CHAPTER 5

Introduction to inverse scattering

Suppose we are given two asymptotically hyperbolic metrics which differ only on a compact set. If the associated scattering operators coincide, one can show that these two metrics coincide up to a diffeomorphism. This result can be extended to manifolds with asymptotically hyperbolic ends when two metrics coincide on one end having a regular infinity. The aim of this chapter is to explain the idea of the proof of these theorems.

1. Local problem on $\mathbf{H}^n$

Recall that in the geodesic polar coordinates centered at $(0,1)$, the metric on $\mathbf{H}^n$ takes the form

$$ds^2 = (dr)^2 + \sinh^2 r\,(d\theta)^2,$$

where $(d\theta)^2$ is the standard metric on S^{n-1} (see formula (1.4) in Chap. 1). Letting $y = 2e^{-r}$ and $x = \theta$, one can rewrite the above metric as

$$ds^2 = \left(\frac{dy}{y}\right)^2 + \left(\frac{1}{y} - \frac{y}{4}\right)^2 (dx)^2, \quad y \in (0,2].$$

Suppose this metric is perturbed so that

$$ds^2 = \frac{(dy)^2 + (dx)^2 + A(x,y,dx,dy)}{y^2},$$

with $A(x,y,dx,dy)$ satisfying the assumption (A-4) of Chap. 3, §3. The theorem we are going to prove is as follows.

Theorem 1.1. *Suppose we are given two Riemannian metrics $G^{(p)}$, $p = 1,2$, on $\mathbf{H}^n$ satisfying the above assumption. Suppose their scattering operators coincide. Suppose furthermore $G^{(1)}$ and $G^{(2)}$ coincide except for a compact set. Then $G^{(1)}$ and $G^{(2)}$ are isometric.*

The proof is done by the following steps. Let $B_a \subset \mathbf{H}^n$ be a ball of radius a with respect to the unperturbed metric centered at $(0,1)$ such that $G^{(1)} = G^{(2)}$ outside B_a. We first take a geodesic sphere $S_a = \partial B_a$, and consider the boundary value problem for the Laplace-Beltrami operators in the interior domain B_a. Then the associated Dirichlet-to-Neumann map (or Neumann-to-Dirichlet map) coincide. We use the boundary control method of Belishev-Kurylev to show that $G^{(1)}$ and $G^{(2)}$ are isometric in B_a (see [**10**] and [**77**]).

2. Scattering operator and N-D map

2.1. Restriction of the generalized eigenfunctions to a surface. Let us start with preparing local regularity estimates for the resolvent $R(k^2 \pm i0)$ constructed in Chap. 2. We first introduce some notation in $\mathbf{R}^n$. Letting $\widehat{f}(\xi)$ be the

Fourier transform of a tempered distribution $f \in \mathcal{S}'(\mathbf{R}^n)$, we introduce the Besov norm $\|f\|_{\widehat{\mathcal{B}}_s}$ by

$$\|f\|_{\widehat{\mathcal{B}}_s} = \sum_{j=0}^{\infty} 2^{j/2} \|\langle\xi\rangle^s \widehat{f}(\xi)\|_{L^2(\Omega_j)}, \tag{2.1}$$

where $s \in \mathbf{R}$ is a real parameter, $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$, $\Omega_0 = \{\xi \in \mathbf{R}^n\,;\,|\xi| < 1\}$, $\Omega_j = \{\xi \in \mathbf{R}^n\,;\,2^{j-1} < |\xi| < 2^j\}$, $(j \geq 1)$. Another Besov norm $\|u\|_{\widehat{\mathcal{B}}^*_{-s}}$ is defined by

$$\|u\|_{\widehat{\mathcal{B}}^*_{-s}} = \left(\sup_{R>1} \frac{1}{R} \int_{|\xi|<R} \langle\xi\rangle^{-2s} |\widehat{u}(\xi)|^2 d\xi\right)^{1/2}, \tag{2.2}$$

which is the norm of the dual space of $\widehat{\mathcal{B}}_s$. The following lemma is proven in Theorems 14.1.1, 14.1.4 of [**55**], Vol 2.

Lemma 2.1. *(1) Let S be an $n-1$-dimensional compact surface in $\mathbf{R}^n$, and τ_S the restriction map $\tau_S : u \to u\big|_S$ on S. Then $\tau_S \in \mathbf{B}(\widehat{\mathcal{B}}_0; L^2(S))$.*
(2) Let H^s be the Sobolev space on $\mathbf{R}^n$, and suppose that a linear operator T satisfies $T \in \mathbf{B}(H^s; H^s)$ for all $s \in \mathbf{R}$. Then $T \in \mathbf{B}(\widehat{\mathcal{B}}_s; \widehat{\mathcal{B}}_s)$ for all $s \in \mathbf{R}$.

By using the partition of unity, one can extend the definition of spaces $\widehat{\mathcal{B}}_s$, $\widehat{\mathcal{B}}^*_{-s}$ on non-compact manifolds, which we shall denote by $\widehat{\mathcal{B}}_{s,loc}$ and $\widehat{\mathcal{B}}^*_{-s,loc}$. We also denote by $\widehat{\mathcal{B}}_{s,comp}$ and $\widehat{\mathcal{B}}^*_{-s,comp}$ the set of the compactly supported distributions in $\widehat{\mathcal{B}}_{s,loc}$ and $\widehat{\mathcal{B}}^*_{-s,loc}$, respectively.

Now for $k > 0$, let $\mathcal{F}^{(+)}(k)$ be the generalized Fourier transformation defined by Chap. 2 (7.1). For a compact hypersurface S in $\mathbf{H}^n$, we define

$$\langle f, g\rangle_S = \int_S f(x,y)\overline{g(x,y)}dS_{x,y},$$

where $dS_{x,y}$ is the measure induced on S.

Lemma 2.2. *Let Ω be a bounded domain in $\mathbf{H}^n$ with smooth boundary $S = \partial\Omega$. Suppose $k^2 \neq 0$ is not a Neumann eigenvalue for H in Ω. If $f \in L^2(S)$ satisfies*

$$\langle f, \partial_\nu \mathcal{F}^{(+)}(k)^*\phi\rangle_S = 0, \quad \forall \phi \in L^2(\mathbf{R}^{n-1}),$$

then $f = 0$, where $\partial_\nu = \dfrac{\partial}{\partial\nu}$ is the normal derivative on S.

Proof. We first study the local regularity of the resolvent. Take $\chi \in C_0^\infty(\mathbf{H}^n)$. Then by the well-known elliptic regularity theorem, $\chi R(k^2 \pm i0)\chi \in \mathbf{B}(H^s; H^{s+2})$, $\forall s \geq 0$. By taking the adjoint, we have $\chi R(k^2 \pm i0)\chi \in \mathbf{B}(H^{-t-2}; H^{-t})$, $\forall t \geq 0$. By interpolation, we then have

$$\chi R(k^2 \pm i0)\chi \in \mathbf{B}(H^m; H^{m+2}), \quad \forall m \in \mathbf{R}.$$

This implies, by Lemma 2.1,

$$\chi R(k^2 \pm i0)\chi \in \mathbf{B}(\widehat{\mathcal{B}}_m, \widehat{\mathcal{B}}_{m+2}), \quad \forall m \in \mathbf{R}. \tag{2.3}$$

For $f \in L^2(S)$, we define

$$(\delta'_S f, g) = \langle f, \partial_\nu g\rangle_S, \quad \forall g \in C_0^\infty(\mathbf{H}^n).$$

Then $\operatorname{supp}\delta'_S f \subset S$ and, by Lemma 2.1 (1), we have $\delta'_S f \in \widehat{\mathcal{B}}^*_{-1,comp}$. Note that $\widehat{\mathcal{B}}^*_{-1,comp} \subset H^{-3/2-\epsilon}$ for any $\epsilon > 0$. For $g \in \mathcal{B}$, due to Theorem 2.1.3 and Lemma 2.1 (2), $\partial_\nu R(k^2 + i0)g$ restricted on S is in $H^{1/2}(S)$. Then, for $f \in L^2(S)$, the mapping

$$\mathcal{B} \ni g \longrightarrow \langle \partial_\nu R(k^2 - i0)g, f\rangle_S$$

is a bounded linear functional. Using the definition of $\delta'_S f$, we have

$$\langle f, \partial_\nu R(k^2 - i0)g\rangle_S = (u, g), \quad \forall g \in \mathcal{B},$$

where $u = R(k^2 + i0)\delta'_S f \in \widehat{\mathcal{B}}^*_{1,loc} \cap \mathcal{B}^* \subset H^{1/2-\epsilon}_{loc} \cap \mathcal{B}$, $\forall \epsilon > 0$. Using the resolvent equation, we see that

$$u = R_0(k^2 + i0)\delta'_S f - R(k^2 + i0)VR_0(k^2 + i0)\delta'_S f, \tag{2.4}$$

where $V = H - H_0$. Note that $R_0(k^2 + i0)\delta'_S f$ can be written as an integral over S

$$R_0(k^2 + i0)\delta'_S f = \int_S \left(\partial_{\nu'} R_0(k^2 + i0)(x, y, x', y')\right) f(x, y') dS_{x',y'}.$$

This is an analogue of the classical double layer potential (see e.g. [**27**]).

To understand the properties of this potential, let S_δ, where $|\delta|$ is sufficiently small, be an equi-distant surface which lies inside Ω for positive δ and inside Ω^c for negative δ. This defines two types of operators K_δ and T_δ, where

$$K_\delta f = R_0(k^2 + i0)\delta'_S f\big|_{S_\delta},$$

$$T_\delta f = \partial_\nu R_0(k^2 + i0)\delta'_S f\big|_{S_\delta}.$$

For $\delta \neq 0$, they are bounded operators on $L^2(S)$, where we use the fact that S_δ is diffeomorphic to S. Moreover, K_δ tends to $K_\pm$ in the strong operator topology on $L^2(S)$, when $\delta \to \pm 0$, and $K_+ - K_- = Id$. This is proven in $\mathbf{R}^3$ for the classes of Hölder continuous functions in Theorem 2.15 and Corollary 2.14 of [**27**]. However, if we take into account that in the Riemannian normal coordinates, $x = (x_1, \cdots, x_n)$, $d^2(x, 0) = |x|^2 + O(|x|^4)$, the method of [**27**] can be extended to the space $L^2(S)$ and general Riemannian manifold $\mathcal{M}$.

Regarding T_δ, it is proven in Theorem 2.23, [**27**], that T_δ tends to $T_\pm$ in the strong operator topology of bounded operators from $C^{1,\alpha}(S)$ to $C^\alpha(S)$, and $T_+ - T_- = 0$. Using duality arguments and the fact that $(T_\delta)^*$ has the same structure as T_δ, we see that T_δ tends to $T_\pm$ in the weak operator topology of boundend operators from $L^2(S)$ to $H^{-s}(S)$, where $s > (n+1)/2$, and $T_+ - T_- = 0$.

Extending formula (7.1) in Chap. 2, we define $\mathcal{F}^{(+)}(k)$ onto $\widehat{\mathcal{B}}^*_{-1,comp}$. Then by Lemma 2.7.3, since $\mathcal{G}^{(+)}(k) = \mathcal{F}^{(+)}(k)$, the behavior of u at infinity is given by

$$R(k^2 + i0)\delta'_S f \simeq C(k)\chi(y)y^{\frac{n-1}{2} - ik}\mathcal{F}^{(+)}(k)\delta'_S f. \tag{2.5}$$

However, by the assumption of the lemma

$$(\delta'_S f, \mathcal{F}^{(+)}(k)^*\phi) = (\mathcal{F}^{(+)}(k)\delta'_S f, \phi)_{L^2(\mathbf{R}^{n-1})} = 0, \quad \forall \phi \in L^2(S^{n-1}).$$

This, together with (2.5), implies

$$\lim_{R\to\infty} \frac{1}{\log R}\int_{1/R}^1 \|u(y)\|^2_{L^2(\mathbf{R}^{n-1})}\frac{dy}{y^n} = 0.$$

Let us note that for any $\varphi \in C_0^\infty(\mathbf{H}^n)$

$$\begin{aligned}((H-k^2)u,\varphi) &= (u,(H-k^2)\varphi) \\ &= \langle f, R(k^2-i0)(H-k^2)\varphi\rangle_S \\ &= \langle f,\varphi\rangle_S,\end{aligned}$$

where we have used the fact that $\varphi = R(k^2-i0)(H-k^2)\varphi$, since φ is compactly supported, hence satisfies the radiation condition. We then have $(H-k^2)u=0$ outside and inside S. Arguing in the same way as in the proof of Theorem 2.2.10 given in Subsection 2.3.2, we have $u=0$ in $\Omega^c := \mathbf{H}^n \setminus \overline{\Omega}$. Thus $T_- f = 0$.

Consider $u_\Omega = u\big|_\Omega$. Then $(H-k^2)u_\Omega = 0$ and $\partial_\nu u_\Omega\big|_\Gamma = T_+ f = 0$. Since k^2 is not a Neumann eigenvalue, $u_\Omega = 0$ in Ω. Therefore $u=0$ globally in $\mathbf{H}^n$, which implies $f=0$. □

By the same arguments, one can prove the following lemma.

Lemma 2.3. *Let Ω be a bounded domain. Suppose $k^2 \neq 0$ is not a Dirichlet eigenvalue for H in Ω. If $f \in L^2(\partial\Omega)$ satisfies*

$$\langle f, \mathcal{F}^{(+)}(k)^*\phi\rangle_{\partial\Omega} = 0, \quad \forall \phi \in L^2(\mathbf{R}^{n-1}),$$

then $f=0$.

2.2. Neumann-to-Dirichlet map. Let Ω be a bounded domain in $\mathbf{H}^n$ with smooth boundary $S=\partial\Omega$, and consider the boundary value problem

$$\begin{cases} (H-k^2)u = 0 & \text{in} \quad \Omega, \\ \partial_\nu u = f \in H^{1/2}(S) & \text{on} \quad S. \end{cases}$$

We denote the corresponding operator in $L^2(\Omega)$ with Neumann boundary condition by H^N, keeping the notation H for the operator in $\mathbf{H}^n$. If k^2 is not an eigenvalue of H^N, this problem has a unique solution u. The operator

$$\Lambda(k) : f \to u\big|_S$$

is called the Neumann-to-Dirichlet map, or simply, N-D map. We consider two operators H_1^N and H_2^N associated with two metrics $G^{(1)}$ and $G^{(2)}$. Let $\widehat{S}_j(k)$ be the S-matrix for H_j.

Theorem 2.4. *Suppose $k^2 \neq 0$ is not an eigenvalue for both of H_1^N and H_2^N. Let $\Lambda_j(k)$ be the N-D map for H_j^N, $j=1,2$. Suppose $G^{(1)} = G^{(2)}$ outside Ω. Then $\widehat{S}_1(k) = \widehat{S}_2(k)$ if and only if $\Lambda_1(k) = \Lambda_2(k)$.*

Proof. Suppose $\Lambda_1(k) = \Lambda_2(k)$. Let $u_j = \mathcal{F}_j^{(+)}(k)^*\phi$ for $\phi \in L^2(\mathbf{R}^{n-1})$. Let u_{in} be the solution to the Neumann problem

$$\begin{cases} (H_2-k^2)u_{in} = 0 & \text{in} \quad \Omega, \\ \partial_\nu u_{in} = \partial_\nu u_1 & \text{on} \quad S. \end{cases}$$

We define a functon u_3 on $\mathbf{H}^n$ by $u_3 = u_{in}$ on Ω and $u_3 = u_1$ on $\Omega^c = \mathbf{H}^n \setminus \overline{\Omega}$. The trace of u_3 computed from outside of S is $u_3\big|_S = u_1\big|_S = \Lambda_1(k)\partial_\nu u_1$, since u_1 satisfies $(H_1-k^2)u_1 = 0$ in $\mathbf{H}^n$, hence in Ω.

On the other hand, the trace computed from inside of S is

$$u_{in}\big|_S = \Lambda_2(k)\partial_\nu u_{in} = \Lambda_2(k)\partial_\nu u_1 = \Lambda_1(k)\partial_\nu u_1.$$

Therefore by our assumption, u_3 and $\partial_\nu u_3$ are continuous across S. Hence $u_3 \in H^2_{loc}$ and satisfies $(H_2 - k^2)u_3 = 0$ on $\mathbf{H}^n$.

Let $u_0 = \mathcal{F}^0(k)^*\phi$. Then $u_3 - u_0$ satisifies the incoming radiation condition, since so does $u_1 - u_0$. Therefore $v = u_3 - u_2 = (u_3 - u_0) - (u_2 - u_0)$ is the solution to the equation $(H_2 - k^2)v = 0$ satisfying the radiation condition. By Lemma 2.2.12, $v = 0$. Observing the behavior of $u_1 = u_2$ near infinity and using Theorem 2.7.9, we have $\widehat{S}_1(k) = \widehat{S}_2(k)$.

Suppose $\widehat{S}_1(k) = \widehat{S}_2(k)$. Let u_j be as above, and put $w = u_1 - u_2$. Then $(H_1 - k^2)w = 0$ in Ω^c. Since $\widehat{S}_1(k) = \widehat{S}_2(k)$, $w \simeq 0$ by virtue of Lemma 2.7.2. Consequently, $w = 0$ by Theorem 2.2.10. Then $u_1 = u_2$ and $\partial_\nu u_1 = \partial_\nu u_2$ on S, i.e.

$$\Lambda_1(k)\partial_\nu \mathcal{F}_1^{(+)}(k)^*\phi = \Lambda_2(k)\partial_\nu \mathcal{F}_2^{(+)}(k)^*\phi = \Lambda_2(k)\partial_\nu \mathcal{F}_1^{(+)}(k)^*\phi.$$

By Lemma 2.1, $\{\partial_\nu \mathcal{F}_1^{(+)}(k)^*\phi\,;\phi \in L^2(\mathbf{R}^{n-1})\}$ is dense in $L^2(S)$, which proves the theorem. □

3. Boundary spectral projection

Our inverse problem is now reduced to determining the metric from the N-D map for a bounded domain. Since the following arguments do not rely on individual nature of the metric, we consider in a general situation. Let Ω be a compact Riemannian manifold with boundary equipped with the metric $ds^2 = g_{ij}(x)dx^i dx^j$. Let Δ_g be the associated Laplace-Beltrami operator, and $\lambda_1 < \lambda_2 < \cdots$ be the Neumann eigenvalues of $-\Delta_g$. We emphasize that we do not count the multiplicities of eigenvalues here. The N-D map is defined as $\Lambda(\lambda) : f \to u\big|_{\partial\Omega}$, where

$$\begin{cases} (-\Delta_g - \lambda)u = 0 & \text{in} \quad \Omega, \\ \partial_\nu u = f \in H^{1/2}(\partial\Omega) & \text{on} \quad \partial\Omega. \end{cases} \tag{3.1}$$

Here we are writing $\Lambda(\lambda)$ instead of $\Lambda(\sqrt{\lambda})$. Note that $\Lambda(\lambda)$ is analytic with respect to $\lambda \in \mathbf{C} \setminus \sigma(-\Delta_g)$. Let $\varphi_{i,1}(x), \cdots, \varphi_{i,m(i)}(x)$ be a complete orthonormal system of eigenvectors associated with λ_i. We first note that the N-D map $\Lambda(\lambda)$ has the following formal integral kernel

$$\Lambda(\lambda; x, y) = \sum_{i=1}^{\infty}\sum_{j=1}^{m(i)} \frac{\varphi_{i,j}(x)\overline{\varphi_{i,j}(y)}}{\lambda_i - \lambda}, \quad x, y \in \partial\Omega. \tag{3.2}$$

In fact, let $\widetilde{f} \in H^2(\Omega)$ be such that $\partial_\nu \widetilde{f} = f$ on $\partial\Omega$. Then $v = u - \widetilde{f}$ solves

$$\begin{cases} (-\Delta_g - \lambda)v = (\Delta_g + \lambda)\widetilde{f} =: F & \text{in} \quad \Omega, \\ \partial_\nu v = 0 \in H^{1/2}(\partial\Omega) & \text{on} \quad \partial\Omega. \end{cases}$$

Therefore, letting (,) be the inner product of $L^2(\Omega)$

$$v = \sum_{i=1}^{\infty} \frac{1}{\lambda_i - \lambda} \sum_{j=1}^{m(i)} (F, \varphi_{i,j})\varphi_{i,j}(x). \tag{3.3}$$

Letting $\langle\ ,\ \rangle$ be the inner product on $L^2(\partial\Omega)$, we have by integration by parts

$$(F, \varphi_{i,j}) = \langle f, \varphi_{i,j}\rangle + (\lambda - \lambda_i)(\widetilde{f}, \varphi_{i,j}),$$

which proves (3.2).

Definition 3.1. The set $\{\lambda_i, \varphi_{i,j}(x)\big|_{\partial\Omega}\,;\, j = 1, \cdots, m(i),\, i = 1, 2, \cdots\}$ is called the boundary spectral data (**BSD**) of the Neumann problem.

Lemma 3.2. *Let $\varphi_{i,1}(x), \cdots, \varphi_{i,m(i)}(x)$ be a complete orthnormal system of eigenvectors associated with λ_i for the Neumann problem. Then $\varphi_{i,j}(x), 1 \leq j \leq m(i)$, are linearly independent in $L^2(\partial\Omega)$. For another complete orthnormal system $\psi_{i,1}(x), \cdots, \psi_{i,m(i)}(x)$, there is a unitary matrix U such that*

$$\Big(\varphi_{i,1}(x), \cdots, \varphi_{i,m(i)}(x)\Big) = \Big(\psi_{i,1}(x), \cdots, \psi_{i,m(i)}(x)\Big)U.$$

Proof. Suppose $\sum_{j=1}^{m(i)} c_j\varphi_{i,j}(x) = 0$ on $\partial\Omega$. Then $u = \sum_{j=1}^{m(i)} c_j\varphi_{i,j}(x)$ satisfies $(-\Delta_g - \lambda_i)u = 0$ in Ω, and $u = \partial_\nu u = 0$ on $\partial\Omega$. By the uniqueness theorem for the Cauchy problem (see e.g. [**101**], p. 373), $u = 0$ in Ω, which implies $c_1 = \cdots = c_{m(i)} = 0$. The 2nd assertion is easy to prove, since $\{\varphi_{i,j}\}$ and $\{\psi_{i,j}\}$ are the orthonomal bases of an $m(i)$-dimensional space. □

Let us give an operator theoretical meaning to (3.2). We need the notion of spectral representation. Let $\widehat{\mathcal{H}} = \oplus_{i=1}^{\infty}\mathbf{C}^{m(i)}$. We define the (discrete) Fourier transformation $\mathcal{F} : L^2(\Omega) \to \widehat{\mathcal{H}}$ by $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \cdots)$ where

$$\mathcal{F}_i : L^2(\Omega) \ni u \to \big((u, \varphi_{i,1}), \cdots, (u, \varphi_{i,m(i)})\big) \in \mathbf{C}^{m(i)}. \tag{3.4}$$

$\mathcal{F}$ is unitary, and diagonalizes the Neumann Laplacian $-\Delta_g$ on $\Omega : \mathcal{F}_i(-\Delta_g u) = \lambda_i\mathcal{F}_i u$. Let P_i be the eigenprojection associated with the eigenvalue λ_i. Then, for $z \notin \sigma(-\Delta_g)$, the resolvent can be written as

$$R_\Omega(z) = \sum_{i=1}^{\infty} \frac{P_i}{\lambda_i - z} = \sum_{i=1}^{\infty} \frac{\mathcal{F}_i^*\mathcal{F}_i}{\lambda_i - z}, \tag{3.5}$$

which converges in the sense of strong limit in $L^2(\Omega)$.

Let $\Gamma = \partial\Omega$, and $r_\Gamma \in \mathbf{B}(H^1(\Omega); H^{1/2}(\Gamma))$ be the trace operator to Γ. Define $\delta_\Gamma \in \mathbf{B}(H^{-1/2}(\Gamma); H^1(\Omega)^*)$ as its adjoint:

$$(\delta_\Gamma f, w)_{L^2(\Omega)} = (f, r_\Gamma w)_{L^2(\Gamma)}, \quad f \in H^{-1/2}(\Gamma), \quad w \in H^1(\Omega).$$

Accordingly, we write as

$$r_\Gamma = \delta_\Gamma^*.$$

Then we have

$$\delta_\Gamma \in \mathbf{B}(H^{-1/2}(\Gamma); H^1(\Omega)^*), \quad \delta_\Gamma^* \in \mathbf{B}(H^1(\Omega); H^{1/2}(\Gamma)). \tag{3.6}$$

Then,

$$\Lambda(z) = \delta_\Gamma^* R_\Omega(z)\delta_\Gamma. \tag{3.7}$$

Let us prove this formula. We first show that the right-hand side is well-defined. Since $R_\Omega(z) \in \mathbf{B}(L^2(\Omega); H^2(\Omega))$, we have $R_\Omega(z) \in \mathbf{B}(H^2(\Omega)^*; L^2(\Omega))$. By an interpolation, we then have $R_\Omega(z) \in \mathbf{B}(H^1(\Omega)^*; H^1(\Omega))$. Using (3.6), we see that $\delta_\Gamma^* R_\Omega(z)\delta_\Gamma \in \mathbf{B}(H^{-1/2}(\Omega); H^{1/2}(\Omega))$.

For $f \in H^{1/2}(\Gamma)$, take $\widetilde{f} \in H^{3/2}(\Omega)$ such that $\partial_\nu\widetilde{f} = f$ on Γ. Let $v = R_\Omega(z)(\Delta_g + z)\widetilde{f}$, and put $u = v + \widetilde{f}$. Then $(-\Delta_g - z)u = 0$ in Ω, and $\partial_\nu u = f$ on Γ. Take $h \in L^2(\Omega)$. Then, by integration by parts,

$$\begin{aligned}(P_i(\Delta_g + z)\widetilde{f}, h)_{L^2(\Omega)} &= (z - \lambda_i)(\widetilde{f}, P_i h)_{L^2(\Omega)} + (f, r_\Gamma P_i h)_{L^2(\Gamma)} \\ &= -(\lambda_i - z)(P_i\widetilde{f}, h)_{L^2(\Omega)} + (P_i\delta_\Gamma f, h)_{L^2(\Omega)}.\end{aligned}$$

This yields

$$P_i u = P_i\widetilde{f} + R_\Omega(z)P_i(\Delta_g + z)\widetilde{f} = \frac{P_i\delta_\Gamma f}{\lambda_i - z}.$$

By (3.5), this implies $u = R_\Omega(z)\delta_\Gamma f$. By taking the trace to Γ, we get (3.7).

By Lemma 3.2, the operator $\delta_\Gamma^* P_i \delta_\Gamma$, whose integral kernel is $\sum_{j=1}^{m(i)} \varphi_{i,j}(x)\overline{\varphi_{i,j}(y)}$ restricted to Γ, is independent of the choice of the eigenvectors. Let us call the set

$$\Big\{\big(\lambda_i, \sum_{j=1}^{m(i)} \varphi_{i,j}(x)\overline{\varphi_{i,j}(y)}\Big|_{x\in\Gamma, y\in\Gamma}\big)\Big\}_{i=1}^{\infty}, \tag{3.8}$$

boundary spectral projection (**BSP**). This is what we actually use in the BC method. BSP is the set of pairs of poles and residues of the N-D map. We then have the following lemma.

Lemma 3.3. *Suppose we are given two metrics on Ω. Then their BSP's coincide if and only if their N-D maps coincide for all λ outside the spectrum.*

In the next chapter, we shall explain how to reconstruct the metric from BSP.

4. Inverse problems for hyperbolic ends

4.1. Exterior boundary value problem. Before entering into the inverse scattering for manifolds with hyperbolic ends, we need to discuss the spectral theory for the exterior boundary value problem. Let Ω be a bounded domain in $\mathbf{H}^n$ with smooth boundary and $\Omega^c := \mathbf{H}^n \setminus \overline{\Omega}$. Let $H^{N,c}$ be H defined in Ω^c with Neumann boundary condition. Namely $D(H^{N,c}) = \{u \in H^2(\Omega^c); \partial_\nu u\big|_{\partial\Omega^c} = 0\}$ and $H^{N,c}u = Hu$ for $u \in D(H^{N,c})$. Then $H^{N,c}$ is self-adjoint. Let $R^c(z) = (H^{N,c} - z)^{-1}$. The theory developed for H in Chap. 2 can be extended to $H^{N,c}$ without any essential change. In fact, let $u(z) = R^c(z)f$, $f \in L^2(\Omega^c)$, for $z \in \mathbf{C}\setminus\mathbf{R}$, and take $\chi \in C^\infty(\mathbf{H}^n)$ such that $\chi = 1$ near infinity, and $\chi = 0$ on a bounded open set containing $\overline{\Omega}$. Then $v(z) = \chi R^c(z)f$ satisfies

$$(H - z)v = [H,\chi]R^c(z)f + \chi f, \quad \text{in} \quad \mathbf{H}^n,$$

where we use that $\omega := \mathrm{supp}[H,\chi] \subset\subset \Omega^c$. Let us show that

$$\|u(z)\|_{\mathcal{B}^*} \leq C(\|u\|_{L^2(\Omega_1)} + \|f\|_{\mathcal{B}}). \tag{4.1}$$

where Ω_1 is a compact set such that $\omega \subset \Omega_1 \subset \Omega^c$. In fact, by elliptic regularity,

$$\|u\|_{H^1(\omega)} \leq C(\|u\|_{L^2(\Omega_1)} + \|f\|_{L^2(\Omega_1)}).$$

The inequality (4.1) then follows from this and (2.6) in Chap. 2.

Having inequality (4.1) in our disposal, we can prove, using the same arguments as for the whole $\mathbf{H}^n$, Lemma 2.2.13 for $R^c(z)$.

Theorem 4.1. *(1)* $\sigma_e(H^{N,c}) = [0,\infty)$, $\sigma_p(H^{N,c}) \cap (0\ ,\infty) = 0$.
(2) For any $\lambda > 0$, $lim_{\epsilon\to 0}R^c(\lambda \pm i\epsilon) =: R^c(\lambda \pm i0)$ exists in $\mathcal{B}^*$ in the weak $*$-sense.
(3) For any compact interval $I \subset (0,\infty)$, there exists a constant $C > 0$ such that

$$\|R^c(\lambda \pm i0)f\|_{\mathcal{B}^*} \leq C\|f\|_{\mathcal{B}}, \quad \forall \lambda \in I.$$

(4) For any $f, g \in \mathcal{B}$, $(0,\infty) \ni \lambda \to (R^c(\lambda \pm i0)f, g)$ is continuous.
(5) For $\lambda > 0$, $R^c(\lambda \pm i0)f$ is a unique solution to the equation

$$\begin{cases} (H - \lambda)u = f \in \mathcal{B} \quad \text{in} \quad \Omega^c, \\ \partial_\nu u = 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

satisfying the outgoting (for $+$) or incoming (for $-$) radiation condition.

The following lemma can now be proved easily by using Theorem 4.1.

Lemma 4.2. *Let $\lambda > 0$ and $f \in H^{1/2}(\partial\Omega)$. Then there exists a unique solution $u_\pm \in \mathcal{B}^*$ to the exterior boundary value problem*

$$\begin{cases} (H-\lambda)u = 0 \quad \text{in} \quad \Omega^c, \\ \partial_\nu u = f \quad \text{on} \quad \partial\Omega \end{cases}$$

satisfying the outgoing or incoming radiation condition.

Using the solutions $u_\pm$ as above, we define the N-D map by $\Lambda^{(\pm)}(\lambda)f = u_\pm\big|_{\partial\Omega}$ in addition to $\Lambda(z)$ for $z \in \mathbf{C} \setminus \sigma(H^{N,c})$. Note that $\Lambda^{(\pm)}(\lambda)$ is the boundary value of $\Lambda(z)$ as $z \to \lambda \pm i0$. Therefore, $\Lambda^{(\pm)}(\lambda)$ defined for $\lambda > 0$ has a unique analytic continuation to $\mathbf{C} \setminus \sigma(H^{N,c})$.

4.2. Inverse scattering at regular ends. Let $\mathcal{M}$ be a manifold satisfying the assumptions (A.1) $\sim$ (A.4) in Chap. 3, §3 with ends of number $N \geq 2$. We assume that at least *one of the ends has a regular infinity.* Let $\mathcal{M}_1$ be such an end. Namely, in the notation of Chap. 3, §2, $\mathcal{M}_1$ is diffeomorphic to $M_1 \times (0,1)$, in other words, $\mathcal{M}_1$ is asymptotically equal to a funnel. Let $\Gamma \subset \mathcal{M}$ be a compact submanifold of codimension 1 such that $\mathcal{M}$ splits into 3 parts Ω, Ω^c, $\partial\Omega = \partial\Omega^c = \Gamma$ in the following way :

$$\mathcal{M} = \Omega \cup \Gamma \cup \Omega^c, \quad \Omega \cap \Gamma = \Omega^c \cap \Gamma = \emptyset,$$

where $\overline{\Omega}$ and $\overline{\Omega^c}$ are assumed to be submanifolds of $\mathcal{M}$ with boundary Γ inheriting the Riemannian metric of $\mathcal{M}$. Assume also that Ω is non-compact and has infinity common to $\mathcal{M}_1$, and has no other infinity, i.e. $\Omega = M_1 \times (0,a)$, $0 < a < 1$. Note that when $N \geq 2$, Ω^c is also non-compact having a finite number of ends which are either regular or cusps. (The case when $N = 1$, which is equivalent to $\overline{\Omega^c}$ being compact, brings about the inverse boundary spectral problem discussed in §3.)

Let H^N be $-\Delta_g - (n-1)^2/4$ in Ω with Neumann boundary condition, and $H^{N,c}$ be the one on Ω^c. Then Theorem 4.1 and Lemma 4.2 also hold for H^N and $H^{N,c}$. Note that if all the ends except for $\mathcal{M}_1$ have cusps, there may be embedded eigenvalues in the essential spectrum of $H^{N,c}$. However, they are discrete with possible accumulation points only at 0 and infinity with rapidly decreasing eigenvectors.

We generalize Lemma 2.1 to the present case. Let $\mathcal{F}^{(\pm)}(k) = (\mathcal{F}^{(\pm)}_1(k), \cdots, \mathcal{F}^{(\pm)}_N(k))$ be the generalized Fourier transformation in $\mathcal{M}$ constructed in Chap. 3, §2, and $\mathbf{h}_\infty$ be defined by (3.47) in Chap. 3.

Lemma 4.3. *Suppose $0 \neq k^2 \notin \sigma_p(H^{N,c})$. If $f \in L^2(\Gamma)$ satisfies*

$$\langle f, \partial_\nu \mathcal{F}^{(+)}(k)^*\phi\rangle_\Gamma = 0, \quad \forall \phi = (\phi_1, 0, \cdots, 0) \in \mathbf{h}_\infty,$$

then $f = 0$.

Proof. Since (2.5) holds in $\mathcal{M}_1$, arguing in the same way as in Lemma 2.1, we have $u = 0$ in Ω. Consider $u^c = u\big|_{\Omega^c}$. Then we have $(H - k^2)u^c = 0$ in Ω^c, and similarly to the proof of Lemma 2.1 $\partial_\nu u^c = 0$ on Γ. Since u^c also satisfies the radiation condition, and $k^2 \notin \sigma_p(H^{N,c})$, we have $u^c = 0$ in Ω^c. This proves the lemma. $\square$

Recall that $H^{N,c}$ has two parts of spectral representations: the generalized Fourier transform, which we denote by $\mathcal{F}_c^{(+)}$ here, corresponding to the absolutely continuous spectrum for $H^{N,c}$, and the discrete Fourier transform, denoted by $\mathcal{F}_p^c$, corresponding to the point specrum for $H^{N,c}$ defined in the same way as in §3.

Lemma 4.4. *The N-D map $\Lambda^c(z)$ corresponding to $H^{N,c}$, which is determined for $z \in \mathbf{C} \setminus \mathbf{R}$, is of the form*

$$\Lambda^c(z) = \int_0^\infty \frac{\delta_\Gamma^* \mathcal{F}_c^{(+)}(k)^* \mathcal{F}_c^{(+)}(k) \delta_\Gamma}{k^2 - z} dk + \sum_i \frac{\delta_\Gamma^* P_i^c \delta_\Gamma}{\lambda_i - z}, \tag{4.2}$$

where the sum over i may be finite or infinite.

Proof. We proceed as in the proof of (3.7). Take $f \in C^\infty(\Gamma)$ and $\widetilde{f} \in C_0^\infty(\overline{\Omega^c})$ such that $\partial_\nu \widetilde{f} = f$ on Γ. Let v solve the boundary value problem

$$\begin{cases} (H - z)v = (-H + z)\widetilde{f} =: F & \text{in} \quad \Omega^c, \\ \partial_\nu v = 0 \quad \text{on} \quad \Gamma. \end{cases}$$

Then v is represented by eigenvectors $\varphi_{i,j}$ and the generalized Fourier transform $\mathcal{F}_c^{(+)}(k)$:

$$v = \int_0^\infty \frac{\mathcal{F}_c^{(+)}(k)^* \mathcal{F}_c^{(+)}(k) F}{k^2 - z} dk + \sum_i \frac{\sum_j (F, \varphi_{i,j})\varphi_{i,j}}{\lambda_i - z}.$$

Take $\phi \in \mathbf{h}_\infty^c$ (see Chap. 3, (3.47), where j varies from 2 to N). Then we have by integration by parts

$$\begin{aligned} (\mathcal{F}_c^{(+)}(k)F, \phi)_{\mathbf{h}_\infty^c} &= ((-H + z)\widetilde{f}, \mathcal{F}_c^{(+)}(k)^*\phi)_{L^2(\Omega^c)} \\ &= (f, \mathcal{F}_c^{(+)}(k)^*\phi)_{L^2(\Gamma)} + (z - k^2)(\widetilde{f}, \mathcal{F}_c^{(+)}(k^2)^*\phi)_{L^2(\Omega^c)} \\ &= (\mathcal{F}_c^{(+)}(k)\delta_\Gamma f, \phi)_{\mathbf{h}_\infty^c} + (z - k^2)(\mathcal{F}_c^{(+)}(k^2)\widetilde{f}, \phi)_{\mathbf{h}_\infty^c}. \end{aligned}$$

This implies

$$\mathcal{F}_c^{(+)}(k)F = \mathcal{F}_c^{(+)}(k)\delta_\Gamma f + (z - k^2)\mathcal{F}_c^{(+)}(k)\widetilde{f}.$$

The term from the point spectrum is dealt with similarly, and the lemma follows from a direct computation. □

Let us call the set

$$\left\{\delta_\Gamma^* \mathcal{F}_c^{(+)}(k)^* \mathcal{F}_c^{(+)}(k)\delta_\Gamma; k > 0\right\} \cup \left\{\left(\lambda_i, \delta_\Gamma^* P_i^c \delta_\Gamma\right); i\right\} \tag{4.3}$$

the boundary spectral projection (**BSP**) for $H^{N,c}$. By (4.2), we have

$$\Lambda^c(z) = \delta_\Gamma^*(H^{N,c} - z)^{-1}\delta_\Gamma. \tag{4.4}$$

Lemma 4.5. *Knowing the N-D map $\Lambda_c^{(+)}(k^2)$ for all k such that $k^2 \notin \sigma_p(H^{N,c})$ is equivalent to knowing BSP for $H^{N,c}$.*

Proof. $\Lambda_c^{(+)}(k^2)$ has a unique analytic continuation $\Lambda^c(z)$ for $z \in \mathbf{C} \setminus \mathbf{R}$, which determines $\Lambda_c^{(-)}(k^2)$ for real $k^2 \notin \sigma_p(H^{N,c})$. By (4.4) and Lemma 3.3.11, we have

$$\Lambda_c^{(+)}(k^2) - \Lambda_c^{(-)}(k^2) = \frac{\pi i}{k}\delta_\Gamma^* \mathcal{F}_c^{(+)}(k)^* \mathcal{F}_c^{(+)}(k)\delta_\Gamma.$$

Therefore we recover $\mathcal{F}_c^{(+)}(k)^*\mathcal{F}_c^{(+)}(k)$ for $k^2 \notin \sigma_p(H^{N,c})$ from $\Lambda_c^{(+)}(k^2)$. By (4.2), we also recover $\lambda_i \in \sigma_p(H^{N,c})$ and $\delta_\Gamma^* P_i^c \delta_\Gamma$ from the poles and residues of $\Lambda^c(z)$. The converse direction is seen by (4.2). □

Since $\mathcal{M}$ has N-ends, the S-matrix for $\mathcal{M}$ is an $N \times N$-matrix:

$$\widehat{S}(k) = \Big(\widehat{S}_{ij}(k)\Big)_{1 \le i,j \le N}.$$

Let $\mathcal{M}^{(j)}, (j = 1,2)$, be manifolds satisfying the assumptions (A.1) $\sim$ (A.4) in Chap. 3, §3. Assume that $\mathcal{M}_1^{(1)}$ and $\mathcal{M}_1^{(2)}$ are isometric, therefore, $\mathcal{M}_1^{(1)} = \mathcal{M}_1^{(2)} = M_1 \times (0,1)$, M_1 being a compact manifold of dimension $n-1$. Letting $\Omega = M_1 \times (0,a)$, we construct Ω_j^c and $H_j^{N,c}$ as above.

Theorem 4.6. *Suppose* $0 \neq k^2 \notin \sigma_p(H_1^{N,c}) \cup \sigma_p(H_2^{N,c})$. *Let* $\Lambda_j^{(+)}(k^2)$ *be the N-D map for* $H_j^{N,c}$. *Then* $\widehat{S}_{11}^{(1)}(k) = \widehat{S}_{11}^{(2)}(k)$ *if and only if* $\Lambda_1^{(+)}(k^2) = \Lambda_2^{(+)}(k^2)$.

The proof is the same as Theorem 2.3.

We now pass to the boundary control method (BC-method) to show that BSP determines the manifold uniquely. The BC-method works for general Riemannian manifold with boundary, if we know the N-D map for all k for the associated Laplace operator. The BC-method was first applied to compact manifolds ([**14**]), and was extended to non-compact manifolds (see e.g. [**78**], [**67**]).

Let us formulate the inverse problem on non-compact Riemannian manifolds. Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be Riemannian manifolds (not necessarily compact) with boundary with metric inherited form the Riemannian metric induced from $\mathcal{N}_j$. We say that $\mathcal{N}_1$ and $\mathcal{N}_2$ have common parts $\Gamma_1 \subset \partial\mathcal{N}_1$ and $\Gamma_2 \subset \partial\mathcal{N}_2$ if there exists an isometry $\Phi : \Gamma_1 \to \Gamma_2$. Let $\Lambda_j(z)$ be the N-D map for the Laplace operator on $\mathcal{N}_j$. Then we define

$$\Lambda_1(z)\Big|_{\Gamma_1} = \Lambda_2(z)\Big|_{\Gamma_2} \Longleftrightarrow \Phi_* \circ \Lambda_1(z)\Big|_{\Gamma_1} = \Lambda_2(z)\Big|_{\Gamma_2} \circ \Phi_*. \tag{4.5}$$

Here $\Lambda_j(z)\big|_{\Gamma_j}$ is defined by

$$\Lambda_j(z)\big|_{\Gamma_j} f = \Lambda_j(z) f\big|_{\Gamma_j}, \quad f \in L^2(\Gamma_j).$$

One can then show that (with some additional assumptions) if $\mathcal{N}_1$ and $\mathcal{N}_2$ have common parts Γ_1 and Γ_2, and (4.5) holds for all $z \notin \mathbf{R}$, then $\mathcal{N}_1$ and $\mathcal{N}_2$ are isometric. In Chapter 6, we shall give the proof of this theorem (Theorem 8.5) for asymptotically hyperbolic manifolds Ω_1^c, Ω_2^c under consideration. Modulus this *theorem*, we have thus proven the following result.

Theorem 4.7. *Let* $\mathcal{M}$ *be a manifold satisfying the assumptions (A.1)* $\sim$ *(A.4) in Chap. 3, §3. We assume that one of the ends has a regular infinity, and denote it by* $\mathcal{M}_1$. *Suppose we are given two metrics* $G^{(j)}$, $j = 1,2$, *on* $\mathcal{M}$ *satisfying (A-3) in Chapt. 3, §3. Assume that* $G^{(1)} = G^{(2)}$ *on* $\mathcal{M}_1$. *If* $\widehat{S}_{11}(k) = \widehat{S}_{11}(k)$ *for all* $k > 0$, *then* $G^{(1)}$ *and* $G^{(2)}$ *are isometric on* $\mathcal{M}$.

We can actually prove a stronger version of Theorem 4.7, which is valid for two manifolds whose structure, in particular the number of ends, are not known a-priori.

Theorem 4.8. *Let $\mathcal{M}^{(j)}$, $j = 1, 2$, be manifolds satisfying the assumptions (A.1) $\sim$ (A.4) in Chap. 3, §3 endowed with metric $G^{(j)}$, $j = 1, 2$. We assume that for both of $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$ one of the ends has a regular infinity, and denote them by $\mathcal{M}_1^{(j)}$, $j = 1, 2$. Assume that $\mathcal{M}_1^{(1)}$ and $\mathcal{M}_1^{(2)}$ are isometric, and $\widehat{S}_{11}(k) = \widehat{S}_{11}(k)$ for all $k > 0$. Then $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$ are isometric.*

4.3. References of inverse scattering on asymptotically hyperbolic manifolds. Melrose's theory of scattering metric studies the spectral properties of the Laplace-Beltrami operator on manifolds whose ends have the metric of the following type

$$ds^2 = \frac{h(x, y, dx, dy)}{y^2}.$$

Each end is assumed to be isomorphic to $X \times (0, 1)$ and $g_0(x, y, dx, dy)$ admits an asymptotic expansion of the form

$$h(x, y, dx, dy) = (dy)^2 + h_0(x, dx) + y\, h_1(x, dx, dy) + y^2 h_2(x, dx, dy) + \cdots ,$$

$h_0(x, dx)$ being a Riemannian metric on the boundary at infinity, X. Mazzeo and Melrose [**96**] developed a pseudo-differential calculus to deal with these manifolds, and proved the existence of analytic continuation of resolvent of the associated Laplace-Beltrami operator into the region $\mathbf{C}\backslash\{\frac{1}{2}(n - \mathbf{N}_0)\}$, $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. Borthwick [**20**] studied the case of variable curvature at the boundary at infinity. Guillarmou [**40**] showed that the resolvent had in general essential singularities at $\{\frac{1}{2}(n - \mathbf{N}_0)\}$. Joshi and Sá Barreto [**73**] proved that the scattering matrix determined the asymptotic expansion of the metric $h(x, y, dx, dy)$ at infinity. Sá Barreto [**120**] proved that the scattering matrix for all energies determined the whole manifold. Let us remark here that, in this setting, two asymptotically hyperbolic manifolds are shown to be isometric, without assuming that the corresponding ends are isometric, if their scattering matrices coincide for all energies. The crucial fact is that the support theorem holds for the Radon transform (see Theorem 7.1 of [**120**], also [**93**]). This support theorem is still open in our assumptions imposed in Chap. 2. Let us also note that the support theorem does not hold for the Radon transform in the Euclidean space (see [**52**] p. 14). See also [**121**].

For the spectral theory of symmetric spaces of higher rank, there is a work [**97**].

Resonance is also an important subject in the inverse scattering theory, and many works are devoted to it. They are summarized in [**45**] or in the book of Borthwick [**21**].

Inverse scattering problem or inverse boundary value problem from a fixed energy is not yet solved completely for the case of the metric. However, in 2-dimensions the inverse boundary value problem is completely solved by Nachman [**109**], Lassas-Uhlmann [**91**], Astala-Paivarinta [**8**] and Astala-Lassas-Paivarinta [**7**]. For higher dimensions, there is a developed theory for isotropic metrics, see the review article of [**127**]. Morever a method was developed to study anisotropic metrics from a known conformal class. See e.g. [**30**].

There is a link between the hyperbolic manifolds and the inverse boundary value problems in the Euclidean space. See [**62**], [**63**], [**64**], [**65**], [**66**]. In [**57**] an application to the numerical computation is given.

In recent years, there appeared a number of interesting papers devoted to scattering and inverse scattering for asymptotically hyperbolic manifolds. See e.g. [**22**], [**41**], [**42**], [**43**], [**129**].