## II. THE RESOLVENT

**13.** We shall follow, with some material deviations, König's exposition of Kronecker's method of solving equations by means of the resolvent. The equations are in general supposed to be non-homogeneous; and homogeneous equations are regarded as a particular case. Thus a homogeneous equation in $n$ variables represents a cone of $n-1$ dimensions with its vertex at the origin. *Homogeneous co-ordinates are excluded.*

The problem is to find all the solutions of any given system of equations $F_1 = F_2 = \dots = F_k = 0$ in $n$ unknowns $x_1, x_2, \dots, x_n$. The unknowns are supposed if necessary to have been subjected to a homogeneous linear substitution beforehand, the object being to make the equations and their solutions of a general character, and to prevent any inconvenient result happening (such as an equation or polynomial being irregular* in any of the variables) which could have been avoided by a linear substitution at the beginning. In theoretical reasoning *this preliminary homogeneous substitution is always to be understood*; but is seldom necessary in dealing with a particular example.

The solutions we shall seek are (i) those, if any, which exist for $x_1$ when $x_2, x_3, \dots, x_n$ have arbitrary values; (ii) those which exist for $x_1, x_2$, not included in (i), when $x_3, \dots, x_n$ have arbitrary values; (iii) those which exist for $x_1, x_2, x_3$, not included in (i) or (ii), when $x_4, \dots, x_n$ have arbitrary values; and so on. A set of solutions for $x_1, x_2, \dots, x_r$ when $x_{r+1}, \dots, x_n$ have arbitrary values is said to be of *rank* $r$, and the *spread* of the points whose coordinates are the solutions is of *rank* $r$ and *dimensions* $n-r$. If there are solutions of rank $r$ and no solutions of rank $< r$ the system of equations $F_1 = F_2 = \dots = F_k = 0$ and the module $(F_1, F_2, \dots, F_k)$ are both said to be of rank $r$.

**14.** The polynomials $F_1, F_2, \dots, F_k$, and also all their factors are regular in $x_1$. Hence their common factor $D$ can be found by the ordinary process of finding the H.C.F. of $F_1, F_2, \dots, F_k$ treated as polynomials in a single variable $x_1$. If $D$ does not involve the variables we take it to be 1. If it does involve the variables the solutions of $D = 0$ treated as an equation for $x_1$ give the first set of solutions of the equations $F_1 = F_2 = \dots = F_k = 0$ mentioned above.

* A polynomial of degree $l$ is said to be regular or irregular in $x_1$ according as the term $x_1{}^l$ is present in it or not.

In the algebraic theory of modules we regard any algebraic equation in one unknown, whether the coefficients involve parameters or not, as completely soluble, i.e. we regard any given non-linear polynomial in one variable as *reducible*. A polynomial in two or more variables is called reducible if it is the product of two polynomials both of which involve the variables. A polynomial which is not reducible is called (absolutely) *irreducible*. Any given polynomial is either irreducible or uniquely expressible as a product of irreducible factors, leaving factors of degree zero out of account. It is assumed that the irreducible factors of any given polynomial are known. Thus the polynomial $D$ above may be supposed to be expressed in its irreducible factors in $x_1, x_2, \ldots, x_n$, and to each irreducible factor corresponds an irreducible or non-degenerate spread.

Put $F_i = D\phi_i$ $(i = 1, 2, \ldots, k)$. Then $\phi_1, \phi_2, \ldots, \phi_k$ have no common factor involving the variables, and the same is true of the two polynomials

$$\lambda_1\phi_1 + \lambda_2\phi_2 + \ldots + \lambda_k\phi_k \text{ and } \mu_1\phi_1 + \mu_2\phi_2 + \ldots + \mu_k\phi_k,$$

where the $\lambda$'s and $\mu$'s are arbitrary quantities. Regarding them as two polynomials in a single variable $x_1$ we calculate their resultant, and arrange it in the form

$$\rho_1 F_1^{(1)} + \rho_2 F_2^{(1)} + \ldots + \rho_{k_1} F_{k_1}^{(1)},$$

where $\rho_1, \rho_2, \ldots, \rho_{k_1}$ are different power products of the $\lambda$'s and $\mu$'s, and $F_1^{(1)}, F_2^{(1)}, \ldots, F_{k_1}^{(1)}$ are polynomials in $x_2, x_3, \ldots, x_n$ not involving the $\lambda$'s and $\mu$'s. Each $F_i^{(1)}$ is regular in $x_2$; for any homogeneous linear substitution beforehand of $x_2, x_3, \ldots, x_n$ among themselves only would be carried through to the $F_i^{(1)}$.

Find the H.C.F. $D^{(1)}$ of $F_1^{(1)}, F_2^{(1)}, \ldots, F_{k_1}^{(1)}$ treated as polynomials in a single variable $x_2$, and put $F_i^{(1)} = D^{(1)}\phi_i^{(1)}$ $(i = 1, 2, \ldots, k_1)$. Then find the resultant of

$$\lambda_1\phi_1^{(1)} + \lambda_2\phi_2^{(1)} + \ldots + \lambda_{k_1}\phi_{k_1}^{(1)} \text{ and } \mu_1\phi_1^{(1)} + \mu_2\phi_2^{(1)} + \ldots + \mu_{k_1}\phi_{k_1}^{(1)}$$

and arrange it in the form

$$\rho_1 F_1^{(2)} + \rho_2 F_2^{(2)} + \ldots + \rho_{k_2} F_{k_2}^{(2)}$$

as before, where $F_1^{(2)}, F_2^{(2)}, \ldots, F_{k_2}^{(2)}$ are polynomials in $x_3, x_4, \ldots, x_n$, which may be assumed regular in $x_3$, and whose H.C.F. $D^{(2)}$ can be found. We thus get the following series in succession:

$$\begin{array}{ll}
F_1, \ F_2, \ \ldots, \ F_k, & \text{with H.C.F. } D, \\
\phi_1, \ \phi_2, \ \ldots, \ \phi_k, & \\
F_1^{(1)}, F_2^{(1)}, \ldots, F_{k_1}^{(1)}, & \text{with H.C.F. } D^{(1)}, \\
\phi_1^{(1)}, \phi_2^{(1)}, \ldots, \phi_{k_1}^{(1)}, & \\
F_1^{(2)}, F_2^{(2)}, \ldots, F_{k_2}^{(2)}, & \text{with H.C.F. } D^{(2)}, \\
\phi_1^{(2)}, \phi_2^{(2)}, \ldots, \phi_{k_2}^{(2)}, & \text{and so on.}
\end{array}$$

Now any solution of $F_1 = F_2 = \ldots = F_k = 0$ is a solution of $D = 0$ or of $\phi_1 = \phi_2 = \ldots = \phi_k = 0$. And any solution of $\phi_1 = \phi_2 = \ldots = \phi_k = 0$ is a solution of $F_1^{(1)} = F_2^{(1)} = \ldots = F_{k_1}^{(1)} = 0$, since $\Sigma \rho_i F_i^{(1)} \equiv 0 \bmod (\Sigma\lambda_i\phi_i, \Sigma\mu_i\phi_i)$, and therefore a solution of $D^{(1)} = 0$ or of $\phi_1^{(1)} = \phi_2^{(1)} = \ldots = \phi_{k_1}^{(1)} = 0$. Hence any solution of $F_1 = F_2 = \ldots = F_k = 0$ is a solution of $D = 0$ or of $D^{(1)} = 0$ or of $\phi_1^{(1)} = \phi_2^{(1)} = \ldots = \phi_{k_1}^{(1)} = 0$. Proceeding in a similar way we find that any solution of $F_1 = \ldots = F_k = 0$ is a solution of $DD^{(1)} \ldots D^{(n-1)} = 0$, since $\phi_1^{(n-1)}, \phi_2^{(n-1)}, \ldots, \phi_{k_{n-1}}^{(n-1)}$ are polynomials in a single variable $x_n$ at most and have no common factor.

*Conversely* if $\xi_3, x_4, \ldots, x_n$ is any solution of $D^{(2)} = 0$ the resultant of $\Sigma\lambda_i\phi_i^{(1)}$ and $\Sigma\mu_i\phi_i^{(1)}$ with respect to $x_2$ vanishes when $x_3 = \xi_3$, and $\Sigma\lambda_i\phi_i^{(1)} = \Sigma\mu_i\phi_i^{(1)} = 0$ have a solution $x_2 = \xi_2$ when $x_3 = \xi_3$; i.e. the equations $\phi_1^{(1)} = \ldots = \phi_{k_1}^{(1)} = 0$, and therefore also the equations $F_1^{(1)} = \ldots = F_{k_1}^{(1)} = 0$, have a solution $\xi_2, \xi_3, x_4, \ldots, x_n$; and, by the same reasoning, the equations $F_1 = F_2 = \ldots = F_k = 0$ have a solution $\xi_1, \xi_2, \xi_3, x_4, \ldots, x_n$. Similarly to any solution of $DD^{(1)} \ldots D^{(n-1)} = 0$, say a solution $\xi_i, x_{i+1}, \ldots, x_n$ of $D^{(i-1)} = 0$, there corresponds a solution $\xi_1, \xi_2, \ldots, \xi_i, x_{i+1}, \ldots, x_n$ of the equations $F_1 = F_2 = \ldots = F_k = 0$. Hence from the solutions of the single equation $DD^{(1)} \ldots D^{(n-1)} = 0$ we can get all the solutions of the system $F_1 = F_2 = \ldots = F_k = 0$, since all the solutions of the latter satisfy the former.

*Definitions.* $DD^{(1)} \ldots D^{(n-1)}$ is called the *complete* (*total*) *resolvent* of the equations $F_1 = F_2 = \ldots = F_k = 0$ and of the module $(F_1, F_2, \ldots, F_k)$. $D^{(i-1)}$ is called the *complete partial resolvent of rank i*, and any whole factor of $D^{(i-1)}$ is called a partial resolvent of rank $i$.

**15.** *The complete resolvent is a member of the module* $(F_1, F_2, \ldots, F_k)$.

For $\quad \Sigma\rho_i F_i^{(1)} \equiv 0 \bmod (\Sigma\lambda_i\phi_i, \Sigma\mu_i\phi_i) = A\,\Sigma\lambda_i\phi_i + B\,\Sigma\mu_i\phi_i,$

where $A$, $B$ are whole functions of $x_1, x_2, \ldots, x_n, \lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_k$. Hence by equating coefficients of the power products $\rho_i$ on both sides, we have

$$F_i^{(1)} \equiv 0 \bmod (\phi_1, \phi_2, \ldots, \phi_k),$$

and $$DF_i^{(1)} = 0 \bmod (F_1, F_2, \ldots, F_k),^*$$

or $$DD^{(1)}\phi_i^{(1)} = 0 \bmod (F_1, F_2, \ldots, F_k).$$

Similarly $\quad DD^{(1)} \ldots D^{(n-1)}\phi_i^{(n-1)} = 0 \bmod (F_1, F_2, \ldots, F_k)$;

and since the $\phi_i^{(n-1)}$ include one variable only (or none at all) and have

* Not $DF_i^{(1)} \equiv 0 \bmod (F_1, F_2, \ldots, F_k)$ because any common factor of $F_1, F_2, \ldots, F_k$ not involving the variables is not included in $D$ and is left out of account.

no common factor, we can choose polynomials $a_i$ in the single variable so that $\Sigma a_i \phi_i^{(n-1)} = 1$. Hence

$$DD^{(1)} \dots D^{(n-1)} = 0 \bmod (F_1, F_2, \dots, F_k).$$

*If the equations* $F_1 = F_2 = \dots = F_k = 0$ *have no finite solution the complete resolvent is equal to* 1 ; *consequently* 1 *is a member of* $(F_1, F_2, \dots, F_k)$, *and every polynomial is a member.*

**16.** We have seen that to every solution $x_i = \xi_i$ of $D^{(i-1)} = 0$ there corresponds a solution $\xi_1, \xi_2, \dots, \xi_i, x_{i+1}, \dots, x_n$ of the equations $F_1 = F_2 = \dots = F_k = 0$. It may happen that there is an earlier complete partial resolvent $D^{(j-1)}$ which vanishes when $x_j = \xi_j, \dots, x_i = \xi_i$. In such a case the solution $\xi_1, \dots, \xi_i, x_{i+1}, \dots, x_n$ of $F_1 = \dots = F_k = 0$ corresponding to a solution of $D^{(i-1)} = 0$ is included in the solutions corresponding to $D^{(j-1)} = 0$, and may be neglected if we are seeking merely the complete solution of $F_1 = F_2 = \dots = F_k = 0$. Such a solution is called an *imbedded* solution. All solutions corresponding to an irreducible factor of $D^{(i-1)}$ will be imbedded if one of them is imbedded.

**17. Examples on the Resolvent.** Geometrically the resolvent enables us to resolve the whole spread represented by any given set of algebraic equations into definite irreducible spreads (§ 21). It has been supposed that the complete resolvent also supplies a definite answer to certain other questions. The following examples disprove this to some extent.

*Example* i. Find the resolvent of $n$ homogeneous equations $F_1 = F_2 = \dots = F_n = 0$ of the same degree $l$ and having no proper solution.

Since there are no solutions of rank $< n$ the complete resolvent is $D^{(n-1)}$. The first derived set of polynomials $F_1^{(1)}, F_2^{(1)}, \dots, F_{k_1}^{(1)}$ are homogeneous and of degree $l^2$, the 2nd set $F_1^{(2)}, F_2^{(2)}, \dots$ are homogeneous and of degree $l^4$, and the $(n-1)$th set $F_1^{(n-1)}, F_2^{(n-1)}, \dots$ are homogeneous and of degree $l^{2^{n-1}}$. This last set involve only one variable $x_n$, and therefore have the common factor $x_n^{l^{2^{n-1}}}$, which is therefore the required complete resolvent.

We should arrive at a similar result if we changed $x_i$ to $x_i + a_i$ $(i = 1, 2, \dots, n)$ beforehand, thus making the polynomials non-homogeneous. The complete resolvent would then be $(x_n + a_n)^{l^{2^{n-1}}}$. The resultant would be $(x_n + a_n)^{l^n}$. The difference in the two results is explained by the fact that the resultant is obtained by a process applying uniformly to all the variables, and the resolvent by a process applied to the variables in succession.

*Example* ii. König (K, p. 219) defines a module or system of equations as being *simple* or *mixed* according as only one or more than one of the complete partial resolvents $D, D^{(1)}, \ldots, D^{(n-1)}$ differs from unity. Kronecker (Kr, p. 31) says that the system of equations $F_1 = F_2 = \ldots = F_k = 0$ is irreducible in this case; and the *Ency. des Sc. Math.* ($W_2$, p. 352) repeats König's definition. We give two examples to show that this definition is a valueless one.

If $u, v, w$ are three linear functions of three or more variables, any polynomial which contains the spread of $u = v = 0$ is of the form $Au + Bv$; if it also contains the spread of $u = w = 0$, $B$ must vanish when $u = w = 0$, hence $B$ must be of the form $Cu + Dw$, and $Au + Bv$ of the form $A'u + B'vw$; if it also contains the spread of $v = w = 0$, $A'$ must be of the form $C'v + D'w$, and $A'u + B'vw$ of the form $C'uv + D'uw + B'vw$. Hence a polynomial which contains all three spreads is a member of the module $(vw, wu, uv)$, and also any member of the module contains the three spreads. This module, although composite, is not mixed in any proper sense of the word.

Besides having partial resolvents of rank 2 corresponding to the three spreads the module has a partial resolvent of rank 3 corresponding to its singular spread $u = v = w = 0$. This last partial resolvent does not correspond to any property of the module which is not included in the properties corresponding to its partial resolvents of rank 2; in other words the partial resolvent of rank 3 is purely redundant.

The resolvent $D^{(1)} D^{(2)}$ can be found as follows: Suppose

$$u = a_0 + a_1 x_1 + a_2 x_2 + \ldots, \quad v = b_0 + b_1 x_1 + b_2 x_2 + \ldots, \quad w = c_0 + c_1 x_1 + c_2 x_2 + \ldots.$$

Then the resultant of $\lambda_1 vw + \lambda_2 wu + \lambda_3 uv$ and $\mu_1 vw + \mu_2 wu + \mu_3 uv$ with respect to $x_1$, apart from a constant factor, is

$$(c_1 v - b_1 w)(a_1 w - c_1 u)(b_1 u - a_1 v) \times \left\{ \frac{c_1 v - b_1 w}{\lambda_2 \mu_3 - \lambda_3 \mu_2} + \frac{a_1 w - c_1 u}{\lambda_3 \mu_1 - \lambda_1 \mu_3} + \frac{b_1 u - a_1 v}{\lambda_1 \mu_2 - \lambda_2 \mu_1} \right\},$$

its four irreducible factors corresponding to the spreads

$$v = w = 0, \quad w = u = 0, \quad u = v = 0,$$

$$(\lambda_2 \mu_3 - \lambda_3 \mu_2) u = (\lambda_3 \mu_1 - \lambda_1 \mu_3) v = (\lambda_1 \mu_2 - \lambda_2 \mu_1) w.$$

Hence $$D^{(1)} = (c_1 v - b_1 w)(a_1 w - c_1 u)(b_1 u - a_1 v);$$

and $$\phi_1^{(1)} = (c_1 v - b_1 w), \quad \phi_2^{(1)} = (a_1 w - c_1 u), \quad \phi_3^{(1)} = (b_1 u - a_1 v),$$

from which we obtain

$$D^{(2)} = (b_1c_2 - b_2c_1)\,u + (c_1a_2 - c_2a_1)\,v + (a_1b_2 - a_2b_1)\,w.$$

*Example* iii. Compare and find the resolvents of the two modules

$$M = (x_1^3,\ x_2^3,\ x_1^2 + x_2^2 + x_1x_2x_3),$$

$$M' = (x_1^3,\ x_1^2x_2,\ x_1x_2^2,\ x_2^3,\ x_1^2 + x_2^2 + x_1x_2x_3).$$

The resolvent of $M'$ will be found by obtaining the resultant with respect to $x_1$ of the two equations

$$\lambda_1x_1^3 + \lambda_2x_1^2x_2 + \lambda_3x_1x_2^2 + \lambda_4x_2^3 + \lambda_5(x_1^2 + x_2^2 + x_1x_2x_3) = 0,$$

and $$\mu_1x_1^3 + \mu_2x_1^2x_2 + \mu_3x_1x_2^2 + \mu_4x_2^3 + \mu_5(x_1^2 + x_2^2 + x_1x_2x_3) = 0.$$

This resultant is the same as that of the first equation and

$$(\lambda_1\mu_5)\,x_1^3 + (\lambda_2\mu_5)\,x_1^2x_2 + (\lambda_3\mu_5)\,x_1x_2^2 + (\lambda_4\mu_5)\,x_2^3 = 0$$

except for a factor $\lambda_5^3$. The roots of the last equation are $\alpha_1x_2,\ \alpha_2x_2,\ \alpha_3x_2$. Hence the resultant, apart from a constant factor, is

$$\Pi\,\{(\lambda_1\alpha^3 + \lambda_2\alpha^2 + \lambda_3\alpha + \lambda_4)\,x_2^3 + \lambda_5(\alpha^2 + 1 + \alpha x_3)\,x_2^2\},\quad (\alpha = \alpha_1,\ \alpha_2,\ \alpha_3)$$

or $$x_2^6\,\Pi\,\{(\lambda_1\alpha^3 + \lambda_2\alpha^2 + \lambda_3\alpha + \lambda_4)\,x_2 + \lambda_5(\alpha^2 + 1 + \alpha x_3)\}.$$

Hence the complete resolvent is $x_2^6$, since no values of $x_2$, $x_3$ independent of the $\lambda$'s and $\mu$'s will make the remaining product of factors of the above resultant vanish.

The complete resolvent of $M$, worked in the same way, is also $x_2^6$; i.e. $M$ and $M'$ have the same complete resolvent, although they are not the same module. $M$, but not $M'$, contains the two modules

$$M'' = (x_3 - 1,\ x_1^2 + x_1x_2 + x_2^2,\ x_1^2x_2 + x_1x_2^2),$$

$$M''' = (x_3 + 1,\ x_1^2 - x_1x_2 + x_2^2,\ x_1^2x_2 - x_1x_2^2),$$

i.e. every member of $M$ is a member of $M''$ and of $M'''$. Thus

$$x_1^3 = x_1(x_1^2 + x_1x_2 + x_2^2) - (x_1^2x_2 + x_1x_2^2),$$

$$x_2^3 = x_2(x_1^2 + x_1x_2 + x_2^2) - (x_1^2x_2 + x_1x_2^2),$$

$$x_1^2 + x_2^2 + x_1x_2x_3 = (x_1^2 + x_1x_2 + x_2^2) + x_1x_2(x_3 - 1).$$

The module $M$ is what is called the L.C.M. of $M'$, $M''$, $M'''$. The two modules $M''$, $M'''$ have $x_1 = x_2 = x_3 - 1 = 0$ and $x_1 = x_2 = x_3 + 1 = 0$ for their spreads, which are imbedded in the spread $x_1 = x_2 = 0$ of the first component of $M$, viz. $M'$.

$M$ is then properly speaking a mixed module although this is not indicated by its complete resolvent $x_2^6$. It has two imbedded spreads, the points $(0,\ 0,\ \pm 1)$. The complete resolvent should have the factors

$x_3 \pm 1$ to indicate these, but it has no such factors. The complete resolvent may indicate imbedded modules which do not exist as in Ex. ii, or it may give no indication of them when they do exist as in Ex. iii.

*Example* iv. It is stated in the *Encyk. der Math. Wiss.* ($W_1$, p. 305) and repeated in ($W_2$, p. 354) that if only one complete partial resolvent $D^{(r)}$ differs from 1, and $D^{(r)}$ has no repeated factor, the module is the product of the prime modules corresponding to the irreducible factors of $D^{(r)}$. The absurdity of this statement is shown by applying it to the module $(u, vw)$, where $u, v, w$ are the same as in Ex. ii. The complete resolvent is $D^{(1)} = (b_1u - a_1v)(c_1u - a_1w)$, and the product of the prime modules $(u, v)$, $(u, w)$ corresponding to its two factors is $(u^2, uv, uw, vw) \neq (u, vw)$.

**18. The *u*-resolvent.** The solutions of $F_1 = F_2 = \dots = F_k = 0$ are obtained in the most useful way by introducing a general unknown $x$ standing for $u_1x_1 + u_2x_2 + \dots + u_nx_n$, where $u_1, u_2, \dots, u_n$ are undetermined coefficients. This is done by putting

$$x_1 = \frac{x - u_2x_2 - \dots - u_nx_n}{u_1}$$

in the system of equations $F_1 = F_2 = \dots = F_k = 0$. We thus get a new system $f_1 = f_2 = \dots = f_k = 0$ in $x, x_2, x_3, \dots, x_n$, where

$$f_i = u_1^{l_i} F_i \left( \frac{x - u_2x_2 - \dots - u_nx_n}{u_1}, x_2, \dots, x_n \right) \quad (i = 1, 2, \dots, k),$$

the multiplier $u_1^{l_i}$ being introduced to make $f_i$ integral in $u_1$. There is evidently a one-one correspondence between the solutions of the two systems, viz. to the solution $\xi_1, \xi_2, \dots, \xi_n$ of $F_1 = F_2 = \dots = F_k = 0$ there corresponds the solution $\xi, \xi_2, \dots, \xi_n$ of $f_1 = f_2 = \dots = f_k = 0$, and *vice versa*, where $\xi = u_1\xi_1 + u_2\xi_2 + \dots + u_n\xi_n$.

*Definition.* The complete resolvent $D_uD_u^{(1)} \dots D_u^{(n-1)}$ $(= F_u)$ of $(f_1, f_2, \dots, f_k)$ obtained by eliminating $x_2, x_3, \dots, x_n$ in succession is called the *complete u-resolvent* of $(F_1, F_2, \dots, F_k)$.

Since $F_u = 0 \bmod (f_1, f_2, \dots, f_k)$, by § 15, we have

$$(F_u)_{x=u_1x_1+\dots+u_nx_n} = 0 \bmod (F_1, F_2, \dots, F_k).$$

$F_u$ is a whole function of $x, x_2, \dots, x_n, u_1, u_2, \dots, u_n$ which resolves into linear factors when regarded as a function of $x$ only. The linear factors of rank $r$, that is, the linear factors of $D_u^{(r-1)}$, are of the type

$$x - u_1\xi_1 - \dots - u_r\xi_r - u_{r+1}x_{r+1} - \dots - u_nx_n$$

where $\xi_1, \ldots, \xi_r, x_{r+1}, \ldots, x_n$ is a solution of $F_1 = F_2 = \ldots = F_k = 0$. For if $x - \xi$ is any linear factor of $D_u^{(r-1)}$ then $\xi$ is a root of $D_u^{(r-1)} = 0$ to which corresponds a solution $\xi, \xi_2, \ldots, \xi_r, x_{r+1}, \ldots, x_n$ of $f_1 = f_2 = \ldots = f_k = 0$ (§ 14) and a solution $\xi_1, \xi_2, \ldots, \xi_r, x_{r+1}, \ldots, x_n$ of $F_1 = F_2 = \ldots = F_k = 0$, where $\xi = u_1\xi_1 + \ldots + u_r\xi_r + u_{r+1}x_{r+1} + \ldots + u_n x_n$.

The linear factors of $F_u$ expressed in the above form supply all the solutions of $f_1 = f_2 = \ldots = f_k = 0$, viz. $\xi, \xi_2, \ldots, \xi_r, x_{r+1}, \ldots, x_n$, and all the solutions of $F_1 = F_2 = \ldots = F_k = 0$, viz. $\xi_1, \xi_2, \ldots, \xi_r, x_{r+1}, \ldots, x_n$, of the several ranks $r = 1, 2, \ldots, n$; but it is only when $\xi_1, \xi_2, \ldots, \xi_r$ are independent of $u_1, u_2, \ldots, u_n$ that we know the solution from merely knowing the factor.

**19.** A linear factor of $F_u$ of rank $r$ such as the above will be called a *true* linear factor if $\xi_1, \xi_2, \ldots, \xi_r$ are independent of $u_1, u_2, \ldots, u_n$, that is, if it is linear in $x, u_1, u_2, \ldots, u_n$.

*If a linear factor of $F_u$ is not a true linear factor the solution supplied by it is an imbedded one.*

Let $x - \xi$ or $x - u_1\xi_1 - \ldots - u_s\xi_s - u_{s+1}x_{s+1} - \ldots - u_n x_n$ be a non-true linear factor of $F_u$, so that $\xi_1, \xi_2, \ldots, \xi_s$ depend on $u_1, u_2, \ldots, u_n$. Then $\xi_1, \xi_2, \ldots, \xi_s, x_{s+1}, \ldots, x_n$ is a solution of $F_1 = F_2 = \ldots = F_k = 0$, and so also is $\eta_1, \eta_2, \ldots, \eta_s, x_{s+1}, \ldots, x_n$ where $\eta_1, \eta_2, \ldots, \eta_s$ are obtained from $\xi_1, \xi_2, \ldots, \xi_s$ by changing $u_1, u_2, \ldots, u_n$ to $v_1, v_2, \ldots, v_n$. Hence $\eta, \eta_2, \ldots, \eta_s, x_{s+1}, \ldots, x_n$ (where $\eta = u_1\eta_1 + \ldots + u_s\eta_s + u_{s+1}x_{s+1} + \ldots + u_n x_n$) is a solution of $f_1 = f_2 = \ldots = f_k = 0$, and therefore makes $F_u$ vanish. But it does not make $D_u^{(s-1)} \ldots D_u^{(n-1)}$ vanish since this does not involve $x_2, \ldots, x_s$, and cannot have a factor $x - \eta$, where $\eta$ involves $v_1, v_2, \ldots, v_n$. Hence it makes some factor $D_u^{(r-1)}$ of $F_u$ of rank $r < s$ vanish. Then $D_u^{(r-1)}$ vanishes when $x, x_{r+1}, \ldots, x_s$ are put equal to $\eta, \eta_{r+1}, \ldots, \eta_s$; and by putting $v_1, v_2, \ldots, v_n$ (of which $D_u^{(r-1)}$ is independent) equal to $u_1, u_2, \ldots, u_n$ it follows that $D_u^{(r-1)}$ vanishes when $x, x_{r+1}, \ldots, x_s$ are put equal to $\xi, \xi_{r+1}, \ldots, \xi_s$. Hence the solution $\xi, \xi_2, \ldots, \xi_s, x_{s+1}, \ldots, x_n$ is an imbedded one (§ 16).

*It follows that all the solutions of $F_1 = F_2 = \ldots = F_k = 0$ are obtainable from true linear factors of $F_u$; and that all the linear factors of the first complete partial u-resolvent (different from* 1*) are true linear factors.*

*It also follows that if there is a spread of rank s which is not imbedded there must be true linear factors of $F_u$ of rank s corresponding to the spread.*

We have not proved that all linear factors of $F_u$ are true linear factors*, and whether this is so or not must be considered doubtful.

**20.** *If an irreducible factor $R_u$ of $F_u$ considered as a whole function of all the quantities $x, x_2, \ldots, x_n, u_1, u_2, \ldots, u_n$ has a true linear factor all its linear factors are true linear factors.*

Let $R_u$ be of rank $r$. Then $R_u$ is independent of $x_1, x_2, \ldots, x_r$ and there is a one-one correspondence between its true linear factors and the sets of values $\xi_1, \xi_2, \ldots, \xi_r$ of $x_1, x_2, \ldots, x_r$ (not involving $u_1, u_2, \ldots, u_n$) for which $(R_u)_{x=u_1x_1+\ldots+u_nx_n}$ vanishes. Let

$$(R_u)_{x=u_1x_1+\ldots+u_nx_n} = \rho_1 R_1 + \rho_2 R_2 + \ldots + \rho_\mu R_\mu,$$

where $\rho_1, \rho_2, \ldots, \rho_\mu$ are different power products of $u_1, u_2, \ldots, u_n$ and $R_1, R_2, \ldots, R_\mu$ are whole functions of $x_1, x_2, \ldots, x_n$ independent of $u_1, u_2, \ldots, u_n$. Then the sets of values $\xi_1, \xi_2, \ldots, \xi_r$ required are the solutions of $R_1 = R_2 = \ldots = R_\mu = 0$ regarded as equations for $x_1, x_2, \ldots, x_r$. These come from the solutions $\xi_1, \xi_2, \ldots, \xi_r, x_{r+1}, \ldots, x_n$ of rank $r$ of the same equations in $x_1, x_2, \ldots, x_n$. Now there is at least one solution of rank $r$, since $R_u$ has a true linear factor; and only a finite number of such solutions altogether, since $R_u$ has only a finite number of such factors. Hence the first complete partial $u$-resolvent (different from 1) of the equations $R_1 = R_2 = \ldots = R_\mu = 0$ is of rank $r$, and resolves completely into true linear factors (§ 19)

$$x - u_1\xi_1 - \ldots - u_r\xi_r - u_{r+1}x_{r+1} - \ldots - u_nx_n.$$

This complete partial $u$-resolvent of rank $r$ is therefore $R_u$ itself (or else a power of $R_u$), which proves the theorem.

If $F_u$ is resolved into factors of the $R_u$ type (irreducible with respect to $x, x_2, \ldots, x_n, u_1, u_2, \ldots, u_n$), and these into irreducible factors as regards $x, x_2, \ldots, x_n$ only, $F_u$ will be resolved into all its irreducible factors. Hence *every irreducible factor of* $F_u$ is a factor of a factor of the $R_u$ type, and *has all or none of its linear factors true linear factors.*

*It follows that any factor of $F_u$ irreducible with respect to $x, x_2, \ldots, x_n$, and having a true linear factor, has all its linear factors true linear factors, and is a whole function of $u_1, u_2, \ldots, u_n$.*

* Kronecker states this as a fact without proving it. König's proof contains an error (K, p. 210). It is not correct to say as he does that $\overline{E}_t^{(h)}\overline{X}_t^{(h)}$ vanishes when $x=\xi_t$, but only when $x, \xi_1, \xi_2, \ldots, \xi_h$ are put equal to $\xi_t, \xi_1', \xi_2', \ldots, \xi_h'$.

**21. The irreducible spreads of a module.** Let $R_u$ be any irreducible factor of $F_u$ of rank $r$ having a true linear factor. We know that

$$R_u = A \prod_{i=1}^{i=d} (x - u_1 x_{1i} - \dots - u_r x_{ri} - u_{r+1}\, x_{r+1} - \dots - u_n x_n).$$

Hence $$(R_u)_{x=u_1x_1+\dots+u_nx_n} = A \prod_{i=1}^{i=d} \{u_1(x_1 - x_{1i}) + \dots + u_r(x_r - x_{ri})\}.$$

To $R_u$ corresponds what is called an *irreducible spread*, viz. the spread of all points $x_{1i}, \dots, x_{ri}, x_{r+1}, \dots, x_n$ in which $x_{r+1}, \dots, x_n$ take all finite values, and $x_{1i}, \dots, x_{ri}$ the $d$ sets of values supplied by the linear factors of $R_u$, which vary as $x_{r+1}, \dots, x_n$ vary.

The degree $d$ of $R_u$ is called the *order* of the irreducible spread.

From the two identities above several useful results can be deduced. It must be remembered that $R_u$ is a known polynomial in $x, x_{r+1}, \dots, x_n, u_1, u_2, \dots, u_n$. No linear factor of $R_u$ can be repeated, unless $x_{r+1}, \dots, x_n$ are given special values; for otherwise $R_u$ and $\frac{\partial R_u}{\partial x}$ would have an H.C.F. involving $x$, and $R_u$ would be the product of two factors. Whatever set of values $x_{r+1}, \dots, x_n$ have, whether general or special, the $d$ sets of corresponding values of $x_1, x_2, \dots, x_r$, viz. $x_{1i}, x_{2i}, \dots, x_{ri}$ are definite and finite, because $R_u$ is regular in $x$.

From the second identity it is seen that $(R_u)_{x=u_1x_1+\dots+u_nx_n}$ is independent of $u_{r+1}, \dots, u_n$, and vanishes identically (i.e. irrespective of $u_1, u_2, \dots, u_n$) at every point of the spread and no other point. *Hence the whole coefficients* of the power products of $u_1, u_2, \dots, u_r$ in $(R_u)_{x=u_1x_1+\dots+u_nx_n}$ all vanish at every point of the spread and do not all vanish at any other point.* These coefficients equated to zero give a system of equations for the spread; but it is not necessary to take them all, and some are simpler than others. The coefficient of $u_r^d$ gives an equation $\phi(x_r, x_{r+1}, \dots, x_n) = A \prod (x_r - x_{ri}) = 0$ for $x_r$, whose roots are the $d$ values of $x_r$ corresponding to given arbitrary values of $x_{r+1}, \dots, x_n$. The coefficient of $u_1 u_r^{d-1}$ gives an equation

$$x_1\phi' - \phi_1 = \phi \Sigma \frac{x_1 - x_{1i}}{x_r - x_{ri}} = 0,$$

where $\phi'$ is $\frac{\partial \phi}{\partial x_r}$ and $\phi_1$, or $\phi \Sigma \frac{x_{1i}}{x_r - x_{ri}}$, is a polynomial in $x_r, x_{r+1}, \dots, x_n$.

* Also these coefficients are members of $(F_1, F_2, \dots, F_k)$ if $(R_u)_{x=u_1x_1+\dots+u_nx_n}$ is a member of $(F_1, F_2, \dots, F_k)$, as it will be proved to be when $(F_1, F_2, \dots, F_k)$ is a prime module (§ 31).

Similarly we have $x_2\phi' - \phi_2 = 0, \ldots, x_{r-1}\phi' - \phi_{r-1} = 0$. The equations

$$\phi = 0, \quad x_1 = \frac{\phi_1}{\phi'}, \quad x_2 = \frac{\phi_2}{\phi'}, \quad \ldots, \quad x_{r-1} = \frac{\phi_{r-1}}{\phi'}$$

are called more particularly the equations of the spread, the first giving the different values of $x_r$ as functions of $x_{r+1}, \ldots, x_n$, and the others giving $x_1, x_2, \ldots, x_{r-1}$ as rational functions of $x_r, x_{r+1}, \ldots, x_n$. If $x_r, x_{r+1}, \ldots, x_n$ have such values that $\phi = \phi' = 0$ then $\phi_1, \phi_2, \ldots, \phi_{r-1}$ all vanish and the expressions above for $x_1, x_2, \ldots, x_{r-1}$ become indeterminate. In such a case the values of $x_1, x_2, \ldots, x_{r-1}$ may be found by taking other equations from $(R_u)_{x=u_1x_1+\ldots+u_nx_n}$ for them.

**22. Geometrical property of an irreducible spread.** An algebraic spread in general is one which is determined by any finite system of algebraic equations, and consists of all points whose coordinates satisfy the equations and no other points. Such a spread has already been shown to consist of a finite number of irreducible spreads each of which is determined by a finite system of equations. The characteristic property of an irreducible spread is that any algebraic spread which contains a part of it, of the same dimensions as the irreducible spread, contains the whole of it.

Let $F_1 = F_2 = \ldots = F_k = 0$ be the equations determining any algebraic spread, and $F_1' = F_2' = \ldots = F'_{k'} = 0$ the equations determining an irreducible spread. The spread they have in common is determined by the combined system of equations $F_1 = F_2 = \ldots = F_k = F_1' = \ldots = F'_{k'} = 0$, and is contained in the irreducible spread and has the same or less dimensions. If it is of the same dimensions as the irreducible spread the complete $u$-resolvent of $F_1 = \ldots = F_k = F_1' = \ldots = F'_{k'} = 0$ will have an irreducible factor $R_u''$ of the same rank as the irreducible factor $R_u'$ of the complete $u$-resolvent of $F_1' = F_2' = \ldots = F'_{k'} = 0$ corresponding to the spread of the same. Also all the roots of $R_u'' = 0$ regarded as an equation for $x$ are roots of $R_u' = 0$. Hence $R_u'$ is divisible by $R_u''$, and since they are both irreducible they must be identical. Hence the spread of $F_1 = \ldots = F_k = F_1' = \ldots = F'_{k'} = 0$ contains the whole of the spread of $F_1' = F_2' = \ldots = F'_{k'} = 0$, and the spread of $F_1 = F_2 = \ldots = F_k = 0$ contains the same. This proves the property stated above.