)\ p(z - a) = \frac{1}{4}\left[\frac{p'(z) + p'(a)}{p(z) - p(a)}\right]^2 - p(z) - p(a).$$

11. Demonstrate the final theorem of the text of § 182.

12. By combining the power series for $p(z)$ and $p'(z)$ show

$$[p'(z)]^2 - 4[p(z)]^3 + 20c_1 p(z) - 28c_2 = Az^2 + \text{higher powers.}$$

Hence infer that the right-hand side must be identically zero.

13. Combine Ex. 12 with Ex. 8 to prove $e + e' + e'' = 0$.

14. With the notations $g_2 = 20c_1$ and $g_3 = -28c_2$ show

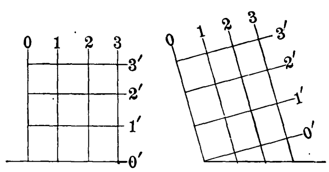
$$p'(z) = \sqrt{4p^3(z) - g_2 p(z) - g_3} \quad \text{or} \quad \frac{dp}{\sqrt{4p^3 - g_2 p - g_3}} = dz.$$

15. If $\zeta(z)$ be defined by $-\frac{d}{dz}\zeta(z) = p(z)$ or $\zeta(z) = -\int p(z)dz$, show that $\zeta(z+\omega) - \zeta(z)$ and $\zeta(z+\omega') - \zeta(z)$ must be merely constants η and η'.

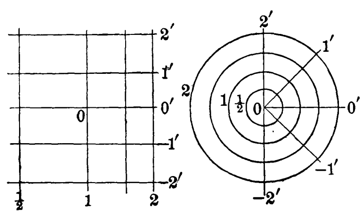
183. Conformal representation. The transformation (§ 178)

$$w = f(z) \quad \text{or} \quad u + iv = u(x, y) + iv(x, y)$$

is conformal between the planes of z and w at all points z at which $f'(z) \neq 0$. The correspondence between the planes may be represented by ruling the z-plane and drawing the corresponding rulings in the w-plane. If in particular the rulings in the z-plane be the lines $x =$ const., $y =$ const., parallel to the axes, those in the w-plane must be two sets of curves which are also orthogonal; in like manner if the z-plane be ruled by circles concentric with the origin and rays issuing from the origin, the w-plane must also be ruled orthogonally; for in both cases the angles between curves must be preserved. It is usually most convenient to consider the w-plane as ruled with the lines $u =$ const., $v =$ const., and hence to have a set of rulings $u(x, y) = c_1$, $v(x, y) = c_2$ in the z-plane. The figures represent several different cases arising from the functions ·



w–*plane* (1) *z*–*plane*



w–*plane* (2) *z*–*plane*

(1) $w = az = (a_1 + a_2 i)(x + iy), \qquad u = a_1 x - a_2 y, \qquad v = a_2 x + a_1 y,$

(2) $w = \log z = \log\sqrt{x^2+y^2} + i\tan^{-1}\frac{y}{x}, \quad u = \log\sqrt{x^2+y^2}, \quad v = \tan^{-1}\frac{y}{x}.$

Consider $w = z^2$, and apply polar coördinates so that

$$w = R(\cos\Phi + i\sin\Phi) = r^2(\cos 2\phi + i\sin 2\phi), \qquad R = r^2, \qquad \Phi = 2\phi.$$

To any point (r, ϕ) in the z-plane corresponds $(R = r^2, \Phi = 2\phi)$ in the w-plane; circles about $z = 0$ become circles about $w = 0$ and rays issuing from $z = 0$ become rays issuing from $w = 0$ at twice the angle. (A figure to scale should be supplied by the reader.) The derivative $w' = 2z$ vanishes at $z = 0$ only. The transformation is conformal for all points except $z = 0$. At $z = 0$ it is clear that the angle between two curves in the z-plane is doubled on passing to the corresponding curves in the w-plane; hence at $z = 0$ the transformation is not conformal. Similar results would be obtained from $w = z^m$ except that the angle between rays issuing from $w = 0$ would be m times the angle between the rays at $z = 0$.

A point in the neighborhood of which a function $w = f(z)$ is analytic but has a vanishing derivative $f'(z)$ is called a *critical point* of $f(z)$; if the derivative $f'(z)$ has a root of multiplicity k at any point, that point is called a *critical point of order* k. Let $z = z_0$ be a critical point of order k. Expand $f'(z)$ as

$$f'(z) = a_k(z - z_0)^k + a_{k+1}(z - z_0)^{k+1} + a_{k+2}(z - z_0)^{k+2} + \cdots;$$

then $f(z) = f(z_0) + \dfrac{a_k}{k+1}(z - z_0)^{k+1} + \dfrac{a_{k+1}}{k+2}(z - z_0)^{k+2} + \cdots,$

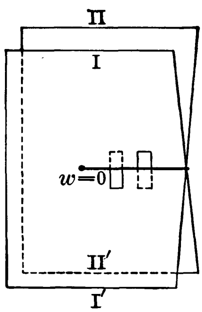
or $$w = w_0 + (z - z_0)^{k+1}E(z) \quad \text{or} \quad w - w_0 = (z - z_0)^{k+1}E(z), \qquad (7)$$

where E is a function that does not vanish at z_0. The point $z = z_0$ goes into $w = w_0$. For a sufficiently small region about z_0 the transformation (7) is sufficiently represented as

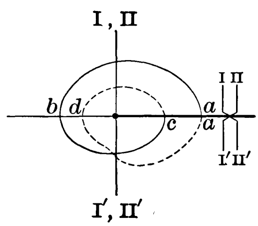
$$w - w_0 = C(z - z_0)^{k+1}, \qquad C = E(z_0).$$

On comparison with the case $w = z^m$, it appears that the angle between two curves meeting at z_0 will be multiplied by $k + 1$ on passing to the corresponding curves meeting at w_0. Hence *at a critical point of the kth order the transformation is not conformal but angles are multiplied by $k + 1$ on passing from the z-plane to the w-plane.*

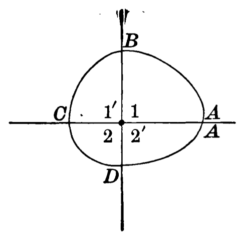
Consider the transformation $w = z^2$ more in detail. To each point z corresponds one and only one point w. To the points z in the first quadrant correspond the points of the first two quadrants in the w-plane, and to the upper half of the z-plane corresponds the whole w-plane. In like manner the lower half of the z-plane will be mapped upon the whole w-plane. Thus in finding the points in the w-plane which correspond to all the points of the z-plane, the w-plane is covered twice. This double counting of the w-plane may be obviated by a simple device. Instead of having one sheet of paper to represent the w-plane, let two sheets be superposed, and let the points corresponding to the upper half of the z-plane be considered as in the upper sheet, while those corresponding to the lower half are considered as in the lower sheet. Now consider the path traced upon the double w-plane when z traces a path in the z-plane. Every time z crosses from the second to



w—surface



w—surface



z—plane

the third quadrant, w passes from the fourth quadrant of the upper sheet into the first of the lower. When z passes from the fourth to the first quadrants, w comes from the fourth quadrant of the lower sheet into the first of the upper.

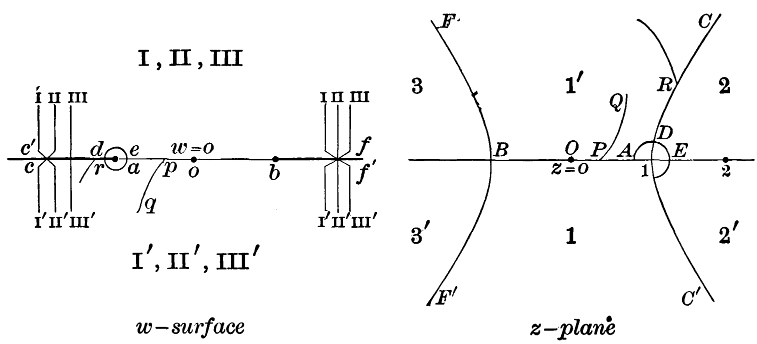
It is convenient to join the two sheets into a single surface so that a continuous path on the z-plane is pictured as a continuous path on the w-surface. This may be done (as indicated at the right of the middle figure) by regarding the lower half of the upper sheet as connected to the upper half of the lower, and the lower half of the lower as connected to the upper half of the upper. The surface therefore cuts through itself along the positive axis of reals, as in the sketch on the left*; the line is called the *junction line* of the surface. The point $w = 0$ which corresponds to the critical point $z = 0$ is called the *branch point* of the surface. Now not only does one point of the z-plane go over into a single point of the w-surface, but to each point of the surface corresponds a single point z; although any two points of the w-surface which are superposed have the same value of w, they correspond to different values of z except in the case of the branch point.

184. The w-surface, which has been obtained as a mere convenience in mapping the z-plane on the w-plane, is of particular value in studying the inverse function $z = \sqrt{w}$. For $\sqrt{w}$ is a multiple valued function and to each value of w correspond two values of z; but if w be

* Practically this may be accomplished for two sheets of paper by pasting gummed strips to the sheets which are to be connected across the cut.

regarded as on the w-surface instead of merely in the w-plane, there is only one value of z corresponding to a point w upon the surface. *Thus the function* $\sqrt{w}$ *which is double valued over the w-plane becomes single valued over the w-surface.* The w-surface is called the *Riemann surface* of the function $z = \sqrt{w}$. The construction of Riemann surfaces is important in the study of multiple valued functions because the surface keeps the different values apart, so that to each point of the surface corresponds only one value of the function. Consider some surfaces. (The student should make a paper model by following the steps as indicated.)

Let $w = z^3 - 3z$ and plot the w-surface. First solve $f'(z) = 0$ to find the critical points z and substitute to find the branch points w. Now if the branch points be considered as removed from the w-plane, the plane is no longer simply connected. It must be made simply connected by drawing proper lines in the figure. This may be accomplished by drawing a line from each branch point to infinity or by connecting the successive branch points to each other and connecting the last one to the point at infinity. These lines are the junction lines. In this particular case the critical points are $z = +1, -1$ and the branch points are $w = -2, +2$, and the junction lines may be taken as the straight lines joining $w = -2$ and $w = +2$ to



infinity and lying along the axis of reals as in the figure. Next spread the requisite number of sheets over the w-plane and cut them along the junction lines. As $w = z^3 - 3z$ is a cubic in z, and to each value of w, except the branch values, there correspond three values of z, three sheets are needed. Now find in the z-plane the *image* of the junction lines. The junction lines are represented by $v = 0$; but $v = 3x^2y - y^3 - 3y$, and hence the line $y = 0$ and the hyperbola $3x^2 - y^2 = 3$ will be the images desired. The z-plane is divided into six pieces which will be seen to correspond to the six half sheets over the w-plane.

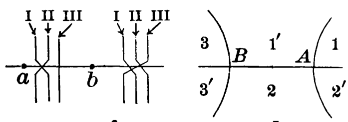
Next z will be made to trace out the images of the junction lines and to turn about the critical points so that w will trace out the junction lines and turn about the branch points in such a manner that the connections between the different sheets may be made. It will be convenient to regard z and w as persons walking along their respective paths so that the terms "right" and "left" have a meaning.

Let z start at $z = 0$ and move forward to $z = 1$; then, as $f'(z)$ is negative, w starts at $w = 0$ and moves back to $w = -2$. Moreover if z turns to the right as at P, so must w turn to the right through the same angle, owing to the conformal property. Thus it appears that not only is OA mapped on oa, but the region 1′ just above OA is mapped on the region I′ just below oa; in like manner OB is mapped on ob. As ab is not a junction line and the sheets have not been cut through along it, the regions 1, 1′ should be assumed to be mapped on the same sheet, say, the uppermost, I, I′. As any point Q in the whole infinite region 1′ may be reached from 0 without crossing any image of ab, it is clear that the whole infinite region 1′ should be considered as mapped on I′; and similarly 1 on I. The converse is also evident, for the same reason.

If, on reaching A, the point z turns to the left through 90° and moves along AC, then w will make a turn to the left of 180°, that is, will keep straight along ac; a turn as at R into 1′ will correspond to a turn as at r into I′. This checks with the statement that all 1′ is mapped on all I′. Suppose that z described a small circuit about $+ 1$. When z reaches D, w reaches d; when z reaches E, w reaches e. But when w crossed ac, it could not have crossed into I, and when it reaches e it cannot be in I; for the points of I are already accounted for as corresponding to points in 1. Hence in crossing ac, w must drop into one of the lower sheets, say the middle, II; and on reaching e it is still in II. It is thus seen that II corresponds to 2. Let z continue around its circuit; then II′ and 2′ correspond. When z crosses AC' from 2′ and moves into 1, the point w crosses ac' and moves from II′ up into I. In fact the upper two sheets are connected along ac just as the two sheets of the surface for $w = z^2$ were connected along their junction.

In like manner suppose that z moves from 0 to -1 and takes a turn about B so that w moves from 0 to 2 and takes a turn about b. When z crosses BF from 1′ to 3, w crosses bf from I′ into the upper half of some sheet, and this must be III for the reason that I and II are already mapped on 1 and 2. Hence I′ and III are connected, and so are I and III′. This leaves II which has been cut along bf, and III cut along ac, which may be reconnected as if they had never been cut. The reason for this appears forcibly if all the points z which correspond to the branch points are added to the diagram. When $w = 2$, the values of z are the critical value -1 (double) and the ordinary value $z = 2$; similarly, $w = -2$ corresponds to $z = -2$. Hence if z describe the half circuit AE so that w gets around to e in II, then if z moves out to $z = 2$, w will move out to $w = 2$, passing by $w = 0$ in the sheet II as z passes through $z = \sqrt{3}$; but as $z = 2$ is not a critical point, $w = 2$ in II cannot be a branch point, and the cut in II may be reconnected.

The w-surface thus constructed for $w = f(z) = z^3 - 3z$ is the Riemann surface for the inverse function $z = f^{-1}(w)$, of which the explicit form cannot be given without solving a cubic. To each point of the surface corresponds one value of z, and to the three superposed values of w correspond three different values of z except at the branch points where two of the sheets come together and give only one value of z while the third sheet gives one other. The Riemann surface could equally well have been constructed by joining the two branch points and then connecting one of them to ∞. The image of $v = 0$ would not have been changed. The connections of the sheets could be established as before, but would be different. If the junction line be $-2, 2, +\infty$, the point $w = 2$ has two junctions running into it, and the connections of the sheets on opposite sides of the point are not independent. It is advisable to arrange the work so that the first branch point which is encircled shall have only one junction running from it. This may be done by taking a very large circuit in z so that w will describe a large circuit and hence cut only one junction line, namely, from 2 to ∞, or by taking a small circuit about $z = 1$ so that w will take a small turn about $w = -2$. Let the latter method be chosen. Let z start from $z = 0$ at O and move to $z = 1$ at A; then w starts at $w = 0$ and moves to $w = -2$. The correspondence between 1′ and I′ is thus established. Let z turn about A; then w turns about $w = -2$ at a. As the line -2 to $-\infty$ or ac is not now a junction line, w moves from I′ into the upper half I, and the region across AC from 1′ should be labeled 1 to correspond. Then 2′, 2 and II′, II may be filled in. The connections of I–II′ and II–I′ are indicated and III–III′ is reconnected, as the branch point is of the first order and only two sheets are involved. Now let z move from $z = 0$ to $z = -1$ and take a turn about B; then w moves from $w = 0$ to $w = 2$ and takes a turn about b. The region next 1′ is marked 3 and I′ is connected to III. Passing from 3 to 3′ for z is equivalent to passing from III to III′ for w between 0 and b where these sheets are connected. From 3′ into 2 for z indicates III′ to II across the junction from $w = 2$ to ∞. This leaves I and II′ to be connected across this junction. The connections are complete. They may be checked by allowing z to describe a large circuit so that the regions 1, 1′, 3, 3′, 2, 2′, 1 are successively traversed. That I, I′, III, III′, II, II′, I is the corresponding succession of sheets is clear from the connections between $w = 2$ and ∞ and the fact that from $w = -2$ to $-\infty$ there is no junction.



w–surface *z–plane*

Consider the function $w = z^6 - 3z^4 + 3z^2$. The critical points are $z = 0, 1, 1, -1, -1$ and the corresponding branch points are $w = 0, 1, 1, 1, 1$. Draw the junction lines from $w = 0$ to $-\infty$ and from $w = 1$ to $+\infty$ along the axis of reals. To find the image of $v = 0$ on the z-plane, polar coördinates may be used.

$$z = r(\cos\phi + i\sin\phi), \qquad w = u + iv = r^6e^{6\phi i} - 3r^4e^{4\phi i} + 3r^2e^{2\phi i}.$$

$$v = 0 = r^2[r^4\sin 6\phi - 3r^2\sin 4\phi + 3\sin 2\phi]$$
$$= r^2\sin 2\phi[r^4(3 - 4\sin 2\phi) - 6r^2\cos\phi + 3].$$

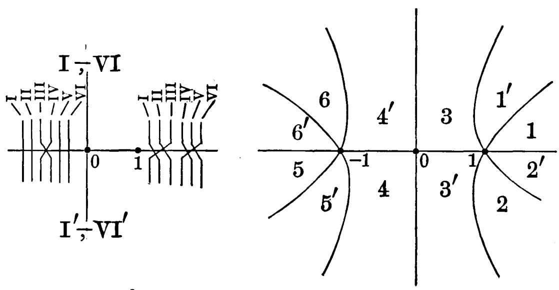
The equation $v = 0$ therefore breaks up into the equation $\sin 2\phi = 0$ and

$$r^2 = \frac{3\cos 2\phi \pm \sqrt{3}\sin 2\phi}{3 - 4\sin^2 2\phi} = \frac{\sqrt{3}}{2}\frac{\sin(60 \pm 2\phi)}{\sin(60 + 2\phi)\sin(60 - 2\phi)} = \frac{\sqrt{3}}{2\sin(60 \pm 2\phi)}.$$

Hence the axes $\phi = 0°$ and $\phi = 90°$ and the two rectangular hyperbolas inclined at angles of $\pm 15°$ are the images of $v = 0$. The z-plane is thus divided into six portions. The function w is of the sixth order and six sheets must be spread over the w-plane and cut along the junction lines.

To connect up the sheets it is merely necessary to get a start. The line $w = 0$ to $w = 1$ is not a junction line and the sheets have not been cut through along it. But when z is small, real, and increasing, w is also small, real, and increasing. Hence to OA corresponds oa in any sheet desired. Moreover the region above OA will correspond to the upper half of the sheet and the region below OA to the lower half. Let the sheet be chosen as III and place the numbers 3 and 3′ so as to correspond with III and III′. Fill in the numbers 4 and 4′ around $z = 0$. When

z turns about the critical point $z = 0$, w turns about $w = 0$, but as angles are doubled it must go around twice and the connections III–IV′, IV–III′ must be made. Fill in more numbers about the critical point $z = 1$ of the second order where angles are tripled. On the w-surface there will be a triple connection III′–II, II′–I, I′–III. In like manner the critical point $z = -1$ may be treated. The surface is complete except for reconnecting sheets I, II, V, VI along $w = 0$ to $w = -\infty$ as if they had never been cut.



w—surface *z—plane*

EXERCISES

1. Plot the corresponding lines for: $(\alpha)\ w = (1 + 2i)z$, $(\beta)\ w = (1 - \frac{1}{2}i)z$.

2. Solve for x and y in (1) and (2) of the text and plot the corresponding lines.

3. Plot the corresponding orthogonal systems of curves in these cases:

$$(\alpha)\ w = \frac{1}{z}, \qquad (\beta)\ w = 1 + z^2, \qquad (\gamma)\ w = \cos z.$$

4. Study the correspondence between z and w near the critical points:

$$(\alpha)\ w = z^3, \qquad (\beta)\ w = 1 - z^2, \qquad (\gamma)\ w = \sin z.$$

5. Upon the w-surface for $w = z^2$ plot the points corresponding to $z = 1,\ 1 + i,\ 2i,\ -\frac{1}{2} + \frac{1}{2}\sqrt{3}\,i,\ -\frac{1}{2},\ -\frac{1}{2}\sqrt{3} - \frac{1}{2}i,\ -i,\ \frac{1}{2} - \frac{1}{2}i$. And in the z-plane plot the points corresponding to $w = \sqrt{2} + \sqrt{2}\,i,\ i,\ -4,\ -\frac{1}{2} - \frac{1}{2}\sqrt{3}\,i,\ 1 - i$, whether in the upper or lower sheet.

6. Construct the w-surface for these functions:

$$(\alpha)\ w = z^3, \qquad (\beta)\ w = z^{-2}, \qquad (\gamma)\ w = 1 + z^2, \qquad (\delta)\ w = (z - 1)^3.$$

In (β) the singular point $z = 0$ should be joined by a cut to $z = \infty$.

7. Construct the Riemann surfaces for these functions:

$$(\alpha)\ w = z^4 - 2z^2, \qquad (\beta)\ w = -z^4 + 4z, \qquad (\gamma)\ w = 2z^5 - 5z^2,$$

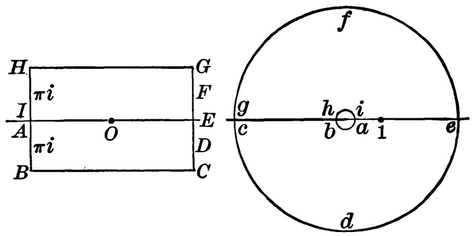
$$(\delta)\ w = z + \frac{1}{z}, \qquad (\epsilon)\ w = z^2 + \frac{1}{z^2}, \qquad (\zeta)\ w = \frac{z^3 + \sqrt{3}\,z}{\sqrt{3}\,z^2 + 1}.$$

185. Integrals and their inversion.

Consider the function

$$z = \int_1^w \frac{dw}{w}, \qquad z = \ln w, \qquad w = \ln^{-1} z,$$

defined by an integral, and let the methods of the theory of functions be applied to the study of the function and its inverse. If w describes a path surrounding the origin, the integral need not vanish; for the integrand is not analytic at $w = 0$. Let a cut be drawn from $w = 0$ to $w = -\infty$. The integral is then a single valued function of w provided the path of integration does not cross the cut. Moreover, it is analytic except at $w = 0$, where the derivative, which is the integrand $1/w$, ceases to be continuous. Let the w-plane as cut be mapped on the z-plane by allowing w to trace the path $1abcdefghi1$, by computing the value of z sufficiently to draw the image, and by applying the principles of conformal representation. When w starts from $w = 1$ and traces $1a$, z starts from $z = 0$ and becomes negatively very large. When w turns to the left to trace ab, z will turn also through 90° to the left. As the integrand along ab is $id\phi$, z must be changing by an amount which is pure imaginary and must reach B when w reaches b. When w traces bc, both w and dw are negative and z must be increasing by real positive quantities, that is, z must trace BC. When w moves along $cdefg$ the same reasoning as for the path ab will show that z moves along $CDEFG$. The remainder of the path may be completed by the reader.



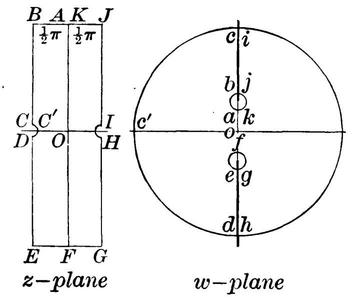
z—plane *w—plane*

It is now clear that the whole w-plane lying between the infinitesimal and infinite circles and bounded by the two edges of the cut is mapped on a strip of width $2\pi i$ bounded upon the right and left by two infinitely distant vertical lines. If w had made a complete turn in the positive direction about $w = 0$ and returned to its starting point, z would have received the increment $2\pi i$. That is to say, the values of z which correspond to the same point w reached by a direct path and by a path which makes k turns about $w = 0$ will differ by $2k\pi i$. Hence when w is regarded inversely as a function of z, the function will be periodic with the period $2\pi i$. It has been seen from the correspondence of $cdefg$ to $CDEFG$ that w becomes infinite when z moves off indefinitely to the right in the strip, and from the correspondence of $BAIH$ with $baih$ that w becomes 0 when z moves off to the left. Hence w must be a rational function of e^z. As w neither becomes infinite nor vanishes for any finite point of the strip, it must reduce merely to Ce^{kz} with k integral. As w has no smaller period than $2\pi i$, it follows that $k = 1$. To determine C, compare the derivative $dw/dz = Ce^z$ at $z = 0$ with its reciprocal $dz/dw = w^{-1}$ at the corresponding point $w = 1$; then $C = 1$. The inverse function $\ln^{-1}z$ is therefore completely determined as e^z.

In like manner consider the integral

$$z=\int_0^w \frac{dw}{1+w^2}, \qquad z=f(w), \qquad w=\phi(z)=f^{-1}(z).$$

Here the points $w= \pm i$ must be eliminated from the w-plane and the plane rendered simply connected by the proper cuts, say, as in the figure. The tracing of the figure may be left to the reader. The chief difficulty may be to show that the integrals along oa and bc are so nearly equal that C lies close to the real axis; no computation is really necessary inasmuch as the integral along oc' would be real and hence C' must lie on the axis. The image of the cut w-plane is a strip of width π. Circuits around either $+i$ or $-i$ add π to z, and hence w as a function of z has the period π. At the ends of the strip, w approaches the finite values $+i$ and $-i$. The function $w=\phi(z)$ has a simple zero when $z=0$ and has no other zero in the strip. At the two points $z= \pm \frac{1}{2}\pi$, the function w becomes infinite, but only one of these points should be considered as in the strip. As the function has only one zero, the point $z=\frac{1}{2}\pi$ must be a pole of the first order. The function is therefore completely determined except for a constant factor which may be fixed by examining the derivative of the function at the origin. Thus



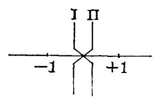
$$w=c\frac{e^{2iz}-1}{e^{2iz}+1}=\frac{1}{i}\frac{e^{iz}-e^{-iz}}{e^{iz}+e^{-iz}}=\tan z, \qquad z=\tan^{-1}w.$$

186. As a third example consider the integral

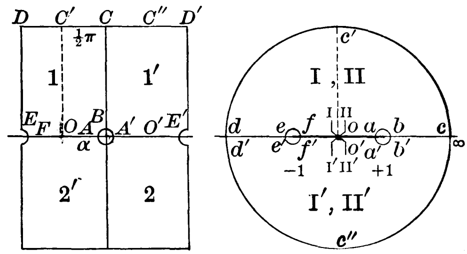
$$z=\int_0^w \frac{dw}{\sqrt{1-w^2}}, \qquad z=f(w), \qquad w=\phi(z)=f^{-1}(z). \tag{8}$$

Here the integrand is double valued in w and consequently there is liable to be confusion of the two values in attempting to follow a path in the w-plane. Hence a two-leaved surface for the integrand will be constructed and the path of integration will be considered to be on the surface. Then to each point of the path there will correspond only one value of the integrand, although to each value of w there correspond two superimposed points in the two sheets of the surface.

As the radical $\sqrt{1-w^2}$ vanishes at $w= \pm 1$ and takes on only the single value 0 instead of two equal and opposite values, the points $w= \pm 1$ are branch points on the surface and they are the only finite branch points. Spread two sheets over the w-plane, mark the branch points $w= \pm 1$, and draw the junction line between them and continue it (provisionally) to $w=\infty$. At $w=-1$ the function $\sqrt{1-w^2}$ may be written $\sqrt{1+w}\,E(w)$, where E denotes a function which does not vanish at $w=-1$. Hence in the neighborhood of $w=-1$ the surface looks like that for $\sqrt{w}$ near $w=0$. This may be accomplished by making the connections across the junction line. At the point $w = +1$ the surface must cut through itself in a similar manner. This will be so provided that the sheets are reconnected across 1∞ as if never cut; if the sheets had been cross-connected along 1∞, each sheet would have been separate, though crossed, over 1, and the branch point would have disappeared. It is noteworthy that if w describes a large circuit including both branch points, the values of $\sqrt{1-w^2}$ are not interchanged; the circuit closes in each sheet without passing into the other. This could be expressed by saying that $w = \infty$ is not a branch point of the function.



Now let w trace out various paths on the surface in the attempt to map the surface on the z-plane by aid of the integral (8). To avoid any difficulties in the way of double or multiple values for z which might arise if w turned about a branch point $w = \pm 1$, let the surface be marked in each sheet over the axis of reals from $-\infty$ to $+1$. Let each of the four half planes be treated separately. Let w start at $w = 0$ in the upper half plane of the upper sheet and let the value of $\sqrt{1-w^2}$ at this point be $+1$; the values of $\sqrt{1-w^2}$ near $w = 0$ in II′ will then be near $+1$ and will be sharply distinguished from the values near -1 which are supposed to correspond to points in I′, II. As w traces oa, the integral z increases from 0 to a definite positive number α. The value of the integral from a to b is infinitesimal. Inasmuch as $w = 1$ is a branch point where two sheets connect, it is natural to assume that as w passes 1 and leaves it on the right, z will turn through half a straight angle. In other words the integral from b to c is naturally presumed to be a large pure imaginary affected with a positive sign. (This fact may easily be checked by examining the change in $\sqrt{1-w^2}$ when w describes a small circle about $w = 1$. In fact if the E-function $\sqrt{1+w}$ be discarded and if $1-w$ be written as $re^{\phi i}$, then $\sqrt{r}e^{\frac{1}{2}\phi i}$ is that value of the radical which is positive when $1-w$ is positive. Now when w describes the small semicircle, ϕ changes from 0° to $-180°$ and hence the value of the radical along bc becomes $-i\sqrt{r}$ and the integrand is a positive pure imaginary.) Hence when w traces bc, z traces BC. At c there is a right-angle turn to the left, and as the value of the integral over the infinite quadrant cc' is $\frac{1}{2}\pi$, the point z will move back through the distance $\frac{1}{2}\pi$. That the point C' thus reached must lie on the pure imaginary axis is seen by noting that the integral taken directly along oc' would be pure imaginary. This shows that $\alpha = \frac{1}{2}\pi$ without any necessity of computing the integral over the interval oa. The rest of the map of I may be filled in at once by symmetry.

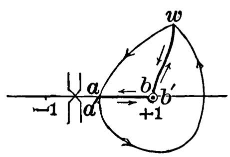


z—plane *w—surface*

To map the rest of the w-surface is now relatively simple. For I′ let w trace $cc''d'$; then z will start at C and trace $CD' = \pi$. When w comes in along the lower side of the cut $d'e'$ in the upper sheet I′, the value of the integrand is identical with the value when this line de regarded as belonging to the upper half plane was described, for the line is not a junction line of the surface. The trace of z is therefore $D'E'$. When w traces $f'o'$ it must be remembered that I′ joins on to II and hence that the values of the integrand are the negative of those along fo. This makes z describe the segment $F'O' = -\alpha = -\frac{1}{2}\pi$. The turn at $E'F'$ checks with the straight angle at the branch point -1. It is further noteworthy that when w returns to o' on I′, z does not return to 0 but takes the value π. This is no contradiction; the one-to-one correspondence which is being established by the integral is between points on the w-surface and points in a certain region of the z-plane, and as there are two points on the surface to each value of w, there will be two points z to each w. Thus far the sheet I has been mapped on the z-plane. To map II let the point w start at o' and drop into the lower sheet and then trace in this sheet the path which lies directly under the path it has traced in I. The integrand now takes on values which are the negatives of those it had previously, and the image on the z-plane is readily sketched in. The figure is self-explanatory. Thus the complete surface is mapped on a strip of width 2π.

To treat the different values which z may have for the same value of w, and in particular to determine the periods of w as the inverse function of z, it is necessary to study the value of the integral along different sorts of paths on the surface. Paths on the surface may be divided into two classes, closed paths and those not closed. A closed path is one which returns to the same point on the surface from which it started; it is not sufficient that it return to the same value of w. Of paths which are not closed on the surface, those which close in w, that is, which return to a point superimposed upon the starting point but in a different sheet, are the most important. These paths, on the particular surface here studied, may be further classified. A path which closes on the surface may either include neither branch point, or may include both branch points or may wind twice around one of the points. A path which closes in w but not on the surface may wind once about one of the branch points. Each of these types will be discussed.

If a closed path contains neither branch point, there is no danger of confusing the two values of the function, the projection of the path on the w-plane gives a region over which the integrand may be considered as single valued and analytic, and hence the value of the circuit integral is 0. If the path surrounds both branch points, there is again no danger of confusing the values of the function, but the projection of the path on the w-plane gives a region at two points of which, namely, the branch points, the integrand ceases to be analytic. The inference is that the value of the integral may not be zero and in fact will not be zero unless the integral around a circuit shrunk close up to the branch points or expanded out to infinity is zero. The integral around $cc'dc''c$ is here equal to 2π; the value of the integral around any path which incloses both branch points once and only once is therefore 2π or -2π according as the path lies in the upper or lower sheet; if the path surrounded the points k times, the value of the integral would be $2k\pi$. It thus appears that w regarded as a function of z has a period 2π. If a path closes in w but not on the surface, let the point where it crosses the junction line be held fast (figure) while the path is shrunk down to $wbaa'b'w$. The value of the integral will not change during this shrinking of the path, for the new and old paths may together be regarded as closed and of the first case considered. Along the paths wba and $a'b'w$ the integrand has opposite signs, but so has dw; around the small circuit the value of the integral is infinitesimal. Hence the value of the integral around the path which closes in w is $2\mathrm{I}$ or $-2\mathrm{I}$ if I is the value from the point a where the path crosses the junction line



to the point w. The same conclusion would follow if the path were considered to shrink down around the other branch point. Thus far the possibilities for z corresponding to any given w are $z + 2k\pi$ and $2m\pi - z$. Suppose finally that a path turns twice around one of the branch points and closes on the surface. By shrinking the path, a new equivalent path is formed along which the integral cancels out term for term except for the small double circuit around ± 1 along which the value of the integral is infinitesimal. Hence the values $z + 2k\pi$ and $2m\pi - z$ are the only values z can have for any given value of w if z be a particular possible value. This makes two and only two values of z in each strip for each value of w, and the function is of the second order.

It thus appears that w, as a function of z, has the period 2π, is single valued, becomes infinite at both ends of the strip, has no singularities within the strip, and has two simple zeros at $z = 0$ and $z = \pi$. Hence w is a rational function of e^{iz} with the numerator $e^{2iz} - 1$ and the denominator $e^{2iz} + 1$. In fact

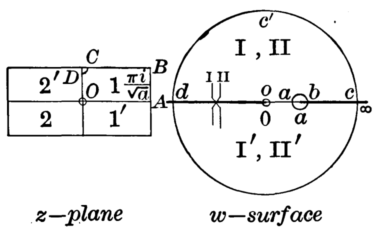
$$w = C\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} = \frac{1}{i}\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} = \sin z.$$

The function, as in the previous cases, has been wholly determined by the general methods of the theory of functions without even computing α.

One more function will be studied in brief. Let

$$z = \int_0^w \frac{dw}{(a - w)\sqrt{w}}, \qquad a > 0, \qquad z = f(w), \qquad w = \phi(z) = f^{-1}(z).$$

Here the Riemann surface has a branch point at $w = 0$ and in addition there is the singular point $w = a$ of the integrand which must be cut out of both sheets. Let the surface be drawn with a junction line from $w = 0$ to $w = -\infty$ and with a cut in each sheet from $w = a$ to $w = \infty$. The map on the z-plane now becomes as indicated in the figure. The different values of z for the same value of w are readily seen to arise when w turns about the point $w = a$ in either sheet or when a path closes in w but not on the surface. These values of z are $z + 2k\pi i/\sqrt{a}$ and $2m\pi i/\sqrt{a} - z$. Hence w as a function of z has the period $2\pi i a^{-\frac{1}{2}}$, has a zero at $z = 0$ and a pole at $z = \pi i/\sqrt{a}$, and approaches the finite value $w = a$ at both ends of the strip. It must be noted, however, that the zero and pole are both necessarily double, for to any ordinary value of w correspond two values of z in the strip. The function is therefore again of the second order, and indeed



$$w = a\frac{(e^{z\sqrt{a}} - 1)^2}{(e^{z\sqrt{a}} + 1)^2} = a \tanh^2 \frac{1}{2} z\sqrt{a}, \qquad z = \frac{2}{\sqrt{a}} \tanh^{-1} \sqrt{\frac{w}{a}}.$$

The success of this method of determining the function $z = f(w)$ defined by an integral, or the inverse $w = f^{-1}(z) = \phi(z)$, has been dependent first upon the ease with which the integral may be used to map the w-plane or w-surface upon the z-plane, and second upon the simplicity of the map, which was such as to indicate that the inverse function was a single valued periodic function. It should be realized that if an attempt were made to apply the methods to integrands which appear equally simple, say to

$$z = \int \sqrt{a^2 - w^2}\,dw, \qquad z = \int (a - w)\,dw/\sqrt{w},$$

the method would lead only with great difficulty, if at all, to the relation between z and w; for the functional relation between z and w is indeed not simple. There is, however, one class of integrals of great importance, namely,

$$z = \int \frac{dw}{\sqrt{(w - \alpha_1)(w - \alpha_2) \cdots (w - \alpha_n)}}$$

for which this treatment is suggestive and useful.

EXERCISES

1. Discuss by the method of the theory of functions these integrals and inverses:

$(\alpha)\ \int_1^w \frac{dw}{2w}$, $(\beta)\ \int_0^w \frac{2\,dw}{1 - w}$, $(\gamma)\ \int_0^w \frac{dw}{1 - w^2}$,

$(\delta)\ \int_0^w \frac{dw}{\sqrt{w^2 - 1}}$, $(\epsilon)\ \int_0^w \frac{dw}{\sqrt{w^2 + 1}}$, $(\zeta)\ \int_\infty^w \frac{dw}{w\sqrt{w^2 + a^2}}$,

$(\eta)\ \int_\infty^w \frac{dw}{w\sqrt{w^2 - a^2}}$, $(\theta)\ \int_0^w \frac{dw}{\sqrt{2aw - w^2}}$, $(\iota)\ \int_1^w \frac{dw}{(w + 1)\sqrt{w^2 - 1}}$.

The results may be checked in each case by actual integration.

2. Discuss $\int_\infty^w \frac{dw}{\sqrt{w(1 - w)(1 + w)}}$ and $\int_0^w \frac{dw}{\sqrt{1 - w^4}}$ (§ 182, and Ex. 10, p. 489).