classes is the part that is common to each (the class of their common elements) and the sum of two classes is the class of all the elements which belong to at least one of them.

P. I.: 1. The product of two propositions is a proposition which implies each of them and which is implied by every proposition which implies both:

2. The sum of two propositions is the proposition which is implied by each of them and which implies every proposition implied by both.

Therefore we can say that the product of two propositions is their weakest common cause, and that their sum is their strongest common consequence, strong and weak being used in a sense that every proposition which implies another is stronger than the latter and the latter is weaker than the one which implies it. Thus it is easily seen that the product of two propositions consists in their *simultaneous affirmation:* "a and b are true", or simply "a and b"; and that their sum consists in their *alternative affirmation*, "either a or b is true", or simply "a or b".

Remark.—Logical addition thus defined is not disjunctive;[1] that is to say, it does not presuppose that the two summands have no element in common.

8. Principles of Simplification and Composition.—The two preceding definitions, or rather the postulates which precede and justify them, yield directly the following formulas:

(1) $$ab < a, \qquad ab < b,$$

(2) $$(x < a)\ (x < b) < (x < ab),$$

(3) $$a < a + b, \qquad b < a + \mathrm{b},$$

(4) $$(a < x)\ (b < x) < (a + b < x).$$

Formulas (1) and (3) bear the name of the *principle of simplification* because by means of them the premises of an

1 [BOOLE, closely following analogy with ordinary mathematics, premised, as a necessary condition to the definition of "$x + y$", that x and y were mutually exclusive. JEVONS, and practically all mathematical logicians after him, advocated, on various grounds, the definition of "logical addition" in a form which does not necessitate mutual exclusiveness.]

argument may be simplified by deducing therefrom weaker propositions, either by deducing one of the factors from a product, or by deducing from a proposition a sum (alternative) of which it is a summand.

Formulas (2) and (4) are called the *principle of composition*, because by means of them two inclusions of the same antecedent or the same consequent may be combined (*composed*). In the first case we have the product of the consequents, in the second, the sum of the antecedents.

The formulas of the principle of composition can be transformed into equalities by means of the principles of the syllogism and of simplification. Thus we have

1 (Syll.) $(x < ab)\ (ab < a) < (x < a)$,

(Syll.) $(x < ab)\ (ab < b) < (x < b)$.

Therefore

(Comp.) $(x < ab) < (x < a) \cdot (x < b)$.

2 (Syll.) $(a < a + b)\ (a + b < x) < (a < x)$,

(Syll.) $(b < a + b)\ (a + b < x) < (b < x)$.

Therefore

(Comp.) $(a + b < x) < (a < x)\ (b < x)$.

If we compare the new formulas with those preceding, which are their converse propositions, we may write

$$(x < ab) = (x < a)\ (x < b),$$

$$(a + b < x) = (a < x)\ (b < x).$$

Thus, to say that x is contained in ab is equivalent to saying that it is contained at the same time in both a and b; and to say that x contains $a + b$ is equivalent to saying that it contains at the same time both a and b.

9. The Laws of Tautology and of Absorption.—Since the definitions of the logical sum and product do not imply any order among the terms added or multiplied, logical addition and multiplication evidently possess commutative and associative properties which may be expressed in the formulas

$ab = ba,$	$a + b = b + a,$
$(ab)\ c = a\ (bc),$	$(a + b) + c = a + (b + c).$