CHAPTER 2

Perturbation of the metric

We shall study in this chapter spectral properties of $-\Delta_g$, where Δ_g is the Laplace-Beltrami operator associated with a Riemannian metric, which is a perturbation of the hyperbolic metric on $\mathbf{H}^n$. We shall prove the limiting absorption principle, construct the generalized Fourier transform and introduce the scattering matrix. To study $\mathbf{H}^n$ in an invariant manner, it is better to employ the ball model and geodesic polar coordinates centered at the origin. However, we use the upper-half space model, since it is of independent interest, necessary in order to make the arguments in Chapter 1 complete by the method adopted here, and also of a preparatory character to deal with hyperbolic ends in Chapter 3.

1. Preliminaries from elliptic partial differential equations

1.1. Regularity theorem. In this section, for the notational convenience, we denote points $x \in \mathbf{R}^n$ by $x = (x_1, \cdots, x_n)$. We consider the differential operator

$$A = \sum_{|\alpha| \leq 2} a_\alpha(x)(-i\partial_x)^\alpha$$

defined on $\mathbf{R}^n$. The coefficients $a_\alpha(x)$ are assumed to satisfy

$$a_\alpha(x) \in C^\infty(\mathbf{R}^n), \quad \partial_x^\beta a_\alpha(x) \in L^\infty(\mathbf{R}^n), \quad \forall\beta,$$

$$\sum_{|\alpha|=2} a_\alpha(x)\xi^\alpha \geq C|\xi|^2, \quad \forall x \in \mathbf{R}^n, \quad \forall \xi \in \mathbf{R}^n,$$

C being a positive constant. A function $u \in L^2_{loc}(\mathbf{R}^n)$ is said to be a weak solution of $Au = f$ if it satisfies

$$\int_{\mathbf{R}^n} u(x)\overline{A^\dagger\varphi(x)}dx = \int_{\mathbf{R}^n} f(x)\overline{\varphi(x)}dx, \quad \forall\varphi \in C_0^\infty(\mathbf{R}^n),$$

where $A^\dagger$ is the formal adjoint of A.

Theorem 1.1. *If $u \in L^2(\mathbf{R}^n)$ is a weak solution of $Au = f$ and $f \in H^m(\mathbf{R}^n)$ for some $m \geq 0$, then $u \in H^{m+2}(\mathbf{R}^n)$, and*

$$\|u\|_{H^{m+2}(\mathbf{R}^n)} \leq C(\|u\|_{L^2(\mathbf{R}^n)} + \|f\|_{H^m(\mathbf{R}^n)}).$$

For the proof see e.g. [**101**]. By using Theorem 1.1, one can prove the following inequality. Let Ω be a bounded open set in $\mathbf{R}^n$ with smooth boundary, and Ω_ϵ an ϵ-neighborhood of Ω. Then

$$\|u\|_{H^{m+2}(\Omega)} \leq C_\epsilon(\|u\|_{L^2(\Omega_\epsilon)} + \|f\|_{H^m(\Omega_\epsilon)}). \tag{1.1}$$

1.2. A-priori estimates in $\mathbf{H}^n$. We next consider $\mathbf{R}^n_+$. We put

$$D_i = x_n \partial_i, \quad 1 \le i \le n, \quad D = (D_1, \cdots, D_n),$$

and let $(\ ,\ )$, $\|\cdot\|$ be the following inner product and the norm:

$$(u, v) = \int_{\mathbf{R}^n_+} u(x)\overline{v(x)} \frac{dx}{(x_n)^n}, \quad \|u\|^2 = (u, u).$$

For operators A and B, $[A, B]$ denotes the commutator $AB - BA$. Straightforward computations show the following lemma.

Lemma 1.2. *(1) For* $j \neq n,\ 1 \le i \le n$,

$$[D_i, D_j] = \delta_{in} D_j.$$

(2) For $u, v \in C_0^\infty(\mathbf{R}^n_+)$,

$$(D_i u, v) = -(u, D_i v) + \delta_{in}(n-1)(u, v).$$

We use the following weight

$$\rho(x) = \log(1 + |x|^2) + \sqrt{1 + (\log x_n)^2}. \tag{1.2}$$

Comparing ρ with ρ_0 in Lemma 1.1.6, there exists a constant $C > 0$ such that

$$C^{-1}(1 + d_h(x)) \le \rho(x) \le C(1 + d_h(x)), \tag{1.3}$$

where $d_h(x)$ is the geodesic distance between x and $(0, 1)$ in the metric $ds^2 = dx^2/x_n^2$, cf. (1.2) of Ch.1. We put

$$\widetilde{D}_i = \widetilde{y}(x_n)\partial_{x_i}, \quad (i = 1, \cdots, n-1), \quad \widetilde{D}_n = D_n, \tag{1.4}$$

where $\widetilde{y}(x_n) \in C^\infty(\mathbf{R})$, $\widetilde{y}(x_n) = 1$ for $x_n < 1$, $\widetilde{y}(x_n) = x_n$ for $x_n > 2$. Then we have for $s \in \mathbf{R}$ and $|\alpha| \ge 1$

$$|\widetilde{D}^\alpha \rho(x)^s| + |D^\alpha \rho(x)^s| \le C_s \rho(x)^{s-1}. \tag{1.5}$$

We consider the differential operator $A = A_0 + A_1$ with

$$A_0 = -D_n^2 + (n-1)D_n - \sum_{i=1}^{n-1} D_i^2,$$

$$A_1 = \sum_{i,j=1}^{n} a_{ij}(x) D_i D_j + \sum_{i=1}^{n} b_i(x) D_i + c(x).$$

We rewrite A as

$$A = P_2(x, D) + P_1(x, D), \quad D = (D_1, \cdots, D_n),$$

where

$$P_2(x, \xi) = |\xi|^2 + \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\overline{\xi_j},$$

$$P_1(x, \xi) = (n-1)\xi_n + \sum_{i=1}^{n} b_i(x)\xi_i + c(x).$$

We assume that the coefficients $a_{ij}(x)$, $b_i(x)$, $c(x)$ are in $C^\infty(\mathbf{R}^n_+; \mathbf{R})$ and satisfy

$$|\widetilde{D}^\alpha a(x)| \le C_\alpha\, \rho(x)^{-\epsilon}, \quad \forall \alpha, \tag{1.6}$$

for some $\epsilon > 0$, where $a(x)$ represents any of $a_{ij}(x)$, $b_i(x)$, $c(x)$. Moreover, a_{ij} is real and symmetric : $a_{ij} = a_{ji}$, and $P_2(x,\xi)$ is *uniformly elliptic*, namely, there exists a constant $C_0 > 0$ such that

$$P_2(x,\xi) \geq C_0|\xi|^2, \quad \forall \xi \in \mathbf{C}^n, \quad \forall x \in \mathbf{R}^n_+. \tag{1.7}$$

Let $\mathcal{B}$ and $\mathcal{B}^*$ be defined as in Chap. 1, §2, with $\mathbf{h} = L^2(\mathbf{R}^{n-1})$. For $s \in \mathbf{R}$, we introduce the function space $\mathcal{X}^s$ as follows

$$\mathcal{X}^s \ni u \Longleftrightarrow \rho(x)^s u(x) \in L^2(\mathbf{H}^n) = L^2\Big(\mathbf{R}^n_+; \frac{dx}{x_n^n}\Big), \tag{1.8}$$

equipped with the norm

$$\|u\|_{\mathcal{X}^s} = \|\rho^s u\|_{L^2(\mathbf{H}^n)}. \tag{1.9}$$

Theorem 1.3. *(1) If $u \in \mathcal{B}^*$ satisfies $(A - z)u = f \in \mathcal{B}^*$ with $z \in \mathbf{C}$, then*

$$\|D_i u\|_{\mathcal{B}^*} \leq C(1 + |z|)^{1/2}(\|u\|_{\mathcal{B}^*} + \|f\|_{\mathcal{B}^*}), \quad 1 \leq i \leq n.$$

(2) Furthermore, if

$$\lim_{R\to\infty} \frac{1}{\log R}\int_{1/R}^R \Big[\|u(\cdot,x_n)\|^2_{L^2(\mathbf{R}^{n-1})} + \|f(\cdot,x_n)\|^2_{L^2(\mathbf{R}^{n-1})}\Big]\frac{dx_n}{(x_n)^n} = 0$$

holds, then, for $1 \leq i \leq n$, we have

$$\lim_{R\to\infty} \frac{1}{\log R}\int_{1/R}^R \|D_i u(\cdot,x_n)\|^2_{L^2(\mathbf{R}^{n-1})}\frac{dx_n}{(x_n)^n} = 0.$$

(3) Assertion (2) also holds with lim *replaced by* lim inf.
(4) If $u, f \in L^2(\mathbf{H}^n)$, then

$$\|D_i u\| \leq C(1 + |z|)^{1/2}(\|u\| + \|f\|), \quad 1 \leq i \leq n, \tag{1.10}$$

$$\|D_i D_j u\| \leq C(1 + |z|)(\|u\| + \|f\|), \quad 1 \leq i, j \leq n. \tag{1.11}$$

(5) If $u, f \in \mathcal{B}^*$,

$$\|D_i D_j u\|_{\mathcal{X}^{-s}} \leq C_s(1 + |z|)(\|u\|_{\mathcal{B}^*} + \|f\|_{\mathcal{B}^*}), \quad 1 \leq i, j \leq n, \tag{1.12}$$

for any $s > 1/2$.
(6) If $u, f \in \mathcal{X}^s$ for some $s \in \mathbf{R}$, then

$$\|D_i u\|_{\mathcal{X}^s} \leq C(1 + |z|)^{1/2}(\|u\|_{\mathcal{X}^s} + \|f\|_{\mathcal{X}^s}), \quad 1 \leq i \leq n, \tag{1.13}$$

$$\|D_i D_j u\|_{\mathcal{X}^s} \leq C(1 + |z|)(\|u\|_{\mathcal{X}^s} + \|f\|_{\mathcal{X}^s}), \quad 1 \leq i, j \leq n. \tag{1.14}$$

In the above estimates in (1), (4), (5) and (6), the constants C and C_s are independnet of $z \in \mathbf{C}$.

We note that assertion (4) is a particular case of assertion (6) with $s = 0$, while assertion (5) follows from (6), if we take into the account that $\mathcal{B}^* \subset \mathcal{X}^{-s}$, $s > 1/2$.

Proof. We take $\chi(t) \in C_0^\infty(\mathbf{R})$ such that $\chi(t) = 1$ $(|t| < 1)$, $\chi(t) = 0$ $(|t| > 2)$, and put

$$\chi_{R,r}(x) = \chi\Big(\frac{\log x_n}{\log R}\Big)\chi\Big(\frac{|x'|}{r}\Big), \quad \chi_R(x_n) = \chi\Big(\frac{\log x_n}{\log R}\Big),$$

where $x' = (x_1, \cdots, x_{n-1})$. Since with $g_{ij} = \delta_{ij} + a_{ij}$,

$$\begin{aligned}(g_{ij}D_iD_ju, \chi^2_{R,r}u) \;=\; & -(g_{ij}D_iu, \chi^2_{R,r}D_iu) \\ & -(D_ju, \big(D_i(g_{ij}\chi^2_{R,r})\big)\, u) + \delta_{in}(n-1)(D_ju, g_{ij}\chi^2_{R,r}u).\end{aligned}$$

Thus, we have

$$
\begin{aligned}
-\sum_{i,j=1}^{n}(g_{ij}D_iD_ju,\chi_{R,r}^2u) &= \sum_{i,j=1}^{n}(g_{ij}\chi_{R,r}D_ju,\chi_{R,r}D_iu)\\
&\quad+\sum_{i,j=1}^{n}(D_ju,\left(D_i(g_{ij}\chi_{R,r}^2)\right)u)\\
&\quad-\sum_{j=1}^{n}\delta_{in}(n-1)(D_ju,g_{nj}\chi_{R,r}^2u).
\end{aligned}
$$

We split the 2nd term of the right-hand side into

$$
-\sum(\chi_{R,r}D_ju,(D_ig_{ij})\chi_{R,r}u)-2\sum(\chi_{R,r}D_ju,g_{ij}(D_i\chi_{R,r})u)
$$

and use the uniform ellipticity (1.7) to see that

$$
\begin{aligned}
C_0\|\chi_{R,r}Du\|^2 &\le \mathrm{Re}\,(Au,\chi_{R,r}^2u)+\epsilon\|\chi_{R,r}Du\|^2\\
&\quad+C_\epsilon(\|\psi_Ru\|^2+\|(D\chi_{R,r})u\|^2).
\end{aligned}
$$

Here ψ_R is defined by

$$
\psi_R(x_n)=\psi(\frac{\log x_n}{\log R}),
$$

where $\psi\in C_0^\infty(\mathbf{R})$, $\psi=1$ on the support of χ. For small $\epsilon>0$, the term $\epsilon\|\chi_{R,r}Du\|^2$ is absorbed by the left-hand side. Therefore, by using the equation $(A-z)u=f$, we have

$$
\|\chi_{R,r}Du\|^2\le C(1+|z|)(\|\psi_Ru\|^2+\|(D\chi_{R,r})u\|^2+\|\psi_Rf\|^2).
$$

We fix R and let $r\to\infty$ to see that $\chi_{R,r}$ can be replaced by χ_R. Moreover

$$
|(D\chi_R)(x_n)|\le\frac{C}{\log R}\psi_R(x_n)\le C\psi_R(x_n)
$$

for $R>e$. Therefore, we have

$$
\|\chi_RDu\|^2\le C(1+|z|)(\|\psi_Ru\|^2+\|\psi_Rf\|^2). \tag{1.15}
$$

Dividing this inequality by $\log R$ and taking the supremum with respect to R, we obtain the assertion (1). Letting $R\to\infty$, we obtain (2) and (3).

Letting $R\to\infty$ in (1.15), we prove (1.10). To prove (1.11), we first observe that the previous considerations do not require (1.6) in full generality, just that $a\in L^\infty(\mathbf{R}^\mathbf{n}_+)$. This makes it possible to consider only the case when u is compactly supported. In fact, in the general case putting $\chi_{R,r}u=v$ we have

$$
(A-z)v=\chi_{R,r}f+[A,\chi_{R,r}]u.
$$

Since $[A,\chi_{R,r}]=\sum_i c_i(x)D_i+d(x)$ and $c_i(x),d(x)$ and $c_i,d\in L^\infty$ independently on $R,r>e$, we can apply (1.10) and (1.11) to see that the right-hand side is in $L^2(\mathbf{R}^\mathbf{n}_+)$ uniformly with respect to R,r.

Now assuming that u is compactly supported, we split u as $u=u_1+u_2+u_3$, where $u_i=\chi_i(\frac{\log x_n}{\log R})u$ so that $\operatorname{supp}u_1\subset\{x_n<2/R\}$, $\operatorname{supp}u_2\subset\{1/R<x_n<2R\}$, $\operatorname{supp}u_3\subset\{x_n>R\}$. Using

$$
\|D_iD_ju\|^2=(D_j^2u,D_i^2u)+(D_ju,[D_j,D_i]D_ju),
$$

we have

$$\sum_{i,j} \|D_i D_j u\|^2 \leq C(\|\sum_i D_i^2 u\|^2 + \sum_i \|D_i u\|^2).$$

We have

$$A_0 u_i = -A_1 u_i + z u_i + f_i, \quad i = 1, 3, \tag{1.16}$$

where

$$\|f_i\| \leq C(\|f\| + \|D_n u\| + \|u\|) \leq C(1 + |z|)^{1/2}(\|f\| + \|u\|),$$

with the last inequality following from (1.10). Since $\|A_0 u_i\|^2 = \sum_{j,k}(D_j^2 u_i, D_k^2 u_i)$, taking the L^2-norm of the both sides of (1.16), and using condition (1.6), we have, for $i = 1, 3$,

$$\sum_{j,k} \|D_j D_k u_i\| \leq \epsilon \sum_{j,k} \|D_j D_k u_i\| + C_\epsilon (1 + |z|)\big(\sum_j \|D_j u_i\| + \|u\| + \|f\|\big),$$

where $\epsilon = \epsilon(R) \to 0$ as $R \to \infty$. Therefore (1.11) holds for $i = 1, 3$ with sufficiently large R. For $i = 2$, we have only to note that u_2 satisfies the following 2nd order elliptic equation with bounded coefficients:

$$\sum_{i,j} \widetilde{a}_{ij}(x)\partial_i\partial_j u_2 + \sum_i \widetilde{a}_i(x)\partial_i u_2 + \widetilde{c}(x)u_2 = f_2$$

and use Theorem 1.1.

To prove (5), we put $v = \rho(x)^{-s}u$ and $g = (A - z)v$. Then Lemma 1.2.7, estimate (1.5) and assertion (1) imply that $v, g \in L^2(\mathbf{H}^n)$. By assertion (4), we then have $D_i v, D_i D_j v \in L^2(\mathbf{H}^n)$, which, in turn, implies that $D_i D_j u \in \mathcal{X}^{-s}$ and the inequality (1.12).

The proof of (1.13) is similar to the proof of (1.10) if w use $\rho(x)^s \chi_{R,r}(x)$ instead of $\chi_{R,r}(x)$.

To prove (1.14), we again consider $v = \rho(x)^{-s}u$, which, due to (1.13) satisfies $(A - z)v = g \in L^2(\mathbf{H}^n)$. Using (1.10) together with (1.13) and (1.5), we arrive at (1.14). □

1.3. Essential self-adjointness. On the upper space $\mathbf{R}^{\mathbf{n}}_+$, we introduce the Riemannian metric

$$ds^2 = \frac{1}{x_n^n} \sum_{i,j=1}^n g_{ij}(x) dx_i dx_j, \tag{1.17}$$

where $g_{ij} = \delta_{ij} + a_{ij}$. Assume that A is symmetric on $C_0^\infty(\mathbf{R}^{\mathbf{n}}_+)$.

Theorem 1.4. *$A\big|_{C_0^\infty(\mathbf{R}^n_+)}$ is essentially self-adjoint.*

Proof. We show that for $u \in L^2(\mathbf{H}^n)$

$$(u, (A - i)\varphi) = 0, \quad \forall \varphi \in C_0^\infty(\mathbf{H}^n) \Longrightarrow u = 0$$

and the same assertion holds with i replaced by $-i$. Applying (1.1), we see that $u \in H^2_{loc}(\mathbf{R}^n_+)$, and $(A + i)u = 0$ holds, moreover, by Theorem 1.3 (4),

$$D_i u, \ D_i D_j u \in L^2(\mathbf{H}^n).$$

Letting

$$\Omega_{r,R} = \{|x'| < r, \ 1/R < x_n < R\}, \quad \Omega_R = \{1/R < x_n < R\},$$

we then have

$$\int_{\Omega_{r,R}} Au\overline{u}d\mu = -i\int_{\Omega_{r,R}} |u|^2 d\mu, \quad d\mu = dx/(x_n)^n.$$

Integrating by parts and taking the imaginary part,

$$\int_{\Omega_{r,R}} |u|^2 d\mu \leq C\sum_i \int_{\partial\Omega_{r,R}} |u||D_i u|dS,$$

where dS is the surface measure associated with hyperbolic metric. Noting that

$$\int_{1/R<x_n<R} |uD_i u|d\mu < \infty,$$

there is a sequence $r_n \to \infty$ such that,

$$\sum_i \int_{\Sigma_{R,n}} |u||D_i u|dS \to 0 \quad \text{as } n \to \infty,$$

where $\Sigma_{R,n} = \{(x', x_n) : |x'| = r_n,\ R^{-1} < x_n < R\}$. Using these r_n's, we see that

$$\int_{\Omega_R} |u|^2 d\mu \leq C\sum_{i=1}^n \left(\int_{x_n=1/R} + \int_{x_n=R}\right) |u||D_i u|\frac{dx'}{(x_n)^{n-1}}. \tag{1.18}$$

We next put

$$f(x_n) = \sum_{i=1}^n \int_{\mathbf{R}^{n-1}} |uD_i u(x', x_n)|\frac{dx'}{(x_n)^{n-1}}.$$

Then, since $u, D_i u \in L^2(\mathbf{H}^n)$, we have

$$\int_0^\infty f(x_n)\frac{dx_n}{x_n} < \infty.$$

Hence, $\liminf_{x_n\to\infty} f(x_n) = 0$ and $\liminf_{x_n\to 0} f(x_n) = 0$. Using this fact, letting R_n tend to infinity along a suitable sequence in (1.18), we have $u = 0$. □

1.4. Rellich's theorem. It is well-known that, for a bounded open set $\Omega \subset \mathbf{R}^n$, the inclusion $H^1(\Omega) \subset L^2(\Omega)$ is compact. This is often stated in the following form and is called Rellich's theorem.

Theorem 1.5. *Let Ω be a bounded open set in $\mathbf{R}^n$, and $m \geq 1$. Then for any bounded sequence $\{f_k\}$ in $H^m(\Omega)$, there exists a subsequence $\{f_{k'}\}$ convergent in $H^{m-1}(\Omega)$.*

For the proof, see e.g. [**101**].

1.5. Unique continuation theorem. Let us assume that on a connected open set $\Omega \subset \mathbf{R}^n$, we are given a differential operator

$$A = \sum_{|\alpha|\leq 2} a_\alpha(x)\partial_x^\alpha,$$

where for $|\alpha| = 1, 2$, $a_\alpha(x) \in C^\infty$, and for $|\alpha| = 0$, $a_\alpha(x) \in L^\infty$, moreover for $|\alpha| = 2$, $a_\alpha(x)$ is real-valued and satisfies

$$\sum_{|\alpha|=2} a_\alpha(x)\xi^\alpha \geq C|\xi|^2, \quad \forall x \in \Omega, \quad \forall \xi \in \mathbf{R}^n,$$

for a constant $C > 0$. Then, if u satisfies $Au = 0$ on Ω, and vanishes on an open subset of Ω, then u vanishes identically on Ω. For the proof, see e.g. [**101**] for a C^∞-coefficient case, and [**6**] for the general case.

2. Basic spectral properties for Laplace-Belrami operators on $\mathbf{H}^n$

2.1. Assumption on the metric. In the sequel, we denote points in $\mathbf{H}^n = \mathbf{R}^n_+$ as (x, y), where $x \in \mathbf{R}^{n-1}$, $y > 0$, and put

$$D_x = y\partial_x, \quad \widetilde{D}_x = \tilde{y}(y)\partial_x, \quad \widetilde{D}_y = D_y = y\partial_y, \tag{2.1}$$

where $\tilde{y}(y) \in C^\infty((0,\infty))$ is a positive function such that $\tilde{y}(y) = 1$ for $y < 1$, $\tilde{y}(y) = y$ for $y > 2$. Recall that we put

$$\rho(x,y) = \log\left(1 + |x|^2 + y^2\right) + \sqrt{1 + |\log y|^2},$$

and have the following inequality

$$C^{-1}(1 + \rho(x,y)) \leq 1 + d_h(x,y) \leq C(1 + \rho(x,y)),$$

$$|\widetilde{D}^\alpha \rho(x,y)^s| + |D^\alpha \rho(x,y)^s| \leq C_s \rho(x,y)^{s-1}, \quad |\alpha| \geq 1, \quad s \in \mathbf{R},$$

where $d_h(x, y)$ is the distance between (x, y) and $(0, 1)$ with respect to the standard hyperbolic metric (Lemma 1.1.6).

To describe the space of metric, we introduce the following class of functions.

Definition 2.1. *For $s \in \mathbf{R}$, let $\mathcal{W}^s$ be the set of real-valued C^∞-functions $f(x,y)$ defined on $\mathbf{R}^{n-1} \times (0,\infty)$ such that for any (multi) index α, β, there exists a constant $C_{\alpha\beta} > 0$ such that*

$$|(\widetilde{D}_x)^\alpha (D_y)^\beta f(x,y)| \leq C_{\alpha\beta}\, \rho(x,y)^{s-\min(|\alpha|+\beta,1)}. \tag{2.2}$$

On the upper half-space $\mathbf{R}^n_+$, we consider the Riemannian metric

$$ds^2 = y^{-2}\Big((dx)^2 + (dy)^2 + A(x,y,dx,dy)\Big), \tag{2.3}$$

where $A(x, y, dx, dy)$ is a symmetric covariant tensor of the form

$$A(x,y,dx,dy) = \sum_{i,j=1}^{n-1} a_{ij}(x,y)dx^i dx^j + 2\sum_{i=1}^{n-1} a_{in}(x,y)dx^i dy + a_{nn}(x,y)(dy)^2.$$

Here each $a_{ij}(x, y)$ $(1 \leq i, j \leq n)$ is assumed to satisfy the following condition:

(C) *There exists a constant $\epsilon > 0$ such that $a_{ij} \in \mathcal{W}^{-1-\epsilon}$ for $y > 1$.*

Let us look at the Laplace-Beltrami operator associated with the above metric ds^2. Let $\mathcal{P}$ the set of differential operators P defined by

$$\mathcal{P} \ni P \Longleftrightarrow P = \sum_{\alpha,\beta}(c_{\alpha\beta} + a_{\alpha\beta})D_x^\alpha D_y^\beta,$$

where $c_{\alpha\beta}$ are constants, $a_{\alpha\beta} \in \mathcal{W}^{-1-\epsilon}$ and the above sum is finite. Then by a direct computation using Lemma 1.2 one can show that $\mathcal{P}$ is an algebra.

We rewrite (2.3) into $ds^2 = g_{ij}(X)dX^i dX^j$, $X = (X_1, \cdots, X_n) = (x, y)$, where $g_{ij}(X) = y^{-2}\big(\delta_{ij} + a_{ij}(x,y)\big)$ and we assume that $a_{ij}\xi_i\xi_j > -|\xi|^2$. Letting $\big(g^{ij}\big) = \big(g_{ij}\big)^{-1}$, we have

$$g^{ij}(x,y) = y^2\big(\delta^{ij} + \widehat{g}^{ij}(x,y)\big),$$

where $\widehat{g}^{ij}(x,y) \in \mathcal{W}^{-1-\epsilon}$. The associated Laplace-Beltrami operator Δ_g is then written as

$$-\Delta_g = D_y^2 - (n-1)D_y + D_x^2 + \sum_{i,j=1}^{n} a^{ij}(x,y)D_iD_j + \sum_{i=1}^{n} b^i(x,y)D_i,$$

where $(D_1, \cdots, D_n) = (y\partial_x, y\partial_y)$ and $a^{ij}(x,y), b^i(x,y) \in \mathcal{W}^{-1-\epsilon}$. Hence $\Delta_g \in \mathcal{P}$.

The operator $-\Delta_g$ is symmetric in $L^2(\mathbf{R}^n_+; \sqrt{g}\ dxdy)$, where $g = \det(g_{ij})$. In order to compare it with the Laplace-Beltrami operator for the standard hyperbolic metric, it is convenient to use the unitary gauge transformation from $L^2(\mathbf{R}^n_+; \sqrt{g}dxdy)$ onto $L^2(\mathbf{R}^n_+; dxdy/y^n)$:

$$u \to (y^{2n}g)^{1/4}u,$$

so that

$$-\Delta_g - \frac{(n-1)^2}{4} \to -(y^{2n}g)^{1/4}\Delta_g(y^{2n}g)^{-1/4} - \frac{(n-1)^2}{4}$$

in $L^2(\mathbf{R}^n_+; dxdy/y^n)$.

2.2. Transformed Laplace-Beltrami operators. We are thus led to the differential operators

$$H = -(y^{2n}g)^{1/4}\Delta_g(y^{2n}g)^{-1/4} - \frac{(n-1)^2}{4} = H_0 + V,$$

$$H_0 = -D_y^2 + (n-1)D_y - D_x^2 - \frac{(n-1)^2}{4}, \quad V = \sum_{|\alpha|\leq 2} a_\alpha(x,y)D^\alpha$$

in $L^2(\mathbf{R}^n_+; dxdy/y^n)$, with the inner product denoted by $(\cdot,\cdot)$. $H\big|_{C_0^\infty(\mathbf{H}^n)}$ is symmetric,

$$(Hf,g) = (f,Hg), \quad \forall f,g \in C_0^\infty(\mathbf{H}^n), \tag{2.4}$$

and uniformly elliptic in the sense of §1. By our assumption a_α satisfies the condition (C).

One should keep in mind that our operator $-H$ is unitarily equivalent to the Riemannian Laplacian Δ_g associated with the metric ds^2 of (2.3) which is shifted by $(n-1)^2/4$. The arguments to be developed in Chapters 2 and 3 are also applicable to the more general operators with perturbation of 1st order differential operators, except for Theorem 2.10. Even in this case, however, Theorem 2.10 still holds except for a discrete set of λ's, which can be proved by the same way as in Theorems 3.3.5 and 3.3.6.

By Theorem 1.4, $H\big|_{C_0^\infty(\mathbf{H}^n)}$ is essentially self-adjoint. Let

$$R_0(z) = (H_0 - z)^{-1},\ R(z) = (H-z)^{-1}.$$

Lemma 2.2. *For $z \notin \mathbf{C} \setminus \mathbf{R}$, $R_0(z)VR(z)$ is compact. Hence*

$$\sigma_d(H) \subset (-\infty, 0), \quad \sigma_e(H) = [0,\infty).$$

Proof. By Theorem 1.3 (4), $VR(z) \in \mathbf{B}(L^2;L^2)$, and $R_0(z)V = (VR_0(\overline{z}))^* \in \mathbf{B}(L^2;L^2)$. We take $\chi(t) \in C_0^\infty(\mathbf{R})$ satisfying $\chi(t) = 1$ $(|t|<1)$, $\chi(t) = 0$ $(|t|>2)$, and put

$$\chi_R(x,y) = \chi\left(\frac{|x|}{R}\right)\chi\left(\frac{\log y}{\log R}\right).$$

Then $\chi_R R(z)$, and henceforth $R_0(z)V\chi_R R(z)$ are compact and, due to the decay assumption of the coefficients, $\|R_0(z)V(1-\chi_R)R(z)\| \to 0$ $(R\to\infty)$. Hence $R_0(z)VR(z)$ is also compact. Since $\sigma(H_0)=\sigma_e(H_0)=[0,\infty)$, the lemma follows from Weyl's theorem ([**62**], p. 26). □

The main purpose of this section is to prove the following theorem.

Theorem 2.3. *(1)* $\sigma_p(H)\cap(0,\infty)=\emptyset$.
(2) For any $\lambda>0$, $\lim_{\epsilon\to0}R(\lambda\pm i\epsilon)=:R(\lambda\pm i0)$ *exists in the weak-$*$ sense, namely*

$$\exists\lim_{\epsilon\to0}(R(\lambda\pm i\epsilon)f,g)=:(R(\lambda\pm i0)f,g),\quad \forall f,g\in\mathcal{B}.$$

(3) For any compact interval $I\subset(0,\infty)$ *there exists a constant* $C>0$ *such that*

$$\|R(\lambda\pm i0)f\|_{\mathcal{B}^*}\le C\|f\|_{\mathcal{B}},\quad \forall\lambda\in I. \tag{2.5}$$

(4) For any $f,g\in\mathcal{B}$, $(0,\infty)\ni\lambda\to(R(\lambda\pm i0)f,g)$ *is continuous.*
(5) Let $E_H(\cdot)$ *be the resolution of the identity for* H. *Then* $E_H((0,\infty))L^2(\mathbf{H}^n)$ *is equal to the absolutely continuous subspace for* H.

Note that the proof of the estimate (2.5) implies the following inequality

$$\|R(z)f\|_{\mathcal{B}^*}\le C\|f\|_{\mathcal{B}},\quad \forall \mathrm{Re}\, z\in I. \tag{2.6}$$

2.3. Resolvent estimates. We shall prove Theorem 2.3 by first establishing some a-priori estimates for solutions to the equation $(H-z)u=f$, and then passing to limiting procedures. Although our method seems to be tricky, the basic idea consists in the following observation. Let us note that by virtue of Lemma 1.4.7, $u^0_\pm=R_0(\lambda\pm i0)f$ behaves like

$$\hat{u}^0_\pm(\xi,y)\sim C_\pm(\xi)y^{(n-1)/2\mp i\sqrt{\lambda}}\quad(y\to0).$$

Therefore, we infer

$$\left(y\partial_y-(\frac{n-1}{2}\mp i\sqrt{\lambda})\right)u^0_\pm=o(y^{(n-1)/2})\quad(y\to0).$$

This suggests the importance of the term $\left(y\partial_y-(\frac{n-1}{2}\mp i\sqrt{\lambda})\right)u^0_\pm$ to derive the estimates for $u^0_\pm$. We put

$$\sigma_\pm=\frac{n-1}{2}\mp i\sqrt{z}.$$

Here for $z=re^{i\theta}, r>0, -\pi<\theta<\pi$, we take the branch of $\sqrt{z}$ as $\sqrt{r}e^{i\theta/2}$.

We begin by estimating $u^0=R_0(\lambda+i0)f$. Let $(\ ,\ )_{\mathbf{h}}$, $\|\cdot\|_{\mathbf{h}}$ denote the inner product and norm of $L^2(\mathbf{R}^{n-1})$, respectively.

Lemma 2.4. *Suppose* u *satisfies* $(H_0-z)u=f$, *and let* $w_\pm=(D_y-\sigma_\pm)u$. *Let* $\varphi(y)\in C^1((0,\infty);\mathbf{R})$ *and* $0<a<b<\infty$. *Then we have*

$$\begin{aligned}
&\int_a^b(D_y\varphi+2\varphi)\|D_xu\|^2_{\mathbf{h}}\frac{dy}{y^n}+\left[\frac{\varphi(\|w_\pm\|^2_{\mathbf{h}}-\|D_xu\|^2_{\mathbf{h}})}{y^{n-1}}\right]_{y=a}^{y=b}\\
=\ &\mp2\,\mathrm{Im}\sqrt{z}\int_a^b\varphi\left(\|w_\pm\|^2_{\mathbf{h}}+\|D_xu\|^2_{\mathbf{h}}\right)\frac{dy}{y^n}\\
&+\int_a^b(D_y\varphi)\|w_\pm\|^2_{\mathbf{h}}\frac{dy}{y^n}-2\,\mathrm{Re}\int_a^b\varphi(f,w_\pm)_{\mathbf{h}}\frac{dy}{y^n}.
\end{aligned}$$

Proof. We rewrite the equation $(H_0 - z)u = f$ as

$$D_y(D_y - \sigma_\pm)u = \sigma_\mp(D_y - \sigma_\pm)u - D_x^2 u - f. \tag{2.7}$$

Taking the inner product of (2.7) and $\varphi w_\pm$, we have

$$\begin{aligned} &\int_a^b \varphi(D_y w_\pm, w_\pm)_{\mathbf{h}} \frac{dy}{y^n} \\ &= \sigma_\mp \int_a^b \varphi \|w_\pm\|_{\mathbf{h}}^2 \frac{dy}{y^n} - \int_a^b \varphi(D_x^2 u, w_\pm)_{\mathbf{h}} \frac{dy}{y^n} - \int_a^b \varphi(f, w_\pm)_{\mathbf{h}} \frac{dy}{y^n}. \end{aligned} \tag{2.8}$$

Take the real part. By integration by parts, the left-hand side is equal to

$$\begin{aligned} &\mathrm{Re} \int_a^b \varphi(D_y w_\pm, w_\pm)_{\mathbf{h}} \frac{dy}{y^n} \\ &= \left[\frac{\varphi\|w_\pm\|_{\mathbf{h}}^2}{2y^{n-1}}\right]_{y=a}^{y=b} - \frac{1}{2}\int_a^b (D_y\varphi)\|w_\pm\|_{\mathbf{h}}^2 \frac{dy}{y^n} + \frac{n-1}{2}\int_a^b \varphi\|w_\pm\|_{\mathbf{h}}^2 \frac{dy}{y^n}. \end{aligned} \tag{2.9}$$

Let us note that using

$$(-D_x^2 u, D_y u)_{\mathbf{h}} = (v, D_y v)_{\mathbf{h}} - \|v\|_{\mathbf{h}}^2, \quad v = \sqrt{D_x^2}u = y\sqrt{-\Delta_x}u,$$

we have

$$\begin{aligned} &-\mathrm{Re} \int_a^b \varphi\big(D_x^2 u, w_\pm\big)_{\mathbf{h}} \frac{dy}{y^n} \\ &= \left[\frac{\varphi\|D_x u\|_{\mathbf{h}}^2}{2y^{n-1}}\right]_{y=a}^{y=b} - \frac{1}{2}\int_a^b (D_y\varphi)\|D_x u\|_{\mathbf{h}}^2 \frac{dy}{y^n} + \left(\frac{n-3}{2} - \mathrm{Re}\,\sigma_\pm\right)\int_a^b \varphi\|D_x u\|_{\mathbf{h}}^2 \frac{dy}{y^n}. \end{aligned}$$

Apply this to the 2nd term of the right-hand side of (2.8). We then have

$$\begin{aligned} &\mathrm{Re} \int_a^b \varphi(D_y w_\pm, w_\pm)_h \frac{dy}{y^n} \\ &= (\mathrm{Re}\,\sigma_\mp)\int_a^b \varphi\|w_\pm\|_h^2 \frac{dy}{y^n} - \mathrm{Re}\int_a^b \varphi(y^2\Delta_h u, w_\pm)_h \frac{dy}{y^n} - \mathrm{Re}\int_a^b \varphi(f, w_\pm)_h \frac{dy}{y^n} \\ &= \left(\frac{n-1}{2} \mp \mathrm{Im}\sqrt{z}\right)\int_a^b \varphi\|w_\pm\|_h^2 \frac{dy}{y^n} + \left[\frac{\varphi\|D_x u\|_h^2}{2y^{n-1}}\right]_{y=a}^{y=b} \\ &\quad -\frac{1}{2}\int_a^b (D_y\varphi)\|D_x u\|_h^2 \frac{dy}{y^n} - (1 \pm \mathrm{Im}\sqrt{z})\int_a^b \varphi\|D_x u\|_h^2 \frac{dy}{y^n} - \mathrm{Re}\int_a^b \varphi(f, w_\pm)_h \frac{dy}{y^n}. \end{aligned} \tag{2.10}$$

Equating (2.9) and (2.10), we obtain the lemma. □

We shall derive estimates of the resolvent $R_0(z) = (H_0 - z)^{-1}$, when $z \in \mathbf{C} \setminus \mathbf{R}$ approaches the real axis.

Lemma 2.5. *Let* $u = R_0(z)f$. *Let* $w_\pm = (D_y - \sigma_\pm)u$, *and put for* $C^1 \ni \varphi \geq 0$ *and constants* $0 < a < b$,

$$L_\pm = \int_a^b \big(D_y\varphi + 2\varphi\big)\|D_x u\|_{\mathbf{h}}^2 \frac{dy}{y^n} + \left[\frac{\varphi(\|w_\pm\|_{\mathbf{h}}^2 - \|D_x u\|_{\mathbf{h}}^2)}{y^{n-1}}\right]_{y=a}^{y=b}, \tag{2.11}$$

$$R_\pm = \int_a^b (D_y\varphi)\|w_\pm\|_{\mathbf{h}}^2 \frac{dy}{y^n} - 2\mathrm{Re}\int_a^b \varphi(f, w_\pm)_{\mathbf{h}} \frac{dy}{y^n}. \tag{2.12}$$

Then we have the following inequality.

$$L_+ \le R_+, \quad L_- \ge R_-, \quad \text{if} \quad \mathrm{Im}\sqrt{z} \ge 0, \tag{2.13}$$

$$L_+ \ge R_+, \quad L_- \le R_-, \quad \text{if} \quad \mathrm{Im}\sqrt{z} \le 0, \tag{2.14}$$

Proof. Using Lemma 2.4, $\varphi \ge 0$, and the sign of $\mathrm{Im}\sqrt{z}$, we obtain the lemma. □

In the following, z varies over the region

$$J_\pm = \{z \in \mathbf{C}\,;\, a \le \mathrm{Re}\, z \le b,\ 0 < \pm \mathrm{Im}\, z < 1\}, \tag{2.15}$$

where $0 < a < b$ are arbitrarily chosen constants.

Lemma 2.6. *Let $u = R_0(z)f$ with $f \in \mathcal{B}$. Then, for any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that*

$$\int_0^\infty \|D_x u\|_{\mathbf{h}}^2 \frac{dy}{y^n} \le \epsilon \|u\|_{\mathcal{B}^*}^2 + C_\epsilon \|f\|_{\mathcal{B}}^2, \quad \forall z \in J_\pm.$$

Proof. Assume that $z \in J_+$. Letting $\varphi = 1$ and using (2.13), we have

$$\int_a^b \|D_x u\|_{\mathbf{h}}^2 \frac{dy}{y^n} \le \left[\frac{\|D_x u\|_{\mathbf{h}}^2 - \|w_+\|_{\mathbf{h}}^2}{2y^{n-1}}\right]_{y=a}^{y=b} + \left|\int_a^b (f, w_+)_{\mathbf{h}} \frac{dy}{y^n}\right|.$$

By Theorem 1.3 (4), $w_+, D_x u \in L^2$ for $z \notin \mathbf{R}$. Hence

$$\liminf_{y\to 0} \frac{\|w_+\|_{\mathbf{h}}^2 + \|D_x u\|_{\mathbf{h}}^2}{y^{n-1}} = 0, \quad \liminf_{y\to\infty} \frac{\|w_+\|_{\mathbf{h}}^2 + \|D_x u\|_{\mathbf{h}}^2}{y^{n-1}} = 0. \tag{2.16}$$

Therefore letting $a \to 0$ and $b \to \infty$ along suitable sequences, we have

$$\int_0^\infty \|D_x u\|_{\mathbf{h}}^2 \frac{dy}{y^n} \le \left|\int_0^\infty (f, w_+)_{\mathbf{h}} \frac{dy}{y^n}\right| \le \epsilon \|w_+\|_{\mathcal{B}^*}^2 + C_\epsilon \|f\|_{\mathcal{B}}^2.$$

Theorem 1.3 (1) yields $\|w_+\|_{\mathcal{B}^*} \le C(\|u\|_{\mathcal{B}^*} + \|f\|_{\mathcal{B}^*})$, which proves the lemma when $z \in J_+$. The case for $z \in J_-$ is proved similarly by using w_-. □

Lemma 2.7. *Let u, f be as in the previous lemma, and $w_\pm = (D_y - \sigma_\pm)u$. Then for any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that, for any $y > 0$,*

$$\frac{\|w_+\|_{\mathbf{h}}^2 - \|D_x u\|_{\mathbf{h}}^2}{y^{n-1}} \le \epsilon \|u\|_{\mathcal{B}^*}^2 + C_\epsilon \|f\|_{\mathcal{B}}^2, \quad \forall z \in J_+,$$

$$\frac{\|w_-\|_{\mathbf{h}}^2 - \|D_x u\|_{\mathbf{h}}^2}{y^{n-1}} \le \epsilon \|u\|_{\mathcal{B}^*}^2 + C_\epsilon \|f\|_{\mathcal{B}}^2, \quad \forall z \in J_-.$$

Proof. As in the previous lemma, assume that $z \in J_+$. Letting $\varphi = 1$ and using (2.13), we have

$$\frac{\|w_+\|_{\mathbf{h}}^2 - \|D_x u\|_{\mathbf{h}}^2}{y^{n-1}}\Big|_{y=b} \le \frac{\|w_+\|_{\mathbf{h}}^2 - \|D_x u\|_{\mathbf{h}}^2}{y^{n-1}}\Big|_{y=a} + C\|f\|_{\mathcal{B}} \|w_+\|_{\mathcal{B}^*}.$$

Using (2.16) and [letting $a \to 0$ along a suitable sequence, we obtain the lemma by Theorem 1.3 (1). □

Lemma 2.8. *Let u, f, $w_\pm$ be as in the previous lemma. Then, for any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that*

$$\|w_+\|_{\mathcal{B}^*} \le \epsilon \|u\|_{\mathcal{B}^*} + C_\epsilon \|f\|_{\mathcal{B}}, \quad \forall z \in J_+,$$

$$\|w_-\|_{\mathcal{B}^*} \le \epsilon \|u\|_{\mathcal{B}^*} + C_\epsilon \|f\|_{\mathcal{B}}, \quad \forall z \in J_-.$$

Proof. We divide the inequality in Lemma 2.7 by y and integrate on $(1/R, R)$. We then use Lemma 2.6 to estimate the integral of $\|D_x u\|_{\mathbf{h}}^2$, and obtain the lemma. □

Lemma 2.9. *There exists a constant $C > 0$ such that*

$$\|R_0(z)f\|_{\mathcal{B}^*} \leq C\|f\|_{\mathcal{B}}, \quad \forall z \in J_{\pm}.$$

Proof. We consider the case that $z \in J_+$, and put $\sqrt{z} = k + i\epsilon$ for $z \in J_+$. Then $\epsilon > 0$ and $k > C$ for some constant $C > 0$. Letting $w_+ = (D_y - \sigma_+)u$, we then have

$$\mathrm{Im}\, D_y(w_+, u)_{\mathbf{h}} = \mathrm{Im}\,(n - 1 + 2ik)(w_+, u)_{\mathbf{h}} - \mathrm{Im}\,(f, u)_{\mathbf{h}}. \tag{2.17}$$

This is a consequence of the formula

$$D_y(w_+, u)_{\mathbf{h}} = (D_y w_+, u)_{\mathbf{h}} + \|w_+\|_{\mathbf{h}}^2 + \left(\frac{n-1}{2} + \epsilon + ik\right)(w_+, u)_{\mathbf{h}}$$

and (2.7). We integrate (2.17). Since

$$\int_a^b D_y(w_+, u)_{\mathbf{h}} \frac{dy}{y^n} = \left[\frac{(w_+, u)_{\mathbf{h}}}{y^{n-1}}\right]_a^b + (n-1)\int_a^b (w_+, u)_{\mathbf{h}} \frac{dy}{y^n},$$

we then have

$$\mathrm{Im}\left[\frac{(w_+, u)_{\mathbf{h}}}{y^{n-1}}\right]_a^b = 2k\,\mathrm{Re}\int_a^b (w_+, u)_{\mathbf{h}} \frac{dy}{y^n} - \mathrm{Im}\int_a^b (f, u)_{\mathbf{h}} \frac{dy}{y^n}. \tag{2.18}$$

Using $w_+ = D_y u - \sigma_+ u$ and integrating by parts, we have

$$\mathrm{Re}\int_a^b (w_+, u)_{\mathbf{h}} \frac{dy}{y^n} = \frac{1}{2}\left[\frac{\|u\|_{\mathbf{h}}^2}{y^{n-1}}\right]_a^b - \epsilon\int_a^b \|u\|_{\mathbf{h}}^2 \frac{dy}{y^n}.$$

Therefore (2.18) is computed as

$$\mathrm{Im}\left[\frac{(w_+, u)_{\mathbf{h}}}{y^{n-1}}\right]_a^b = k\left[\frac{\|u\|_{\mathbf{h}}^2}{y^{n-1}}\right]_a^b - 2\epsilon k\int_a^b \|u\|_{\mathbf{h}}^2 \frac{dy}{y^n} - \mathrm{Im}\int_a^b (f, u)_{\mathbf{h}} \frac{dy}{y^n},$$

which implies

$$\mathrm{Im}\left[\frac{(w_+, u)_{\mathbf{h}}}{y^{n-1}}\right]_a^b \leq k\left[\frac{\|u\|_{\mathbf{h}}^2}{y^{n-1}}\right]_a^b + C\|f\|_{\mathcal{B}}\|u\|_{\mathcal{B}^*}.$$

Note that for $z \notin \mathbf{R}$, w_+ and u are in $L^2((0,\infty); L^2(\mathbf{R}^{n-1}); dy/y^n)$. Hence, there exists a sequence $b_1 < b_2 < \cdots \to \infty$ such that

$$\frac{|(w_+, u)_{\mathbf{h}}(b_m)| + \|u(b_m)\|_{\mathbf{h}}^2}{b_m^{n-1}} \to 0.$$

For w_+, we take $a = y < b = b_m$ to have

$$\frac{\|u(y)\|_{\mathbf{h}}^2}{y^{n-1}} \leq C_k\left(\frac{\|w_+(y)\|_{\mathbf{h}}^2}{y^{n-1}} + \frac{|(w_+, u)_{\mathbf{h}}(b_m)| + \|u(b_m)\|_{\mathbf{h}}^2}{b_m^{n-1}} + \|f\|_{\mathcal{B}}\|u\|_{\mathcal{B}^*}\right).$$

Letting $m \to \infty$, we see that

$$\frac{\|u(y)\|_{\mathbf{h}}^2}{y^{n-1}} \leq C\left(\frac{\|w_+(y)\|_{\mathbf{h}}^2}{y^{n-1}} + \|f\|_{\mathcal{B}}\|u\|_{\mathcal{B}^*}\right).$$

Dividing by y and integrating from $1/R$ to R, we have

$$\frac{1}{\log R}\int_{1/R}^{R}\|u(y)\|_{\mathbf{h}}^2\frac{dy}{y^n}\le\frac{C}{\log R}\int_{1/R}^{R}\|w_+(y)\|_{\mathbf{h}}^2\frac{dy}{y^n}+C\|f\|_{\mathcal{B}}\|u\|_{\mathcal{B}^*},$$

which implies

$$\|u\|_{\mathcal{B}^*}^2\le C\|w_+\|_{\mathcal{B}^*}^2+C\|f\|_{\mathcal{B}}\|u\|_{\mathcal{B}^*}.$$

This, together with Lemma 2.8, yields

$$\|u\|_{\mathcal{B}^*}\le C\|f\|_{\mathcal{B}},\quad \forall z\in J_+.$$

Similarly, we can prove the lemma for $z\in J_-$. □

Lemma 2.9 completes the proof of Theorem 1.4.2.

2.4. Radiation conditions and uniqueness theorem. The following theorem specifies the fastest decay order of non-trivial solutions to the Helmholtz equation $(H-\lambda)u=0$.

Theorem 2.10. *Let $\lambda>0$. If $u\in\mathcal{B}^*$ satisfies $(H-\lambda)u=0$ for $0<y<y_0$ with some $y_0>0$, and*

$$\liminf_{R\to\infty}\frac{1}{\log R}\int_{1/R}^{1}\|u(y)\|_{L^2(\mathbf{R}^{n-1})}^2\frac{dy}{y^n}=0,$$

then $u=0$ for $0<y<y_0$.

We should stress that we have only to assume the equation $(H-\lambda)u=0$ to be satisfied near $y=0$. The proof is given in the next section.

Corollary 2.11. $\sigma_p(H)\cap\big(0,\infty\big)=\emptyset.$

We say that $u\in\mathcal{B}^*$ satisfies the *outgoing radiation condition* (for σ_+), or *incoming radiation condition* (for σ_-), if the following two conditions (2.19) and (2.20) are fulfilled:

$$\lim_{R\to\infty}\frac{1}{\log R}\int_{1/R}^{1}\|(D_y-\sigma_\pm(\lambda))u(y)\|_{L^2(\mathbf{R}^{n-1})}^2\frac{dy}{y^n}=0, \tag{2.19}$$

$$\sigma_\pm(\lambda)=\frac{n-1}{2}\mp i\sqrt{\lambda}.$$

$$\lim_{R\to\infty}\frac{1}{\log R}\int_{1}^{R}\|u(y)\|_{L^2(\mathbf{R}^{n-1})}^2\frac{dy}{y^n}=0. \tag{2.20}$$

Lemma 2.12. *Assume that $\lambda>0$ and $u\in\mathcal{B}^*$ satisfies the equation $(H-\lambda)u=0$, and the outgoing or incoming radiation condition. Then $u=0$.*

Proof. We assume that u satisfies the outgoing radiation condition. We take $0\le\rho(t)\in C_0^\infty(\mathbf{R})$ satisfying $\operatorname{supp}\rho\subset(-1,1)$, $\int_{-1}^{1}\rho(t)dt=1$, and put

$$\varphi_R(y)=\chi\Big(\frac{\log y}{\log R}\Big),\quad \chi(t)=\int_{-\infty}^{t}\rho(s)ds.$$

Let $(\ , \ )_{\mathbf{h}}$ and $\|\cdot\|_{\mathbf{h}}$ denote the inner product and the norm of $L^2(\mathbf{R}^{n-1})$, respectively. We multiply the equation $(H-\lambda)u = 0$ by $\varphi_R(y)\overline{u}$ and integrate over $\mathbf{R}^{n-1}\times(0,R)$ to obtain

$$
\begin{aligned}
0 &= \mathrm{Im}\int_0^R \big((-D_y^2+(n-1)D_y+V)u, \varphi_R u\big)_{\mathbf{h}}\frac{dy}{y^n}\\
&= -\mathrm{Im}\frac{(D_y u,u)_{\mathbf{h}}}{y^{n-1}}\Big|_{y=R} + \mathrm{Im}\frac{1}{\log R}\int_0^R \rho(\frac{\log y}{\log R})\,(D_y u,u)_{\mathbf{h}}\frac{dy}{y^n}\\
&\quad + \mathrm{Im}\int_0^R (Vu,\varphi_R u)_{\mathbf{h}}\frac{dy}{y^n}.
\end{aligned}
\tag{2.21}
$$

Observe that (2.20) implies, due to Theorem 1.3 (2), that

$$
\lim_{R\to\infty}\frac{1}{\log R}\int_1^R ||D_y u||^2_{L^2(\mathbf{R}^{n-1})}\frac{dy}{y^n} = 0. \tag{2.22}
$$

Indeed, let $\psi(y)\in C^\infty(\mathbf{R}_+)$, $\psi = 1$ for $y>1$ and $\psi = 0$ for $y<1/2$. Then, with $v=\psi u$,

$$
(H-\lambda)v = f := [H,\psi]\,u \in \mathcal{B},
$$

due to Theorem 1.3 (1) and the fact, that $\mathrm{supp}(f)\subset\{1/2<y<1\}$. Thus, v satisfies conditions of Theorem 1.3 (2), which implies (2.22).

Conditions (2.20), (2.22) yield that

$$
\lim_{R\to\infty}\frac{1}{\log R}\int_{1/R}^R ||(D_y-\sigma_\pm)u(y)||_{L^2(\mathbf{R}^{n-1})}\frac{dy}{y^n} = 0. \tag{2.23}
$$

Also (2.20), (2.22) imply that

$$
\lim\inf_{y\to\infty}\frac{|(D_y u,u)_{y=a}|}{y^{n-1}} = 0.
$$

We also see that

$$
\mathrm{Im}\int_0^R (Vu,\varphi_R u)_{\mathbf{h}}\frac{dy}{y^n} \to \mathrm{Im}\int_0^\infty (Vu,u)_{\mathbf{h}}\frac{dy}{y^n} = 0.
$$

Indeed, $\int_0^\infty \big|(Vu,u)_{\mathbf{h}} d\big|y/y^n<\infty$, since $Vu\in\mathcal{X}^s$, $1/2<s<(1+\epsilon)/2$ due to (2.4) and Theorem 1.3 (5). As V is symmetric, this gives the result.

Hence, by (2.21), there is a subsequence $R_1<R_2<\cdots\to\infty$ such that

$$
\mathrm{Im}\frac{1}{\log R_j}\int_0^\infty \rho(\frac{\log y}{\log R_j})(D_y u,u)_{\mathbf{h}}\frac{dy}{y^n}\to 0.
$$

Combining this equation with (2.23), we have

$$
\lim_{j\to\infty}\frac{\sqrt{\lambda}}{\log R_j}\int_0^\infty \big(\rho(\frac{\log y}{\log R_j})u,u\big)_{\mathbf{h}}\frac{dy}{y^n} = 0,\quad \forall\rho\in C_0^\infty(\mathbf{R}).
$$

This implies that

$$
\lim_{j\to\infty}\frac{1}{\log R'_j}\int_{1/R'_j}^{R'_j}\|u(y)\|^2_{\mathbf{h}}\frac{dy}{y^n} = 0
$$

along a suitable sequence $R'_1<R'_2<\cdots\to\infty$. The lemma then follows from Theorem 2.10. □

2.5. Proof of Theorem 2.3. The assertion (1) has been proved in Corollary 2.11. Let ϵ be as in the condition (C) in Subsection 2.1, and take s such that

$$\frac{1}{2} < s < \frac{1+\epsilon}{2}.$$

Take a compact interval $I \subset (0,\infty)$ arbitrarily, and put

$$J = \{\lambda \pm i\epsilon\,;\, \lambda \in I,\ 0 < \epsilon < 1\}.$$

Lemma 2.13. *(1) There exists a constant $C > 0$ such that*

$$\sup_{z\in J} \|R(z)f\|_{\mathcal{X}^{-s}} \le C\|f\|_{\mathcal{B}}, \tag{2.24}$$

$$\sup_{z\in J} \|R(z)f\|_{\mathcal{B}^*} \le C\|f\|_{\mathcal{B}}. \tag{2.25}$$

(2) For any $\lambda > 0$ and $f \in \mathcal{B}$, the strong limit $\lim_{\epsilon\to 0} R(\lambda \pm i\epsilon)f =: R(\lambda \pm i0)f$ *exists in $\mathcal{X}^{-s}$. Moreover, $R(\lambda \pm i0)f \in \mathcal{B}^*$, and* $\lim_{\epsilon\to 0}(R(\lambda \pm i\epsilon)f, g) = (R(\lambda \pm i0)f, g)$ *for any $g \in \mathcal{B}$.*
(3) For any $f, g \in \mathcal{B}$, $R(\lambda \pm i0)f$ is an $\mathcal{X}^{-s}$-valued strongly continuous function of $\lambda > 0$, and $(R(\lambda \pm i0)f, g)$ is a continuous function of $\lambda > 0$.

Proof. If (2.24) does not hold, there exist $z_n \in J$ and $f_n \in \mathcal{B}$ satisfying

$$\|f_n\|_{\mathcal{B}} \to 0, \quad \|u_n\|_{\mathcal{X}^{-s}} = 1, \quad u_n = R(z_n)f_n.$$

These imply that

$$(H_0 - z_n)u_n = f_n - Vu_n, \tag{2.26}$$

and we can assume without loss of generality that $z_n \to \lambda \in I$. By Theorem 1.3 (6),

$$||D^\alpha u_n||_{\chi^{-s}} \le C, \quad |\alpha| \le 2.$$

Therefore, by the condition (C), $Vu_n \in \mathcal{B}$ and

$$||Vu_n||_{\mathcal{B}} \le C.$$

Returning to (2.26), this implies, due to Lemma 2.9, that

$$||u_n||_{\mathcal{B}^*} \le C. \tag{2.27}$$

Therefore, there exists a subsequence, which we continue to denote by u_n, such that $u_n \to u$ weakly in $\mathcal{X}^{-s}$.

On the other hand, applying Theorem 1.3 (4), we see that, with $|\alpha| \le 2$ and $1/2 < t, t' < s$,

$$||D^\alpha u_n||_{\chi^{-t}} \le C\left(||u_n||_{\chi^{-t}} + ||f_n||_{\chi^{-t}}\right) \le C; \tag{2.28}$$

$$||D^\alpha (u_n - u_m)||_{\chi^{-t'}} \le C\Big(||u_n - u_m||_{\chi^{-t'}} + ||f_n - f_m||_{\chi^{-t'}} + |z_n - z_m|\Big). \tag{2.29}$$

These imply, using Rellich's theorem, that there exists a subsequence such that $D^\alpha u_n \to D^\alpha u$ in χ^{-s}, $|\alpha| \le 2$ and, in particular, $||u||_{\chi^{-s}} = 1$. Then

$$u = -R_0(\lambda \pm i0)Vu, \quad Vu \in \mathcal{B},$$

and, by Corollary 1.4.8 (2) and Lemma 1.4.9, u satisfies the radiation condition. Thus, by Lemma 2.12, $u = 0$, contradicting $||u||_{\chi^{-s}} = 1$. This completes the proof of (2.24).

To prove (2.25), we have only to use the resolvent equation $R(z) = R_0(z) - R_0(z)VR(z)$, (2.24) and Lemma 2.9.

The assertion (2), (3) can be proved by the similar manner. □

The assertions (2), (3), (4) of Theorem 2.3 are now easily derived from Lemma 2.13 and the resolvent equation $R(z) = R_0(z) - R_0(z)VR(z)$. To this end, we use Theorem 1.3 (6) with $s < (1+\epsilon)/2$, (C) in the decay assumption of the metric in subsection 2.1 and Theorem 1.4.2 (3).

For the proof of (5), see [**58**] or [**62**], p. 49. □

The following lemma is a consequence of the above proof.

Lemma 2.14. *For any $f \in \mathcal{B}$ and $\lambda > 0$, $u = R(\lambda \pm i0)f$ satisfies the equation $(H - \lambda)u = f$, and the radiation condition. Conversely, any solution $u \in \mathcal{B}^*$ of the above equation satisfying the radiation condition is unique and is given by $u = R(\lambda \pm i0)f$.*

3. Growth order of solutions to reduced wave equations

3.1. Abstract differential equations. Let X be a Hilbert space and consider the following differential equation for an X-valued function $u(t)$:

$$-u''(t) + B(t)u(t) + V(t)u(t) - Eu(t) = P(t)u(t), \quad t > 0, \tag{3.1}$$

$E > 0$ being a constant. The following assumptions are imposed.

(A-1) $B(t)$ is a non-negative self-adjoint operator valued function with domain $D(B(t)) = D \subset X$ independent of $t > 0$. For each $x \in D$, the map $(0,\infty) \ni t \to B(t)x \in X$ is C^1, and there exist constants $t_0 > 0$ and $\delta > 0$ such that

$$t\frac{dB(t)}{dt} + (1+\delta)B(t) \leq 0, \quad \forall t > t_0. \tag{3.2}$$

(A-2) For any fixed t, $V(t)$ is bounded self-adjoint on X and satisfies

$$V(t) \in C^1((0,\infty); \mathbf{B}(X)), \tag{3.3}$$

$$\frac{1}{t}\|V(t)\| + \|\frac{dV(t)}{dt}\| \leq C(1+t)^{-1-\epsilon}, \quad \forall t \geq 1, \tag{3.4}$$

for some constants $C, \epsilon > 0$.
(A-3) For any fixed t, $P(t)$ is a closed (not necessarily self-adjoint) operator on X with domain $D(P(t)) \supset D$ satisfying

$$P(t)^*P(t) \leq C(1+t)^{-2-2\epsilon}\big(B(t)+1\big). \tag{3.5}$$

Moreover,

$$\mathrm{Re}\, P(t) := \frac{1}{2}\left(P(t) + P(t)^*\right)$$

is a bounded operator on X and satsifies

$$\|\mathrm{Re}\, P(t)\| \leq C(1+t)^{-1-\epsilon}, \quad \forall t > 0. \tag{3.6}$$

Theorem 3.1. *Under the above assumptions (A-1), (A-2), (A-3), if*

$$\liminf_{t\to\infty}(\|u'(t)\|_X + \|u(t)\|_X) = 0$$

holds, there exists $t_1 > 0$ such that $u(t) = 0$, $\forall t > t_1$.

The proof below is a modification of the method in [**118**] p. 29. In the following, $\|\cdot\|_X$ is simply written as $\|\cdot\|$. We put

$$(Ku)(t) = \|u'(t)\|^2 + E\|u(t)\|^2 - (B(t)u(t), u(t)) - (V(t)u(t), u(t)).$$

Lemma 3.2. *There exist constants* $C_1, T_1 > 0$ *such that*

$$\frac{d}{dt}(Ku)(t) \geq -C_1(1+t)^{-1-\epsilon}(Ku)(t), \quad \forall t > T_1.$$

Proof. By choosing ϵ small enough, we can assume that, in addition to (A-2) and (A-3),

$$\|V'(t)\| \leq C(1+t)^{-1-2\epsilon}. \tag{3.7}$$

By the equation (3.1)

$$\begin{aligned}\frac{d}{dt}(Ku)(t) &= 2\mathrm{Re}\Big[(u'', u') + E(u, u') - (Bu, u') - (Vu, u')\Big] - ((B'+V')u, u) \\ &= -2\mathrm{Re}\,(Pu, u') - ((B'+V')u, u).\end{aligned}$$

By (3.5)

$$\|Pu\| \leq C\big(1+t)^{-1-\epsilon}(\sqrt{(Bu,u)} + \|u\|\big). \tag{3.8}$$

By (3.7), there exists $t_0 = t_0(\epsilon) > 0$ such that for $t > t_0$

$$|(V'(t)u, u)| \leq \frac{\epsilon}{2}(1+t)^{-1-\epsilon}\|u\|^2.$$

By (3.2)

$$-(B'u, u) \geq \frac{1+\delta}{t}(Bu, u).$$

Putting the above estimates together we have that there is $C_\epsilon > 0$ such that for $t > t_0$

$$\begin{aligned}\frac{d}{dt}(Ku)(t) &\geq -Ct^{-1-\epsilon}(\|u'\|^2 + \|u\|\|u'\| + \frac{\epsilon}{2}\|u\|^2) + \frac{1}{t}(Bu, u) \\ &\geq -C_\epsilon t^{-1-\epsilon}\|u'\|^2 - C\epsilon t^{-1-\epsilon}\|u\|^2 + \frac{1}{t}(Bu, u).\end{aligned}$$

We rewrite the right-hand side as

$$\begin{aligned}&-C_\epsilon t^{-1-\epsilon}(\|u'\|^2 + E\|u\|^2) + (C_\epsilon E - C\epsilon)t^{-1-\epsilon}\|u\|^2 + \frac{1}{t}(Bu, u) \\ = \;&-C_\epsilon t^{-1-\epsilon}(Ku)(t) \\ &+(C_\epsilon E - C\epsilon)t^{-1-\epsilon}\|u\|^2 - C_\epsilon t^{-1-\epsilon}(Vu, u) + \big(\frac{1}{t} - \frac{C_\epsilon}{t^{1+\epsilon}}\big)(Bu, u).\end{aligned}$$

Choose C_ϵ large enough so that $C_\epsilon E - C\epsilon \geq \frac{1}{2}C_\epsilon E$. Using (3.4), choose $t_0 = t_0(\epsilon, C_\epsilon)$ such that, for $t > t_0$, $\frac{E}{2}\|u\|^2 - (Vu, u) \geq 0$, and $1 - Ct^{-\epsilon} > 0$. Thus, the 3rd line is non-negative for $t > t_0$. Hence the lemma is proved. □

Let $m > 0$ be an integer and put

$$(Nu)(t) = t\left[K(e^{d(t)}u) + \frac{m^2 - \log t}{t^{2\alpha}}\|e^{d(t)}u\|^2\right],$$

$$\frac{1}{3} < \alpha < \frac{1}{2}, \quad d(t) = \frac{m}{1-\alpha}t^{1-\alpha}.$$

Lemma 3.3. *If* $\operatorname{supp} u(t)$ *is unbounded, there exist constants* $m_1 \geq 1$, $T_2 \geq T_1$ *such that*

$$(Nu)(t) \geq 0, \quad \forall t \geq T_2, \quad \forall m \geq m_1.$$

Proof. Letting $w(t) = e^{d(t)}u(t)$, we have

$$\begin{aligned}
\frac{d}{dt}(Nu) &= Kw + t\frac{d}{dt}(Kw) + (1-2\alpha)\frac{m^2 - \log t}{t^{2\alpha}}\|w\|^2 \\
&\quad - t^{-2\alpha}\|w\|^2 + 2(m^2 - \log t)t^{1-2\alpha}\mathrm{Re}\,(w', w) \\
&= \|w'\|^2 + \Big(E + (1-2\alpha)\frac{m^2 - \log t}{t^{2\alpha}} - t^{-2\alpha}\Big)\|w\|^2 \\
&\quad - (Bw, w) - (Vw, w) + t\frac{d}{dt}(Kw) \\
&\quad + 2t^{1-2\alpha}(m^2 - \log t)\mathrm{Re}\,(w', w).
\end{aligned} \tag{3.9}$$

By direct computation,

$$\begin{aligned}
w' &= e^d u' + mt^{-\alpha}w, \\
w'' &= e^d u'' + mt^{-\alpha}e^d u' + mt^{-\alpha}w' - \alpha m t^{-\alpha-1}w \\
&= Bw + Vw - Ew + 2mt^{-\alpha}w' \\
&\quad - \left[P + (\alpha m t^{-\alpha-1} + m^2 t^{-2\alpha})\right] w.
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{d}{dt}(Kw) &= 2\mathrm{Re}\,(w'' + Ew - Vw - Bw, w') - \big((B' + V')w, w\big) \\
&= 4mt^{-\alpha}\|w'\|^2 - 2(\alpha m t^{-\alpha-1} + m^2 t^{-2\alpha})\mathrm{Re}\,(w, w') \\
&\quad - \big((B' + V')w, w\big) - 2\mathrm{Re}\,(Pw, w').
\end{aligned} \tag{3.10}$$

By (3.9) and (3.10) we have

$$\begin{aligned}
&\frac{d}{dt}(Nu) \\
&= (4mt^{1-\alpha} + 1)\|w'\|^2 + \{E + (1-2\alpha)t^{-2\alpha}(m^2 - \log t) - t^{-2\alpha}\}\|w\|^2 \\
&\quad - 2(\alpha m t^{-\alpha} + t^{1-2\alpha}\log t)\ \mathrm{Re}\ (w, w') - ((V + tV')w, w) \\
&\quad - ((tB' + B)w, w) - 2t\mathrm{Re}\,(Pw, w') \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

For large $t > 0$, I_1 is estimated from below as

$$I_1 \geq (4mt^{1-\alpha} + 1)\|w'\|^2 + \big(\frac{E}{2} + (1-2\alpha)t^{-2\alpha}m^2\big)\|w\|^2.$$

By (3.4), I_2 is estimated from below as

$$\begin{aligned}
I_2 &\geq -2(\alpha m t^{-\alpha} + t^{1-2\alpha}\log t)\|w\|\|w'\| - Ct^{-\epsilon}\|w\|^2 \\
&\geq -\epsilon m^2 t^{-2\alpha}\|w\|^2 - C_\epsilon\|w'\|^2 \\
&\quad -2t^{1-2\alpha}\log t\|w\|\|w'\| - Ct^{-\epsilon}\|w\|^2.
\end{aligned}$$

By (3.2), I_3 is stimated from below as

$$I_3 \geq \delta(Bw, w) - 2t\|Pw\| \cdot \|w'\|.$$

Using (3.8), we estimate the 2nd term as

$$2t\|Pw\| \cdot \|w'\| \leq \frac{1}{2}\|w'\|^2 + Ct^{-\epsilon}((Bw, w) + \|w\|^2).$$

Therefore for large t, we have

$$I_3 \geq -\frac{1}{2}\|w'\|^2 - Ct^{-\epsilon}\|w\|^2.$$

Putting the above estimates together, we then have

$$\frac{d}{dt}(Nu) \geq \frac{7}{2}mt^{1-\alpha}\|w'\|^2 + \frac{E}{3}\|w\|^2 - 2t^{1-2\alpha}\log t\,\|w\|\|w'\|.$$

Finally, we use the inequality

$$t^{1-2\alpha}\log t\|w\|\|w'\| \leq \epsilon t^{1-\alpha}\|w'\|^2 + C_\epsilon t^{1-3\alpha}(\log t)^2\|w\|^2$$

and $1 - 3\alpha < 0$. Then there is $t_0 > 0$ independent of m such that

$$\frac{d}{dt}(Nu)(t) \geq 3mt^{1-\alpha}\|w'\|^2 + \frac{E}{4}\|w\|^2 \geq 0 \tag{3.11}$$

for $t > t_0$.

On the other hand, $Nu(t)$ can be rewritten as

$$\begin{aligned}(Nu)(t) = te^{2d}\big[&\|mt^{-\alpha}u + u'\|^2 + E\|u\|^2 \\ &- (Bu, u) - (Vu, u) + t^{-2\alpha}(m^2 - \log t)\big]\|u\|^2 \\ = te^{2d}\big[&2t^{-2\alpha}\|u\|^2m^2 + 2t^{-\alpha}\mathrm{Re}\,(u, u')m \\ &+ (Ku - t^{-2\alpha}\|u\|^2\log t)\big].\end{aligned} \tag{3.12}$$

By the assumption of the lemma, $\operatorname{supp} u(t)$ is unbounded. Therefore, there is $T_2 > t_0$ such that $\|u(T_2)\| > 0$. By choosing m_1 large enough, we then have

$$(Nu)(T_2) > 0, \quad \forall m > m_1. \tag{3.13}$$

The inequalities (3.11) and (3.13) prove the lemma. □

Proof of Theorem 3.1. We show that if $\operatorname{supp} u(t)$ is unbounded,

$$\liminf_{t\to\infty}(\|u'(t)\|^2 + \|u(t)\|^2) > 0 \tag{3.14}$$

holds. We first consider the case in which there exists a sequence $t_n \to \infty$ such that $(Ku)(t_n) > 0$ $(n = 1, 2, \cdots)$. Let T_1 be as in Lemma 3.2. Then for some $T > T_1$, $(Ku)(T) > 0$. We show that $(Ku)(t) \geq 0$, $\forall t > T$. In fact Lemma 3.2 implies

$$\frac{d}{dt}\left\{\exp\left(C_1\int_T^t(1+s)^{-1-\epsilon}ds\right)(Ku)(t)\right\} \geq 0, \quad \forall t > T.$$

Hence,

$$(Ku)(t) \geq \exp\left(-C_1\int_T^t(1+s)^{-1-\epsilon}ds\right)(Ku)(T), \quad \forall t > T.$$

This then implies that, for $t > t(E)$,

$$\begin{aligned}\|u'(t)\|^2 + E\|u(t)\|^2 &= Ku(t) + (B(t)u(t), u(t)) + (V(t)u(t), u(t)) \\ &\geq \exp\left(-C_1\int_T^t(1+s)^{-1-\epsilon}ds\right)(Ku)(T) \\ &\quad - CEt^{-\epsilon}\|u(t)\|^2.\end{aligned}$$

Therefore, we arrive at

$$\liminf_{t\to\infty}(\|u'(t)\|^2 + \|u(t)\|^2) \geq \frac{1}{2}\exp\left(-C_1\int_T^\infty (1+s)^{-1-\epsilon}ds\right)(Ku)(T) > 0.$$

We next consider the case in which $(Ku)(t) \leq 0$ for all t large enough. Lemma 3.3 and (3.12) show that, for large t,

$$2t^{-2\alpha}\|u(t)\|^2 m^2 + 2t^{-\alpha}\mathrm{Re}\,(u(t), u'(t))m - t^{-2\alpha}\|u(t)\|^2\log t \geq 0,$$

which together with

$$\frac{d}{dt}\|u(t)\|^2 = 2\mathrm{Re}\,(u(t), u'(t)),$$

yields, for large $t > 0$, that

$$\frac{d}{dt}\|u(t)\|^2 \geq t^{-\alpha}\left(\frac{1}{m}\log t - 2m\right)\|u(t)\|^2 \geq 0. \tag{3.15}$$

Since the support of $u(t)$ is unbounded, by choosing T large enough so that $\|u(T)\| > 0$. In view of (3.15), we then have

$$\|u(t)\| \geq \|u(T)\| > 0, \quad \forall t > T,$$

which proves (3.14). □

3.2. Canonical form. In order to apply Theorem 3.1 to the operator H in the previous section, we transform the metric ds^2 into the following canonical form.

Theorem 3.4. *Let ds^2 be the Riemannian metric satisfying the condition (C). Choose a sufficiently small $y_0 > 0$. Then there exists a diffeomorphism $(x, y) \to (\overline{x}, \overline{y})$ in the region $0 < y < y_0$ such that*

$$|\partial_{\overline{x}}^\alpha D_{\overline{y}}^\beta(\overline{x} - x)| \leq C_{\alpha\beta}(1 + d_h(x, y))^{-\min(|\alpha|+\beta,1)-1-\epsilon/2}, \quad \forall\alpha, \beta,$$

$$|\partial_{\overline{x}}^\alpha D_{\overline{y}}^\beta\Big(\frac{\overline{y} - y}{\overline{y}}\Big)| \leq C_{\alpha\beta}(1 + d_h(x, y))^{-\min(|\alpha|+\beta,1)-1-\epsilon/2}, \quad \forall\alpha, \beta,$$

and in the $(\overline{x}, \overline{y})$ coordinate system, the Riemannian metric takes the form

$$ds^2 = (\overline{y})^{-2}\left((d\overline{x})^2 + (d\overline{y})^2 + \sum_{i,j=1}^{n-1} b_{ij}(\overline{x}, \overline{y})d\overline{x}^i d\overline{x}^j\right).$$

Here $b_{ij}(\overline{x}^i, \overline{x}^j)$ satisfies the condition (C) with ϵ replaced by $\epsilon/2$.

The point is that there is no cross term $d\overline{x}^i d\overline{y}$. The proof is a slight modification of the one given in Chap. 4, §2. This theorem also holds for the asymptotically hyperbolic ends with regular infinity to be discussed in Chap. 3, §2.

Let us prove Theorem 2.10. In the coordinate system of Theorem 3.4, (denoting $(\overline{x}, \overline{y})$ by (x, y)), the equation $(-\Delta_g - \frac{(n-1)^2}{4} - \lambda)u = 0$ becomes

$$\Big(-\frac{1}{\sqrt{g}}\partial_y\big(\sqrt{g}g^{nn}\partial_y\big) - \sum_{i,j=1}^{n-1}\frac{1}{\sqrt{g}}\partial_{x_i}\big(\sqrt{g}g^{ij}\partial_{x_j}\big) - \frac{(n-1)^2}{4} - \lambda\Big)u = 0.$$

This is rewritten as

$$\Big(-D_y^2 + hD_y - \sum_{i,j=1}^{n-1} D_{x_i}h^{ij}D_{x_j} - \frac{(n-1)^2}{4} + Q - \lambda\Big)u = 0,$$

where $Q = \sum_{i=1}^{n-1} b_i(x,y)D_i + c(x,y)$. Here $h-(n-1), h^{ij}-\delta^{ij}$ and Q satisfy the condition (C), since for y close to 0, $d_h(x,y)$ and $\rho(x,y)$ are equivalent. Putting $t = -\log y$ and $u = v\exp(-\frac{1}{2}\int_{t_0}^t h(x,e^s)ds)$, we have

$$(-\partial_t^2 + B(t) - \lambda)v = P(t)v,$$

where

$$B(t) = -e^{-2t}\sum_{i,j=1}^{n-1} \partial_{x_i}(\delta_{ij} + a_{ij}(t,x))\partial_{x_j},$$

$$P(t) = -e^{-t}\sum_{i=1}^{n-1} b_i(t,x)\partial_{x_i} + c(t,x),$$

and, for large $t > 0$, a_{ij}, b_i, c satisfy

$$|\partial_x^\alpha \partial_t^\beta m(t,x)| \leq C_{\alpha\beta}(1+t)^{-\beta-1-\epsilon}, \quad \forall\alpha,\beta.$$

We have, therefore, for large $t > 0$

$$tB'(t) + 2B(t) = -\sum i,j = 1^{n-1}\partial_{x_i}e^{-2t}\{(-2t+2)(\delta_{ij}+a_{ij}) + \partial_t a_{ij})\}\partial_{x_j} \leq 0,$$

Hence, with $X = L^2(\mathbf{R}^{n-1})$, the assumption (3.2) is satisfied. Rewriting $P(t)^*P(t)$ as

$$P(t)^*P(t) = \sum_{|\alpha|\leq 2} a_\alpha(t,x)(D_x)^\alpha, \quad D_x = e^{-t}\partial_x,$$

we have, for any $\varphi \in C_0^\infty(\mathbf{R}^{n-1})$,

$$\begin{aligned}(P(t)^*P(t)\varphi,\varphi) &\leq C(1+t)^{-2-2\epsilon}\Big(e^{-2t}\|\partial_x\varphi\|^2 + \|\varphi\|^2\Big) \\ &\leq C(1+t)^{-2-2\epsilon}\Big((B(t)\varphi,\varphi) + (\varphi,\varphi)\Big),\end{aligned}$$

which proves (3.5). Note that as $t \to \infty$, $y \to 0$ and

(3.16) $$\exp(-\frac{1}{2}\int_{t_0}^t h(x,e^s)ds) = y^{(n-1)/2}\Big(1 + O(|\log y|^{-1})\Big).$$

Our next goal is to show that the condition in Theorem 3.1 is satisfied. To this end, we return to the proof of Theorem 1.3 (2). Take $\chi(t) \in C_0^\infty(\mathbf{R})$ such that $\chi(t) = 1$ for $-1 < t < -1/2$, and $\chi(t) = 0$ for $t < -2$ or $t > -1/4$. Take $\psi \in C_0^\infty(\mathbf{R})$ such that $\psi = 1$ on $\operatorname{supp}\chi$, and $\psi(t) = 0$ for $t > 0$ or $t < -3$. Then the estimate (1.15) is valid for this choice of χ and ψ. Following the arguments after this inequality, we obtain

$$\liminf_{R\to\infty}\frac{1}{\log R}\int_{R^{-1}}^{R^{-1/2}} \|D_i u(y)\|^2 \frac{dy}{y^n} = 0$$

if the condition of Theorem 2.10 is satisfied. This implies that

$$\liminf_{y\to 0}\frac{\|D_y u(y)\|^2 + \|u(y)|^2}{y^{n-1}} = 0.$$

Since $t = -\log y$, it follows from this formula together with (3.16) that

$$\liminf_{t\to\infty}(\|v'(t)\| + \|v(t)\|) = 0.$$

Therefore, by Theorem 3.1, $v(t) = 0$ for large t, i.e. $u(y) = 0$ for small y. By the unique continuation theorem, this in turn imples that $u(y) = 0$ for $y < y_0$. □

3.3. Asymptotically Euclidean metric. Let us remark that Theorem 3.1 also applies to asymptotically Euclidean metrics on $\mathbf{R}^n$. In fact, given a metric $g_{ij}(x)$ satisfying

$$|\partial_x^\alpha (g_{ij}(x) - \delta_{ij})| \leq C_\alpha (1 + |x|)^{-|\alpha|-1-\epsilon_0}, \quad \forall \alpha,$$

one can construct a diffeomorphism near infinity such that this metric is transformed into

$$(dr)^2 + r^2 h(r, \omega, d\omega), \quad r > r_0, \quad \omega \in S^{n-1},$$

where $h(r, \omega, d\omega)$ is a positive definite metric on S^{n-1}, and behaves like $h_0(\omega, d\omega)$ at infinity, where $h_0(\omega, d\omega)$ is the standard metric on S^{n-1} (see Appendix A, §2).

4. Abstract theory for spectral representations

4.1. Basic ideas. Let $H = \int_{-\infty}^{\infty} \lambda dE(\lambda)$ be a self-adjoint operator on a Hilbert space $\mathcal{H}$, and I an open interval contained in $\sigma_{ac}(H)$. Let $\mathbf{h}$ be an auxiliary Hilbert space and $\widehat{\mathcal{H}} = L^2(I; \mathbf{h}; \rho(\lambda)d\lambda)$ the Hilbert space of all $\mathbf{h}$-valued L^2-functions on I with respect to the measure $\rho(\lambda)d\lambda$. By a spectral representation of H on I, we mean a unitary operator $U : E(I)\mathcal{H} \to \widehat{\mathcal{H}}$ such that

$$(UHf)(\lambda) = \lambda (Uf)(\lambda), \quad \forall f \in D(H), \quad \forall \lambda \in I.$$

We mainly consider the following situation. There exist Banach spaces $\mathcal{H}_+, \mathcal{H}_-$ such that $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$ and for $\lambda \in I$, $\lim_{\epsilon \downarrow 0}(H - \lambda \mp i\epsilon)^{-1}$ exists as a bounded operator in $\mathbf{B}(\mathcal{H}_+; \mathcal{H}_-)$. For the limits $(H - (\lambda \pm i0))^{-1}$ one can associate the operators $U_\pm(\lambda) \in \mathbf{B}(\mathcal{H}_+; \mathbf{h})$ and the spectral representations $U_\pm$ satisfying

$$(U_\pm f)(\lambda) = U_\pm(\lambda) f, \quad \forall \lambda \in I, \quad \forall f \in \mathcal{H}_+.$$

Then there is a unitary operator $\widehat{S}(\lambda)$ on $\mathbf{h}$ such that

$$U_+(\lambda) = \widehat{S}(\lambda) U_-(\lambda), \quad \forall \lambda \in I.$$

This $\widehat{S}(\lambda)$ is called the scattering matrix or S-matrix. The two limits $\lim_{\epsilon \downarrow 0}(H - \lambda \mp i\epsilon)^{-1}$ appear naturally in computing the limit $\lim_{t \to \pm\infty} e^{-itH}$. Hence, the S-marix is closely related with the asymptotic behavior of solutions to the time-dependent Schrödinger equation $i\partial_t u = Hu$. However, the scattering matrix depends on the spectral representations $U_\pm$ so that there exist apparently different S-matrices for the same operator H. In this and the next sections, we shall introduce three kinds of S-matrices and study their relationships in the case of $\mathbf{R}^n$ and $\mathbf{H}^n$. We begin with an abstract framework.

4.2. Stationary wave operators. Assume that we are given a Hilbert space $\mathcal{H}$ and Banach spaces $\mathcal{H}_\pm$ with norms $\| \cdot \|$, and $\| \cdot \|_\pm$ satisfying

$$\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-, \quad \|f\|_- \leq \|f\| \leq \|f\|_+, \quad \forall f \in \mathcal{H}_+.$$

We also assume that the above inclusions are dense, and that the inner product $(\, , \,)$ of $\mathcal{H}$ is naturally identified with the coupling of $\mathcal{H}_+$ and $\mathcal{H}_-$. This means that there exists an isometry $T : \mathcal{H}_- \to (\mathcal{H}_+)^*$ such that

$$\langle f, Tu \rangle = (f, u), \quad \forall f \in \mathcal{H}_+, \quad \forall u \in \mathcal{H},$$

where $\langle f, v \rangle$ denotes the value $v(f)$ of $v \in (\mathcal{H}_+)^*$ for $f \in \mathcal{H}_+$. In this case we simply write $\mathcal{H}_- = (\mathcal{H}_+)^*$.

Let $H_j, j = 1, 2$, be self-adjoint operators on $\mathcal{H}$ such that $D(H_1) = D(H_2)$. For $j = 1, 2$, we put $R_j(z) = (H_j - z)^{-1}$. Since $D(H_1) = D(H_2)$, we have

$$(H_2 - H_1)R_j(z) \in \mathbf{B}(\mathcal{H};\mathcal{H}), \quad z \notin \mathbf{R}. \tag{4.1}$$

Now for $j = 1, 2$, we assume the following:

(A-1) *For any* $\varphi(\lambda) \in C_0^\infty(\mathbf{R})$, $\varphi(H_j)\mathcal{H}_+ \subset \mathcal{H}_+$.

(A-2) *There exists an open set* $I \subset \mathbf{R}$ *such that* $\sigma_p(H_j) \cap I = \emptyset$, *and the following strong limit exists*

$$\lim_{\epsilon\to 0} R_j(\lambda \pm i\epsilon) =: R_j(\lambda \pm i0) \in \mathbf{B}(\mathcal{H}_+;\mathcal{H}_-), \quad \forall\lambda \in I.$$

Moreover for any $f \in \mathcal{H}_+$, $I \ni \lambda \to R_j(\lambda \pm i0)f \in \mathcal{H}_-$ *is strongly continuous.*

(A-3) *We put* $G_{jk}(z) = (H_j - z)R_k(z)$ *for* $z \notin \mathbf{R}$, *and assume that for* $\lambda \in I$, $\epsilon > 0$ *there exists a strong limit*

$$\lim_{\epsilon\to 0} G_{jk}(\lambda \pm i\epsilon) =: G_{jk}(\lambda \pm i0) \in \mathbf{B}(\mathcal{H}_+;\mathcal{H}_+).$$

Furthermore for any $f \in \mathcal{H}_+$, $I \ni \lambda \to G_{jk}(\lambda \pm i0)f \in \mathcal{H}_+$ *is strongly continuous.*

We first introduce an operator which shows the similarity of H_1 and H_2. Let $E_j(\lambda)$ be the spectral measure for H_j, and for $\lambda \in I$, put

$$E_j'(\lambda) = \frac{1}{2\pi i}\left(R_j(\lambda + i0) - R_j(\lambda - i0)\right).$$

By the assumption (A-2), $E_j'(\lambda) \in \mathbf{B}(\mathcal{H}_+;\mathcal{H}_-)$. Now for any compact interval $e \subset I$ and $f \in \mathcal{H}_+$, we define

$$\Omega_{jk}^{(\pm)}(e)f = \int_e E_j'(\lambda)G_{jk}(\lambda \pm i0)f d\lambda.$$

This is called the stationary wave operator. By the above assumptions, $\Omega_{jk}^{(\pm)}(e) \in \mathbf{B}(\mathcal{H}_+;\mathcal{H}_-)$. However, we have the following stronger results. Let us recall one terminology. For two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, closed subspaces $S_1 \subset \mathcal{H}_1$ and $S_2 \subset \mathcal{H}_2$ and $U \in \mathbf{B}(\mathcal{H}_1;\mathcal{H}_2)$, we say that U is a partial isometry from $\mathcal{H}_1$ to $\mathcal{H}_2$ with initial set S_1 and final set S_2 if $U : S_1 \to S_2$ is unitary and $U : S_1^\perp \to 0$. U is a partial isometry if and only if U^*U and UU^* are orthogonal projections onto its initial set S_1 and final set S_2, respectively.

Theorem 4.1. *Let* e *be any compact interval in* I.
(1) $\Omega_{jk}^{(\pm)}(e)$ *is uniquely extended to a bounded operator on* $\mathcal{H}$, *and is a partial isometry with initial set* $E_k(e)\mathcal{H}$ *and final set* $E_j(e)\mathcal{H}$.
(2) $(\Omega_{jk}^{(\pm)}(e))^* = \Omega_{kj}^{(\pm)}(e)$, *where* * *means the adjoint in* $\mathcal{H}$.
(3) $\Omega_{jk}^{(\pm)}(e)$ *intertwines* H_j *and* H_k. *That is, for any bounded Borel function* $\varphi(\lambda)$,

$$\varphi(H_j)\Omega_{jk}^{(\pm)}(e) = \Omega_{jk}^{(\pm)}(e)\varphi(H_k).$$

Theorem 4.1 is proved through a series of Lemmas.

Lemma 4.2. *Let* $f(\lambda), g(\lambda)$ *be* $\mathcal{H}_+$*-valued bounded measurable functions on* I, *and* e, e' *compact intervals in* I. *We put*

$$\varphi = \int_e E_j'(\lambda)f(\lambda)d\lambda, \quad \psi = \int_{e'} E_j'(\lambda)g(\lambda)d\lambda.$$

Then $\varphi, \psi \in \mathcal{H}$ *and*

$$(\varphi, \psi) = \int_{e\cap e'} (E_j'(\lambda) f(\lambda), g(\lambda)) d\lambda.$$

Proof. If $f(\lambda), g(\lambda)$ are constant functions f and g, by Stone's formula, $\varphi = E_j(e)f, \psi = E_j(e')g$. Hence,

$$(\varphi, \psi) = (E_j(e \cap e')f, g) = \int_{e\cap e'} (E_j'(\lambda) f, g) d\lambda.$$

If $f(\lambda)$, $g(\lambda)$ are step functions, i.e. $f(\lambda) = \sum_n \chi_n(\lambda) f_n, g(\lambda) = \sum_n \chi_n(\lambda) g_n$, $\chi_n(\lambda)$ being a characteristic function of the interval e_n, φ and ψ are written as

$$\varphi = \sum_n E_j(e \cap e_n) f_n, \quad \psi = \sum_n E_j(e' \cap e_n) g_n.$$

Therefore,

$$\begin{aligned}(\varphi, \psi) &= \sum_{m,n} (E_j(e \cap e' \cap e_m \cap e_n) f_m, g_n) \\ &= \sum_{m,n} \int_{e\cap e' \cap e_m \cap e_n} (E_j'(\lambda) f_m, g_n) d\lambda \\ &= \int_{e\cap e'} (E_j'(\lambda) f(\lambda), g(\lambda)) d\lambda.\end{aligned}$$

Hence, the lemma holds for step functions.

Let $f(\lambda), g(\lambda)$ be bounded measurable functions, i.e. we can approximate them by step functions $f_m(\lambda), g_n(\lambda)$ such that

$$\lim_{m\to\infty} \|f(\lambda) - f_m(\lambda)\|_+ = 0 \quad a.e. \tag{4.2}$$

and similarly for g. We put

$$\varphi_m = \int_e E_j'(\lambda) f_m(\lambda) d\lambda, \quad \psi_n = \int_{e'} E_j'(\lambda) g_n(\lambda) d\lambda.$$

Then we have

$$\|\varphi_m - \varphi_{m'}\|^2 = \int_e (E_j'(\lambda)(f_m(\lambda) - f_{m'}(\lambda), f_m(\lambda) - f_{m'}(\lambda)) d\lambda \to 0,$$

when $m, m' \to \infty$. Indeed, assumption (A-2) and boundedness of f imply that the integrand is uniformly bounded with respect to m, m'. Also (4.2) implies that this integrand tends to 0 a.e. By Lebesgue's theorem, the result follows.

Thus, the sequence $\{\varphi_m\}$ converges to φ in $\mathcal{H}$ and similaly, $\{\psi_n\}$ converges to ψ. Moreover, letting $m, n \to \infty$ in the formula

$$(\varphi_m, \psi_n) = \int_{e\cap e'} (E_j'(\lambda) f_m(\lambda), g_n(\lambda)) d\lambda,$$

we complete the proof of the lemma. □

Lemma 4.3. *If* $f, g \in \mathcal{H}_+$ *and* e, e' *are compact intervals in* I, *we have*

$$\Omega_{jk}^{(\pm)}(e) f, \ \Omega_{jk}^{(\pm)}(e') g \in \mathcal{H},$$

$$(\Omega_{jk}^{(\pm)}(e) f, \Omega_{jk}^{(\pm)}(e') g) = (E_k(e \cap e') f, g).$$

Proof. By Lemma 4.2

$$(\Omega_{jk}^{(\pm)}(e)f, \Omega_{jk}^{(\pm)}(e')g) = \int_{e\cap e'} (E_j'(\lambda)G_{jk}(\lambda \pm i0)f, G_{jk}(\lambda \pm i0)g)d\lambda.$$

Using the resolvent equation, we have

$$\begin{aligned} G_{jk}^*(\lambda \pm i\epsilon)\frac{1}{2\pi i}[R_j(\lambda + i\epsilon) - R_j(\lambda - i\epsilon)]G_{jk}(\lambda \pm i\epsilon) \\ = \frac{1}{2\pi i}[R_k(\lambda + i\epsilon) - R_k(\lambda - i\epsilon)]. \end{aligned} \tag{4.3}$$

Hence,

$$\begin{aligned} &\left(\frac{1}{2\pi i}[R_j(\lambda + i\epsilon) - R_j(\lambda - i\epsilon)]G_{jk}(\lambda \pm i\epsilon)f, G_{jk}(\lambda \pm i\epsilon)g\right) \\ &\qquad = \left(\frac{1}{2\pi i}[R_k(\lambda + i\epsilon) - R_k(\lambda - i\epsilon)]f, g\right). \end{aligned}$$

Letting $\epsilon \to 0$, we finally obtain

$$(E_j'(\lambda)G_{jk}(\lambda \pm i0)f, G_{jk}(\lambda \pm i0)g) = (E_k'(\lambda)f, g), \tag{4.4}$$

which proves the lemma. □

By Lemma 4.3, $\Omega_{jk}^{(\pm)}(e)$ is a partial isometry on $\mathcal{H}$ with initial set $E_k(e)\mathcal{H}$.

Lemma 4.4. *For any compact interval $e \subset I$, we have $(\Omega_{jk}^{(\pm)}(e))^* = \Omega_{kj}^{(\pm)}(e)$.*

Proof. Since $G_{kj}^*(z)G_{jk}^*(z) = 1$, by multiplying (4.3) by $G_{kj}^*(\lambda \pm i\epsilon)$, we have

$$\begin{aligned} &\frac{1}{2\pi i}[R_j(\lambda + i\epsilon) - R_j(\lambda - i\epsilon)]G_{jk}(\lambda \pm i\epsilon) \\ &= G_{kj}^*(\lambda \pm i\epsilon)\frac{1}{2\pi i}[R_k(\lambda + i\epsilon) - R_k(\lambda - i\epsilon)] \end{aligned}$$

Letting $\epsilon \to 0$, we have for $f, g \in \mathcal{H}_+$

$$(f, E_j'(\lambda)G_{jk}(\lambda \pm i0)g) = (E_k'(\lambda)G_{kj}(\lambda \pm i0)f, g), \tag{4.5}$$

which proves the lemma. □

This lemma implies that the final set of $\Omega_{jk}^{(\pm)}(e)$ is the initial set of $\Omega_{kj}^{(\pm)}(e)$, i.e. $\Omega_{jk}^{(\pm)}(e)$ is a partial isometry with initial set $E_k(e)\mathcal{H}$ and final set $E_j(e)\mathcal{H}$.

Lemma 4.5. *For any compact intervals $e, e' \subset I$, we have $E_j(e')\Omega_{jk}^{(\pm)}(e) = \Omega_{jk}^{(\pm)}(e)E_k(e')$.*

Proof. Lemma 4.2 yields for $f, g \in \mathcal{H}_+$

$$\begin{aligned} (E_j(e')\Omega_{jk}^{(\pm)}(e)f, g) &= (\Omega_{jk}^{(\pm)}(e)f, E_j(e')g) \\ &= \int_{e\cap e'} (E_j'(\lambda)G_{jk}(\lambda \pm i0)f, g)d\lambda. \end{aligned}$$

By (4.5) the right-hand side is equal to

$$\begin{aligned} \int_{e\cap e'} (f, E_k'(\lambda)G_{kj}(\lambda \pm i0)g)d\lambda &= (f, E_k(e')\Omega_{kj}^{(\pm)}(e)g) \\ &= (\Omega_{jk}^{(\pm)}(e)E_k(e')f, g), \end{aligned}$$

which proves the lemma. □

The assertion (3) of Theorem 4.1 is a direct consequence of the above lemma. Approximating I by compact intervals, we define $\Omega_{jk}^{(\pm)}(I)$.

4.3. Time-dependent wave operators. We consider the relation between stationary and time-dependent wave operators. We impose a new assumption.

(A-4) *For any open set $e \subset I$, there is a set $\mathcal{D}_e \subset \mathcal{H}_+ \cap E_1(e)\mathcal{H}$, which is assumed to be dense in $E_1(e)\mathcal{H}$, such that for any $f \in \mathcal{D}_e$*

$$\int_{-\infty}^{\infty} \|(H_2 - H_1)e^{-itH_1}f\|dt < \infty.$$

Theorem 4.6. *Under the assumptions (A-1) $\sim$ (A-4), for any open set $e \subset I$, the strong limit*

$$\mathrm{s} - \lim_{t \to \pm\infty} e^{itH_2}e^{-itH_1}E_1(e) =: W_{21}^{(\pm)}(e)$$

in $\mathcal{H}$ exists and $\Omega_{21}^{(\pm)}(e) = W_{21}^{(\pm)}(e)$.

Proof. The assumption (A-4) implies that, for $f \in \mathcal{D}_e$,

$$\int_{-\infty}^{\infty} \|\frac{d}{dt}\left(e^{itH_2}e^{-itH_1}f\right)\|dt < \infty$$

holds. Hence there exist the limits $\mathrm{s} - \lim_{t\to\pm\infty} e^{itH_2}e^{-itH_1}E_1(e)f$ and, therefore, by the density of $\mathcal{D}_e$ the existence of $W_{21}^{(\pm)}(e)$.

To prove $\Omega_{21}^{(\pm)}(e) = W_{21}^{(\pm)}(e)$ for any open set $e \subset I$, it suffices, due to Lemma 4.3, to consider relatively compact sets e.

Let $V_{21} = H_2 - H_1$. For $f \in \mathcal{D}_e$ we have

$$W_{21}^{(+)}(e)f = f + i\int_0^{\infty} e^{itH_2}V_{21}e^{-itH_1}f dt.$$

Hence, for $f \in \mathcal{D}_e, g \in \mathcal{H}$,

$$(W_{21}^{(+)}(e)f, g) = (f, g) + \lim_{\epsilon\to 0} i\int_0^{\infty} (e^{itH_2}V_{21}e^{-itH_1}f, e^{-2\epsilon t}g)dt. \tag{4.6}$$

Using the following relations

$$R(\lambda + i\epsilon) = i\int_0^{\infty} e^{it(\lambda+i\epsilon-H)}dt, \quad R(\lambda - i\epsilon) = -i\int_{-\infty}^{0} e^{it(\lambda-i\epsilon-H)}dt$$

and Plancherel's formula for the Fourier transform, we have for $f \in \mathcal{D}_e$ and $g \in \mathcal{H}$

$$\begin{aligned} & i\int_0^{\infty}(e^{itH_2}V_{21}e^{-itH_1}f, e^{-2\epsilon t}g)dt \\ & = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}(V_{21}R_1(\lambda+i\epsilon)f, R_2(\lambda+i\epsilon)g)d\lambda. \end{aligned} \tag{4.7}$$

Here we should note that $\|V_{21}R(\cdot + i\epsilon)f\|_{\mathcal{H}}, \|R_2(\cdot + i\epsilon)g\|_{\mathcal{H}} \in L^2(\mathbf{R})$, hence the integral of the right-hand side is absolutely convergent. To see this, we have only to note that

$$\|R_j(\lambda + i\epsilon)h\|^2 = \int_{-\infty}^{\infty} \frac{1}{(\mu-\lambda)^2 + \epsilon^2} d_\mu(E_j(\mu)h, h),$$

$$V_{21}R_1(\lambda + i\epsilon)f = V_{21}(H_1 + i)^{-1}R_1(\lambda + i\epsilon)(H_1 + i)f,$$

and $(H_1 + i)f \in E_1(e)\mathcal{H}$, also $V_{21}(H_1+i)^{-1} \in \mathbf{B}(\mathcal{H};\mathcal{H})$ by (4.1).

We now let

$$\delta_2(\lambda,\epsilon) = \frac{1}{2\pi i}\big(R_2(\lambda+i\epsilon) - R_2(\lambda - i\epsilon)\big),$$

and prove that, if $f \in \mathcal{D}_e$ and g is such that $d_\mu(E_2(\mu)g,g)$ is compactly supported,

$$(4.8)\quad i\int_0^\infty (e^{itH_2}V_{21}e^{-itH_1}f, e^{-2\epsilon t}g)dt = \lim_{N\to\infty}\int_{-N}^{N}(\delta_2(\lambda,\epsilon)V_{21}R_1(\lambda+i\epsilon)f,g)d\lambda.$$

Indeed, by using the identity $R_2(z) - R_1(z) = -R_2(z)V_{21}R_1(z)$, we have

$$\begin{aligned} &-\frac{1}{2\pi i}\int_{-N}^{N}(V_{21}R_1(\lambda+i\epsilon)f, R_2(\lambda+i\epsilon)g)d\lambda \\ &= \int_{-N}^{N}(\delta_2(\lambda,\epsilon)V_{21}R_1(\lambda+i\epsilon)f,g)d\lambda \\ &+ \frac{1}{2\pi i}\int_{-N}^{N}((R_2(\lambda+i\epsilon) - R_1(\lambda+i\epsilon))f,g)d\lambda. \end{aligned}$$

However, $\dfrac{1}{2\pi i}\displaystyle\int_{-N}^{N}(R_j(\lambda+i\epsilon)f,g)d\lambda \to \dfrac{1}{2}(f,g)$ when $N\to\infty$. In fact,

$$(R_j(z)f,g) = \int_{-\infty}^{\infty}\frac{1}{\mu - z}d_\mu(E_j(\mu)f, E_j(\mu)g),$$

where the domain of integration is bounded by our assumptions on f and g. Therefore

$$\frac{1}{2\pi}\int_{-N}^{N}(R_j(\lambda+i\epsilon)f,g)d\lambda = \frac{1}{2\pi}\int_{-\infty}^{\infty}\ln\left(\frac{-N-\mu+i\epsilon}{N-\mu+i\epsilon}\right)d_\mu(E_j(\mu)f,E_j(\mu)g).$$

Since $\ln\left(\dfrac{-N-\mu+i\epsilon}{N-\mu+i\epsilon}\right) \to \pi i$ for any μ, the result follows by Lebesgue's dominated convergence theorem.

Let us take bounded open intervals J, J_1 such that

$$(4.9)\qquad e \subset \overline{e} \subset J \subset \overline{J} \subset J_1 \subset \overline{J_1} \subset I,$$

and $g = \varphi(H_2)h$ for some $\varphi(\lambda) \in C_0^\infty(J)$ and $h \in \mathcal{H}_+$. Such g's are dense in $E_2(I)\mathcal{H}$. Then we have

$$(4.10)\qquad (\delta_2(\cdot,\epsilon)V_{21}R_1(\cdot+i\epsilon)f,g) \in L^1(\mathbf{R}),\quad \epsilon>0,$$

$$(4.11)\qquad \lim_{\epsilon\to 0}\int_{-\infty}^{\infty}(\delta_2(\lambda,\epsilon)V_{21}R_1(\lambda+i\epsilon)f,g)d\lambda = (\Omega_{21}^{(+)}(e)f,g) - (f,g).$$

In fact, since $V_{21}R_1(\lambda+i\epsilon) = G_{21}(\lambda+i\epsilon) - 1$, we have

$$(\delta_2(\lambda,\epsilon)V_{21}R_1(\lambda+i\epsilon)f,g) = (\delta_2(\lambda,\epsilon)G_{21}(\lambda+i\epsilon)f,g) - (f,\delta_2(\lambda,\epsilon)g).$$

Then the 2nd term of the right-hand side is written as

$$(4.12)\qquad (f,\delta_2(\lambda,\epsilon)g) = \frac{\epsilon}{\pi}\int_{-\infty}^{\infty}\frac{1}{(\mu-\lambda)^2+\epsilon^2}d_\mu(f,E_2(\mu)g).$$

If $\lambda \notin \overline{J_1}$, the right-hand side is dominated from above by $C\epsilon(1+|\lambda|^2)^{-1}$. On the other hand, assumptions (A-1), (A-2) imply that the left-hand side is bounded for

$\lambda \in \overline{J_1}$ uniformly with respect to ϵ. Therefore $(f, \delta_2(\cdot, \epsilon)g) \in L^1(\mathbf{R})$, and by Stone's theorem

$$\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} (f, \delta_2(\lambda, \epsilon)g) d\lambda = (f, g). \tag{4.13}$$

By the resolvent equation, $R_1(z) = R_1(i)(1 + (z - i)R_1(z))$. Then we have

$$G_{21}(z) = (H_2 - H_1)R_1(i) \left(1 + (z - i)R_1(z)\right) + 1.$$

Since $f \in E_1(e)\mathcal{H}$, we have $\|(\lambda + i\epsilon - i)R_1(\lambda + i\epsilon)f\| \leq C_f$ uniformly for $\lambda \notin \overline{J_1}$ and $\epsilon > 0$. Hence so is $\|G_{21}(\lambda + i\epsilon)f\|$. Then formula (4.12) implies that if $\lambda \notin \overline{J_1}$,

$$|(G_{21}(\lambda + i\epsilon)f, \delta_2(\lambda, \epsilon)g)| \leq C\epsilon(1 + |\lambda|^2)^{-1},$$

which implies

$$\int_{\mathbf{R}\setminus\overline{J_1}} (\delta_2(\lambda, \epsilon)G_{21}(\lambda + i\epsilon)f, g) d\lambda \to 0, \quad \epsilon \to 0. \tag{4.14}$$

Since $f \in E_1(e)H$,

$$\int_{J_1} E_2'(\lambda)G_{21}(\lambda + i\epsilon)f d\lambda \to \Omega_{21}^{(+)}(e)f.$$

Together with (4.14), this implies that

$$\int_{\mathbf{R}} (\delta_2(\lambda, \epsilon)G_{21}(\lambda + i\epsilon)f, g)\, d\lambda \to \left(\Omega_{21}^{(+)}(e)f, g\right) \tag{4.15}$$

Equations (4.13) and (4.15) prove (4.11). By (4.6), (4.8) and (4.11) we get $W_{21}^{(+)}(e) = \Omega_{21}^{(+)}$ when e is a relatively compact interval in I.

For an open subset $e \subset I$, we have only to apprriximate e by a finite number of relatively compact intervals. The proof for $W_{21}^{(-)}(e) = \Omega_{21}^{(-)}(e)$ is similar. □

4.4. Spectral representation. Let us recall that for a self-adjoint operator $H = \int_{-\infty}^{\infty} \lambda dE(\lambda)$, we take an open interval I in $\sigma_{ac}(H)$. We take an auxiliary Hilbert space $\mathbf{h}$ and a measure $\rho(\lambda)d\lambda$ on I, $\rho(\lambda) \in L^1(I; d\lambda)$, and put

$$\widehat{\mathcal{H}}(I) = L^2(I; \mathbf{h}; \rho(\lambda)d\lambda).$$

A unitary operator U from $E(I)\mathcal{H}$ onto $\widehat{\mathcal{H}}(I)$ satisfying

$$(UHf)(\lambda) = \lambda(Uf)(\lambda), \quad \lambda \in I, \quad f \in D(H)$$

is called a spectral representaion of H on I. By the functional calculus,

$$(U\varphi(H)f)(\lambda) = \varphi(\lambda)(Uf)(\lambda) \tag{4.16}$$

holds for any bounded Borel function φ and $f \in E(I)\mathcal{H}$. In fact, (4.16) is first proven for the resolvent $\varphi(H) = (H - z)^{-1}$, next for the spectral measure $E(\mu)$ by using Stone's formula, and then for any bounded Borel function.

Let $\mathcal{H}_+, \mathcal{H}_-$ be Banach spaces satisfying the assumptions in subsection 4.2. We assume that, for $\lambda \in I$, there exists a bounded operator $U(\lambda) \in \mathbf{B}(\mathcal{H}_+; \mathbf{h})$, which is stronlgy continuous in λ, such that

$$(Uf)(\lambda) = U(\lambda)f, \quad \lambda \in I, \quad f \in \mathcal{H}_+.$$

Then $U(\lambda)^* \in \mathbf{B}(\mathbf{h}; \mathcal{H}_-)$. Let us show that for $\Phi \in \widehat{H}(I)$

$$U^*\Phi = \int_I U(\lambda)^*\Phi(\lambda)\rho(\lambda)d\lambda \in E(I)\mathcal{H}. \tag{4.17}$$

Indeed, let us first assume that $\operatorname{supp}\Phi \subset J$, where $\overline{J}$ is a compact set in I. Then, for $f \in \mathcal{H}_+$, we have

$$\Big(\int_I U(\lambda)^*\Phi(\lambda)\rho(\lambda)d\lambda, f\Big) = \int_I (\Phi(\lambda), U(\lambda)f)_{\mathbf{h}}\rho(\lambda)d\lambda = (\Phi, Uf)_{\widehat{\mathcal{H}}} = (U^*\Phi, f).$$

As U^* is partial isometry, the right-hand side can be extended to $f \in \mathcal{H}$, which together with Riesz' theorem implies (4.17) for Φ with $\operatorname{supp}\Phi \subset J$. Since J is arbitrary, and $I \subset \sigma_{ac}(H)$, (4.17) is exteded onto $\widehat{\mathcal{H}}(I)$.

As a consequence, we have the inversion formula for $f \in E(I)\mathcal{H}$

$$f = \int_I U(\lambda)^*(Uf)(\lambda)\rho(\lambda)d\lambda. \tag{4.18}$$

In fact, for $g \in \mathcal{H}_+$,

$$(f, g)_{\mathcal{H}} = (Uf, Ug)_{\widehat{\mathcal{H}}} = \int_I ((Uf)(\lambda), U(\lambda)g)_{\mathbf{h}}\rho(\lambda)d\lambda.$$

Hence we have

$$(f, g)_{\mathcal{H}} = \int_I (U(\lambda)^*(Uf)(\lambda), g)\rho(\lambda)d\lambda,$$

which proves (4.18) by virtue of (4.17).

We need a new assumption:

(A-5) *There exists a subspace $\mathcal{D} \subset D(H)\cap\mathcal{H}_+$ such that $\mathcal{D}$ as well as $H\mathcal{D}$ are dense in $\mathcal{H}_+$ and $D(H)$.*

Then, for $\psi \in \mathbf{h}, f \in \mathcal{D}$,

$$(U(\lambda)^*\psi, (H-\lambda)f) = 0$$

holds, since $U(\lambda)Hf = \lambda U(\lambda)f$. Therefore, $U(\lambda)^* \in \mathbf{B}(\mathbf{h}; \mathcal{H}_-)$ satisfies the equation

$$(H-\lambda)U(\lambda)^* = 0,$$

and is called the *eigenoperator* of H. Here the self-adjoint operator H in $\mathcal{H}$ is extended to $\mathcal{H}_-$ via the relation

$$(Hu, f) = (u, Hf), \quad u \in \mathcal{H}_-, \quad f \in \mathcal{D}. \tag{4.19}$$

We now discuss the perturbation theory for spectral representations. For H_1 we assume that

(A-6) *For any $\lambda \in I$ there exists $U_1(\lambda) \in \mathbf{B}(\mathcal{H}_+; \mathbf{h})$ such that for $f, g \in \mathcal{H}_+$*

$$(E_1'(\lambda)f, g) = \rho(\lambda)(U_1(\lambda)f, U_1(\lambda)g)_{\mathbf{h}}.$$

Moreover, U_1 defined by $(U_1f)(\lambda) = U_1(\lambda)f$ is uniquely extended to a unitary operator from $E_1(I)\mathcal{H}$ to $\widehat{H}(I)$.

By this assumption, we have for $f \in D(H_1)$

$$(U_1H_1f)(\lambda) = \lambda(U_1f)(\lambda) a.e.. \tag{4.20}$$

In fact, let $f \in \mathcal{D}$. Since $R_1(z)H_1 = 1 + zR_1(z)$, we have $E_1'(\lambda)H_1 f = \lambda E_1'(\lambda)f$. The assumption (A-6) then implies

$$\begin{aligned}(E_1(I)H_1 f, g) &= \int_I \lambda\left((U_1 f)(\lambda), (U_1 g)(\lambda)\right)_{\mathbf{h}} \rho(\lambda) d\lambda \\ &= \int_I ((U_1 H_1 f)(\lambda), (U_1 g)(\lambda))_{\mathbf{h}} \rho(\lambda)\, d\lambda,\end{aligned}$$

which proves (4.20) for $f \in \mathcal{D}$. Since $\mathcal{D}$ is dense in $D(H_1)$ we obtain (4.20). Therefore, $U_1(\lambda)^* \in \mathbf{B}(\mathbf{h}; \mathcal{H}_-)$ is an eigenoperator of H_1:

$$(H_1 - \lambda)U_1(\lambda)^* = 0.$$

We construct the spectral representation of H_2 by using that of H_1. Starting from U_1, we will construct two operators $U_2^{(+)}, U_2^{(-)}$ corresponding to wave operators $W_{21}^{(+)}, W_{21}^{(-)}$. For $\lambda \in I$, we define

$$U_2^{(\pm)}(\lambda) = U_1(\lambda)G_{12}(\lambda \pm i0).$$

For $f \in \mathcal{H}_+$, we put $(U_2^{(\pm)} f)(\lambda) = U_2^{(\pm)}(\lambda)f$. Then we have the following theorem.

Theorem 4.7. *Under the assumptions (A-1) $\sim$ (A-6), we have*

$$(E_2'(\lambda)f, g) = \rho(\lambda)(U_2^{(\pm)}(\lambda)f, U_2^{(\pm)}(\lambda)g)_{\mathbf{h}}, \quad f, g \in \mathcal{H}_+.$$

Moreover $U_2^{(\pm)} = U_1 \left(W_{21}^{(\pm)}(I)\right)^*$, *and* $U_2^{(\pm)}$ *is a spectral representation for* H_2.

Proof. The first half of the theorem follows from (4.4) and (A-6). By virtue of (4.5) and (A-6) we have

$$\begin{aligned}(E_2'(\lambda)G_{21}(\lambda \pm i0)f, g) &= (f, E_1'(\lambda)G_{12}(\lambda \pm i0)g) \\ &= \rho(\lambda)(U_1(\lambda)f, U_2^{(\pm)}(\lambda)g)_{\mathbf{h}}.\end{aligned}$$

Integration with respect to λ then yields, in view of Theorem 4.6, that

$$(W_{21}^{(\pm)}(I)f, g) = (U_1 f, U_2^{(\pm)} g)_{\widehat{\mathcal{H}}},$$

hence $W_{21}^{(\pm)}(I) = (U_2^{(\pm)})^* U_1$. We have, therefore, $U_2^{(\pm)} = U_1 \left(W_{21}^{(\pm)}(I)\right)^*$. Since $\operatorname{Ran} W_{21}^{(\pm)}(I) = E_2(I)\mathcal{H}$ and $W_{21}^{(\pm)}\varphi(H_1) = \varphi(H_2)W_{21}^{(\pm)}$ for any bounded Borel function $\varphi(\lambda)$, $U_2^{(\pm)}$ is a partial isometry with initial set $E_2(I)\mathcal{H}$ and final set $\widehat{\mathcal{H}}(I)$. Moreover $U_2^{(\pm)}\varphi(H_2) = \varphi(\lambda)U_2^{(\pm)}$ for any bounded Borel function. Therefore, $U_2^{(\pm)}$ is a spectral representation for H_2. □

By the relation $U_2^{(\pm)}(\lambda)^* = (1 - R_2(\lambda \mp i0)V)U_1(\lambda)^*$, $V = H_2 - H_1$, we have

$$(H_2 - \lambda)U_2^{(\pm)}(\lambda)^* = 0.$$

Hence $U_2^{(\pm)}(\lambda)^*$ is an eigenoperator of H_2. Let us summarize the results obtained so far. Let $E_2(\lambda)$ be the resolution of identity for H_2.

Theorem 4.8. *(1) Let* $V_{21} = H_2 - H_1$ *and put*

$$U_2^{(\pm)}(\lambda) = U_1(\lambda)(1 - V_{21}R(\lambda \pm i0)) = U_1(\lambda)G_{12}(\lambda \pm i0).$$

Then $U_2^{(\pm)}(\lambda) \in \mathbf{B}(\mathcal{H}_+;\mathbf{h})$ *for* $\lambda \in I$.
(2) $U_2^{(\pm)}(\lambda)^* \in \mathbf{B}(\mathbf{h};\mathcal{H}_-)$ *is an eigenoperator of* H *with eigenvalue* $\lambda \in I$ *in the following sense*

$$((H_2-\lambda)f, U_2^{(\pm)}(\lambda)^*\varphi) = 0$$

for any $f \in \mathcal{H}_+$ *such that* $H_2 f \in \mathcal{H}_+$ *and* $\varphi \in \mathbf{h}$. *Moreover,*

$$(U_2^{(\pm)}H_2 f)(\lambda) = \lambda(U_2^{(\pm)}f)(\lambda), \quad f \in D(H_2), \quad \lambda \in I.$$

(3) The operator $U_2^{(\pm)}$ *defined by* $(U_2^{(\pm)}f)(\lambda) = U_2^{(\pm)}(\lambda)f$ *for* $f \in \mathcal{H}_+$ *is uniquely extended to a partial isometry with the initial set* $E_2(I)\mathcal{H}$ *and the final set* $\widehat{\mathcal{H}}(I)$.
(4) For any $\Phi \in \widehat{\mathcal{H}}(I)$ *and any compact interval* $e \subset I$,

$$\int_e U_2^{(\pm)}(\lambda)^*\Phi(\lambda)\rho(\lambda)d\lambda \in \mathcal{H}.$$

(5) For any $f \in E_2(I)\mathcal{H}$, *the following inversion formula holds:*

$$f = \mathrm{s}-\lim_{n\to\infty}\int_{I_n} U_2^{(\pm)}(\lambda)^*(U_2^{(\pm)}f)(\lambda)\rho(\lambda)d\lambda,$$

where $I_n = [a_n, b_n], a < a_n < b_n < b$, $a_n \to a, b_n \to b$ *and* $I = (a,b)$.

Proof. We have only to show the assertions (4) and (5). Let $I_e(\Phi)$ be the integral in (4). We first assume that $\operatorname{supp}\Phi(\lambda)$ is a compact set e in I. We take $f \in E_2(I)\mathcal{H}$ such that $U_2^{(\pm)}f = \Phi$. Then for any $g \in \mathcal{H}_+$, we have

$$\begin{aligned}(f,g) &= (U_2^{(\pm)}f, U_2^{(\pm)}g)\\ &= \int_e ((U_2^{(\pm)}f)(\lambda),(U_2^{(\pm)}g)(\lambda))\rho(\lambda)d\lambda = \int_e \left(\Phi(\lambda),(U_2^{(\pm)}g)(\lambda)\right)\rho(\lambda)d\lambda\\ &= \int_e (U_2^{(\pm)}(\lambda)^*\Phi(\lambda),g)\rho(\lambda)d\lambda = (I_e(\Phi),g).\end{aligned}$$

We have, therefore, $I_e(\Phi) = f \in \mathcal{H}$. This implies also that, for any $f \in E_2(I)\mathcal{H}$ and a compact interval $e \subset I$,

$$E_2(e)f = \int_e U_2^{(\pm)}(\lambda)^*(U_2^{(\pm)}f)(\lambda)\rho(\lambda)d\lambda,$$

since $(U_2^{(\pm)}E_2(e)f)(\lambda) = \chi_e(\lambda)(U_2^{(\pm)}f)(\lambda)$, where $\chi_e(\lambda)$ is the characteristic function of e. Therefore

$$\left\|\int_e U_2^{(\pm)}(\lambda)^*(U_2^{(\pm)}f)(\lambda)\rho(\lambda)d\lambda\right\| \to 0$$

if the measure of e tends to 0. This proves (5). □

4.5. S-matrix. The scattering operator for H_1, H_2 (on I) is defined by

$$S = (W_{21}^{(+)}(I))^*W_{21}^{(-)}(I).$$

This is unitary on $E_1(I)\mathcal{H}$. Let us rewrite it by using the spectral representation. We define

$$\widehat{S} = U_1 S U_1^*.$$

Letting $V_{21} = H_2 - H_1$, we also put

$$\widehat{S}(\lambda) = 1 - 2\pi i\rho^2(\lambda)A(\lambda),$$

$$A(\lambda) = U_1(\lambda)V_{21}U_1(\lambda)^* - U_1(\lambda)V_{21}R_2(\lambda+i0)V_{21}U_1(\lambda)^*.$$

Then $\widehat{S}(\lambda) \in \mathbf{B}(\mathbf{h};\mathbf{h})$ and is called the S-matrix or the scattering matrix.

Theorem 4.9. *$\widehat{S}(\lambda)$ is unitary on $\mathbf{h}$, and for any $\hat{f} \in \widehat{\mathcal{H}}$*

$$(\widehat{S}\hat{f})(\lambda) = \widehat{S}(\lambda)\hat{f}(\lambda)$$

holds. Here the right-hand side means that we fix λ arbitrarily, regard $\hat{f}(\lambda)$ as an element of $\mathbf{h}$ and apply $\widehat{S}(\lambda)$.

Proof. Noting that

$$W_{21}^{(\pm)}(I) = E_1(I) + i\int_0^{\pm\infty} e^{isH_2}V_{21}e^{-isH_1}E_1(I)ds,$$

we have

$$W_{21}^{(+)}(I) - W_{21}^{(-)}(I) = i\int_{-\infty}^{\infty} e^{itH_2}V_{21}e^{-itH_1}E_1(I)dt.$$

By the definition of S, we have

$$(S-1)E_1(I) = (W_{21}^{(+)})^*(W_{21}^{(-)}(I) - W_{21}^{(+)}(I)).$$

Letting $f = E_1(I)f, g = E_1(I)g$, we then have

$$\begin{aligned}
&(Sf,g) - (f,g) \\
&= -i\int_{-\infty}^{\infty}(e^{itH_2}V_{21}e^{-itH_1}f, W_{21}^{(+)}(I)g)dt \\
&= -i\int_{-\infty}^{\infty}(V_{21}e^{-itH_1}f, e^{-itH_1}g)dt \\
&\quad - \int_0^{\infty} ds\int_{-\infty}^{\infty}(V_{21}e^{-itH_1}f, e^{isH_2}V_{21}e^{-i(s+t)H_1}g)dt,
\end{aligned} \tag{4.21}$$

where we have used $e^{-itH_2}W_{21}^{(+)}(I) = W_{21}^{(+)}(I)e^{-itH_1}$. Letting $\hat{f}(\lambda) = U_1(\lambda)f, \hat{g}(\lambda) = U_1(\lambda)g$, we obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty}(V_{21}e^{-isH_2}V_{21}e^{-itH_1}f, e^{-i(s+t)H_1}g)dt \\
= &\int_{-\infty}^{\infty}dt\int_I(U_1(\lambda)V_{21}e^{-isH_2}V_{21}e^{-itH_1}f, e^{-i(s+t)\lambda}\hat{g}(\lambda))_{\mathbf{h}}\rho(\lambda)d\lambda.
\end{aligned}$$

Inserting $e^{-\epsilon|t|}$ and letting $\epsilon \to 0$, this converges to

$$\begin{aligned}
&2\pi\int_I(U_1(\lambda)V_{21}e^{-is(H_2-\lambda)}V_{21}E_1'(\lambda)f, \hat{g}(\lambda))_{\mathbf{h}}\rho(\lambda)d\lambda \\
= \ &2\pi\int_I(U_1(\lambda)V_{21}e^{-is(H_2-\lambda)}V_{21}U_1(\lambda)^*\hat{f}(\lambda), \hat{g}(\lambda))_{\mathbf{h}}\rho(\lambda)^2d\lambda,
\end{aligned}$$

where we have used $E_1'(\lambda) = \rho(\lambda)U_1(\lambda)^*U_1(\lambda)$. Therefore, the last term of the most right-hand side of (4.21) is equal to

$$-2\pi\int_0^{\infty}ds\int_I(U_1(\lambda)V_{21}e^{-is(H_2-\lambda)}V_{21}U_1(\lambda)^*\hat{f}(\lambda), \hat{g}(\lambda))_{\mathbf{h}}\rho(\lambda)^2d\lambda.$$

Inserting $e^{-\epsilon s}$ and letting $\epsilon \to 0$, this converges to

$$2\pi i\int_I(U_1(\lambda)V_{21}R_2(\lambda+i0)V_{21}U_1(\lambda)^*\hat{f}(\lambda), \hat{g}(\lambda))_{\mathbf{h}}\rho(\lambda)^2d\lambda.$$

Similarly the first term of the most right-hand side of (4.21) is rewritten as

$$-2\pi i\int_I (U_1(\lambda)V_{21}U_1(\lambda)^*\hat{f}(\lambda),\hat{g}(\lambda))_{\mathbf{h}}\rho(\lambda)^2 d\lambda.$$

This proves the representation of $\widehat{S}$. Since $\widehat{S}$ is unitary on $\widehat{\mathcal{H}}$, so is $\widehat{S}(\lambda)$ on $\mathbf{h}$. □

Lemma 4.10. *For any $\lambda \in I$, we have*

$$U_2^{(+)}(\lambda) = \widehat{S}(\lambda)U_2^{(-)}(\lambda).$$

Proof. By Theorem 4.7, we have

$$\left(W_{12}^{(+)}(I)\right)^* = (U_1)^*\, U_2^{(+)}, \quad W_{12}^{(-)} = \left(U_2^{(-)}\right)^* U_1.$$

Therefore by the definition of $\widehat{S}$, we have

$$\widehat{S}U_2^{(-)} = U_2^{(+)},$$

which proves the lemma. □

5. Examples of spectral representations

5.1. Spectral representation on $\mathbf{R}^n$. Let us apply the results in the previous section to Schrödinger operators $H_0 = -\Delta$ and

$$H = -\sum_{i,j=1}^n a_{ij}(x)\partial_i\partial_j + \sum_{i=1}^n a_i(x)\partial_i + a_0(x)$$

on $\mathbf{R}^n$, where $\partial_i = \partial/\partial x_i$. Let $\mathcal{H} = L^2(\mathbf{R}^n; dx)$ and assume that H is formally self-adjoint and uniformly elliptic on $\mathbf{R}^n$, i.e. there exists a constant $C_0 > 0$ such that

$$C^{-1}|\xi|^2 \le \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \le C|\xi|^2, \quad \forall x, \xi \in \mathbf{R}^n.$$

The coefficients $a_{ij}(x) - \delta_{ij}$ and $a_i(x)$ of H are assumed to be smooth and satisfy

$$|\partial^\alpha a(x)| \le C_\alpha (1 + |x|)^{-1-\epsilon-|\alpha|}, \quad \forall\alpha, \quad \forall x \in \mathbf{R}^n$$

for a constant $\epsilon > 0$. For $s \in \mathbf{R}$ we define the space $L^{2,s}$ by

$$L^{2,s} \ni f \Longleftrightarrow \|f\|_s^2 = \int_{\mathbf{R}^n} (1 + |x|^2)^s |f(x)|^2 dx < \infty.$$

Let $s > 1/2$ be arbitraily fixed. Then, by choosing $\mathcal{H}_\pm = L^{2,\pm s}$, the assumptions (A-1) $\sim$ (A-3) are satisfied for $H_1 = H_0$, $H_2 = H$ and $I = (0, \infty)$. We should remark that by this choice of $\mathcal{H}_\pm$, the boundary value of the resolvent $R_j(\lambda \pm i0)f$ is strongly continuous in $L^{2,-s}$ as a function of $\lambda > 0$. These facts are well-known and are proved in e.g. [**62**], where they are proved for the potential perturbation of $-\Delta$, however, the proof also works for the case of the 2nd order variable coefficients. Let us also note that Theorem 3.1 can also be applied in this case.

As a spectral representation for H_0, we employ the usual Fourier transformation:

$$\left(U_0(\lambda)f\right)(\omega) = (2\pi)^{-1/2}\int_{\mathbf{R}^n} e^{-i\sqrt{\lambda}\omega\cdot x} f(x)dx,$$

and $\mathbf{h} = L^2(S^{n-1})$ and $\rho(\lambda) = \frac{1}{2}\lambda^{(n-2)/2}$. Then the assumption (A-4) is also satisfied. Let $R(z) = (H - z)^{-1}$ and $V = H - H_0$. Then

$$U_{\pm}(\lambda) = U_0(\lambda)(1 - VR(\lambda \pm i0))$$

gives the spectral representation for H.

5.2. Spectral representations on $\mathbf{H}^n$. Let $\mathcal{H} = L^2(\mathbf{R}^n_+; dxdy/y^n)$ and consider the operators H_0 and H introduced in §2. Let $L^{2,s}$ be defined by Definition 2.6 of Chap. 1. Let $\mathcal{H}_{\pm} = L^{2,\pm s}$ with $1/2 < s < (1+\epsilon)/2$ and $H_1 = H_0$, $H_2 = H$ and $I = (0, \infty)$. First we check (A-1). Let $\langle \log y \rangle^s = (1 + |\log y|^2)^{s/2}$. We show that there exists a constant C_s independent of $z \notin \mathbf{R}$ such that

$$\|\langle \log y \rangle^s (H_j - z)^{-1} \langle \log y \rangle^{-s}\| \leq C_s |\mathrm{Im}\, z|^{-2}(1 + |z|). \tag{5.1}$$

Once we have proven (5.1), we can use an abstract theorem from functional analysis (see Lemma 3.1 in Chap. 3, where σ can be an arbitrary negative number) to show

$$\langle \log y \rangle^s \varphi(H_j) \langle \log y \rangle^{-s} \in \mathbf{B}(\mathcal{H}; \mathcal{H}), \quad \forall \varphi \in C_0^\infty(\mathbf{R}),$$

which yields (A-1).

Let us prove (5.1). We have

$$\begin{aligned}&\langle \log y \rangle^s (H_j - z)^{-1} \langle \log y \rangle^{-s} \\ &= (H_j - z)^{-1} + (H_j - z)^{-1}[H_j, \langle \log y \rangle^s](H_j - z)^{-1}\langle \log y \rangle^{-s}.\end{aligned}$$

Since $[H_j, \langle \log y \rangle^s]$ is a 1st order differential operator with respect to D_x, D_y with bounded coefficients, one can show

$$\|[H_j, \langle \log y \rangle^s](H_j - z)^{-1}\| \leq C_s |\mathrm{Im}\, z|^{-1}(1 + |z|)$$

by using Theorem 1.3 (4) and the standard estimate of the resolvent. The inequality (5.1) imediately follows from this.

Theorem 2.3 together with Lemma 1.2.7 justify (A-2). As above, by this choice of $L^{2,\pm s}$ the strong continuity of $R_j(\lambda \pm i0)f$ with respect to λ is guaranteed.

To prove (A-4) for a proper $\mathcal{D}_e$, $e = (a^2, b^2)$, $0 < a < b < \infty$, we first observe that it is sufficient to show that, for $1 < s < 1 + \epsilon$ and $f \in \mathcal{D}_e$,

$$\int_{-\infty}^{\infty} \Big(||e^{-itH_0} f||_{\chi^{-s}} + \sum_j ||D_j e^{-itH_0} f||_{\chi^{-s}} + \sum_{j,l} ||D_j D_l e^{-itH_0} f||_{\chi^{-s}} \Big) dt < \infty.$$

Assuming that $H_0 \mathcal{D}_e \subset \mathcal{D}_e$, and utilising Theorem 1.3 (6), we can confine to the proof that

$$\int_{-\infty}^{\infty} \left(||e^{-itH_0} f||_{\chi^{-s}} + ||e^{-itH_0} H_0 f||_{\chi^{-s}} \right) dt < \infty, \quad f \in \mathcal{D}_e.$$

Let

$$\mathcal{D}_e = \left\{ f : \phi(k, \xi) = (F_0 \mathcal{F}_0^{(+)} f)(k, \xi) \in C_0^\infty((a, b) \times \mathbf{R}^{n-1}) \right\}.$$

Since then $(F_0 \mathcal{F}_0^{(+)} H_0 f)(k, \xi) = k^2 \phi(k, \xi) \in C_0^\infty((a, b) \times \mathbf{R}^{n-1})$, we have $H_0 \mathcal{D}_e \subset \mathcal{D}_e$, it suffices to show that

$$\int_{-\infty}^{\infty} ||e^{-itH_0} f||_{\chi^{-s}} dt < \infty \tag{5.2}$$

This is proved in the same way as in Theorem 1.5.5. In fact, letting $u(t,\xi,y) = F_0 e^{-itH_0} f$, we have

$$u(t,\xi,y) = \int_0^\infty \frac{(2k\sinh(k\pi))^{1/2}}{\pi} \left(\frac{|\xi|}{2}\right)^{ik} y^{(n-1)/2} K_{ik}(|\xi|y) e^{-itk^2} \phi(k,\xi) dk,$$

(cf. Chap. 1, (5.2)). Then, similar to Chap. 1, (5.3), we show that, for any $\sigma > 0$,

$$\int_\delta^\infty \|u(t,\cdot,y)\|^2_{L^2(\mathbf{R}^{n-1})} \frac{dy}{y^n} \leq C_N(1+|t|)^{-N}, \quad \forall N > 0 \tag{5.3}$$

To consider the behavior of $u(t,\cdot,y)$ for $0 < y < \sigma$, we, similar to the proof of Theorem 1.5.5, use the decomposition

$$u(t,\xi,y) = u_0^{(+)}(t,\xi,y) + u_0^{(-)}(t,\xi,y) + u_1(t,\xi,y),$$

which have the same representations as in Theorem 1.5.5 with, however, e^{-ikt} replaced by e^{-ik^2t}. Since, for $k \in (a,b)$ and bounded $|\xi|, y$, we have

$$|r_1(k,|\xi|,y)| \leq C|\xi|y, \quad |\partial_k^2 r_1(k,|\xi|,y)| \leq C\log(|\xi|y)|\xi|y,$$

(see (3.1), (3.2)), we see that, for $y < \sigma$,

$$|u_1(t,\xi,y)| \leq C_\phi y^{(n+1)/2}(1+|\log(y)|)(1+|t|)^{-2}.$$

This implies that

$$\int_{-\infty}^\infty \left(\int_0^\sigma ||u_1(t,\cdot,y)||_{L^2(\mathbf{R}(n-1)} \frac{dy}{y^n}\right)^{1/2} dt < \infty. \tag{5.4}$$

Using (5.5), we see that, for $t > \frac{2|\log(y)|}{a}$ and $t < \frac{|\log(y)|}{2b}$,

$$|u_0^\pm(t,\xi,y)| \leq C_\phi y^{(n-1)/2}(1+|t|)^{-2}, \tag{5.5}$$

which implies that

$$\int_{-\infty}^\infty \left(\int_0^\infty ||u_0^\pm(t,\cdot,y)\Theta_{a,b}(y,t)||_{L^2(\mathbf{R}^{n-1})}(1+|\log(y)|)^{-2s}\frac{dy}{y^n}\right)^{1/2} dt < \infty$$

for $s > 1$. Here $\Theta_{a,b}(y,t) = 1$, if $t > \frac{2|\log(y)|}{a}$ and $t < \frac{|\log(y)|}{2b}$, and 0 otherwise. As for the remaining part, we have, by the stationary phase method, that, for $\frac{1}{2b} < \frac{|t|}{|\log(y)|} < \frac{2}{a}$,

$$|u_0^\pm(t,\xi,y)| \leq C_\phi y^{(n-1)/2}\left(|t| + |\log(y)|\right)^{-1/2}$$

Taking into account that the domain of integration with respect to ξ is bounded, we obtain that

$$\int_0^\sigma |\log(y)|^{-2s}||u_0^\pm(t,\cdot,y)||_{L^2(\mathbf{R}^{n-1})}\left(1-\Theta_{a,b}(y,t)\right)\frac{dy}{y^n} \leq C_\phi(1+|t|)^{-2s}.$$

This estimate, together with (5.4), shows that

$$\int_{-\infty}^\infty ||u(t,\cdot,y)H(\sigma-y)||_{\mathcal{X}^{-s}} dt < \infty,$$

which, due to (5.3), implies (5.2).

As for the spectral representation, we put

$$(U_0(\lambda)f)(x) = \frac{\left(2\sqrt{\lambda}\sinh(\sqrt{\lambda}\pi)\right)^{1/2}}{\pi}(2\pi)^{-(n-1)/2}$$
$$\times \iint_{\mathbf{R}^{n-1}\times(0,\infty)} e^{ix\cdot\xi}\left(\frac{|\xi|}{2}\right)^{-i\sqrt{\lambda}} y^{(n-1)/2}K_{i\sqrt{\lambda}}(|\xi|y)\hat{f}(\xi,y)\frac{d\xi dy}{y^n}.$$

and $\mathbf{h} = L^2(\mathbf{R}^{n-1})$, $\rho(\lambda) = \frac{1}{2}\lambda^{-1/2}$. Then the assumptions (A-5), (A-6) are fulfilled. Taking

$$U_{\pm}(\lambda) = U_0(\lambda)(1 - VR(\lambda \pm i0))$$

gives, due to Theorem 4.7, the spectral representation for H, where $R(z) = (H - z)^{-1}$ and $V = H - H_0$.

5.3. Absolutely continuous subspace. Let us recall the well-known classification of the spectra of self-adjoint operators. Let $H = \int_{-\infty}^{\infty}\lambda dE_H(\lambda)$ be a self-adjoint operator in a Hilbert space $\mathcal{H}$. Then for any $u \in \mathcal{H}$, $(E_H(I)u, u)$, where I is any Borel set in $\mathbf{R}$, defines a Borel measure on $\mathbf{R}$. Then the *absolutely continuous subspace* for H is defined by

(5.6)
$$\mathcal{H}_{ac}(H) = \{u \in \mathcal{H}\,;\,(E_H(\cdot)u, u) \text{ is absolutely continuous with respect to } d\lambda\}.$$

This is a closed subspace in $\mathcal{H}$. The importance of this subspace is that it is usually stable under the perturbation appearing in scattering phenomena (see e.g. [**80**]).

Let $R_H(z) = (H - z)^{-1}$, and I be an open interval in $\sigma(H)$. If the limiting absorption principle holds on I, i.e. the condition (A-2) in §4 is guaranteed on I, we have

(5.7)
$$E_H(I)\mathcal{H} \subset \mathcal{H}_{ac}(H).$$

In fact, for u in a dense subset of $\mathcal{H}$, we have by Stone's formula

$$(E_H(B)u, u) = \frac{1}{2\pi i}\int_B ((R_H(\lambda + i0) - R_H(\lambda - i0))u, u)\,d\lambda,$$

for any Borel set B in I, which yields (5.7). Therefore, for our case of $H = -\Delta_g$ for the asymptotically Euclidean metric, or $H = -\Delta_g - (n-1)^2/4$ for the asymptotically hyperbolic metric,

$$E_H((0,\infty))\mathcal{H} = \mathcal{H}_{ac}(H).$$

In this case, we often say that the continuous spectrum of H is absolutely continuous, or H has no singular continuos spectrum.

The spectral representation $U^{(\pm)}$ is then a unitary operator from $\mathcal{H}_{ac}(H)$ to the representation space $L^2((0,\infty)); \mathbf{h}; \rho(\lambda)d\lambda)$, where $\mathbf{h} = L^2(S^{n-1})$ for the Euclidean metric, and $\mathbf{h} = L^2(\mathbf{R}^{n-1})$ for the hyperbolic metric.

6. Geometric S-matrix

In §4 and §5, we have constructed two Fourier transforms $U_{\pm}$ for $H = H_0 + V$, however only one Fourier transform U_0 is adopted for H_0. As a matter of fact, it is natural to associate two kinds of Fourier transforms also with H_0. To see this let us recall that the Green operator for $-\Delta - \lambda$ on $\mathbf{R}^3$ is written as

$$(-\Delta - \lambda \mp i0)^{-1}f = \frac{1}{4\pi}\int_{\mathbf{R}^3}\frac{e^{\pm i\sqrt{\lambda}|x-y|}}{|x-y|}f(y)dy.$$

Noting the asymptotic expansion $|x-y| \sim r - \omega\cdot y$ ($\omega = x/r$) as $r = |x| \to \infty$, we have for $f \in C_0^\infty(\mathbf{R}^3)$

$$(-\Delta - \lambda \mp i0)^{-1} f \sim \frac{e^{\pm i\sqrt{\lambda}r}}{4\pi r}\int_{\mathbf{R}^3} e^{\mp i\sqrt{\lambda}\omega\cdot y} f(y)dy, \quad (r \to \infty).$$

This suggests that we have two Fourier transforms

$$\left(U_0^{(\pm)}(\lambda)f\right)(\omega) = (2\pi)^{-n/2}\int_{\mathbf{R}^n} e^{\mp i\sqrt{\lambda}\omega\cdot y} f(y)dy$$

for $H_0 = -\Delta$ in $\mathbf{R}^n$. They are related as

$$U_0^{(+)}(\lambda) = JU_0^{(-)}(\lambda),$$

where J is the unitary operator on $L^2(S^{n-1})$ defined by

$$J : \varphi(\omega) \to \varphi(-\omega). \tag{6.1}$$

In the case of the hyperbolic space $\mathbf{H}^n$, two Fourier transforms for $H_0 = -\Delta_g$ are defined by

$$\begin{aligned}\left(U_0^{(\pm)}(\lambda)f\right)(x) &= \frac{\left(2\sqrt{\lambda}\sinh(\sqrt{\lambda}\pi)\right)^{1/2}}{\pi}(2\pi)^{-(n-1)/2}\\ &\times \iint_{\mathbf{R}^{n-1}\times(0,\infty)} e^{ix\cdot\xi}\left(\frac{|\xi|}{2}\right)^{\mp i\sqrt{\lambda}} y^{(n-1)/2} K_{i\sqrt{\lambda}}(|\xi|y)\hat{f}(\xi,y)\frac{d\xi dy}{y^n}.\end{aligned}$$

They are related as

$$U_0^{(+)}(\lambda) = J(\lambda)U_0^{(-)}(\lambda),$$

$$J(\lambda) = F_0^*\left(\frac{|\xi|}{2}\right)^{-2i\sqrt{\lambda}}F_0.$$

Let us return to the abstract theory in §4. Assume that we have two spectral representatios $\mathcal{F}_0^{(\pm)}$ for H_0. Define

$$\begin{aligned}\mathcal{F}^{(\pm)}(\lambda) &= \mathcal{F}^0(\lambda)(1 - VR(\lambda \pm i0)),\\ \mathcal{F}^0(\lambda) &= \mathcal{F}_0^{(+)}(\lambda),\\ \mathcal{G}^{(\pm)}(\lambda) &= \mathcal{F}_0^{(\pm)}(\lambda)(1 - VR(\lambda \pm i0)).\end{aligned}$$

Note that

$$\mathcal{G}^{(+)}(\lambda) = \mathcal{F}^{(+)}(\lambda).$$

Then by Theorem 4.7, $\mathcal{F}^{(\pm)}$, $\mathcal{G}^{(\pm)}$ give spectral representations for H. The S-matrix in §4 is defined through $\mathcal{F}^{(\pm)}(\lambda)$. Namely

$$\begin{aligned}\widehat{S} &= \mathcal{F}^{(+)}\left(\mathcal{F}^{(-)}\right)^*,\\ \widehat{S}(\lambda) &= 1 - 2\pi i\,\rho(\lambda)\mathcal{F}^0(\lambda)(V - VR(\lambda + i0)V)\mathcal{F}^0(\lambda)^*,\\ &= 1 - 2\pi i\,\rho(\lambda)\mathcal{F}^{(+)}(\lambda)V\mathcal{F}^0(\lambda)^*.\end{aligned}$$

Here we introduce a new assumption.

(A-7) *There exists a unitary operator $J(\lambda)$ on $\mathbf{h}$ satisftying*

$$\mathcal{F}_0^{(+)}(\lambda) = J(\lambda)\mathcal{F}_0^{(-)}(\lambda).$$

We define a unitary operator J on $L^2(I;\mathbf{h};d\lambda)$ by

$$\left(Jf\right)(\lambda) = J(\lambda)f(\lambda).$$

Then we have

$$\mathcal{F}^{(-)}(\lambda) = J(\lambda)\mathcal{G}^{(-)}(\lambda), \quad \mathcal{F}^{(-)} = J\mathcal{G}^{(-)}.$$

We define a new scattering operator by

$$\widehat{S}_{geo} = \mathcal{G}^{(+)}\left(\mathcal{G}^{(-)}\right)^*,$$

and a new scattering matrix by

$$\begin{aligned} \widehat{S}_{geo}(\lambda) &= \widehat{S}(\lambda)J(\lambda) \\ &= J(\lambda) - 2\pi i\,\rho(\lambda)\mathcal{F}^{(+)}(\lambda)V\mathcal{F}_0^{(-)}(\lambda)^*. \end{aligned} \tag{6.2}$$

We call $\widehat{S}_{geo}(\lambda)$ the *geometric scattering matrix*. Since $\mathcal{F}^{(+)} = \mathcal{G}^{(+)}$, we have

$$\widehat{S}_{geo} = \widehat{S}J,$$

and the following theorem holds.

Theorem 6.1. *$\widehat{S}_{geo}(\lambda)$ is unitary on* $\mathbf{h}$, *and*

$$\left(\widehat{S}_{geo}\hat{f}\right)(\lambda) = \widehat{S}_{geo}(\lambda)\hat{f}(\lambda), \quad \forall \hat{f} \in \widehat{\mathcal{H}}, \quad \forall \lambda \in I.$$

The reason why $\widehat{S}_{geo}(\lambda)$ is called the *geometric* S-matrix is as follows. Suppose we are given a Schrödinger operator H on a Riemannian manifold $\mathcal{M}$. In some cases, we can associate a *boundary at infinity* $\partial_\infty\mathcal{M}$ for $\mathcal{M}$, and construct the spectral representation $\mathcal{F}^{(\pm)}(\lambda)$ as above with $\mathbf{h} = L^2(\partial_\infty\mathcal{M})$, and prove the asymptotic expansion

$$R(\lambda \pm i0)f \simeq C_\pm(\lambda)a(\rho)e^{\pm iS(\rho,\lambda)}\mathcal{F}^{(\pm)}(\lambda)f, \quad (\rho \to \infty)$$

at infinity in an appropriate topology. Here, $R(z) = (H - z)^{-1}$ and ρ is a geodesic distance from a fixed point x_0 of $\mathcal{M}$. Moreover the solutions of the equation $(H - \lambda)u = 0$ belonging to a certain class admit the following asymptotic expansion at infinity

$$u \simeq C_-(\lambda)a(\rho)e^{-iS(\rho,\lambda)}\varphi_- + C_+(\lambda)a(\rho)e^{+iS(\rho,\lambda)}\varphi_+,$$
$$\varphi_+ = \widehat{S}_{geo}(\lambda)\varphi_-,$$

(see e.g. [**99**]). The geometric S-matrix is non-trivial even for the case $V = 0$, since $\widehat{S}_{geo}(\lambda) = J(\lambda)$. We shall discuss these facts in the next section for the case of $\mathbf{R}^n$ and $\mathbf{H}^n$.

7. Helmholtz equation and geometric S-matrix

7.1. The case of $\mathbf{H}^n$. We incoporate the results in Chap. 1 §4 and Chap. 2 §5. For $k > 0$ we define $\mathcal{F}_0^{(\pm)}(k)$ by Chap. 1 (4.2) and put

$$\mathcal{F}^0(k) = \mathcal{F}_0^{(+)}(k),$$

$$\mathcal{F}^{(\pm)}(k) = \mathcal{F}^0(k)\big(1 - VR((k \pm i0)^2)\big), \tag{7.1}$$

and $\mathcal{H}_\pm = L^{2,\pm s}$ for $s > 1/2$. Note that we write $(k \pm i0)^2$ instead of $k^2 \pm i0$. Later this choice will turn out to be convenient. Then $\mathcal{F}^0(k) \in \mathbf{B}(L^{2,s}; L^2(\mathbf{R}^{n-1}))$, and Theorem 4.7, together with the results of section 5.2, implies

$$\frac{k}{\pi i}\left(\left[R(k^2 + i0) - R(k^2 - i0)\right]f, f\right) = \|\mathcal{F}^{(\pm)}(k)f\|^2_{L^2(\mathbf{R}^{n-1})},$$

where $R(z) = (H - z)^{-1}$. Therefore by Theorem 2.3, for any $0 < a < b < \infty$ there exists a constant $C > 0$ such that

$$\|\mathcal{F}^{(\pm)}(k)f\|_{L^2(\mathbf{R}^{n-1})} \leq C\|f\|_{\mathcal{B}}. \quad a < \forall k < b, \tag{7.2}$$

By the argument in §4, we have the following theorem. Let $E(\lambda)$ be the resolution of identity for H.

Theorem 7.1. *(1) $\mathcal{F}^{(\pm)}$ defined by $\big(\mathcal{F}^{(\pm)}f\big)(k) = \mathcal{F}^{(\pm)}(k)f$ is uniquely extended to a unitary operator from $E((0,\infty))L^2(\mathbf{H}^n)$ to $L^2((0,\infty); L^2(R^{n-1}); dk)$. Moreover,*

$$\left(\mathcal{F}^{(\pm)}Hf\right)(k) = k^2\left(\mathcal{F}^{(\pm)}f\right)(k), \quad \forall k > 0, \quad \forall f \in D(H).$$

(2) For $f \in E((0,\infty))L^2(\mathbf{H}^n)$, the inversion formula holds:

$$f = \mathrm{s} - \lim_{N\to\infty} \int_{1/N}^{N} \mathcal{F}^{(\pm)}(k)^*(\mathcal{F}^{(\pm)}f)(k)dk.$$

(3) $\mathcal{F}^{(\pm)}(k)^* \in \mathbf{B}(L^2(\mathbf{R}^{n-1}); \mathcal{B}^*)$ is an eigenoperator of H in the sense that

$$(H - k^2)\mathcal{F}^{(\pm)}(k)^*\phi = 0, \quad \forall \phi \in L^2(\mathbf{R}^{n-1}).$$

(4) The wave operators

$$W_{\pm} = \mathrm{s} - \lim_{t\to\pm\infty} e^{itH}e^{-itH_0}$$

exist and $W_{\pm} = (\mathcal{F}^{(\pm)})^*\mathcal{F}^0$.
(5) The S-matrix is written as

$$\widehat{S}(k) = 1 - \frac{\pi i}{k}\mathcal{F}^{(+)}(k)V\mathcal{F}^0(k)^*, \tag{7.3}$$

and satisfies

$$\mathcal{F}^{(+)}(k) = \widehat{S}(k)\mathcal{F}^{(-)}(k). \tag{7.4}$$

We next consider the geometric scattering matrix for H. For $k > 0$ we define

$$\mathcal{G}^{(\pm)}(k) = \mathcal{F}_0^{(\pm)}(k)\big(1 - VR((k \pm i0)^2)\big). \tag{7.5}$$

As above, $\mathcal{G}^{(\pm)}(k) \in \mathbf{B}(\mathcal{B}; L^2(\mathbf{R}^{n-1}))$ and $\mathcal{G}^{(\pm)}$ give other spectral representations for H. Note that, letting F_0 be the Fourier transform on $\mathbf{R}^{n-1}$, we have

$$\mathcal{F}_0^{(+)}(k) = J(k)\mathcal{F}_0^{(-)}(k),$$

$$J(k) = F_0^*\Big(\frac{|\xi|}{2}\Big)^{-2ik}F_0. \tag{7.6}$$

We extend Theorem 1.4.7 for H. For $u, v \in \mathcal{B}^*$, we define

$$u \simeq v \iff \lim_{R\to\infty}\frac{1}{\log R}\int_{1/R}^{R}\|u(y) - v(y)\|^2_{L^2(\mathbf{R}^{n-1})}\frac{dy}{y^n} = 0.$$

Lemma 7.2. *Let $\chi(y) = 1$ $(y < 1/2)$, $\chi(y) = 0$ $(y > 1)$, and $\omega_{\pm}(k)$ be as in Chap. 1 (4.15). Then for any $\varphi \in L^2(\mathbf{R}^{n-1})$ and $k > 0$*

$$\begin{aligned}\mathcal{F}_0^{(+)}(k)^*\varphi \simeq\ & \frac{k}{\pi i}\omega_+(k)\chi(y)y^{(n-1)/2-ik}\varphi \\ & - \frac{k}{\pi i}\omega_-(k)\chi(y)y^{(n-1)/2+ik}J(k)^*\varphi,\end{aligned}$$

$$\mathcal{F}_0^{(-)}(k)^*\varphi \simeq \frac{k}{\pi i}\omega_+(k)\chi(y)y^{(n-1)/2-ik}J(k)\varphi - \frac{k}{\pi i}\omega_-(k)\chi(y)y^{(n-1)/2+ik}\varphi.$$

Proof. In view of Chap. 1 (4.14), we have only to compute the behavior of the left-hand side as $y \to 0$ for $\widehat{\varphi} \in C_0^\infty(\mathbf{R}^{n-1})$. We use Chap.1 (3.6) in the expression Chap.1 (4.10) and compute directly to prove the lemma. □

Lemma 7.3. *Let $\chi(y)$ and $\omega_\pm(k)$ be as in the previous lemma. Then, for $f \in \mathcal{B}$ and $k > 0$,*

$$R(k^2 \pm i0)f \simeq \omega_\pm(k)\chi(y)y^{(n-1)/2\mp ik}\mathcal{G}^{(\pm)}(k)f.$$

Proof. The lemma follows from the resolvent equation

$$R(k^2 \pm i0) = R_0(k^2 \pm i0) - R_0(k^2 \pm i0)VR(k^2 \pm i0),$$

Lemmas 4.7, 4.9 of Chap.1 and (7.5). □

By (6.2), the geometric scattering matrix is defined to be

$$\widehat{S}_{geo}(k) = J(k) - \frac{\pi i}{k}\mathcal{F}^{(+)}(k)V\mathcal{F}_0^{(-)}(k)^*.$$

Lemma 7.4. *For $\varphi \in L^2(\mathbf{R}^{n-1})$*

$$\mathcal{G}^{(-)}(k)^*\varphi \simeq \frac{k}{\pi i}\omega_+(k)\chi(y)y^{(n-1)/2-ik}\widehat{S}_{geo}(k)\varphi - \frac{k}{\pi i}\omega_-(k)\chi(y)y^{(n-1)/2+ik}\varphi.$$

Proof. By (7.5)

$$\mathcal{G}^{(-)}(k)^*\varphi = \mathcal{F}_0^{(-)}(k)^*\varphi - R(k^2 + i0)V\mathcal{F}_0^{(-)}(k)^*\varphi.$$

Since $\mathcal{F}^{(+)}(k) = \mathcal{G}^{(+)}(k)$, we obtain, by Lemmas 7.2 and 7.3, that

$$\begin{aligned}\mathcal{G}^{(-)}(k)^*\varphi &\simeq \frac{k}{\pi i}\omega_+(k)\chi(y)y^{(n-1)/2-ik}J(k)\varphi \\ &- \frac{k}{\pi i}\omega_-(k)\chi(y)y^{(n-1)/2+ik}\varphi - \omega_+(k)\chi(y)y^{(n-1)/2-ik}\left[J(k) - \widehat{S}_{geo}(k)\right]\varphi \\ &\simeq \frac{k}{\pi i}\omega_+(k)\chi(y)y^{(n-1)/2-ik}\widehat{S}_{geo}(k)\varphi - \frac{k}{\pi i}\omega_-(k)\chi(y)y^{(n-1)/2+ik}\varphi. \quad \square\end{aligned}$$

Lemma 7.5. *There exists a constant $C = C(k) > 0$ such that for any $\varphi \in L^2(\mathbf{R}^{n-1})$*

$$\lim_{R\to\infty}\frac{1}{\log R}\int_{1/R}^{R}\|\mathcal{G}^{(-)}(k)^*\varphi\|^2_{L^2(\mathbf{R}^{n-1})}\frac{dy}{y^n} = C\|\varphi\|^2_{L^2(\mathbf{R}^{n-1})}.$$

Proof. We put $a_+ = \widehat{S}_{geo}(k)\varphi$, $a_- = \varphi$. Then by Lemma 7.4 $\|\mathcal{G}^{(-)}(k)^*\varphi\|^2_{L^2(\mathbf{R}^{n-1})}$ behaves like

$$|C_+(k)|^2y^{n-1}\|a_+\|^2_{\mathbf{h}} + |C_-(k)|^2y^{n-1}\|a_-\|^2_{\mathbf{h}} + C_+(k)\overline{C_-(k)}y^{n-1-2ik}(a_+,a_-)_{\mathbf{h}} + C_-(k)\overline{C_+(k)}y^{n-1+2ik}(a_-,a_+)_{\mathbf{h}},$$

where $C_\pm(k)$ are constants. Simple computation shows that the 3rd and 4th terms tend to 0. As $\widehat{S}_{geo}(\lambda)$ is unitary, the lemma follows. □

Together with (7.2), this implies

Corollary 7.6. *There is a constant $C > 0$ such that*

$$C^{-1}\|\varphi\|_{L^2(\mathbf{R}^{n-1})} \leq \|\mathcal{G}^{(\pm)}(k)^*\varphi\|_{\mathcal{B}^*} \leq C\|\varphi\|_{L^2(\mathbf{R}^{n-1})}.$$

Lemma 7.7. *If $u \in \mathcal{B}^*$, $(H - k^2)u = 0$, $f \in \mathcal{B}$, and either $\mathcal{G}^{(+)}(k)f = 0$ or $\mathcal{G}^{(-)}(k)f = 0$ holds, then $(u, f) = 0$.*

Proof. The same as Lemma 1.4.10. □

These preparations are sufficient to extend Theorem 1.4.3 to H.

Theorem 7.8. *For $k > 0$*

$$\{u \in \mathcal{B}^* \, ; \, (H - k^2)u = 0\} = \mathcal{G}^{(\pm)}(k)^*\big(L^2(\mathbf{R}^{n-1})\big).$$

Theorem 7.9. *If $u \in \mathcal{B}^*$ satisfies $(H - k^2)u = 0$ for $k > 0$, there exist $\varphi_\pm \in L^2(\mathbf{R}^{n-1})$ such that*

$$u \simeq \frac{k}{\pi i}\omega_+(k)\chi(y)y^{(n-1)/2-ik}\varphi_+ - \frac{k}{\pi i}\omega_-(k)\chi(y)y^{(n-1)/2+ik}\varphi_-.$$

Moreover,

$$\varphi_+ = \widehat{S}_{geo}(k)\varphi_-.$$

Proof. By Theorem 7.8, u can be written as $u = \mathcal{G}^{(-)}(k)^*\psi$. Using Lemma 7.4, we prove the theorem. □

Theorem 7.10. *For any $\varphi_- \in L^2(\mathbf{R}^{n-1})$, there exist unique $u \in \mathcal{B}^*$ and $\varphi_+ \in L^2(\mathbf{R}^{n-1})$ such that the equation $(H - k^2)u = 0$ and the expansion in Theorem 7.9 hold.*

Proof. The existence of such φ_+ and u follows from Theorem 7.9. We prove the uniqueness. If $\varphi_- = 0$, we have $u \simeq C(k)\chi(y)y^{(n-1)/2-ik}\varphi_+$, hence u satisfies the radiation conditions (2.19), (2.20). Then $u = 0$ by Lemma 2.12, which also proves $\varphi_+ = 0$. □

7.2. The case of $\mathbf{R}^n$. It is worthwhile to give a brief look at the case of $\mathbf{R}^n$. We define the weighted L^2 space $L^{2,s}$ and the Besov type space $\mathcal{B}$ by

$$L^{2,s} \ni u \Longleftrightarrow \|u\|_s^2 = \int_{\mathbf{R}^n} (1 + |x|)^{2s}|u(x)|^2 dx < \infty,$$

$$\|u\|_{\mathcal{B}} = \sum_{j=0}^{\infty} 2^{j/2}\|u\|_{L^2(\Omega_j)} < \infty,$$

$$\Omega_j = \{x \in \mathbf{R}^n; r_{j-1} < |x| < r_j\},$$

where $r_j = 2^j$ $(j \geq 0)$, $r_{-1} = 0$. The dual space of $\mathcal{B}$ has the following equivalent norm

$$\|u\|_{\mathcal{B}^*}^2 = \sup_{R>1} \frac{1}{R}\int_{|x|<R} |u(x)|^2 dx.$$

Let H be as in subsection 5.1, $\mathbf{h} = L^2(S^{n-1})$, and put for $k > 0$

$$\left(\mathcal{F}_0^{(\pm)}(k)f\right)(\omega) = (2\pi)^{-n/2}\int_{\mathbf{R}^n} e^{\mp ik\omega\cdot x}f(x)dx,$$

$$\mathcal{F}^0(k) = \mathcal{F}_0^{(+)}(k),$$

$$\mathcal{F}^{(\pm)}(k) = \mathcal{F}^0(k)\big(1 - VR((k \pm i0)^2)\big),$$

$$\mathcal{G}^{(\pm)}(k) = \mathcal{F}_0^{(\pm)}(k)\big(1 - VR((k \pm i0)^2)\big).$$

Then the results in §5 and §6 can be applied to H. Let $E(\lambda)$ be the resolution of identity for H.

Theorem 7.11. *(1) $\mathcal{F}^{(\pm)}$ defined by $\big(\mathcal{F}^{(\pm)}f\big)(k) = \mathcal{F}^{(\pm)}(k)f$ is uniquely extended to a unitary operator from $E((0,\infty))L^2(\mathbf{R}^n)$ to $L^2((0,\infty); L^2(S^{n-1}); k^{n-1}dk)$. Moreover*

$$\left(\mathcal{F}^{(\pm)}Hf\right)(k) = k^2\left(\mathcal{F}^{(\pm)}f\right)(k), \quad \forall k > 0, \quad \forall f \in D(H).$$

(2) For $f \in E((0,\infty))L^2(\mathbf{R}^n)$, the inversion formula holds:

$$f = \mathrm{s}-\lim_{N\to\infty}\int_{1/N}^{N}\mathcal{F}^{(\pm)}(k)^*(\mathcal{F}^{(\pm)}f)(k)k^{n-1}dk.$$

(3) $\mathcal{F}^{(\pm)}(k)^* \in \mathbf{B}(L^2(S^{n-1}); \mathcal{B}^*)$ is an eigenoperator of H in the sense that

$$(H - k^2)\mathcal{F}^{(\pm)}(k)^*\phi = 0, \quad \forall \phi \in L^2(S^{n-1}).$$

8. Modified Radon transform

8.1. Extension of the Fourier transform. In order to construct the modified Radon transform associated with H in §2, we extend the definition of the generalized Fourier transform for all $k \in \mathbf{R}$. Let us repeat the definitions of the Fourier transforms introduced so far:

$$\left(\mathcal{F}_0^{(\pm)}(k)f\right)(x) = \sqrt{\frac{2}{\pi}}\,k\sqrt{\frac{\sinh(k\pi)}{k\pi}} \times F_0^*\left(\left(\frac{|\xi|}{2}\right)^{\mp ik}\int_0^\infty y^{(n-1)/2}K_{ik}(|\xi|y)\widehat{f}(\xi,y)\frac{dy}{y^n}\right),$$

$$\begin{aligned}
\mathcal{F}^0(k) &= \mathcal{F}_0^{(+)}(k),\\
\mathcal{F}_0(k) &= \frac{1}{\sqrt{2}}\Omega(k)\mathcal{F}^0(k),\\
\Omega(k) &= \frac{-i}{\Gamma(1-ik)}\sqrt{\frac{k\pi}{\sinh(k\pi)}},\\
J(k) &= F_0^*\left(\frac{|\xi|}{2}\right)^{-2ik}F_0,
\end{aligned} \tag{8.1}$$

F_0 being the Fourier transformation on $\mathbf{R}^{n-1}$. We have also defined

$$\mathcal{F}^{(\pm)}(k) = \mathcal{F}_0^{(+)}(k)(1 - VR((k \pm i0)^2)).$$

Note that the operators $\mathcal{F}_0^{(\pm)}(k)$, $\mathcal{F}^{(\pm)}(k)$ can be extended using the above formulae for $0 \neq k \in \mathbf{R}$ and, by (3.26) of Chap. 1,

$$\mathcal{F}_0^{(+)}(k) = -\mathcal{F}_0^{(-)}(-k) = J(k)\mathcal{F}_0^{(-)}(k) = -J(k)\mathcal{F}_0^{(+)}(-k),$$
$$\mathcal{F}^{(+)}(k) = -J(k)\mathcal{F}^{(-)}(-k).$$

We now define a new Fourier transformation $\mathcal{F}_\pm(k)$ by

$$\mathcal{F}_\pm(k) = \frac{1}{\sqrt{2}}\Omega(\pm k)\mathcal{F}^{(\pm)}(k), \quad 0 \neq k \in \mathbf{R}, \tag{8.2}$$

and put $(\mathcal{F}_\pm f)(k) = \mathcal{F}_\pm(k)f$. Let $\widehat{S}(k)$ be the S-matrix defined by (7.3). Then by (7.4), we have

$$\mathcal{F}_+(k) = \frac{\Gamma(1+ik)}{\Gamma(1-ik)}\widehat{S}(k)\mathcal{F}_-(k), \quad k > 0.$$

By definition we also have

$$\mathcal{F}_+(-k) = -J(-k)\mathcal{F}_-(k).$$

The following Theorem can be proved easily from the above formulas.

Theorem 8.1. *(1) $\mathcal{F}_\pm : L^2(\mathbf{H}^n) \to L^2(\mathbf{R}; L^2(\mathbf{R}^{n-1}); dk)$ is a partial isometry with initial set $E((0,\infty))L^2(\mathbf{H}^n)$, $E(\lambda)$ being the resolution of identity for H, and*

$$(\mathcal{F}_\pm H f)(k) = k^2(\mathcal{F}_\pm f)(k), \quad k \in \mathbf{R}, \quad f \in D(H).$$

(2) For $k > 0$, we have

$$\mathcal{F}_+(k) = -\frac{\Gamma(1+ik)}{\Gamma(1-ik)}\widehat{S}(k)J(k)\mathcal{F}_+(-k).$$

Consequently, the range of $\mathcal{F}_\pm$ has the following characterization:

$$g \in \operatorname{Ran}\mathcal{F}_+ \Longleftrightarrow g(k) = -\frac{\Gamma(1+ik)}{\Gamma(1-ik)}\widehat{S}(k)J(k)g(-k), \quad k > 0,$$

$$g \in \operatorname{Ran}\mathcal{F}_- \Longleftrightarrow J(k)g(-k) = -\frac{\Gamma(1+ik)}{\Gamma(1-ik)}\widehat{S}(k)g(k), \quad k > 0.$$

Note that the above relation is rewritten as

$$g \in \operatorname{Ran}\mathcal{F}_+ \Longleftrightarrow g(k) = -\frac{\Gamma(1+ik)}{\Gamma(1-ik)}\widehat{S}_{geo}(k)g(-k), \quad k > 0.$$

We put

$$\mathcal{H}_{>0} = L^2((0,\infty); L^2(\mathbf{R}^{n-1}); dk), \quad \mathcal{H}_{<0} = L^2((-\infty,0); L^2(\mathbf{R}^{n-1}); dk), \tag{8.3}$$

and let r_+ and r_- be the projections onto $\mathcal{H}_{>0}$ and $\mathcal{H}_{<0}$, respectively.

Lemma 8.2.

$$W_+ = 2(\mathcal{F}_+)^* r_+ \mathcal{F}_0, \quad W_- = 2(\mathcal{F}_+)^* r_- \mathcal{F}_0, \tag{8.4}$$

$$W_+ = 2(\mathcal{F}_-)^* G r_- \mathcal{F}_0, \quad W_- = 2(\mathcal{F}_-)^* G r_+ \mathcal{F}_0, \tag{8.5}$$

where G is the operator of multiplication by $\dfrac{\Gamma(1-ik)}{\Gamma(1+ik)}$.

Proof. Recall that $|\Omega(k)| = 1$ and $J(k)$ is unitary on $L^2(\mathbf{R}^{n-1})$. By Theorem 7.1(4), using $\mathcal{F}^0(-k) = -J(-k)\mathcal{F}^0(k)$ and $\mathcal{F}^{(-)}(-k) = -J(-k)\mathcal{F}^{(+)}(k)$, we have, for $f, g \in \mathcal{B}$,

$$\begin{aligned}(W_- f, g) &= (\mathcal{F}^0 f, \mathcal{F}^{(-)} g)\\ &= \int_0^\infty (\mathcal{F}^0(k) f, \mathcal{F}^{(-)}(k) g) dk\\ &= \int_{-\infty}^0 (J(-k)\mathcal{F}^0(k) f, J(-k)\mathcal{F}^{(+)}(k) g) dk\\ &= \int_{-\infty}^0 (\Omega(k)\mathcal{F}^0(k) f, \Omega(k)\mathcal{F}^{(+)}(k) g) dk\\ &= 2\int_{-\infty}^0 (\mathcal{F}_0(k) f, \mathcal{F}_+(k) g) dk\\ &= (2(\mathcal{F}_+)^* r_- \mathcal{F}_0 f, g),\end{aligned}$$

which proves (8.4) for W_-. By the similar and simpler manner, one can prove (8.4) for W_+. Using $\mathcal{F}^0(-k) = -J(-k)\mathcal{F}^0(k)$ and $\mathcal{F}^{(+)}(-k) = -J(-k)\mathcal{F}^{(-)}(k)$, we have for $f, g \in \mathcal{B}$

$$\begin{aligned}(W_+ f, g) &= (\mathcal{F}^0 f, \mathcal{F}^{(+)} g)\\ &= \int_0^\infty (\mathcal{F}^0(k) f, \mathcal{F}^{(+)}(k) g) dk\\ &= \int_{-\infty}^0 (J(-k)\mathcal{F}^0(k) f, J(-k)\mathcal{F}^{(-)}(k) g) dk\\ &= \int_{-\infty}^0 \frac{\Omega(-k)}{\Omega(k)} (\Omega(k)\mathcal{F}^0(k) f, \Omega(-k)\mathcal{F}^{(-)}(k) g) dk\\ &= 2\int_{-\infty}^0 \frac{\Omega(-k)}{\Omega(k)} (\mathcal{F}_0(k) f, \mathcal{F}_-(k) g) dk\\ &= (2(\mathcal{F}_-)^* G r_- \mathcal{F}_0 f, g),\end{aligned}$$

which proves (8.5) for W_+. Similarly, we can prove (8.5) for W_-. □

We define operators $\hat{I}$ and U on $L^2(\mathbf{R}; L^2(\mathbf{R}^{n-1}; dk)$ by

$$(\hat{I} f)(k) = f(-k),$$
$$(U f)(k) = \frac{\Gamma(1-ik)}{\Gamma(1+ik)} \big(F_0^* \left(\frac{|\xi|}{2}\right)^{2ik} F_0 f\big)(k).$$

Direct computation shows the following relations:

$$\hat{I} r_+ = r_- \hat{I},\quad \hat{I} U \hat{I} = U^{-1},\quad U r_\pm = r_\pm U. \tag{8.6}$$

Lemma 8.3.

$$\mathcal{F}_0(\mathcal{F}_0)^* = \frac{1}{2}(I + \hat{I} U). \tag{8.7}$$

Proof. Let $\Pi = (I + \hat{I} U)/2$. Then by (8.6), one can show $\Pi^* = \Pi^2 = \Pi$. Moreover, $g = \Pi f$ satisfies $\hat{I} g = U g$. Therefore by Lemma 1.5.2 (3), Π is the projection onto the range of $\mathcal{F}_0$. □

Lemma 8.4.

$$\mathcal{F}_+ = r_+\mathcal{F}_0(W_+)^* + r_-\mathcal{F}_0(W_-)^*, \tag{8.8}$$

$$\mathcal{F}_- = Gr_+\mathcal{F}_0(W_-)^* + Gr_-\mathcal{F}_0(W_+)^*. \tag{8.9}$$

Proof. By (8.4) and (8.7),

$$\begin{aligned}\mathcal{F}_0(W_+)^* &= 2\mathcal{F}_0(\mathcal{F}_0)^* r_+\mathcal{F}_+ \\ &= r_+\mathcal{F}_+ + \hat{I}Ur_+\mathcal{F}_+.\end{aligned}$$

Since $\hat{I}Ur_+ = r_-\hat{I}U$ by (8.6), multiplying both sides by r_+, we obtain

$$r_+\mathcal{F}_0(W_+)^* = r_+\mathcal{F}_+.$$

Similarly, we have

$$r_-\mathcal{F}_0(W_-)^* = r_-\mathcal{F}_+.$$

Adding these two equalities, we obtain (8.8). The formula (8.9) is proved in a similar manner. □

8.2. Modified Radon transform. We now define the modified Radon transform for H.

Definition 8.5. For $s \in \mathbf{R}$, we define

$$(\mathcal{R}_\pm f)(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iks}(\mathcal{F}_\pm f)(k)dk.$$

Theorem 8.6. *$\mathcal{R}_\pm$ is a partial isometry from $L^2(\mathbf{H}^n)$ to $L^2(\mathbf{R}; L^2(\mathbf{R}^{n-1}); dk)$ with initial set $E((0,\infty))L^2(\mathbf{H}^n)$. The Fourier transform of the final set of $\mathcal{R}_\pm$ is characterized by Theorem 8.1 (2). Moreover*

$$\mathcal{R}_\pm H = -\partial_s^2\mathcal{R}_\pm.$$

The scattering operator can also be defined by the Radon transform.

Definition 8.7. We define the scattering operator $\mathcal{S}_R$ by

$$\mathcal{S}_R = \mathcal{R}_+(\mathcal{R}_-)^*.$$

Lemma 8.8. *The scattering operator $\mathcal{S}_R$ is a partial isometry with initial set* $\mathrm{Ran}\,\mathcal{R}_-$ *and final set* $\mathrm{Ran}\,\mathcal{R}_+$. *The relation between $S = (W_+)^*W_-$ and $\mathcal{S}_R$ is given by the following formula. Let $\mathcal{F}_1$ be the 1-dimensional Fourier transformation. Then*

$$\mathcal{F}_1\mathcal{S}_R(\mathcal{F}_1)^* = r_+\mathcal{F}_0S(\mathcal{F}_0)^*r_+G^* + r_-\mathcal{F}_0S^*(\mathcal{F}_0)^*r_-G^* + \frac{1}{2}\hat{I}UG^*.$$

Proof. The first half of the lemma follows from the definition. Since $\mathcal{F}_1\mathcal{S}_R(\mathcal{F}_1)^* = \mathcal{F}_+(\mathcal{F}_-)^*$, the second half follows from Lemma 8.4 and direct computation. □

8.3. Asymptotic profiles of solutions to the wave equation. We compute the asymptotic profile of the solution

$$u(t) = \cos(t\sqrt{H})f + \sin(t\sqrt{H})\sqrt{H}^{-1}g$$

to the wave equation

$$\begin{cases}\partial_t^2 u + Hu = 0, \\ u\big|_{t=0} = f, \quad \partial_t u\big|_{t=0} = g.\end{cases}$$

Theorem 8.9. *For any $f \in E((0,\infty))L^2(\mathbf{H}^n)$, we have as $t \to \infty$*

$$\left\| \cos(t\sqrt{H})f - \frac{y^{(n-1)/2}}{\sqrt{2}}(\mathcal{R}_+ f)(-\log y - t, x)\right\|_{L^2(\mathbf{H}^n)} \to 0,$$

$$\left\| \sin(t\sqrt{H})f - \frac{iy^{(n-1)/2}}{\sqrt{2}}(\mathcal{R}_+ sgn(-i\partial_s)f)(-\log y - t, x)\right\|_{L^2(\mathbf{H}^n)} \to 0,$$

where sgn is defined in Theorem 1.5.5.

Proof. Using the relations

$$\mathcal{F}^{(+)}(k)^* = \mathcal{F}^0(k)^* - R((k-i0)^2)V\mathcal{F}^{(+)}(k)^*,$$

we have by the spectral representation theorem

$$\begin{aligned} e^{-it\sqrt{H}}f &= \int_0^\infty e^{-itk}\mathcal{F}^{(+)}(k)^*\left(\mathcal{F}^{(+)}f\right)(k)dk \\ &= \int_0^\infty e^{-itk}\mathcal{F}^0(k)^*\left(\mathcal{F}^{(+)}f\right)(k)dk \\ &\quad - \int_0^\infty e^{-itk}R(k^2-i0)V\mathcal{F}^0(k)^*\left(\mathcal{F}^{(+)}f\right)(k)dk. \end{aligned} \tag{8.10}$$

By the same computation as in the proof of Theorem 1.5.5, the first term of the right-hand side of (8.10) tends to

$$\frac{y^{(n-1)/2}}{\sqrt{\pi}}\int_0^\infty e^{ik(-\log y - t)}\left(\mathcal{F}_+ f\right)(k)dk$$

as $t \to \infty$.

We need the following lemma to deal with the 2nd term of the right-hand side of (8.10).

Lemma 8.10. *Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$. For $\psi(k) \in C_0((0,\infty);\mathcal{H})$ we put*

$$\Psi_\pm(t) = \int_0^\infty e^{\pm ikt}\psi(k)dk.$$

Then for any $\epsilon > 0$

$$\left\|\int_0^\infty (A - k \mp i\epsilon)^{-1}e^{\pm ikt}\psi(k)dk\right\| \leq \int_t^\infty \|\Psi_\pm(s)\|ds$$

holds. Similarly letting

$$\Phi_\pm(t) = \int_{-\infty}^0 e^{\mp ikt}\psi(k)dk$$

for $\psi(k) \in C_0((-\infty,0);\mathcal{H})$, we have for any $\epsilon > 0$

$$\left\|\int_{-\infty}^0 (A + k \pm i\epsilon)^{-1}e^{\mp ikt}\psi(k)dk\right\| \leq \int_{-\infty}^t \|G_\mp(s)\|ds.$$

Proof. By virtue of the identity

$$(A - k \mp i\epsilon)^{-1} = \pm i\int_0^\infty e^{\mp is(A-k\mp i\epsilon)}ds,$$

we have

$$\int_0^\infty (A - k \mp i\epsilon)^{-1}e^{\pm ikt}\psi(k)dk = \pm i\int_0^\infty e^{\mp is(A\mp i\epsilon)}\Psi_\pm(s+t)ds,$$

which proves the first half of the lemma. We also have

$$(A + k \mp i\epsilon)^{-1} = \pm i \int_{-\infty}^{0} e^{\pm is(A+k\mp i\epsilon)} ds$$

which proves the second half. □

Proof of Theorem 8.9 (continued). Letting $\sqrt{H} = A$, we have

$$(H - k^2 \mp i0)^{-1} = (A - k \mp i0)^{-1}(A + k)^{-1}.$$

Therefore, to show that the 2nd term of the right-hand side of (8.10) tends to 0, letting

$$\psi(k) = (A + k)^{-1} V \mathcal{F}_0(k)^* \left(\mathcal{F}^{(+)} f\right)(k),$$

$$\Psi(t) = \int_0^\infty e^{-ikt} \psi(k) dk,$$

we have only to prove

$$\int_0^\infty \|\Psi(t)\| dt < \infty.$$

Take $g \in L^2(\mathbf{H}^n)$, and consider

$$(\Psi(t), g) = \int_0^\infty e^{-ikt} (V\mathcal{F}_0(k)^* \big(\mathcal{F}^{(+)} f\big)(k), (A + k)^{-1} g) dk.$$

Arguing in the same way as the proof of (A-4) in Subsection 5.2. we have

$$|(\Psi(t), g)| \le C(1 + t)^{-1-\epsilon} \|g\|,$$

implying that $\|\Psi(t)\| \le C(1 + t)^{-1-\epsilon}$. We have thus derived that

$$\left\| e^{-it\sqrt{H}} f - \frac{y^{(n-1)/2}}{\sqrt{\pi}} \int_0^\infty e^{ik(-\log y - t)} \left(\mathcal{F}_+ f\right)(k) dk \right\| \to 0 \tag{8.11}$$

as $t \to \infty$.

By using the relation

$$\mathcal{F}^{(-)}(k)^* \mathcal{F}^{(-)}(k) = \mathcal{F}^{(+)}(-k)^* \mathcal{F}^{(+)}(-k),$$

we have as above

$$\begin{aligned} e^{-it\sqrt{H}} f &= \int_0^\infty e^{-itk} \mathcal{F}^{(-)}(k)^* \left(\mathcal{F}^{(-)} f\right)(k) dk \\ &= \int_{-\infty}^0 e^{itk} \mathcal{F}^{(+)}(k)^* \left(\mathcal{F}^{(+)} f\right)(k) dk \\ &= \int_{-\infty}^0 e^{itk} \mathcal{F}^0(k)^* \left(\mathcal{F}^{(+)} f\right)(k) dk \\ &\quad - \int_{-\infty}^0 e^{itk} R(k^2 + i0) V \mathcal{F}^0(k)^* \left(\mathcal{F}^{(+)} f\right)(k) dk. \end{aligned}$$

Arguing as above, we can derive

$$\left\| e^{-it\sqrt{H}} f - \frac{y^{(n-1)/2}}{\sqrt{\pi}} \int_{-\infty}^0 e^{ik(-\log y + t)} \left(\mathcal{F}_+ f\right)(k) dk \right\| \to 0 \tag{8.12}$$

as $t \to -\infty$. Theorem 8.9 then follows from (8.11) and (8.12). □

8.4. Invariance principle. Suppose for two self-adjoint operators A and B, the wave operator

$$W_\pm = \mathrm{s} - \lim_{t\to\pm\infty} e^{itA}e^{-itB}P_{ac}(B),$$

exists, where $P_{ac}(B)$ denotes the projection onto the absolutely continuous subspace for B. Then, for a suitable Borel function $\phi(s)$ on $\mathbf{R}$, the wave operator

$$W_\pm^{(\phi)} = \mathrm{s} - \lim_{t\to\pm\infty} e^{it\phi(A)}e^{-it\phi(B)}P_{ac}(B),$$

exists and $W_\pm = W_\pm^{(\phi)}$. This fact is called *invariance principle*, and is proved in a general setting (see e.g. pp. 545, 579 of [**80**]). We are interested in the case where $\phi(s) = \sqrt{s}$. Then $W_\pm$ is the wave operator for the Schrödinger equation, and $W_\pm^{(\phi)}$ is the wave operator for the wave equation.

Under the assumptions in the present chapter, we can prove this invariance principle directly for the above operators H and H_0 on $\mathbf{H}^n$. In fact, letting

$$H_+ = E_H((0,\infty))H,$$

where $E_H(\lambda)$ is the spectral resolution for H, the existence of the strong limit

$$\mathrm{s} - \lim_{t\to\pm\infty} e^{it\sqrt{H_+}}e^{-it\sqrt{H_0}}P_{ac}(H_0) \tag{8.13}$$

can be proven by the same argument as that for the wave operator

$$W_\pm = \mathrm{s} - \lim_{t\to\pm\infty} e^{itH}e^{-itH_0}.$$

Observing the proof of Theorem 8.9 (see the arguments after (8.10)), we see that for $f \in \mathcal{H}_{ac}(H) = E_H((0,\infty))L^2(\mathbf{H}^n)$ (see Chap. 2, Subsection 5.3)

$$\left\| e^{-it\sqrt{H}}f - \int_0^\infty e^{-itk}\mathcal{F}^0(k)^*(\mathcal{F}^{(\pm)}f)(k)dk \right\| \to 0,$$

as $t \to \infty$, which implies that

$$\mathrm{s} - \lim_{t\to\pm\infty} e^{it\sqrt{H_+}}e^{-it\sqrt{H_0}}P_{ac}(H_0) = \left(\mathcal{F}^{(+)}\right)^*\mathcal{F}^0 = W_+.$$

Note that, since $E_H((0,\infty)) = P_{ac}(H)$, we have

$$\mathrm{s} - \lim_{t\to\pm\infty} e^{it\sqrt{H_+}}e^{-it\sqrt{H_0}}P_{ac}(H_0) = \mathrm{s} - \lim_{t\to\pm\infty} P_{ac}(H)e^{it\sqrt{H}}e^{-it\sqrt{H_0}}P_{ac}(H_0). \tag{8.14}$$

We have thus proven the following theorem.

Theorem 8.11. *Let H and H_0 be as in Subsection 2.2. Then the wave operator for the wave equation*

$$\mathrm{s} - \lim_{t\to\pm\infty} e^{it\sqrt{H_+}}e^{-it\sqrt{H_0}}P_{ac}(H_0)$$

exists and is equal to the wave operator for the Schrödinger equation

$$\mathrm{s} - \lim_{t\to\pm\infty} e^{itH}e^{-itH_0}P_{ac}(H_0).$$

In particular, this theorem implies that the scattering matrix for the Schrödinger equation and that for the wave equation coincide.