Chapter 6

Some facts from analytic number theory

The aim of this chapter is to prove Claim 5.5 which was used in proving the main theorem, i.e., the Bohr-Jessen limit theorem. Following Matsumoto [26], we show a general theorem, i.e., Carlson's mean value theorem $\left[\text{cf. Theorem 6.3}\right]$, and then apply it to prove this claim.

For the proof of this claim and Carlson's mean value theorem, we study the following matters:

- Square mean value estimate of the Riemann zeta function, in other words, asymptotics of $\int_1^T |\zeta(\sigma+\sqrt{-1}t)|^2 dt$ as $T \to \infty$,
- Exponential decay of $\Gamma^{(l)}(\sigma+\sqrt{-1}t)$ as $|t| \to \infty$, where $\Gamma^{(l)}$ is the lth derivative of the gamma function.

These are discussed in Sections 6.1 and 6.2 respectively. After that, in Section 6.3, we present Carlson's mean value theorem and give its proof. Finally, in Section 6.4, we prove Claim 5.5, which is quickly finished owing to considerable efforts up to then.

6.1 Square mean value estimate of $\zeta(s)$

We begin with an easy part of the square mean value estimate.

Claim 6.1 $$\left|\int_1^T \left|\zeta(\sigma+\sqrt{-1}t)\right|^2 dt - T\zeta(2\sigma)\right|$$
$$\leq \zeta(2\sigma) + 8\zeta(2\sigma-1) - 8\zeta'(2\sigma-1) + 4\frac{\zeta(\sigma)^2}{\log 2}, \quad T \geq 1, \sigma > 1.$$

Proof. Fix $\sigma > 1$. For each $t \in \mathbb{R}$,

$$\begin{aligned}
\left|\zeta(\sigma+\sqrt{-1}t)\right|^2 &= \zeta(\sigma+\sqrt{-1}t)\overline{\zeta(\sigma+\sqrt{-1}t)} \\
&= \sum_{m=1}^{\infty} \frac{1}{m^{\sigma+\sqrt{-1}t}} \sum_{n=1}^{\infty} \frac{1}{n^{\sigma-\sqrt{-1}t}} \\
&= \sum_{n,m=1}^{\infty} \frac{n^{\sqrt{-1}t} m^{-\sqrt{-1}t}}{n^\sigma m^\sigma}
\end{aligned}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^{2\sigma}} + \sum_{\substack{n,m\geq 1;\\ n\neq m}} \frac{e^{\sqrt{-1}t(\log n - \log m)}}{n^{\sigma} m^{\sigma}}$$
$$= \zeta(2\sigma) + \sum_{n>m\geq 1} \frac{1}{n^{\sigma} m^{\sigma}} \left(e^{\sqrt{-1}t \log \frac{n}{m}} + e^{-\sqrt{-1}t \log \frac{n}{m}}\right)$$
$$= \zeta(2\sigma) + 2 \sum_{n>m\geq 1} \frac{1}{n^{\sigma} m^{\sigma}} \cos\Bigl(t \log \frac{n}{m}\Bigr).$$

Integration in $t \in [1, T]$ gives that

$$\int_1^T \bigl|\zeta(\sigma + \sqrt{-1}t)\bigr|^2 dt = (T-1)\zeta(2\sigma) + 2 \sum_{n>m\geq 1} \frac{1}{n^{\sigma} m^{\sigma}} \int_1^T \cos\Bigl(t \log \frac{n}{m}\Bigr) dt$$
$$= (T-1)\zeta(2\sigma) + 2 \sum_{n>m\geq 1} \frac{1}{n^{\sigma} m^{\sigma}} \frac{\sin(T \log \frac{n}{m}) - \sin(\log \frac{n}{m})}{\log \frac{n}{m}}.$$

From this, it follows that

$$\Bigl|\int_1^T \bigl|\zeta(\sigma + \sqrt{-1}t)\bigr|^2 dt - T\zeta(2\sigma)\Bigr|$$
$$= \left|-\zeta(2\sigma) + 2 \sum_{n>m\geq 1} \frac{1}{n^{\sigma} m^{\sigma}} \frac{\sin(T \log \frac{n}{m}) - \sin(\log \frac{n}{m})}{\log \frac{n}{m}}\right|$$
$$\leq \zeta(2\sigma) + 2 \sum_{n>m\geq 1} \frac{1}{n^{\sigma} m^{\sigma}} \frac{|\sin(T \log \frac{n}{m}) - \sin(\log \frac{n}{m})|}{\log \frac{n}{m}}$$
$$\leq \zeta(2\sigma) + 4 \sum_{n>m\geq 1} \frac{1}{n^{\sigma} m^{\sigma}} \frac{1}{\log \frac{n}{m}}. \tag{6.1}$$

Let us estimate from above the series of the 2nd term in R.H.S. First, we divide this into two terms:

$$\sum_{n>m\geq 1} \frac{1}{n^{\sigma} m^{\sigma}} \frac{1}{\log \frac{n}{m}} = \sum_{m=1}^{\infty} \frac{1}{m^{\sigma}} \sum_{n>m} \frac{1}{n^{\sigma}} \frac{1}{\log \frac{n}{m}}$$
$$= \sum_{m=1}^{\infty} \frac{1}{m^{\sigma}} \sum_{m<n\leq 2m} \frac{1}{n^{\sigma}} \frac{1}{\log \frac{n}{m}} + \sum_{m=1}^{\infty} \frac{1}{m^{\sigma}} \sum_{n>2m} \frac{1}{n^{\sigma}} \frac{1}{\log \frac{n}{m}}$$
$$=: \text{the 1st term} + \text{the 2nd term}.$$

From the implications

$$n > 2m \Rightarrow \frac{n}{m} > 2 \Rightarrow \log \frac{n}{m} > \log 2 \Rightarrow \frac{1}{\log \frac{n}{m}} < \frac{1}{\log 2},$$

the 2nd term is etstimated as

$$\text{the 2nd term} \leq \frac{1}{\log 2} \sum_{m=1}^{\infty} \frac{1}{m^{\sigma}} \sum_{n>2m} \frac{1}{n^{\sigma}} \leq \frac{1}{\log 2} \sum_{m=1}^{\infty} \frac{1}{m^{\sigma}} \sum_{n=1}^{\infty} \frac{1}{n^{\sigma}} = \frac{\zeta(\sigma)^2}{\log 2}.$$

From the inequalities

$$\log(1+x) = \int_0^1 \big(\log(1+sx)\big)' ds = \int_0^1 \frac{x}{1+sx} ds \geq \frac{x}{2} \quad (0 \leq x \leq 1) \tag{6.2}$$

and

$$\sum_{k=1}^{m} \frac{1}{k} = 1 + \sum_{k=2}^{m} \int_{k-1}^{k} \frac{dt}{k} \leq 1 + \sum_{k=2}^{m} \int_{k-1}^{k} \frac{dt}{t} = 1 + \int_1^m \frac{dt}{t} = 1 + \log m, \tag{6.3}$$

the 1st term is estimated as

$$\begin{aligned}
\text{the 1st term} &= \sum_{m=1}^{\infty} \frac{1}{m^\sigma} \sum_{m<n\leq 2m} \frac{1}{n^\sigma} \frac{1}{\log(1+\frac{n}{m}-1)} \\
&\leq \sum_{m=1}^{\infty} \frac{1}{m^\sigma} \sum_{m<n\leq 2m} \frac{1}{n^\sigma} \frac{2}{\frac{n}{m}-1} \\
&= 2\sum_{m=1}^{\infty} \frac{m}{m^\sigma} \sum_{m<n\leq 2m} \frac{1}{n^\sigma} \frac{1}{n-m} \\
&\leq 2\sum_{m=1}^{\infty} \frac{1}{m^{2\sigma-1}} \sum_{m<n\leq 2m} \frac{1}{n-m} \qquad \left[\because m<n \Rightarrow \tfrac{1}{n^\sigma} < \tfrac{1}{m^\sigma}\right] \\
&= 2\sum_{m=1}^{\infty} \frac{1}{m^{2\sigma-1}} \sum_{k=1}^{m} \frac{1}{k} \\
&\leq 2\sum_{m=1}^{\infty} \frac{1+\log m}{m^{2\sigma-1}} \\
&= 2\Big(\sum_{m=1}^{\infty} \frac{1}{m^{2\sigma-1}} + \sum_{m=1}^{\infty} \frac{\log m}{m^{2\sigma-1}}\Big) \\
&= 2\big(\zeta(2\sigma-1) - \zeta'(2\sigma-1)\big).
\end{aligned}$$

Combining these with (6.1), we have the assertion of the claim at once. ■

Square mean value estimate for $\frac{1}{2} < \sigma \leq 1$ does not go well as above. We need the following theorem:

Theorem 6.1 *For $C > 1$, $x \geq 1$ and $s = \sigma + \sqrt{-1}t \neq 1$ with $\sigma > 0$, $|t| \leq \frac{2\pi x}{C}$, the following estimate holds:*

$$\begin{aligned}
&\Big|\zeta(s) - \Big(\sum_{n\leq x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s}\Big)\Big| \\
&\leq x^{-\sigma}\left(1 + \frac{3}{\pi}\log 3 + \frac{5}{12}\sigma + \frac{3}{2\pi}\Big(1+\frac{\sigma}{\pi}\Big)\frac{C}{C-1}\right).
\end{aligned}$$

For the proof, we present a lemma and a claim:

Lemma 6.1 *Let $N \in \mathbb{N}$. On $\{s \in \mathbb{C}; \operatorname{Re} s > 0\} \setminus \{1\}$,*

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} - \frac{N^{1-s}}{1-s} - s\int_N^\infty \frac{\{x\} - \frac{1}{2}}{x^{s+1}} dx - \frac{1}{2} N^{-s}.$$

Proof. Fix $N \in \mathbb{N}$. Let $s \in \mathbb{C}$, $\operatorname{Re} s > 1$. Theorem 4.1 with $f(x) = x^{-s}$, $a = N$, $b = M$ (where $M \in \mathbb{N}$, $M > N$) and $n = 1$ gives that

$$\begin{aligned}\sum_{k=N+1}^{M} \frac{1}{k^s} &= \int_N^M x^{-s} dx - \Big(\frac{\overline{B_1}(M)}{M^s} - \frac{\overline{B_1}(N)}{N^s}\Big) - s\int_N^M \frac{\overline{B_1}(x)}{x^{s+1}} dx \\ &= \frac{M^{1-s} - N^{1-s}}{1-s} + \frac{1}{2}\Big(\frac{1}{M^s} - \frac{1}{N^s}\Big) - s\int_N^M \frac{\{x\} - \frac{1}{2}}{x^{s+1}} dx \\ &\qquad \big[\text{cf. Claim 4.2(i)}\big].\end{aligned}$$

By letting $M \to \infty$,

$$\sum_{k=N+1}^{\infty} \frac{1}{k^s} = \frac{-N^{1-s}}{1-s} - \frac{1}{2}\frac{1}{N^s} - s\int_N^\infty \frac{\{x\} - \frac{1}{2}}{x^{s+1}} dx,$$

and thus

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} - \frac{N^{1-s}}{1-s} - s\int_N^\infty \frac{\{x\} - \frac{1}{2}}{x^{s+1}} dx - \frac{1}{2} N^{-s}.$$

By 1°(b) in the proof of Theorem 4.2, the function of R.H.S. is meromorphic on $\{s \in \mathbb{C}; \operatorname{Re} s > 0\}$, is holomorphic except $s = 1$, and has a simple pole at $s = 1$, with residue 1. Therefore, by the uniqueness theorem, the identity above is valid on $\{s \in \mathbb{C}; \operatorname{Re} s > 0\} \setminus \{1\}$. ∎

Claim 6.2 *Let $-\infty < a < b < \infty$. Let $f : [a, b] \to \mathbb{R}$ be of class C^1, f' nonincreasing, $g : [a, b] \to [0, \infty)$ of class C^1 and nonincreasing, and $|g'|$ nonincreasing. For $\alpha = f'(b)$, $\beta = f'(a)$, $0 < \eta < 1$, it holds that*

$$\begin{aligned}&\Big|\sum_{a<n\le b} g(n) e^{\sqrt{-1}2\pi f(n)} - \sum_{\lfloor\alpha-\eta\rfloor \le \nu < \beta+\eta} \int_a^b g(x) e^{\sqrt{-1}2\pi(f(x)-\nu x)} dx\Big| \\ &\le g(a) + \frac{|g'(a)|}{6} \\ &\quad + \frac{3|g'(a)|}{4\pi^2}\Bigg(\mathbf{1}_{\beta-\lfloor\alpha-\eta\rfloor\ge\frac{1}{2}} \frac{1}{\beta - \lfloor\alpha-\eta\rfloor}\Big(\frac{1}{1+\lfloor\beta\rfloor-\beta} + \log\big(\lfloor 2\beta\rfloor - \beta - \lfloor\alpha-\eta\rfloor\big) \\ &\qquad\qquad - \log(1 + \lfloor\beta\rfloor - \beta)\Big) + \frac{\pi^2}{3}\Bigg) \\ &\quad + \frac{3g(a)}{2\pi}\log\Big(2\big(\lfloor\beta+\eta\rfloor - \lfloor\alpha-\eta\rfloor\big) + 1\Big) \\ &\quad + \frac{3g(a)}{2\pi}\log\Big(2\big(\lfloor\beta\rfloor - \lfloor\alpha-\eta\rfloor\big) + 3\Big) \\ &\quad + \frac{3g(a)}{2\pi}\Bigg(\frac{1}{1+\lfloor\beta\rfloor-\beta} + \log\Big(2\big(\lfloor\beta\rfloor - \lfloor\alpha-\eta\rfloor\big) + 3\Big)\Bigg).\end{aligned}$$

This claim is a key lemma connecting the *exponential sum*

$$\sum_{a<n\le b} g(n)e^{\sqrt{-1}2\pi f(n)}$$

with the *exponential integral*

$$\int_a^b g(x)e^{\sqrt{-1}2\pi(f(x)-\nu x)}dx.$$

We here write R.H.S. above in a naked form so as to reveal the dependence of parameters α, β, η. It enables us to apply this key lemma to various cases.

Recognizing Claim 6.2, let us prove Theorem 6.1:

Proof of Theorem 6.1. Fix $C > 1$ and $x \ge 1$. Let $s = \sigma + \sqrt{-1}t \ne 1$ satisfy $\sigma > 0$, $|t| \le \frac{2\pi x}{C}$. We divide the proof into two steps:

$\underline{1^{\text{o}}}$ For $N \in \mathbb{N} \cap (x, \infty)$,

$$\left|\sum_{x<n\le N}\frac{1}{n^s} - \frac{N^{1-s}-x^{1-s}}{1-s}\right| \le x^{-\sigma}\left(1+\frac{3}{\pi}\log 3+\frac{5}{12}\sigma+\frac{3}{2\pi}\left(1+\frac{\sigma}{\pi}\right)\frac{C}{C-1}\right).$$

$\because$ Apply Claim 6.2. Let $t \ne 0$, and take $a = x$, $b = N$,

$$f(u) = \begin{cases} \dfrac{t}{2\pi}\log u & \text{if } t > 0, \\ \dfrac{-t}{2\pi}\log u & \text{if } t < 0, \end{cases} \qquad g(u) = u^{-\sigma}.$$

Clearly f, g satisfy the assumptions in Claim 6.2. Since $\alpha = f'(b) = \frac{|t|}{2\pi}\frac{1}{b} = \frac{|t|}{2\pi}\frac{1}{N} > 0$, $\beta = f'(a) = \frac{|t|}{2\pi}\frac{1}{a} = \frac{|t|}{2\pi}\frac{1}{x} \le \frac{1}{C} < 1$, we can take $0 < \eta < 1$ so that $0 < \beta + \eta < 1$, $0 < \alpha - \eta < 1$. Then Claim 6.2 gives that

$$\begin{aligned}
&\left|\sum_{x<n\le N}\frac{e^{\sqrt{-1}|t|\log n}}{n^\sigma} - \int_x^N \frac{e^{\sqrt{-1}|t|\log u}}{u^\sigma}du\right| \\
&\le x^{-\sigma} + \frac{\sigma}{6}x^{-\sigma-1} \\
&\quad + \frac{3\sigma x^{-\sigma-1}}{4\pi^2}\left(\mathbf{1}_{\frac{|t|}{2\pi}\frac{1}{x}\ge\frac{1}{2}}\frac{1}{\frac{|t|}{2\pi}\frac{1}{x}}\left(\frac{1}{1-\frac{|t|}{2\pi}\frac{1}{x}} + \log\left(1-\frac{|t|}{2\pi}\frac{1}{x}\right) - \log\left(1-\frac{|t|}{2\pi}\frac{1}{x}\right)\right) + \frac{\pi^2}{3}\right) \\
&\quad + \frac{3x^{-\sigma}}{2\pi}\log 3 + \frac{3x^{-\sigma}}{2\pi}\left(\frac{1}{1-\frac{|t|}{2\pi}\frac{1}{x}} + \log 3\right) \\
&\le x^{-\sigma} + \frac{\sigma}{6}x^{-\sigma-1} \\
&\quad + \frac{3\sigma x^{-\sigma-1}}{4\pi^2}\left(2\frac{C}{C-1} + \frac{\pi^2}{3}\right) + \frac{3x^{-\sigma}}{2\pi}\left(\frac{C}{C-1} + 2\log 3\right) \\
&\qquad \left[\because \frac{|t|}{2\pi}\frac{1}{x} \le \frac{1}{C} \Rightarrow \frac{1}{1-\frac{|t|}{2\pi}\frac{1}{x}} \le \frac{1}{1-\frac{1}{C}} = \frac{C}{C-1}\right]
\end{aligned}$$

$$
\begin{aligned}
&= x^{-\sigma}\left(1+\frac{\sigma}{6}\frac{1}{x}+\frac{3\sigma}{4\pi^2}\frac{1}{x}\left(2\frac{C}{C-1}+\frac{\pi^2}{3}\right)+\frac{3}{2\pi}\left(\frac{C}{C-1}+2\log 3\right)\right)\\
&\le x^{-\sigma}\left(1+\frac{\sigma}{6}+\frac{3\sigma}{4\pi^2}\left(2\frac{C}{C-1}+\frac{\pi^2}{3}\right)+\frac{3}{2\pi}\left(\frac{C}{C-1}+2\log 3\right)\right)\\
&\qquad \left[\because\ x \ge 1 \Rightarrow \tfrac{1}{x} \le 1\right]\\
&= x^{-\sigma}\left(1+\frac{\sigma}{6}+\frac{3\sigma}{2\pi^2}\frac{C}{C-1}+\frac{\sigma}{4}+\frac{3}{2\pi}\frac{C}{C-1}+\frac{3}{\pi}\log 3\right)\\
&= x^{-\sigma}\left(1+\frac{3}{\pi}\log 3+\frac{5}{12}\sigma+\frac{3}{2\pi}\left(1+\frac{\sigma}{\pi}\right)\frac{C}{C-1}\right).
\end{aligned}
$$

Noting that

$$
\begin{aligned}
\sum_{x<n\le N}\frac{e^{\sqrt{-1}|t|\log n}}{n^\sigma} &= \sum_{x<n\le N}\frac{n^{\sqrt{-1}|t|}}{n^\sigma}\\
&= \sum_{x<n\le N}\frac{1}{n^{\sigma-\sqrt{-1}|t|}}\\
&= \begin{cases} \overline{\displaystyle\sum_{x<n\le N}\frac{1}{n^{\sigma+\sqrt{-1}t}}} & \text{if } t>0,\\ \displaystyle\sum_{x<n\le N}\frac{1}{n^{\sigma+\sqrt{-1}t}} & \text{if } t<0,\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\int_x^N \frac{e^{\sqrt{-1}|t|\log u}}{u^\sigma}du &= \int_x^N \frac{u^{\sqrt{-1}|t|}}{u^\sigma}du\\
&= \int_x^N u^{-\sigma+\sqrt{-1}|t|}du\\
&= \left[\frac{u^{1-\sigma+\sqrt{-1}|t|}}{1-\sigma+\sqrt{-1}|t|}\right]_x^N\\
&= \frac{N^{1-(\sigma-\sqrt{-1}|t|)}-x^{1-(\sigma-\sqrt{-1}|t|)}}{1-(\sigma-\sqrt{-1}|t|)}\\
&= \begin{cases} \overline{\dfrac{N^{1-(\sigma+\sqrt{-1}t)}-x^{1-(\sigma+\sqrt{-1}t)}}{1-(\sigma+\sqrt{-1}t)}} & \text{if } t>0,\\ \dfrac{N^{1-(\sigma+\sqrt{-1}t)}-x^{1-(\sigma+\sqrt{-1}t)}}{1-(\sigma+\sqrt{-1}t)} & \text{if } t<0,\end{cases}
\end{aligned}
$$

we have

$$
\begin{aligned}
&\left|\sum_{x<n\le N}\frac{1}{n^{\sigma+\sqrt{-1}t}}-\frac{N^{1-(\sigma+\sqrt{-1}t)}-x^{1-(\sigma+\sqrt{-1}t)}}{1-(\sigma+\sqrt{-1}t)}\right|\\
&\quad\le x^{-\sigma}\left(1+\frac{3}{\pi}\log 3+\frac{5}{12}\sigma+\frac{3}{2\pi}\left(1+\frac{\sigma}{\pi}\right)\frac{C}{C-1}\right).
\end{aligned}
$$

The estimate for $t = 0$ follows by letting $t \to 0$ in the above. Thus we obtain the assertion of 1°.

2° From Lemma 6.1, it follows that for $N \in \mathbb{N} \cap (x, \infty)$,

$$\begin{aligned}\zeta(s) &= \sum_{n=1}^{N} \frac{1}{n^s} - \frac{N^{1-s}}{1-s} - s\int_N^\infty \frac{\{u\}-\frac{1}{2}}{u^{s+1}}du - \frac{1}{2}N^{-s} \\ &= \sum_{n\le x}\frac{1}{n^s} + \sum_{x<n\le N}\frac{1}{n^s} - \frac{N^{1-s}-x^{1-s}}{1-s} \\ &\quad - \frac{x^{1-s}}{1-s} - s\int_N^\infty \frac{\{u\}-\frac{1}{2}}{u^{s+1}}du - \frac{1}{2}N^{-s}.\end{aligned}$$

By 1°,

$$\begin{aligned}&\left|\zeta(s) - \Big(\sum_{n\le x}\frac{1}{n^s} - \frac{x^{1-s}}{1-s}\Big)\right| \\ &= \left|\sum_{x<n\le N}\frac{1}{n^s} - \frac{N^{1-s}-x^{1-s}}{1-s} - s\int_N^\infty \frac{\{u\}-\frac{1}{2}}{u^{s+1}}du - \frac{1}{2}N^{-s}\right| \\ &\le \left|\sum_{x<n\le N}\frac{1}{n^s} - \frac{N^{1-s}-x^{1-s}}{1-s}\right| + \left|s\int_N^\infty \frac{\{u\}-\frac{1}{2}}{u^{s+1}}du + \frac{1}{2}N^{-s}\right| \\ &\le x^{-\sigma}\left(1 + \frac{3}{\pi}\log 3 + \frac{5}{12}\sigma + \frac{3}{2\pi}\Big(1+\frac{\sigma}{\pi}\Big)\frac{C}{C-1}\right) \\ &\quad + \left|s\int_N^\infty \frac{\{u\}-\frac{1}{2}}{u^{s+1}}du + \frac{1}{2}N^{-s}\right|.\end{aligned}$$

Letting $N \to \infty$ yields the assertion of the theorem. ■

Following Matsumoto [26], we give a proof of the claim in question, which requires considerable efforts.

Proof of Claim 6.2. Fix $0 < {}^\forall\eta < 1$, and let $k := \lfloor \alpha - \eta \rfloor$. Note that $\eta \le \alpha - k < \eta + 1$. $h(x) := f(x) - kx$ is of class C^1, $h' = f' - k$. h' is nonincreasing, $h'(a) = f'(a) - k = \beta - k$, $h'(b) = f'(b) - k = \alpha - k \in [\eta, \eta + 1)$. And

$$\begin{aligned}&\Bigg|\sum_{a<n\le b} g(n)e^{\sqrt{-1}2\pi h(n)} \\ &- \Big(\int_a^b g(x)e^{\sqrt{-1}2\pi h(x)}dx + \sum_{1\le\nu<h'(a)+\eta}\int_a^b g(x)e^{\sqrt{-1}2\pi(h(x)-\nu x)}dx\Big)\Bigg| \\ &= \Bigg|\sum_{a<n\le b} g(n)e^{\sqrt{-1}2\pi(f(n)-kn)} \\ &\quad - \Big(\int_a^b g(x)e^{\sqrt{-1}2\pi(f(x)-kx)}dx + \sum_{1\le\nu<\beta-k+\eta}\int_a^b g(x)e^{\sqrt{-1}2\pi(f(x)-kx-\nu x)}dx\Big)\Bigg|\end{aligned}$$

$$= \Big| \sum_{a<n\le b} g(n) e^{-\sqrt{-1}2\pi k n} e^{\sqrt{-1}2\pi f(n)}$$
$$- \Big(\int_a^b g(x) e^{\sqrt{-1}2\pi(f(x)-kx)} dx + \sum_{1\le \nu<\beta-k+\eta} \int_a^b g(x) e^{\sqrt{-1}2\pi(f(x)-(k+\nu)x)} dx \Big) \Big|$$
$$= \Big| \sum_{a<n\le b} g(n) e^{\sqrt{-1}2\pi f(n)} - \sum_{k\le \nu<\beta+\eta} \int_a^b g(x) e^{\sqrt{-1}2\pi(f(x)-\nu x)} dx \Big|$$
$$\big[\because\ 1 \le \nu < \beta - k + \eta \Leftrightarrow k+1 \le \nu + k < \beta + \eta\big].$$

In what follows, suppose

$$f : [a,b] \to \mathbb{R} \text{ is of class } C^1 \text{ s.t. } \begin{cases} \bullet\ f' \text{ is nonincreasing,} \\ \bullet\ f'(b) \in [\eta, \eta+1). \end{cases} \tag{6.4}$$

Needless to say, $g : [a,b] \to [0,\infty)$ is of class C^1 and $g, |g'|$ are nonincreasing. We divide the proof into eleven steps:

1° Since, by Theorem 4.1,

$$\sum_{a<n\le b} g(n) e^{\sqrt{-1}2\pi f(n)}$$
$$= \int_a^b g(x) e^{\sqrt{-1}2\pi f(x)} dx$$
$$+ \frac{-1}{1}\Big(\overline{B_1}(b) g(b) e^{\sqrt{-1}2\pi f(b)} - \overline{B_1}(a) g(a) e^{\sqrt{-1}2\pi f(a)} \Big)$$
$$+ \int_a^b \overline{B_1}(x) \Big(g'(x) e^{\sqrt{-1}2\pi f(x)} + g(x) e^{\sqrt{-1}2\pi f(x)} \big(\sqrt{-1}2\pi\big) f'(x) \Big) dx$$
$$= \int_a^b g(x) e^{\sqrt{-1}2\pi f(x)} dx$$
$$+ \Big(\{a\} - \frac{1}{2}\Big) g(a) e^{\sqrt{-1}2\pi f(a)} - \Big(\{b\} - \frac{1}{2}\Big) g(b) e^{\sqrt{-1}2\pi f(b)}$$
$$+ \int_a^b \Big(\{x\} - \frac{1}{2}\Big) \big(g'(x) + \sqrt{-1}2\pi g(x) f'(x) \big) e^{\sqrt{-1}2\pi f(x)} dx,$$

we have

$$\Big| \sum_{a<n\le b} g(n) e^{\sqrt{-1}2\pi f(n)} - \int_a^b g(x) e^{\sqrt{-1}2\pi f(x)} dx$$
$$- \int_a^b \Big(\{x\} - \frac{1}{2}\Big) \big(g'(x) + \sqrt{-1}2\pi g(x) f'(x) \big) e^{\sqrt{-1}2\pi f(x)} dx \Big|$$
$$= \Big| \Big(\{a\} - \frac{1}{2}\Big) g(a) e^{\sqrt{-1}2\pi f(a)} - \Big(\{b\} - \frac{1}{2}\Big) g(b) e^{\sqrt{-1}2\pi f(b)} \Big|$$
$$\le \Big|\{a\} - \frac{1}{2}\Big| g(a) + \Big|\{b\} - \frac{1}{2}\Big| g(b)$$

$$\leq \frac{1}{2}\big(g(a)+g(b)\big)$$
$$\leq g(a) \quad \big[\because g \text{ is nonincreasing}\big].$$

2° (i) $\{x\} - \dfrac{1}{2} = -\dfrac{1}{\pi}\displaystyle\sum_{\nu=1}^{\infty}\frac{\sin 2\pi\nu x}{\nu} \quad ({}^{\forall}x \in \mathbb{R}\setminus\mathbb{Z}).$

(ii) $\left|-\dfrac{1}{\pi}\displaystyle\sum_{\nu=1}^{n}\frac{\sin 2\pi\nu x}{\nu}\right| < 2 + \dfrac{1}{\pi} \quad ({}^{\forall}n \geq 1, {}^{\forall}x \in \mathbb{R}).$

$\because$ (i) $\mathbb{R} \ni x \mapsto \{x\} - \frac{1}{2} \in \mathbb{R}$ is periodic, with period 1, and is of bounded variation on every finite interval. From the general theory of Fourier series [cf. Katznelson [19, Corollary to Theorem II.2.2]],

$$\lim_{n\to\infty}\sum_{|\nu|\leq n} c_\nu e^{\sqrt{-1}2\pi\nu x} = \begin{cases} \{x\} - \frac{1}{2}, & x \in \mathbb{R}\setminus\mathbb{Z}, \\ 0, & x \in \mathbb{Z}. \end{cases}$$

Here c_ν are the Fourier coefficients of this function: When $\nu = 0$,

$$c_0 = \int_0^1 \Big(\{x\} - \frac{1}{2}\Big)dx = \int_0^1 \Big(x - \frac{1}{2}\Big)dx = 0;$$

when $\nu \neq 0$,

$$\begin{aligned}
c_\nu &= \int_0^1 \Big(\{x\} - \frac{1}{2}\Big)e^{-\sqrt{-1}2\pi\nu x}dx \\
&= \int_0^1 \Big(x - \frac{1}{2}\Big)e^{-\sqrt{-1}2\pi\nu x}dx \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} y e^{-\sqrt{-1}2\pi\nu(y+\frac{1}{2})}dy \quad \big[\because \text{change of variable: } y = x - \tfrac{1}{2}\big] \\
&= e^{-\sqrt{-1}\pi\nu}\int_{-\frac{1}{2}}^{\frac{1}{2}} \big(y\cos 2\pi\nu y - \sqrt{-1}\,y\sin 2\pi\nu y\big)dy \\
&= (-1)^\nu\big(-\sqrt{-1}\big)2\int_0^{\frac{1}{2}} y\sin 2\pi\nu y\,dy \\
&\qquad \big[\because y\cos 2\pi\nu y \text{ is odd, } y\sin 2\pi\nu y \text{ is even}\big] \\
&= (-1)^\nu\big(-\sqrt{-1}\big)2\int_0^{\frac{1}{2}} y\Big(-\frac{\cos 2\pi\nu y}{2\pi\nu}\Big)'dy \\
&= (-1)^\nu\big(-\sqrt{-1}\big)2\bigg(\Big[y\Big(-\frac{\cos 2\pi\nu y}{2\pi\nu}\Big)\Big]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}}\Big(-\frac{\cos 2\pi\nu y}{2\pi\nu}\Big)dy\bigg) \\
&\qquad \big[\because \text{integration by parts}\big] \\
&= (-1)^\nu\big(-\sqrt{-1}\big)2\bigg(\frac{1}{2}\Big(-\frac{\cos \pi\nu}{2\pi\nu}\Big) + \Big[\frac{\sin 2\pi\nu y}{(2\pi\nu)^2}\Big]_0^{\frac{1}{2}}\bigg) \\
&= \frac{\sqrt{-1}(-1)^\nu(-1)^\nu}{2\pi\nu}
\end{aligned}$$

$$= \frac{\sqrt{-1}}{2\pi\nu}.$$

Thus, we have that for $x \in \mathbb{R} \setminus \mathbb{Z}$,

$$\begin{aligned}
\{x\} - \frac{1}{2} &= \lim_{n\to\infty} \sum_{1\le|\nu|\le n} \frac{\sqrt{-1}}{2\pi\nu} e^{\sqrt{-1}2\pi\nu x} \\
&= \lim_{n\to\infty} \sum_{\nu=1}^{n} \frac{\sqrt{-1}}{2\pi\nu} \left(e^{\sqrt{-1}2\pi\nu x} - e^{-\sqrt{-1}2\pi\nu x}\right) \\
&= \lim_{n\to\infty} \sum_{\nu=1}^{n} \frac{\sqrt{-1}}{2\pi\nu} 2\sqrt{-1}\sin 2\pi\nu x \\
&= -\frac{1}{\pi} \sum_{\nu=1}^{\infty} \frac{\sin 2\pi\nu x}{\nu}.
\end{aligned}$$

(ii) For simplicity, put

$$\left.\begin{aligned}
\overline{D_n}(x)^{\dagger 1} &:= \sum_{\nu=1}^{n} \sin 2\pi\nu x, \\
\rho_n(x) &:= -\frac{1}{\pi} \sum_{\nu=1}^{n} \frac{\sin 2\pi\nu x}{\nu}
\end{aligned}\right\} \quad (x \in \mathbb{R}).$$

Since, by the addition theorem,

$$\begin{aligned}
(\sin\pi x)\overline{D_n}(x) &= \sum_{\nu=1}^{n} \sin 2\pi\nu x \sin\pi x \\
&= \sum_{\nu=1}^{n} \frac{1}{2}\Big(\cos(2\pi\nu x - \pi x) - \cos(2\pi\nu x + \pi x)\Big) \\
&= \frac{1}{2} \sum_{\nu=1}^{n} \Big(\cos(2\nu - 1)\pi x - \cos(2\nu + 1)\pi x\Big) \\
&= \frac{1}{2} \Big(\sum_{\nu=0}^{n-1} \cos(2\nu + 1)\pi x - \sum_{\nu=1}^{n} \cos(2\nu + 1)\pi x\Big) \\
&= \frac{1}{2}\big(\cos\pi x - \cos(2n+1)\pi x\big) \\
&= \frac{1}{2}\Big(\cos\big((n+1)\pi x - n\pi x\big) - \cos\big((n+1)\pi x + n\pi x\big)\Big) \\
&= \sin(n+1)\pi x \sin n\pi x,
\end{aligned}$$

it follows that for $x \in \mathbb{R} \setminus \mathbb{Z}$,

$$|\overline{D_n}(x)| = \left|\frac{\sin(n+1)\pi x \sin n\pi x}{\sin\pi x}\right| \le \frac{1}{|\sin\pi x|}.$$

†1 This $\overline{D_n}(x)$ is not the conjugate of $D_n(x)$.

Now, for $l, m \in \mathbb{N}$ with $m \geq l$ and $x \in \mathbb{R} \setminus \mathbb{Z}$, this estimate tells us that

$$\begin{aligned}
\Big|\sum_{\nu=l}^{m} \frac{\sin 2\pi\nu x}{\nu}\Big| &= \Big|\sum_{\nu=l}^{m} \frac{1}{\nu}\big(\overline{D_\nu}(x) - \overline{D_{\nu-1}}(x)\big)\Big| \quad \big[\text{where } \overline{D_0}(x) := 0\big] \\
&= \Big|\sum_{\nu=l}^{m} \frac{1}{\nu}\overline{D_\nu}(x) - \sum_{\nu=l-1}^{m-1} \frac{1}{\nu+1}\overline{D_\nu}(x)\Big| \\
&= \Big|-\frac{1}{l}\overline{D_{l-1}}(x) + \sum_{\nu=l}^{m-1}\Big(\frac{1}{\nu} - \frac{1}{\nu+1}\Big)\overline{D_\nu}(x) + \frac{1}{m}\overline{D_m}(x)\Big| \\
&\leq \frac{1}{l}\big|\overline{D_{l-1}}(x)\big| + \sum_{\nu=l}^{m-1}\Big(\frac{1}{\nu} - \frac{1}{\nu+1}\Big)\big|\overline{D_\nu}(x)\big| + \frac{1}{m}\big|\overline{D_m}(x)\big| \\
&\leq \Big(\frac{1}{l} + \sum_{\nu=l}^{m-1}\Big(\frac{1}{\nu} - \frac{1}{\nu+1}\Big) + \frac{1}{m}\Big)\frac{1}{|\sin\pi x|} \\
&= \frac{2}{l}\frac{1}{|\sin\pi x|}.
\end{aligned}$$

Thus, in case $0 < |x| \leq \frac{1}{2}$,

$$\begin{aligned}
|\rho_n(x)| &= \Big|-\frac{1}{\pi}\sum_{\substack{1\leq\nu\leq n;\\ \nu\leq\frac{1}{|x|}}} \frac{\sin 2\pi\nu x}{\nu} - \frac{1}{\pi}\sum_{\substack{1\leq\nu\leq n;\\ \nu>\frac{1}{|x|}}} \frac{\sin 2\pi\nu x}{\nu}\Big| \\
&\leq \frac{1}{\pi}\sum_{\substack{1\leq\nu\leq n;\\ \nu\leq\frac{1}{|x|}}} \frac{|\sin 2\pi\nu x|}{\nu} + \frac{1}{\pi}\Big|\sum_{\substack{1\leq\nu\leq n;\\ \nu>\frac{1}{|x|}}} \frac{\sin 2\pi\nu x}{\nu}\Big| \\
&\leq \frac{1}{\pi}\sum_{\substack{1\leq\nu\leq n;\\ \nu\leq\frac{1}{|x|}}} \frac{2\pi\nu|x|}{\nu} + \frac{1}{\pi}\Big|\sum_{\nu=\lfloor 1/|x|\rfloor+1}^{n} \frac{\sin 2\pi\nu x}{\nu}\Big| \\
&\qquad \big[\because\ |\sin y| \leq |y| \ \ (y \in \mathbb{R})\big] \\
&\leq 2|x|\frac{1}{|x|} + \frac{1}{\pi}\frac{2}{\lfloor\frac{1}{|x|}\rfloor + 1}\frac{1}{|\sin\pi x|} \\
&\leq 2 + \frac{1}{\pi}\frac{2}{\lfloor\frac{1}{|x|}\rfloor + 1}\frac{1}{2|x|} \\
&\qquad \left[\begin{aligned} \because\ |x| \leq \tfrac{1}{2} &\Rightarrow |\pi x| \leq \tfrac{\pi}{2} \\
&\Rightarrow |\sin\pi x| \geq \tfrac{2}{\pi}|\pi x| \\
&\qquad \left[\because \text{ Jordan's inequality, i.e., } \tfrac{2}{\pi}y \leq \sin y \leq y \ \ (0 \leq y \leq \tfrac{\pi}{2})\right] \\
&\qquad = 2|x| \\
&\Rightarrow \tfrac{1}{|\sin\pi x|} \leq \tfrac{1}{2|x|}
\end{aligned}\right]
\end{aligned}$$

$$= 2 + \frac{1}{\pi} \frac{1}{|x|\left(\left\lfloor \frac{1}{|x|} \right\rfloor + 1\right)}$$

$$< 2 + \frac{1}{\pi}$$

$$\left[\because \left\lfloor \frac{1}{|x|} \right\rfloor + 1 > \frac{1}{|x|} \ge \left\lfloor \frac{1}{|x|} \right\rfloor \Rightarrow |x|\left(\left\lfloor \frac{1}{|x|} \right\rfloor + 1\right) > 1 \atop \Rightarrow \frac{1}{|x|\left(\lfloor \frac{1}{|x|} \rfloor + 1\right)} < 1 \right].$$

In conjunction with $\rho_n(0) = 0$,

$$|\rho_n(x)| < 2 + \frac{1}{\pi}, \quad -\frac{1}{2} \le x \le \frac{1}{2}.$$

For a general $x \in \mathbb{R}$, take $m \in \mathbb{Z}$ so that $|x - m| \le \frac{1}{2}$. Since $\rho_n(x - m) = \rho_n(x)$, we have

$$|\rho_n(x)| = |\rho_n(x - m)| < 2 + \frac{1}{\pi}.$$

<u>3°</u> By 2° and the bounded convergence theorem,

$$\int_a^b \left(\{x\} - \frac{1}{2}\right)\left(g'(x) + \sqrt{-1}2\pi g(x) f'(x)\right) e^{\sqrt{-1}2\pi f(x)} dx$$

$$= -\frac{1}{\pi} \sum_{\nu=1}^{\infty} \frac{1}{\nu} \int_a^b \sin 2\pi\nu x \left(g'(x) + \sqrt{-1}2\pi g(x) f'(x)\right) e^{\sqrt{-1}2\pi f(x)} dx$$

$$= -\frac{1}{\pi} \sum_{\nu=1}^{\infty} \frac{1}{\nu} \int_a^b \frac{e^{\sqrt{-1}2\pi\nu x} - e^{-\sqrt{-1}2\pi\nu x}}{2\sqrt{-1}} \left(g'(x) + \sqrt{-1}2\pi g(x) f'(x)\right) e^{\sqrt{-1}2\pi f(x)} dx$$

$$= -\frac{1}{2\pi\sqrt{-1}} \sum_{\nu=1}^{\infty} \left(\frac{1}{\nu} \int_a^b \left(g'(x) + \sqrt{-1}2\pi g(x) f'(x)\right) e^{\sqrt{-1}2\pi(f(x)+\nu x)} dx \right.$$
$$\left. - \frac{1}{\nu} \int_a^b \left(g'(x) + \sqrt{-1}2\pi g(x) f'(x)\right) e^{\sqrt{-1}2\pi(f(x)-\nu x)} dx \right)$$

$$= \lim_{N\to\infty} \left(-\frac{1}{2\pi\sqrt{-1}} \sum_{\nu=1}^{N} \frac{1}{\nu} \int_a^b g'(x) e^{\sqrt{-1}2\pi(f(x)+\nu x)} dx \right.$$
$$+ \frac{1}{2\pi\sqrt{-1}} \sum_{\nu=1}^{N} \frac{1}{\nu} \int_a^b g'(x) e^{\sqrt{-1}2\pi(f(x)-\nu x)} dx$$
$$- \sum_{\nu=1}^{N} \frac{1}{\nu} \int_a^b g(x) f'(x) e^{\sqrt{-1}2\pi(f(x)+\nu x)} dx$$
$$\left. + \sum_{\nu=1}^{N} \frac{1}{\nu} \int_a^b g(x) f'(x) e^{\sqrt{-1}2\pi(f(x)-\nu x)} dx \right)$$

$$= \lim_{N\to\infty} \left(-\sum_{\nu=1}^{N} \frac{1}{\nu} \int_a^b g(x) f'(x) e^{\sqrt{-1}2\pi(f(x)+\nu x)} dx \right.$$

$$+\sum_{\substack{\nu\in\mathbb{N};\\ N\geq\nu\geq\beta+\eta}} \frac{1}{\nu}\int_a^b g(x)f'(x)e^{\sqrt{-1}2\pi(f(x)-\nu x)}dx$$
$$+\sum_{\substack{\nu\in\mathbb{N};\\ 1\leq\nu<\beta+\eta}} \frac{1}{\nu}\int_a^b g(x)f'(x)e^{\sqrt{-1}2\pi(f(x)-\nu x)}dx$$
$$-\sum_{\substack{\nu\in\mathbb{N};\\ 1\leq\nu<\beta+\eta}} \int_a^b g(x)e^{\sqrt{-1}2\pi(f(x)-\nu x)}dx$$
$$+\sum_{\substack{\nu\in\mathbb{N};\\ 1\leq\nu<\beta+\eta}} \int_a^b g(x)e^{\sqrt{-1}2\pi(f(x)-\nu x)}dx$$
$$-\frac{1}{2\pi\sqrt{-1}}\sum_{\nu=1}^{N}\frac{1}{\nu}\int_a^b g'(x)e^{\sqrt{-1}2\pi(f(x)+\nu x)}dx$$
$$+\frac{1}{2\pi\sqrt{-1}}\sum_{\substack{\nu\in\mathbb{N};\\ N\geq\nu\geq\beta+\eta}} \frac{1}{\nu}\int_a^b g'(x)e^{\sqrt{-1}2\pi(f(x)-\nu x)}dx$$
$$+\frac{1}{2\pi\sqrt{-1}}\sum_{\substack{\nu\in\mathbb{N};\\ 1\leq\nu<\beta+\eta}} \frac{1}{\nu}\int_a^b g'(x)e^{\sqrt{-1}2\pi(f(x)-\nu x)}dx\Bigg).$$

4° Noting that $f'(x)+\nu \geq \alpha+\nu \geq \eta+1 > 0$ ($^\forall \nu \geq 1$) by $f'(x) \geq f'(b) = \alpha \geq \eta > 0$ $\big[$cf. (6.4)$\big]$, we rewrite

$$\begin{aligned}
&\int_a^b g(x)f'(x)e^{\sqrt{-1}2\pi(f(x)+\nu x)}dx\\
&= \int_a^b g(x)f'(x)\cos 2\pi\big(f(x)+\nu x\big)dx\\
&\quad + \sqrt{-1}\int_a^b g(x)f'(x)\sin 2\pi\big(f(x)+\nu x\big)dx\\
&= \int_a^b \frac{g(x)f'(x)}{f'(x)+\nu}\big(f'(x)+\nu\big)\cos 2\pi\big(f(x)+\nu x\big)dx\\
&\quad + \sqrt{-1}\int_a^b \frac{g(x)f'(x)}{f'(x)+\nu}\big(f'(x)+\nu\big)\sin 2\pi\big(f(x)+\nu x\big)dx.
\end{aligned}$$

Here $\frac{gf'}{f'+\nu}$ is nonincreasing. Because, for $a \leq x_1 \leq x_2 \leq b$,

$$\begin{aligned}
&\frac{gf'}{f'+\nu}(x_1) - \frac{gf'}{f'+\nu}(x_2)\\
&= \big(g(x_1)-g(x_2)\big)\frac{f'}{f'+\nu}(x_1) + g(x_2)\Big(\frac{f'}{f'+\nu}(x_1) - \frac{f'}{f'+\nu}(x_2)\Big)\\
&\geq g(x_2)\nu\Big(\frac{1}{f'(x_2)+\nu} - \frac{1}{f'(x_1)+\nu}\Big)
\end{aligned}$$

$$
\begin{aligned}
&= g(x_2)\frac{\nu}{(f'(x_1)+\nu)(f'(x_2)+\nu)}\big(f'(x_1)-f'(x_2)\big)\\
&\geq 0.
\end{aligned}
$$

By this property and the nonnegativity of $\frac{gf'}{f'+\nu}$, the second mean value theorem for integrals [cf. Claim A.11] gives that

$$
\begin{aligned}
&a \leq {}^{\exists}\xi_{\cos}^{(\nu)},\ {}^{\exists}\xi_{\sin}^{(\nu)} \leq b\\
&\text{s.t.}\ \int_a^b \frac{gf'}{f'+\nu}(x)\big(f'(x)+\nu\big)\cos 2\pi\big(f(x)+\nu x\big)dx\\
&\quad= \frac{gf'}{f'+\nu}(a)\int_a^{\xi_{\cos}^{(\nu)}}\big(f'(x)+\nu\big)\cos 2\pi\big(f(x)+\nu x\big)dx\\
&\quad= \frac{gf'}{f'+\nu}(a)\Big[\frac{1}{2\pi}\sin 2\pi\big(f(x)+\nu x\big)\Big]_a^{\xi_{\cos}^{(\nu)}}\\
&\quad= \frac{g(a)f'(a)}{f'(a)+\nu}\frac{1}{2\pi}\Big(\sin 2\pi\big(f(\xi_{\cos}^{(\nu)})+\nu\xi_{\cos}^{(\nu)}\big)-\sin 2\pi\big(f(a)+\nu a\big)\Big),\\
&\int_a^b \frac{gf'}{f'+\nu}(x)\big(f'(x)+\nu\big)\sin 2\pi\big(f(x)+\nu x\big)dx\\
&\quad= \frac{gf'}{f'+\nu}(a)\int_a^{\xi_{\sin}^{(\nu)}}\big(f'(x)+\nu\big)\sin 2\pi\big(f(x)+\nu x\big)dx\\
&\quad= \frac{gf'}{f'+\nu}(a)\Big[-\frac{1}{2\pi}\cos 2\pi\big(f(x)+\nu x\big)\Big]_a^{\xi_{\sin}^{(\nu)}}\\
&\quad= \frac{g(a)f'(a)}{f'(a)+\nu}\Big(-\frac{1}{2\pi}\Big)\Big(\cos 2\pi\big(f(\xi_{\sin}^{(\nu)})+\nu\xi_{\sin}^{(\nu)}\big)-\cos 2\pi\big(f(a)+\nu a\big)\Big).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\Big|\sum_{\nu=1}^{N}\frac{1}{\nu}\int_a^b g(x)f'(x)e^{\sqrt{-1}2\pi(f(x)+\nu x)}dx\Big|\\
&= \Big|\sum_{\nu=1}^{N}\frac{1}{\nu}\Big(\int_a^b \frac{gf'}{f'+\nu}(x)\big(f'(x)+\nu\big)\cos 2\pi\big(f(x)+\nu x\big)dx\\
&\qquad\qquad + \sqrt{-1}\int_a^b \frac{gf'}{f'+\nu}(x)\big(f'(x)+\nu\big)\sin 2\pi\big(f(x)+\nu x\big)dx\Big)\Big|\\
&= \Big|\sum_{\nu=1}^{N}\frac{1}{\nu}\frac{g(a)f'(a)}{f'(a)+\nu}\frac{1}{2\pi}\Big(\sin 2\pi\big(f(\xi_{\cos}^{(\nu)})+\nu\xi_{\cos}^{(\nu)}\big)\\
&\qquad\qquad - \sin 2\pi\big(f(a)+\nu a\big)\\
&\qquad\qquad - \sqrt{-1}\cos 2\pi\big(f(\xi_{\sin}^{(\nu)})+\nu\xi_{\sin}^{(\nu)}\big)\\
&\qquad\qquad + \sqrt{-1}\cos 2\pi\big(f(a)+\nu a\big)\Big)\Big|
\end{aligned}
$$

$$
\begin{aligned}
&\le \sum_{\nu=1}^{N} \frac{1}{\nu} \frac{g(a) f'(a)}{f'(a)+\nu} \frac{1}{2\pi} \cdot 3 \\
&= \frac{3g(a)}{2\pi} \sum_{\nu=1}^{N} \frac{f'(a)}{\nu(f'(a)+\nu)} \\
&= \frac{3g(a)}{2\pi} \sum_{\nu=1}^{N} \frac{\beta}{\nu(\beta+\nu)} \\
&\le \frac{3g(a)}{2\pi} \sum_{\nu=1}^{\infty} \frac{\beta}{\nu(\beta+\nu)}.
\end{aligned}
$$

<u>5°</u> Let $\nu \in \mathbb{N}$ with $\nu \ge \beta + \eta$. Then

$$
f'(x) - \nu \le f'(a) - \nu = \beta - \nu \le -\eta < 0 \quad (x \in [a,b]),
$$

and $\frac{g(-f')}{f'-\nu}$ is nonincreasing. Because, for $a \le x_1 \le x_2 \le b$,

$$
\begin{aligned}
&\frac{g(-f')}{f'-\nu}(x_1) - \frac{g(-f')}{f'-\nu}(x_2) \\
&= \big(g(x_1) - g(x_2)\big) \frac{-f'}{f'-\nu}(x_1) + g(x_2) \Big(\frac{-f'}{f'-\nu}(x_1) - \frac{-f'}{f'-\nu}(x_2) \Big) \\
&\ge g(x_2) \nu \Big(\frac{1}{f'(x_2)-\nu} - \frac{1}{f'(x_1)-\nu} \Big) \\
&= g(x_2) \frac{\nu}{(f'(x_1)-\nu)(f'(x_2)-\nu)} \big(f'(x_1) - f'(x_2)\big) \\
&\ge 0.
\end{aligned}
$$

By this property and the nonnegativity of $\frac{g(-f')}{f'-\nu}$, the second mean value theorem for integrals gives that

$$
\begin{aligned}
&\int_a^b g(x) f'(x) e^{\sqrt{-1} 2\pi (f(x) - \nu x)} dx \\
&= \int_a^b g(x) f'(x) \cos 2\pi \big(f(x) - \nu x\big) dx \\
&\quad + \sqrt{-1} \int_a^b g(x) f'(x) \sin 2\pi \big(f(x) - \nu x\big) dx \\
&= - \int_a^b \frac{g(-f')}{f'-\nu}(x) \big(f'(x) - \nu\big) \cos 2\pi \big(f(x) - \nu x\big) dx \\
&\quad - \sqrt{-1} \int_a^b \frac{g(-f')}{f'-\nu}(x) \big(f'(x) - \nu\big) \sin 2\pi \big(f(x) - \nu x\big) dx \\
&= - \frac{g(-f')}{f'-\nu}(a) \int_a^{\xi_{\cos}^{(\nu)}} \big(f'(x) - \nu\big) \cos 2\pi \big(f(x) - \nu x\big) dx \\
&\quad - \sqrt{-1} \frac{g(-f')}{f'-\nu}(a) \int_a^{\xi_{\sin}^{(\nu)}} \big(f'(x) - \nu\big) \sin 2\pi \big(f(x) - \nu x\big) dx
\end{aligned}
$$

$$\big[\text{for some } \xi_{\cos}^{(\nu)}, \xi_{\sin}^{(\nu)} \in [a,b] \text{ which are different things from those in } 4^{\circ}\big]$$
$$= \frac{g(a)f'(a)}{f'(a)-\nu}\bigg(\Big[\frac{1}{2\pi}\sin 2\pi\big(f(x)-\nu x\big)\Big]_a^{\xi_{\cos}^{(\nu)}} - \sqrt{-1}\Big[\frac{1}{2\pi}\cos 2\pi\big(f(x)-\nu x\big)\Big]_a^{\xi_{\sin}^{(\nu)}}\bigg)$$
$$= g(a)\frac{\beta}{\beta-\nu}\frac{1}{2\pi}\Big(\sin 2\pi\big(f(\xi_{\cos}^{(\nu)})-\nu\xi_{\cos}^{(\nu)}\big) - \sin 2\pi\big(f(a)-\nu a\big)$$
$$-\sqrt{-1}\cos 2\pi\big(f(\xi_{\sin}^{(\nu)})-\nu\xi_{\sin}^{(\nu)}\big) + \sqrt{-1}\cos 2\pi\big(f(a)-\nu a\big)\Big).$$

Thus

$$\Bigg|\sum_{\substack{\nu\in\mathbb{N};\\ N\ge\nu\ge\beta+\eta}} \frac{1}{\nu}\int_a^b g(x)f'(x)e^{\sqrt{-1}2\pi(f(x)-\nu x)}dx\Bigg|$$
$$= \Bigg|\sum_{\substack{\nu\in\mathbb{N};\\ N\ge\nu\ge\beta+\eta}} \frac{1}{\nu}g(a)\frac{\beta}{\beta-\nu}\frac{1}{2\pi}\Big(\sin 2\pi\big(f(\xi_{\cos}^{(\nu)})-\nu\xi_{\cos}^{(\nu)}\big) - \sqrt{-1}\cos 2\pi\big(f(\xi_{\sin}^{(\nu)})-\nu\xi_{\sin}^{(\nu)}\big) + \sqrt{-1}e^{\sqrt{-1}2\pi(f(a)-\nu a)}\Big)\Bigg|$$
$$\le \frac{3g(a)}{2\pi}\sum_{\substack{\nu\in\mathbb{N};\\ N\ge\nu\ge\beta+\eta}}\frac{\beta}{\nu(\nu-\beta)}$$
$$\le \frac{3g(a)}{2\pi}\sum_{\substack{\nu\in\mathbb{N};\\ \nu\ge\beta+\eta}}\frac{\beta}{\nu(\nu-\beta)}.$$

6° For $\nu\in\mathbb{N}$ with $1\le\nu<\beta+\eta$,

$$\int_a^b g(x)f'(x)e^{\sqrt{-1}2\pi(f(x)-\nu x)}dx$$
$$= \int_a^b g(x)\big(f'(x)-\nu\big)e^{\sqrt{-1}2\pi(f(x)-\nu x)}dx + \nu\int_a^b g(x)e^{\sqrt{-1}2\pi(f(x)-\nu x)}dx$$
$$= \int_a^b g(x)\Big(\frac{1}{2\pi\sqrt{-1}}e^{\sqrt{-1}2\pi(f(x)-\nu x)}\Big)'dx + \nu\int_a^b g(x)e^{\sqrt{-1}2\pi(f(x)-\nu x)}dx$$
$$= \Big[\frac{g(x)}{2\pi\sqrt{-1}}e^{\sqrt{-1}2\pi(f(x)-\nu x)}\Big]_a^b - \frac{1}{2\pi\sqrt{-1}}\int_a^b g'(x)e^{\sqrt{-1}2\pi(f(x)-\nu x)}dx$$
$$+\nu\int_a^b g(x)e^{\sqrt{-1}2\pi(f(x)-\nu x)}dx \quad [\because \text{ integration by parts}]$$
$$= \frac{1}{2\pi\sqrt{-1}}\Big(g(b)e^{\sqrt{-1}2\pi(f(b)-\nu b)} - g(a)e^{\sqrt{-1}2\pi(f(a)-\nu a)}\Big)$$
$$-\frac{1}{2\pi\sqrt{-1}}\int_a^b g'(x)e^{\sqrt{-1}2\pi(f(x)-\nu x)}dx + \nu\int_a^b g(x)e^{\sqrt{-1}2\pi(f(x)-\nu x)}dx.$$

Thus

$$\Bigl| \sum_{\substack{\nu\in\mathbb{N};\\ 1\le\nu<\beta+\eta}} \frac{1}{\nu}\int_a^b g(x)f'(x)e^{\sqrt{-1}2\pi(f(x)-\nu x)}dx - \sum_{\substack{\nu\in\mathbb{N};\\ 1\le\nu<\beta+\eta}} \int_a^b g(x)e^{\sqrt{-1}2\pi(f(x)-\nu x)}dx\Bigr|$$

$$= \Bigl| \sum_{\substack{\nu\in\mathbb{N};\\ 1\le\nu<\beta+\eta}} \Bigl(\frac{1}{\nu}\int_a^b g(x)f'(x)e^{\sqrt{-1}2\pi(f(x)-\nu x)}dx - \int_a^b g(x)e^{\sqrt{-1}2\pi(f(x)-\nu x)}dx\Bigr)\Bigr|$$

$$= \Bigl| \sum_{\substack{\nu\in\mathbb{N};\\ 1\le\nu<\beta+\eta}} \frac{1}{\nu}\frac{1}{2\pi\sqrt{-1}}\Bigl(g(b)e^{\sqrt{-1}2\pi(f(b)-\nu b)} - g(a)e^{\sqrt{-1}2\pi(f(a)-\nu a)} - \int_a^b g'(x)e^{\sqrt{-1}2\pi(f(x)-\nu x)}dx\Bigr)\Bigr|$$

$$\le \frac{1}{2\pi}\sum_{\substack{\nu\in\mathbb{N};\\ 1\le\nu<\beta+\eta}} \frac{1}{\nu}\Bigl(g(b)+g(a)+\int_a^b |g'(x)|dx\Bigr)$$

$$= \frac{g(a)}{\pi}\sum_{\substack{\nu\in\mathbb{N};\\ 1\le\nu<\beta+\eta}} \frac{1}{\nu}$$

$\bigl[\because$ Since $g'\le 0$, $\int_a^b |g'(x)|dx = \int_a^b(-g'(x))dx = -g(b)+g(a)\bigr]$.

7° $|g'|$ is nonincreasing and nonnegative, and $\frac{1}{f'+\nu}$ is nondecreasing and nonnegative. Thus the second mean value theorem for integrals gives that

$$\Bigl|\sum_{\nu=1}^{N}\frac{1}{\nu}\int_a^b g'(x)e^{\sqrt{-1}2\pi(f(x)+\nu x)}dx\Bigr|$$

$$= \Bigl|\sum_{\nu=1}^{N}\frac{-1}{\nu}\Bigl(\int_a^b |g'(x)|\cos 2\pi\bigl(f(x)+\nu x\bigr)dx + \sqrt{-1}\int_a^b |g'(x)|\sin 2\pi\bigl(f(x)+\nu x\bigr)dx\Bigr)\Bigr|$$

$$= \Bigl|\sum_{\nu=1}^{N}\frac{-1}{\nu}\Bigl(|g'(a)|\int_a^{\xi_{\cos}^{(\nu)}} \cos 2\pi\bigl(f(x)+\nu x\bigr)dx + \sqrt{-1}|g'(a)|\int_a^{\xi_{\sin}^{(\nu)}} \sin 2\pi\bigl(f(x)+\nu x\bigr)dx\Bigr)\Bigr|$$

$$= |g'(a)|\Bigl|\sum_{\nu=1}^{N}\frac{1}{\nu}\Bigl(\int_a^{\xi_{\cos}^{(\nu)}} \frac{1}{f'(x)+\nu}\bigl(f'(x)+\nu\bigr)\cos 2\pi\bigl(f(x)+\nu x\bigr)dx + \sqrt{-1}\int_a^{\xi_{\sin}^{(\nu)}} \frac{1}{f'(x)+\nu}\bigl(f'(x)+\nu\bigr)\sin 2\pi\bigl(f(x)+\nu x\bigr)dx\Bigr)\Bigr|$$

$$= |g'(a)|\Bigl|\sum_{\nu=1}^{N}\frac{1}{\nu}\Bigl(\frac{1}{f'(\xi_{\cos}^{(\nu)})+\nu}\int_{\xi_{\cos}^{(\nu)\prime}}^{\xi_{\cos}^{(\nu)}} \bigl(f'(x)+\nu\bigr)\cos 2\pi\bigl(f(x)+\nu x\bigr)dx$$

$$+ \frac{\sqrt{-1}}{f'(\xi_{\sin}^{(\nu)}) + \nu} \int_{\xi_{\sin}^{(\nu)\prime}}^{\xi_{\sin}^{(\nu)}} \big(f'(x) + \nu\big) \sin 2\pi \big(f(x) + \nu x\big) dx \Big) \Big|$$

$$\big[\text{for some } \xi_{\cos}^{(\nu)\prime} \in [a, \xi_{\cos}^{(\nu)}],\ \xi_{\sin}^{(\nu)\prime} \in [a, \xi_{\sin}^{(\nu)}]\big]$$

$$= |g'(a)| \Big| \sum_{\nu=1}^{N} \frac{1}{\nu} \Big(\frac{1}{f'(\xi_{\cos}^{(\nu)}) + \nu} \cdot \frac{1}{2\pi} \Big(\sin 2\pi \big(f(\xi_{\cos}^{(\nu)}) + \nu \xi_{\cos}^{(\nu)}\big) - \sin 2\pi \big(f(\xi_{\cos}^{(\nu)\prime}) + \nu \xi_{\cos}^{(\nu)\prime}\big)\Big)$$

$$+ \frac{\sqrt{-1}}{f'(\xi_{\sin}^{(\nu)}) + \nu} \cdot \frac{-1}{2\pi} \Big(\cos 2\pi \big(f(\xi_{\sin}^{(\nu)}) + \nu \xi_{\sin}^{(\nu)}\big) - \cos 2\pi \big(f(\xi_{\sin}^{(\nu)\prime}) + \nu \xi_{\sin}^{(\nu)\prime}\big)\Big)\Big) \Big|$$

$$\leq |g'(a)| \sum_{\nu=1}^{N} \frac{1}{\nu} \Big(\frac{1}{f'(\xi_{\cos}^{(\nu)}) + \nu} \frac{1}{\pi} + \frac{1}{f'(\xi_{\sin}^{(\nu)}) + \nu} \frac{1}{\pi} \Big)$$

$$\leq \frac{2}{\pi} |g'(a)| \sum_{\nu=1}^{N} \frac{1}{\nu^2}$$

$$\big[\because f'(\xi_{\cos}^{(\nu)}) + \nu,\ f'(\xi_{\sin}^{(\nu)}) + \nu \geq f'(b) + \nu \geq \eta + \nu > \nu\big]$$

$$\leq \frac{2|g'(a)|}{\pi} \sum_{\nu=1}^{\infty} \frac{1}{\nu^2}$$

$$= \frac{\pi}{3} |g'(a)| \quad [\because \text{Claim A.10}].$$

8° For $\nu \in \mathbb{N}$ with $\nu \geq \beta + \eta$, $\frac{-1}{f' - \nu}$ is nonincreasing and nonnegative. By the second mean value theorem for integrals,

$$\Big| \sum_{\substack{\nu \in \mathbb{N};\\ N \geq \nu \geq \beta + \eta}} \frac{1}{\nu} \int_a^b g'(x) e^{\sqrt{-1} 2\pi (f(x) - \nu x)} dx \Big|$$

$$= \Big| \sum_{\substack{\nu \in \mathbb{N};\\ N \geq \nu \geq \beta + \eta}} \frac{-1}{\nu} \Big(\int_a^b |g'(x)| \cos 2\pi \big(f(x) - \nu x\big) dx + \sqrt{-1} \int_a^b |g'(x)| \sin 2\pi \big(f(x) - \nu x\big) dx \Big) \Big|$$

$$= \Big| \sum_{\substack{\nu \in \mathbb{N};\\ N \geq \nu \geq \beta + \eta}} \frac{-1}{\nu} \Big(|g'(a)| \int_a^{\xi_{\cos}^{(\nu)}} \cos 2\pi \big(f(x) - \nu x\big) dx + \sqrt{-1} |g'(a)| \int_a^{\xi_{\sin}^{(\nu)}} \sin 2\pi \big(f(x) - \nu x\big) dx \Big) \Big|$$

$$= |g'(a)| \Big| \sum_{\substack{\nu \in \mathbb{N};\\ N \geq \nu \geq \beta + \eta}} \frac{1}{\nu} \Big(\int_a^{\xi_{\cos}^{(\nu)}} \frac{-1}{f'(x) - \nu} \big(f'(x) - \nu\big) \cos 2\pi \big(f(x) - \nu x\big) dx + \sqrt{-1} \int_a^{\xi_{\sin}^{(\nu)}} \frac{-1}{f'(x) - \nu} \big(f'(x) - \nu\big) \sin 2\pi \big(f(x) - \nu x\big) dx \Big) \Big|$$

$$= |g'(a)| \Bigg| \sum_{\substack{\nu\in\mathbb{N};\\ N\ge\nu\ge\beta+\eta}} \frac{1}{\nu}\Bigg(\frac{-1}{f'(a)-\nu}\int_a^{\xi_{\cos}^{(\nu)\prime}} \big(f'(x)-\nu\big)\cos 2\pi\big(f(x)-\nu x\big)dx + \sqrt{-1}\,\frac{-1}{f'(a)-\nu}\int_a^{\xi_{\sin}^{(\nu)\prime}} \big(f'(x)-\nu\big)\sin 2\pi\big(f(x)-\nu x\big)dx\Bigg)\Bigg|$$

$\big[$where $a \le {}^{\exists}\xi_{\cos}^{(\nu)\prime} \le \xi_{\cos}^{(\nu)}$, $a \le {}^{\exists}\xi_{\sin}^{(\nu)\prime} \le \xi_{\sin}^{(\nu)}\big]$

$$= |g'(a)| \Bigg| \sum_{\substack{\nu\in\mathbb{N};\\ N\ge\nu\ge\beta+\eta}} \frac{1}{\nu}\frac{-1}{f'(a)-\nu}\Bigg(\frac{1}{2\pi}\Big(\sin 2\pi\big(f(\xi_{\cos}^{(\nu)\prime})-\nu\xi_{\cos}^{(\nu)\prime}\big) - \sin 2\pi(f(a)-\nu a)\Big) - \frac{\sqrt{-1}}{2\pi}\Big(\cos 2\pi\big(f(\xi_{\sin}^{(\nu)\prime})-\nu\xi_{\sin}^{(\nu)\prime}\big) - \cos 2\pi(f(a)-\nu a)\Big)\Bigg)\Bigg|$$

$$\le |g'(a)| \sum_{\substack{\nu\in\mathbb{N};\\ N\ge\nu\ge\beta+\eta}} \frac{1}{\nu(\nu-\beta)}\frac{3}{2\pi}$$

$$\le \frac{3|g'(a)|}{2\pi} \sum_{\substack{\nu\in\mathbb{N};\\ \nu\ge\beta+\eta}} \frac{1}{\nu(\nu-\beta)}.$$

As for the sum over $\nu \in \mathbb{N}$ with $1 \le \nu < \beta + \eta$,

$$\Bigg| \sum_{\substack{\nu\in\mathbb{N};\\ 1\le\nu<\beta+\eta}} \frac{1}{\nu}\int_a^b g'(x)e^{\sqrt{-1}2\pi(f(x)-\nu x)}dx\Bigg| \le \sum_{\substack{\nu\in\mathbb{N};\\ 1\le\nu<\beta+\eta}} \frac{1}{\nu}\int_a^b |g'(x)|dx = \sum_{\substack{\nu\in\mathbb{N};\\ 1\le\nu<\beta+\eta}} \frac{1}{\nu}\big(g(a)-g(b)\big) \le g(a) \sum_{\substack{\nu\in\mathbb{N};\\ 1\le\nu<\beta+\eta}} \frac{1}{\nu}.$$

9° By 3° to 8°,

$$\Bigg|\int_a^b \Big(\{x\}-\frac{1}{2}\Big)\big(g'(x)+\sqrt{-1}2\pi g(x)f'(x)\big)e^{\sqrt{-1}2\pi f(x)}dx - \sum_{\substack{\nu\in\mathbb{N};\\ 1\le\nu<\beta+\eta}} \int_a^b g(x)e^{\sqrt{-1}2\pi(f(x)-\nu x)}dx\Bigg|$$

$$\le \frac{3g(a)}{2\pi}\sum_{\nu=1}^{\infty}\frac{\beta}{\nu(\beta+\nu)} + \frac{3g(a)}{2\pi}\sum_{\substack{\nu\in\mathbb{N};\\ \nu\ge\beta+\eta}} \frac{\beta}{\nu(\nu-\beta)}$$

$$
\begin{aligned}
&+ \frac{g(a)}{\pi} \sum_{\substack{\nu\in\mathbb{N};\\ 1\le\nu<\beta+\eta}} \frac{1}{\nu} \\
&+ \frac{1}{6}|g'(a)| + \frac{3|g'(a)|}{4\pi^2} \sum_{\substack{\nu\in\mathbb{N};\\ \nu\ge\beta+\eta}} \frac{1}{\nu(\nu-\beta)} \\
&+ \frac{g(a)}{2\pi} \sum_{\substack{\nu\in\mathbb{N};\\ 1\le\nu<\beta+\eta}} \frac{1}{\nu}.
\end{aligned}
$$

In conjunction with 1°,

$$
\begin{aligned}
&\left| \sum_{a<n\le b} g(n)e^{\sqrt{-1}2\pi f(n)} - \sum_{\substack{\nu\in\mathbb{N}\cup\{0\};\\ 0\le\nu<\beta+\eta}} \int_a^b g(x)e^{\sqrt{-1}2\pi(f(x)-\nu x)}dx \right| \\
&\le g(a) + \frac{1}{6}|g'(a)| + \frac{3|g'(a)|}{4\pi^2} \sum_{\substack{\nu\in\mathbb{N};\\ \nu\ge\beta+\eta}} \frac{1}{\nu(\nu-\beta)} \\
&\quad + \frac{3g(a)}{2\pi} \sum_{\substack{\nu\in\mathbb{N};\\ 1\le\nu<\beta+\eta}} \frac{1}{\nu} \\
&\quad + \frac{3g(a)}{2\pi} \sum_{\nu=1}^{\infty} \frac{\beta}{\nu(\beta+\nu)} + \frac{3g(a)}{2\pi} \sum_{\substack{\nu\in\mathbb{N};\\ \nu\ge\beta+\eta}} \frac{\beta}{\nu(\nu-\beta)}.
\end{aligned}
$$

<u>10°</u> (i) $\displaystyle\sum_{\nu=1}^{m} \frac{1}{\nu} \le \log(2m+1)$ $(m \in \mathbb{N}\cup\{0\})$[†2]. Thus $\displaystyle\sum_{\substack{\nu\in\mathbb{N};\\ 1\le\nu<\beta+\eta}} \frac{1}{\nu} \le \log\big(2\lfloor\beta+\eta\rfloor+1\big)$.

(ii) $\displaystyle\sum_{\nu=1}^{\infty} \frac{\beta}{\nu(\nu+\beta)} \le \log\big(2\lfloor\beta\rfloor+3\big)$.

(iii) $\displaystyle\sum_{\substack{\nu\in\mathbb{N};\\ \nu\ge\beta+\eta}} \frac{\beta}{\nu(\nu-\beta)} \le \frac{1}{\lfloor\beta\rfloor+1-\beta} + \log\big(2\lfloor\beta\rfloor+3\big)$.

(iv) $\displaystyle\sum_{\substack{\nu\in\mathbb{N};\\ \nu\ge\beta+\eta}} \frac{1}{\nu(\nu-\beta)} \le \mathbf{1}_{\beta\ge\frac{1}{2}} \frac{1}{\beta}\Big(\frac{1}{1+\lfloor\beta\rfloor-\beta} + \log\big(\lfloor 2\beta\rfloor-\beta\big) - \log\big(1+\lfloor\beta\rfloor-\beta\big)\Big) + \frac{\pi^2}{3}$.

∵ (i) Since, for $\nu \in \mathbb{N}$,

$$
\int_{\nu-\frac{1}{2}}^{\nu+\frac{1}{2}} \frac{dx}{x} - \int_{\nu-\frac{1}{2}}^{\nu+\frac{1}{2}} \frac{dx}{\nu}
$$

[†2] In case $m \ge 2$, since $\frac{2m+1}{m} = 2 + \frac{1}{m} \le 2 + \frac{1}{2} = 2.5 < 2.718 < e$, we have $\log \frac{2m+1}{m} < \log e = 1$, so that $\log(2m+1) < 1 + \log m$. This, together with 10°(i), implies (6.3).

$$
\begin{aligned}
&= \int_{\nu-\frac{1}{2}}^{\nu}\Big(\frac{1}{x}-\frac{1}{\nu}\Big)dx - \int_{\nu}^{\nu+\frac{1}{2}}\Big(\frac{1}{\nu}-\frac{1}{x}\Big)dx \\
&= \int_0^{\frac{1}{2}}\Big(\frac{1}{y+\nu-\frac{1}{2}}-\frac{1}{\nu}\Big)dy - \int_0^{\frac{1}{2}}\Big(\frac{1}{\nu}-\frac{1}{y+\nu}\Big)dy \\
&= \int_0^{\frac{1}{2}} \frac{\frac{1}{2}-y}{\big(\nu-(\frac{1}{2}-y)\big)\nu}dy - \int_0^{\frac{1}{2}}\frac{y}{\nu(y+\nu)}dy \\
&= \int_0^{\frac{1}{2}} \frac{y}{\nu(\nu-y)}dy - \int_0^{\frac{1}{2}}\frac{y}{\nu(y+\nu)}dy \\
&= \int_0^{\frac{1}{2}} \frac{y}{\nu}\frac{y+\nu-\nu+y}{\nu^2-y^2}dy \\
&= \int_0^{\frac{1}{2}} \frac{2y^2}{\nu(\nu^2-y^2)}dy \\
&> 0,
\end{aligned}
$$

it follows that

$$
\begin{aligned}
\sum_{\nu=1}^{m}\frac{1}{\nu} = \sum_{\nu=1}^{m}\int_{\nu-\frac{1}{2}}^{\nu+\frac{1}{2}}\frac{dx}{\nu} < \sum_{\nu=1}^{m}\int_{\nu-\frac{1}{2}}^{\nu+\frac{1}{2}}\frac{dx}{x} &= \int_{\frac{1}{2}}^{m+\frac{1}{2}}\frac{dx}{x} \\
&= \big[\log x\big]_{\frac{1}{2}}^{m+\frac{1}{2}} = \log(2m+1).
\end{aligned}
$$

(ii) By (i),

$$
\begin{aligned}
\sum_{\nu=1}^{\infty}\frac{\beta}{\nu(\nu+\beta)} &= \sum_{\nu=1}^{\infty}\Big(\frac{1}{\nu}-\frac{1}{\nu+\beta}\Big) \\
&= \lim_{N\to\infty}\Big(\sum_{\nu=1}^{N}\frac{1}{\nu}-\sum_{\nu=1}^{N}\frac{1}{\nu+\beta}\Big) \\
&\le \lim_{N\to\infty}\Big(\sum_{\nu=1}^{N}\frac{1}{\nu}-\sum_{\nu=1}^{N}\frac{1}{\nu+\lfloor\beta\rfloor+1}\Big) \\
&\qquad \Big[\because\ \beta<\lfloor\beta\rfloor+1 \Rightarrow \tfrac{1}{\nu+\beta} > \tfrac{1}{\nu+\lfloor\beta\rfloor+1}\Big] \\
&= \lim_{N\to\infty}\Big(\sum_{\nu=1}^{N}\frac{1}{\nu}-\sum_{\nu=\lfloor\beta\rfloor+2}^{N+\lfloor\beta\rfloor+1}\frac{1}{\nu}\Big) \\
&= \lim_{N\to\infty}\Big(\sum_{\nu=1}^{\lfloor\beta\rfloor+1}\frac{1}{\nu}-\sum_{\nu=N+1}^{N+\lfloor\beta\rfloor+1}\frac{1}{\nu}\Big) \\
&= \lim_{N\to\infty}\Big(\sum_{\nu=1}^{\lfloor\beta\rfloor+1}\frac{1}{\nu}-\sum_{\nu=1}^{\lfloor\beta\rfloor+1}\frac{1}{\nu+N}\Big) \\
&= \sum_{\nu=1}^{\lfloor\beta\rfloor+1}\frac{1}{\nu}
\end{aligned}
$$

$$\leq \log\bigl(2(\lfloor\beta\rfloor + 1) + 1\bigr)$$
$$= \log\bigl(2\lfloor\beta\rfloor + 3\bigr).$$

(iii) Note that $\lfloor\beta\rfloor + 2 > \beta + \eta$ by $\lfloor\beta\rfloor + 2 - (\beta + \eta) = \lfloor\beta\rfloor + 1 - \beta + 1 - \eta > 0$. In case $\lfloor\beta\rfloor + 1 \geq \beta + \eta$,

$$\begin{aligned}
&\sum_{\substack{\nu\in\mathbb{N};\\ \nu\geq\beta+\eta}} \frac{\beta}{\nu(\nu-\beta)} \\
&= \sum_{\nu=\lfloor\beta\rfloor+1}^{\infty} \frac{\beta}{\nu(\nu-\beta)} \\
&= \sum_{\nu=\lfloor\beta\rfloor+1}^{\infty} \Bigl(\frac{1}{\nu-\beta} - \frac{1}{\nu}\Bigr) \\
&= \lim_{N\to\infty}\Bigl(\sum_{\nu=\lfloor\beta\rfloor+1}^{N} \frac{1}{\nu-\beta} - \sum_{\nu=\lfloor\beta\rfloor+1}^{N} \frac{1}{\nu}\Bigr) \\
&= \lim_{N\to\infty}\Bigl(\frac{1}{\lfloor\beta\rfloor+1-\beta} + \sum_{\nu=\lfloor\beta\rfloor+2}^{N} \frac{1}{\nu-\beta} - \sum_{\nu=\lfloor\beta\rfloor+1}^{N} \frac{1}{\nu}\Bigr) \\
&\leq \lim_{N\to\infty}\Bigl(\frac{1}{\lfloor\beta\rfloor+1-\beta} + \sum_{\nu=\lfloor\beta\rfloor+2}^{N} \frac{1}{\nu-\lfloor\beta\rfloor-1} - \sum_{\nu=\lfloor\beta\rfloor+1}^{N} \frac{1}{\nu}\Bigr) \\
&\qquad \Bigl[\because\ \beta < \lfloor\beta\rfloor + 1 \Rightarrow \nu - \beta > \nu - \lfloor\beta\rfloor - 1 \Rightarrow \tfrac{1}{\nu-\beta} < \tfrac{1}{\nu-\lfloor\beta\rfloor-1}\Bigr] \\
&= \lim_{N\to\infty}\Bigl(\frac{1}{\lfloor\beta\rfloor+1-\beta} + \sum_{\nu=1}^{N-\lfloor\beta\rfloor-1} \frac{1}{\nu} - \sum_{\nu=\lfloor\beta\rfloor+1}^{N} \frac{1}{\nu}\Bigr) \\
&= \lim_{N\to\infty}\Bigl(\frac{1}{\lfloor\beta\rfloor+1-\beta} + \sum_{\nu=1}^{\lfloor\beta\rfloor} \frac{1}{\nu} + \sum_{\nu=\lfloor\beta\rfloor+1}^{N-\lfloor\beta\rfloor-1} \frac{1}{\nu} - \sum_{\nu=\lfloor\beta\rfloor+1}^{N-\lfloor\beta\rfloor-1} \frac{1}{\nu} - \sum_{\nu=N-\lfloor\beta\rfloor}^{N} \frac{1}{\nu}\Bigr) \\
&= \lim_{N\to\infty}\Bigl(\frac{1}{\lfloor\beta\rfloor+1-\beta} + \sum_{\nu=1}^{\lfloor\beta\rfloor} \frac{1}{\nu} - \sum_{\nu=0}^{\lfloor\beta\rfloor} \frac{1}{N-\nu}\Bigr) \\
&= \frac{1}{\lfloor\beta\rfloor+1-\beta} + \sum_{\nu=1}^{\lfloor\beta\rfloor} \frac{1}{\nu} \\
&\leq \frac{1}{\lfloor\beta\rfloor+1-\beta} + \log\bigl(2\lfloor\beta\rfloor + 1\bigr).
\end{aligned}$$

In case $\lfloor\beta\rfloor + 1 < \beta + \eta$,

$$\sum_{\substack{\nu\in\mathbb{N};\\ \nu\geq\beta+\eta}} \frac{\beta}{\nu(\nu-\beta)} = \sum_{\nu=\lfloor\beta\rfloor+2}^{\infty} \frac{\beta}{\nu(\nu-\beta)}$$

$$
\begin{aligned}
&= \sum_{\nu=\lfloor\beta\rfloor+2}^{\infty}\Big(\frac{1}{\nu-\beta}-\frac{1}{\nu}\Big) \\
&= \lim_{N\to\infty}\Big(\sum_{\nu=\lfloor\beta\rfloor+2}^{N}\frac{1}{\nu-\beta}-\sum_{\nu=\lfloor\beta\rfloor+2}^{N}\frac{1}{\nu}\Big) \\
&\le \lim_{N\to\infty}\Big(\sum_{\nu=\lfloor\beta\rfloor+2}^{N}\frac{1}{\nu-\lfloor\beta\rfloor-1}-\sum_{\nu=\lfloor\beta\rfloor+2}^{N}\frac{1}{\nu}\Big) \\
&= \lim_{N\to\infty}\Big(\sum_{\nu=1}^{N-\lfloor\beta\rfloor-1}\frac{1}{\nu}-\sum_{\nu=\lfloor\beta\rfloor+2}^{N}\frac{1}{\nu}\Big) \\
&= \lim_{N\to\infty}\Big(\sum_{\nu=1}^{\lfloor\beta\rfloor+1}\frac{1}{\nu}-\sum_{\nu=N-\lfloor\beta\rfloor}^{N}\frac{1}{\nu}\Big) \\
&= \sum_{\nu=1}^{\lfloor\beta\rfloor+1}\frac{1}{\nu} \\
&\le \log\big(2\lfloor\beta\rfloor+3\big).
\end{aligned}
$$

(iv) First

$$
\sum_{\substack{\nu\in\mathbb{N};\\ \nu\ge\beta+\eta}}\frac{1}{\nu(\nu-\beta)} = \sum_{\substack{\nu\in\mathbb{N};\\ \beta+\eta\le\nu\le2\beta}}\frac{1}{\nu(\nu-\beta)} + \sum_{\substack{\nu\in\mathbb{N};\\ \nu>2\beta}}\frac{1}{\nu(\nu-\beta)}
$$
$$
=: \text{the 1st term} + \text{the 2nd term}.
$$

From the implications

$$
\nu>2\beta \Rightarrow \frac{\nu}{2}>\beta \Rightarrow \nu-\beta=\frac{\nu}{2}+\frac{\nu}{2}-\beta>\frac{\nu}{2} \Rightarrow \frac{1}{\nu-\beta}<\frac{2}{\nu}
$$
$$
\Rightarrow \frac{1}{\nu(\nu-\beta)}<\frac{2}{\nu^2},
$$

the 2nd term is etstimated as

$$
\text{the 2nd term} \le \sum_{\substack{\nu\in\mathbb{N};\\ \nu>2\beta}}\frac{2}{\nu^2} \le 2\sum_{\nu=1}^{\infty}\frac{1}{\nu^2}=\frac{\pi^2}{3}.
$$

For the 1st term, let $\beta\ge\frac{1}{2}$ from the implications

$$
\beta<\frac{1}{2} \Rightarrow 2\beta<1 \Rightarrow \text{the 1st term} = 0.
$$

Then, from the implications

$$
\beta+\eta\le\nu\le2\beta \Rightarrow \beta<\nu\le2\beta \Rightarrow \frac{1}{\nu}<\frac{1}{\beta},\ \ \lfloor\beta\rfloor<\nu\le\lfloor2\beta\rfloor,
$$

it follows that

$$
\begin{aligned}
&\text{the 1st term}\\
&\le \frac{1}{\beta}\sum_{\lfloor\beta\rfloor<\nu\le\lfloor2\beta\rfloor}\frac{1}{\nu-\beta}\quad\big[\text{Note that }\lfloor2\beta\rfloor-\lfloor\beta\rfloor\ge 1\big]\\
&=\frac{1}{\beta}\sum_{\nu=\lfloor\beta\rfloor+1}^{\lfloor2\beta\rfloor}\frac{1}{\nu-\beta}\\
&=\frac{1}{\beta}\sum_{\lambda=1}^{\lfloor2\beta\rfloor-\lfloor\beta\rfloor}\frac{1}{\lambda+\lfloor\beta\rfloor-\beta}\\
&=\frac{1}{\beta}\bigg(\frac{1}{1+\lfloor\beta\rfloor-\beta}+\sum_{\lambda=2}^{\lfloor2\beta\rfloor-\lfloor\beta\rfloor}\int_{\lambda-1}^{\lambda}\frac{dx}{\lambda+\lfloor\beta\rfloor-\beta}\bigg)\\
&\le\frac{1}{\beta}\bigg(\frac{1}{1+\lfloor\beta\rfloor-\beta}+\int_{1}^{\lfloor2\beta\rfloor-\lfloor\beta\rfloor}\frac{dx}{x+\lfloor\beta\rfloor-\beta}\bigg)\\
&=\frac{1}{\beta}\bigg(\frac{1}{1+\lfloor\beta\rfloor-\beta}+\Big[\log\big(x+\lfloor\beta\rfloor-\beta\big)\Big]_1^{\lfloor2\beta\rfloor-\lfloor\beta\rfloor}\bigg)\\
&=\frac{1}{\beta}\bigg(\frac{1}{1+\lfloor\beta\rfloor-\beta}+\log\big(\lfloor2\beta\rfloor-\lfloor\beta\rfloor+\lfloor\beta\rfloor-\beta\big)-\log\big(1+\lfloor\beta\rfloor-\beta\big)\bigg)\\
&=\frac{1}{\beta}\bigg(\frac{1}{1+\lfloor\beta\rfloor-\beta}+\log\big(\lfloor2\beta\rfloor-\beta\big)-\log\big(1+\lfloor\beta\rfloor-\beta\big)\bigg).
\end{aligned}
$$

Thus we have the assertion (iv).

11º By 9º and 10º,

$$
\begin{aligned}
&\bigg|\sum_{a<n\le b}g(n)e^{\sqrt{-1}2\pi f(n)}-\sum_{\substack{\nu\in\mathbb{N}\cup\{0\};\\0\le\nu<\beta+\eta}}\int_a^b g(x)e^{\sqrt{-1}2\pi(f(x)-\nu x)}dx\bigg|\\
&\le g(a)+\frac{1}{6}|g'(a)|\\
&\quad+\frac{3|g'(a)|}{4\pi^2}\bigg(\mathbf{1}_{\beta\ge\frac12}\frac{1}{\beta}\Big(\frac{1}{1+\lfloor\beta\rfloor-\beta}+\log\big(\lfloor2\beta\rfloor-\beta\big)-\log\big(1+\lfloor\beta\rfloor-\beta\big)\Big)+\frac{\pi^2}{3}\bigg)\\
&\quad+\frac{3g(a)}{2\pi}\log\big(2\lfloor\beta+\eta\rfloor+1\big)\\
&\quad+\frac{3g(a)}{2\pi}\log\big(2\lfloor\beta\rfloor+3\big)+\frac{3g(a)}{2\pi}\bigg(\frac{1}{\lfloor\beta\rfloor+1-\beta}+\log\big(2\lfloor\beta\rfloor+3\big)\bigg).
\end{aligned}
$$

■

We now present the square mean value estimate of $\zeta(\cdot)$ for $\frac{1}{2}<\sigma\le 1$.

Claim 6.3 *For* $\frac{1}{2}<\sigma\le 1$,

$$\bigg|\int_1^T\big|\zeta(\sigma+\sqrt{-1}t)\big|^2dt-T\zeta(2\sigma)\bigg|=O\big(T^{2-2\sigma}\log T\big)+O\big(T^{\frac12}\big)$$

$$= \begin{cases} O\big(T^{2-2\sigma}\log T\big), & \frac{1}{2} < \sigma \le \frac{3}{4}, \\ O\big(T^{\frac{1}{2}}\big), & \frac{3}{4} < \sigma \le 1. \end{cases}$$

Proof. We divide the proof into four steps:

$\underline{1^{\circ}}$ Let $\sigma > 0$ and $t \ge 1$. Theorem 6.1 with $C = 2\pi$, $x = t$, $s = \sigma + \sqrt{-1}t$ gives that

$$\begin{aligned} &\left|\zeta(\sigma+\sqrt{-1}t) - \Big(\sum_{n\le t}\frac{1}{n^{\sigma+\sqrt{-1}t}} - \frac{t^{1-(\sigma+\sqrt{-1}t)}}{1-(\sigma+\sqrt{-1}t)}\Big)\right| \\ &\le t^{-\sigma}\left(1 + \frac{3}{\pi}\log 3 + \frac{5}{12}\sigma + \frac{3}{2\pi}\Big(1+\frac{\sigma}{\pi}\Big)\frac{2\pi}{2\pi-1}\right) \\ &= t^{-\sigma}\left(1 + \frac{3}{\pi}\log 3 + \frac{5}{12}\sigma + \frac{3}{2\pi-1}\Big(1+\frac{\sigma}{\pi}\Big)\right). \end{aligned}$$

For simplicity, put

$$r(\sigma+\sqrt{-1}t) := \zeta(\sigma+\sqrt{-1}t) - \sum_{n\le t}\frac{1}{n^{\sigma+\sqrt{-1}t}}. \tag{6.5}$$

Then

$$\begin{aligned} &|r(\sigma+\sqrt{-1}t)| \\ &= \left|\zeta(\sigma+\sqrt{-1}t) - \Big(\sum_{n\le t}\frac{1}{n^{\sigma+\sqrt{-1}t}} - \frac{t^{1-(\sigma+\sqrt{-1}t)}}{1-(\sigma+\sqrt{-1}t)}\Big) - \frac{t^{1-(\sigma+\sqrt{-1}t)}}{1-(\sigma+\sqrt{-1}t)}\right| \\ &\le \left|\zeta(\sigma+\sqrt{-1}t) - \Big(\sum_{n\le t}\frac{1}{n^{\sigma+\sqrt{-1}t}} - \frac{t^{1-(\sigma+\sqrt{-1}t)}}{1-(\sigma+\sqrt{-1}t)}\Big)\right| + \left|\frac{t^{1-\sigma}t^{-\sqrt{-1}t}}{1-\sigma-\sqrt{-1}t}\right| \\ &\le t^{-\sigma}\left(1 + \frac{3}{\pi}\log 3 + \frac{5}{12}\sigma + \frac{3}{2\pi-1}\Big(1+\frac{\sigma}{\pi}\Big)\right) + \frac{t^{1-\sigma}}{\sqrt{(1-\sigma)^2+t^2}} \\ &= t^{-\sigma}\left(1 + \frac{3}{\pi}\log 3 + \frac{5}{12}\sigma + \frac{3}{2\pi-1}\Big(1+\frac{\sigma}{\pi}\Big) + \frac{t}{\sqrt{(1-\sigma)^2+t^2}}\right) \\ &\le t^{-\sigma}\left(2 + \frac{3}{\pi}\log 3 + \frac{5}{12}\sigma + \frac{3}{2\pi-1}\Big(1+\frac{\sigma}{\pi}\Big)\right). \end{aligned}$$

$\underline{2^{\circ}}$ Let $\frac{1}{2} < \sigma \le 1$. For $T \ge 1$,

$$\begin{aligned} &\int_1^T \Big|\sum_{n\le t}\frac{1}{n^{\sigma+\sqrt{-1}t}}\Big|^2 dt \\ &= \int_1^T \sum_{n\le t}\frac{1}{n^{\sigma+\sqrt{-1}t}}\sum_{m\le t}\frac{1}{m^{\sigma-\sqrt{-1}t}}dt \\ &= \int_1^T \sum_{n,m\le t}\frac{e^{\sqrt{-1}t\log\frac{m}{n}}}{n^\sigma m^\sigma}dt \\ &= \int_1^T \sum_{n,m\le T}\mathbf{1}_{t\ge n\vee m}\frac{e^{\sqrt{-1}t\log\frac{m}{n}}}{n^\sigma m^\sigma}dt \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n,m \le T} \frac{1}{n^\sigma m^\sigma} \int_{n \vee m}^{T} e^{\sqrt{-1} t \log \frac{m}{n}} dt \\
&= \sum_{n \le T} \frac{1}{n^{2\sigma}} (T-n) + \sum_{\substack{n,m \le T; \\ n \ne m}} \frac{1}{n^\sigma m^\sigma} \int_{n \vee m}^{T} e^{\sqrt{-1} t \log \frac{m}{n}} dt \\
&= T \sum_{n \le T} \frac{1}{n^{2\sigma}} - \sum_{n \le T} \frac{1}{n^{2\sigma - 1}} \\
&\quad + \sum_{\substack{n,m \le T; \\ m > n}} \frac{1}{n^\sigma m^\sigma} \int_{n \vee m}^{T} e^{\sqrt{-1} t \log \frac{m}{n}} dt \\
&\quad + \sum_{\substack{n,m \le T; \\ n > m}} \frac{1}{n^\sigma m^\sigma} \int_{n \vee m}^{T} e^{-\sqrt{-1} t \log \frac{n}{m}} dt \\
&= T \Big(\zeta(2\sigma) - \sum_{n > T} \frac{1}{n^{2\sigma}} \Big) - \sum_{n \le T} \frac{1}{n^{2\sigma-1}} \\
&\quad + 2 \sum_{\substack{n,m \le T; \\ m > n}} \frac{1}{n^\sigma m^\sigma} \int_{n \vee m}^{T} \cos\Big(t \log \frac{m}{n} \Big) dt \\
&= T \zeta(2\sigma) - T \sum_{n > T} \frac{1}{n^{2\sigma}} - \sum_{n \le T} \frac{1}{n^{2\sigma-1}} \\
&\quad + 2 \sum_{\substack{n,m \le T; \\ m > n}} \frac{1}{n^\sigma m^\sigma} \frac{\sin\big(T \log \frac{m}{n}\big) - \sin\big((n \vee m) \log \frac{m}{n}\big)}{\log \frac{m}{n}}. \qquad (6.6)
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\left| \int_1^T \Big| \sum_{n \le t} \frac{1}{n^{\sigma + \sqrt{-1} t}} \Big|^2 dt - T \zeta(2\sigma) \right| \\
&\le T \sum_{n > T} \frac{1}{n^{2\sigma}} + \sum_{n \le T} \frac{1}{n^{2\sigma-1}} + 4 \sum_{\substack{n,m \le T; \\ m > n}} \frac{1}{n^\sigma m^\sigma} \frac{1}{\log \frac{m}{n}} \\
&= T \sum_{n > T} \frac{1}{n^{2\sigma}} + \sum_{n \le T} \frac{1}{n^{2\sigma-1}} + 4 \sum_{\substack{n,m \le T; \\ n < m \le 2n}} \frac{1}{n^\sigma m^\sigma} \frac{1}{\log \frac{m}{n}} + 4 \sum_{\substack{n,m \le T; \\ m > 2n}} \frac{1}{n^\sigma m^\sigma} \frac{1}{\log \frac{m}{n}} \\
&\le T \sum_{n > T} \frac{1}{n^{2\sigma}} + \sum_{n \le T} \frac{1}{n^{2\sigma-1}} + 4 \sum_{n \le T} \frac{1}{n^\sigma} \sum_{n < m \le 2n} \frac{1}{m^\sigma} \frac{1}{\log(1 + \frac{m}{n} - 1)} \\
&\quad + \frac{4}{\log 2} \sum_{n,m \le T} \frac{1}{n^\sigma} \frac{1}{m^\sigma} \\
&\le T \sum_{n > T} \frac{1}{n^{2\sigma}} + \sum_{n \le T} \frac{1}{n^{2\sigma-1}} + 8 \sum_{n \le T} \frac{1}{n^{2\sigma-1}} \sum_{n < m \le 2n} \frac{1}{m-n} + \frac{4}{\log 2} \Big(\sum_{n \le T} \frac{1}{n^\sigma} \Big)^2 \\
&\quad \Big[\because \text{By (6.2)}, \log(1 + \tfrac{m}{n} - 1) \ge \tfrac{1}{2}(\tfrac{m}{n} - 1) = \tfrac{m-n}{2n} \Big]
\end{aligned}
$$

$$\le T\sum_{n>T}\frac{1}{n^{2\sigma}}+9\sum_{n\le T}\frac{1}{n^{2\sigma-1}}+8\sum_{n\le T}\frac{\log n}{n^{2\sigma-1}}+\frac{4}{\log 2}\Big(\sum_{n\le T}\frac{1}{n^{\sigma}}\Big)^2 \tag{6.7}$$

$\big[\because$ By (6.3), $\sum_{n<m\le 2n}\frac{1}{m-n}=\sum_{k=1}^{n}\frac{1}{k}\le 1+\log n\big]$.

<u>3°</u> Let $0<a\le 1<b<\infty$. As $x\to\infty$,

$$\sum_{n\le x}\frac{1}{n^a}\sim\begin{cases}\dfrac{x^{1-a}}{1-a}, & 0<a<1,\\ \log x, & a=1,\end{cases}$$

$$\sum_{n\le x}\frac{\log n}{n^a}\sim\begin{cases}\dfrac{x^{1-a}\log x}{1-a}, & 0<a<1,\\ \dfrac{1}{2}(\log x)^2, & a=1,\end{cases}$$

$$\sum_{n>x}\frac{1}{n^b}\sim\frac{x^{1-b}}{b-1}.$$

$\because$ By Theorem 4.1,

$$\begin{aligned}\sum_{n\le x}\frac{1}{n^a}&=\int_1^x t^{-a}dt-\big(\overline{B_1}(x)x^{-a}-\overline{B_1}(1-)\big)+\int_1^x\overline{B_1}(t)(-a)t^{-a-1}dt\\&=\int_1^x t^{-a}dt-\Big(\frac{\{x\}-\frac12}{x^a}-\frac12\Big)-a\int_1^x\frac{\{t\}-\frac12}{t^{a+1}}dt\\&=\int_1^x t^{-a}dt+O(1),\end{aligned}$$

$$\begin{aligned}\sum_{n\le x}\frac{\log n}{n^a}&=\int_1^x\frac{\log t}{t^a}dt-\overline{B_1}(x)\frac{\log x}{x^a}+\int_1^x\overline{B_1}(t)t^{-a-1}(1-a\log t)dt\\&=\int_1^x\frac{\log t}{t^a}dt-\Big(\{x\}-\frac12\Big)\frac{\log x}{x^a}+\int_1^x\Big(\{t\}-\frac12\Big)\frac{1-a\log t}{t^{a+1}}dt\\&=\int_1^x\frac{\log t}{t^a}dt+O(1),\end{aligned}$$

$$\begin{aligned}\sum_{n>x}\frac{1}{n^b}&=\int_x^\infty t^{-b}dt+\overline{B_1}(x)x^{-b}+\int_x^\infty\overline{B_1}(t)(-b)t^{-b-1}dt\\&=\int_x^\infty t^{-b}dt+\frac{\{x\}-\frac12}{x^b}-b\int_x^\infty\frac{\{t\}-\frac12}{t^{b+1}}dt\\&=\frac{x^{1-b}}{b-1}+\frac{\{x\}-\frac12}{x^b}-b\int_x^\infty\frac{\{t\}-\frac12}{t^{b+1}}dt\\&=x^{1-b}\Big(\frac{1}{b-1}+\frac{\{x\}-\frac12}{x}-bx^{b-1}\int_x^\infty\frac{\{t\}-\frac12}{t^{b+1}}dt\Big)\\&=x^{1-b}\Big(\frac{1}{b-1}+\frac{\{x\}-\frac12}{x}-\frac{b}{x}\int_1^\infty\frac{\{xs\}-\frac12}{s^{b+1}}ds\Big)\end{aligned}$$

$$\left[\because \text{ change of variable: } s = \tfrac{t}{x}\right]$$
$$= x^{1-b}\Big(\frac{1}{b-1} + o(1)\Big)$$
$$\sim \frac{x^{1-b}}{b-1}.$$

In case $a = 1$,

$$\int_1^x t^{-a}dt = \int_1^x \frac{dt}{t} = \log x,$$
$$\int_1^x \frac{\log t}{t}dt = \int_1^x \Big(\frac{1}{2}(\log t)^2\Big)'dt = \frac{1}{2}(\log x)^2.$$

In case $0 < a < 1$,

$$\int_1^x t^{-a}dt = \Big[\frac{t^{1-a}}{1-a}\Big]_1^x = \frac{x^{1-a}}{1-a} - \frac{1}{1-a} \sim \frac{x^{1-a}}{1-a},$$
$$\lim_{x\to\infty} \frac{\int_1^x \frac{\log t}{t^a}dt}{x^{1-a}\log x} = \lim_{x\to\infty} \frac{x^{-a}\log x}{(1-a)x^{-a}\log x + x^{1-a}\cdot \frac{1}{x}} \quad [\because \text{ L'Hospital's theorem}]$$
$$= \lim_{x\to\infty} \frac{\log x}{(1-a)\log x + 1}$$
$$= \frac{1}{1-a}.$$

Thus we have the assertion of 3°.

4° Since, by 3°,

$$T\sum_{n>T} \frac{1}{n^{2\sigma}} = T\frac{T^{1-2\sigma}}{2\sigma-1}\big(1+o(1)\big) = \frac{T^{2-2\sigma}}{2\sigma-1}\big(1+o(1)\big),$$
$$\sum_{n\le T} \frac{1}{n^{2\sigma-1}} = \begin{cases} \dfrac{T^{2-2\sigma}}{2-2\sigma}\big(1+o(1)\big), & \frac{1}{2} < \sigma < 1, \\ (\log T)\big(1+o(1)\big), & \sigma = 1, \end{cases}$$
$$\sum_{n\le T} \frac{\log n}{n^{2\sigma-1}} = \begin{cases} \dfrac{T^{2-2\sigma}\log T}{2-2\sigma}\big(1+o(1)\big), & \frac{1}{2} < \sigma < 1, \\ \dfrac{1}{2}(\log T)^2\big(1+o(1)\big), & \sigma = 1, \end{cases}$$
$$\Big(\sum_{n\le T} \frac{1}{n^{\sigma}}\Big)^2 = \begin{cases} \dfrac{T^{2-2\sigma}}{(1-\sigma)^2}\big(1+o(1)\big), & \frac{1}{2} < \sigma < 1, \\ (\log T)^2\big(1+o(1)\big), & \sigma = 1, \end{cases}$$

it follows from (6.7) that

$$\left|\int_1^T \Big|\sum_{n\le t} \frac{1}{n^{\sigma+\sqrt{-1}t}}\Big|^2 dt - T\zeta(2\sigma)\right|$$

$$\leq \begin{cases} \dfrac{T^{2-2\sigma}}{2\sigma-1}\big(1+o(1)\big)+9\dfrac{T^{2-2\sigma}}{2-2\sigma}\big(1+o(1)\big)+8\dfrac{T^{2-2\sigma}\log T}{2-2\sigma}\big(1+o(1)\big) \\ \quad +\dfrac{4}{\log 2}\dfrac{T^{2-2\sigma}}{(1-\sigma)^2}\big(1+o(1)\big)=O\big(T^{2-2\sigma}\log T\big), \quad \frac{1}{2}<\sigma<1, \\ 1+o(1)+9(\log T)\big(1+o(1)\big)+4(\log T)^2\big(1+o(1)\big) \\ \quad +\dfrac{4}{\log 2}(\log T)^2\big(1+o(1)\big)=O\big((\log T)^2\big), \quad \sigma=1, \end{cases} \tag{6.8}$$

which implies that

$$\int_1^T \Big|\sum_{n\leq t}\frac{1}{n^{\sigma+\sqrt{-1}t}}\Big|^2 dt = O(T). \tag{6.9}$$

By 1°,

$$\begin{aligned} &\int_1^T \big|r(\sigma+\sqrt{-1}t)\big|^2 dt \\ &\leq \left(2+\frac{3}{\pi}\log 3+\frac{5}{12}\sigma+\frac{3}{2\pi-1}\Big(1+\frac{\sigma}{\pi}\Big)\right)^2 \int_1^T t^{-2\sigma}dt \\ &= \left(2+\frac{3}{\pi}\log 3+\frac{5}{12}\sigma+\frac{3}{2\pi-1}\Big(1+\frac{\sigma}{\pi}\Big)\right)^2 \frac{1-\big(\frac{1}{T}\big)^{2\sigma-1}}{2\sigma-1} \\ &= O(1). \end{aligned} \tag{6.10}$$

This, together with (6.9), implies that

$$\begin{aligned} &\int_1^T \Big|\sum_{n\leq t}\frac{1}{n^{\sigma+\sqrt{-1}t}}\Big|\big|r(\sigma+\sqrt{-1}t)\big|dt \\ &\leq \sqrt{\int_1^T \Big|\sum_{n\leq t}\frac{1}{n^{\sigma+\sqrt{-1}t}}\Big|^2 dt}\sqrt{\int_1^T \big|r(\sigma+\sqrt{-1}t)\big|^2 dt} \\ &= O\big(T^{\frac{1}{2}}\big). \end{aligned} \tag{6.11}$$

Now, by (6.5),

$$\begin{aligned} &\Big|\int_1^T \big|\zeta(\sigma+\sqrt{-1}t)\big|^2 dt - T\zeta(2\sigma)\Big| \\ &= \Big|\int_1^T \Big|\sum_{n\leq t}\frac{1}{n^{\sigma+\sqrt{-1}t}}+r(\sigma+\sqrt{-1}t)\Big|^2 dt - T\zeta(2\sigma)\Big| \\ &= \Big|\int_1^T \Big(\Big|\sum_{n\leq t}\frac{1}{n^{\sigma+\sqrt{-1}t}}\Big|^2+\sum_{n\leq t}\frac{1}{n^{\sigma+\sqrt{-1}t}}\cdot\overline{r(\sigma+\sqrt{-1}t)} \\ &\qquad +\overline{\sum_{n\leq t}\frac{1}{n^{\sigma+\sqrt{-1}t}}}\cdot r(\sigma+\sqrt{-1}t)+|r(\sigma+\sqrt{-1}t)|^2\Big)dt - T\zeta(2\sigma)\Big| \end{aligned}$$

$$
\begin{aligned}
&= \Bigg| \int_1^T \Big| \sum_{n \le t} \frac{1}{n^{\sigma+\sqrt{-1}t}} \Big|^2 dt - T\zeta(2\sigma) \\
&\quad + \int_1^T \sum_{n \le t} \frac{1}{n^{\sigma+\sqrt{-1}t}} \cdot \overline{r(\sigma+\sqrt{-1}t)}dt \\
&\quad + \int_1^T \overline{\sum_{n \le t} \frac{1}{n^{\sigma+\sqrt{-1}t}}} \cdot r(\sigma+\sqrt{-1}t)dt + \int_1^T |r(\sigma+\sqrt{-1}t)|^2 dt \Bigg| \\
&\le \Bigg| \int_1^T \Big| \sum_{n \le t} \frac{1}{n^{\sigma+\sqrt{-1}t}} \Big|^2 dt - T\zeta(2\sigma) \Bigg| \\
&\quad + 2\int_1^T \Big| \sum_{n \le t} \frac{1}{n^{\sigma+\sqrt{-1}t}} \Big| \, |r(\sigma+\sqrt{-1}t)| dt + \int_1^T |r(\sigma+\sqrt{-1}t)|^2 dt.
\end{aligned}
$$

Therefore, combining this with (6.8), (6.11) and (6.10), we have

$$
\begin{aligned}
&\Big| \int_1^T \big| \zeta(\sigma+\sqrt{-1}t) \big|^2 dt - T\zeta(2\sigma) \Big| \\
&\le O\big(T^{2-2\sigma} \log T\big) + O\big((\log T)^2\big) + O\big(T^{\frac{1}{2}}\big) + O(1) \\
&= O\big(T^{2-2\sigma} \log T\big) + O\big(T^{\frac{1}{2}}\big).
\end{aligned}
$$

■

6.2 Stirling's formula and estimate of $\Gamma^{(l)}(\sigma+\sqrt{-1}t)$

The aim of this section is to show the exponential decay of $\Gamma^{(l)}(\sigma+\sqrt{-1}t)$ as $|t| \to \infty$, where $\Gamma^{(l)}$ is the lth derivative of the gamma function Γ.

We begin with the following theorem:

Theorem 6.2 (i) *For each $s \in \mathbb{C} \setminus (-\infty, 0]$,*

$$
\int_{-\frac{3}{2}-\sqrt{-1}\infty}^{-\frac{3}{2}+\sqrt{-1}\infty} \frac{\pi s^z}{z \sin \pi z} \zeta(z) dz := \lim_{R\to\infty} \int_{-\frac{3}{2}-\sqrt{-1}R}^{-\frac{3}{2}+\sqrt{-1}R} \frac{\pi s^z}{z \sin \pi z} \zeta(z) dz
$$

is convergent. And its convergence is uniform on $\{s \in \mathbb{C}; |s| \ge \varepsilon, |\arg s| \le \pi - \delta\}$ for $^\forall \varepsilon > 0$ and $0 < {}^\forall \delta < \pi$. Thus

$$
\mathbb{C} \setminus (-\infty, 0] \ni s \mapsto \int_{-\frac{3}{2}-\sqrt{-1}\infty}^{-\frac{3}{2}+\sqrt{-1}\infty} \frac{\pi s^z}{z \sin \pi z} \zeta(z) dz \in \mathbb{C}
$$

is holomorphic, and its nth derivative ($n \in \mathbb{N}$) is

$$
\begin{aligned}
&\frac{d^n}{ds^n} \int_{-\frac{3}{2}-\sqrt{-1}\infty}^{-\frac{3}{2}+\sqrt{-1}\infty} \frac{\pi s^z}{z \sin \pi z} \zeta(z) dz \\
&= \int_{-\frac{3}{2}-\sqrt{-1}\infty}^{-\frac{3}{2}+\sqrt{-1}\infty} \frac{\pi z(z-1)\cdots\big(z-(n-1)\big)s^{z-n}}{z \sin \pi z} \zeta(z) dz.
\end{aligned}
$$

(ii) *On* $\mathbb{C}\setminus(-\infty,0]$,

$$\log\Gamma(s)=\left(s-\frac{1}{2}\right)\log s-s+\frac{1}{2}\log 2\pi+\frac{1}{12}\frac{1}{s}$$
$$-\frac{1}{2\pi\sqrt{-1}}\int_{-\frac{3}{2}-\sqrt{-1}\infty}^{-\frac{3}{2}+\sqrt{-1}\infty}\frac{\pi s^z}{z\sin\pi z}\zeta(z)dz$$

holds.

Remark 6.1 (i) By Lemma A.1 and Claim A.9, $\Gamma(\cdot)$ is holomorphic and has no zeros on $\mathbb{C}\setminus(-\infty,0]$. Also $\mathbb{C}\setminus(-\infty,0]$ is a simply connected domain of $\mathbb{C}$. Thus, $\log\Gamma$ of L.H.S. above is the function defined by (3.4) with $a=1$, i.e., $\log\Gamma(s)=\int_1^s\frac{\Gamma'(z)}{\Gamma(z)}dz$, $s\in\mathbb{C}\setminus(-\infty,0]$.
(ii) By 1° in the proof of Theorem 6.2(i),

$$\begin{aligned}
&\big|\text{the last term of R.H.S. above}\big|\\
&=\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\pi s^{-\frac{3}{2}+\sqrt{-1}v}}{(-\frac{3}{2}+\sqrt{-1}v)\sin\pi(-\frac{3}{2}+\sqrt{-1}v)}\zeta\left(-\frac{3}{2}+\sqrt{-1}v\right)dv\right|\\
&\le\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{\pi s^{-\frac{3}{2}+\sqrt{-1}v}}{(-\frac{3}{2}+\sqrt{-1}v)\sin\pi(-\frac{3}{2}+\sqrt{-1}v)}\zeta\left(-\frac{3}{2}+\sqrt{-1}v\right)\right|dv\\
&\le\left(\frac{1}{|s|}\right)^{\frac{3}{2}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{\frac{9}{4}+v^2}}\left(\frac{41}{12}+\frac{1}{24}\left(\frac{5}{2}+|v|\right)+\frac{1}{24}\left(\frac{5}{2}+|v|\right)^3\right)e^{-\delta|v|}dv\\
&=O\left(\left(\frac{1}{|s|}\right)^{\frac{3}{2}}\right)\quad\text{on }\left\{s\in\mathbb{C}\setminus\{0\};|\arg s|\le\pi-\delta\right\}\text{ (where }0<\delta<\pi).
\end{aligned}$$

This tells us that Theorem 6.2(ii) is a refinement of Stirling's formula.

Following Whittaker-Watson ([34, Chapter XIII]), we prove this theorem. To this end, we present two lemmas:

Lemma 6.2 *For* $\sigma\ge-\frac{3}{2}$, $|\sigma+\sqrt{-1}t-1|\ge\frac{1}{3}$,

$$\left|\zeta(\sigma+\sqrt{-1}t)\right|\le\frac{41}{12}+\frac{1}{24}\left(\frac{5}{2}+|t|\right)+\frac{1}{24}\left(\frac{5}{2}+|t|\right)^3.$$

Proof. First, by the definition of Bernoulli polynomial [cf. Definition 4.2],

$$\begin{aligned}
B_3(x)=\sum_{k=0}^{3}\binom{3}{k}B_{3-k}x^k&=B_3+3B_2x+3B_1x^2+B_0x^3\\
&=\frac{1}{2}x-\frac{3}{2}x^2+x^3=x\left(x-\frac{1}{2}\right)(x-1).
\end{aligned}$$

Putting this into (4.2), we see that for $\mathrm{Re}\,s>-2$, $s\ne1$,

$$\zeta(s)=\frac{1}{s-1}+\frac{1}{2}+\frac{s}{12}+\frac{1}{6}s(s+1)(s+2)\int_1^{\infty}\frac{\{x\}\left(\{x\}-\frac{1}{2}\right)\left(1-\{x\}\right)}{x^{s+3}}dx.\quad(6.12)$$

Taking the absolute value, we have

$$
\begin{aligned}
|\zeta(s)| &\le \frac{1}{|s-1|} + \frac{1}{2} + \frac{|s|}{12} + \frac{1}{6}|s||s+1||s+2| \int_1^\infty \frac{\big|\{x\}\big(\{x\}-\frac{1}{2}\big)\big(1-\{x\}\big)\big|}{x^{\operatorname{Re} s+3}} dx \\
&\le \frac{1}{|s-1|} + \frac{1}{2} + \frac{|s|}{12} + \frac{1}{48}\frac{|s||s+1||s+2|}{\operatorname{Re} s + 2} \\
&\qquad \Big[\because \text{ For } 0 \le x \le 1,\ \big|x(x-\tfrac{1}{2})(1-x)\big| \le \tfrac{1}{8}\Big].
\end{aligned}
$$

In case $s = \sigma + \sqrt{-1}t$, $-\frac{3}{2} \le \sigma \le \frac{3}{2}$, $|\sigma + \sqrt{-1}t - 1| \ge \frac{1}{3}$, since

$$
\begin{aligned}
|s| &\le |\sigma| + |t| \le \frac{3}{2} + |t|, \\
|s+1| &\le \frac{5}{2} + |t|, \\
|s+2| &\le \frac{7}{2} + |t|, \\
\operatorname{Re} s + 2 &= \sigma + 2 \ge \frac{1}{2},
\end{aligned}
$$

the estimate above implies that

$$
\begin{aligned}
|\zeta(\sigma+\sqrt{-1}t)| &\le 3 + \frac{1}{2} + \frac{\frac{3}{2}+|t|}{12} + \frac{1}{24}\Big(\frac{3}{2}+|t|\Big)\Big(\frac{5}{2}+|t|\Big)\Big(\frac{7}{2}+|t|\Big) \\
&= \frac{7}{2} + \frac{1}{12}\Big(\frac{5}{2}+|t|-1\Big) + \frac{1}{24}\Big(\frac{5}{2}+|t|\Big)\Big(\Big(\frac{5}{2}+|t|\Big)^2 - 1\Big) \\
&= \frac{7}{2} - \frac{1}{12} + \frac{1}{12}\Big(\frac{5}{2}+|t|\Big) + \frac{1}{24}\Big(\frac{5}{2}+|t|\Big)^3 - \frac{1}{24}\Big(\frac{5}{2}+|t|\Big) \\
&= \frac{41}{12} + \frac{1}{24}\Big(\frac{5}{2}+|t|\Big) + \frac{1}{24}\Big(\frac{5}{2}+|t|\Big)^3.
\end{aligned}
$$

On the other hand, in case $s = \sigma + \sqrt{-1}t$, $\sigma \ge \frac{3}{2}$,

$$
\begin{aligned}
|\zeta(\sigma+\sqrt{-1}t)| = \Big|\sum_{n=1}^\infty \frac{1}{n^{\sigma+\sqrt{-1}t}}\Big| &\le \sum_{n=1}^\infty \Big|\frac{1}{n^{\sigma+\sqrt{-1}t}}\Big| \\
&= \sum_{n=1}^\infty \frac{1}{n^\sigma} \\
&\le \sum_{n=1}^\infty \frac{1}{n^{\frac{3}{2}}} \\
&= 1 + \sum_{n=2}^\infty \int_{n-1}^n \frac{dx}{n^{\frac{3}{2}}} \\
&\le 1 + \sum_{n=2}^\infty \int_{n-1}^n x^{-\frac{3}{2}} dx \\
&= 1 + \int_1^\infty \big(-2x^{-\frac{1}{2}}\big)' dx
\end{aligned}
$$

$$= 1 + 2 = 3 < \frac{41}{12}.$$

Therefore we obtain the assertion of the lemma. ■

Lemma 6.3 (i) $\lim_{s\to 1}\big(\zeta(s) - \frac{1}{s-1}\big) = \gamma$. *Here* γ *is Euler's constant, i.e.,*

$$\gamma = \lim_{n\to\infty}\Big(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\Big).$$

(ii) $\zeta(-1) = -\frac{1}{12}$.

(iii) $\zeta(0) = -\frac{1}{2}$.

(iv) $\zeta'(0) = -\frac{1}{2}\log 2\pi$.

Proof. First, recall the following identities:

(a) $\zeta(s) = 2\Gamma(1-s)\sin\Big(\frac{\pi}{2}s\Big)(2\pi)^{s-1}\zeta(1-s)$ $\big[$cf. Theorem 4.3(i)$\big]$.

(b)
$$\begin{aligned}
\zeta(s) &= \frac{1}{s-1} - \sum_{k=1}^{1}\frac{(-1)^k}{k!}B_k(1)(-s)(-s-1)\cdots\big(-s-(k-2)\big) \\
&\quad + \frac{(-1)^2}{1!}(-s)(-s-1)\cdots\big(-s-(1-1)\big)\int_1^\infty \overline{B_1}(x)x^{-s-1}dx \\
&= \frac{1}{s-1} - \frac{-1}{1}B_1(1) - s\int_1^\infty \overline{B_1}(x)x^{-s-1}dx \\
&= \frac{1}{s-1} + \frac{1}{2} - s\int_1^\infty \frac{\{x\}-\frac{1}{2}}{x^{s+1}}dx, \quad \mathrm{Re}\, s > 0,\ s \neq 1 \quad \big[\text{cf. (4.1)}\big].
\end{aligned}$$

(i) By (b),

$$\begin{aligned}
\lim_{s\to 1}\Big(\zeta(s) - \frac{1}{s-1}\Big) &= \lim_{s\to 1}\Big(\frac{1}{2} - s\int_1^\infty \frac{\{x\}-\frac{1}{2}}{x^{s+1}}dx\Big) \\
&= \frac{1}{2} - \int_1^\infty \frac{\{x\}-\frac{1}{2}}{x^2}dx \\
&= \frac{1}{2} + \frac{1}{2}\int_1^\infty \frac{dx}{x^2} - \lim_{N\to\infty}\int_1^{N+1}\frac{x-\lfloor x\rfloor}{x^2}dx \\
&= \frac{1}{2}\Big(1 + \Big[-\frac{1}{x}\Big]_1^\infty\Big) - \lim_{N\to\infty}\Big([\log x]_1^{N+1} - \sum_{n=1}^{N} n\int_n^{n+1}\frac{dx}{x^2}\Big) \\
&= 1 - \lim_{N\to\infty}\Big(\log(N+1) - \sum_{n=1}^{N} n\Big(-\frac{1}{n+1} + \frac{1}{n}\Big)\Big) \\
&= \lim_{N\to\infty}\Big(1 + \sum_{n=1}^{N} n\frac{1}{n(n+1)} - \log(N+1)\Big) \\
&= \lim_{N\to\infty}\Big(\sum_{n=1}^{N+1}\frac{1}{n} - \log(N+1)\Big) \\
&= \gamma.
\end{aligned}$$

Also by differentiating (b) in s,

$$\begin{aligned}\zeta'(s) &= \frac{-1}{(s-1)^2} - \int_1^\infty \frac{\{x\}-\frac{1}{2}}{x^{s+1}}dx - s\int_1^\infty \frac{\{x\}-\frac{1}{2}}{x^{s+1}}\log\frac{1}{x}dx \\ &= -\frac{1}{(s-1)^2} - \int_1^\infty \frac{\{x\}-\frac{1}{2}}{x^{s+1}}dx + s\int_1^\infty \frac{\{x\}-\frac{1}{2}}{x^{s+1}}\log x dx;\end{aligned}$$

by letting $s \to 1$ here,

$$\lim_{s\to 1}\Big(\zeta'(s) + \frac{1}{(s-1)^2}\Big) = -\int_1^\infty \frac{\{x\}-\frac{1}{2}}{x^2}dx + \int_1^\infty \frac{\{x\}-\frac{1}{2}}{x^2}\log x dx.$$

(ii) Putting $s = -1$ in (a) yields that

$$\begin{aligned}\zeta(-1) &= 2\Gamma(2)\sin\Big(-\frac{\pi}{2}\Big)(2\pi)^{-2}\zeta(2) \\ &= 2\cdot(-1)\cdot\frac{1}{4\pi^2}\frac{\pi^2}{6} \quad \Big[\because\ \Gamma(2) = 1\cdot\Gamma(1) = 1,\ \zeta(2) = \tfrac{\pi^2}{6}\Big] \\ &= -\frac{1}{12}.\end{aligned}$$

(iii) When $s \neq 0$, (a) is rewritten as

$$\zeta(s) = 2\Gamma(1-s)\frac{\pi}{2}\frac{\sin\big(\frac{\pi}{2}s\big)}{\frac{\pi}{2}s}(2\pi)^{s-1}\big(-(1-s-1)\zeta(1-s)\big).$$

Since $\lim_{s\to 0}(1-s-1)\zeta(1-s) = 1$ (by (i)) and $\lim_{s\to 0}\frac{\sin(\frac{\pi}{2}s)}{\frac{\pi}{2}s} = \lim_{z\to 0}\frac{\sin z}{z} = 1$, it is seen that

$$\begin{aligned}\zeta(0) &= \lim_{s\to 0} 2\Gamma(1-s)\frac{\pi}{2}\frac{\sin\big(\frac{\pi}{2}s\big)}{\frac{\pi}{2}s}(2\pi)^{s-1}\big(-(1-s-1)\zeta(1-s)\big) \\ &= 2\Gamma(1)\frac{\pi}{2}(2\pi)^{-1}(-1) \\ &= 2\cdot\frac{\pi}{2}\cdot\frac{1}{2\pi}\cdot(-1) \quad \big[\text{cf. } \Gamma(1) = 1\big] \\ &= -\frac{1}{2}.\end{aligned}$$

(iv) By differentiating (a) in s,

$$\begin{aligned}\zeta'(s) &= -2\Gamma'(1-s)\sin\Big(\frac{\pi}{2}s\Big)(2\pi)^{s-1}\zeta(1-s) \\ &\quad + 2\Gamma(1-s)\cos\Big(\frac{\pi}{2}s\Big)\frac{\pi}{2}(2\pi)^{s-1}\zeta(1-s) \\ &\quad + 2\Gamma(1-s)\sin\Big(\frac{\pi}{2}s\Big)(2\pi)^{s-1}\log(2\pi)\zeta(1-s) \\ &\quad - 2\Gamma(1-s)\sin\Big(\frac{\pi}{2}s\Big)(2\pi)^{s-1}\zeta'(1-s) \\ &= -2\Gamma'(1-s)\frac{\pi}{2}\frac{\sin(\frac{\pi}{2}s)}{\frac{\pi}{2}s}(2\pi)^{s-1}\big(-(1-s-1)\zeta(1-s)\big)\end{aligned}$$

$$
\begin{aligned}
&+2\Gamma(1-s)\frac{\pi}{2}\frac{\sin(\frac{\pi}{2}s)}{\frac{\pi}{2}s}(2\pi)^{s-1}\log(2\pi)\big(-(1-s-1)\zeta(1-s)\big)\\
&+2\Gamma(1-s)(2\pi)^{s-1}\bigg(\cos\Big(\frac{\pi}{2}s\Big)\frac{\pi}{2}\Big(\zeta(1-s)+\frac{1}{s}-\frac{1}{s}\Big)\\
&\qquad -\sin\Big(\frac{\pi}{2}s\Big)\Big(\zeta'(1-s)+\frac{1}{s^2}-\frac{1}{s^2}\Big)\bigg)\\
={}&\big(-2\Gamma'(1-s)+2\Gamma(1-s)\log(2\pi)\big)\frac{\pi}{2}(2\pi)^{s-1}\\
&\times\frac{\sin(\frac{\pi}{2}s)}{\frac{\pi}{2}s}\big(-(1-s-1)\zeta(1-s)\big)\\
&+2\Gamma(1-s)(2\pi)^{s-1}\bigg(\frac{\pi}{2}\cos\Big(\frac{\pi}{2}s\Big)\Big(\zeta(1-s)-\frac{1}{1-s-1}\Big)\\
&\qquad -\sin\Big(\frac{\pi}{2}s\Big)\Big(\zeta'(1-s)+\frac{1}{(1-s-1)^2}\Big)\\
&\qquad +\Big(\frac{\pi}{2}\Big)^2\frac{\sin\frac{\pi}{2}s-\frac{\pi}{2}s\cos\frac{\pi}{2}s}{(\frac{\pi}{2}s)^2}\bigg).
\end{aligned}
$$

From the proofs of (i) and (iii), and the convergence

$$
\lim_{s\to 0}\frac{\sin\frac{\pi}{2}s-\frac{\pi}{2}s\cos\frac{\pi}{2}s}{(\frac{\pi}{2}s)^2}=\lim_{z\to 0}\frac{\sin z-z\cos z}{z^2}=0,
$$

it follows that

$$
\begin{aligned}
\zeta'(0)&=\lim_{s\to 0}\zeta'(s)\\
&=\big(-2\Gamma'(1)+2\log(2\pi)\big)\frac{\pi}{2}\cdot\frac{1}{2\pi}\cdot(-1)+2\cdot\frac{1}{2\pi}\cdot\frac{\pi}{2}\cdot\gamma\\
&=\frac{1}{2}\big(\Gamma'(1)+\gamma\big)-\frac{1}{2}\log 2\pi.
\end{aligned}
\tag{6.13}
$$

Recall Gauss's product formula [cf. Claim A.9(ii)]:

$$
\Gamma(s)=\lim_{n\to\infty}\frac{n!n^s}{s(s+1)\cdots(s+n)},\qquad s\in\mathbb{C}\setminus\{0,-1,-2,\dots\}.
\tag{6.14}
$$

By letting $s>0$ and taking the logarithm,

$$
\begin{aligned}
\log\Gamma(s)&=\lim_{n\to\infty}\big(\log n!+s\log n-\log s-\log(s+1)-\cdots-\log(s+n)\big)\\
&=\lim_{n\to\infty}\bigg(s\Big(\log n-\sum_{k=1}^{n}\frac{1}{k}\Big)-\log s-\sum_{k=1}^{n}\Big(\log(s+k)-\log k-\frac{s}{k}\Big)\bigg)\\
&=\lim_{n\to\infty}\bigg(-s\Big(\sum_{k=1}^{n}\frac{1}{k}-\log n\Big)-\log s-\sum_{k=1}^{n}\Big(\log\Big(1+\frac{s}{k}\Big)-\frac{s}{k}\Big)\bigg)\\
&=-\gamma s-\log s-\sum_{k=1}^{\infty}\Big(\log\Big(1+\frac{s}{k}\Big)-\frac{s}{k}\Big).
\end{aligned}
\tag{6.15}
$$

Since

$$\left|\left(\log\left(1+\frac{s}{k}\right)-\frac{s}{k}\right)'\right| = \left|\frac{\frac{1}{k}}{1+\frac{s}{k}}-\frac{1}{k}\right| = \left|\frac{1}{k+s}-\frac{1}{k}\right| = \frac{s}{k(k+s)} \le \frac{s}{k^2},$$

a termwise differentiation of R.H.S. in (6.15) is permissible, so that

$$\begin{aligned}\frac{\Gamma'(s)}{\Gamma(s)} &= -\gamma - \frac{1}{s} - \sum_{k=1}^{\infty}\left(\frac{\frac{1}{k}}{1+\frac{s}{k}} - \frac{1}{k}\right)\\ &= -\gamma - \frac{1}{s} - \sum_{k=1}^{\infty}\left(\frac{1}{k+s} - \frac{1}{k}\right).\end{aligned}$$

In particular, putting $s = 1$ yields that

$$\begin{aligned}\Gamma'(1) &= -\gamma - 1 - \sum_{k=1}^{\infty}\left(\frac{1}{k+1} - \frac{1}{k}\right)\\ &= -\gamma - 1 + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right)\\ &= -\gamma - 1 + 1 = -\gamma.\end{aligned}$$

This, together with (6.13), implies that $\zeta'(0) = -\frac{1}{2}\log 2\pi$. ■

Proof of Theorem 6.2(i). First, for $n \in \mathbb{N} \cup \{0\}$,

$$\frac{d^n}{ds^n}\left(\frac{\pi s^z}{z\sin\pi z}\zeta(z)\right) = \frac{\pi z(z-1)\cdots\big(z-(n-1)\big)s^{z-n}}{z\sin\pi z}\zeta(z), \qquad s \in \mathbb{C}\setminus(-\infty,0],\ \operatorname{Re} z = -\tfrac{3}{2}.$$

Fix $\varepsilon > 0$ and $0 < \delta < \pi$, and put

$$E_{\varepsilon,\delta} := \left\{s = re^{\sqrt{-1}\theta};\, r \ge \varepsilon,\ |\theta| \le \pi - \delta\right\} \subset \mathbb{C}\setminus(-\infty,0].$$

We divide the proof into two steps:

<u>1°</u> For $n \in \mathbb{N}\cup\{0\}$, $s \in E_{\varepsilon,\delta}$ and $\operatorname{Re} z = -\frac{3}{2}$,

$$\begin{aligned}&\left|\frac{\pi z(z-1)\cdots\big(z-(n-1)\big)s^{z-n}}{z\sin\pi z}\zeta(z)\right|\\ &\le 2\pi\left(\frac{1}{\varepsilon}\right)^{\frac{3}{2}+n}\left(\frac{\prod_{k=0}^{n-1}\left((\frac{3}{2}+k)^2+(\operatorname{Im} z)^2\right)}{\frac{9}{4}+(\operatorname{Im} z)^2}\right)^{\frac{1}{2}}\\ &\quad\times\left(\frac{41}{12}+\frac{1}{24}\left(\frac{5}{2}+|\operatorname{Im} z|\right)+\frac{1}{24}\left(\frac{5}{2}+|\operatorname{Im} z|\right)^3\right)e^{-\delta|\operatorname{Im} z|}.\end{aligned}$$

∵ Let $s = re^{\sqrt{-1}\theta}$ $(r \ge \varepsilon, |\theta| \le \pi - \delta)$ and $z = -\frac{3}{2} + \sqrt{-1}v$ $(v \in \mathbb{R})$. Since

$$\sin\pi z = \sin\pi\left(-\frac{3}{2}+\sqrt{-1}v\right) = \sin\left(-\frac{3}{2}\pi + \sqrt{-1}\pi v\right)$$

$$
\begin{aligned}
&= \sin\Bigl(-\frac{3}{2}\pi\Bigr)\cos\bigl(\sqrt{-1}\pi v\bigr) \\
&= \cos\bigl(\sqrt{-1}\pi v\bigr) \\
&= \frac{1}{2}\bigl(e^{\sqrt{-1}\sqrt{-1}\pi v} + e^{-\sqrt{-1}\sqrt{-1}\pi v}\bigr) \\
&= \frac{1}{2}\bigl(e^{\pi v} + e^{-\pi v}\bigr),
\end{aligned}
$$

$$
\begin{aligned}
s^{z-n} = e^{(z-n)\log s} &= e^{(-\frac{3}{2}-n+\sqrt{-1}v)(\log r+\sqrt{-1}\theta)} \\
&= e^{(-\frac{3}{2}-n)\log r-\theta v+\sqrt{-1}\left((-\frac{3}{2}-n)\theta+v\log r\right)} \\
&= r^{-\frac{3}{2}-n}e^{-\theta v}e^{\sqrt{-1}\left((-\frac{3}{2}-n)\theta+v\log r\right)},
\end{aligned}
$$

$$
\begin{aligned}
\bigl|z(z-1)\cdots\bigl(z-(n-1)\bigr)\bigr| = \prod_{k=0}^{n-1}|z-k| &= \prod_{k=0}^{n-1}\Bigl|-\frac{3}{2}-k+\sqrt{-1}v\Bigr| \\
&= \Biggl(\prod_{k=0}^{n-1}\Bigl(\Bigl(\frac{3}{2}+k\Bigr)^2+v^2\Bigr)\Biggr)^{\frac{1}{2}},
\end{aligned}
$$

$$
\bigl|\zeta(z)\bigr| = \Bigl|\zeta\Bigl(-\frac{3}{2}+\sqrt{-1}v\Bigr)\Bigr| \le \frac{41}{12} + \frac{1}{24}\Bigl(\frac{5}{2}+|v|\Bigr) + \frac{1}{24}\Bigl(\frac{5}{2}+|v|\Bigr)^3 \quad \bigl[\because \text{ Lemma 6.2}\bigr],
$$

we have the following estimate:

$$
\begin{aligned}
&\left|\frac{\pi z(z-1)\cdots\bigl(z-(n-1)\bigr)s^{z-n}}{z\sin\pi z}\zeta(z)\right| \\
&= \frac{\pi\bigl|z(z-1)\cdots\bigl(z-(n-1)\bigr)\bigr||s^{z-n}|}{|z||\sin\pi z|}|\zeta(z)| \\
&\le \frac{\pi\left(\prod_{k=0}^{n-1}\left((\frac{3}{2}+k)^2+v^2\right)\right)^{\frac{1}{2}}}{\left(\frac{9}{4}+v^2\right)^{\frac{1}{2}}\frac{1}{2}(e^{\pi v}+e^{-\pi v})} r^{-\frac{3}{2}-n}e^{-\theta v}\left(\frac{41}{12}+\frac{1}{24}\Bigl(\frac{5}{2}+|v|\Bigr)+\frac{1}{24}\Bigl(\frac{5}{2}+|v|\Bigr)^3\right) \\
&= 2\pi\Bigl(\frac{1}{r}\Bigr)^{\frac{3}{2}+n}\left(\frac{\prod_{k=0}^{n-1}\left((\frac{3}{2}+k)^2+v^2\right)}{\frac{9}{4}+v^2}\right)^{\frac{1}{2}}\left(\frac{41}{12}+\frac{1}{24}\Bigl(\frac{5}{2}+|v|\Bigr)+\frac{1}{24}\Bigl(\frac{5}{2}+|v|\Bigr)^3\right) \\
&\quad\times\frac{e^{-\theta v}}{e^{\pi v}+e^{-\pi v}}.
\end{aligned}
$$

Here noting that

$$
\begin{aligned}
\frac{e^{-\theta v}}{e^{\pi v}+e^{-\pi v}} = \frac{e^{-\theta v}}{e^{\pi|v|}+e^{-\pi|v|}} &\le \frac{e^{|\theta||v|}}{e^{\pi|v|}+e^{-\pi|v|}} \quad \bigl[\because\ -\theta v \le |\theta||v|\bigr] \\
&= \frac{e^{-\pi|v|}e^{|\theta||v|}}{1+e^{-2\pi|v|}} \\
&= \frac{e^{-(\pi-|\theta|)|v|}}{1+e^{-2\pi|v|}}
\end{aligned}
$$

$$\leq \frac{e^{-\delta|v|}}{1+e^{-2\pi|v|}} \quad \big[\because |\theta| \leq \pi - \delta \Rightarrow \pi - |\theta| \geq \delta\big]$$
$$\leq e^{-\delta|v|},$$

we obtain

$$\left|\frac{\pi z(z-1)\cdots\big(z-(n-1)\big)s^{z-n}}{z \sin \pi z}\zeta(z)\right|$$
$$\leq 2\pi\Big(\frac{1}{\varepsilon}\Big)^{\frac{3}{2}+n}\left(\frac{\prod_{k=0}^{n-1}\big((\frac{3}{2}+k)^2+v^2\big)}{\frac{9}{4}+v^2}\right)^{\frac{1}{2}}\left(\frac{41}{12}+\frac{1}{24}\Big(\frac{5}{2}+|v|\Big)+\frac{1}{24}\Big(\frac{5}{2}+|v|\Big)^3\right)$$
$$\times e^{-\delta|v|}.$$

2° For each $n \in \mathbb{N} \cup \{0\}$, it is clear that

$$\int_{-\infty}^{\infty}\left(\frac{\prod_{k=0}^{n-1}\big((\frac{3}{2}+k)^2+v^2\big)}{\frac{9}{4}+v^2}\right)^{\frac{1}{2}}\left(\frac{41}{12}+\frac{1}{24}\Big(\frac{5}{2}+|v|\Big)+\frac{1}{24}\Big(\frac{5}{2}+|v|\Big)^3\right)e^{-\delta|v|}dv < \infty.$$

This tells us that

$$\int_{-\frac{3}{2}-\sqrt{-1}\infty}^{-\frac{3}{2}+\sqrt{-1}\infty}\frac{\pi s^z}{z \sin \pi z}\zeta(z)dz$$
$$= \int_{-\infty}^{\infty}\frac{\pi s^{-\frac{3}{2}+\sqrt{-1}v}}{(-\frac{3}{2}+\sqrt{-1}v)\sin \pi(-\frac{3}{2}+\sqrt{-1}v)}\zeta\Big(-\frac{3}{2}+\sqrt{-1}v\Big)\sqrt{-1}dv$$

is convergent for each $s \in \mathbb{C} \setminus (-\infty, 0]$ and that it is infinitely differentiable in s under the integral sign, and its nth derivative is

$$\frac{d^n}{ds^n}\int_{-\frac{3}{2}-\sqrt{-1}\infty}^{-\frac{3}{2}+\sqrt{-1}\infty}\frac{\pi s^z}{z \sin \pi z}\zeta(z)dz$$
$$= \int_{-\frac{3}{2}-\sqrt{-1}\infty}^{-\frac{3}{2}+\sqrt{-1}\infty}\frac{\pi z(z-1)\cdots\big(z-(n-1)\big)s^{z-n}}{z \sin \pi z}\zeta(z)dz. \qquad \blacksquare$$

Proof of Theorem 6.2(ii). By Theorem 6.2(i), the function of R.H.S. in the identity in question is holomorphic on $\mathbb{C} \setminus (-\infty, 0]$. Thus, by the uniqueness theorem, it suffices to verify this identity for $0 < s < 1$. Fix $0 < s < 1$. We divide the proof into six steps:

1° From (3.3) and (3.16), it is seen that

$$\sum_{k=1}^{\infty}\Big(\log\Big(1+\frac{s}{k}\Big)-\frac{s}{k}\Big) = \sum_{k=1}^{\infty}-\Big(-\log\Big(1-\Big(\frac{-s}{k}\Big)\Big)-\Big(\frac{-s}{k}\Big)\Big)$$
$$= \sum_{k=1}^{\infty}\Big(-\sum_{m=2}^{\infty}\frac{1}{m}\Big(\frac{-s}{k}\Big)^m\Big)$$
$$= \sum_{k=1}^{\infty}\sum_{m=2}^{\infty}\frac{(-1)^{m-1}}{m}\frac{s^m}{k^m},$$

$$
\begin{aligned}
\sum_{k=1}^{\infty}\sum_{m=2}^{\infty}\left|\frac{(-1)^{m-1}}{m}\frac{s^m}{k^m}\right| &= \sum_{k=1}^{\infty}\sum_{m=2}^{\infty}\frac{1}{m}\Big(\frac{s}{k}\Big)^m \\
&= \sum_{k=1}^{\infty}\Big(-\log\Big(1-\frac{s}{k}\Big)-\frac{s}{k}\Big) \\
&\le \sum_{k=1}^{\infty}\frac{\big(\frac{s}{k}\big)^2}{2\big(1-\frac{s}{k}\big)} \\
&\le \frac{s^2}{2(1-s)}\sum_{k=1}^{\infty}\frac{1}{k^2} \\
&\qquad \left[\because\ k \ge 1 \Rightarrow \frac{s}{k} \le s \Rightarrow 1-\frac{s}{k} \ge 1-s > 0 \Rightarrow \frac{1}{1-\frac{s}{k}} \le \frac{1}{1-s}\right] \\
&< \infty.
\end{aligned}
$$

Thus the order of summations on k and m is interchangeable, so that

$$
\begin{aligned}
\sum_{k=1}^{\infty}\Big(\log\Big(1+\frac{s}{k}\Big)-\frac{s}{k}\Big) &= \sum_{m=2}^{\infty}\frac{(-1)^{m-1}}{m}s^m\sum_{k=1}^{\infty}\frac{1}{k^m} \\
&= \sum_{m=2}^{\infty}\frac{(-1)^{m-1}}{m}s^m\zeta(m).
\end{aligned}
$$

Putting this into (6.15), we have

$$
\log\Gamma(s) = -\gamma s - \log s - \sum_{m=2}^{\infty}\frac{(-1)^{m-1}}{m}s^m\zeta(m).
$$

2° For $N \in \mathbb{N}$, let C_N be a contour as in Figure 6.1. The function $z \mapsto \frac{\pi s^z}{z\sin\pi z}\zeta(z)$ is meromorphic on $\mathbb{C}$, is holomorphic on $\mathbb{C}\setminus\mathbb{Z}$ and has a pole at each point of $\mathbb{Z}$. By the residue theorem,

$$
\frac{1}{2\pi\sqrt{-1}}\int_{C_N}\frac{\pi s^z}{z\sin\pi z}\zeta(z)dz = \sum_{m=2}^{N+1}\mathrm{Res}(m).
$$

Throughout the proof of Theorem 6.2(ii), $\mathrm{Res}(m)$ is the residue of $\frac{\pi s^z}{z\sin\pi z}\zeta(z)$ at $z = m$. Since, for $m \ge 2$,

$$
\begin{aligned}
(z-m)\frac{\pi s^z}{z\sin\pi z}\zeta(z) &= \pi(z-m)\frac{s^z}{z\sin\big(\pi(z-m)+m\pi\big)}\zeta(z) \\
&= \pi(z-m)\frac{s^z}{z\sin\pi(z-m)\cos m\pi}\zeta(z) \\
&= \frac{\pi(z-m)}{\sin\pi(z-m)}\frac{(-1)^m}{z}s^z\zeta(z) \\
&\to \frac{(-1)^m}{m}s^m\zeta(m) \quad \text{as } z \to m,
\end{aligned}
$$

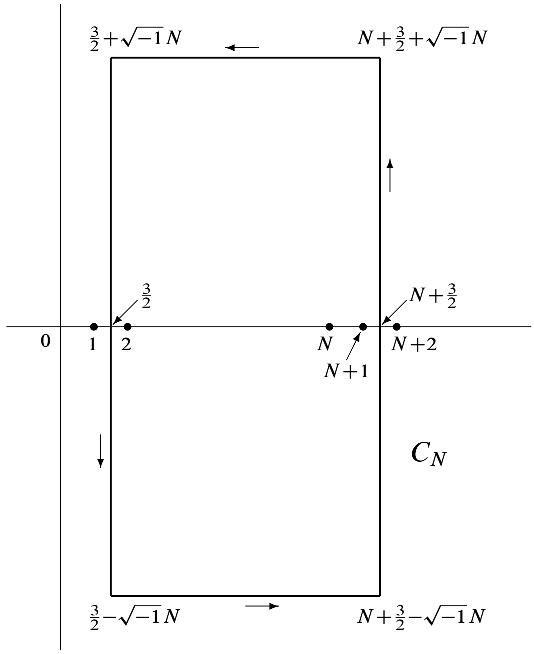

Figure 6.1: C_N

the identity above is

$$\frac{1}{2\pi\sqrt{-1}}\int_{C_N}\frac{\pi s^z}{z\sin\pi z}\zeta(z)dz=\sum_{m=2}^{N+1}\frac{(-1)^m}{m}s^m\zeta(m).$$

Letting $N\to\infty$, we have by 1° that

$$\log\Gamma(s)=-\gamma s-\log s+\lim_{N\to\infty}\frac{1}{2\pi\sqrt{-1}}\int_{C_N}\frac{\pi s^z}{z\sin\pi z}\zeta(z)dz.$$

3° We divide C_N into four contours $C_{N,1}$, $C_{N,2}^+$, $C_{N,3}$, $C_{N,1}^-$:

- $C_{N,1}$ is a segment from $N+\frac{3}{2}-\sqrt{-1}N$ to $N+\frac{3}{2}+\sqrt{-1}N$,
- $C_{N,2}^+$ is a segment form $\frac{3}{2}+\sqrt{-1}N$ to $N+\frac{3}{2}+\sqrt{-1}N$,
- $C_{N,3}$ is a segment from $\frac{3}{2}-\sqrt{-1}N$ to $\frac{3}{2}+\sqrt{-1}N$,
- $C_{N,2}^-$ is a segment from $\frac{3}{2}-\sqrt{-1}N$ to $N+\frac{3}{2}-\sqrt{-1}N$.

Then

$$\begin{aligned}&\frac{1}{2\pi\sqrt{-1}}\int_{C_N}\frac{\pi s^z}{z\sin\pi z}\zeta(z)dz\\&=\frac{1}{2\pi\sqrt{-1}}\int_{C_{N,1}}\frac{\pi s^z}{z\sin\pi z}\zeta(z)dz-\frac{1}{2\pi\sqrt{-1}}\int_{C_{N,2}^+}\frac{\pi s^z}{z\sin\pi z}\zeta(z)dz\end{aligned}$$

$$- \frac{1}{2\pi\sqrt{-1}} \int_{C_{N,3}} \frac{\pi s^z}{z \sin \pi z} \zeta(z) dz + \frac{1}{2\pi\sqrt{-1}} \int_{C_{N,2}^-} \frac{\pi s^z}{z \sin \pi z} \zeta(z) dz$$
$$=: I_{N,1} - I_{N,2}^+ - I_{N,3} + I_{N,2}^-.$$

$I_{N,1}$ is

$$I_{N,1} = \frac{1}{2\pi\sqrt{-1}} \int_{-1}^{1} \frac{\pi s^{N+\frac{3}{2}+\sqrt{-1}Nv}}{\left(N+\frac{3}{2}+\sqrt{-1}Nv\right) \sin \pi \left(N+\frac{3}{2}+\sqrt{-1}Nv\right)} \times \zeta\left(N+\frac{3}{2}+\sqrt{-1}Nv\right)\sqrt{-1}N dv$$
$$\left[\because \text{ change of variable: } z = N+\tfrac{3}{2}+\sqrt{-1}Nv\right].$$

Noting that

$$\begin{aligned}
&\sin \pi \left(N+\frac{3}{2}+\sqrt{-1}Nv\right) \\
&= \sin\left(\pi\left(N+\frac{3}{2}\right)+\sqrt{-1}\pi N v\right) \\
&= \sin \pi \left(N+\frac{3}{2}\right) \cos \sqrt{-1}\pi N v + \cos \pi \left(N+\frac{3}{2}\right) \sin \sqrt{-1}\pi N v \\
&= (-1)^{N-1} \cos \sqrt{-1}\pi N v \\
&\quad \left[\begin{array}{ll} \because & \sin \pi\left(N+\frac{3}{2}\right) = \sin \pi N \cos \frac{3}{2}\pi + \cos \pi N \sin \frac{3}{2}\pi = (-1)^{N-1}, \\ & \cos \pi\left(N+\frac{3}{2}\right) = \cos \pi N \cos \frac{3}{2}\pi - \sin \pi N \cos \frac{3}{2}\pi = 0 \end{array}\right] \\
&= (-1)^{N-1}\frac{1}{2}\left(e^{\sqrt{-1}\sqrt{-1}\pi N v} + e^{-\sqrt{-1}\sqrt{-1}\pi N v}\right) \\
&= (-1)^{N-1}\frac{1}{2}\left(e^{\pi N v} + e^{-\pi N v}\right),
\end{aligned}$$

we see that

$$\begin{aligned}
|I_{N,1}| &= \left| \frac{1}{2} \int_{-1}^{1} \frac{N}{N+\frac{3}{2}+\sqrt{-1}Nv} \frac{s^{N+\frac{3}{2}} s^{\sqrt{-1}Nv}}{\frac{1}{2}\left(e^{\pi N v}+e^{-\pi N v}\right)} \zeta\left(N+\frac{3}{2}+\sqrt{-1}Nv\right) dv \right| \\
&\le \int_{-1}^{1} \frac{N}{\left|N+\frac{3}{2}+\sqrt{-1}Nv\right|} \frac{s^{N+\frac{3}{2}}}{e^{\pi N v}+e^{-\pi N v}} \left|\zeta\left(N+\frac{3}{2}+\sqrt{-1}Nv\right)\right| dv \\
&\le s^{N+\frac{3}{2}} \zeta\left(N+\frac{3}{2}\right) \int_{-1}^{1} \frac{1}{\sqrt{(1+\frac{3}{2N})^2+v^2}} \frac{1}{e^{\pi N v}+e^{-\pi N v}} dv \\
&\to 0 \quad \text{as } N \to \infty.
\end{aligned}$$

$I_{N,2}^{\pm}$ is

$$I_{N,2}^{\pm} = \frac{1}{2\pi\sqrt{-1}} \int_{\frac{3}{2}}^{N+\frac{3}{2}} \frac{\pi s^{u \pm \sqrt{-1}N}}{(u \pm \sqrt{-1}N) \sin \pi (u \pm \sqrt{-1}N)} \zeta(u \pm \sqrt{-1}N) du$$
$$\left[\because \text{ change of variable: } z = u \pm \sqrt{-1}N\right].$$

Since

$$\begin{aligned}\sin\pi\big(u\pm\sqrt{-1}N\big) &= \frac{1}{2\sqrt{-1}}\Big(e^{\sqrt{-1}\pi(u\pm\sqrt{-1}N)}-e^{-\sqrt{-1}\pi(u\pm\sqrt{-1}N)}\Big)\\ &= \frac{1}{2\sqrt{-1}}\Big(e^{\sqrt{-1}\pi u}e^{\pm\sqrt{-1}\sqrt{-1}\pi N}-e^{-\sqrt{-1}\pi u}e^{\pm\sqrt{-1}(-\sqrt{-1})\pi N}\Big)\\ &= \frac{1}{2\sqrt{-1}}\Big(e^{\sqrt{-1}\pi u}e^{-(\pm\pi N)}-e^{-\sqrt{-1}\pi u}e^{\pm\pi N}\Big),\end{aligned}$$

and thus

$$\begin{aligned}\big|\sin\pi\big(u\pm\sqrt{-1}N\big)\big| &= \frac{1}{2}\big|e^{\sqrt{-1}\pi u}e^{-(\pm\pi N)}-e^{-\sqrt{-1}\pi u}e^{\pm\pi N}\big|\\ &\geq \frac{1}{2}\big|e^{-(\pm\pi N)}-e^{\pm\pi N}\big|\\ &= \frac{1}{2}\big(e^{\pi N}-e^{-\pi N}\big),\end{aligned}$$

we see that

$$\begin{aligned}\big|I_{N,2}^{\pm}\big| &\leq \frac{1}{2\pi}\int_{\frac{3}{2}}^{N+\frac{3}{2}}\frac{\pi s^u}{|u\pm\sqrt{-1}N||\sin\pi(u\pm\sqrt{-1}N)|}|\zeta(u\pm\sqrt{-1}N)|du\\ &\leq \frac{1}{2}\int_{\frac{3}{2}}^{N+\frac{3}{2}}\frac{s^u\zeta(u)}{\sqrt{u^2+N^2}\frac{1}{2}(e^{\pi N}-e^{-\pi N})}du\\ &\leq \frac{\zeta(\frac{3}{2})}{e^{\pi N}-e^{-\pi N}}\to 0 \quad \text{as } N\to\infty.\end{aligned}$$

Therefore we have

$$\lim_{N\to\infty}\frac{1}{2\pi\sqrt{-1}}\int_{C_N}\frac{\pi s^z}{z\sin\pi z}\zeta(z)dz = -\lim_{N\to\infty}\frac{1}{2\pi\sqrt{-1}}\int_{C_{N,3}}\frac{\pi s^z}{z\sin\pi z}\zeta(z)dz,$$

which, together with 2°, implies that

$$\log\Gamma(s) = -\gamma s-\log s-\lim_{N\to\infty}\frac{1}{2\pi\sqrt{-1}}\int_{\frac{3}{2}-\sqrt{-1}N}^{\frac{3}{2}+\sqrt{-1}N}\frac{\pi s^z}{z\sin\pi z}\zeta(z)dz.$$

4° For $R\geq 1$, let D_R be a contour as in Figure 6.2. By the residue theorem,

$$\frac{1}{2\pi\sqrt{-1}}\int_{D_R}\frac{\pi s^z}{z\sin\pi z}\zeta(z)dz = \sum_{m=-1}^{1}\mathrm{Res}(m).$$

We divide D_R into four contours $D_{R,1}$, $D_{R,2}^{+}$, $D_{R,2}^{-}$, $D_{R,3}$:

- $D_{R,1}$ is a segment from $\frac{3}{2}-\sqrt{-1}R$ to $\frac{3}{2}+\sqrt{-1}R$,
- $D_{R,2}^{\pm}$ is a segment from $-\frac{3}{2}\pm\sqrt{-1}R$ to $\frac{3}{2}\pm\sqrt{-1}R$,
- $D_{R,3}$ is a segment from $-\frac{3}{2}-\sqrt{-1}R$ to $-\frac{3}{2}+\sqrt{-1}R$.

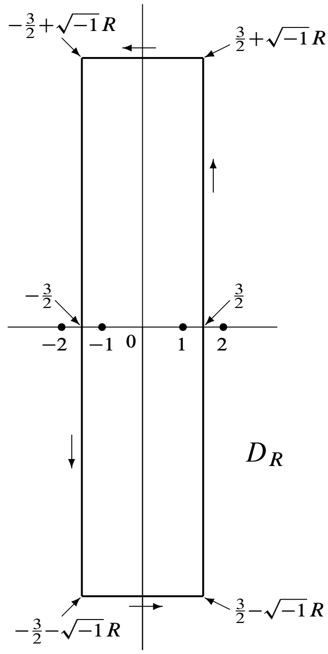

Figure 6.2: D_R

Then

$$\begin{aligned}\text{L.H.S.} = &\frac{1}{2\pi\sqrt{-1}}\int_{D_{R,1}} \frac{\pi s^z}{z\sin\pi z}\zeta(z)dz\\ &- \frac{1}{2\pi\sqrt{-1}}\int_{D_{R,2}^+} \frac{\pi s^z}{z\sin\pi z}\zeta(z)dz\\ &- \frac{1}{2\pi\sqrt{-1}}\int_{D_{R,3}} \frac{\pi s^z}{z\sin\pi z}\zeta(z)dz\\ &+ \frac{1}{2\pi\sqrt{-1}}\int_{D_{R,2}^-} \frac{\pi s^z}{z\sin\pi z}\zeta(z)dz.\end{aligned}$$

Since

$$\begin{aligned}&\left|\frac{1}{2\pi\sqrt{-1}}\int_{D_{R,2}^{\pm}} \frac{\pi s^z}{z\sin\pi z}\zeta(z)dz\right|\\ &= \left|\frac{1}{2\pi\sqrt{-1}}\int_{-\frac{3}{2}}^{\frac{3}{2}} \frac{\pi s^{u\pm\sqrt{-1}R}}{(u\pm\sqrt{-1}R)\sin\pi(u\pm\sqrt{-1}R)}\zeta(u\pm\sqrt{-1}R)du\right|\\ &\qquad \left[\because \text{ change of variable: } z = u\pm\sqrt{-1}R\right]\\ &\le \frac{1}{2}\int_{-\frac{3}{2}}^{\frac{3}{2}} \frac{s^u|\zeta(u\pm\sqrt{-1}R)|}{|u\pm\sqrt{-1}R||\sin\pi(u\pm\sqrt{-1}R)|}du\end{aligned}$$

$$\leq \frac{1}{2}\int_{-\frac{3}{2}}^{\frac{3}{2}} \frac{s^u}{\sqrt{u^2+R^2}} \frac{|\zeta(u \pm \sqrt{-1}R)|}{\frac{1}{2}(e^{\pi R} - e^{-\pi R})} du \quad \left[\text{cf. the estimate of } |I_{N,2}^{\pm}| \text{ in } 3^{\circ}\right]$$

$$\leq \frac{1}{e^{\pi R} - e^{-\pi R}} \frac{3}{R}\left(\frac{41}{12} + \frac{1}{24}\left(\frac{5}{2} + R\right) + \frac{1}{24}\left(\frac{5}{2} + R\right)^3\right) \quad \left[\because \text{ Lemma 6.2}\right]$$

$$\to 0 \quad \text{as } R \to \infty,$$

we see that

$$\sum_{m=-1}^{1} \operatorname{Res}(m) = \lim_{R\to\infty} \frac{1}{2\pi\sqrt{-1}} \int_{D_{R,1}} \frac{\pi s^z}{z \sin \pi z} \zeta(z) dz - \lim_{R\to\infty} \frac{1}{2\pi\sqrt{-1}} \int_{D_{R,3}} \frac{\pi s^z}{z \sin \pi z} \zeta(z) dz,$$

which, together with 3°, implies that

$$\log \Gamma(s) = -\gamma s - \log s - \sum_{m=-1}^{1} \operatorname{Res}(m) - \lim_{R\to\infty} \frac{1}{2\pi\sqrt{-1}} \int_{D_{R,3}} \frac{\pi s^z}{z \sin \pi z} \zeta(z) dz$$

$$= -\gamma s - \log s - \sum_{m=-1}^{1} \operatorname{Res}(m) - \frac{1}{2\pi\sqrt{-1}} \int_{-\frac{3}{2}-\sqrt{-1}\infty}^{-\frac{3}{2}+\sqrt{-1}\infty} \frac{\pi s^z}{z \sin \pi z} \zeta(z) dz.$$

5° (i) $\operatorname{Res}(-1) = -\frac{1}{12}\frac{1}{s}$.
(ii) $\operatorname{Res}(0) = -\frac{1}{2}\log s - \frac{1}{2}\log 2\pi$.
(iii) $\operatorname{Res}(1) = -\gamma s + s(1 - \log s)$.

∵ (i) Since, as $z \to -1$,

$$(z+1)\frac{\pi s^z}{z \sin \pi z}\zeta(z) = \pi(z+1)\frac{s^z}{z \sin\bigl(\pi(z+1) - \pi\bigr)}\zeta(z)$$

$$= \pi(z+1)\frac{s^z}{z \sin \pi(z+1) \cos \pi}\zeta(z)$$

$$= \frac{\pi(z+1)}{\sin \pi(z+1)} \frac{s^z}{-z}\zeta(z)$$

$$\to \frac{s^{-1}}{1}\zeta(-1)$$

$$= -\frac{1}{12}\frac{1}{s} \quad \left[\because \text{ Lemma 6.3(ii)}\right],$$

we have $\operatorname{Res}(-1) = -\frac{1}{12}\frac{1}{s}$.
(ii) By Lemma 6.3(iii) and (iv), the Taylor expansion of $\zeta(\cdot)$ about $z = 0$ is

$$\zeta(z) = -\frac{1}{2} - \frac{1}{2}(\log 2\pi)z + b_2 z^2 + b_3 z^3 + \cdots.$$

Denoting the Laurent expansion of $\frac{\pi s^z}{z \sin \pi z}$ about $z = 0$ by

$$\frac{\pi s^z}{z \sin \pi z} = \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 + \cdots,$$

we have

$$\begin{aligned}\frac{\pi s^z}{z \sin \pi z}\zeta(z) &= \Big(\sum_{i=-2}^{\infty} a_i z^i\Big)\Big(\sum_{j=0}^{\infty} b_j z^j\Big) \quad \big[\text{where } b_0 = -\tfrac{1}{2}, b_1 = -\tfrac{1}{2}\log 2\pi\big]\\ &= \sum_{\substack{i \geq -2,\\ j \geq 0}} a_i b_j z^{i+j},\end{aligned}$$

which gives that

$$\mathrm{Res}(0) = a_{-2}b_1 + a_{-1}b_0 = a_{-2}\Big(-\frac{1}{2}\log 2\pi\Big) + a_{-1}\Big(-\frac{1}{2}\Big).$$

We find a_{-2} and a_{-1}:

$$\begin{aligned} a_{-2} &= \lim_{z\to 0} \frac{\pi z}{\sin \pi z} s^z = 1,\\ a_{-1} &= \Big(\frac{\pi z}{\sin \pi z} s^z\Big)'\Big|_{z=0}\\ &= \lim_{z\to 0}\Big(\frac{\pi \sin \pi z - \pi z \cdot \cos \pi z \cdot \pi}{\sin^2 \pi z} s^z + \frac{\pi z}{\sin \pi z} s^z \log s\Big)\\ &= \pi \lim_{w\to 0} \frac{\sin w - w \cos w}{\sin^2 w} + \log s\\ &= \pi \lim_{w\to 0}\Big(\frac{w}{\sin w}\Big)^2 \frac{\sin w - w\cos w}{w^2} + \log s\\ &= \log s.\end{aligned}$$

Thus $\mathrm{Res}(0) = -\frac{1}{2}\log 2\pi - \frac{1}{2}\log s$.
(iii) By Lemma 6.3(i), the Laurent expansion of $\zeta(\cdot)$ about $z = 1$ is

$$\zeta(z) = \frac{1}{z-1} + \gamma + d_1(z-1) + d_2(z-1)^2 + \cdots.$$

Denoting the Laurent expansion of $\frac{\pi s^z}{z \sin \pi z}$ about $z = 1$ by

$$\frac{\pi s^z}{z \sin \pi z} = \frac{c_{-1}}{z-1} + c_0 + c_1(z-1) + c_2(z-1)^2 + \cdots,$$

we have

$$\begin{aligned}\frac{\pi s^z}{z \sin \pi z}\zeta(z) &= \Big(\sum_{i=-1}^{\infty} c_i (z-1)^i\Big)\Big(\sum_{j=-1}^{\infty} d_j (z-1)^j\Big) \quad \big[\text{where } d_{-1} = 1, d_0 = \gamma\big]\\ &= \sum_{i,j\geq -1} c_i d_j (z-1)^{i+j},\end{aligned}$$

which gives that

$$\mathrm{Res}(1) = c_{-1}d_0 + c_0 d_{-1} = c_{-1}\gamma + c_0.$$

We find c_{-1} and c_0:

$$\begin{aligned}
c_{-1} &= \lim_{z\to 1}(z-1)\frac{\pi s^z}{z\sin \pi z}\\
&= \lim_{z\to 1}\frac{\pi(z-1)}{z}\frac{s^z}{\sin\bigl(\pi(z-1)+\pi\bigr)}\\
&= \lim_{z\to 1}\frac{\pi(z-1)}{z}\frac{s^z}{\sin\pi(z-1)\cos\pi}\\
&= \lim_{z\to 1}\frac{\pi(z-1)}{\sin\pi(z-1)}\frac{s^z}{z}(-1)\\
&= \frac{s}{1}(-1) = -s,\\
c_0 &= \Bigl((z-1)\frac{\pi s^z}{z\sin\pi z}\Bigr)'\Big|_{z=1}\\
&= \lim_{z\to 1}\Bigl(\frac{\pi(z-1)}{\sin\pi(z-1)}\Bigl(-\frac{s^z}{z}\Bigr)\Bigr)'\\
&= \lim_{z\to 1}\Bigl(\frac{\pi\sin\pi(z-1)-\pi(z-1)\cos\pi(z-1)\cdot\pi}{\sin^2\pi(z-1)}\cdot\Bigl(-\frac{s^z}{z}\Bigr)\\
&\qquad + \frac{\pi(z-1)}{\sin\pi(z-1)}\cdot -\frac{s^z(\log s)z-s^z}{z^2}\Bigr)\\
&= \pi\lim_{z\to 1}\frac{\sin\pi(z-1)-\pi(z-1)\cos\pi(z-1)}{\sin^2\pi(z-1)}(-s)\\
&\quad + \lim_{z\to 1}\frac{\pi(z-1)}{\sin\pi(z-1)}(-s)(\log s-1)\\
&= s(1-\log s).
\end{aligned}$$

Thus $\mathrm{Res}(1) = -s\gamma + s(1-\log s)$.

<u>6°</u> By 4° and 5°,

$$\begin{aligned}
\log\Gamma(s) &= -\gamma s - \log s\\
&\quad - \Bigl(-\frac{1}{12}\frac{1}{s} - \frac{1}{2}\log s - \frac{1}{2}\log 2\pi - \gamma s + s(1-\log s)\Bigr)\\
&\quad - \frac{1}{2\pi\sqrt{-1}}\int_{-\frac{3}{2}-\sqrt{-1}\infty}^{-\frac{3}{2}+\sqrt{-1}\infty}\frac{\pi s^z}{z\sin\pi z}\zeta(z)dz\\
&= -\log s + \frac{1}{12}\frac{1}{s} + \frac{1}{2}\log s + \frac{1}{2}\log 2\pi + s\log s - s\\
&\quad - \frac{1}{2\pi\sqrt{-1}}\int_{-\frac{3}{2}-\sqrt{-1}\infty}^{-\frac{3}{2}+\sqrt{-1}\infty}\frac{\pi s^z}{z\sin\pi z}\zeta(z)dz\\
&= \Bigl(s-\frac{1}{2}\Bigr)\log s - s + \frac{1}{2}\log 2\pi + \frac{1}{12}\frac{1}{s}\\
&\quad - \frac{1}{2\pi\sqrt{-1}}\int_{-\frac{3}{2}-\sqrt{-1}\infty}^{-\frac{3}{2}+\sqrt{-1}\infty}\frac{\pi s^z}{z\sin\pi z}\zeta(z)dz. \qquad \blacksquare
\end{aligned}$$

Claim 6.4 *For $-\infty<\sigma_1<\sigma_2<\infty$, put*

$$
\begin{aligned}
A(\sigma_1,\sigma_2) := \sqrt{2\pi}\, e^{-\sigma_1} \exp\Bigg\{ & \big(|\sigma_1|\vee|\sigma_2|\big)\Big(|\sigma_1|\vee|\sigma_2|+\frac{3}{2}\Big)+\frac{1}{12} \\
&+\int_{-\infty}^{\infty}\Big(\frac{1}{\frac{9}{4}+v^2}\Big)^{\frac{1}{2}}\bigg(\frac{41}{12}+\frac{1}{24}\Big(\frac{5}{2}+|v|\Big)+\frac{1}{24}\Big(\frac{5}{2}+|v|\Big)^3\bigg) \\
&\times e^{-\left(\pi-\cot^{-1}(\sigma_1\wedge 0)\right)|v|}dv\Bigg\}, \qquad (6.16)
\end{aligned}
$$

$$
\begin{aligned}
B(\sigma_1,\sigma_2) := {} & |\sigma_1|\vee|\sigma_2|+\frac{\pi}{2}+\frac{7}{12} \\
&+\int_{-\infty}^{\infty}\bigg(\frac{41}{12}+\frac{1}{24}\Big(\frac{5}{2}+|v|\Big)+\frac{1}{24}\Big(\frac{5}{2}+|v|\Big)^3\bigg)e^{-\left(\pi-\cot^{-1}(\sigma_1\wedge 0)\right)|v|}dv, \\
& \qquad (6.17)
\end{aligned}
$$

$$
\begin{aligned}
C_n(\sigma_1) := {} & n!\Big(\frac{1}{n}+\frac{1}{2}+\frac{n+1}{12}\Big) \\
&+\int_{-\infty}^{\infty}\bigg(\frac{\prod_{k=0}^{n}\big((\frac{3}{2}+k)^2+v^2\big)}{\frac{9}{4}+v^2}\bigg)^{\frac{1}{2}}\bigg(\frac{41}{12}+\frac{1}{24}\Big(\frac{5}{2}+|v|\Big)+\frac{1}{24}\Big(\frac{5}{2}+|v|\Big)^3\bigg) \\
&\times e^{-\left(\pi-\cot^{-1}(\sigma_1\wedge 0)\right)|v|}dv, \quad n=1,2,\ldots \qquad (6.18)
\end{aligned}
$$

Define $c_{l,k}=c_{l,k}(\sigma_1)$, $0\le k\le l<\infty$ by

$$
\begin{aligned}
c_{0,0} &= 1, \\
c_{1,0} &= 0,\ c_{1,1}=1, \\
c_{l+1,k} &= \begin{cases}
\displaystyle\sum_{i=0}^{l-1}\binom{l}{i}C_{l-i}(\sigma_1)c_{i,0}, & k=0, \\
\displaystyle c_{l,k-1}+\sum_{i=k}^{l-1}\binom{l}{i}C_{l-i}(\sigma_1)c_{i,k}, & 1\le k\le l, \\
c_{l,l}, & k=l+1.
\end{cases}
\end{aligned}
$$

Then

$$
\begin{aligned}
\big|\Gamma^{(l)}(\sigma\pm\sqrt{-1}t)\big| \le \sum_{k=0}^{l} c_{l,k}\big(\log t+B(\sigma_1,\sigma_2)\big)^k A(\sigma_1,\sigma_2)t^{\sigma_2-\frac{1}{2}}e^{-t\frac{\pi}{2}}, \\
l=0,1,2,\ldots,\ \sigma_1\le\sigma\le\sigma_2,\ t\ge 1.
\end{aligned}
$$

Proof. First, Theorem 6.2 gives that for $s\in\mathbb{C}\setminus(-\infty,0]$,

$$
\frac{\Gamma'(s)}{\Gamma(s)}=\log s+\Big(s-\frac{1}{2}\Big)\frac{1}{s}-1-\frac{1}{12}\frac{1}{s^2}-\frac{1}{2\pi\sqrt{-1}}\int_{-\frac{3}{2}-\sqrt{-1}\infty}^{-\frac{3}{2}+\sqrt{-1}\infty}\frac{\pi z s^{z-1}}{z\sin\pi z}\zeta(z)dz
$$

$$= \log s - \frac{1}{2}\frac{1}{s} - \frac{1}{12}\frac{1}{s^2} - \frac{1}{2\pi\sqrt{-1}}\int_{-\frac{3}{2}-\sqrt{-1}\infty}^{-\frac{3}{2}+\sqrt{-1}\infty} \frac{\pi z s^{z-1}}{z \sin \pi z}\zeta(z)dz. \tag{6.19}$$

Since, for $n \in \mathbb{N}$,

$$\begin{aligned}
\big(\log s\big)^{(n)} &= -(-1)^n \frac{n!}{n}\Big(\frac{1}{s}\Big)^n, \\
\Big(\frac{1}{s}\Big)^{(n)} &= (-1)^n n! \Big(\frac{1}{s}\Big)^{n+1}, \\
\Big(\frac{1}{s^2}\Big)^{(n)} &= (-1)^n (n+1)! \Big(\frac{1}{s}\Big)^{n+2},
\end{aligned}$$

it follows that

$$\begin{aligned}
\Big(\frac{\Gamma'}{\Gamma}\Big)^{(n)}(s) &= -(-1)^n \frac{n!}{n}\Big(\frac{1}{s}\Big)^n - \frac{1}{2}(-1)^n n!\Big(\frac{1}{s}\Big)^{n+1} - \frac{1}{12}(-1)^n (n+1)!\Big(\frac{1}{s}\Big)^{n+2} \\
&\quad - \frac{1}{2\pi\sqrt{-1}}\int_{-\frac{3}{2}-\sqrt{-1}\infty}^{-\frac{3}{2}+\sqrt{-1}\infty} \frac{\pi z(z-1)\cdots(z-n)s^{z-n-1}}{z\sin\pi z}\zeta(z)dz \\
&= -(-1)^n n!\Big(\frac{1}{n}\Big(\frac{1}{s}\Big)^n + \frac{1}{2}\Big(\frac{1}{s}\Big)^{n+1} + \frac{n+1}{12}\Big(\frac{1}{s}\Big)^{n+2}\Big) \\
&\quad - \frac{1}{2\pi\sqrt{-1}}\int_{-\frac{3}{2}-\sqrt{-1}\infty}^{-\frac{3}{2}+\sqrt{-1}\infty} \frac{\pi z(z-1)\cdots(z-n)s^{z-n-1}}{z\sin\pi z}\zeta(z)dz.
\end{aligned} \tag{6.20}$$

In the following, fix $-\infty < \sigma_1 < \sigma_2 < \infty$. We divide the proof into two steps:

$\underline{1^{\circ}}$ Let $s = \sigma \pm \sqrt{-1}t$ ($\sigma_1 \le \sigma \le \sigma_2, t \ge 1$). Then

$$|s| = \sqrt{\sigma^2 + t^2} \ge t \ge 1, \quad \arg s = \pm \cot^{-1}\Big(\frac{\sigma}{t}\Big).$$

Here $\cot^{-1}$ is the inverse function of $\cot : (0,\pi) \to (-\infty,\infty)$. Since $\frac{\sigma}{t} \ge \frac{\sigma_1}{t} \ge \sigma_1 \wedge 0$ $\big[\because$ When $\sigma_1 \ge 0$, it is clear. When $\sigma_1 < 0$, $\frac{\sigma_1}{t} - \sigma_1 = (-\sigma_1)\big(1 - \frac{1}{t}\big) \ge 0\big]$ and $\cot^{-1}$ is decreasing, $\cot^{-1}\big(\frac{\sigma}{t}\big) \le \cot^{-1}(\sigma_1 \wedge 0) < \pi$.

From 1° in the proof of Theorem 6.2(i), it is seen that for $m \in \mathbb{N} \cup \{0\}$ and $z = -\frac{3}{2} + \sqrt{-1}v$ ($v \in \mathbb{R}$),

$$\begin{aligned}
&\Big|\frac{\pi z(z-1)\cdots\big(z-(m-1)\big)s^{z-m}}{z\sin\pi z}\zeta(z)\Big| \\
&\le 2\pi\Big(\frac{1}{|s|}\Big)^{\frac{3}{2}+m}\Big(\frac{\prod_{k=0}^{m-1}\big((\frac{3}{2}+k)^2+v^2\big)}{\frac{9}{4}+v^2}\Big)^{\frac{1}{2}}\Big(\frac{41}{12} + \frac{1}{24}\Big(\frac{5}{2}+|v|\Big) + \frac{1}{24}\Big(\frac{5}{2}+|v|\Big)^3\Big) \\
&\quad \times e^{-\big(\pi - |\arg s|\big)|v|} \\
&\le 2\pi\Big(\frac{\prod_{k=0}^{m-1}\big((\frac{3}{2}+k)^2+v^2\big)}{\frac{9}{4}+v^2}\Big)^{\frac{1}{2}}\Big(\frac{41}{12} + \frac{1}{24}\Big(\frac{5}{2}+|v|\Big) + \frac{1}{24}\Big(\frac{5}{2}+|v|\Big)^3\Big) \\
&\quad \times e^{-\big(\pi - \cot^{-1}(\sigma_1\wedge 0)\big)|v|},
\end{aligned}$$

which implies that

$$\begin{aligned}
&\left|\frac{1}{2\pi\sqrt{-1}}\int_{-\frac{3}{2}-\sqrt{-1}\infty}^{-\frac{3}{2}+\sqrt{-1}\infty}\frac{\pi z(z-1)\cdots\big(z-(m-1)\big)s^{z-m}}{z\sin\pi z}\zeta(z)dz\right| \\
&\le \int_{-\infty}^{\infty}\left(\frac{\prod_{k=0}^{m-1}\big((\frac{3}{2}+k)^2+v^2\big)}{\frac{9}{4}+v^2}\right)^{\frac{1}{2}}\left(\frac{41}{12}+\frac{1}{24}\Big(\frac{5}{2}+|v|\Big)+\frac{1}{24}\Big(\frac{5}{2}+|v|\Big)^3\right) \\
&\qquad \times e^{-\left(\pi-\cot^{-1}(\sigma_1\wedge 0)\right)|v|}dv \\
&< \infty.
\end{aligned}$$

Theorem 6.2(ii) gives that

$$\begin{aligned}
&\Gamma(\sigma\pm\sqrt{-1}t) \\
&= e^{\log\Gamma(\sigma\pm\sqrt{-1}t)} \\
&= \exp\Big\{\Big(\sigma\pm\sqrt{-1}t-\frac{1}{2}\Big)\big(\log(\sigma\pm\sqrt{-1}t)-\log(\pm\sqrt{-1}t)\big) \\
&\qquad +\Big(\sigma\pm\sqrt{-1}t-\frac{1}{2}\Big)\log(\pm\sqrt{-1}t) \\
&\qquad -(\sigma\pm\sqrt{-1}t)+\frac{1}{2}\log 2\pi+\frac{1}{12}\frac{1}{\sigma\pm\sqrt{-1}t} \\
&\qquad -\frac{1}{2\pi\sqrt{-1}}\int_{-\frac{3}{2}-\sqrt{-1}\infty}^{-\frac{3}{2}+\sqrt{-1}\infty}\frac{\pi(\sigma\pm\sqrt{-1}t)^z}{z\sin\pi z}\zeta(z)dz\Big\} \\
&= \exp\Big\{\Big(\sigma\pm\sqrt{-1}t-\frac{1}{2}\Big)\int_0^1\frac{\sigma}{\sigma u\pm\sqrt{-1}t}du \\
&\qquad +\Big(\sigma\pm\sqrt{-1}t-\frac{1}{2}\Big)\Big(\log t\pm\sqrt{-1}\frac{\pi}{2}\Big) \\
&\qquad -(\sigma\pm\sqrt{-1}t)+\frac{1}{2}\log 2\pi+\frac{1}{12}\frac{1}{\sigma\pm\sqrt{-1}t} \\
&\qquad -\frac{1}{2\pi\sqrt{-1}}\int_{-\frac{3}{2}-\sqrt{-1}\infty}^{-\frac{3}{2}+\sqrt{-1}\infty}\frac{\pi(\sigma\pm\sqrt{-1}t)^z}{z\sin\pi z}\zeta(z)dz\Big\}
\end{aligned}$$

$$\left[\begin{aligned}
&\because \text{ By (3.1),} \\
&\quad \log(\sigma\pm\sqrt{-1}t)-\log(\pm\sqrt{-1}t) = \int_{\pm\sqrt{-1}t}^{\sigma\pm\sqrt{-1}t}\frac{dw}{w} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\quad = \int_0^1\frac{\sigma}{\sigma u\pm\sqrt{-1}t}du. \\
&\text{Also } \log(\pm\sqrt{-1}t)=\log\big(te^{\pm\sqrt{-1}\frac{\pi}{2}}\big)=\log t\pm\sqrt{-1}\frac{\pi}{2}
\end{aligned}\right]$$

$$\begin{aligned}
&= \exp\Big\{\Big(\sigma-\frac{1}{2}\Big)\log t-t\frac{\pi}{2}-\sigma+\frac{1}{2}\log 2\pi \\
&\qquad +\pm\sqrt{-1}\Big(\frac{\pi}{2}\Big(\sigma-\frac{1}{2}\Big)+t\log t-t\Big) \\
&\qquad +\Big(\sigma\pm\sqrt{-1}t-\frac{1}{2}\Big)\int_0^1\frac{\sigma}{\sigma u\pm\sqrt{-1}t}du
\end{aligned}$$

$$
\begin{aligned}
&\qquad + \frac{1}{12}\frac{1}{\sigma \pm \sqrt{-1}t} \\
&\qquad - \frac{1}{2\pi\sqrt{-1}} \int_{-\frac{3}{2}-\sqrt{-1}\infty}^{-\frac{3}{2}+\sqrt{-1}\infty} \frac{\pi(\sigma \pm \sqrt{-1}t)^z}{z \sin \pi z} \zeta(z) dz \Bigg\} \\
&= \sqrt{2\pi} e^{-\sigma} t^{\sigma - \frac{1}{2}} e^{-t\frac{\pi}{2}} \\
&\quad \times e^{\pm\sqrt{-1}\left(\frac{\pi}{2}(\sigma - \frac{1}{2}) + t \log t - t\right)} \\
&\quad \times \exp\Bigg\{ \left(\sigma \pm \sqrt{-1}t - \frac{1}{2}\right) \int_0^1 \frac{\sigma}{\sigma u \pm \sqrt{-1}t} du \\
&\qquad + \frac{1}{12}\frac{1}{\sigma \pm \sqrt{-1}t} \\
&\qquad - \frac{1}{2\pi\sqrt{-1}} \int_{-\frac{3}{2}-\sqrt{-1}\infty}^{-\frac{3}{2}+\sqrt{-1}\infty} \frac{\pi(\sigma \pm \sqrt{-1}t)^z}{z \sin \pi z} \zeta(z) dz \Bigg\}.
\end{aligned}
$$

By taking the absolute value,

$$
\begin{aligned}
&\left|\Gamma(\sigma \pm \sqrt{-1}t)\right| \\
&= \sqrt{2\pi} e^{-\sigma} t^{\sigma - \frac{1}{2}} e^{-t\frac{\pi}{2}} \\
&\quad \times \Bigg| \exp\Bigg\{ \left(\sigma \pm \sqrt{-1}t - \frac{1}{2}\right) \int_0^1 \frac{\sigma}{\sigma u \pm \sqrt{-1}t} du \\
&\qquad + \frac{1}{12}\frac{1}{\sigma \pm \sqrt{-1}t} \\
&\qquad - \frac{1}{2\pi\sqrt{-1}} \int_{-\frac{3}{2}-\sqrt{-1}\infty}^{-\frac{3}{2}+\sqrt{-1}\infty} \frac{\pi(\sigma \pm \sqrt{-1}t)^z}{z \sin \pi z} \zeta(z) dz \Bigg\} \Bigg| \\
&\le \sqrt{2\pi} e^{-\sigma_1} t^{\sigma_2 - \frac{1}{2}} e^{-t\frac{\pi}{2}} \\
&\quad \times \exp\Bigg\{ \left|\sigma \pm \sqrt{-1}t - \frac{1}{2}\right| \int_0^1 \frac{|\sigma|}{|\sigma u \pm \sqrt{-1}t|} du \\
&\qquad + \frac{1}{12}\frac{1}{|\sigma \pm \sqrt{-1}t|} \\
&\qquad + \left| \frac{1}{2\pi\sqrt{-1}} \int_{-\frac{3}{2}-\sqrt{-1}\infty}^{-\frac{3}{2}+\sqrt{-1}\infty} \frac{\pi(\sigma \pm \sqrt{-1}t)^z}{z \sin \pi z} \zeta(z) dz \right| \Bigg\} \\
&\quad \left[\because \text{ For } w \in \mathbb{C},\ |e^w| \le e^{|w|}\right] \\
&\le \sqrt{2\pi} e^{-\sigma_1} t^{\sigma_2 - \frac{1}{2}} e^{-t\frac{\pi}{2}} \\
&\quad \times \exp\Bigg\{ (|\sigma_1| \vee |\sigma_2|)\left(|\sigma_1| \vee |\sigma_2| + \frac{3}{2}\right) + \frac{1}{12} \\
&\qquad + \int_{-\infty}^{\infty} \left(\frac{1}{\frac{9}{4} + v^2}\right)^{\frac{1}{2}} \left(\frac{41}{12} + \frac{1}{24}\left(\frac{5}{2} + |v|\right) + \frac{1}{24}\left(\frac{5}{2} + |v|\right)^3\right)
\end{aligned}
$$

$$\times e^{-\left(\pi-\cot^{-1}(\sigma_1\wedge 0)\right)|v|}dv\Big\}$$

$$\left[\begin{array}{l} \because \text{ Since } |\sigma u \pm \sqrt{-1}t| = \sqrt{\sigma^2u^2+t^2} \geq t \geq 1 \text{ and } |\sigma| \leq |\sigma_1| \vee |\sigma_2|, \\ |\sigma \pm \sqrt{-1}t - \frac{1}{2}| \int_0^1 \frac{|\sigma|}{|\sigma u \pm \sqrt{-1}t|} du \leq (|\sigma| + |t| + \frac{1}{2})|\sigma|\frac{1}{t} \\ \qquad = |\sigma|\big((|\sigma|+\frac{1}{2})\frac{1}{t}+1\big) \\ \qquad \leq |\sigma|(|\sigma|+\frac{3}{2}) \\ \qquad \leq (|\sigma_1|\vee|\sigma_2|)(|\sigma_1|\vee|\sigma_2|+\frac{3}{2}) \end{array}\right]$$

$$= A(\sigma_1,\sigma_2)t^{\sigma_2-\frac{1}{2}}e^{-t\frac{\pi}{2}}. \tag{6.21}$$

By (6.19),

$$\begin{aligned} \frac{\Gamma'}{\Gamma}(\sigma \pm \sqrt{-1}t) &= \log(\sigma \pm \sqrt{-1}t) - \log(\pm\sqrt{-1}t) + \log(\pm\sqrt{-1}t) \\ &\quad - \frac{1}{2}\frac{1}{\sigma \pm \sqrt{-1}t} - \frac{1}{12}\frac{1}{(\sigma \pm \sqrt{-1}t)^2} \\ &\quad - \frac{1}{2\pi\sqrt{-1}}\int_{-\frac{3}{2}-\sqrt{-1}\infty}^{-\frac{3}{2}+\sqrt{-1}\infty} \frac{\pi z(\sigma \pm \sqrt{-1}t)^{z-1}}{z\sin\pi z}\zeta(z)dz \\ &= \int_0^1 \frac{\sigma}{\sigma u \pm \sqrt{-1}t}du + \log t \pm \sqrt{-1}\frac{\pi}{2} \\ &\quad - \frac{1}{2}\frac{1}{\sigma \pm \sqrt{-1}t} - \frac{1}{12}\frac{1}{(\sigma \pm \sqrt{-1}t)^2} \\ &\quad - \frac{1}{2\pi\sqrt{-1}}\int_{-\frac{3}{2}-\sqrt{-1}\infty}^{-\frac{3}{2}+\sqrt{-1}\infty} \frac{\pi z(\sigma \pm \sqrt{-1}t)^{z-1}}{z\sin\pi z}\zeta(z)dz, \end{aligned}$$

and thus

$$\begin{aligned} &\left|\frac{\Gamma'}{\Gamma}(\sigma \pm \sqrt{-1}t)\right| \\ &\leq |\sigma_1|\vee|\sigma_2| + \log t + \frac{\pi}{2} + \frac{1}{2} + \frac{1}{12} \\ &\quad + \int_{-\infty}^{\infty}\left(\frac{41}{12} + \frac{1}{24}\Big(\frac{5}{2}+|v|\Big) + \frac{1}{24}\Big(\frac{5}{2}+|v|\Big)^3\right)e^{-\left(\pi-\cot^{-1}(\sigma_1\wedge 0)\right)|v|}dv \\ &= \log t + B(\sigma_1,\sigma_2). \end{aligned} \tag{6.22}$$

By (6.20),

$$\begin{aligned} &\left|\Big(\frac{\Gamma'}{\Gamma}\Big)^{(n)}(\sigma \pm \sqrt{-1}t)\right| \\ &= \Bigg|-(-1)^n n!\Bigg(\frac{1}{n}\Big(\frac{1}{\sigma \pm \sqrt{-1}t}\Big)^n + \frac{1}{2}\Big(\frac{1}{\sigma \pm \sqrt{-1}t}\Big)^{n+1} + \frac{n+1}{12}\Big(\frac{1}{\sigma \pm \sqrt{-1}t}\Big)^{n+2}\Bigg) \\ &\quad - \frac{1}{2\pi\sqrt{-1}}\int_{-\frac{3}{2}-\sqrt{-1}\infty}^{-\frac{3}{2}+\sqrt{-1}\infty} \frac{\pi z(z-1)\cdots(z-n)(\sigma \pm \sqrt{-1}t)^{z-n-1}}{z\sin\pi z}\zeta(z)dz\Bigg| \end{aligned}$$

$$
\begin{aligned}
&\leq n!\Big(\frac{1}{n}\Big(\frac{1}{|\sigma\pm\sqrt{-1}t|}\Big)^{n}+\frac{1}{2}\Big(\frac{1}{|\sigma\pm\sqrt{-1}t|}\Big)^{n+1}+\frac{n+1}{12}\Big(\frac{1}{|\sigma\pm\sqrt{-1}t|}\Big)^{n+2}\Big)\\
&\quad+\Big|\frac{1}{2\pi\sqrt{-1}}\int_{-\frac{3}{2}-\sqrt{-1}\infty}^{-\frac{3}{2}+\sqrt{-1}\infty}\frac{\pi z(z-1)\cdots(z-n)(\sigma\pm\sqrt{-1}t)^{z-n-1}}{z\sin\pi z}\zeta(z)dz\Big|\\
&\leq n!\Big(\frac{1}{n}+\frac{1}{2}+\frac{n+1}{12}\Big)\\
&\quad+\int_{-\infty}^{\infty}\Big(\frac{\prod_{k=0}^{n}\big((\frac{3}{2}+k)^2+v^2\big)}{\frac{9}{4}+v^2}\Big)^{\frac{1}{2}}\Big(\frac{41}{12}+\frac{1}{24}\Big(\frac{5}{2}+|v|\Big)+\frac{1}{24}\Big(\frac{5}{2}+|v|\Big)^3\Big)\\
&\qquad\times e^{-\left(\pi-\cot^{-1}(\sigma_1\wedge 0)\right)|v|}dv\\
&=C_n(\sigma_1). \qquad (6.23)
\end{aligned}
$$

2° By (6.21), the assertion of the claim holds for $l=0$. By (6.22) and (6.21), so does for $l=1$ also. Let $j\geq 1$ and assume that the assertion holds for all l up to j. Then, noting that

$$
\begin{aligned}
\Gamma^{(j+1)}=\big(\Gamma'\big)^{(j)}=\Big(\frac{\Gamma'}{\Gamma}\Gamma\Big)^{(j)}&=\sum_{k=0}^{j}\binom{j}{k}\Big(\frac{\Gamma'}{\Gamma}\Big)^{(k)}\Gamma^{(j-k)}\\
&=\frac{\Gamma'}{\Gamma}\Gamma^{(j)}+\sum_{k=1}^{j}\binom{j}{k}\Big(\frac{\Gamma'}{\Gamma}\Big)^{(k)}\Gamma^{(j-k)},
\end{aligned}
$$

we see that

$$
\begin{aligned}
&\big|\Gamma^{(j+1)}(\sigma\pm\sqrt{-1}t)\big|\\
&=\Big|\frac{\Gamma'}{\Gamma}(\sigma\pm\sqrt{-1}t)\Gamma^{(j)}(\sigma\pm\sqrt{-1}t)\\
&\quad+\sum_{k=1}^{j}\binom{j}{k}\Big(\frac{\Gamma'}{\Gamma}\Big)^{(k)}(\sigma\pm\sqrt{-1}t)\Gamma^{(j-k)}(\sigma\pm\sqrt{-1}t)\Big|\\
&\leq\Big|\frac{\Gamma'}{\Gamma}(\sigma\pm\sqrt{-1}t)\Big|\big|\Gamma^{(j)}(\sigma\pm\sqrt{-1}t)\big|\\
&\quad+\sum_{k=1}^{j}\binom{j}{k}\Big|\Big(\frac{\Gamma'}{\Gamma}\Big)^{(k)}(\sigma\pm\sqrt{-1}t)\Big|\big|\Gamma^{(j-k)}(\sigma\pm\sqrt{-1}t)\big|\\
&\leq\big(\log t+B(\sigma_1,\sigma_2)\big)\sum_{k=0}^{j}c_{j,k}\big(\log t+B(\sigma_1,\sigma_2)\big)^{k}A(\sigma_1,\sigma_2)t^{\sigma_2-\frac{1}{2}}e^{-t\frac{\pi}{2}}\\
&\quad+\sum_{k=1}^{j}\binom{j}{k}C_k(\sigma_1)\sum_{i=0}^{j-k}c_{j-k,i}\big(\log t+B(\sigma_1,\sigma_2)\big)^{i}A(\sigma_1,\sigma_2)t^{\sigma_2-\frac{1}{2}}e^{-t\frac{\pi}{2}}\\
&=\Big(\sum_{k=1}^{j+1}c_{j,k-1}\big(\log t+B(\sigma_1,\sigma_2)\big)^{k}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k=0}^{j-1}\binom{j}{k}C_{j-k}(\sigma_1)\sum_{i=0}^{k}c_{k,i}\big(\log t+B(\sigma_1,\sigma_2)\big)^i\Bigg)A(\sigma_1,\sigma_2)t^{\sigma_2-\frac{1}{2}}e^{-t\frac{\pi}{2}}\\
&=\Bigg(\sum_{k=1}^{j+1}c_{j,k-1}\big(\log t+B(\sigma_1,\sigma_2)\big)^k\\
&+\sum_{i=0}^{j-1}\Bigg(\sum_{k=i}^{j-1}\binom{j}{k}C_{j-k}(\sigma_1)c_{k,i}\Bigg)\big(\log t+B(\sigma_1,\sigma_2)\big)^i\Bigg)A(\sigma_1,\sigma_2)t^{\sigma_2-\frac{1}{2}}e^{-t\frac{\pi}{2}}\\
&=\Bigg(\sum_{k=0}^{j-1}\binom{j}{k}C_{j-k}(\sigma_1)c_{k,0}\\
&+\sum_{i=1}^{j}\Bigg(c_{j,i-1}+\sum_{k=i}^{j-1}\binom{j}{k}C_{j-k}(\sigma_1)c_{k,i}\Bigg)\big(\log t+B(\sigma_1,\sigma_2)\big)^i\\
&+c_{j,j}\big(\log t+B(\sigma_1,\sigma_2)\big)^{j+1}\Bigg)A(\sigma_1,\sigma_2)t^{\sigma_2-\frac{1}{2}}e^{-t\frac{\pi}{2}}\\
&=\Bigg(c_{j+1,0}+\sum_{i=1}^{j}c_{j+1,i}\big(\log t+B(\sigma_1,\sigma_2)\big)^i\\
&+c_{j+1,j+1}\big(\log t+B(\sigma_1,\sigma_2)\big)^{j+1}\Bigg)A(\sigma_1,\sigma_2)t^{\sigma_2-\frac{1}{2}}e^{-t\frac{\pi}{2}}\\
&=\sum_{i=0}^{j+1}c_{j+1,i}\big(\log t+B(\sigma_1,\sigma_2)\big)^iA(\sigma_1,\sigma_2)t^{\sigma_2-\frac{1}{2}}e^{-t\frac{\pi}{2}}.
\end{aligned}
$$

This tells us that the assertion is true for $l = j + 1$.

Therefore the assertion of the claim holds for $^\forall l \geq 0$. ■

Remark 6.2 Our method used in the proof of this claim seems to be a little simpler in comparison with one stated in [26, Chapter 4], because ours is direct.

6.3 Carlson's mean value theorem

For this mean value theorem, we begin with the following lemma:

Lemma 6.4 *For $c > 0$ and $x > 0$,*

$$
e^{-x}=\frac{1}{2\pi\sqrt{-1}}\int_{c-\sqrt{-1}\infty}^{c+\sqrt{-1}\infty}\Gamma(s)x^{-s}ds:=\lim_{T\to\infty}\frac{1}{2\pi\sqrt{-1}}\int_{c-\sqrt{-1}T}^{c+\sqrt{-1}T}\Gamma(s)x^{-s}ds.
$$

Proof. Fix $c > 0$ and $x > 0$. Put

$$
h(v):=e^{-e^{-v}}e^{-cv},\quad v\in\mathbb{R}. \tag{6.24}
$$

Note that $h'(v) = h(v)(e^{-v} - c)$, $\int_{\mathbb{R}}|h(v)|dv = \int_{\mathbb{R}}h(v)dv = \Gamma(c)$, $\int_{\mathbb{R}}|h'(v)|dv = \int_0^\infty e^{-x}x^{c-1}|x-c|dx \leq 2c\Gamma(c) < \infty$ and $\lim\limits_{v\to\pm\infty}h(v) = 0$. For $t \in \mathbb{R}$, we have an identity

$$\begin{aligned}
\Gamma(c+\sqrt{-1}t) &= \int_0^\infty e^{-u}u^{c+\sqrt{-1}t-1}du \\
&= \int_{-\infty}^\infty e^{-e^{-v}}e^{-(c+\sqrt{-1}t)v}dv \quad [\because \text{ change of variable: } v=-\log u] \\
&= \int_{-\infty}^\infty e^{-\sqrt{-1}tv}h(v)dv. \qquad (6.25)
\end{aligned}$$

Let $x = e^{-y}$ ($y \in \mathbb{R}$) and $T > 0$. Then

$$\begin{aligned}
&\frac{1}{2\pi\sqrt{-1}}\int_{c-\sqrt{-1}T}^{c+\sqrt{-1}T}\Gamma(s)x^{-s}ds \\
&= \frac{1}{2\pi\sqrt{-1}}\int_{c-\sqrt{-1}T}^{c+\sqrt{-1}T}\Gamma(s)e^{sy}ds \\
&= \frac{1}{2\pi\sqrt{-1}}\int_{-T}^{T}\Gamma(c+\sqrt{-1}t)e^{(c+\sqrt{-1}t)y}\sqrt{-1}dt \\
&\qquad [\because \text{ change of variable: } s=c+\sqrt{-1}t] \\
&= \frac{1}{2\pi}\int_{-T}^{T}e^{cy}e^{\sqrt{-1}ty}dt\int_{-\infty}^\infty e^{-\sqrt{-1}tv}h(v)dv \\
&= \frac{e^{cy}}{2\pi}\int_{-\infty}^\infty h(v)dv\int_{-T}^{T}e^{\sqrt{-1}t(y-v)}dt \\
&= \frac{e^{cy}}{2\pi}\int_{-\infty}^\infty h(v)dv 2\int_0^T\cos t(y-v)dt \\
&= \frac{e^{cy}}{\pi}\int_{-\infty}^\infty h(v)\frac{\sin T(y-v)}{y-v}dv \\
&\qquad \left[\text{When } y-v=0, \text{ we understand that } \tfrac{\sin T(y-v)}{y-v}=T\right] \\
&= \frac{e^{cy}}{\pi}\int_{-\infty}^\infty h(w+y)\frac{\sin Tw}{w}dw \quad [\because \text{ change of variable: } w=v-y] \\
&= \frac{e^{cy}}{\pi}\int_{-\infty}^\infty h(w+y)\frac{(1-\cos Tw)'}{Tw}dw \\
&= \frac{e^{cy}}{\pi}\left\{\left[h(w+y)\frac{1-\cos Tw}{Tw}\right]_{-\infty}^{\infty}-\int_{-\infty}^\infty h'(w+y)\frac{1-\cos Tw}{Tw}dw\right. \\
&\qquad \left.+\int_{-\infty}^\infty h(w+y)\frac{1-\cos Tw}{Tw^2}dw\right\} \quad [\because \text{ integration by parts}] \\
&= \frac{e^{cy}}{\pi}\left\{-\int_{-\infty}^\infty h'(w+y)\frac{1-\cos Tw}{Tw}dw+\int_{-\infty}^\infty h\Big(\frac{r}{T}+y\Big)\frac{1-\cos r}{r^2}dr\right\} \\
&\qquad [\because \text{ change of variable: } r=Tw].
\end{aligned}$$

We let $T \to \infty$. Since $|h'(w+y)\frac{1-\cos Tw}{Tw}| \le |h'(w+y)|$, $\int_{-\infty}^\infty |h'(w+y)|dw < \infty$, $\lim\limits_{T\to\infty} h'(w+y)\frac{1-\cos Tw}{Tw} = 0$, $\sup\limits_{r\in\mathbb{R}}\left|h(\frac{r}{T}+y)\right| = \sup\limits_{v\in\mathbb{R}} h(v) < \infty$, $\int_{-\infty}^\infty \frac{1-\cos r}{r^2}dr = \pi$ and

$\lim_{T\to\infty} h(\frac{r}{T} + y) = h(y)$, it follows from Lebesgue's convergence theorem that

$$\lim_{T\to\infty} \int_{-\infty}^{\infty} h'(w + y) \frac{1 - \cos Tw}{Tw} dw = 0,$$
$$\lim_{T\to\infty} \int_{-\infty}^{\infty} h\Big(\frac{r}{T} + y\Big) \frac{1 - \cos r}{r^2} dr = h(y) \int_{-\infty}^{\infty} \frac{1 - \cos r}{r^2} dr = \pi h(y).$$

Thus we have

$$\lim_{T\to\infty} \frac{1}{2\pi\sqrt{-1}} \int_{c-\sqrt{-1}T}^{c+\sqrt{-1}T} \Gamma(s) x^{-s} ds = \frac{e^{cy}}{\pi} \cdot \pi h(y) = e^{cy} e^{-e^{-y}} e^{-cy} = e^{-x}. \qquad \blacksquare$$

Corollary 6.1 *Let $s \in \mathbb{C}$, $c > \operatorname{Re} s$ and $x > 0$.*
(i) *For $T > 0$,*

$$\Big| \frac{1}{2\pi\sqrt{-1}} \int_{c-\sqrt{-1}T}^{c+\sqrt{-1}T} \Gamma(z - s) x^{-(z-s)} dz \Big|$$
$$\leq \frac{1}{\pi} \Big(\big(2(c - \operatorname{Re} s) + |\operatorname{Im} s|\big) \Gamma(c - \operatorname{Re} s) + e^{-(c-\operatorname{Re} s)} (c - \operatorname{Re} s)^{c-\operatorname{Re} s} \pi \Big) x^{-(c-\operatorname{Re} s)}.$$

(ii) $\displaystyle \lim_{T\to\infty} \frac{1}{2\pi\sqrt{-1}} \int_{c-\sqrt{-1}T}^{c+\sqrt{-1}T} \Gamma(z - s) x^{-(z-s)} dz = e^{-x}.$

Proof. Fix $s \in \mathbb{C}$, $c > \operatorname{Re} s$ and $x > 0$.
(i) First

$$\frac{1}{2\pi\sqrt{-1}} \int_{c-\sqrt{-1}T}^{c+\sqrt{-1}T} \Gamma(z - s) x^{-(z-s)} dz$$
$$= \frac{1}{2\pi\sqrt{-1}} \int_{-T}^{T} \Gamma(c + \sqrt{-1}u - s) x^{-(c+\sqrt{-1}u-s)} \sqrt{-1} du$$
$$\big[\because \text{ change of variable: } z = c + \sqrt{-1}u\big]$$
$$= \frac{1}{2\pi} \int_{-T}^{T} \Gamma\big(c - \operatorname{Re} s + \sqrt{-1}(u - \operatorname{Im} s)\big) x^{-\left(c-\operatorname{Re} s+\sqrt{-1}(u-\operatorname{Im} s)\right)} du$$
$$= \frac{1}{2\pi} \int_{-T-\operatorname{Im} s}^{T-\operatorname{Im} s} \Gamma\big(c - \operatorname{Re} s + \sqrt{-1}v\big) x^{-\left(c-\operatorname{Re} s+\sqrt{-1}v\right)} dv$$
$$\big[\because \text{ change of variable: } v = u - \operatorname{Im} s\big]$$
$$= \frac{x^{-(c-\operatorname{Re} s)}}{2\pi} \int_{-T-\operatorname{Im} s}^{T-\operatorname{Im} s} \Gamma\big(c - \operatorname{Re} s + \sqrt{-1}v\big) x^{-\sqrt{-1}v} dv.$$

For simplicity, let $h(w) = e^{-e^{-w}} e^{-(c-\operatorname{Re} s)w}$ ($w \in \mathbb{R}$) and $x = e^{-y}$ ($y \in \mathbb{R}$). The identity above is

$$\frac{1}{2\pi\sqrt{-1}} \int_{c-\sqrt{-1}T}^{c+\sqrt{-1}T} \Gamma(z - s) x^{-(z-s)} dz$$

$$
\begin{aligned}
&= \frac{e^{(c-\operatorname{Re} s)y}}{2\pi} \int_{-T-\operatorname{Im} s}^{T-\operatorname{Im} s} e^{\sqrt{-1}yv} dv \int_{-\infty}^{\infty} e^{-\sqrt{-1}vw} h(w) dw \quad [\text{cf. } (6.25)] \\
&= \frac{e^{(c-\operatorname{Re} s)y}}{2\pi} \int_{-\infty}^{\infty} h(w) dw \int_{-T-\operatorname{Im} s}^{T-\operatorname{Im} s} e^{\sqrt{-1}v(y-w)} dv \\
&= \frac{e^{(c-\operatorname{Re} s)y}}{2\pi} \int_{-\infty}^{\infty} h(w) dw \Big(\int_{-T}^{T} e^{\sqrt{-1}v(y-w)} dv \\
&\qquad - \int_{-T}^{-T-\operatorname{Im} s} e^{\sqrt{-1}v(y-w)} dv \\
&\qquad - \int_{T-\operatorname{Im} s}^{T} e^{\sqrt{-1}v(y-w)} dv \Big) \\
&= \frac{e^{(c-\operatorname{Re} s)y}}{2\pi} \Big(\int_{-\infty}^{\infty} h(w) dw \int_{-T}^{T} e^{\sqrt{-1}v(y-w)} dv \\
&\qquad - \int_{-\infty}^{\infty} h(w) dw \int_{-T}^{-T-\operatorname{Im} s} e^{\sqrt{-1}v(y-w)} dv \\
&\qquad - \int_{-\infty}^{\infty} h(w) dw \int_{T-\operatorname{Im} s}^{T} e^{\sqrt{-1}v(y-w)} dv \Big) \\
&= \frac{e^{(c-\operatorname{Re} s)y}}{2\pi} \Big(-2 \int_{-\infty}^{\infty} h'(w+y) \frac{1-\cos Tw}{Tw} dw \\
&\qquad + 2 \int_{-\infty}^{\infty} h\Big(\frac{r}{T} + y\Big) \frac{1-\cos r}{r^2} dr \\
&\qquad - \int_{-\infty}^{\infty} h(w) dw \int_{-T}^{-T-\operatorname{Im} s} e^{\sqrt{-1}v(y-w)} dv \\
&\qquad - \int_{-\infty}^{\infty} h(w) dw \int_{T-\operatorname{Im} s}^{T} e^{\sqrt{-1}v(y-w)} dv \Big). \qquad (6.26)
\end{aligned}
$$

Taking the absolute value yields that

$$
\begin{aligned}
&\left| \frac{1}{2\pi\sqrt{-1}} \int_{c-\sqrt{-1}T}^{c+\sqrt{-1}T} \Gamma(z-s) x^{-(z-s)} dz \right| \\
&\le \frac{e^{(c-\operatorname{Re} s)y}}{2\pi} \Big(2 \int_{-\infty}^{\infty} |h'(w+y)| \left| \frac{1-\cos Tw}{Tw} \right| dw \\
&\qquad + 2 \int_{-\infty}^{\infty} h\Big(\frac{r}{T} + y\Big) \frac{1-\cos r}{r^2} dr \\
&\qquad + \int_{-\infty}^{\infty} h(w) dw \left| \int_{-T}^{-T-\operatorname{Im} s} dv \right| \\
&\qquad + \int_{-\infty}^{\infty} h(w) dw \left| \int_{T-\operatorname{Im} s}^{T} dv \right| \Big) \\
&\le \frac{e^{(c-\operatorname{Re} s)y}}{2\pi} \Big(2 \cdot 2(c-\operatorname{Re} s)\Gamma(c-\operatorname{Re} s) + 2e^{-(c-\operatorname{Re} s)}(c-\operatorname{Re} s)^{c-\operatorname{Re} s}\pi \\
&\qquad + 2|\operatorname{Im} s|\Gamma(c-\operatorname{Re} s) \Big)
\end{aligned}
$$

$$\left[\because \max_{w\in\mathbb{R}} h(w) = \max_{t\in(0,\infty)} e^{-t} t^{c-\operatorname{Re} s} = e^{-(c-\operatorname{Re} s)}(c-\operatorname{Re} s)^{c-\operatorname{Re} s}\right]$$
$$= \frac{1}{\pi}\Big(\big(2(c-\operatorname{Re} s) + |\operatorname{Im} s|\big)\Gamma(c-\operatorname{Re} s) + e^{-(c-\operatorname{Re} s)}(c-\operatorname{Re} s)^{c-\operatorname{Re} s}\pi\Big)x^{-(c-\operatorname{Re} s)}.$$

(ii) From the proof of Lemma 6.4,

$$\lim_{T\to\infty}\int_{-\infty}^{\infty} h'(w+y)\frac{1-\cos Tw}{Tw}dw = 0,$$
$$\lim_{T\to\infty}\int_{-\infty}^{\infty} h\Big(\frac{r}{T}+y\Big)\frac{1-\cos r}{r^2}dr = h(y)\pi = e^{-e^{-y}}e^{-(c-\operatorname{Re} s)y}\pi$$
$$= e^{-x}e^{-(c-\operatorname{Re} s)y}\pi.$$

Also, since

$$\int_{-\infty}^{\infty} h(w)dw\int_{\pm T}^{\pm T-\operatorname{Im} s} e^{\sqrt{-1}v(y-w)}dv$$
$$= \int_{-\infty}^{\infty} h(w)\frac{e^{-\sqrt{-1}(\operatorname{Im} s)(y-w)}-1}{\sqrt{-1}(y-w)}e^{\sqrt{-1}(\pm T)(y-w)}dw$$
$$= \int_{-\infty}^{\infty} h(u+y)\frac{e^{\sqrt{-1}(\operatorname{Im} s)u}-1}{-\sqrt{-1}u}e^{-\sqrt{-1}(\pm T)u}du,$$

$$\int_{-\infty}^{\infty}\left|h(u+y)\frac{e^{\sqrt{-1}(\operatorname{Im} s)u}-1}{-\sqrt{-1}u}\right|du = \int_{-\infty}^{\infty} h(u+y)\left|\frac{e^{\sqrt{-1}(\operatorname{Im} s)u}-1}{-\sqrt{-1}u}\right|du$$
$$\le |\operatorname{Im} s|\int_{-\infty}^{\infty} h(u+y)du$$
$$= |\operatorname{Im} s|\Gamma(c-\operatorname{Re} s) < \infty,$$

it follows from Riemann-Lebesgue's theorem that

$$\lim_{T\to\infty}\int_{-\infty}^{\infty} h(w)dw\int_{\pm T}^{\pm T-\operatorname{Im} s} e^{\sqrt{-1}v(y-w)}dv = 0.$$

We let $T\to\infty$ in (6.26). By these convergences, we have

$$\lim_{T\to\infty}\frac{1}{2\pi\sqrt{-1}}\int_{c-\sqrt{-1}T}^{c+\sqrt{-1}T}\Gamma(z-s)x^{-(z-s)}dz = \frac{e^{(c-\operatorname{Re} s)y}}{2\pi}2e^{-x}e^{-(c-\operatorname{Re} s)y}\pi$$
$$= e^{-x}.$$ ■

Theorem 6.3 (Carlson's mean value theorem) *Let* $\sum_{n=1}^{\infty}\frac{a_n}{n^s}$ *be a Dirichlet series absolutely convergent on* $\{s\in\mathbb{C}; \operatorname{Re} s>1\}$. *Suppose it is analytically continuable to a meromorphic function* $f(\cdot)$ *on* $\{s\in\mathbb{C}; \operatorname{Re} s\ge\alpha\}$ *(where* $-\infty<\alpha<1$*),* $f(\cdot)$ *is holomorphic except* $s=1$ *which is a removable singularity or a pole of* $f(\cdot)$*, and* $f(s)=O\big((|\operatorname{Im} s|+2)^C\big)$ *except some neighborhood of* $s=1$ *where* C *is a positive constant. If*

$$\int_{-T}^{T}\big|f(\alpha+\sqrt{-1}t)\big|^2dt = O(T) \quad \text{as } T\to\infty,$$

the following holds:

(i) *For* $\alpha < {}^\forall\sigma < \infty$, $\displaystyle\sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma}} < \infty$.

(ii) *For* $\alpha < {}^\forall\sigma_1 < {}^\forall\sigma_2 < \infty$,

$$\sup_{\sigma_1 \le \sigma \le \sigma_2} \left| \frac{1}{T} \int_2^T \left| f(\sigma + \sqrt{-1}t) \right|^2 dt - \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma}} \right| \to 0 \quad \text{as } T \to \infty.$$

Proof. Fix $\alpha < \sigma_1 < \sigma_2 < \infty$. Let $c > (1 - \sigma_1) \vee 0$, $\lambda > (\sigma_2 - \alpha) \vee 2$ and $0 < \delta \le 1$. We divide the proof into seven steps:

1° For $s = \sigma + \sqrt{-1}t$ ($\sigma_1 \le \sigma \le \sigma_2$, $t \ge 1$),

$$\frac{1}{2\pi\sqrt{-1}\lambda} \int_{c-\sqrt{-1}\infty}^{c+\sqrt{-1}\infty} \Gamma\Big(\frac{z}{\lambda}\Big) f(s+z)\delta^{-z} dz = \sum_{n=1}^{\infty} \frac{a_n}{n^s} e^{-(n\delta)^\lambda}.$$

∵ First, for $T > 0$,

$$\begin{aligned}
&\frac{1}{2\pi\sqrt{-1}\lambda} \int_{c-\sqrt{-1}T}^{c+\sqrt{-1}T} \Gamma\Big(\frac{z}{\lambda}\Big) f(s+z)\delta^{-z} dz \\
&= \frac{1}{2\pi\sqrt{-1}\lambda} \int_{c-\sqrt{-1}T}^{c+\sqrt{-1}T} \Gamma\Big(\frac{z}{\lambda}\Big) \sum_{n=1}^{\infty} \frac{a_n}{n^{s+z}} \delta^{-z} dz \\
&= \sum_{n=1}^{\infty} \frac{a_n}{n^s} \frac{1}{2\pi\sqrt{-1}\lambda} \int_{c-\sqrt{-1}T}^{c+\sqrt{-1}T} \Gamma\Big(\frac{z}{\lambda}\Big)(n\delta)^{-z} dz \\
&\qquad \big[\because \operatorname{Re}(s+z) = \sigma + c \ge \sigma_1 + c > \sigma_1 + 1 - \sigma_1 = 1\big] \\
&= \sum_{n=1}^{\infty} \frac{a_n}{n^s} \frac{1}{2\pi\sqrt{-1}} \int_{\frac{c}{\lambda}-\sqrt{-1}\frac{T}{\lambda}}^{\frac{c}{\lambda}+\sqrt{-1}\frac{T}{\lambda}} \Gamma(w)(n\delta)^{-\lambda w} dw \\
&\qquad \big[\because \text{change of variable: } w = \tfrac{z}{\lambda}\big].
\end{aligned}$$

By Corollary 6.1(ii),

$$\begin{aligned}
&\lim_{T\to\infty} \frac{1}{2\pi\sqrt{-1}} \int_{\frac{c}{\lambda}-\sqrt{-1}\frac{T}{\lambda}}^{\frac{c}{\lambda}+\sqrt{-1}\frac{T}{\lambda}} \Gamma(w)(n\delta)^{-\lambda w} dw \\
&= \lim_{T\to\infty} \frac{1}{2\pi\sqrt{-1}} \int_{\frac{c}{\lambda}-\sqrt{-1}\frac{T}{\lambda}}^{\frac{c}{\lambda}+\sqrt{-1}\frac{T}{\lambda}} \Gamma(w)\big((n\delta)^\lambda\big)^{-w} dw \\
&= e^{-(n\delta)^\lambda}, \quad n \in \mathbb{N}.
\end{aligned}$$

On the other hand, by Corollary 6.1(i),

$$\begin{aligned}
&\left| \frac{1}{2\pi\sqrt{-1}} \int_{\frac{c}{\lambda}-\sqrt{-1}\frac{T}{\lambda}}^{\frac{c}{\lambda}+\sqrt{-1}\frac{T}{\lambda}} \Gamma(w)(n\delta)^{-\lambda w} dw \right| \\
&\le \frac{1}{\pi}\left(2\frac{c}{\lambda}\Gamma\Big(\frac{c}{\lambda}\Big) + e^{-\frac{c}{\lambda}}\Big(\frac{c}{\lambda}\Big)^{\frac{c}{\lambda}} \pi \right)(n\delta)^{-c}, \quad T > 0.
\end{aligned}$$

Also

$$\sum_{n=1}^{\infty}\left|\frac{a_n}{n^s}\right|(n\delta)^{-c}=\delta^{-c}\sum_{n=1}^{\infty}\frac{|a_n|}{n^{\sigma+c}}<\infty \quad \left[\because\ \sigma+c\geq\sigma_1+c>1\right].$$

Hence it follows from Lebesgue's convergence theorem that

$$\begin{aligned}
&\lim_{T\to\infty}\frac{1}{2\pi\sqrt{-1}\lambda}\int_{c-\sqrt{-1}T}^{c+\sqrt{-1}T}\Gamma\left(\frac{z}{\lambda}\right)f(s+z)\delta^{-z}dz\\
&=\sum_{n=1}^{\infty}\frac{a_n}{n^s}\lim_{T\to\infty}\frac{1}{2\pi\sqrt{-1}}\int_{\frac{c}{\lambda}-\sqrt{-1}\frac{T}{\lambda}}^{\frac{c}{\lambda}+\sqrt{-1}\frac{T}{\lambda}}\Gamma(w)(n\delta)^{-\lambda w}dw\\
&=\sum_{n=1}^{\infty}\frac{a_n}{n^s}e^{-(n\delta)^{\lambda}}.
\end{aligned}$$

$\underline{2^{\circ}}$ $\displaystyle\sup_{\sigma_1\leq\sigma\leq\sigma_2}\left|\frac{1}{T}\int_2^T\left|\sum_{n=1}^{\infty}\frac{a_n}{n^{\sigma+\sqrt{-1}t}}e^{-(n\delta)^{\lambda}}\right|^2dt-\sum_{n=1}^{\infty}\frac{|a_n|^2}{n^{2\sigma}}e^{-2(n\delta)^{\lambda}}\right|\to 0$ as $T\to\infty$.

$\because$ Let $\sigma_1\leq\sigma\leq\sigma_2$ and $T\geq 2$. First

$$\begin{aligned}
&\left|\sum_{n=1}^{\infty}\frac{a_n}{n^{\sigma+\sqrt{-1}t}}e^{-(n\delta)^{\lambda}}\right|^2\\
&=\sum_{n=1}^{\infty}\frac{a_n}{n^{\sigma+\sqrt{-1}t}}e^{-(n\delta)^{\lambda}}\sum_{m=1}^{\infty}\frac{\overline{a_m}}{m^{\sigma-\sqrt{-1}t}}e^{-(m\delta)^{\lambda}}\\
&=\sum_{n,m=1}^{\infty}\frac{a_n\overline{a_m}}{n^{\sigma+\sqrt{-1}t}m^{\sigma-\sqrt{-1}t}}e^{-(n\delta)^{\lambda}}e^{-(m\delta)^{\lambda}}\\
&=\sum_{n=1}^{\infty}\frac{|a_n|^2}{n^{2\sigma}}e^{-2(n\delta)^{\lambda}}\\
&\quad+\sum_{\substack{n,m\geq 1;\\ n\neq m}}a_n\overline{a_m}\frac{e^{-(n\delta)^{\lambda}}e^{-(m\delta)^{\lambda}}}{n^{\sigma}m^{\sigma}}e^{\sqrt{-1}t(\log m-\log n)}\\
&=\sum_{n=1}^{\infty}\frac{|a_n|^2}{n^{2\sigma}}e^{-2(n\delta)^{\lambda}}\\
&\quad+\sum_{n>m\geq 1}a_n\overline{a_m}\frac{e^{-(n\delta)^{\lambda}}e^{-(m\delta)^{\lambda}}}{n^{\sigma}m^{\sigma}}e^{-\sqrt{-1}t\log\frac{n}{m}}\\
&\quad+\sum_{m>n\geq 1}a_n\overline{a_m}\frac{e^{-(n\delta)^{\lambda}}e^{-(m\delta)^{\lambda}}}{n^{\sigma}m^{\sigma}}e^{\sqrt{-1}t\log\frac{m}{n}}\\
&=\sum_{n=1}^{\infty}\frac{|a_n|^2}{n^{2\sigma}}e^{-2(n\delta)^{\lambda}}\\
&\quad+\sum_{m>n\geq 1}\frac{e^{-(n\delta)^{\lambda}}e^{-(m\delta)^{\lambda}}}{n^{\sigma}m^{\sigma}}\left(a_n\overline{a_m}e^{\sqrt{-1}t\log\frac{m}{n}}+\overline{a_n}a_me^{-\sqrt{-1}t\log\frac{m}{n}}\right).
\end{aligned}$$

Integration in $t \in [2, T]$ is

$$\begin{aligned}
&\int_2^T \Big|\sum_{n=1}^{\infty} \frac{a_n}{n^{\sigma+\sqrt{-1}t}} e^{-(n\delta)^\lambda}\Big|^2 dt \\
&= (T-2)\sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma}} e^{-2(n\delta)^\lambda} \\
&\quad + \sum_{m>n\geq 1} \frac{e^{-(n\delta)^\lambda} e^{-(m\delta)^\lambda}}{n^\sigma m^\sigma} \bigg(a_n \overline{a_m} \frac{e^{\sqrt{-1}T \log \frac{m}{n}} - e^{\sqrt{-1}2 \log \frac{m}{n}}}{\sqrt{-1} \log \frac{m}{n}} \\
&\qquad\qquad + \overline{a_n} a_m \frac{e^{-\sqrt{-1}T \log \frac{m}{n}} - e^{-\sqrt{-1}2 \log \frac{m}{n}}}{-\sqrt{-1} \log \frac{m}{n}} \bigg) \\
&= (T-2)\sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma}} e^{-2(n\delta)^\lambda} \\
&\quad + \sum_{m>n\geq 1} \frac{e^{-(n\delta)^\lambda} e^{-(m\delta)^\lambda}}{n^\sigma m^\sigma} 2 \operatorname{Re}\bigg(a_n \overline{a_m} \frac{e^{\sqrt{-1}T \log \frac{m}{n}} - e^{\sqrt{-1}2 \log \frac{m}{n}}}{\sqrt{-1} \log \frac{m}{n}} \bigg).
\end{aligned}$$

Hence

$$\begin{aligned}
&\bigg| \frac{1}{T} \int_2^T \Big|\sum_{n=1}^{\infty} \frac{a_n}{n^{\sigma+\sqrt{-1}t}} e^{-(n\delta)^\lambda}\Big|^2 dt - \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma}} e^{-2(n\delta)^\lambda} \bigg| \\
&= \frac{1}{T} \bigg| -2 \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma}} e^{-2(n\delta)^\lambda} \\
&\quad + \sum_{m>n\geq 1} \frac{e^{-(n\delta)^\lambda} e^{-(m\delta)^\lambda}}{n^\sigma m^\sigma} 2 \operatorname{Re}\bigg(a_n \overline{a_m} \frac{e^{\sqrt{-1}T \log \frac{m}{n}} - e^{\sqrt{-1}2 \log \frac{m}{n}}}{\sqrt{-1} \log \frac{m}{n}} \bigg) \bigg| \\
&\leq \frac{1}{T} \bigg(2 \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma}} e^{-2(n\delta)^\lambda} + 4 \sum_{m>n\geq 1} \frac{|a_n||a_m| e^{-(n\delta)^\lambda} e^{-(m\delta)^\lambda}}{n^\sigma m^\sigma} \frac{1}{\log \frac{m}{n}} \bigg) \\
&= \frac{1}{T} \bigg(2 \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma}} e^{-2(n\delta)^\lambda} \\
&\quad + 4 \sum_{n=1}^{\infty} \sum_{n<m\leq 2n} \frac{|a_n| e^{-(n\delta)^\lambda}}{n^\sigma} \frac{|a_m| e^{-(m\delta)^\lambda}}{m^\sigma} \frac{1}{\log \frac{m}{n}} \\
&\quad + 4 \sum_{n=1}^{\infty} \sum_{m>2n} \frac{|a_n| e^{-(n\delta)^\lambda}}{n^\sigma} \frac{|a_m| e^{-(m\delta)^\lambda}}{m^\sigma} \frac{1}{\log \frac{m}{n}} \bigg) \\
&\leq \frac{1}{T} \bigg(2 \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma}} e^{-2(n\delta)^\lambda} \\
&\quad + 4 \sup_{m\geq 1} \big(|a_m| e^{-(m\delta)^\lambda} \big) \sum_{n=1}^{\infty} \frac{|a_n| e^{-(n\delta)^\lambda}}{n^{2\sigma}} \sum_{k=1}^{n} \frac{1}{\log(1+\frac{k}{n})}
\end{aligned}$$

$$+\frac{4}{\log 2}\sum_{n=1}^{\infty}\sum_{m>2n}\frac{|a_n|e^{-(n\delta)^\lambda}}{n^\sigma}\frac{|a_m|e^{-(m\delta)^\lambda}}{m^\sigma}\Bigg)$$

$\Big[$ ∵ Let $l = \lceil\frac{1}{\sigma_2-\alpha}\rceil \in \mathbb{N}$. Then $\lambda l = \lambda\lceil\frac{1}{\sigma_2-\alpha}\rceil \geq \lambda\frac{1}{\sigma_2-\alpha} > 1$.
Since $e^x > \frac{x^l}{l!}$ $(x>0)$, and thus $e^{-x} < \frac{l!}{x^l}$,

$$\begin{aligned}\sup_{m\geq 1}\big(|a_m|e^{-(m\delta)^\lambda}\big) &\leq \sup_{m\geq 1}\big(|a_m|\tfrac{l!}{(m\delta)^{\lambda l}}\big)\\ &= \tfrac{l!}{\delta^{\lambda l}}\sup_{m\geq 1}\tfrac{|a_m|}{m^{\lambda l}}\\ &< \infty \Big]\end{aligned}$$

$$\begin{aligned}\leq \frac{1}{T}\Bigg(&2\sum_{n=1}^{\infty}\frac{|a_n|^2}{n^{2\sigma}}e^{-2(n\delta)^\lambda}\\ &+8\sup_{m\geq 1}\big(|a_m|e^{-(m\delta)^\lambda}\big)\sum_{n=1}^{\infty}\frac{|a_n|e^{-(n\delta)^\lambda}}{n^{2\sigma-1}}\sum_{k=1}^{n}\frac{1}{k}\\ &+\frac{4}{\log 2}\Big(\sum_{n=1}^{\infty}\frac{|a_n|e^{-(n\delta)^\lambda}}{n^\sigma}\Big)^2\Bigg)\end{aligned}$$

$\big[$ ∵ $\log(1+\frac{k}{n}) \geq \frac{k}{2n}$ $(k=1,\ldots,n)$ [cf. (6.2)]$\big]$

$$\begin{aligned}\leq \frac{1}{T}\Bigg(&2\sum_{n=1}^{\infty}\frac{|a_n|^2}{n^{2\sigma}}e^{-2(n\delta)^\lambda}\\ &+8\sup_{m\geq 1}\big(|a_m|e^{-(m\delta)^\lambda}\big)\sum_{n=1}^{\infty}\frac{|a_n|e^{-(n\delta)^\lambda}}{n^{2\sigma-1}}\big(1+\log n\big)\\ &+\frac{4}{\log 2}\Big(\sum_{n=1}^{\infty}\frac{|a_n|e^{-(n\delta)^\lambda}}{n^\sigma}\Big)^2\Bigg)\end{aligned}$$

$$\begin{aligned}\leq \frac{1}{T}\Bigg(&2\sum_{n=1}^{\infty}\frac{|a_n|^2}{n^{2\sigma_1}}e^{-2(n\delta)^\lambda}\\ &+8\sup_{m\geq 1}\big(|a_m|e^{-(m\delta)^\lambda}\big)\sum_{n=1}^{\infty}\frac{|a_n|e^{-(n\delta)^\lambda}}{n^{2\sigma_1-1}}\big(1+\log n\big)\\ &+\frac{4}{\log 2}\Big(\sum_{n=1}^{\infty}\frac{|a_n|e^{-(n\delta)^\lambda}}{n^{\sigma_1}}\Big)^2\Bigg).\end{aligned}$$

Letting $T\to\infty$, we have the assertion of 2°.

3° (i) For $\sigma_1 \leq \sigma \leq \sigma_2$,

$$\int_{\alpha-\sigma-\sqrt{-1}\infty}^{\alpha-\sigma+\sqrt{-1}\infty}\Gamma\Big(\frac{z}{\lambda}\Big)f(\sigma+\sqrt{-1}t+z)\delta^{-z}dz$$

is convergent for each $t\in\mathbb{R}$, and is continuous as a function of t.
(ii) For $T\geq 2$,

$$\sup_{\sigma_1\leq\sigma\leq\sigma_2}\int_2^T\left|\frac{1}{2\pi\sqrt{-1}\lambda}\int_{\alpha-\sigma-\sqrt{-1}\infty}^{\alpha-\sigma+\sqrt{-1}\infty}\Gamma\Big(\frac{z}{\lambda}\Big)f(\sigma+\sqrt{-1}t+z)\delta^{-z}dz\right|^2dt$$

$$\leq T\Big(\frac{\delta^{\sigma_1-\alpha}}{2\pi}\Big)^2 2\lambda\Big(C\big(\tfrac{\alpha-\sigma_2}{\lambda},\tfrac{\alpha-\sigma_1}{\lambda}\big)+\frac{2}{\pi}A\big(\tfrac{\alpha-\sigma_2}{\lambda},\tfrac{\alpha-\sigma_1}{\lambda}\big)\Big)$$
$$\times\Bigg(A\big(\tfrac{\alpha-\sigma_2}{\lambda},\tfrac{\alpha-\sigma_1}{\lambda}\big)C_1^2\int_{-\infty}^{\infty}e^{-|v|\frac{\pi}{4\lambda}}\big(2+|v|\big)^{2C}dv$$
$$+\Big(C\big(\tfrac{\alpha-\sigma_2}{\lambda},\tfrac{\alpha-\sigma_1}{\lambda}\big)+\frac{2}{\pi}A\big(\tfrac{\alpha-\sigma_2}{\lambda},\tfrac{\alpha-\sigma_1}{\lambda}\big)\Big)C_2 2\lambda\Big(2+\frac{\lambda}{2}\Big)\Bigg).$$

Here C_1 and C_2 are constants defined by (6.30) and (6.31), respectively; $C\big(\frac{\alpha-\sigma_2}{\lambda},\frac{\alpha-\sigma_1}{\lambda}\big)$ is a constant in (6.27) with $a=\frac{\alpha-\sigma_2}{\lambda}$, $b=\frac{\alpha-\sigma_1}{\lambda}$; $A\big(\frac{\alpha-\sigma_2}{\lambda},\frac{\alpha-\sigma_1}{\lambda}\big)$ is a constant in (6.16) where σ_1 and σ_2 are replaced by $\frac{\alpha-\sigma_2}{\lambda}$ and $\frac{\alpha-\sigma_1}{\lambda}$, respectively.

∵ (i) $\Gamma(\cdot)$ is holomorphic on $\{z\in\mathbb{C};-1<\operatorname{Re}z<0\}$. For $-1<a<b<0$, put

$$C(a,b):=\max_{\substack{a\leq u\leq b,\\|v|\leq 1}}\big|\Gamma(u+\sqrt{-1}v)\big|<\infty. \tag{6.27}$$

Since $-1<\frac{\alpha-\sigma_2}{\lambda}\leq\frac{\alpha-\sigma}{\lambda}\leq\frac{\alpha-\sigma_1}{\lambda}<0$ ($\sigma_1\leq\sigma\leq\sigma_2$), Claim 6.4 gives that for $\sigma_1\leq\sigma\leq\sigma_2$, $v\in\mathbb{R}$,

$$\begin{aligned}\Big|\Gamma\Big(\frac{\alpha-\sigma}{\lambda}+\sqrt{-1}v\Big)\Big| &\leq \mathbf{1}_{|v|\leq 1}C\big(\tfrac{\alpha-\sigma_2}{\lambda},\tfrac{\alpha-\sigma_1}{\lambda}\big)\\ &\quad+\mathbf{1}_{|v|>1}A\big(\tfrac{\alpha-\sigma_2}{\lambda},\tfrac{\alpha-\sigma_1}{\lambda}\big)|v|^{\frac{\alpha-\sigma_1}{\lambda}-\frac{1}{2}}e^{-|v|\frac{\pi}{2}}\\ &\leq \mathbf{1}_{|v|\leq 1}C\big(\tfrac{\alpha-\sigma_2}{\lambda},\tfrac{\alpha-\sigma_1}{\lambda}\big)\\ &\quad+\mathbf{1}_{|v|>1}A\big(\tfrac{\alpha-\sigma_2}{\lambda},\tfrac{\alpha-\sigma_1}{\lambda}\big)e^{-|v|\frac{\pi}{2}}\quad\big[∵\ \tfrac{\alpha-\sigma_1}{\lambda}-\tfrac{1}{2}<0\big].\end{aligned} \tag{6.28}$$

By assumption,

$\mathbb{R}\ni v\mapsto f(\alpha+\sqrt{-1}v)\in\mathbb{C}$ is continuous (in fact, it is expanded in a power series about each point of $\mathbb{R}$), (6.29)

$$C_1:=\sup\Big\{\frac{|f(z)|}{(2+|\operatorname{Im}z|)^C};\operatorname{Re}z\geq\alpha,\ |z-1|\geq\frac{1-\alpha}{2}\Big\}<\infty, \tag{6.30}$$

$$C_2:=\sup_{T>0}\frac{1}{T}\int_{-T}^{T}\big|f(\alpha+\sqrt{-1}v)\big|^2dv<\infty. \tag{6.31}$$

Thus, for $z=\alpha-\sigma+\sqrt{-1}v$ ($\sigma_1\leq\sigma\leq\sigma_2$, $v\in\mathbb{R}$) and $t\in\mathbb{R}$, we have the following estimate:

$$\begin{aligned}&\Big|\Gamma\Big(\frac{\alpha-\sigma+\sqrt{-1}v}{\lambda}\Big)f(\sigma+\sqrt{-1}t+\alpha-\sigma+\sqrt{-1}v)\delta^{-(\alpha-\sigma+\sqrt{-1}v)}\Big|\\ &=\Big|\Gamma\Big(\frac{\alpha-\sigma+\sqrt{-1}v}{\lambda}\Big)\Big|\big|f(\alpha+\sqrt{-1}(t+v))\big|\delta^{\sigma-\alpha}\\ &\leq\delta^{\sigma-\alpha}\Big(\mathbf{1}_{|v|\leq\lambda}C\big(\tfrac{\alpha-\sigma_2}{\lambda},\tfrac{\alpha-\sigma_1}{\lambda}\big)+\mathbf{1}_{|v|>\lambda}A\big(\tfrac{\alpha-\sigma_2}{\lambda},\tfrac{\alpha-\sigma_1}{\lambda}\big)e^{-|v|\frac{\pi}{2\lambda}}\Big)C_1\big(2+|t|+|v|\big)^C,\end{aligned}$$

from which, the assertion (i) follows at once.

(ii) Let $\sigma_1\leq\sigma\leq\sigma_2$ and $T\geq 1$. First, for $1\leq t\leq T$,

$$\Big|\frac{1}{2\pi\sqrt{-1}\lambda}\int_{\alpha-\sigma-\sqrt{-1}\infty}^{\alpha-\sigma+\sqrt{-1}\infty}\Gamma\Big(\frac{z}{\lambda}\Big)f(\sigma+\sqrt{-1}t+z)\delta^{-z}dz\Big|^2$$

$$= \left| \frac{1}{2\pi\sqrt{-1}\lambda} \int_{-\infty}^{\infty} \Gamma\Big(\frac{\alpha - \sigma + \sqrt{-1}v}{\lambda}\Big) f(\sigma + \sqrt{-1}t + \alpha - \sigma + \sqrt{-1}v) \right.$$
$$\left. \times\, \delta^{-(\alpha-\sigma+\sqrt{-1}v)}\sqrt{-1}\,dv \right|^2$$
$$= \left| \frac{\delta^{\sigma-\alpha}}{2\pi\lambda} \int_{-\infty}^{\infty} \Gamma\Big(\frac{\alpha - \sigma}{\lambda} + \sqrt{-1}\frac{v}{\lambda}\Big) f\big(\alpha + \sqrt{-1}(v+t)\big)\delta^{-\sqrt{-1}v}\,dv \right|^2$$
$$= \Big(\frac{\delta^{\sigma-\alpha}}{2\pi\lambda}\Big)^2 \left| \int_{-\infty}^{\infty} \Gamma\Big(\frac{\alpha - \sigma}{\lambda} + \sqrt{-1}\frac{v-t}{\lambda}\Big) f(\alpha + \sqrt{-1}v)\delta^{-\sqrt{-1}(v-t)}\,dv \right|^2$$
$$\le \Big(\frac{\delta^{\sigma-\alpha}}{2\pi\lambda}\Big)^2 \Big(\int_{-\infty}^{\infty} \Big|\Gamma\Big(\frac{\alpha - \sigma}{\lambda} + \sqrt{-1}\frac{v-t}{\lambda}\Big)\Big| \big| f(\alpha + \sqrt{-1}v)\big|\,dv \Big)^2$$
$$\le \Big(\frac{\delta^{\sigma-\alpha}}{2\pi\lambda}\Big)^2 \int_{-\infty}^{\infty} \Big|\Gamma\Big(\frac{\alpha - \sigma}{\lambda} + \sqrt{-1}\frac{v-t}{\lambda}\Big)\Big|\,dv$$
$$\times \int_{-\infty}^{\infty} \Big|\Gamma\Big(\frac{\alpha - \sigma}{\lambda} + \sqrt{-1}\frac{v-t}{\lambda}\Big)\Big| \big| f(\alpha + \sqrt{-1}v)\big|^2\,dv$$
[$\because$ Schwarz inequality]
$$= \Big(\frac{\delta^{\sigma-\alpha}}{2\pi\lambda}\Big)^2 \lambda \int_{-\infty}^{\infty} \Big|\Gamma\Big(\frac{\alpha - \sigma}{\lambda} + \sqrt{-1}v\Big)\Big|\,dv$$
$$\times \int_{-\infty}^{\infty} \Big|\Gamma\Big(\frac{\alpha - \sigma}{\lambda} + \sqrt{-1}\frac{v-t}{\lambda}\Big)\Big| \big| f(\alpha + \sqrt{-1}v)\big|^2\,dv$$
$$\le \Big(\frac{\delta^{\sigma-\alpha}}{2\pi\lambda}\Big)^2 \lambda \Big(2C\big(\tfrac{\alpha-\sigma_2}{\lambda}, \tfrac{\alpha-\sigma_1}{\lambda}\big) + 2A\big(\tfrac{\alpha-\sigma_2}{\lambda}, \tfrac{\alpha-\sigma_1}{\lambda}\big)\frac{2}{\pi}\Big)$$
$$\times \int_{-\infty}^{\infty} \Big|\Gamma\Big(\frac{\alpha - \sigma}{\lambda} + \sqrt{-1}\frac{v-t}{\lambda}\Big)\Big| \big| f(\alpha + \sqrt{-1}v)\big|^2\,dv \quad [\text{cf. } (6.28)]$$
$$\le \Big(\frac{\delta^{\sigma_1-\alpha}}{2\pi\lambda}\Big)^2 2\lambda \Big(C\big(\tfrac{\alpha-\sigma_2}{\lambda}, \tfrac{\alpha-\sigma_1}{\lambda}\big) + \frac{2}{\pi} A\big(\tfrac{\alpha-\sigma_2}{\lambda}, \tfrac{\alpha-\sigma_1}{\lambda}\big)\Big)$$
$$\times \int_{-\infty}^{\infty} \Big|\Gamma\Big(\frac{\alpha - \sigma}{\lambda} + \sqrt{-1}\frac{v-t}{\lambda}\Big)\Big| \big| f(\alpha + \sqrt{-1}v)\big|^2\,dv$$
[$\because$ $\delta^{\sigma-\alpha} \le \delta^{\sigma_1-\alpha}$ by $0 < \delta \le 1$, $\sigma_1 - \alpha \le \sigma - \alpha$].

Integration in $t \in [2, T]$ yields that

$$\int_2^T \left| \frac{1}{2\pi\sqrt{-1}\lambda} \int_{\alpha-\sigma-\sqrt{-1}\infty}^{\alpha-\sigma+\sqrt{-1}\infty} \Gamma\Big(\frac{z}{\lambda}\Big) f(\sigma + \sqrt{-1}t + z)\delta^{-z}\,dz \right|^2 dt$$
$$\le \Big(\frac{\delta^{\sigma_1-\alpha}}{2\pi\lambda}\Big)^2 2\lambda \Big(C\big(\tfrac{\alpha-\sigma_2}{\lambda}, \tfrac{\alpha-\sigma_1}{\lambda}\big) + \tfrac{2}{\pi} A\big(\tfrac{\alpha-\sigma_2}{\lambda}, \tfrac{\alpha-\sigma_1}{\lambda}\big)\Big)$$
$$\times \int_2^T dt \int_{-\infty}^{\infty} \Big|\Gamma\Big(\frac{\alpha - \sigma}{\lambda} + \sqrt{-1}\frac{v-t}{\lambda}\Big)\Big| \big| f(\alpha + \sqrt{-1}v)\big|^2\,dv$$
$$= \Big(\frac{\delta^{\sigma_1-\alpha}}{2\pi\lambda}\Big)^2 2\lambda \Big(C\big(\tfrac{\alpha-\sigma_2}{\lambda}, \tfrac{\alpha-\sigma_1}{\lambda}\big) + \tfrac{2}{\pi} A\big(\tfrac{\alpha-\sigma_2}{\lambda}, \tfrac{\alpha-\sigma_1}{\lambda}\big)\Big)$$
$$\times \Big(\int_2^T dt \int_{|v| \ge \lambda + 2T} \Big|\Gamma\Big(\frac{\alpha - \sigma}{\lambda} + \sqrt{-1}\frac{v-t}{\lambda}\Big)\Big| \big| f(\alpha + \sqrt{-1}v)\big|^2\,dv$$

$$+\int_2^T dt\int_{|v|<\lambda+2T}\Big|\Gamma\Big(\frac{\alpha-\sigma}{\lambda}+\sqrt{-1}\frac{v-t}{\lambda}\Big)\Big|\big|f(\alpha+\sqrt{-1}v)\big|^2dv\Big)$$

$$\le\Big(\frac{\delta^{\sigma_1-\alpha}}{2\pi\lambda}\Big)^2 2\lambda\Big(C\big(\tfrac{\alpha-\sigma_2}{\lambda},\tfrac{\alpha-\sigma_1}{\lambda}\big)+\tfrac{2}{\pi}A\big(\tfrac{\alpha-\sigma_2}{\lambda},\tfrac{\alpha-\sigma_1}{\lambda}\big)\Big)$$

$$\times\Big(\int_2^T dt\int_{|v|\ge\lambda+2T}A\big(\tfrac{\alpha-\sigma_2}{\lambda},\tfrac{\alpha-\sigma_1}{\lambda}\big)e^{-|v|\frac{\pi}{4\lambda}}C_1^2(2+|v|)^{2C}dv$$

$$+\int_{|v|<\lambda+2T}\big|f(\alpha+\sqrt{-1}v)\big|^2dv\int_2^T\Big|\Gamma\Big(\frac{\alpha-\sigma}{\lambda}+\sqrt{-1}\frac{v-t}{\lambda}\Big)\Big|dt\Big)$$

$$\left[\begin{array}{l}\because\ \text{For } 2\le t\le T \text{ and } |v|\ge\lambda+2T,\\ \big|\frac{v-t}{\lambda}\big|\ge\frac{|v|-t}{\lambda}\begin{cases}\ge\frac{|v|-T}{\lambda}\ge\frac{\lambda+T}{\lambda}>1,\\ =\frac{|v|+|v|-2t}{2\lambda}\ge\frac{|v|+|v|-2T}{2\lambda}\ge\frac{|v|+\lambda}{2\lambda}>\frac{|v|}{2\lambda}\end{cases}\end{array}\right]$$

$$\le\Big(\frac{\delta^{\sigma_1-\alpha}}{2\pi\lambda}\Big)^2 2\lambda\Big(C\big(\tfrac{\alpha-\sigma_2}{\lambda},\tfrac{\alpha-\sigma_1}{\lambda}\big)+\tfrac{2}{\pi}A\big(\tfrac{\alpha-\sigma_2}{\lambda},\tfrac{\alpha-\sigma_1}{\lambda}\big)\Big)$$

$$\times\Big(A\big(\tfrac{\alpha-\sigma_2}{\lambda},\tfrac{\alpha-\sigma_1}{\lambda}\big)C_1^2(T-2)\int_{-\infty}^{\infty}e^{-|v|\frac{\pi}{4\lambda}}(2+|v|)^{2C}dv$$

$$+\lambda\int_{-\infty}^{\infty}\Big|\Gamma\Big(\frac{\alpha-\sigma}{\lambda}+\sqrt{-1}t\Big)\Big|dt\int_{-\lambda-2T}^{\lambda+2T}\big|f(\alpha+\sqrt{-1}v)\big|^2dv\Big)$$

$$\le\Big(\frac{\delta^{\sigma_1-\alpha}}{2\pi\lambda}\Big)^2 2\lambda\Big(C\big(\tfrac{\alpha-\sigma_2}{\lambda},\tfrac{\alpha-\sigma_1}{\lambda}\big)+\tfrac{2}{\pi}A\big(\tfrac{\alpha-\sigma_2}{\lambda},\tfrac{\alpha-\sigma_1}{\lambda}\big)\Big)$$

$$\times\Big(A\big(\tfrac{\alpha-\sigma_2}{\lambda},\tfrac{\alpha-\sigma_1}{\lambda}\big)C_1^2T\int_{-\infty}^{\infty}e^{-|v|\frac{\pi}{4\lambda}}(2+|v|)^{2C}dv$$

$$+2\lambda\Big(C\big(\tfrac{\alpha-\sigma_2}{\lambda},\tfrac{\alpha-\sigma_1}{\lambda}\big)+\tfrac{2}{\pi}A\big(\tfrac{\alpha-\sigma_2}{\lambda},\tfrac{\alpha-\sigma_1}{\lambda}\big)\Big)C_2(\lambda+2T)\Big)$$

$$\le T\Big(\frac{\delta^{\sigma_1-\alpha}}{2\pi\lambda}\Big)^2 2\lambda\Big(C\big(\tfrac{\alpha-\sigma_2}{\lambda},\tfrac{\alpha-\sigma_1}{\lambda}\big)+\tfrac{2}{\pi}A\big(\tfrac{\alpha-\sigma_2}{\lambda},\tfrac{\alpha-\sigma_1}{\lambda}\big)\Big)$$

$$\times\Big(A\big(\tfrac{\alpha-\sigma_2}{\lambda},\tfrac{\alpha-\sigma_1}{\lambda}\big)C_1^2\int_{-\infty}^{\infty}e^{-|v|\frac{\pi}{4\lambda}}(2+|v|)^{2C}dv$$

$$+\Big(C\big(\tfrac{\alpha-\sigma_2}{\lambda},\tfrac{\alpha-\sigma_1}{\lambda}\big)+\tfrac{2}{\pi}A\big(\tfrac{\alpha-\sigma_2}{\lambda},\tfrac{\alpha-\sigma_1}{\lambda}\big)\Big)C_2 2\lambda\Big(2+\frac{\lambda}{2}\Big)\Big).$$

<u>4°</u> Fix $\sigma_1\le\sigma\le\sigma_2$ and $t\ge 2$, and let $s=\sigma+\sqrt{-1}t$. $\Gamma\big(\frac{z}{\lambda}\big)f(s+z)\delta^{-z}$ is meromorphic on $\{z\in\mathbb{C};\operatorname{Re}z\ge\alpha-\sigma\}$ and is holomorphic except $z=0,\ 1-s\ (\ne 0)$. $z=0$ is a simple pole and $z=1-s$ is a removable singularity or a pole of this function. For $R>t$, we consider a contour E_R as in Figure 6.3. By the residue theorem,

$$\frac{1}{2\pi\sqrt{-1}\lambda}\int_{E_R}\Gamma\Big(\frac{z}{\lambda}\Big)f(s+z)\delta^{-z}dz=\frac{1}{\lambda}\operatorname{Res}(0)+\frac{1}{\lambda}\operatorname{Res}(1-s).$$

We divide L.H.S. into four terms:

$$\text{L.H.S.}=\frac{1}{2\pi\sqrt{-1}\lambda}\int_{c-\sqrt{-1}R}^{c+\sqrt{-1}R}\Gamma\Big(\frac{z}{\lambda}\Big)f(s+z)\delta^{-z}dz$$

$$-\frac{1}{2\pi\sqrt{-1}\lambda}\int_{\alpha-\sigma+\sqrt{-1}R}^{c+\sqrt{-1}R}\Gamma\Big(\frac{z}{\lambda}\Big)f(s+z)\delta^{-z}dz$$

$$- \frac{1}{2\pi\sqrt{-1}\lambda} \int_{\alpha-\sigma-\sqrt{-1}R}^{\alpha-\sigma+\sqrt{-1}R} \Gamma\Big(\frac{z}{\lambda}\Big) f(s+z)\delta^{-z} dz$$
$$+ \frac{1}{2\pi\sqrt{-1}\lambda} \int_{\alpha-\sigma-\sqrt{-1}R}^{c-\sqrt{-1}R} \Gamma\Big(\frac{z}{\lambda}\Big) f(s+z)\delta^{-z} dz.$$

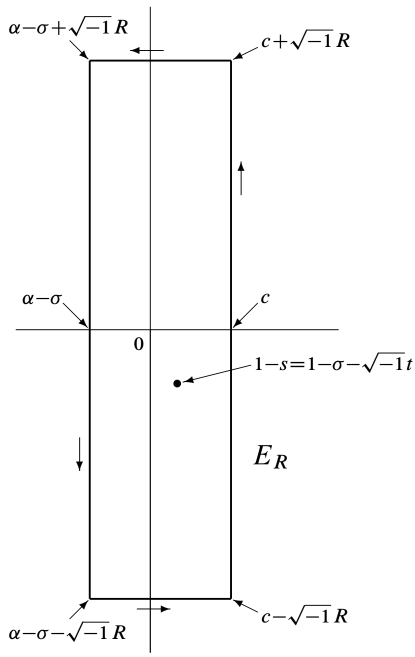

Figure 6.3: E_R

For $R > \lambda \vee \big(t + \frac{1-\alpha}{2}\big)$,

$$\left|\frac{1}{2\pi\sqrt{-1}\lambda} \int_{\alpha-\sigma\pm\sqrt{-1}R}^{c\pm\sqrt{-1}R} \Gamma\Big(\frac{z}{\lambda}\Big) f(s+z)\delta^{-z} dz\right|$$
$$= \left|\frac{1}{2\pi\sqrt{-1}\lambda} \int_{\alpha-\sigma}^{c} \Gamma\Big(\frac{u \pm \sqrt{-1}R}{\lambda}\Big) f(\sigma + \sqrt{-1}t + u \pm \sqrt{-1}R)\delta^{-(u\pm\sqrt{-1}R)} du\right|$$
$\big[\because$ change of variable: $z = u \pm \sqrt{-1}R\big]$
$$\le \frac{1}{2\pi\lambda} \int_{\alpha-\sigma}^{c} \Big|\Gamma\Big(\frac{u}{\lambda} \pm \sqrt{-1}\frac{R}{\lambda}\Big)\Big| \big|f\big(\sigma + u + \sqrt{-1}(t \pm R)\big)\big| \delta^{-u} du$$
$$\le \frac{1}{2\pi\lambda} \int_{\alpha-\sigma}^{c} A\big(\tfrac{\alpha-\sigma_2}{\lambda}, \tfrac{c}{\lambda}\big)\Big(\frac{R}{\lambda}\Big)^{\frac{c}{\lambda}-\frac{1}{2}} e^{-\frac{R}{\lambda}\frac{\pi}{2}} C_1\big(2 + |t \pm R|\big)^C \delta^{-u} du$$
$\big[$cf. (6.21) and (6.30)$\big]$
$$= \frac{1}{2\pi\lambda} A\big(\tfrac{\alpha-\sigma_2}{\lambda}, \tfrac{c}{\lambda}\big)\Big(\frac{R}{\lambda}\Big)^{\frac{c}{\lambda}-\frac{1}{2}} e^{-\frac{R}{\lambda}\frac{\pi}{2}} C_1\big(2 + |t \pm R|\big)^C \int_{\alpha-\sigma}^{c} \delta^{-u} du$$
$\to 0$ as $R \to \infty$.

This, together with 1° and 3°, implies that

$$\frac{1}{\lambda}\operatorname{Res}(0)+\frac{1}{\lambda}\operatorname{Res}(1-s)=\sum_{n=1}^{\infty}\frac{a_n}{n^s}e^{-(n\delta)^\lambda}-\frac{1}{2\pi\sqrt{-1}\lambda}\int_{\alpha-\sigma-\sqrt{-1}\infty}^{\alpha-\sigma+\sqrt{-1}\infty}\Gamma\Big(\frac{z}{\lambda}\Big)f(s+z)\delta^{-z}dz.$$

We find $\operatorname{Res}(0)$ and $\operatorname{Res}(1-s)$. From

$$\lim_{z\to 0}z\Gamma\Big(\frac{z}{\lambda}\Big)f(s+z)\delta^{-z}=\lim_{z\to 0}\lambda\Gamma\Big(\frac{z}{\lambda}+1\Big)f(s+z)\delta^{-z}=\lambda f(s),$$

it is clear that $\operatorname{Res}(0)=\lambda f(s)$. Let the Laurent expansion of $f(\cdot)$ about $z=1$ be

$$f(z)=\sum_{j=-k}^{\infty}b_j(z-1)^j,$$

where k is a nonnegative integer, and $b_{-k}\neq 0$ if $k>0$. Then the Laurent expansion of $f(s+\cdot)$ about $z=1-s$ is

$$f(s+z)=\sum_{j=-k}^{\infty}b_j\big(z-(1-s)\big)^j.$$

Since the Taylor expansion of $\Gamma\big(\frac{z}{\lambda}\big)\delta^{-z}$ about $z=1-s$ is

$$\Gamma\Big(\frac{z}{\lambda}\Big)\delta^{-z}=\sum_{i=0}^{\infty}\frac{1}{i!}\frac{d^i}{dz^i}\Big(\Gamma\Big(\frac{z}{\lambda}\Big)\delta^{-z}\Big)\Big|_{z=1-s}\big(z-(1-s)\big)^i,$$

it is seen that

$$\begin{aligned}\operatorname{Res}(1-s)&=\sum_{\substack{i\ge 0,j\ge -k;\\ i+j=-1}}\frac{1}{i!}\frac{d^i}{dz^i}\Big(\Gamma\Big(\frac{z}{\lambda}\Big)\delta^{-z}\Big)\Big|_{z=1-s}b_j\\&=\sum_{i=0}^{k-1}\frac{1}{i!}\frac{d^i}{dz^i}\Big(\Gamma\Big(\frac{z}{\lambda}\Big)\delta^{-z}\Big)\Big|_{z=1-s}b_{-i-1}\\&=\sum_{i=0}^{k-1}\frac{1}{i!}\sum_{l=0}^{i}\binom{i}{l}\frac{1}{\lambda^l}\Gamma^{(l)}\Big(\frac{1-s}{\lambda}\Big)\delta^{s-1}\Big(\log\frac{1}{\delta}\Big)^{i-l}b_{-i-1}\\&=\delta^{s-1}\sum_{l=0}^{k-1}\frac{1}{l!}\Big(\frac{1}{\lambda}\Big)^l\Gamma^{(l)}\Big(\frac{1-s}{\lambda}\Big)\sum_{i=l}^{k-1}\frac{b_{-i-1}}{(i-l)!}\Big(\log\frac{1}{\delta}\Big)^{i-l}.\end{aligned}\tag{6.32}$$

Therefore

$$f(\sigma+\sqrt{-1}t)=\sum_{n=1}^{\infty}\frac{a_n}{n^{\sigma+\sqrt{-1}t}}e^{-(n\delta)^\lambda}$$

$$- \frac{1}{2\pi\sqrt{-1}\lambda} \int_{\alpha-\sigma-\sqrt{-1}\infty}^{\alpha-\sigma+\sqrt{-1}\infty} \Gamma\Big(\frac{z}{\lambda}\Big) f\big(\sigma + \sqrt{-1}t + z\big)\delta^{-z} dz$$

$$- \delta^{\sigma-1+\sqrt{-1}t} \sum_{l=0}^{k-1} \frac{1}{l!}\Big(\frac{1}{\lambda}\Big)^{l+1} \Gamma^{(l)}\Big(\frac{1-\sigma-\sqrt{-1}t}{\lambda}\Big) \sum_{i=l}^{k-1} \frac{b_{-i-1}}{(i-l)!}\Big(\log\frac{1}{\delta}\Big)^{i-l}. \quad (6.33)$$

$\underline{5^{\text{o}}}$ Let $\mathrm{Res}(1-s)$ be as in (6.32). Then

$$\sup_{\sigma_1 \le \sigma \le \sigma_2} \int_2^\infty \big|\mathrm{Res}(1-\sigma-\sqrt{-1}t)\big|^2 dt$$

$$\le \lambda \delta^{2(\sigma_1-1)} \Bigg\{ \Bigg(\sum_{l=0}^{k-1} \frac{1}{l!}\Big(\frac{1}{\lambda}\Big)^l \max\Big\{ \big|\Gamma^{(l)}(u+\sqrt{-1}v)\big| ; \begin{array}{l} \frac{1-\sigma_2}{\lambda} \le u \le \frac{1-\sigma_1}{\lambda}, \\ -1 \le v \le -\frac{2}{\lambda} \end{array} \Big\}$$

$$\times \sum_{i=l}^{k-1} \frac{|b_{-i-1}|}{(i-l)!}\Big(\log\frac{1}{\delta}\Big)^{i-l} \Bigg)^2$$

$$+ \int_1^\infty \Bigg(\sum_{l=0}^{k-1} \frac{1}{l!}\Big(\frac{1}{\lambda}\Big)^l \Bigg(\sum_{j=0}^{l} c_{l,j}\big(\tfrac{1-\sigma_2}{\lambda}\big)\Big(\log t + B\big(\tfrac{1-\sigma_2}{\lambda}, \tfrac{1-\sigma_1}{\lambda}\big)\Big)^j \Bigg)$$

$$\times A\big(\tfrac{1-\sigma_2}{\lambda}, \tfrac{1-\sigma_1}{\lambda}\big) t^{\frac{1-\sigma_1}{\lambda}-\frac{1}{2}} e^{-t\frac{\pi}{2}} \sum_{i=l}^{k-1} \frac{|b_{-i-1}|}{(i-l)!}\Big(\log\frac{1}{\delta}\Big)^{i-l} \Bigg)^2 dt \Bigg\}.$$

∵ $\mathrm{Res}(1-s) = 0$ when $k=0$, and so we suppose $k \ge 1$. Let $\sigma_1 \le \sigma \le \sigma_2$ and $t \ge 2$. Since $\frac{1-\sigma_2}{\lambda} \le \frac{1-\sigma}{\lambda} \le \frac{1-\sigma_1}{\lambda}$, Claim 6.4 gives that for $l = 0, 1, \dots, k-1$,

$$\Big|\Gamma^{(l)}\Big(\frac{1-\sigma}{\lambda} - \sqrt{-1}\frac{t}{\lambda}\Big)\Big|$$

$$\le \mathbf{1}_{t\le\lambda} \max\Big\{ \big|\Gamma^{(l)}(u+\sqrt{-1}v)\big| ; \begin{array}{l} \frac{1-\sigma_2}{\lambda} \le u \le \frac{1-\sigma_1}{\lambda}, \\ -1 \le v \le -\frac{2}{\lambda} \end{array} \Big\}$$

$$+ \mathbf{1}_{t>\lambda} \sum_{j=0}^{l} c_{l,j}\big(\tfrac{1-\sigma_2}{\lambda}\big)\Big(\log\frac{t}{\lambda} + B\big(\tfrac{1-\sigma_2}{\lambda}, \tfrac{1-\sigma_1}{\lambda}\big)\Big)^j A\big(\tfrac{1-\sigma_2}{\lambda}, \tfrac{1-\sigma_1}{\lambda}\big)\Big(\frac{t}{\lambda}\Big)^{\frac{1-\sigma_1}{\lambda}-\frac{1}{2}} e^{-\frac{t}{\lambda}\frac{\pi}{2}}.$$

Using this estimate in (6.32), we have

$$\int_2^\infty \big|\mathrm{Res}(1-\sigma-\sqrt{-1}t)\big|^2 dt$$

$$= \int_2^\infty \Bigg| \delta^{\sigma-1+\sqrt{-1}t} \sum_{l=0}^{k-1} \frac{1}{l!}\Big(\frac{1}{\lambda}\Big)^l \Gamma^{(l)}\Big(\frac{1-\sigma}{\lambda} - \sqrt{-1}\frac{t}{\lambda}\Big) \sum_{i=l}^{k-1} \frac{b_{-i-1}}{(i-l)!}\Big(\log\frac{1}{\delta}\Big)^{i-l} \Bigg|^2 dt$$

$$\le \delta^{2(\sigma-1)} \int_2^\infty \Bigg(\sum_{l=0}^{k-1} \frac{1}{l!}\Big(\frac{1}{\lambda}\Big)^l \Big|\Gamma^{(l)}\Big(\frac{1-\sigma}{\lambda} - \sqrt{-1}\frac{t}{\lambda}\Big)\Big| \sum_{i=l}^{k-1} \frac{|b_{-i-1}|}{(i-l)!}\Big(\log\frac{1}{\delta}\Big)^{i-l} \Bigg)^2 dt$$

$$\le \delta^{2(\sigma_1-1)} \Bigg\{ (\lambda-2) \Bigg(\sum_{l=0}^{k-1} \frac{1}{l!}\Big(\frac{1}{\lambda}\Big)^l \max\Big\{ \big|\Gamma^{(l)}(u+\sqrt{-1}v)\big| ; \begin{array}{l} \frac{1-\sigma_2}{\lambda} \le u \le \frac{1-\sigma_1}{\lambda}, \\ -1 \le v \le -\frac{2}{\lambda} \end{array} \Big\}$$

$$\times \sum_{i=l}^{k-1} \frac{|b_{-i-1}|}{(i-l)!}\Big(\log\frac{1}{\delta}\Big)^{i-l} \Bigg)^2$$

$$+ \int_\lambda^\infty \Big(\sum_{l=0}^{k-1} \frac{1}{l!} \Big(\frac{1}{\lambda}\Big)^l \Big(\sum_{j=0}^{l} c_{l,j}\big(\tfrac{1-\sigma_2}{\lambda}\big) \Big(\log\frac{t}{\lambda} + B\big(\tfrac{1-\sigma_2}{\lambda}, \tfrac{1-\sigma_1}{\lambda}\big)\Big)^j \Big)$$
$$\times A\big(\tfrac{1-\sigma_2}{\lambda}, \tfrac{1-\sigma_1}{\lambda}\big)\Big(\frac{t}{\lambda}\Big)^{\frac{1-\sigma_1}{\lambda}-\frac{1}{2}} e^{-\frac{t}{\lambda}\frac{\pi}{2}}$$
$$\times \sum_{i=l}^{k-1} \frac{|b_{-i-1}|}{(i-l)!} \Big(\log\frac{1}{\delta}\Big)^{i-l} \Big)^2 dt \Big\},$$

which shows the assertion of 5°.

6° From Minkowski's inequality, (6.33), 3°(ii) and 5°, it is seen that for $\sigma_1 \le \sigma \le \sigma_2$, $T \ge 2$,

$$\Big| \Big(\frac{1}{T}\int_2^T |f(\sigma+\sqrt{-1}t)|^2 dt\Big)^{\frac12} - \Big(\frac{1}{T}\int_2^T \Big|\sum_{n=1}^\infty \frac{a_n}{n^{\sigma+\sqrt{-1}t}} e^{-(n\delta)^\lambda}\Big|^2 dt\Big)^{\frac12} \Big|$$
$$\le \Big(\frac{1}{T}\int_2^T \Big| f(\sigma+\sqrt{-1}t) - \sum_{n=1}^\infty \frac{a_n}{n^{\sigma+\sqrt{-1}t}} e^{-(n\delta)^\lambda}\Big|^2 dt\Big)^{\frac12}$$
$$= \Big(\frac{1}{T}\int_2^T \Big| -\frac{1}{2\pi\sqrt{-1}\lambda}\int_{\alpha-\sigma-\sqrt{-1}\infty}^{\alpha-\sigma+\sqrt{-1}\infty} \Gamma\Big(\frac{z}{\lambda}\Big) f(\sigma+\sqrt{-1}t+z)\delta^{-z}dz$$
$$- \frac{1}{\lambda}\operatorname{Res}(1-\sigma-\sqrt{-1}t)\Big|^2 dt\Big)^{\frac12}$$
$$\le \Big(\frac{1}{T}\int_2^T \Big| \frac{1}{2\pi\sqrt{-1}\lambda}\int_{\alpha-\sigma-\sqrt{-1}\infty}^{\alpha-\sigma+\sqrt{-1}\infty} \Gamma\Big(\frac{z}{\lambda}\Big) f(\sigma+\sqrt{-1}t+z)\delta^{-z}dz\Big|^2 dt\Big)^{\frac12}$$
$$+ \frac{1}{\lambda}\Big(\frac{1}{T}\int_2^T \big|\operatorname{Res}(1-\sigma-\sqrt{-1}t)\big|^2 dt\Big)^{\frac12}$$
$$\le \Big(\Big(\frac{\delta^{\sigma_1-\alpha}}{2\pi}\Big)^2 2\lambda\Big(C\big(\tfrac{\alpha-\sigma_2}{\lambda}, \tfrac{\alpha-\sigma_1}{\lambda}\big) + \frac{2}{\pi}A\big(\tfrac{\alpha-\sigma_2}{\lambda}, \tfrac{\alpha-\sigma_1}{\lambda}\big)\Big)$$
$$\times \Big(A\big(\tfrac{\alpha-\sigma_2}{\lambda}, \tfrac{\alpha-\sigma_1}{\lambda}\big)C_1^2 \int_{-\infty}^\infty e^{-|v|\frac{\pi}{4\lambda}}\big(2+|v|\big)^{2C} dv$$
$$+ \Big(C\big(\tfrac{\alpha-\sigma_2}{\lambda}, \tfrac{\alpha-\sigma_1}{\lambda}\big) + \frac{2}{\pi}A\big(\tfrac{\alpha-\sigma_2}{\lambda}, \tfrac{\alpha-\sigma_1}{\lambda}\big)\Big)C_2 2\lambda\Big(2+\frac{\lambda}{2}\Big)\Big)\Big)^{\frac12}$$
$$+ \frac{1}{\lambda}\Big(\frac{\lambda}{T}\delta^{2(\sigma_1-1)}\Big\{\Big(\sum_{l=0}^{k-1}\frac{1}{l!}\Big(\frac{1}{\lambda}\Big)^l \max\Big\{\big|\Gamma^{(l)}(u+\sqrt{-1}v)\big|; \begin{matrix}\frac{1-\sigma_2}{\lambda} \le u \le \frac{1-\sigma_1}{\lambda}, \\ -1 \le v \le -\frac{2}{\lambda}\end{matrix}\Big\}$$
$$\times \sum_{i=l}^{k-1}\frac{|b_{-i-1}|}{(i-l)!}\Big(\log\frac{1}{\delta}\Big)^{i-l}\Big)^2$$
$$+ \int_1^\infty \Big(\sum_{l=0}^{k-1}\frac{1}{l!}\Big(\frac{1}{\lambda}\Big)^l \Big(\sum_{j=0}^{l} c_{l,j}\big(\tfrac{1-\sigma_2}{\lambda}\big)\Big(\log t + B\big(\tfrac{1-\sigma_2}{\lambda}, \tfrac{1-\sigma_1}{\lambda}\big)\Big)^j\Big)$$

$$\times A\left(\tfrac{1-\sigma_2}{\lambda}, \tfrac{1-\sigma_1}{\lambda}\right) t^{\frac{1-\sigma_1}{\lambda}-\frac{1}{2}} e^{-t\frac{\pi}{2}} \sum_{i=l}^{k-1} \frac{|b_{-i-1}|}{(i-l)!}\left(\log\frac{1}{\delta}\right)^{i-l}\Bigg)^2 dt\Bigg\}\Bigg)^{\frac{1}{2}}. \tag{6.34}$$

Since, from the proof of 2°,

$$\sup_{T\geq 2}\sup_{\sigma_1\leq\sigma\leq\sigma_2}\frac{1}{T}\int_2^T\Big|\sum_{n=1}^{\infty}\frac{a_n}{n^{\sigma+\sqrt{-1}t}}e^{-(n\delta)^{\lambda}}\Big|^2 dt < \infty,$$

this, together with (6.34), implies that

$$\sup_{T\geq 2}\sup_{\sigma_1\leq\sigma\leq\sigma_2}\frac{1}{T}\int_2^T |f(\sigma+\sqrt{-1}t)|^2 dt < \infty. \tag{6.35}$$

By (6.34) again,

$$\begin{aligned}
&\Big(\frac{1}{T}\int_2^T\Big|\sum_{n=1}^{\infty}\frac{a_n}{n^{\sigma+\sqrt{-1}t}}e^{-(n\delta)^{\lambda}}\Big|^2 dt\Big)^{\frac{1}{2}} \\
&\leq \sup_{T\geq 2}\sup_{\sigma_1\leq\sigma\leq\sigma_2}\Big(\frac{1}{T}\int_2^T |f(\sigma+\sqrt{-1}t)|^2 dt\Big)^{\frac{1}{2}} \\
&\quad + \Bigg(\Big(\frac{\delta^{\sigma_1-\alpha}}{2\pi}\Big)^2 2\lambda\Big(C\big(\tfrac{\alpha-\sigma_2}{\lambda}, \tfrac{\alpha-\sigma_1}{\lambda}\big) + \frac{2}{\pi}A\big(\tfrac{\alpha-\sigma_2}{\lambda}, \tfrac{\alpha-\sigma_1}{\lambda}\big)\Big) \\
&\qquad \times \Bigg(A\big(\tfrac{\alpha-\sigma_2}{\lambda}, \tfrac{\alpha-\sigma_1}{\lambda}\big)C_1^2\int_{-\infty}^{\infty} e^{-|v|\frac{\pi}{4\lambda}}\big(2+|v|\big)^{2C} dv \\
&\qquad\quad + \Big(C\big(\tfrac{\alpha-\sigma_2}{\lambda}, \tfrac{\alpha-\sigma_1}{\lambda}\big) + \frac{2}{\pi}A\big(\tfrac{\alpha-\sigma_2}{\lambda}, \tfrac{\alpha-\sigma_1}{\lambda}\big)\Big)C_2 2\lambda\Big(2+\frac{\lambda}{2}\Big)\Bigg)\Bigg)^{\frac{1}{2}} \\
&\quad + \frac{1}{\lambda}\Bigg(\frac{\lambda}{T}\delta^{2(\sigma_1-1)}\Bigg\{\Bigg(\sum_{l=0}^{k-1}\frac{1}{l!}\Big(\frac{1}{\lambda}\Big)^l \max\Big\{\big|\Gamma^{(l)}(u+\sqrt{-1}v)\big|; \begin{array}{l}\frac{1-\sigma_2}{\lambda}\leq u\leq\frac{1-\sigma_1}{\lambda},\\ -1\leq v\leq -\frac{2}{\lambda}\end{array}\Big\} \\
&\qquad\qquad \times \sum_{i=l}^{k-1}\frac{|b_{-i-1}|}{(i-l)!}\Big(\log\frac{1}{\delta}\Big)^{i-l}\Bigg)^2 \\
&\qquad + \int_1^{\infty}\Bigg(\sum_{l=0}^{k-1}\frac{1}{l!}\Big(\frac{1}{\lambda}\Big)^l\Big(\sum_{j=0}^{l} c_{l,j}\big(\tfrac{1-\sigma_2}{\lambda}\big)\Big(\log t + B\big(\tfrac{1-\sigma_2}{\lambda}, \tfrac{1-\sigma_1}{\lambda}\big)\Big)^j\Big) \\
&\qquad\qquad \times A\big(\tfrac{1-\sigma_2}{\lambda}, \tfrac{1-\sigma_1}{\lambda}\big)t^{\frac{1-\sigma_1}{\lambda}-\frac{1}{2}}e^{-t\frac{\pi}{2}}\sum_{i=l}^{k-1}\frac{|b_{-i-1}|}{(i-l)!}\Big(\log\frac{1}{\delta}\Big)^{i-l}\Bigg)^2 dt\Bigg\}\Bigg)^{\frac{1}{2}}.
\end{aligned} \tag{6.36}$$

Letting $T\to\infty$, we have by 2° that

$$\Big(\sum_{n=1}^{\infty}\frac{|a_n|^2}{n^{2\sigma}}e^{-2(n\delta)^{\lambda}}\Big)^{\frac{1}{2}}$$

$$\le \sup_{T\ge 2}\sup_{\sigma_1\le\sigma\le\sigma_2}\Big(\frac{1}{T}\int_2^T |f(\sigma+\sqrt{-1}t)|^2dt\Big)^{\frac{1}{2}}$$
$$+\bigg(\Big(\frac{\delta^{\sigma_1-\alpha}}{2\pi}\Big)^2 2\lambda\Big(C\big(\tfrac{\alpha-\sigma_2}{\lambda},\tfrac{\alpha-\sigma_1}{\lambda}\big)+\frac{2}{\pi}A\big(\tfrac{\alpha-\sigma_2}{\lambda},\tfrac{\alpha-\sigma_1}{\lambda}\big)\Big)$$
$$\times\bigg(A\big(\tfrac{\alpha-\sigma_2}{\lambda},\tfrac{\alpha-\sigma_1}{\lambda}\big)C_1^2\int_{-\infty}^{\infty}e^{-|v|\frac{\pi}{4\lambda}}\big(2+|v|\big)^{2C}dv$$
$$+\Big(C\big(\tfrac{\alpha-\sigma_2}{\lambda},\tfrac{\alpha-\sigma_1}{\lambda}\big)+\frac{2}{\pi}A\big(\tfrac{\alpha-\sigma_2}{\lambda},\tfrac{\alpha-\sigma_1}{\lambda}\big)\Big)C_2 2\lambda\Big(2+\frac{\lambda}{2}\Big)\bigg)\bigg)^{\frac{1}{2}}.$$

Since $\lim_{\delta\searrow 0}\delta^{\sigma_1-\alpha}=0$ by $\sigma_1-\alpha>0$, it follows from the monotone convergence theorem that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{|a_n|^2}{n^{2\sigma}} &= \sum_{n=1}^{\infty}\lim_{\delta\searrow 0}(\nearrow)\frac{|a_n|^2}{n^{2\sigma}}e^{-2(n\delta)^\lambda}\\ &=\lim_{\delta\searrow 0}\sum_{n=1}^{\infty}\frac{|a_n|^2}{n^{2\sigma}}e^{-2(n\delta)^\lambda}\\ &\le \sup_{T\ge 2}\sup_{\sigma_1\le\sigma\le\sigma_2}\frac{1}{T}\int_2^T|f(\sigma+\sqrt{-1}t)|^2dt\\ &<\infty,\end{aligned}$$

which is the assertion (i) of the theorem.

<u>7°</u> First, by (6.34), (6.35) and (6.36),

$$\sup_{\sigma_1\le\sigma\le\sigma_2}\bigg|\frac{1}{T}\int_2^T|f(\sigma+\sqrt{-1}t)|^2dt-\frac{1}{T}\int_2^T\Big|\sum_{n=1}^{\infty}\frac{a_n}{n^{\sigma+\sqrt{-1}t}}e^{-(n\delta)^\lambda}\Big|^2dt\bigg|$$
$$=\sup_{\sigma_1\le\sigma\le\sigma_2}\bigg|\Big(\frac{1}{T}\int_2^T|f(\sigma+\sqrt{-1}t)|^2dt\Big)^{\frac{1}{2}}-\Big(\frac{1}{T}\int_2^T\Big|\sum_{n=1}^{\infty}\frac{a_n}{n^{\sigma+\sqrt{-1}t}}e^{-(n\delta)^\lambda}\Big|^2dt\Big)^{\frac{1}{2}}\bigg|$$
$$\times\bigg(\Big(\frac{1}{T}\int_2^T|f(\sigma+\sqrt{-1}t)|^2dt\Big)^{\frac{1}{2}}+\Big(\frac{1}{T}\int_2^T\Big|\sum_{n=1}^{\infty}\frac{a_n}{n^{\sigma+\sqrt{-1}t}}e^{-(n\delta)^\lambda}\Big|^2dt\Big)^{\frac{1}{2}}\bigg)$$
$$\le\sup_{\sigma_1\le\sigma\le\sigma_2}\bigg|\Big(\frac{1}{T}\int_2^T|f(\sigma+\sqrt{-1}t)|^2dt\Big)^{\frac{1}{2}}-\Big(\frac{1}{T}\int_2^T\Big|\sum_{n=1}^{\infty}\frac{a_n}{n^{\sigma+\sqrt{-1}t}}e^{-(n\delta)^\lambda}\Big|^2dt\Big)^{\frac{1}{2}}\bigg|$$
$$\times\bigg(\sup_{T\ge 2}\sup_{\sigma_1\le\sigma\le\sigma_2}\Big(\frac{1}{T}\int_2^T|f(\sigma+\sqrt{-1}t)|^2dt\Big)^{\frac{1}{2}}$$
$$+\sup_{\sigma_1\le\sigma\le\sigma_2}\Big(\frac{1}{T}\int_2^T\Big|\sum_{n=1}^{\infty}\frac{a_n}{n^{\sigma+\sqrt{-1}t}}e^{-(n\delta)^\lambda}\Big|^2dt\Big)^{\frac{1}{2}}\bigg)\underset{\substack{\text{first } T\to\infty\\ \text{second } \delta\searrow 0}}{\to}0.$$

Next, by 2° and the assertion (i),

$$\sup_{\sigma_1\le\sigma\le\sigma_2}\bigg|\frac{1}{T}\int_2^T\Big|\sum_{n=1}^{\infty}\frac{a_n}{n^{\sigma+\sqrt{-1}t}}e^{-(n\delta)^\lambda}\Big|^2dt-\sum_{n=1}^{\infty}\frac{|a_n|^2}{n^{2\sigma}}\bigg|$$

$$= \sup_{\sigma_1 \le \sigma \le \sigma_2} \left| \frac{1}{T} \int_2^T \left| \sum_{n=1}^{\infty} \frac{a_n}{n^{\sigma+\sqrt{-1}t}} e^{-(n\delta)^\lambda} \right|^2 dt - \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma}} e^{-2(n\delta)^\lambda} \right.$$
$$\left. - \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma}} \left(1 - e^{-2(n\delta)^\lambda}\right) \right|$$
$$\le \sup_{\sigma_1 \le \sigma \le \sigma_2} \left| \frac{1}{T} \int_2^T \left| \sum_{n=1}^{\infty} \frac{a_n}{n^{\sigma+\sqrt{-1}t}} e^{-(n\delta)^\lambda} \right|^2 dt - \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma}} e^{-2(n\delta)^\lambda} \right|$$
$$+ \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma_1}} \left(1 - e^{-2(n\delta)^\lambda}\right) \underset{\substack{\text{first } T \to \infty \\ \text{second } \delta \searrow 0}}{\longrightarrow} 0.$$

Therefore

$$\sup_{\sigma_1 \le \sigma \le \sigma_2} \left| \frac{1}{T} \int_2^T |f(\sigma + \sqrt{-1}t)|^2 dt - \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma}} \right| \to 0 \quad \text{as } T \to \infty. \qquad \blacksquare$$

6.4 Proof of Claim 5.5

For $N \in \mathbb{N}$, put

$$f_N(s) := \frac{\zeta}{\zeta_N}(s) - 1, \quad \operatorname{Re} s > 0.$$

f_N is meromorphic on $\{s \in \mathbb{C}; \operatorname{Re} s > 0\}$ and is holomorphic except $s = 1$ which is a simple pole of f_N. By 1° in the proof of Claim 4.3,

$$f_N(s) = \sum_{\substack{n \ge 2; \\ p_1 \nmid n, \dots, p_N \nmid n}} \frac{1}{n^s}, \quad \operatorname{Re} s > 1.$$

This tells us that $f_N(\cdot)$ is expanded in a Dirichlet series on $\{s \in \mathbb{C}; \operatorname{Re} s > 1\}$ and its convergence is absolute.

Clearly, for $\operatorname{Re} s \ge \frac{1}{2}$, $s \ne 1$,

$$|f_N(s)| = \left| \zeta(s) \prod_{i=1}^{N} \left(1 - \frac{1}{p_i^s}\right) - 1 \right| \le |\zeta(s)| \prod_{i=1}^{N} \left(1 + \frac{1}{p_i^{\operatorname{Re} s}}\right) + 1$$
$$\le |\zeta(s)| \prod_{i=1}^{N} \left(1 + \frac{1}{p_i^{\frac{1}{2}}}\right) + 1. \tag{6.37}$$

Lemma 6.2, together with this, gives that for $\operatorname{Re} s \ge \frac{1}{2}$, $|s - 1| \ge \frac{1}{3}$,

$$|f_N(s)| \le \left(\frac{41}{12} + \frac{1}{24}\left(\frac{5}{2} + |\operatorname{Im} s|\right) + \frac{1}{24}\left(\frac{5}{2} + |\operatorname{Im} s|\right)^3 \right) \prod_{i=1}^{N} \left(1 + \frac{1}{p_i^{\frac{1}{2}}}\right) + 1,$$

from which, it is easily seen that

$$\sup \left\{ \frac{|f_N(s)|}{(2 + |\operatorname{Im} s|)^3}; \operatorname{Re} s \ge \frac{1}{2},\ |s - 1| \ge \frac{1}{3} \right\} < \infty.$$

For $\frac{1}{2} < \alpha < 1$, Claim 6.3 gives that

$$\begin{aligned}
&\int_{1\le|t|\le T} |\zeta(\alpha+\sqrt{-1}t)|^2 dt \\
&= \int_1^T |\zeta(\alpha+\sqrt{-1}t)|^2 dt + \int_{-T}^{-1} |\zeta(\alpha+\sqrt{-1}t)|^2 dt \\
&= \int_1^T \bigl(|\zeta(\alpha+\sqrt{-1}t)|^2 + |\zeta(\alpha-\sqrt{-1}t)|^2\bigr) dt \\
&= \int_1^T \bigl(|\zeta(\alpha+\sqrt{-1}t)|^2 + \bigl|\zeta\bigl(\overline{\alpha+\sqrt{-1}t}\bigr)\bigr|^2\bigr) dt \\
&= 2\int_1^T |\zeta(\alpha+\sqrt{-1}t)|^2 dt \quad \bigl[\text{cf. Remark 4.1}\bigr] \\
&\sim 2T\zeta(2\alpha) \quad \text{as } T\to\infty.
\end{aligned}$$

By (6.37), this implies that

$$\begin{aligned}
&\int_{-T}^{T} |f_N(\alpha+\sqrt{-1}t)|^2 dt \\
&\le \int_{-T}^{T} \Bigl(|\zeta(\alpha+\sqrt{-1}t)| \prod_{i=1}^{N}\Bigl(1+\frac{1}{p_i^{\frac{1}{2}}}\Bigr) + 1\Bigr)^2 dt \\
&\le 2\int_{-T}^{T} |\zeta(\alpha+\sqrt{-1}t)|^2 dt \prod_{i=1}^{N}\Bigl(1+\frac{1}{p_i^{\frac{1}{2}}}\Bigr)^2 + 4T \\
&= 2\Bigl(\int_{1\le|t|\le T} |\zeta(\alpha+\sqrt{-1}t)|^2 dt + \int_{|t|\le 1} |\zeta(\alpha+\sqrt{-1}t)|^2 dt\Bigr) \\
&\qquad \times \prod_{i=1}^{N}\Bigl(1+\frac{1}{p_i^{\frac{1}{2}}}\Bigr)^2 + 4T \\
&= 2\cdot 2T\bigl(\zeta(2\alpha)+o(1)\bigr)\prod_{i=1}^{N}\Bigl(1+\frac{1}{p_i^{\frac{1}{2}}}\Bigr)^2 + 4T \\
&= O(T) \quad \text{as } T\to\infty.
\end{aligned}$$

By putting all together, it turns out that $f_N(\cdot)$ satisfies the assumptions in Theorem 6.3. We thus apply this theorem to have that for $\frac{1}{2} < {}^\forall\alpha < 1$, ${}^\forall N \in \mathbb{N}$ and $\alpha < {}^\forall\sigma_1 < {}^\forall\sigma_2 < \infty$,

$$\sup_{\sigma_1\le\sigma\le\sigma_2}\Bigl|\frac{1}{T}\int_2^T |f_N(\sigma+\sqrt{-1}t)|^2 dt - \sum_{\substack{n\ge 2;\\ p_1\nmid n,\ldots,p_N\nmid n}} \frac{1}{n^{2\sigma}}\Bigr| \to 0 \quad \text{as } T\to\infty.$$

Now, fix $\frac{1}{2} < {}^\forall\alpha < 1$ and ${}^\forall\eta > 0$. Since $n > p_N$ provided $n \ge 2$ and $p_1 \nmid n, \ldots, p_N \nmid n$,

$$\sum_{\substack{n\ge 2;\\ p_1\nmid n,\ldots,p_N\nmid n}} \frac{1}{n^{2\alpha}} \le \sum_{n>p_N} \frac{1}{n^{2\alpha}}.$$

Take $N_1(\alpha,\eta) \in \mathbb{N}$ so that

$$\sum_{n > p_{N_1(\alpha,\eta)}} \frac{1}{n^{2\alpha}} < \frac{\eta}{4}. \tag{6.38}$$

From what we saw above, it follows that for $^\forall N \geq N_1(\alpha,\eta)$, $\alpha < {}^\forall\sigma_1 < {}^\forall\sigma_2 < \infty$,

$$\begin{aligned}
&{}^\exists T_1(N,\alpha,\sigma_1,\sigma_2,\eta) \geq 2\\
&\text{s.t.} \quad \sup_{\sigma_1 \leq \sigma \leq \sigma_2} \left| \frac{1}{T}\int_2^T |f_N(\sigma+\sqrt{-1}t)|^2 dt - \sum_{\substack{n\geq 2;\\ p_1\nmid n,\ldots,p_N\nmid n}} \frac{1}{n^{2\sigma}} \right| < \frac{\eta}{4},\\
&\qquad\qquad\qquad\qquad\qquad\qquad {}^\forall T \geq T_1(N,\alpha,\sigma_1,\sigma_2,\eta).
\end{aligned}$$

In conjunction with (6.38), we see that for $^\forall T \geq T_1(N,\alpha,\sigma_1,\sigma_2,\eta)$

$$\begin{aligned}
&\sup_{\sigma_1 \leq \sigma \leq \sigma_2} \frac{1}{T}\int_2^T |f_N(\sigma+\sqrt{-1}t)|^2 dt\\
&= \sup_{\sigma_1 \leq \sigma \leq \sigma_2} \frac{1}{T}\left(\int_2^T |f_N(\sigma+\sqrt{-1}t)|^2 dt - \sum_{\substack{n\geq 2;\\ p_1\nmid n,\ldots,p_N\nmid n}} \frac{1}{n^{2\sigma}} + \sum_{\substack{n\geq 2;\\ p_1\nmid n,\ldots,p_N\nmid n}} \frac{1}{n^{2\sigma}} \right)\\
&\leq \sup_{\sigma_1 \leq \sigma \leq \sigma_2} \left| \frac{1}{T}\int_2^T |f_N(\sigma+\sqrt{-1}t)|^2 dt - \sum_{\substack{n\geq 2;\\ p_1\nmid n,\ldots,p_N\nmid n}} \frac{1}{n^{2\sigma}} \right| + \sum_{\substack{n\geq 2;\\ p_1\nmid n,\ldots,p_N\nmid n}} \frac{1}{n^{2\alpha}}\\
&< \frac{\eta}{4} + \frac{\eta}{4} = \frac{\eta}{2}.
\end{aligned}$$

On the other hand, since

$$\begin{aligned}
\sup_{\sigma_1 \leq \sigma \leq \sigma_2} \frac{1}{T}\int_{\frac{1}{2}}^2 |f_N(\sigma+\sqrt{-1}t)|^2 dt &\leq \frac{1}{T}\int_{\frac{1}{2}}^2 \max_{\sigma_1 \leq \sigma \leq \sigma_2} |f_N(\sigma+\sqrt{-1}t)|^2 dt\\
&\to 0 \quad \text{as } T \to \infty,
\end{aligned}$$

$^\exists T_2(N,\sigma_1,\sigma_2,\eta) \geq 2$ s.t.

$$\sup_{\sigma_1 \leq \sigma \leq \sigma_2} \frac{1}{T}\int_{\frac{1}{2}}^2 |f_N(\sigma+\sqrt{-1}t)|^2 dt < \frac{\eta}{2}, \quad {}^\forall T \geq T_2(N,\sigma_1,\sigma_2,\eta).$$

Combining these, we have that for $^\forall T \geq T_1(N,\alpha,\sigma_1,\sigma_2,\eta) \vee T_2(N,\sigma_1,\sigma_2,\eta)$,

$$\begin{aligned}
&\sup_{\sigma_1 \leq \sigma \leq \sigma_2} \frac{1}{T}\int_{\frac{1}{2}}^T |f_N(\sigma+\sqrt{-1}t)|^2 dt\\
&= \sup_{\sigma_1 \leq \sigma \leq \sigma_2} \left(\frac{1}{T}\int_{\frac{1}{2}}^2 |f_N(\sigma+\sqrt{-1}t)|^2 dt + \frac{1}{T}\int_2^T |f_N(\sigma+\sqrt{-1}t)|^2 dt \right)\\
&\leq \sup_{\sigma_1 \leq \sigma \leq \sigma_2} \frac{1}{T}\int_{\frac{1}{2}}^2 |f_N(\sigma+\sqrt{-1}t)|^2 dt + \sup_{\sigma_1 \leq \sigma \leq \sigma_2} \frac{1}{T}\int_2^T |f_N(\sigma+\sqrt{-1}t)|^2 dt
\end{aligned}$$

$$< \frac{\eta}{2} + \frac{\eta}{2} = \eta.$$

Let $\frac{1}{2} < \sigma_0 \le 1 < \sigma_1$ and $0 < \delta < \sigma_0 - \frac{1}{2}$. In the argument above, let $\alpha = \frac{\frac{1}{2}+\sigma_0-\delta}{2}$, $\sigma_1 = \sigma_0 - \delta$ and $\sigma_2 = \sigma_1 + \delta$, and put

$$\begin{aligned}
N_0(\sigma_0, \delta, \eta) &:= N_1\Big(\frac{\frac{1}{2} + \sigma_0 - \delta}{2}, \eta\Big) \quad \text{for } \eta > 0, \\
T_0(N, \sigma_0, \sigma_1, \delta, \eta) &:= T_1\Big(N, \frac{\frac{1}{2} + \sigma_0 - \delta}{2}, \sigma_0 - \delta, \sigma_1 + \delta, \eta\Big) \\
&\qquad \vee T_2\Big(N, \sigma_0 - \delta, \sigma_1 + \delta, \eta\Big) \quad \text{for } N \ge N_0(\sigma_0, \delta, \eta).
\end{aligned}$$

Then we obtain

$$\int_{\frac{1}{2}}^{T} \Big|\frac{\zeta}{\zeta_N}(\sigma + \sqrt{-1}t) - 1\Big|^2 dt = \int_{\frac{1}{2}}^{T} |f_N(\sigma + \sqrt{-1}t)|^2 dt \le \eta T,$$
$$^{\forall}T \ge T_0(N, \sigma_0, \sigma_1, \delta, \eta),\ \sigma_0 - \delta \le {}^{\forall}\sigma \le \sigma_1 + \delta,$$

which is the assertion of the claim.