ADVANCED CALCULUS

INTRODUCTORY REVIEW

CHAPTER I

REVIEW OF FUNDAMENTAL RULES

1. On differentiation. If the function $f(x)$ is interpreted as the curve $y = f(x)$,* the quotient of the increments Δy and Δx of the dependent and independent variables measured from (x_0, y_0) is

$$\frac{y - y_0}{x - x_0} = \frac{\Delta y}{\Delta x} = \frac{\Delta f(x)}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}, \tag{1}$$

and represents the *slope of the secant* through the points $P(x_0, y_0)$ and $P'(x_0 + \Delta x, y_0 + \Delta y)$ on the curve. The limit approached by the quotient $\Delta y/\Delta x$ when P remains fixed and $\Delta x \doteq 0$ is the *slope of the tangent* to the curve at the point P. *This limit,*

$$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \doteq 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = f'(x_0), \tag{2}$$

is called the *derivative* of $f(x)$ for the value $x = x_0$. As the derivative may be computed for different points of the curve, it is customary to speak of the derivative as itself a function of x and write

$$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \doteq 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x). \tag{3}$$

There are numerous notations for the derivative, for instance

$$f'(x) = \frac{df(x)}{dx} = \frac{dy}{dx} = D_x f = D_x y = y' = Df = Dy.$$

* Here and throughout the work, where figures are not given, the reader should draw graphs to illustrate the statements. Training in making one's own illustrations, whether graphical or analytic, is of great value.

The first five show distinctly that the independent variable is x, whereas the last three do not explicitly indicate the variable and should not be used unless there is no chance of a misunderstanding.

2. The fundamental formulas of differential calculus are derived directly from the application of the definition (2) or (3) and from a few fundamental propositions in limits. First may be mentioned

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}, \quad \text{where } z = \phi(y) \text{ and } y = f(x). \tag{4}$$

$$\frac{dx}{dy} = \frac{df^{-1}(y)}{dy} = \frac{1}{\dfrac{df(x)}{dx}} = \frac{1}{\dfrac{dy}{dx}}. \tag{5}$$

$$D(u \pm v) = Du \pm Dv, \qquad D(uv) = uDv + vDu. \tag{6}$$

$$D\left(\frac{u}{v}\right) = \frac{vDu - uDv}{v^2},^* \qquad D(x^n) = nx^{n-1}. \tag{7}$$

It may be recalled that (4), which is the rule for differentiating a function of a function, follows from the application of the theorem that the limit of a product is the product of the limits to the fractional identity $\frac{\Delta z}{\Delta x} = \frac{\Delta z}{\Delta y}\frac{\Delta y}{\Delta x}$; whence

$$\lim_{\Delta x \doteq 0} \frac{\Delta z}{\Delta x} = \lim_{\Delta x \doteq 0} \frac{\Delta z}{\Delta y} \cdot \lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta y \doteq 0}{}^{\dagger} \frac{\Delta z}{\Delta y} \cdot \lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x},$$

which is equivalent to (4). Similarly, if $y = f(x)$ and if x, as the inverse function of y, be written $x = f^{-1}(y)$ from analogy with $y = \sin x$ and $x = \sin^{-1} y$, the relation (5) follows from the fact that $\Delta x/\Delta y$ and $\Delta y/\Delta x$ are reciprocals. The next three result from the immediate application of the theorems concerning limits of sums, products, and quotients (§ 21). The rule for differentiating a power is derived in case n is integral by the application of the binomial theorem.

$$\frac{\Delta y}{\Delta x} = \frac{(x + \Delta x)^n - x^n}{\Delta x} = nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}\Delta x + \cdots + (\Delta x)^{n-1},$$

and the limit when $\Delta x \doteq 0$ is clearly nx^{n-1}. The result may be extended to rational values of the index n by writing $n = \frac{p}{q}$, $y = x^{\frac{p}{q}}$, $y^q = x^p$ and by differentiating both sides of the equation and reducing. To prove that (7) still holds when n is irrational, it would be *necessary to have a workable definition of irrational numbers* and to develop the properties of such numbers in greater detail than seems wise at this point. The formula is therefore assumed in accordance with *the principle of permanence of form* (§ 178), just as formulas like $a^m a^n = a^{m+n}$ of the theory of exponents, which may readily be proved for rational bases and exponents, are assumed without proof to hold also for irrational bases and exponents. See, however, §§ 18–25 and the exercises thereunder.

* It is frequently better to regard the quotient as the product $u \cdot v^{-1}$ and apply (6).

† For when $\Delta x \doteq 0$, then $\Delta y \doteq 0$ or $\Delta y/\Delta x$ could not approach a limit.

3. Second may be mentioned the formulas for the derivatives of the trigonometric and the inverse trigonometric functions.

$$D\sin x = \cos x, \qquad D\cos x = -\sin x, \tag{8}$$

or

$$D\sin x = \sin(x + \tfrac{1}{2}\pi), \qquad D\cos x = \cos(x + \tfrac{1}{2}\pi), \tag{8'}$$

$$D\tan x = \sec^2 x, \qquad D\cot x = -\csc^2 x, \tag{9}$$

$$D\sec x = \sec x\tan x, \qquad D\csc x = -\csc x\cot x, \tag{10}$$

$$D\operatorname{vers} x = \sin x, \quad \text{where} \quad \operatorname{vers} x = 1 - \cos x = 2\sin^2\tfrac{1}{2}x, \tag{11}$$

$$D\sin^{-1}x = \frac{\pm 1}{\sqrt{1-x^2}}, \qquad \begin{cases} + \text{ in quadrants I, IV,} \\ - \text{ " " II, III,} \end{cases} \tag{12}$$

$$D\cos^{-1}x = \frac{\pm 1}{\sqrt{1-x^2}}, \qquad \begin{cases} - \text{ in quadrants I, II,} \\ + \text{ " " III, IV,} \end{cases} \tag{13}$$

$$D\tan^{-1}x = \frac{1}{1+x^2}, \qquad D\cot^{-1}x = -\frac{1}{1+x^2}, \tag{14}$$

$$D\sec^{-1}x = \frac{\pm 1}{x\sqrt{x^2-1}}, \qquad \begin{cases} + \text{ in quadrants I, III,} \\ - \text{ " " II, IV,} \end{cases} \tag{15}$$

$$D\csc^{-1}x = \frac{\pm 1}{x\sqrt{x^2-1}}, \qquad \begin{cases} - \text{ in quadrants I, III,} \\ + \text{ " " II, IV,} \end{cases} \tag{16}$$

$$D\operatorname{vers}^{-1}x = \frac{\pm 1}{\sqrt{2x-x^2}}, \qquad \begin{cases} + \text{ in quadrants I, II,} \\ - \text{ " " III, IV.} \end{cases} \tag{17}$$

It may be recalled that to differentiate $\sin x$ the definition is applied. Then

$$\frac{\Delta\sin x}{\Delta x} = \frac{\sin(x+\Delta x) - \sin x}{\Delta x} = \frac{\sin\Delta x}{\Delta x}\cos x - \frac{1-\cos\Delta x}{\Delta x}\sin x.$$

It now is merely a question of evaluating the two limits which thus arise, namely,

$$\lim_{\Delta x \doteq 0}\frac{\sin\Delta x}{\Delta x} \quad \text{and} \quad \lim_{\Delta x \doteq 0}\frac{1-\cos\Delta x}{\Delta x}. \tag{18}$$

From the properties of the circle it follows that these are respectively 1 and 0. Hence the derivative of $\sin x$ is $\cos x$. The derivative of $\cos x$ may be found in like manner or from the identity $\cos x = \sin(\frac{1}{2}\pi - x)$. The results for all the other trigonometric functions are derived by expressing the functions in terms of $\sin x$ and $\cos x$. And to treat the inverse functions, it is sufficient to recall the general method in (5). Thus

$$\text{if} \quad y = \sin^{-1}x, \qquad \text{then} \quad \sin y = x.$$

Differentiate both sides of the latter equation and note that $\cos y = \pm\sqrt{1-\sin^2 y} = \pm\sqrt{1-x^2}$ and the result for $D\sin^{-1}x$ is immediate. To ascertain which sign to use with the radical, it is sufficient to note that $\pm\sqrt{1-x^2}$ is $\cos y$, which is positive when the angle $y = \sin^{-1}x$ is in quadrants I and IV, negative in II and III. Similarly for the other inverse functions.

EXERCISES *

1. Carry through the derivation of (7) when $n = p/q$, and review the proofs of typical formulas selected from the list (5)–(17). Note that the formulas are often given as $D_x u^n = nu^{n-1} D_x u$, $D_x \sin u = \cos u \, D_x u$, $\cdots$, and may be derived in this form directly from the definition (3).

2. Derive the two limits necessary for the differentiation of $\sin x$.

3. Draw graphs of the inverse trigonometric functions and label the portions of the curves which correspond to quadrants I, II, III, IV. Verify the sign in (12)–(17) from the slope of the curves.

4. Find $D \tan x$ and $D \cot x$ by applying the definition (3) directly.

5. Find $D \sin x$ by the identity $\sin u - \sin v = 2 \cos \frac{u+v}{2} \sin \frac{u-v}{2}$.

6. Find $D \tan^{-1} x$ by the identity $\tan^{-1} u - \tan^{-1} v = \tan^{-1} \frac{u-v}{1+uv}$ and (3).

7. Differentiate the following expressions:

(α) $\csc 2x - \cot 2x$, (β) $\frac{1}{3}\tan^3 x - \tan x + x$, ($\gamma$) $x \cos^{-1} x - \sqrt{1-x^2}$,

(δ) $\sec^{-1} \frac{1}{\sqrt{1-x^2}}$, ($\epsilon$) $\sin^{-1} \frac{x}{\sqrt{1+x^2}}$, ($\zeta$) $x\sqrt{a^2-x^2} + a^2 \sin^{-1} \frac{x}{a}$,

(η) $a \operatorname{vers}^{-1} \frac{x}{a} - \sqrt{2ax - x^2}$, ($\theta$) $\cot^{-1} \frac{2ax}{x^2-a^2} - 2\tan^{-1} \frac{x}{a}$.

What trigonometric identities are suggested by the answers for the following:

(α) $\sec^2 x$, (δ) $\frac{1}{\sqrt{1-x^2}}$, (ϵ) $\frac{1}{1+x^2}$, (θ) 0?

8. In B. O. Peirce's "Short Table of Integrals" (revised edition) differentiate the right-hand members to confirm the formulas: Nos. 31, 45–47, 91–97, 125, 127–128, 131–135, 161–163, 214–216, 220, 260–269, 294–298, 300, 380–381, 386–394.

9. If x is measured in degrees, what is $D \sin x$?

4. The logarithmic, exponential, and hyperbolic functions. The next set of formulas to be cited are

$$D \log_e x = \frac{1}{x}, \qquad D \log_a x = \frac{\log_a e}{x}, \tag{19}$$

$$De^x = e^x, \qquad Da^x = a^x \log_e a.\dagger \tag{20}$$

It may be recalled that the procedure for differentiating the logarithm is

$$\frac{\Delta \log_a x}{\Delta x} = \frac{\log_a (x + \Delta x) - \log_a x}{\Delta x} = \frac{1}{\Delta x} \log_a \frac{x + \Delta x}{x} = \frac{1}{x} \log_a \left(1 + \frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}.$$

* The student should keep on file his solutions of at least the important exercises; many subsequent exercises and considerable portions of the text depend on previous exercises.

† As is customary, the subscript e will hereafter be omitted and the symbol log will denote the logarithm to the base e; any base other than e must be specially designated as such. This observation is particularly necessary with reference to the common base 10 used in computation.

If now $x/\Delta x$ be set equal to h, the problem becomes that of evaluating

$$\lim_{h=\infty}\left(1+\frac{1}{h}\right)^h = e = 2.71828\cdots,^* \qquad \log_{10} e = 0.434294\cdots; \tag{21}$$

and hence if e be chosen as the base of the system, $D\log x$ takes the simple form $1/x$. The exponential functions e^x and a^x may be regarded as the inverse functions of $\log x$ and $\log_a x$ in deducing (21). Further it should be noted that it is frequently useful to take the logarithm of an expression before differentiating. This is known as *logarithmic differentiation* and is used for products and complicated powers and roots. Thus

$$\text{if} \quad y = x^x, \qquad \text{then} \quad \log y = x\log x,$$

$$\text{and} \quad \frac{1}{y}y' = 1+\log x \qquad \text{or} \quad y' = x^x(1+\log x).$$

It is the expression y'/y which is called the *logarithmic derivative* of y. An especially noteworthy property of the function $y = Ce^x$ is that the function and its derivative are equal, $y' = y$; and more generally *the function $y = Ce^{kx}$ is proportional to its derivative, $y' = ky$.*

5. The *hyperbolic functions* are the hyperbolic sine and cosine,

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2}; \tag{22}$$

and the related functions $\tanh x$, $\coth x$, $\operatorname{sech} x$, $\operatorname{csch} x$, derived from them by the same ratios as those by which the corresponding trigonometric functions are derived from $\sin x$ and $\cos x$. From these definitions in terms of exponentials follow the formulas:

$$\cosh^2 x - \sinh^2 x = 1, \qquad \tanh^2 x + \operatorname{sech}^2 x = 1, \tag{23}$$

$$\sinh(x \pm y) = \sinh x\cosh y \pm \cosh x \sinh y, \tag{24}$$

$$\cosh(x \pm y) = \cosh x\cosh y \pm \sinh x\sinh y, \tag{25}$$

$$\cosh\frac{x}{2} = +\sqrt{\frac{\cosh x + 1}{2}}, \qquad \sinh\frac{x}{2} = \pm\sqrt{\frac{\cosh x - 1}{2}}, \tag{26}$$

$$D\sinh x = \cosh x, \qquad D\cosh x = \sinh x, \tag{27}$$

$$D\tanh x = \operatorname{sech}^2 x, \qquad D\coth x = -\operatorname{csch}^2 x, \tag{28}$$

$$D\operatorname{sech} x = -\operatorname{sech} x\tanh x, \qquad D\operatorname{csch} x = -\operatorname{csch} x\coth x. \tag{29}$$

The inverse functions are expressible in terms of logarithms. Thus

$$y = \sinh^{-1} x, \qquad x = \sinh y = \frac{e^{2y}-1}{2e^y},$$

$$e^{2y} - 2xe^y - 1 = 0, \qquad e^y = x \pm \sqrt{x^2+1}.$$

* The treatment of this limit is far from complete in the majority of texts. Reference for a careful presentation may, however, be made to Granville's "Calculus," pp. 31–34, and Osgood's "Calculus," pp. 78–82. See also Ex. 1, (β), in § 165 below.

Here only the positive sign is available, for e^y is never negative. Hence

$$\sinh^{-1} x = \log\left(x + \sqrt{x^2+1}\right), \qquad \text{any } x, \tag{30}$$

$$\cosh^{-1} x = \log\left(x \pm \sqrt{x^2-1}\right), \qquad x > 1, \tag{31}$$

$$\tanh^{-1} x = \frac{1}{2}\log\frac{1+x}{1-x}, \qquad x^2 < 1, \tag{32}$$

$$\coth^{-1} x = \frac{1}{2}\log\frac{x+1}{x-1}, \qquad x^2 > 1, \tag{33}$$

$$\operatorname{sech}^{-1} x = \log\left(\frac{1}{x} \pm \sqrt{\frac{1}{x^2} - 1}\right), \qquad x < 1, \tag{34}$$

$$\operatorname{csch}^{-1} x = \log\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} + 1}\right), \qquad \text{any } x, \tag{35}$$

$$D\sinh^{-1} x = \frac{+1}{\sqrt{x^2+1}}, \qquad D\cosh^{-1} x = \frac{\pm 1}{\sqrt{x^2-1}}, \tag{36}$$

$$D\tanh^{-1} x = \frac{1}{1-x^2} = D\coth^{-1} x = \frac{1}{1-x^2}, \tag{37}$$

$$D\operatorname{sech}^{-1} x = \frac{\pm 1}{x\sqrt{1-x^2}}, \qquad D\operatorname{csch}^{-1} x = \frac{-1}{x\sqrt{1+x^2}}. \tag{38}$$

EXERCISES

1. Show by logarithmic differentiation that

$$D(uvw\cdots) = \left(\frac{u'}{u} + \frac{v'}{v} + \frac{w'}{w} + \cdots\right)(uvw\cdots),$$

and hence derive the rule: To differentiate a product differentiate each factor alone and add all the results thus obtained.

2. Sketch the graphs of the hyperbolic functions, interpret the graphs as those of the inverse functions, and verify the range of values assigned to x in (30)–(35).

3. Prove sundry of formulas (23)–(29) from the definitions (22).

4. Prove sundry of (30)–(38), checking the signs with care. In cases where double signs remain, state when each applies. Note that in (31) and (34) *the double sign may be placed before the* log *for the reason that the two expressions are reciprocals.*

5. Derive a formula for $\sinh u \pm \sinh v$ by applying (24); find a formula for $\tanh \frac{1}{2}x$ analogous to the trigonometric formula $\tan \frac{1}{2}x = \sin x/(1 + \cos x)$.

6. *The gudermannian.* The function $\phi = \operatorname{gd} x$, defined by the relations

$$\sinh x = \tan\phi, \quad \phi = \operatorname{gd} x = \tan^{-1}\sinh x, \quad -\tfrac{1}{2}\pi < \phi < +\tfrac{1}{2}\pi,$$

is called the gudermannian of x. Prove the set of formulas:

$$\cosh x = \sec\phi, \quad \tanh x = \sin\phi, \quad \operatorname{csch} x = \cot\phi, \quad \text{etc.};$$

$$D\operatorname{gd} x = \operatorname{sech} x, \quad x = \operatorname{gd}^{-1}\phi = \log\tan\left(\tfrac{1}{2}\phi + \tfrac{1}{4}\pi\right), \quad D\operatorname{gd}^{-1}\phi = \sec\phi.$$

7. Substitute the functions of ϕ in Ex. 6 for their hyperbolic equivalents in (23), (26), (27), and reduce to simple known trigonometric formulas.

8. Differentiate the following expressions:

(α) $(x+1)^2(x+2)^{-3}(x+3)^{-4}$, ($\beta$) $x^{\log x}$, (γ) $\log_x(x+1)$,
(δ) $x + \log\cos(x - \frac{1}{4}\pi)$, ($\epsilon$) $2\tan^{-1}e^x$, (ζ) $x - \tanh x$,
(η) $x\tanh^{-1}x + \frac{1}{2}\log(1-x^2)$, ($\theta$) $\dfrac{e^{ax}(a\sin mx - m\cos mx)}{m^2+a^2}$.

9. Check sundry formulas of Peirce's "Table," pp. 1–61, 81–82.

6. Geometric properties of the derivative. As the quotient (1) and its limit (2) give the slope of a secant and of the tangent, it appears from graphical considerations that when the derivative is positive the function is increasing with x, but decreasing when the derivative is negative.* Hence *to determine the regions in which a function is increasing or decreasing, one may find the derivative and determine the regions in which it is positive or negative.*

One must, however, be careful not to apply this rule too blindly; for in so simple a case as $f(x) = \log x$ it is seen that $f'(x) = 1/x$ is positive when $x > 0$ and negative when $x < 0$, and yet $\log x$ has no graph when $x < 0$ and is not considered as decreasing. Thus the formal derivative may be real when the function is not real, and it is therefore best to make a rough sketch of the function to corroborate the evidence furnished by the examination of $f'(x)$.

If x_0 is a value of x such that immediately† upon one side of $x = x_0$ the function $f(x)$ is increasing whereas immediately upon the other side it is decreasing, the ordinate $y_0 = f(x_0)$ will be a maximum or minimum or $f(x)$ will become positively or negatively infinite at x_0. If the case where $f(x)$ becomes infinite be ruled out, one may say that *the function will have a minimum or maximum at x_0 according as the derivative changes from negative to positive or from positive to negative when x, moving in the positive direction, passes through the value x_0.* Hence *the usual rule for determining maxima and minima is to find the roots of $f'(x) = 0$.*

This rule, again, must not be applied blindly. For first, $f'(x)$ may vanish where there is no maximum or minimum as in the case $y = x^3$ at $x = 0$ where the derivative does not change sign; or second, $f'(x)$ may change sign by becoming infinite as in the case $y = x^{\frac{2}{3}}$ at $x = 0$ where the curve has a vertical cusp, point down, and a minimum; or third, the function $f(x)$ may be restricted to a given range of values $a \leqq x \leqq b$ for x and then the values $f(a)$ and $f(b)$ of the function at the ends of the interval will in general be maxima or minima without implying that the derivative vanish. Thus although the derivative is highly useful in determining maxima and minima, it should not be trusted to the complete exclusion of the corroborative evidence furnished by a rough sketch of the curve $y = f(x)$.

* The construction of illustrative figures is again left to the reader.

† The word "immediately" is necessary because the maxima or minima may be merely *relative*; in the case of several maxima and minima in an interval, some of the maxima may actually be less than some of the minima.

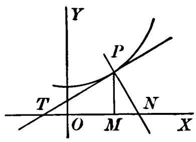
7. The derivative may be used to express the *equations of the tangent and normal*, the *values of the subtangent and subnormal*, and so on.

Equation of tangent, $y - y_0 = y_0'(x - x_0)$, (39)

Equation of normal, $(y - y_0)y_0' + (x - x_0) = 0$, (40)

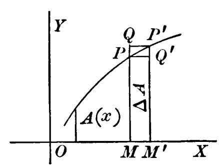
$$TM = \text{subtangent} = y_0/y_0', \quad MN = \text{subnormal} = y_0 y_0', \tag{41}$$

$$OT = x\text{-intercept of tangent} = x_0 - y_0/y_0', \text{ etc.} \tag{42}$$



The derivation of these results is sufficiently evident from the figure. It may be noted that the subtangent, subnormal, etc., are numerical values for a given point of the curve but may be regarded as functions of x like the derivative.

In geometrical and physical problems it is frequently necessary to apply the definition of the derivative to finding the derivative of an unknown function. For instance if A denote the area under a curve and measured from a fixed ordinate to a variable ordinate, A is surely a function $A(x)$ of the abscissa x of the variable ordinate. If the curve is rising, as in the figure, then



$$MPQ'M' < \Delta A < MQP'M', \text{ or } y\Delta x < \Delta A < (y + \Delta y)\Delta x.$$

Divide by Δx and take the limit when $\Delta x \doteq 0$. There results

$$\lim_{\Delta x \doteq 0} y \leqq \lim_{\Delta x \doteq 0} \frac{\Delta A}{\Delta x} \leqq \lim_{\Delta x \doteq 0} (y + \Delta y).$$

Hence $$\lim_{\Delta x \doteq 0} \frac{\Delta A}{\Delta x} = \frac{dA}{dx} = y. \tag{43}$$

Rolle's Theorem and the *Theorem of the Mean* are two important theorems on derivatives which will be treated in the next chapter but may here be stated as evident from their geometric interpretation. Rolle's Theorem states that: *If a function has a derivative at every*

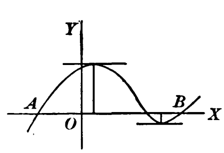


FIG. 1

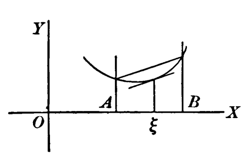


FIG. 2

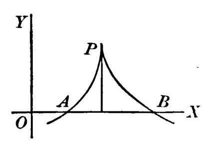


FIG. 3

point of an interval and if the function vanishes at the ends of the interval, then there is at least one point within the interval at which the derivative vanishes. This is illustrated in Fig. 1, in which there are two such points. The Theorem of the Mean states that: *If a function* *has a derivative at each point of an interval, there is at least one point in the interval such that the tangent to the curve $y = f(x)$ is parallel to the chord of the interval.* This is illustrated in Fig. 2 in which there is only one such point.

Again care must be exercised. In Fig. 3 the function vanishes at A and B but there is no point at which the slope of the tangent is zero. This is not an exception or contradiction to Rolle's Theorem for the reason that the function does not satisfy the conditions of the theorem. In fact at the point P, although there is a tangent to the curve, there is no derivative ; the quotient (1) formed for the point P becomes negatively infinite as $\Delta x \doteq 0$ from one side, positively infinite as $\Delta x \doteq 0$ from the other side, and therefore does not approach a definite limit as is required in the definition of a derivative. The hypothesis of the theorem is not satisfied and there is no reason that the conclusion should hold.

EXERCISES

1. Determine the regions in which the following functions are increasing or decreasing, sketch the graphs, and find the maxima and minima:

(α) $\frac{1}{3}x^3 - x^2 + 2$, (β) $(x+1)^{\frac{2}{3}}(x-5)^3$, (γ) $\log(x^2-4)$,
(δ) $(x-2)\sqrt{x-1}$, (ϵ) $-(x+2)\sqrt{12-x^2}$, (ζ) $x^3 + ax + b$.

2. The ellipse is $r = \sqrt{x^2+y^2} = e(d+x)$ referred to an origin at the focus. Find the maxima and minima of the focal radius r, and state why $D_x r = 0$ does not give the solutions while $D_\phi r = 0$ does [the polar form of the ellipse being $r = k(1 - e\cos\phi)^{-1}$].

3. Take the ellipse as $x^2/a^2 + y^2/b^2 = 1$ and discuss the maxima and minima of the central radius $r = \sqrt{x^2+y^2}$. Why does $D_x r = 0$ give half the result when r is expressed as a function of x, and why will $D_\lambda r = 0$ give the whole result when $x = a\cos\lambda$, $y = b\sin\lambda$ and the ellipse is thus expressed in terms of the eccentric angle ?

4. If $y = P(x)$ is a polynomial in x such that the equation $P(x) = 0$ has multiple roots, show that $P'(x) = 0$ for each multiple root. What more complete relationship can be stated and proved ?

5. Show that the triple relation $27b^2 + 4a^3 \gtreqless 0$ determines completely the nature of the roots of $x^3 + ax + b = 0$, and state what corresponds to each possibility.

6. Define *the angle θ between two intersecting curves.* Show that

$$\tan\theta = [f'(x_0) - g'(x_0)] \div [1 + f'(x_0)g'(x_0)]$$

if $y = f(x)$ and $y = g(x)$ cut at the point (x_0, y_0).

7. Find the subnormal and subtangent of the three curves

(α) $y^2 = 4px$, (β) $x^2 = 4py$, (γ) $x^2 + y^2 = a^2$.

8. *The pedal curve.* The locus of the foot of the perpendicular dropped from a fixed point to a variable tangent of a given curve is called the pedal of the given curve with respect to the given point. Show that if the fixed point is the origin, the pedal of $y = f(x)$ may be obtained by eliminating x_0, y_0, y_0' from the equations

$$y - y_0 = y_0'(x - x_0), \quad yy_0' + x = 0, \quad y_0 = f(x_0), \quad y_0' = f'(x_0).$$

Find the pedal (α) of the hyperbola with respect to the center and (β) of the parabola with respect to the vertex and (γ) the focus. Show (δ) that the pedal of the parabola with respect to any point is a cubic.

9. If the curve $y = f(x)$ be revolved about the x-axis and if $V(x)$ denote the volume of revolution thus generated when measured from a fixed plane perpendicular to the axis out to a variable plane perpendicular to the axis, show that $D_x V = \pi y^2$.

10. More generally if $A(x)$ denote the area of the section cut from a solid by a plane perpendicular to the x-axis, show that $D_x V = A(x)$.

11. If $A(\phi)$ denote the sectorial area of a plane curve $r = f(\phi)$ and be measured from a fixed radius to a variable radius, show that $D_\phi A = \frac{1}{2} r^2$.

12. If ρ, h, p are the density, height, pressure in a vertical column of air, show that $dp/dh = -\rho$. If $\rho = kp$, show $p = Ce^{-kh}$.

13. Draw a graph to illustrate an apparent exception to the Theorem of the Mean analogous to the apparent exception to Rolle's Theorem, and discuss.

14. Show that the analytic statement of the Theorem of the Mean for $f(x)$ is that a value $x = \xi$ intermediate to a and b may be found such that

$$f(b) - f(a) = f'(\xi)(b - a), \qquad a < \xi < b.$$

15. Show that the semiaxis of an ellipse is a mean proportional between the x-intercept of the tangent and the abscissa of the point of contact.

16. Find the values of the length of the tangent (α) from the point of tangency to the x-axis, (β) to the y-axis, (γ) the total length intercepted between the axes. Consider the same problems for the normal (figure on page 8).

17. Find the angle of intersection of (α) $y^2 = 2\,mx$ and $x^2 + y^2 = a^2$, (β) $x^2 = 4\,ay$ and $y = \dfrac{8\,a^3}{x^2 + 4\,a^2}$, (γ) $\dfrac{x^2}{a^2 - \lambda^2} + \dfrac{y^2}{b^2 - \lambda^2} = 1$ for $0 < \lambda < b$ and $b < \lambda < a$.

18. A constant length is laid off along the normal to a parabola. Find the locus.

19. The length of the tangent to $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ intercepted by the axes is constant.

20. The triangle formed by the asymptotes and any tangent to a hyperbola has constant area.

21. Find the length PT of the tangent to $x = \sqrt{c^2 - y^2} + c\,\text{sech}^{-1}(y/c)$.

22. Find the greatest right cylinder inscribed in a given right cone.

23. Find the cylinder of greatest lateral surface inscribed in a sphere.

24. From a given circular sheet of metal cut out a sector that will form a cone (without base) of maximum volume.

25. Join two points A, B in the same side of a line to a point P of the line in such a way that the distance $PA + PB$ shall be least.

26. Obtain the formula for the distance from a point to a line as the minimum distance.

27. *Test for maximum or minimum.* (α) If $f(x)$ vanishes at the ends of an interval and is positive within the interval and if $f'(x) = 0$ has only one root in the interval, that root indicates a maximum. Prove this by Rolle's Theorem. Apply it in Exs. 22–24. (β) If $f(x)$ becomes indefinitely great at the ends of an interval and $f'(x) = 0$ has only one root in the interval, that root indicates a minimum.

Prove by Rolle's Theorem, and apply in Exs. 25–26. These rules or various modifications of them generally suffice in practical problems to distinguish between maxima and minima without examining either the changes in sign of the first derivative or the sign of the second derivative; for generally there is only one root of $f'(x) = 0$ in the region considered.

28. Show that $x^{-1}\sin x$ from $x = 0$ to $x = \frac{1}{2}\pi$ steadily decreases from 1 to $2/\pi$.

29. If $0 < x < 1$, show (α) $0 < x - \log(1+x) < \frac{1}{2}x^2$, ($\beta$) $\frac{\frac{1}{2}x^2}{1+x} < x - \log(1+x)$.

30. If $0 > x > -1$, show that $\frac{1}{2}x^2 < x - \log(1+x) < \frac{\frac{1}{2}x^2}{1+x}$.

8. Derivatives of higher order. The derivative of the derivative (regarded as itself a function of x) is the second derivative, and so on to the nth derivative. Customary notations are:

$$f''(x) = \frac{d^2f(x)}{dx^2} = \frac{d^2y}{dx^2} = D_x^2 f = D_x^2 y = y'' = D^2 f = D^2 y,$$

$$f'''(x),\ f^{\text{iv}}(x),\ \cdots,\ f^{(n)}(x);\quad \frac{d^3y}{dx^3},\ \frac{d^4y}{dx^4},\ \cdots,\ \frac{d^ny}{dx^n},\ \cdots.$$

The nth derivative of the sum or difference is the sum or difference of the nth derivatives. For the nth derivative of the product there is a special formula known as *Leibniz's Theorem*. It is

$$D^n(uv) = D^n u \cdot v + nD^{n-1}uDv + \frac{n(n-1)}{2!}D^{n-2}uD^2v + \cdots + uD^nv. \tag{44}$$

This result may be written in symbolic form as

$$\text{Leibniz's Theorem} \quad D^n(uv) = (Du + Dv)^n, \tag{44'}$$

where it is to be understood that in expanding $(Du + Dv)^n$ the term $(Du)^k$ is to be replaced by $D^k u$ and $(Du)^0$ by $D^0 u = u$. In other words the powers refer to repeated differentiations.

A proof of (44) by induction will be found in § 27. The following proof is interesting on account of its ingenuity. Note first that from

$$D(uv) = uDv + vDu, \quad D^2(uv) = D(uDv) + D(vDu),$$

and so on, it appears that $D^2(uv)$ consists of a sum of terms, in each of which there are two differentiations, with numerical coefficients independent of u and v. In like manner it is clear that

$$D^n(uv) = C_0 D^n u \cdot v + C_1 D^{n-1}uDv + \cdots + C_{n-1}DuD^{n-1}v + C_n uD^n v$$

is a sum of terms, in each of which there are n differentiations, with coefficients C independent of u and v. To determine the C's any suitable functions u and v, say,

$$u = e^x, \quad v = e^{ax}, \quad uv = e^{(1+a)x}, \quad D^k e^{ax} = a^k e^{ax},$$

may be substituted. If the substitution be made and $e^{(1+a)x}$ be canceled,

$$e^{-(1+a)x}D^n(uv) = (1+a)^n = C_0 + C_1 a + \cdots + C_{n-1}a^{n-1} + C_n a^n,$$

and hence the C's are the coefficients in the binomial expansion of $(1 + a)^n$.

Formula (4) for the derivative of a function of a function may be extended to higher derivatives by repeated application. More generally *any desired change of variable may be made by the repeated use of* (4) *and* (5). For if x and y be expressed in terms of known functions of new variables u and v, it is always possible to obtain the derivatives $D_x y$, $D_x^2 y$, $\cdots$ in terms of $D_u v$, $D_u^2 v$, $\cdots$, and thus any expression $F(x, y, y', y'', \cdots)$ may be changed into an equivalent expression $\Phi(u, v, v', v'', \cdots)$ in the new variables. In each case that arises the transformations should be carried out by repeated application of (4) and (5) rather than by substitution in any general formulas.

The following typical cases are illustrative of the method of change of variable. Suppose only the dependent variable y is to be changed to z defined as $y=f(z)$. Then

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{dz}{dx}\frac{dy}{dz}\right)=\frac{d^2z}{dx^2}\frac{dy}{dz}+\frac{dz}{dx}\left(\frac{d}{dx}\frac{dy}{dz}\right)$$

$$=\frac{d^2z}{dx^2}\frac{dy}{dz}+\frac{dz}{dx}\left(\frac{d}{dz}\frac{dy}{dz}\frac{dz}{dx}\right)=\frac{d^2z}{dx^2}\frac{dy}{dz}+\left(\frac{dz}{dx}\right)^2\frac{d^2y}{dz^2}.$$

As the derivatives of $y=f(z)$ are known, the derivative d^2y/dx^2 has been expressed in terms of z and derivatives of z with respect to x. The third derivative would be found by repeating the process. If the problem were to change the independent variable x to z, defined by $x=f(z)$,

$$\frac{dy}{dx}=\frac{dy}{dz}\frac{dz}{dx}=\frac{dy}{dz}\left(\frac{dx}{dz}\right)^{-1},\qquad \frac{d^2y}{dx^2}=\frac{d}{dx}\left[\frac{dy}{dz}\left(\frac{dx}{dz}\right)^{-1}\right].$$

$$\frac{d^2y}{dx^2}=\frac{d^2y}{dz^2}\frac{dz}{dx}\left(\frac{dx}{dz}\right)^{-1}-\frac{dy}{dz}\left(\frac{dx}{dz}\right)^{-2}\frac{dz}{dx}\frac{d^2x}{dz^2}=\left[\frac{d^2y}{dz^2}\frac{dx}{dz}-\frac{d^2x}{dz^2}\frac{dy}{dz}\right]\div\left(\frac{dx}{dz}\right)^3.$$

The change is thus made as far as derivatives of the second order are concerned. If the change of both dependent and independent variables was to be made, the work would be similar. Particularly useful changes are to find the derivatives of y by x when y and x are expressed parametrically as functions of t, or when both are expressed in terms of new variables r, ϕ as $x=r\cos\phi$, $y=r\sin\phi$. For these cases see the exercises.

9. The *concavity of a curve* $y=f(x)$ is given by the table:

if $f''(x_0)>0$,	the curve is concave up at $x=x_0$,
if $f''(x_0)<0$,	the curve is concave down at $x=x_0$,
if $f''(x_0)=0$,	an inflection point at $x=x_0$. (?)

Hence the *criterion for distinguishing between maxima and minima:*

if $f'(x_0)=0$ and $f''(x_0)>0$,	a minimum at $x=x_0$,
if $f'(x_0)=0$ and $f''(x_0)<0$,	a maximum at $x=x_0$,
if $f'(x_0)=0$ and $f''(x_0)=0$,	neither max. nor min. (?)

The question points are necessary in the third line because the statements are not always true unless $f'''(x_0) \neq 0$ (see Ex. 7 under § 39).

It may be recalled that the reason that the curve is concave up in case $f''(x_0) > 0$ is because the derivative $f'(x)$ is then an increasing function in the neighborhood of $x = x_0$; whereas if $f''(x_0) < 0$, the derivative $f'(x)$ is a decreasing function and the curve is convex up. It should be noted that concave up is not the same as concave toward the x-axis, except when the curve is below the axis. With regard to the use of the second derivative as a criterion for distinguishing between maxima and minima, it should be stated that in practical examples the criterion is of relatively small value. It is usually shorter to discuss the change of sign of $f'(x)$ directly, — and indeed in most cases either a rough graph of $f(x)$ or the physical conditions of the problem which calls for the determination of a maximum or minimum will immediately serve to distinguish between them (see Ex. 27 above).

The second derivative is fundamental in dynamics. By definition the *average velocity* $\bar{v}$ of a particle is the ratio of the space traversed to the time consumed, $\bar{v} = s/t$. The *actual velocity* v at any time is the limit of this ratio when the interval of time is diminished and approaches zero as its limit. Thus

$$\bar{v} = \frac{\Delta s}{\Delta t} \quad \text{and} \quad v = \lim_{\Delta t \doteq 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}. \tag{45}$$

In like manner if a particle describes a straight line, say the x-axis, the *average acceleration* $\bar{f}$ is the ratio of the increment of velocity to the increment of time, and the *actual acceleration* f at any time is the limit of this ratio as $\Delta t \doteq 0$. Thus

$$\bar{f} = \frac{\Delta v}{\Delta t} \quad \text{and} \quad f = \lim_{\Delta t \doteq 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} = \frac{d^2x}{dt^2}. \tag{46}$$

By *Newton's Second Law of Motion, the force acting on the particle is equal to the rate of change of momentum* with the time, momentum being defined as the product of the mass and velocity. Thus

$$F = \frac{d(mv)}{dt} = m\frac{dv}{dt} = mf = m\frac{d^2x}{dt^2}, \tag{47}$$

where it has been assumed in differentiating that the mass is constant, as is usually the case. Hence (47) appears as the fundamental equation for rectilinear motion (see also §§ 79, 84). It may be noted that

$$F = mv\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}mv^2\right) = \frac{dT}{dx}, \tag{47'}$$

where $T = \frac{1}{2}mv^2$ denotes by definition the *kinetic energy* of the particle For comments see Ex. 6 following.

EXERCISES

1. State and prove the extension of Leibniz's Theorem to products of three or more factors. Write out the square and cube of a trinomial.

2. Write, by Leibniz's Theorem, the second and third derivatives:

(α) $e^x \sin x$, (β) $\cosh x \cos x$, (γ) $x^2 e^x \log x$.

3. Write the nth derivatives of the following functions, of which the last three should first be simplified by division or separation into partial fractions.

(α) $\sqrt{x+1}$, (β) $\log(ax+b)$, (γ) $(x^2+1)(x+1)^{-3}$,
(δ) $\cos ax$, (ϵ) $e^x \sin x$, (ζ) $(1-x)/(1+x)$,
(η) $\dfrac{1}{x^2-1}$, (θ) $\dfrac{x^3+x+1}{x-1}$, (ι) $\left(\dfrac{ax+1}{ax-1}\right)^2$.

4. If y and x are each functions of t, show that

$$\frac{d^2y}{dx^2} = \frac{\dfrac{dx}{dt}\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt}\dfrac{d^2x}{dt^2}}{\left(\dfrac{dx}{dt}\right)^3} = \frac{x'y'' - y'x''}{x'^3},$$

$$\frac{d^3y}{dx^3} = \frac{x'(x'y''' - y'x''') - 3x''(x'y'' - y'x'')}{x'^5}.$$

5. Find the inflection points of the curve $x = 4\phi - 2\sin\phi$, $y = 4 - 2\cos\phi$.

6. Prove (47′). Hence infer that the force which is the time-derivative of the momentum mv by (47) is also the space-derivative of the kinetic energy.

7. If A denote the area under a curve, as in (43), find $dA/d\theta$ for the curves

(α) $y = a(1-\cos\theta)$, $x = a(\theta - \sin\theta)$, (β) $x = a\cos\theta$, $y = b\sin\theta$.

8. Make the indicated change of variable in the following equations:

(α) $\dfrac{d^2y}{dx^2} + \dfrac{2x}{1+x^2}\dfrac{dy}{dx} + \dfrac{y}{(1+x^2)^2} = 0$, $x = \tan z$. *Ans.* $\dfrac{d^2y}{dz^2} + y = 0$.

(β) $(1-x^2)\left[\dfrac{d^2y}{dx^2} - \dfrac{1}{y}\left(\dfrac{dy}{dx}\right)^2\right] - x\dfrac{dy}{dx} + y = 0$, $y = e^v$, $x = \sin u$.

Ans. $\dfrac{d^2v}{du^2} + 1 = 0$.

9. *Transformation to polar coördinates.* Suppose that $x = r\cos\phi$, $y = r\sin\phi$. Then

$$\frac{dx}{d\phi} = \frac{dr}{d\phi}\cos\phi - r\sin\phi, \qquad \frac{dy}{d\phi} = \frac{dr}{d\phi}\sin\phi + r\cos\phi,$$

and so on for higher derivatives. Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2} = \dfrac{r^2 + 2(D_\phi r)^2 - rD_\phi^2 r}{(\cos\phi\, D_\phi r - r\sin\phi)^3}$.

10. Generalize formula (5) for the differentiation of an inverse function. Find d^2x/dy^2 and d^3x/dy^3. Note that these may also be found from Ex. 4.

11. A point describes a circle with constant speed. Find the velocity and acceleration of the projection of the point on any fixed diameter.

12. Prove $\dfrac{d^2y}{dx^2} = 2uv^3 + 4v^4\left(\dfrac{dv}{du}\right)^{-1} - v^5\dfrac{d^2v}{du^2}\left(\dfrac{dv}{du}\right)^{-3}$ if $x = \dfrac{1}{v}$, $y = uv$.

10. The indefinite integral. To integrate a function $f(x)$ is to *find a function $F(x)$ the derivative of which is $f(x)$*. The integral $F(x)$ is not uniquely determined by the integrand $f(x)$; for any two functions which differ merely by an additive constant have the same derivative. In giving formulas for integration the constant may be omitted and understood; but in applications of integration to actual problems it should always be inserted and must usually be determined to fit the requirements of special conditions imposed upon the problem and known as the *initial conditions*.

It must not be thought that the constant of integration always appears added to the function $F(x)$. It may be combined with $F(x)$ so as to be somewhat disguised. Thus

$$\log x, \qquad \log x + C, \qquad \log Cx, \qquad \log (x/C)$$

are all integrals of $1/x$, and all except the first have the constant of integration C, although only in the second does it appear as formally additive. To illustrate the determination of the constant by initial conditions, consider the problem of finding the area under the curve $y = \cos x$. By (43)

$$D_xA = y = \cos x \quad \text{and hence} \quad A = \sin x + C.$$

If the area is to be measured from the ordinate $x = 0$, then $A = 0$ when $x = 0$, and by direct substitution it is seen that $C = 0$. Hence $A = \sin x$. But if the area be measured from $x = -\frac{1}{2}\pi$, then $A = 0$ when $x = -\frac{1}{2}\pi$ and $C = 1$. Hence $A = 1 + \sin x$. In fact the area under a curve is not definite until the ordinate from which it is measured is specified, and the constant is needed to allow the integral to fit this initial condition.

11. The fundamental formulas of integration are as follows:

$$\int \frac{1}{x} = \log x, \qquad \int x^n = \frac{1}{n+1} x^n \text{ if } n \neq -1, \qquad (48)$$

$$\int e^x = e^x, \qquad \int a^x = a^x/\log a, \qquad (49)$$

$$\int \sin x = -\cos x, \qquad \int \cos x = \sin x, \qquad (50)$$

$$\int \tan x = -\log \cos x, \qquad \int \cot x = \log \sin x, \qquad (51)$$

$$\int \sec^2 x = \tan x, \qquad \int \csc^2 x = -\cot x, \qquad (52)$$

$$\int \tan x \sec x = \sec x, \qquad \int \cot x \csc x = -\csc x, \qquad (53)$$

with formulas similar to (50)–(53) for the hyperbolic functions. Also

$$\int \frac{1}{1+x^2} = \tan^{-1} x \text{ or } -\cot^{-1} x, \quad \int \frac{1}{1-x^2} = \tanh^{-1} x \text{ or } \coth^{-1} x, \quad (54)$$

$$\int\frac{1}{\sqrt{1-x^2}}=\sin^{-1}x \text{ or } -\cos^{-1}x,\quad \int\frac{\pm 1}{\sqrt{1+x^2}}=\pm\sinh^{-1}x, \tag{55}$$

$$\int\frac{1}{x\sqrt{x^2-1}}=\sec^{-1}x \text{ or } -\csc^{-1}x,\quad \int\frac{\pm 1}{x\sqrt{1-x^2}}=\mp\operatorname{sech}^{-1}x, \tag{56}$$

$$\int\frac{\pm 1}{\sqrt{x^2-1}}=\pm\cosh^{-1}x,\qquad \int\frac{\pm 1}{x\sqrt{1+x^2}}=\mp\operatorname{csch}^{-1}x, \tag{57}$$

$$\int\frac{1}{\sqrt{2x-x^2}}=\operatorname{vers}^{-1}x,\quad \int\sec x=\operatorname{gd}^{-1}x=\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right). \tag{58}$$

For the integrals expressed in terms of the inverse hyperbolic functions, the logarithmic equivalents are sometimes preferable. This is not the case, however, in the many instances in which the problem calls for immediate solution with regard to x. Thus if $y=\int(1+x^2)^{-\frac{1}{2}}=\sinh^{-1}x+C$, then $x=\sinh(y-C)$, and the solution is effected and may be translated into exponentials. This is not so easily accomplished from the form $y=\log\left(x+\sqrt{1+x^2}\right)+C$. For this reason and because the inverse hyperbolic functions are briefer and offer striking analogies with the inverse trigonometric functions, it has been thought better to use them in the text and allow the reader to make the necessary substitutions from the table (30)–(35) in case the logarithmic form is desired.

12. In addition to these special integrals, which are consequences of the corresponding formulas for differentiation, there are the general rules of integration which arise from (4) and (6).

$$\int\frac{dz}{dy}\frac{dy}{dx}=\int\frac{dz}{dx}=z, \tag{59}$$

$$\int(u+v-w)=\int u+\int v-\int w, \tag{60}$$

$$uv=\int uv'+\int u'v. \tag{61}$$

Of these rules the second needs no comment and the third will be treated later. Especial attention should be given to the first. For instance suppose it were required to integrate $2\log x/x$. This does not fall under any of the given types; but

$$\frac{2}{x}\log x=\frac{d(\log x)^2}{d\log x}\frac{d\log x}{dx}=\frac{dz}{dy}\frac{dy}{dx}.$$

Here $(\log x)^2$ takes the place of z and $\log x$ takes the place of y. The integral is therefore $(\log x)^2$ as may be verified by differentiation. In general, it may be possible to see that a given integrand is separable into two factors, of which one is integrable when considered as a function of some function of x, while the other is the derivative of that function. Then (59) applies. Other examples are:

$$\int e^{\sin x}\cos x,\qquad \int\tan^{-1}x/(1+x^2),\qquad \int x^2\sin(x^3).$$

In the first, $z = e^y$ is integrable and as $y = \sin x$, $y' = \cos x$; in the second, $z = y$ is integrable and as $y = \tan^{-1}x$, $y' = (1 + x^2)^{-1}$; in the third $z = \sin y$ is integrable and as $y = x^3$, $y' = 3x^2$. The results are

$$e^{\sin x}, \qquad \tfrac{1}{2}(\tan^{-1}x)^2, \qquad -\tfrac{1}{3}\cos(x^3).$$

This method of integration at sight covers such a large percentage of the cases that arise in geometry and physics that it must be thoroughly mastered.*

EXERCISES

1. Verify the fundamental integrals (48)–(58) and give the hyperbolic analogues of (50)–(53).

2. Tabulate the integrals here expressed in terms of inverse hyperbolic functions by means of the corresponding logarithmic equivalents.

3. Write the integrals of the following integrands at sight:

(α) $\sin ax$, (β) $\cot(ax + b)$, (γ) $\tanh 3x$,

(δ) $\dfrac{1}{a^2 + x^2}$, (ε) $\dfrac{1}{\sqrt{x^2 - a^2}}$, (ζ) $\dfrac{1}{\sqrt{2ax - x^2}}$,

(η) $\dfrac{1}{x \log x}$, (θ) $\dfrac{e^{\frac{1}{x}}}{x^2}$, (ι) $\dfrac{x}{x^2 + a^2}$,

(κ) $x^3\sqrt{ax^2 + b}$, (λ) $\tan x \sec^2 x$, (μ) $\tan x \log \sin x$,

(ν) $\dfrac{(x^{-1} - 1)^5}{x^2}$, (ο) $\dfrac{\tanh^{-1}x}{1 - x^2}$, (π) $\dfrac{2 + \log x}{x}$,

(ρ) $a^{1 + \sin x}\cos x$, (σ) $\dfrac{\sin x}{\sqrt{\cos x}}$, (τ) $\dfrac{1}{\sqrt{1 - x^2}\sin^{-1}x}$.

4. Integrate after making appropriate changes such as $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$ or $\sec^2 x = 1 + \tan^2 x$, division of denominator into numerator, resolution of the product of trigonometric functions into a sum, completing the square, and so on.

(α) $\cos^2 2x$, (β) $\sin^4 x$. (γ) $\tan^4 x$,

(δ) $\dfrac{1}{x^2 + 3x + 25}$, (ε) $\dfrac{2x + 1}{x + 2}$, (ζ) $\dfrac{1 - \sin x}{\operatorname{vers} x}$,

(η) $\dfrac{x + 3}{4x^2 - 5x + 1}$, (θ) $\dfrac{e^{2x} + e^x}{e^{2x} + 1}$, (ι) $\dfrac{1}{\sqrt{2ax + x^2}}$,

(κ) $\sin 5x \cos 2x + 1$, (λ) $\sinh mx \sinh nx$, (μ) $\cos x \cos 2x \cos 3x$,

(ν) $\sec^5 x \tan x - \sqrt{2x}$, (ο) $\dfrac{cx + d}{x^2 + ax + b}$, (π) $-\dfrac{x^{m-1}}{(ax^m + b)^p}$.

* The use of differentials (§ 35) is perhaps more familiar than the use of derivatives.

$$z(x) = \int \frac{dz}{dx}dx = \int \frac{dz}{dy}\frac{dy}{dx}dx = \int \frac{dz}{dy}dy = z[y(x)].$$

Then $$\int \frac{2}{x}\log x\, dx = \int 2\log x\, d\log x = (\log x)^2.$$

The use of this notation is left optional with the reader; it has some advantages and some disadvantages. The essential thing is to keep clearly in mind the fact that the problem is to be inspected with a view to detecting the function which will differentiate into the given integrand.

5. How are the following types integrated?

(α) $\sin^m x \cos^n x$, m or n odd, or m and n even,
(β) $\tan^n x$ or $\cot^n x$ when n is an integer,
(γ) $\sec^n x$ or $\csc^n x$ when n is even,
(δ) $\tan^m x \sec^n x$ or $\cot^m x \csc^n x$, n even.

6. Explain the alternative forms in (54)–(56) with all detail possible.

7. Find (α) the area under the parabola $y^2 = 4px$ from $x = 0$ to $x = a$; also (β) the corresponding volume of revolution. Find (γ) the total volume of an ellipsoid of revolution (see Ex. 9, p. 10).

8. Show that the area under $y = \sin mx \sin nx$ or $y = \cos mx \cos nx$ from $x = 0$ to $x = \pi$ is zero if m and n are unequal integers but $\frac{1}{2}\pi$ if they are equal.

9. Find the sectorial area of $r = a \tan \phi$ between the radii $\phi = 0$ and $\phi = \frac{1}{4}\pi$.

10. Find the area of the (α) lemniscate $r^2 = a^2 \cos 2\phi$ and (β) cardioid $r = 1 - \cos\phi$.

11. By Ex. 10, p. 10, find the volumes of these solids. Be careful to choose the parallel planes so that $A(x)$ may be found easily.

(α) The part cut off from a right circular cylinder by a plane through a diameter of one base and tangent to the other. *Ans.* $2/3\pi$ of the whole volume.

(β) How much is cut off from a right circular cylinder by a plane tangent to its lower base and inclined at an angle θ to the plane of the base?

(γ) A circle of radius $b < a$ is revolved, about a line in its plane at a distance a from its center, to generate a ring. The volume of the ring is $2\pi^2 ab^2$.

(δ) The axes of two equal cylinders of revolution of radius r intersect at right angles. The volume common to the cylinders is $16r^3/3$.

12. If the cross section of a solid is $A(x) = a_0x^3 + a_1x^2 + a_2x + a_3$, a cubic in x, the volume of the solid between two parallel planes is $\frac{1}{6}h(B + 4M + B')$ where h is the altitude and B and B' are the bases and M is the middle section.

13. Show that $\int \frac{1}{1+x^2} = \tan^{-1}\frac{x+c}{1-cx}$.

13. Aids to integration. The majority of cases of integration which arise in simple applications of calculus may be treated by the method of § 12. Of the remaining cases a large number cannot be integrated at all in terms of the functions which have been treated up to this point. Thus it is impossible to express $\int \frac{1}{\sqrt{(1-x^2)(1-a^2x^2)}}$ in terms of elementary functions. One of the chief reasons for introducing a variety of new functions in higher analysis is to have means for effecting the integrations called for by important applications. The discussion of this matter cannot be taken up here. The problem of integration from an elementary point of view calls for the tabulation of some devices which will accomplish the integration for a wide variety of integrands integrable in terms of elementary functions. The devices which will be treated are:

Integration by parts,	Resolution into partial fractions,
Various substitutions,	Reference to tables of integrals.

Integration by parts is an application of (61) when written as

$$\int uv' = uv - \int u'v. \tag{61'}$$

That is, it may happen that the integrand can be written as the product uv' of two factors, where v' is integrable and where $u'v$ is also integrable. Then uv' is integrable. For instance, $\log x$ is not integrated by the fundamental formulas; but

$$\int \log x = \int \log x \cdot 1 = x \log x - \int x/x = x \log x - x.$$

Here $\log x$ is taken as u and 1 as v', so that v is x, u' is $1/x$, and $u'v = 1$ is immediately integrable. This method applies to the inverse trigonometric and hyperbolic functions. Another example is

$$\int x \sin x = - x \cos x + \int \cos x = \sin x - x \cos x.$$

Here if $x = u$ and $\sin x = v'$, both v' and $u'v = - \cos x$ are integrable. If the choice $\sin x = u$ and $x = v'$ had been made, v' would have been integrable but $u'v = \frac{1}{2} x^2 \cos x$ would have been less simple to integrate than the original integrand. Hence in applying integration by parts it is necessary to *look ahead* far enough to see that both v' and $u'v$ are integrable, or at any rate that v' is integrable and the integral of $u'v$ is simpler than the original integral.*

Frequently integration by parts has to be applied several times in succession. Thus

$$\begin{aligned} \int x^2 e^x &= x^2 e^x - \int 2 x e^x && \text{if } u = x^2,\ v' = e^x, \\ &= x^2 e^x - 2\left[x e^x - \int e^x\right] && \text{if } u = x,\ v' = e^x, \\ &= x^2 e^x - 2 x e^x + 2 e^x. \end{aligned}$$

Sometimes it may be applied in such a way as to lead back to the given integral and thus afford an equation from which that integral can be obtained by solution. For example,

$$\begin{aligned} \int e^x \cos x &= e^x \cos x + \int e^x \sin x && \text{if } u = \cos x,\ v' = e^x, \\ &= e^x \cos x + \left[e^x \sin x - \int e^x \cos x\right] && \text{if } u = \sin x,\ v' = e^x, \\ &= e^x (\cos x + \sin x) - \int e^x \cos x. \end{aligned}$$

Hence $$\int e^x \cos x = \tfrac{1}{2} e^x (\cos x + \sin x).$$

* The method of differentials may again be introduced if desired.

14. For the *integration of a rational fraction* $f(x)/F(x)$ where f and F are polynomials in x, the fraction is first resolved into *partial fractions*. This is accomplished as follows. First if f is not of lower degree than F, divide F into f until the remainder is of lower degree than F. The fraction f/F is thus resolved into the sum of a polynomial (the quotient) and a fraction (the remainder divided by F) of which the numerator is of lower degree than the denominator. As the polynomial is integrable, it is merely necessary to consider fractions f/F where f is of lower degree than F. Next it is a fundamental theorem of algebra that a polynomial F may be resolved into linear and quadratic factors

$$F(x) = k(x-a)^\alpha (x-b)^\beta (x-c)^\gamma \cdots (x^2+mx+n)^\mu (x^2+px+q)^\nu \cdots,$$

where $a, b, c, \cdots$ are the real roots of the equation $F(x)=0$ and are of the respective multiplicities $\alpha, \beta, \gamma, \cdots$, and where the quadratic factors when set equal to zero give the pairs of conjugate imaginary roots of $F=0$, the multiplicities of the imaginary roots being $\mu, \nu, \cdots$. It is then a further theorem of algebra that the fraction f/F may be written as

$$\frac{f(x)}{F(x)} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_\alpha}{(x-a)^\alpha} + \frac{B_1}{x-b} + \cdots + \frac{B_\beta}{(x-b)^\beta} + \cdots$$
$$+ \frac{M_1 x + N_1}{x^2+mx+n} + \frac{M_2 x + N_2}{(x^2+mx+n)^2} + \cdots + \frac{M_\mu x + N_\mu}{(x^2+mx+n)^\mu} + \cdots,$$

where there is for each irreducible factor of F a term corresponding to the highest power to which that factor occurs in F and also a term corresponding to every lesser power. The coefficients $A, B, \cdots, M, N, \cdots$ may be obtained by clearing of fractions and equating coefficients of like powers of x, and solving the equations; or they may be obtained by clearing of fractions, substituting for x as many different values as the degree of F, and solving the resulting equations.

When f/F has thus been resolved into partial fractions, the problem has been reduced to the integration of each fraction, and this does not present serious difficulty. The following two examples will illustrate the method of resolution into partial fractions and of integration. Let it be required to integrate

$$\int \frac{x^2+1}{x(x-1)(x-2)(x^2+x+1)} \quad \text{and} \quad \int \frac{2x^3+6}{(x-1)^2(x-3)^3}.$$

The first fraction is expansible into partial fractions in the form

$$\frac{x^2+1}{x(x-1)(x-2)(x^2+x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2} + \frac{Dx+E}{x^2+x+1}.$$

Hence $\quad x^2+1 = A(x-1)(x-2)(x^2+x+1) + Bx(x-2)(x^2+x+1)$
$$+ Cx(x-1)(x^2+x+1) + (Dx+E)x(x-1)(x-2).$$

Rather than multiply out and equate coefficients, let $0, 1, 2, -1, -2$ be substituted. Then

$$1 = 2A, \quad 2 = -3B, \quad 5 = 14C, \quad D - E = 1/21, \quad E - 2D = 1/7,$$

$$\int \frac{x^2+1}{x(x-1)(x-2)(x^2+x+1)} = \int \frac{1}{2x} - \int \frac{2}{3(x-1)} + \int \frac{5}{14(x-2)} - \int \frac{4x+5}{21(x^2+x+1)}$$
$$= \frac{1}{2}\log x - \frac{2}{3}\log(x-1) + \frac{5}{14}\log(x-2) - \frac{2}{21}\log(x^2+x+1) - \frac{2}{7\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}.$$

In the second case the form to be assumed for the expansion is

$$\frac{2x^3+6}{(x-1)^2(x-3)^3}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x-3)}+\frac{D}{(x-3)^2}+\frac{E}{(x-3)^3}.$$

$$2x^3+6=A(x-1)(x-3)^3+B(x-3)^3+C(x-1)^2(x-3)^2$$
$$+D(x-1)^2(x-3)+E(x-1)^2.$$

The substitution of 1, 3, 0, 2, 4 gives the equations

$$8=-8B,\qquad 60=4E,\qquad 9A+3C-D+12=0,$$
$$A-C+D+6=0,\qquad A+3C+3D=0.$$

The solutions are $-9/4$, -1, $+9/4$, $-3/2$, 15, and the integral becomes

$$\int\frac{2x^3+6}{(x-1)^2(x-3)^3}=-\frac{9}{4}\log(x-1)+\frac{1}{x-1}+\frac{9}{4}\log(x-3)$$
$$+\frac{3}{2(x-3)}-\frac{15}{2(x-3)^2}.$$

The importance of the fact that the method of partial fractions shows that *any rational fraction may be integrated* and, moreover, that the integral may at most consist of a rational part plus the logarithm of a rational fraction plus the inverse tangent of a rational fraction should not be overlooked. Taken with the method of substitution it establishes very wide categories of integrands which are integrable in terms of elementary functions, and effects their integration even though by a somewhat laborious method.

15. The *method of substitution* depends on the identity

$$\int_x f(x)=\int_y f[\phi(y)]\frac{dx}{dy}\qquad\text{if}\qquad x=\phi(y),\tag{59'}$$

which is allied to (59). To show that the integral on the right with respect to y is the integral of $f(x)$ with respect to x it is merely necessary to show that its derivative with respect to x is $f(x)$. By definition of integration,

$$\frac{d}{dy}\int_y f[\phi(y)]\frac{dx}{dy}=f[\phi(y)]\frac{dx}{dy}$$

and

$$\frac{d}{dx}\int_y f[\phi(y)]\frac{dx}{dy}=f[\phi(y)]\frac{dx}{dy}\cdot\frac{dy}{dx}=f[\phi(y)]$$

by (4). The identity is therefore proved. The method of integration by substitution is in fact seen to be merely such a systematization of the method based on (59) and set forth in § 12 as will make it practicable for more complicated problems. Again, differentials may be used if preferred.

Let R denote a rational function. To effect the integration of

$$\int\sin x\,R(\sin^2 x,\ \cos x),\qquad\text{let}\quad\cos x=y,\qquad\text{then}\quad\int_y -R(1-y^2,\ y);$$

$$\int\cos x\,R(\cos^2 x,\ \sin x),\qquad\text{let}\quad\sin x=y,\qquad\text{then}\quad\int_y R(1-y^2,\ y);$$

$$\int R\left(\frac{\sin x}{\cos x}\right)=\int R(\tan x),\qquad\text{let}\quad\tan x=y,\qquad\text{then}\quad\int_y\frac{R(y)}{1+y^2};$$

$$\int R(\sin x,\ \cos x),\qquad\text{let}\quad\tan\frac{x}{2}=y,\qquad\text{then}\quad\int_y R\left(\frac{2y}{1+y^2},\ \frac{1-y^2}{1+y^2}\right)\frac{2}{1+y^2}.$$

The last substitution renders any rational function of $\sin x$ and $\cos x$ rational in the variable y; it should not be used, however, if the previous ones are applicable — it is almost certain to give a more difficult final rational fraction to integrate.

A large number of geometric problems give integrands which are rational in x and in some one of the radicals $\sqrt{a^2+x^2}$, $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$. These may be converted into trigonometric or hyperbolic integrands by the following substitutions:

$$\int R(x, \sqrt{a^2-x^2}) \qquad x = a\sin y, \qquad \int_y R(a\sin y, a\cos y)\, a\cos y;$$

$$\int R(x, \sqrt{a^2+x^2}) \quad \begin{cases} x = a\tan y, & \int_y R(a\tan y, a\sec y)\, a\sec^2 y \\ x = a\sinh y, & \int_y R(a\sinh y, a\cosh y)\, a\cosh y; \end{cases}$$

$$\int R(x, \sqrt{x^2-a^2}) \quad \begin{cases} x = a\sec y, & \int_y R(a\sec y, a\tan y)\, a\sec y\tan y \\ x = a\cosh y, & \int_y R(a\cosh y, a\sinh y)\, a\sinh y. \end{cases}$$

It frequently turns out that the integrals on the right are easily obtained by methods already given; otherwise they can be treated by the substitutions above.

In addition to these substitutions there are a large number of others which are applied under specific conditions. Many of them will be found among the exercises. Moreover, it frequently happens that an integrand, which does not come under any of the standard types for which substitutions are indicated, is none the less integrable by some substitution which the form of the integrand will suggest.

Tables of integrals, giving the integrals of a large number of integrands, have been constructed by using various methods of integration. B. O. Peirce's "Short Table of Integrals" may be cited. If the particular integrand which is desired does not occur in the Table, it may be possible to devise some substitution which will reduce it to a tabulated form. In the Table are also given a large number of reduction formulas (for the most part deduced by means of integration by parts) which accomplish the successive simplification of integrands which could perhaps be treated by other methods, but only with an excessive amount of labor. Several of these reduction formulas are cited among the exercises. Although the Table is useful in performing integrations and indeed makes it to a large extent unnecessary to learn the various methods of integration, the exercises immediately below, which are constructed for the purpose of illustrating methods of integration, should be done without the aid of a Table.

EXERCISES

1. Integrate the following by parts:

$$(\alpha)\ \int x\cosh x, \qquad (\beta)\ \int \tan^{-1} x, \qquad (\gamma)\ \int x^m \log x,$$

$$(\delta)\ \int \frac{\sin^{-1}x}{x^2}, \qquad (\epsilon)\ \int \frac{xe^x}{(1+x)^2}, \qquad (\zeta)\ \int \frac{1}{x(x^2-a^2)^{\frac{3}{2}}}.$$

2. If $P(x)$ is a polynomial and $P'(x)$, $P''(x)$, $\cdots$ its derivatives, show

$$(\alpha)\ \int P(x)\, e^{ax} = \frac{1}{a} e^{ax}\left[P(x) - \frac{1}{a}P'(x) + \frac{1}{a^2}P''(x) - \cdots\right],$$

$$(\beta)\ \int P(x)\cos ax = \frac{1}{a}\sin ax\left[P(x) - \frac{1}{a^2}P''(x) + \frac{1}{a^4}P^{\text{iv}}(x) - \cdots\right]$$
$$+ \frac{1}{a}\cos ax\left[\frac{1}{a}P'(x) - \frac{1}{a^3}P'''(x) + \frac{1}{a^5}P^{\text{v}}(x) - \cdots\right],$$

and (γ) derive a similar result for the integrand $P(x)\sin ax$.

3. By successive integration by parts and subsequent solution, show

$$(\alpha)\ \int e^{ax}\sin bx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2},$$

$$(\beta)\ \int e^{ax}\cos bx = \frac{e^{ax}(b\sin bx + a\cos bx)}{a^2+b^2},$$

$$(\gamma)\ \int xe^{2x}\cos x = \tfrac{1}{25}e^{2x}[5x(\sin x + 2\cos x) - 4\sin x - 3\cos x].$$

4. Prove by integration by parts the reduction formulas

$$(\alpha)\ \int \sin^m x\cos^n x = \frac{\sin^{m+1}x\cos^{n-1}x}{m+n} + \frac{n-1}{m+n}\int \sin^m x\cos^{n-2}x,$$

$$(\beta)\ \int \tan^m x\sec^n x = \frac{\tan^{m-1}x\sec^n x}{m+n-1} - \frac{m-1}{m+n-1}\int \tan^{n-2}x\sec^n x,$$

$$(\gamma)\ \int \frac{1}{(x^2+a^2)^n} = \frac{1}{2(n-1)a^2}\left[\frac{x}{(x^2+a^2)^{n-1}} + (2n-3)\int\frac{1}{(x^2+a^2)^{n-1}}\right],$$

$$(\delta)\ \int \frac{x^m}{(\log x)^n} = -\frac{x^{m+1}}{(n-1)(\log x)^{n-1}} + \frac{m+1}{n-1}\int\frac{x^m}{(\log x)^{n-1}}.$$

5. Integrate by decomposition into partial fractions:

$$(\alpha)\ \int\frac{x^2-3x+3}{(x-1)(x-2)},\qquad (\beta)\ \int\frac{1}{a^4-x^4},\qquad (\gamma)\ \int\frac{1}{1+x^4},$$

$$(\delta)\ \int\frac{x^2}{(x+2)^2(x+1)},\qquad (\epsilon)\ \int\frac{4x^2-3x+1}{2x^5+x^3},\qquad (\zeta)\ \int\frac{1}{x(1+x^2)^2}.$$

6. Integrate by trigonometric or hyperbolic substitution:

$$(\alpha)\ \int\sqrt{a^2-x^2},\qquad (\beta)\ \int\sqrt{x^2-a^2},\qquad (\gamma)\ \int\sqrt{a^2+x^2},$$

$$(\delta)\ \int\frac{1}{(a-x^2)^{\frac{3}{2}}},\qquad (\epsilon)\ \int\frac{\sqrt{x^2-a^2}}{x},\qquad (\zeta)\ \int\frac{(a^{\frac{2}{3}}-x^{\frac{2}{3}})^{\frac{3}{2}}}{x^{\frac{1}{3}}}.$$

7. Find the areas of these curves and their volumes of revolution:

$$(\alpha)\ x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}},\qquad (\beta)\ a^4y^2=a^2x^4-x^6,\qquad (\gamma)\ \left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1.$$

8. Integrate by converting to a rational algebraic fraction:

$$(\alpha)\ \int\frac{\sin 3x}{a^2\cos^2x+b^2\sin^2x},\qquad (\beta)\ \int\frac{\cos 3x}{a^2\cos^2x+b^2\sin^2x},\qquad (\gamma)\ \int\frac{\sin 2x}{a^2\cos^2x+b^2\sin^2x},$$

$$(\delta)\ \int\frac{1}{a+b\cos x},\qquad (\epsilon)\ \int\frac{1}{a+b\cos x+c\sin x},\qquad (\zeta)\ \int\frac{1-\cos x}{1+\sin x}.$$

9. Show that $\int R(x, \sqrt{a+bx+cx^2})$ may be treated by trigonometric substitution; distinguish between $b^2-4ac \gtreqless 0$.

10. Show that $\int R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right)$ is made rational by $y^n=\frac{ax+b}{cx+d}$. Hence infer that $\int R(x, \sqrt{(x-\alpha)(x-\beta)})$ is rationalized by $y^2=\frac{x-\beta}{x-\alpha}$. This accomplishes the integration of $R(x, \sqrt{a+bx+cx^2})$ when the roots of $a+bx+cx^2=0$ are real, that is, when $b^2-4ac>0$.

11. Show that $\int R\left[x, \left(\frac{ax+b}{cx+d}\right)^m, \left(\frac{ax+b}{cx+d}\right)^n, \cdots\right]$, where the exponents m, n, $\cdots$ are rational, is rationalized by $y^k = \frac{ax+b}{cx+d}$ if k is so chosen that km, kn, $\cdots$ are integers.

12. Show that $\int (a + by)^p y^q$ may be rationalized if p or q or $p + q$ is an integer. By setting $x^n = y$ show that $\int x^m (a + bx^n)^p$ may be reduced to the above type and hence is integrable when $\frac{m+1}{n}$ or p or $\frac{m+1}{n} + p$ is integral.

13. If the roots of $a + bx + cx^2 = 0$ are imaginary, $\int R(x, \sqrt{a + bx + cx^2})$ may be rationalized by $y = \sqrt{a + bx + cx^2} \mp x\sqrt{c}$.

14. Integrate the following.

$(\alpha)\ \int \frac{x^3}{\sqrt{x-1}}$, $\quad (\beta)\ \int \frac{1+\sqrt[3]{x}}{1+\sqrt[4]{x}}$, $\quad (\gamma)\ \int \frac{x}{\sqrt[3]{1+x} - \sqrt{1+x}}$,

$(\delta)\ \int \frac{e^{2x}}{\sqrt[4]{e^x + 1}}$, $\quad (\epsilon)\ \int \frac{x^4}{\sqrt{(1-x^2)^3}}$, $\quad (\zeta)\ \int \frac{1}{(x-d)\sqrt{a + bx + cx^2}}$,

$(\eta)\ \int \frac{1}{x(1+x^2)^{\frac{3}{2}}}$, $\quad (\theta)\ \int \frac{\sqrt{2x^2 + x}}{x^2}$, $\quad (\lambda)\ \int \frac{x^8}{\sqrt{1-x^3}} + \frac{\sqrt{1-x^3}}{x}$.

15. In view of Ex. 12 discuss the integrability of:

$(\alpha)\ \int \sin^m x \cos^n x$, let $\sin x = \sqrt{y}$, $\qquad (\beta)\ \int \frac{x^m}{\sqrt{ax - x^2}}$ $\begin{cases} \text{let } x = ay^2, \\ \text{or } \sqrt{ax - x^2} = xy. \end{cases}$

16. Apply the reduction formulas, Table, p. 66, to show that the final integral for

$$\int \frac{x^m}{\sqrt{1-x^2}} \quad \text{is} \quad \int \frac{1}{\sqrt{1-x^2}} \quad \text{or} \quad \int \frac{x}{\sqrt{1-x^2}} \quad \text{or} \quad \int \frac{1}{x\sqrt{1-x^2}}$$

according as m is even or odd and positive or odd and negative.

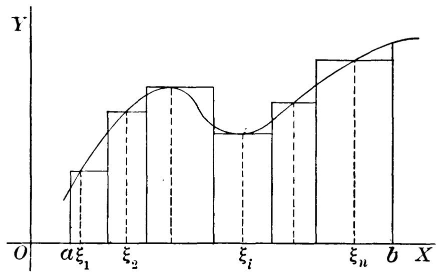
17. Prove sundry of the formulas of Peirce's Table.

18. Show that if $R(x, \sqrt{a^2 - x^2})$ contains x only to odd powers, the substitution $z = \sqrt{a^2 - x^2}$ will rationalize the expression. Use Exs. 1 (ξ) and 6 (ε) to compare the labor of this algebraic substitution with that of the trigonometric or hyperbolic.

16. Definite integrals. If an interval from $x = a$ to $x = b$ be divided into n successive intervals $\Delta x_1, \Delta x_2, \cdots, \Delta x_n$ and the value $f(\xi_i)$ of a function $f(x)$ be computed from some point ξ_i in each interval Δx_i and be multiplied by Δx_i, then *the limit of the sum*

$$\lim_{\substack{\Delta x_i \doteq 0 \\ n = \infty}} [f(\xi_1)\Delta x_1 + f(\xi_2)\Delta x_2 + \cdots + f(\xi_n)\Delta x_n] = \int_a^b f(x)\,dx, \qquad (62)$$

when each interval becomes infinitely short and their number n becomes infinite, is known as *the definite integral* of $f(x)$ from a to b, and is designated as indicated. If $y = f(x)$ be graphed, the sum will be represented by the area under a broken line, and it is clear that the limit of the sum, that is, the integral, will be represented by the *area under the curve* $y = f(x)$ and between the ordinates $x = a$ and $x = b$. Thus the definite integral, defined arithmetically by (62), may be connected with a geometric concept which can serve to suggest properties of the integral much as the interpretation of the derivative as the slope of the tangent served as a useful geometric representation of the arithmetical definition (2).



For instance, if a, b, c are successive values of x, then

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx \tag{63}$$

is the equivalent of the fact that the area from a to c is equal to the sum of the areas from a to b and b to c. Again, if Δx be considered positive when x moves from a to b, it must be considered negative when x moves from b to a and hence from (62)

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx. \tag{64}$$

Finally, if M be the maximum of $f(x)$ in the interval, the area under the curve will be less than that under the line $y = M$ through the highest point of the curve; and if m be the minimum of $f(x)$, the area under the curve is greater than that under $y = m$. Hence

$$m(b-a) < \int_a^b f(x)\,dx < M(b-a). \tag{65}$$

There is, then, some intermediate value $m < \mu < M$ such that the integral is equal to $\mu(b-a)$; and if the line $y = \mu$ cuts the curve in a point whose abscissa is ξ intermediate between a and b, then

$$\int_a^b f(x)\,dx = \mu(b-a) = (b-a)f(\xi). \tag{65'}$$

This is the fundamental *Theorem of the Mean* for definite integrals.

The definition (62) may be applied directly to the evaluation of the definite integrals of the simplest functions. Consider first $1/x$ and let a, b be positive with a less than b. Let the interval from a to b be divided into n intervals Δx_i which are in geometrical progression in the ratio r so that $x_1 = a$, $x_2 = ar$, $\cdots$, $x_{n+1} = ar^n$

and $\Delta x_1 = a(r-1)$, $\Delta x_2 = ar(r-1)$, $\Delta x_3 = ar^2(r-1)$, $\cdots$, $\Delta x_n = ar^{n-1}(r-1)$;

whence $\quad b - a = \Delta x_1 + \Delta x_2 + \cdots + \Delta x_n = a(r^n - 1)$ and $r^n = b/a$.

Choose the points ξ_i in the intervals Δx_i as the initial points of the intervals. Then

$$\frac{\Delta x_1}{\xi_1} + \frac{\Delta x_2}{\xi_2} + \cdots + \frac{\Delta x_n}{\xi_n} = \frac{a(r-1)}{a} + \frac{ar(r-1)}{ar} + \cdots + \frac{ar^{n-1}(r-1)}{ar^{n-1}} = n(r-1).$$

But $$r = \sqrt[n]{b/a} \quad \text{or} \quad n = \log(b/a) \div \log r.$$

Hence $$\frac{\Delta x_1}{\xi_1} + \frac{\Delta x_2}{\xi_2} + \cdots + \frac{\Delta x_n}{\xi_n} = n(r-1) = \log\frac{b}{a}\cdot\frac{r-1}{\log r} = \log\frac{b}{a}\cdot\frac{h}{\log(1+h)}.$$

Now if n becomes infinite, r approaches 1, and h approaches 0. But the limit of $\log(1+h)/h$ as $h \doteq 0$ is by definition the derivative of $\log(1+x)$ when $x = 0$ and is 1. Hence

$$\int_a^b \frac{dx}{x} = \lim_{n=\infty}\left[\frac{\Delta x_1}{\xi_1} + \frac{\Delta x_2}{\xi_2} + \cdots + \frac{\Delta x_n}{\xi_n}\right] = \log\frac{b}{a} = \log b - \log a.$$

As another illustration let it be required to evaluate the integral of $\cos^2 x$ from 0 to $\frac{1}{2}\pi$. Here let the intervals Δx_i be equal and their number odd. Choose the ξ's as the initial points of their intervals. The sum of which the limit is desired is

$$\sigma = \cos^2 0 \cdot \Delta x + \cos^2 \Delta x \cdot \Delta x + \cos^2 2\,\Delta x \cdot \Delta x + \cdots$$
$$+ \cos^2 (n-2)\,\Delta x \cdot \Delta x + \cos^2(n-1)\,\Delta x \cdot \Delta x.$$

But $\quad n\Delta x = \frac{1}{2}\pi$, and $(n-1)\,\Delta x = \frac{1}{2}\pi - \Delta x$, $(n-2)\,\Delta x = \frac{1}{2}\pi - 2\,\Delta x$, $\cdots$,

and $$\cos(\tfrac{1}{2}\pi - y) = \sin y \quad \text{and} \quad \sin^2 y + \cos^2 y = 1.$$

Hence $$\sigma = \Delta x\left[\cos^2 0 + \cos^2 \Delta x + \cos^2 2\,\Delta x + \cdots + \sin^2 2\,\Delta x + \sin^2 \Delta x\right]$$
$$= \Delta x\left[1 + \frac{n-1}{2}\right].$$

Hence $$\int_0^{\frac{\pi}{2}} \cos^2 x\,dx = \lim_{\Delta x \doteq 0}\left[\tfrac{1}{2}n\,\Delta x + \tfrac{1}{2}\Delta x\right] = \lim_{\Delta x \doteq 0}\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\Delta x\right) = \tfrac{1}{4}\pi.$$

Indications for finding the integrals of other functions are given in the exercises.

It should be noticed that the variable x which appears in the expression of the definite integral really has nothing to do with the value of the integral but merely serves as a symbol useful in forming the sum in (62). What is of importance is the function f and the limits a, b of the interval over which the integral is taken.

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(y)\,dy = \int_a^b f(*)\,d*.$$

The variable in the integrand disappears in the integration and leaves the value of the integral as a function of the limits a and b alone.

17. If the lower limit of the integral be fixed, the value

$$\int_a^b f(x)\,dx = \Phi(b)$$

of the integral is a function of the upper limit regarded as variable. To find the derivative $\Phi'(b)$, form the quotient (2),

$$\frac{\Phi(b+\Delta b)-\Phi(b)}{\Delta b}=\frac{\displaystyle\int_a^{b+\Delta b} f(x)\,dx-\int_a^b f(x)\,dx}{\Delta b}.$$

By applying (63) and (65′), this takes the simpler form

$$\frac{\Phi(b+\Delta b)-\Phi(b)}{\Delta b}=\frac{\displaystyle\int_b^{b+\Delta b} f(x)\,dx}{\Delta b}=\frac{1}{\Delta b}\cdot f(\xi)\,\Delta b,$$

where ξ is intermediate between b and $b+\Delta b$. Let $\Delta b \doteq 0$. Then ξ approaches a and $f(\xi)$ approaches $f(a)$. Hence

$$\Phi'(b)=\frac{d}{db}\int_a^b f(x)\,dx=f(b). \tag{66}$$

If preferred, the variable b may be written as x, and

$$\Phi(x)=\int_a^x f(x)\,dx, \qquad \Phi'(x)=\frac{d}{dx}\int_a^x f(x)\,dx=f(x). \tag{66′}$$

This equation will establish the relation between the definite integral and the indefinite integral. For by definition, the indefinite integral $F(x)$ of $f(x)$ is any function such that $F'(x)$ equals $f(x)$. As $\Phi'(x)=f(x)$ it follows that

$$\int_a^x f(x)\,dx=F(x)+C. \tag{67}$$

Hence except for an additive constant, the indefinite integral of f is the definite integral of f from a fixed lower limit to a variable upper limit. As the definite integral vanishes when the upper limit coincides with the lower, the constant C is $-F(a)$ and

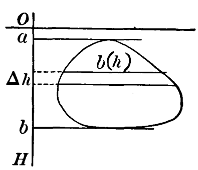
$$\int_a^b f(x)\,dx=F(b)-F(a). \tag{67′}$$

Hence, *the definite integral of $f(x)$ from a to b is the difference between the values of any indefinite integral $F(x)$ taken for the upper and lower limits of the definite integral;* and if the indefinite integral of f is known, the definite integral may be obtained without applying the definition (62) to f.

The great importance of definite integrals to geometry and physics lies in that fact that *many quantities* connected with geometric figures or physical bodies *may be expressed simply for small portions* of the figures or bodies and may then be obtained as the sum of those quantities taken over all the small portions, or rather, as the *limit of that sum when the portions become smaller and smaller.* Thus the area under a curve cannot in the first instance be evaluated; but if only that portion of the curve which lies over a small interval Δx be considered and the rectangle corresponding to the ordinate $f(\xi)$ be drawn, it is clear that the area of the rectangle is $f(\xi)\Delta x$, that the area of all the rectangles is the sum $\Sigma f(\xi)\Delta x$ taken from a to b, that when the intervals Δx approach zero the limit of their sum is the area under the curve; and hence that area may be written as the definite integral of $f(x)$ from a to b.*

In like manner consider *the mass of a rod* of variable density and suppose the rod to lie along the x-axis so that the density may be taken as a function of x. In any small length Δx of the rod the density is nearly constant and the mass of that part is approximately equal to the product $\rho\Delta x$ of the density $\rho(x)$ at the initial point of that part times the length Δx of the part. In fact it is clear that the mass will be intermediate between the products $m\Delta x$ and $M\Delta x$, where m and M are the minimum and maximum densities in the interval Δx. In other words the mass of the section Δx will be exactly equal to $\rho(\xi)\Delta x$ where ξ is some value of x in the interval Δx. The mass of the whole rod is therefore the sum $\Sigma\rho(\xi)\Delta x$ taken from one end of the rod to the other, and if the intervals be allowed to approach zero, the mass may be written as the integral of $\rho(x)$ from one end of the rod to the other.†

Another problem that may be treated by these methods is that of finding the *total pressure* on a vertical area submerged in a liquid, say, in water. Let w be the weight of a column of water of cross section 1 sq. unit and of height 1 unit. (If the unit is a foot, $w = 62.5$ lb.) At a point h units below the surface of the water the pressure is wh and upon a small area near that depth the pressure is approximately whA if A be the area. The pressure on the area A is exactly equal to $w\xi A$ if ξ is some depth intermediate between that of the top and that of the bottom of the area. Now let the finite area be ruled into strips of height Δh. Consider the product $whb(h)\Delta h$ where $b(h) = f(h)$ is the breadth of the area at the depth h. This



* The ξ's may evidently be so chosen that the finite sum $\Sigma f(\xi)\Delta x$ is exactly equal to the area under the curve; but still it is necessary to let the intervals approach zero and thus replace the sum by an integral because the values of ξ which make the sum equal to the area are unknown.

† This and similar problems, here treated by using the Theorem of the Mean for integrals, may be treated from the point of view of differentiation as in § 7 or from that of Duhamel's or Osgood's Theorem as in §§ 34, 35. It should be needless to state that in any particular problem some one of the three methods is likely to be somewhat preferable to either of the others. The reason for laying such emphasis upon the Theorem of the Mean here and in the exercises below is that the theorem is in itself very important and needs to be thoroughly mastered.

is approximately the pressure on the strip as it is the pressure at the top of the strip multiplied by the approximate area of the strip. Then $w\xi b(\xi)\Delta h$, where ξ is some value between h and $h + \Delta h$, is the actual pressure on the strip. (It is sufficient to write the pressure as approximately $whb(h)\Delta h$ and not trouble with the ξ.) The total pressure is then $\Sigma w\xi b(\xi)\Delta h$ or better the limit of that sum. Then

$$P = \lim \sum w\xi b(\xi)\,dh = \int_a^b whb(h)\,dh,$$

where a is the depth of the top of the area and b that of the bottom. To evaluate the pressure it is merely necessary to find the breadth b as a function of h and integrate.

EXERCISES

1. If k is a constant, show $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$.

2. Show that $\int_a^b (u \pm v)\,dx = \int_a^b u dx \pm \int_a^b v dx$.

3. If, from a to b, $\psi(x) < f(x) < \phi(x)$, show $\int_a^b \psi(x)\,dx < \int_a^b f(x)\,dx < \int_a^b \phi(x)\,dx$.

4. Suppose that the minimum and maximum of the quotient $Q(x) = f(x)/\phi(x)$ of two functions in the interval from a to b are m and M, and let $\phi(x)$ be positive so that

$$m < Q(x) = \frac{f(x)}{\phi(x)} < M \quad \text{and} \quad m\phi(x) < f(x) < M\phi(x)$$

are true relations. Show by Exs. 3 and 1 that

$$m < \frac{\int_a^b f(x)\,dx}{\int_a^b \phi(x)\,dx} < M \quad \text{and} \quad \frac{\int_a^b f(x)\,dx}{\int_a^b \phi(x)\,dx} = \mu = Q(\xi) = \frac{f(\xi)}{\phi(\xi)},$$

where ξ is some value of x between a and b.

5. If m and M are the minimum and maximum of $f(x)$ between a and b and if $\phi(x)$ is always positive in the interval, show that

$$m\int_a^b \phi(x)\,dx < \int_a^b f(x)\,\phi(x)\,dx < M\int_a^b \phi(x)\,dx$$

and

$$\int_a^b f(x)\,\phi(x)\,dx = \mu\int_a^b \phi(x)\,dx = f(\xi)\int_a^b \phi(x)\,dx.$$

Note that the integrals of $[M - f(x)]\,\phi(x)$ and $[f(x) - m]\,\phi(x)$ are positive and apply Ex. 2.

6. Evaluate the following by the direct application of (62):

$$(\alpha)\ \int_a^b x dx = \frac{b^2 - a^2}{2}, \qquad (\beta)\ \int_a^b e^x dx = e^b - e^a.$$

Take equal intervals and use the rules for arithmetic and geometric progressions.

7. Evaluate $(\alpha)\ \int_a^b x^m dx = \frac{1}{m+1}(b^{m+1} - a^{m+1})$, $\quad (\beta)\ \int_a^b c^x dx = \frac{1}{\log c}(c^b - c^a)$.
In the first the intervals should be taken in geometric progression with $r^n = b/a$.

8. Show directly that (α) $\int_0^\pi \sin^2 x dx = \frac{1}{2}\pi$, ($\beta$) $\int_0^\pi \cos^n x dx = 0$, if n is odd.

9. With the aid of the trigonometric formulas

$$\cos x + \cos 2x + \cdots + \cos(n-1)x = \tfrac{1}{2}[\sin nx \cot \tfrac{1}{2}x - 1 - \cos nx],$$
$$\sin x + \sin 2x + \cdots + \sin(n-1)x = \tfrac{1}{2}[(1 - \cos nx)\cot \tfrac{1}{2}x - \sin nx],$$

show (α) $\int_a^b \cos x dx = \sin b - \sin a$, ($\beta$) $\int_a^b \sin x dx = \cos a - \cos b$.

10. A function is said to be *even* if $f(-x) = f(x)$ and *odd* if $f(-x) = -f(x)$. Show (α) $\int_{-a}^{+a} f(x)\,dx = 2\int_0^a f(x)\,dx$, f even, (β) $\int_{-a}^{+a} f(x)\,dx = 0$, f odd.

11. Show that if an integral is regarded as a function of the lower limit, the upper limit being fixed, then

$$\Phi'(a) = \frac{d}{da}\int_a^b f(x)\,dx = -f(a), \quad \text{if } \Phi(a) = \int_a^b f(x)\,dx.$$

12. Use the relation between definite and indefinite integrals to compare

$$\int_a^b f(x)\,dx = (b-a)f(\xi) \quad \text{and} \quad F(b) - F(a) = (b-a)F'(\xi),$$

the Theorem of the Mean for derivatives and for definite integrals.

13. From consideration of Exs. 12 and 4 establish *Cauchy's Formula*

$$\frac{\Delta F}{\Delta \Phi} = \frac{F(b) - F(a)}{\Phi(b) - \Phi(a)} = \frac{F'(\xi)}{\Phi'(\xi)}, \qquad a < \xi < b,$$

which states that the quotient of the increments ΔF and $\Delta \Phi$ of two functions, in any interval in which the derivative $\Phi'(x)$ does not vanish, is equal to the quotient of the derivatives of the functions for some interior point of the interval. What would the application of the Theorem of the Mean for derivatives to numerator and denominator of the left-hand fraction give, and wherein does it differ from Cauchy's Formula?

14. Discuss the volume of revolution of $y = f(x)$ as the limit of the sum of thin cylinders and compare the results with those found in Ex. 9, p. 10.

15. Show that the mass of a rod running from a to b along the x-axis is $\frac{1}{2}k(b^2 - a^2)$ if the density varies as the distance from the origin (k is a factor of proportionality).

16. Show (α) that the mass in a rod running from a to b is the same as the area under the curve $y = \rho(x)$ between the ordinates $x = a$ and $x = b$, and explain why this should be seen intuitively to be so. Show (β) that if the density in a plane slab bounded by the x-axis, the curve $y = f(x)$, and the ordinates $x = a$ and $x = b$ is a function $\rho(x)$ of x alone, the mass of the slab is $\int_a^b y\rho(x)\,dx$; also (γ) that the mass of the corresponding volume of revolution is $\int_a^b \pi y^2 \rho(x)\,dx$.

17. An isosceles triangle has the altitude a and the base $2b$. Find (α) the mass on the assumption that the density varies as the distance from the vertex (measured along the altitude). Find (β) the mass of the cone of revolution formed by revolving the triangle about its altitude if the law of density is the same.

18. In a plane, the *moment of inertia I of a particle of mass m with respect to a point* is defined as the product mr^2 of the mass by the square of its distance from the point. Extend this definition from particles to bodies.

(α) Show that the moments of inertia of a rod running from a to b and of a circular slab of radius a are respectively

$$I = \int_a^b x^2 \rho(x)\, dx \quad \text{and} \quad I = \int_0^a 2\pi r^3 \rho(r)\, dr, \qquad \rho \text{ the density,}$$

if the point of reference for the rod is the origin and for the slab is the center.

(β) Show that for a rod of length $2l$ and of uniform density, $I = \frac{1}{3}Ml^2$ with respect to the center and $I = \frac{4}{3}Ml^2$ with respect to the end, M being the total mass of the rod.

(γ) For a uniform circular slab with respect to the center $I = \frac{1}{4}Ma^2$.

(δ) For a uniform rod of length $2l$ with respect to a point at a distance d from its center is $I = M(\frac{1}{3}l^2 + d^2)$. Take the rod along the axis and let the point be (α, β) with $d^2 = \alpha^2 + \beta^2$.

19. A rectangular gate holds in check the water in a reservoir. If the gate is submerged over a vertical distance H and has a breadth B and the top of the gate is a units below the surface of the water, find the pressure on the gate. At what depth in the water is the point where the pressure is the mean pressure over the gate?

20. A dam is in the form of an isosceles trapezoid 100 ft. along the top (which is at the water level) and 60 ft. along the bottom and 30 ft. high. Find the pressure in tons.

21. Find the pressure on a circular gate in a water main if the radius of the circle is r and the depth of the center of the circle below the water level is $d \geqq r$.

22. In space, *moments of inertia* are defined *relative to an axis* and in the formula $I = mr^2$, for a single particle, r is the perpendicular distance from the particle to the axis.

(α) Show that if the density in a solid of revolution generated by $y = f(x)$ varies only with the distance along the axis, the moment of inertia about the axis of revolution is $I = \int_a^b \frac{1}{2}\pi y^4 \rho(x)\, dx$. Apply Ex. 18 after dividing the solid into disks.

(β) Find the moment of inertia of a sphere about a diameter in case the density is constant; $I = \frac{2}{5}Ma^2 = \frac{8}{15}\pi\rho a^5$.

(γ) Apply the result to find the moment of inertia of a spherical shell with external and internal radii a and b; $I = \frac{2}{5}M(a^5 - b^5)/(a^3 - b^3)$. Let $b \doteq a$ and thus find $I = \frac{2}{3}Ma^2$ as the moment of inertia of a spherical surface (shell of negligible thickness).

(δ) For a cone of revolution $I = \frac{3}{10}Ma^2$ where a is the radius of the base.

23. If the force of attraction exerted by a mass m upon a point is $kmf(r)$ where r is the distance from the mass to the point, show that the attraction exerted at the origin by a rod of density $\rho(x)$ running from a to b along the x-axis is

$$A = \int_a^b kf(x)\rho(x)\, dx, \quad \text{and that} \quad A = kM/ab, \qquad M = \rho(b - a),$$

is the attraction of a uniform rod if the law is the Law of Nature, that is, $f(r) = 1/r^2$.

24. Suppose that the density ρ in the slab of Ex. 16 were a function $\rho(x, y)$ of both x and y. Show that the mass of a small slice over the interval $\Delta \bar{x}_i$ would be of the form

$$\Delta x \int_0^{y=f(\xi)} \rho(x, y)\, dy = \Phi(\xi)\, \Delta x \quad \text{and that} \quad \int_a^b \Phi(x)\, \Delta x = \int_a^b \left[\int_0^{y=f(x)} \rho(x, y)\, dy \right] dx$$

would be the expression for the total mass and would require an integration with respect to y in which x was held constant, a substitution of the limits $f(x)$ and 0 for y, and then an integration with respect to x from a to b.

25. Apply the considerations of Ex. 24 to finding moments of inertia of
(α) a uniform triangle $y = mx$, $y = 0$, $x = a$ with respect to the origin,
(β) a uniform rectangle with respect to the center,
(γ) a uniform ellipse with respect to the center.

26. Compare Exs. 24 and 16 to treat the volume under the surface $z = \rho(x, y)$ and over the area bounded by $y = f(x)$, $y = 0$, $x = a$, $x = b$. Find the volume
(α) under $z = xy$ and over $y^2 = 4px$, $y = 0$, $x = 0$, $x = b$,
(β) under $z = x^2 + y^2$ and over $x^2 + y^2 = a^2$, $y = 0$. $x = 0$, $x = Q$,
(γ) under $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ and over $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $y = 0$, $x = 0$, $x = a$.

27. Discuss sectorial area $\frac{1}{2}\int r^2 d\phi$ in polar coördinates as the limit of the sum of small sectors running out from the pole.

28. Show that the moment of inertia of a uniform circular sector of angle α and radius a is $\frac{1}{4}\rho\alpha a^4$. Hence infer $I = \frac{1}{4}\rho \int_{\alpha_0}^{\alpha_1} r^4 d\phi$ in polar coördinates.

29. Find the moment of inertia of a uniform (α) lemniscate $r^2 = a^2 \cos^2 2\phi$ and (β) cardioid $r = a(1 - \cos\phi)$ with respect to the pole. Also of (γ) the circle $r = 2a\cos\phi$ and (δ) the rose $r = a \sin 2\phi$ and (ϵ) the rose $r = a \sin 3\phi$.