Chapter 1

Almost periodic functions

Following Yosida [35], let us view the almost periodic function due to H. Bohr. For English literatures, cf. Besicovitch [1], Bohr [3], Katznelson [19] and so on. It should be noted that the almost periodic function in the sense of Bohr is generalized to the one in the sense of Besicovitch [cf. [1]], and moreover to the one on locally compact abelian groups [cf. [19, 28]]. Our proofs given below are a freshman or sophomore level in college except for using Bochner's theorem in Lemma 1.2.

1.1 Definition and some properties

First of all, put

$$C(\mathbb{R};\mathbb{C}) := \{f : \mathbb{R} \to \mathbb{C} \text{ is continuous}\},$$
$$C(\mathbb{C};\mathbb{C}) := \{f : \mathbb{C} \to \mathbb{C} \text{ is continuous}\}.$$

Definition 1.1 For $f \in C(\mathbb{R};\mathbb{C})$,

$$f \text{ is an } \textit{almost periodic function}$$
$$\underset{\text{def}}{\iff} {}^\forall\varepsilon > 0,\ {}^\exists l = l(\varepsilon) > 0 \text{ s.t. } \begin{cases} {}^\forall\alpha \in \mathbb{R},\ {}^\exists\tau \in (\alpha, \alpha + l) \\ \text{s.t. } \sup\limits_{t\in\mathbb{R}} |f(t+\tau) - f(t)| \le \varepsilon. \end{cases}$$

Definition 1.2 For $\varepsilon > 0$ and $f \in C(\mathbb{R};\mathbb{C})$, put

$$A(\varepsilon, f) := \left\{\tau \in \mathbb{R}; \sup_{t\in\mathbb{R}} |f(t+\tau) - f(t)| \le \varepsilon\right\}.$$

By this, the appearance of Definition 1.1 becomes simple as

$$f \text{ is almost periodic}$$
$$\underset{\text{def}}{\iff} {}^\forall\varepsilon > 0,\ {}^\exists l = l(\varepsilon) > 0 \text{ s.t. } (\alpha, \alpha + l) \cap A(\varepsilon, f) \ne \emptyset \ \ ({}^\forall\alpha \in \mathbb{R}).$$

Example 1.1 If $f \in C(\mathbb{R};\mathbb{C})$ has period $a > 0$, i.e., satisfies that $f(t + a) = f(t)$ for ${}^\forall t \in \mathbb{R}$, then f is almost periodic.

Proof. For $\alpha \in \mathbb{R}$, the following implications hold:

$$\begin{aligned} m := \left\lfloor \frac{\alpha}{a} \right\rfloor + 1 \Rightarrow m - 1 &= \left\lfloor \frac{\alpha}{a} \right\rfloor \\ &\Rightarrow m - 1 \le \frac{\alpha}{a} < m \\ &\Rightarrow ma - a \le \alpha < ma \\ &\Rightarrow ma \le \alpha + a,\ \alpha < ma \\ &\Rightarrow \alpha < ma \le \alpha + a \\ &\Rightarrow ma \in (\alpha, \alpha + a]. \end{aligned}$$

Since

$$\sup_{t\in\mathbb{R}} |f(t + ma) - f(t)| = \sup_{t\in\mathbb{R}} |f(t) - f(t)| = 0,$$

we have $(\alpha, \alpha + a] \cap A(\varepsilon, f) \neq \emptyset,\ {}^\forall\varepsilon > 0.$ ■

Definition 1.3 $AP(\mathbb{R})$:= the set of all almost periodic functions.

Claim 1.1 *For* $f \in AP(\mathbb{R})$*,*
(i) $\sup_{t\in\mathbb{R}} |f(t)| < \infty$,
(ii) f *is uniformly continuous on* $\mathbb{R}$*,*
(iii) $\varphi \circ f \in AP(\mathbb{R}),\ {}^\forall\varphi \in C(\mathbb{C};\mathbb{C})$.

Proof. Fix $f \in AP(\mathbb{R})$.
(i) For $\varepsilon = 1$,

$${}^\exists l = l(1) > 0 \text{ s.t. } \begin{cases} {}^\forall\alpha \in \mathbb{R},\ {}^\exists\tau \in (\alpha - l, \alpha) \\ \text{s.t. } |f(t + \tau) - f(t)| \le 1 \ \ ({}^\forall t \in \mathbb{R}). \end{cases}$$

Since $|f(\alpha - \tau + \tau) - f(\alpha - \tau)| \le 1$ with $t = \alpha - \tau$,

$$|f(\alpha)| \le 1 + |f(\alpha - \tau)| \le 1 + \max_{0\le t\le l} |f(t)|.$$

Thus

$$\sup_{\alpha\in\mathbb{R}} |f(\alpha)| \le 1 + \max_{0\le t\le l} |f(t)|.$$

(ii) For ${}^\forall\varepsilon > 0$,

$${}^\exists l = l(\varepsilon) > 0 \text{ s.t. } \begin{cases} {}^\forall\alpha \in \mathbb{R},\ {}^\exists\tau \in (\alpha, \alpha + l) \\ \text{s.t. } |f(t + \tau) - f(t)| \le \varepsilon \ \ ({}^\forall t \in \mathbb{R}). \end{cases}$$

Since f is uniformly continuous over $[-l, l + 1]$,

$$0 < {}^\exists\delta < 1 \text{ s.t. } \begin{cases} s, t \in [-l, l + 1],\ |s - t| \le \delta \\ \Rightarrow |f(s) - f(t)| \le \varepsilon. \end{cases}$$

Let $s, t \in \mathbb{R}$ with $|s - t| \le \delta$. Without loss of generality, we may assume that $s \le t$. For $n = \lfloor \frac{s}{l} \rfloor \in \mathbb{Z}$, we take $\tau \in (nl, nl + l)$ so that $|f(u + \tau) - f(u)| \le \varepsilon$ $({}^\forall u \in \mathbb{R})$. Then

$$|f(s) - f(s - \tau)| \le \varepsilon,\ |f(t) - f(t - \tau)| \le \varepsilon.$$

Since $nl \le s < (n+1)l$,

$$nl \le s \le t \le s + \delta < s + 1 < (n+1)l + 1 = nl + l + 1.$$

From this and $-nl - l < -\tau < -nl$, it is clear that $-l < s - \tau \le t - \tau < l + 1$, so that

$$|f(s-\tau) - f(t-\tau)| \le \varepsilon.$$

Therefore, putting three estimates together, we have

$$\begin{aligned} |f(s) - f(t)| &= |f(s) - f(s-\tau) + f(s-\tau) + f(t-\tau) + f(t-\tau) - f(t)| \\ &\le |f(s) - f(s-\tau)| + |f(s-\tau) + f(t-\tau)| + |f(t-\tau) - f(t)| \\ &\le 3\varepsilon. \end{aligned}$$

(iii) By (i), we can take $M > 0$ such that $\sup_{s\in\mathbb{R}} |f(s)| \le M$. Since φ is uniformly continuous over $[-M, M]$,

$$^\forall\varepsilon > 0,\ ^\exists\delta > 0 \ \text{ s.t. } \begin{cases} u, v \in [-M, M],\ |u - v| \le \delta \\ \Rightarrow |\varphi(u) - \varphi(v)| \le \varepsilon. \end{cases}$$

For this $\delta > 0$,

$$^\exists l > 0 \ \text{ s.t. } \begin{cases} ^\forall\alpha \in \mathbb{R},\ ^\exists\tau \in (\alpha, \alpha + l) \\ \text{ s.t. } |f(t+\tau) - f(t)| \le \delta \ \ (^\forall t \in \mathbb{R}). \end{cases}$$

Then, since $f(t+\tau), f(t) \in [-M, M]$, we have

$$\big|\varphi\big(f(t+\tau)\big) - \varphi\big(f(t)\big)\big| \le \varepsilon.$$

This shows that $(\alpha, \alpha + l) \cap A(\varepsilon, \varphi \circ f) \ne \emptyset$. ■

Claim 1.2 *For $f \in C(\mathbb{R}; \mathbb{C})$,*

$$f \in AP(\mathbb{R}) \underset{\text{iff}}{\Longleftrightarrow} \begin{array}{l} \textit{for every real sequence } \{a_n\}_{n=1}^\infty,\ ^\exists\{n_k\}\textit{: a subsequence s.t.} \\ f(\cdot + a_{n_k}) \textit{ is uniformly convergent on } \mathbb{R} \textit{ as } k \to \infty. \end{array}$$

Proof. "$\Rightarrow$" We divide the proof into two steps.

1° For $^\forall f \in AP(\mathbb{R})$, $^\forall\varepsilon > 0$ and $^\forall$real sequence $\{a_n\}_{n=1}^\infty$,

$$^\exists\{n_k\}_{k=1}^\infty\text{: a subsequence, } ^\exists\gamma \in \mathbb{R} \ \text{ s.t. } \varlimsup_{k\to\infty} \sup_{t\in\mathbb{R}} \big|f(t + a_{n_k}) - f(t+\gamma)\big| \le \varepsilon.$$

∵ Fix $f \in AP(\mathbb{R})$, $\varepsilon > 0$ and real sequence $\{a_n\}_{n=1}^\infty$. Take $l > 0$ so that $(\alpha, \alpha + l) \cap A(\varepsilon, f) \ne \emptyset$ $(^\forall\alpha \in \mathbb{R})$. Since $a_n \in \mathbb{R} = \sum_{m\in\mathbb{Z}} \big[ml, (m+1)l\big)$,

$$^\exists m_n \in \mathbb{Z} \ \text{ s.t. } m_n l \le a_n < m_n l + l.$$

For $\alpha = m_n l$, let $\tau_n \in (m_n l, m_n l + l)$ be such that $\sup_{t\in\mathbb{R}} |f(t + \tau_n) - f(t)| \le \varepsilon$. Then

$$|f(t + a_n) - f(t + a_n - \tau_n)| \le \varepsilon \ \ (^\forall t \in \mathbb{R}).$$

Since $-l < a_n - \tau_n < l$ by $-m_n l - l < -\tau_n < -m_n l$, it follows that $^\exists\{n_k\}_{k=1}^\infty$: a subsequence, $^\exists\gamma \in \mathbb{R}$ s.t. $a_{n_k} - \tau_{n_k} \to \gamma$ as $k \to \infty$. By Claim 1.1(ii), $f(\cdot + a_{n_k} - \tau_{n_k}) \rightrightarrows f(\cdot + \gamma)$ as $k \to \infty$, and thus

$$\varlimsup_{k\to\infty} \sup_{t\in\mathbb{R}} |f(t + a_{n_k}) - f(t+\gamma)| \le \varepsilon.$$

2° Let $f \in AP(\mathbb{R})$. Suppose there exists a real sequence $\{a_n\}_{n=1}^{\infty}$ such that for any subsequence $\{n_k\}_{k=1}^{\infty}$, $f(\cdot + a_{n_k})$ is not uniformly convergent on $\mathbb{R}$ as $k \to \infty$. Since the whole sequence $f(\cdot + a_n)$ is not uniformly convergent on $\mathbb{R}$,

$$
{}^{\exists}\varepsilon_0 > 0 \ \text{ s.t. } \begin{cases} {}^{\forall}N \in \mathbb{N},\ {}^{\exists}n, {}^{\exists}m > N \\ \text{s.t. } \sup\limits_{t \in \mathbb{R}} |f(t + a_n) - f(t + a_m)| > \varepsilon_0. \end{cases} \tag{1.1}
$$

By 1°,

$$
{}^{\exists}\{n_k\}_{k=1}^{\infty}\text{: a subsequence, } {}^{\exists}\gamma \in \mathbb{R} \ \text{ s.t. } \ \varlimsup_{k\to\infty} \sup_{t\in\mathbb{R}} \big| f(t + a_{n_k}) - f(t + \gamma) \big| \le \frac{\varepsilon_0}{3}.
$$

Then

$$
\begin{aligned}
&\varlimsup_{k,l\to\infty} \sup_{t\in\mathbb{R}} \big| f(t + a_{n_k}) - f(t + a_{n_l}) \big| \\
&\le \varlimsup_{k,l\to\infty} \Big(\sup_{t\in\mathbb{R}} \big| f(t + a_{n_k}) - f(t + \gamma) \big| + \sup_{t\in\mathbb{R}} \big| f(t + a_{n_l}) - f(t + \gamma) \big| \Big) \\
&\le \frac{2\varepsilon_0}{3}.
\end{aligned}
$$

This contradicts (1.1).

"⇐" We show the contraposition. Let $f \in C(\mathbb{R};\mathbb{C}) \setminus AP(\mathbb{R})$. By definition,

$$
{}^{\exists}\varepsilon_0 > 0 \ \text{ s.t. } \begin{cases} {}^{\forall}l > 0,\ {}^{\exists}\alpha < {}^{\exists}\beta \\ \text{s.t. } \beta - \alpha = l,\ (\alpha, \beta) \cap A(\varepsilon_0, f) = \emptyset. \end{cases}
$$

Fix ${}^{\forall}a \in \mathbb{R}$, and put $a_1 = a$. For $l_1 > 2|a_1|$,

$$
{}^{\exists}\alpha_1 < {}^{\exists}\beta_1 \ \text{ s.t. } \begin{cases} \beta_1 - \alpha_1 = l_1, \\ (\alpha_1, \beta_1) \cap A(\varepsilon_0, f) = \emptyset. \end{cases}
$$

Put $a_2 = \frac{\alpha_1 + \beta_1}{2}$. Then

$$
\begin{aligned}
a_2 - a_1 - \alpha_1 &= \frac{\alpha_1 + \beta_1}{2} - \alpha_1 - a_1 = \frac{\beta_1 - \alpha_1}{2} - a_1 \ge \frac{l_1}{2} - |a_1| > 0, \\
\beta_1 - (a_2 - a_1) &= \beta_1 - \frac{\alpha_1 + \beta_1}{2} + a_1 = \frac{\beta_1 - \alpha_1}{2} + a_1 \ge \frac{l_1}{2} - |a_1| > 0,
\end{aligned}
$$

so $a_2 - a_1 \in (\alpha_1, \beta_1)$. Next, for $l_2 > 2\big(|a_1| + |a_2|\big)$,

$$
{}^{\exists}\alpha_2 < {}^{\exists}\beta_2 \ \text{ s.t. } \begin{cases} \beta_2 - \alpha_2 = l_2, \\ (\alpha_2, \beta_2) \cap A(\varepsilon_0, f) = \emptyset. \end{cases}
$$

Put $a_3 = \frac{\alpha_2 + \beta_2}{2}$. Then, for $i = 1, 2$,

$$
\begin{aligned}
a_3 - a_i - \alpha_2 &= \frac{\alpha_2 + \beta_2}{2} - \alpha_2 - a_i = \frac{\beta_2 - \alpha_2}{2} - a_i \ge \frac{l_2}{2} - |a_i| > 0, \\
\beta_2 - (a_3 - a_i) &= \beta_2 - \frac{\alpha_2 + \beta_2}{2} + a_i = \frac{\beta_2 - \alpha_2}{2} + a_i \ge \frac{l_2}{2} - |a_i| > 0,
\end{aligned}
$$

so $a_3 - a_1, a_3 - a_2 \in (\alpha_2, \beta_2)$. Repeat this process and let us get $a_1, \ldots, a_k$. For $l_k > 2\big(|a_1| + \cdots + |a_k|\big)$,

$$^{\exists}\alpha_k < {}^{\exists}\beta_k \ \text{ s.t. } \begin{cases} \beta_k - \alpha_k = l_k, \\ (\alpha_k, \beta_k) \cap A(\varepsilon_0, f) = \emptyset. \end{cases}$$

Put $a_{k+1} = \frac{\alpha_k + \beta_k}{2}$. Then, for $i = 1, \ldots, k$,

$$a_{k+1} - a_i - \alpha_k = \frac{\alpha_k + \beta_k}{2} - \alpha_k - a_i = \frac{\beta_k - \alpha_k}{2} - a_i \geq \frac{l_k}{2} - |a_i| > 0,$$
$$\beta_k - (a_{k+1} - a_i) = \beta_k - \frac{\alpha_k + \beta_k}{2} + a_i = \frac{\beta_k - \alpha_k}{2} + a_i \geq \frac{l_k}{2} - |a_i| > 0,$$

so $a_{k+1} - a_1, \ldots, a_{k+1} - a_k \in (\alpha_k, \beta_k)$. Thus we see the existence of a real sequence $\{a_n\}_{n=1}^{\infty}$ satisfying $a_i - a_j \notin A(\varepsilon_0, f)$ $(i > j \geq 1)$.

Now

$$\sup_{t \in \mathbb{R}} |f(t + a_i) - f(t + a_j)| = \sup_{t \in \mathbb{R}} |f(t + a_i - a_j) - f(t)| \geq \varepsilon_0 \ \ (i > j \geq 1).$$

This tells us that $f(\cdot + a_{n_k})$ is not uniformly convergent on $\mathbb{R}$ as $k \to \infty$ for any subsequence $\{n_k\}_{k=1}^{\infty}$. ∎

Claim 1.3 *$AP(\mathbb{R})$ is a self-adjoint algebra of complex continuous functions on $\mathbb{R}$. Namely, for $f, g \in AP(\mathbb{R})$ and $\xi, \eta \in \mathbb{C}$,*

$$\xi f + \eta g, \ fg, \ \overline{f} \in AP(\mathbb{R}).$$

Here $\overline{f}(t) := \overline{f(t)}$. Moreover, for $\alpha \in \mathbb{R} \setminus \{0\}$ and $\beta \in \mathbb{R}$, $f(\alpha t + \beta) \in AP(\mathbb{R})$ provided that $f \in AP(\mathbb{R})$.

Proof. Let $f, g \in AP(\mathbb{R})$, $\alpha \in \mathbb{R} \setminus \{0\}$ and $\beta \in \mathbb{R}$. For any real sequence $\{a_n\}_{n=1}^{\infty}$,

$$^{\exists}\{n_k\}\text{: a subsequence s.t. } \begin{cases} f(\cdot + a_{n_k}),\ f(\cdot + \alpha a_{n_k} + \beta) \text{ and } g(\cdot + a_{n_k}) \text{ are uniformly} \\ \text{convergent on } \mathbb{R} \text{ as } k \to \infty, \text{ respectively.} \end{cases}$$

Then, so are $\xi f(\cdot + a_{n_k}) + \eta g(\cdot + a_{n_k})$ $(\xi, \eta \in \mathbb{C})$, $f(\cdot + a_{n_k}) g(\cdot + a_{n_k})$, $\overline{f(\cdot + a_{n_k})}$ and $f\big(\alpha(\cdot + a_{n_k}) + \beta\big)$. Thus, by Claim 1.2, we have the assertion of the claim. ∎

Claim 1.4 *Let $\{f_k\}_{k=1}^{\infty} \subset AP(\mathbb{R})$ and $f \in C(\mathbb{R}; \mathbb{C})$. If $f_k \rightrightarrows f$ as $k \to \infty$, then $f \in AP(\mathbb{R})$.*

Proof. Take a real sequence $\{a_n\}_{n=1}^{\infty}$ arbitrarily and fix it. By Claim 1.2,

$$^{\exists}\{n(1, j)\}_{j=1}^{\infty}\text{: a subsequence of } \{1, 2, 3, \ldots\}$$
$$\text{s.t. } f_1(\cdot + a_{n(1,j)}) \text{ is uniformly convergent on } \mathbb{R} \text{ as } j \to \infty,$$
$$^{\exists}\{n(2, j)\}_{j=1}^{\infty}\text{: a subsequence of } \{n(1, j)\}_{j=1}^{\infty}$$
$$\text{s.t. } f_2(\cdot + a_{n(2,j)}) \text{ is uniformly convergent on } \mathbb{R} \text{ as } j \to \infty,$$
$$\vdots$$

$$\begin{aligned}&{}^{\exists}\{n(i,j)\}_{j=1}^{\infty}\text{: a subsequence of } \{n(i-1,j)\}_{j=1}^{\infty}\\ &\quad \text{s.t. } f_i(\cdot + a_{n(i,j)}) \text{ is uniformly convergent on } \mathbb{R} \text{ as } j\to\infty,\\ &\qquad\vdots\end{aligned}$$

Put $n_k := n(k,k)$ $(k \geq 1)$. $\{n_k\}_{k=1}^{\infty}$ is a subsequence of $\{1,2,3,\ldots\}$, and $f(\cdot + a_{n_k})$ is uniformly convergent on $\mathbb{R}$ as $k\to\infty$. For if we take $\{J_k(i)\}_{i=1}^{\infty}$ so that $n(k+1,i) = n(k, J_k(i))$ $(i = 1,2,\ldots)$, and if, for $k,l > j$, we rewrite

$$\begin{aligned} n(k,k) &= n(k-1, J_{k-1}(k))\\ &= n\big(k-2, J_{k-2}(J_{k-1}(k))\big)\\ &= \cdots\\ &= n\big(j, J_j(J_{j+1}(\cdots(J_{k-1}(k))\cdots))\big),\\ n(l,l) &= n\big(j, J_j(J_{j+1}(\cdots(J_{l-1}(l))\cdots))\big), \end{aligned}$$

then

$$\begin{aligned} \sup_{t\in\mathbb{R}}\big|f(t+a_{n_k}) - f(t+a_{n_l})\big| &= \sup_{t\in\mathbb{R}}\Big|f\Big(t+a_{n\left(j,J_j(J_{j+1}(\cdots(J_{k-1}(k))\cdots))\right)}\Big)\\ &\qquad - f\Big(t+a_{n\left(j,J_j(J_{j+1}(\cdots(J_{l-1}(l))\cdots))\right)}\Big)\Big|\\ &= \sup_{t\in\mathbb{R}}\Big|f\Big(t+a_{n\left(j,J_j(J_{j+1}(\cdots(J_{k-1}(k))\cdots))\right)}\Big)\\ &\qquad - f_j\Big(t+a_{n\left(j,J_j(J_{j+1}(\cdots(J_{k-1}(k))\cdots))\right)}\Big)\\ &\qquad + f_j\Big(t+a_{n\left(j,J_j(J_{j+1}(\cdots(J_{k-1}(k))\cdots))\right)}\Big)\\ &\qquad - f_j\Big(t+a_{n\left(j,J_j(J_{j+1}(\cdots(J_{l-1}(l))\cdots))\right)}\Big)\\ &\qquad + f_j\Big(t+a_{n\left(j,J_j(J_{j+1}(\cdots(J_{l-1}(l))\cdots))\right)}\Big)\\ &\qquad - f\Big(t+a_{n\left(j,J_j(J_{j+1}(\cdots(J_{l-1}(l))\cdots))\right)}\Big)\Big|\\ &\leq 2\sup_{s\in\mathbb{R}}|f(s) - f_j(s)|\\ &\quad + \sup_{h\geq k, i\geq l}\sup_{t\in\mathbb{R}}\big|f_j(t+a_{n(j,h)}) - f_j(t+a_{n(j,i)})\big|\\ &\quad \left[\because\ \begin{array}{l} J_j(J_{j+1}(\cdots(J_{k-1}(k))\cdots)) \geq k,\\ J_j(J_{j+1}(\cdots(J_{l-1}(l))\cdots)) \geq l \end{array}\right]. \end{aligned}$$

Since the 2nd term $\to 0$ as $k,l\to\infty$, and the 1st term $\to 0$ as $j\to\infty$, we have $f(\cdot + a_{n_k}) - f(\cdot + a_{n_l}) \rightrightarrows 0$ as $k,l\to\infty$. ■

1.2 Mean values

Theorem 1.1 *For each $f \in AP(\mathbb{R})$,*

$$\frac{1}{T}\int_0^T f(t)\,dt$$

is convergent as $T \to \infty$. *If we denote this limit by* $M(f)$, *then*

$$\frac{1}{T}\int_a^{T+a} f(t)dt$$

is uniformly convergent in $a \in \mathbb{R}$ *to* $M(f)$.

Proof. Fix $^\forall \varepsilon > 0$.

$$^\exists l = l(\varepsilon) > 0 \ \text{ s.t. } \ (\alpha, \alpha + l) \cap A(\varepsilon, f) \neq \emptyset \ \ (^\forall \alpha \in \mathbb{R}).$$

Let $T_0 := l\big((\frac{1}{\varepsilon}\sup_{t\in\mathbb{R}}|f(t)|) \vee 1\big)$, and $T \geq T_0$. For $^\forall a \in \mathbb{R}$,

$$^\exists \tau \in (a, a+l) \ \text{ s.t. } \ \sup_{t\in\mathbb{R}}|f(t+\tau) - f(t)| \leq \varepsilon.$$

Then

$$\begin{aligned}
&\Big|\frac{1}{T}\int_a^{T+a} f(t)dt - \frac{1}{T}\int_0^T f(t)dt\Big| \\
&= \frac{1}{T}\Big|\int_a^{\tau} f(t)dt + \int_{\tau}^{T+\tau} f(t)dt - \int_{T+a}^{T+\tau} f(t)dt - \int_0^T f(t)dt\Big| \\
&= \frac{1}{T}\Big|\int_a^{\tau} f(t)dt + \int_0^T \big(f(t+\tau) - f(t)\big)dt - \int_{T+a}^{T+\tau} f(t)dt\Big| \\
&\leq \frac{1}{T}\Big(\int_a^{\tau} |f(t)|dt + \int_0^T \big|f(t+\tau) - f(t)\big|dt + \int_{T+a}^{T+\tau} |f(t)|dt\Big) \\
&\leq \frac{1}{T}\Big(2(\tau - a)\sup_{t\in\mathbb{R}}|f(t)| + T\varepsilon\Big) \\
&\leq \frac{2l}{T}\sup_{t\in\mathbb{R}}|f(t)| + \varepsilon \quad \big[\because\ 0 < \tau - a < l\big] \\
&\leq 3\varepsilon \quad \big[\because\ T \geq T_0 \geq \tfrac{l}{\varepsilon}\sup_{t\in\mathbb{R}}|f(t)| \Rightarrow \tfrac{2l}{T}\sup_{t\in\mathbb{R}}|f(t)| \leq 2\varepsilon\big].
\end{aligned} \tag{1.2}$$

Particularly, putting $a = (k-1)T$ $(k = 1, \ldots, n)$ in the above, we see that

$$\Big|\frac{1}{T}\int_{(k-1)T}^{kT} f(t)dt - \frac{1}{T}\int_0^T f(t)dt\Big| \leq 3\varepsilon.$$

Adding this inequality over k, and then multiplying it by $\frac{1}{n}$, we have

$$\begin{aligned}
&\Big|\frac{1}{nT}\int_0^{nT} f(t)dt - \frac{1}{T}\int_0^T f(t)dt\Big| \\
&= \Big|\frac{1}{n}\sum_{k=1}^{n}\Big(\frac{1}{T}\int_{(k-1)T}^{kT} f(t)dt - \frac{1}{T}\int_0^T f(t)dt\Big)\Big| \\
&\leq \frac{1}{n}\sum_{k=1}^{n}\Big|\frac{1}{T}\int_{(k-1)T}^{kT} f(t)dt - \frac{1}{T}\int_0^T f(t)dt\Big| \\
&\leq 3\varepsilon.
\end{aligned}$$

For $T_1, T_2 \geq T_0$, put

$$T_i^{(m)} := \frac{\lceil mT_i \rceil}{m} \quad (i = 1, 2).$$

Clearly $T_i^{(m)} \in \mathbb{Q}$, $T_i^{(m)} \geq T_i$ and $\lim_{m\to\infty} T_i^{(m)} = T_i$. By noting that $\frac{T_2^{(m)}}{T_1^{(m)}} = \frac{n_2}{n_1}$ for some $n_1, n_2 \in \mathbb{N}$, the estimate above yields that

$$\begin{aligned}
&\Big|\frac{1}{T_1^{(m)}}\int_0^{T_1^{(m)}} f(t)dt - \frac{1}{T_2^{(m)}}\int_0^{T_2^{(m)}} f(t)dt\Big| \\
&= \Big|\frac{1}{T_1^{(m)}}\int_0^{T_1^{(m)}} f(t)dt - \frac{1}{n_2T_1^{(m)}}\int_0^{n_2T_1^{(m)}} f(t)dt \\
&\quad + \frac{1}{n_1T_2^{(m)}}\int_0^{n_1T_2^{(m)}} f(t)dt - \frac{1}{T_2^{(m)}}\int_0^{T_2^{(m)}} f(t)dt\Big| \\
&\leq \Big|\frac{1}{T_1^{(m)}}\int_0^{T_1^{(m)}} f(t)dt - \frac{1}{n_2T_1^{(m)}}\int_0^{n_2T_1^{(m)}} f(t)dt\Big| \\
&\quad + \Big|\frac{1}{n_1T_2^{(m)}}\int_0^{n_1T_2^{(m)}} f(t)dt - \frac{1}{T_2^{(m)}}\int_0^{T_2^{(m)}} f(t)dt\Big| \\
&\leq 6\varepsilon.
\end{aligned}$$

Letting $m \to \infty$, we have

$$\Big|\frac{1}{T_1}\int_0^{T_1} f(t)dt - \frac{1}{T_2}\int_0^{T_2} f(t)dt\Big| \leq 6\varepsilon.$$

This shows the convergence of $\frac{1}{T}\int_0^T f(t)dt$ as $T \to \infty$. Moreover, from (1.2), it follows that $\frac{1}{T}\int_a^{T+a} f(t)dt$ is uniformly convergent in $a \in \mathbb{R}$ to this limit. ■

Definition 1.4 We call $M(f)$ the *mean value* of f.

Claim 1.5 *Let $f \in AP(\mathbb{R})$ and $f \geq 0$. Then*

$$M(f) = 0 \underset{\textit{iff}}{\Longleftrightarrow} f = 0.$$

Proof. "$\Leftarrow$" is trivial.

"$\Rightarrow$" We show the contraposition. Suppose $f \neq 0$. Then $^\exists t_0 \in \mathbb{R}$, $^\exists\varepsilon > 0$ s.t. $f(t_0) \geq 2\varepsilon$. By the continuity of f, $^\exists a < t_0 < {}^\exists b$ s.t. $f(t) \geq \varepsilon$ for $t \in (a, b)$. By the almost periodicity of f, $^\exists l = l(\frac{\varepsilon}{2}) > 0$ s.t. $(\alpha, \alpha + l) \cap A(\frac{\varepsilon}{2}, f) \neq \emptyset$ $(^\forall\alpha \in \mathbb{R})$. Put $L := l \vee (b - a)$. For each $n \in \mathbb{N}$,

$$^\exists\tau_n \in \big(2(n-1)L - a, 2(n-1)L - a + L\big) \text{ s.t. } \sup_{t\in\mathbb{R}} |f(t + \tau_n) - f(t)| \leq \frac{\varepsilon}{2}.$$

Noting that, for $^\forall t \in (a, b)$,

$$f(t + \tau_n) = f(t) + f(t + \tau_n) - f(t) \geq f(t) - |f(t + \tau_n) - f(t)|$$

$$\geq \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2},$$

we see that

$$\int_{2(n-1)L}^{2nL} f(s)ds = \int_{2(n-1)L-\tau_n}^{2nL-\tau_n} f(t+\tau_n)dt \quad [\because \text{ change of variable: } s - \tau_n = t]$$

$$\geq \int_a^b f(t+\tau_n)dt$$

$$\left[\begin{array}{l} \because \text{ by } 2(n-1)L - a < \tau_n < 2(n-1)L - a + L, \\ \qquad 2(n-1)L - \tau_n < a, \\ \qquad 2nL - \tau_n > 2nL - 2(n-1)L + a - L \\ \qquad\qquad = L + a \\ \qquad\qquad \geq b - a + a = b \end{array}\right]$$

$$\geq (b-a)\frac{\varepsilon}{2}.$$

Adding this inequality over $n \in \{1, \ldots, N\}$, we have

$$\frac{1}{2NL}\int_0^{2NL} f(s)ds = \frac{1}{2NL}\sum_{n=1}^{N}\int_{2(n-1)L}^{2nL} f(s)ds \geq \frac{1}{2NL}\sum_{n=1}^{N}(b-a)\frac{\varepsilon}{2}$$
$$= \frac{\varepsilon(b-a)}{4L}.$$

Finally, letting $N \to \infty$, we obtain

$$M(f) = \lim_{N\to\infty}\frac{1}{2NL}\int_0^{2NL} f(s)ds \geq \frac{\varepsilon(b-a)}{4L} > 0. \qquad \blacksquare$$

1.3 Convolutions

For $f, g \in AP(\mathbb{R})$ and $u \in \mathbb{R}$, $f(u-\cdot)g(\cdot) \in AP(\mathbb{R})$ by Claim 1.3. Thus, by Theorem 1.1,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(u-t)g(t)dt = M\big(f(u-\cdot)g(\cdot)\big).$$

Moreover the following holds:

Claim 1.6 (i) *As* $T \to \infty$,

$$\frac{1}{T}\int_a^{T+a} f(u-t)g(t)dt \to M\big(f(u-\cdot)g(\cdot)\big) \quad \textit{uniformly in } a, u \in \mathbb{R}.$$

(ii) $\mathbb{R} \ni u \mapsto M\big(f(u-\cdot)g(\cdot)\big) \in \mathbb{C}$ *is almost periodic.*

Proof. (i) For $\varepsilon > 0$, put

$$A(\varepsilon, f, g) := \Big\{\tau \in \mathbb{R}; \sup_{t\in\mathbb{R}}\big(|f(-t-\tau) - f(-t)| + |g(t+\tau) - g(t)|\big) \leq \varepsilon\Big\}.$$

We divide the proof into two steps:

<u>1°</u> $^\forall\varepsilon > 0, {}^\exists l = l(\varepsilon) > 0$ s.t. $(a, a+l) \cap A(\varepsilon, f, g) \neq \emptyset$ $(^\forall a \in \mathbb{R})$.

∵ We prove it by a reduction to absurdity. Its negation is as follows:

$$^{\exists}\varepsilon_0 > 0 \ \text{ s.t. } \begin{cases} ^{\forall}l > 0, \ ^{\exists}\alpha < {}^{\exists}\beta \\ \text{s.t. } \ \beta - \alpha = l, \ (\alpha, \beta) \cap A(\varepsilon_0, f, g) = \emptyset. \end{cases}$$

We do the similar argument as in the proof of "⇐" in Claim 1.2. Fix $^{\forall}a \in \mathbb{R}$, and put $a_1 = a$. For $l_1 > 2|a_1|$,

$$^{\exists}\alpha_1 < {}^{\exists}\beta_1 \ \text{ s.t. } \begin{cases} \beta_1 - \alpha_1 = l_1, \\ (\alpha_1, \beta_1) \cap A(\varepsilon_0, f, g) = \emptyset. \end{cases}$$

Then, for $a_2 = \frac{\alpha_1+\beta_1}{2}$, $a_2 - a_1 \in (\alpha_1, \beta_1)$. Next, for $l_2 > 2\big(|a_1| + |a_2|\big)$,

$$^{\exists}\alpha_2 < {}^{\exists}\beta_2 \ \text{ s.t. } \begin{cases} \beta_2 - \alpha_2 = l_2, \\ (\alpha_2, \beta_2) \cap A(\varepsilon_0, f, g) = \emptyset. \end{cases}$$

Then, for $a_3 = \frac{\alpha_2+\beta_2}{2}$, $a_3 - a_1, a_3 - a_2 \in (\alpha_2, \beta_2)$. We repeat this process to find a real sequence $\{a_n\}_{n=1}^{\infty}$ such that $a_i - a_j \notin A(\varepsilon_0, f, g)$ $(i > j \geq 1)$. Then

$$\begin{aligned} &\sup_{t\in\mathbb{R}}\big(|f(-t-a_i+a_j) - f(-t)| + |g(t+a_i-a_j) - g(t)|\big) > \varepsilon_0 \\ &\qquad\qquad\qquad\qquad\qquad \| \\ &\sup_{t\in\mathbb{R}}\big(|f(-t-a_i) - f(-t-a_j)| + |g(t+a_i) - g(t+a_j)|\big) \ \ (i \neq j). \end{aligned}$$

On the other hand, since $f, g \in AP(\mathbb{R})$, it follows from Claim 1.2 that

$$^{\exists}\{n_k\}_{k=1}^{\infty}\text{: a subsequence } \text{ s.t. } \begin{cases} f(\cdot - a_{n_k}) \text{ and } g(\cdot + a_{n_k}) \text{ are uniformly con-} \\ \text{vergent on } \mathbb{R} \text{ as } k \to \infty \text{, respectively.} \end{cases}$$

This implies that

$$\sup_{t\in\mathbb{R}}\big(|f(-t-a_{n_k}) - f(-t-a_{n_l})| + |g(t+a_{n_k}) - g(t+a_{n_l})|\big) < \varepsilon_0 \ \ (k, l \gg 1),$$

which contradicts the above estimate. Therefore the assertion of 1° must hold.

<u>2°</u> Fix $^{\forall}\varepsilon > 0$. By 1°,

$$^{\exists}l = l(\varepsilon) > 0 \ \text{ s.t. } \begin{cases} ^{\forall}a \in \mathbb{R}, \ ^{\exists}\tau \in (a, a+l) \\ \text{s.t. } \ \sup_{t\in\mathbb{R}}\Big(\big(\sup_{s\in\mathbb{R}}|g(s)|\big)|f(-t-\tau) - f(-t)| \\ \qquad\qquad + \big(\sup_{s\in\mathbb{R}}|f(s)|\big)|g(t+\tau) - g(t)|\Big) \leq \varepsilon. \end{cases}$$

Let $T_0 := l\big(\frac{1}{\varepsilon}(\sup_{s\in\mathbb{R}}|f(s)|)(\sup_{s\in\mathbb{R}}|g(s)|) \vee 1\big)$, and $T \geq T_0$. Fix $^{\forall}u \in \mathbb{R}$. If, for $^{\forall}a \in \mathbb{R}$, $\tau \in (a, a+l)$ is taken as above,

$$\begin{aligned} &\Big|\frac{1}{T}\int_a^{T+a} f(u-t)g(t)dt - \frac{1}{T}\int_0^T f(u-t)g(t)dt\Big| \\ &= \frac{1}{T}\Big|\int_a^{\tau} f(u-t)g(t)dt + \int_{\tau}^{T+\tau} f(u-t)g(t)dt \end{aligned}$$

$$
\begin{aligned}
&\quad - \int_{T+a}^{T+\tau} f(u-t)g(t)dt - \int_0^T f(u-t)g(t)dt \Big| \\
&= \frac{1}{T}\Big| \int_a^\tau f(u-t)g(t)dt \\
&\quad + \int_0^T \big(f(u-t-\tau)g(t+\tau) - f(u-t)g(t)\big)dt \\
&\quad - \int_{T+a}^{T+\tau} f(u-t)g(t)dt \Big| \\
&\le \frac{1}{T}\Big(\int_a^\tau |f(u-t)|\,|g(t)|dt \\
&\quad + \int_0^T \big|\big(f(u-t-\tau) - f(u-t)\big)g(t+\tau) \\
&\qquad + f(u-t)\big(g(t+\tau) - g(t)\big)\big| dt \\
&\quad + \int_{T+a}^{T+\tau} |f(u-t)|\,|g(t)|dt \Big) \\
&\le \frac{1}{T}\Bigg(2(\tau - a)\big(\sup_{s\in\mathbb{R}} |f(s)|\big)\big(\sup_{s\in\mathbb{R}} |g(s)|\big) \\
&\quad + \int_0^T \Big(\big(\sup_{s\in\mathbb{R}} |g(s)|\big)\big| f\big(-(t-u)-\tau\big) - f\big(-(t-u)\big)\big| \\
&\qquad + \big(\sup_{s\in\mathbb{R}} |f(s)|\big)\big| g(t+\tau) - g(t)\big|\Big) dt \Bigg) \\
&\le \frac{1}{T}\Big(2l\big(\sup_{s\in\mathbb{R}} |f(s)|\big)\big(\sup_{s\in\mathbb{R}} |g(s)|\big) + 2\varepsilon T \Big) \\
&\le 4\varepsilon. \qquad (1.3)
\end{aligned}
$$

For $n \in \mathbb{N}$,

$$
\begin{aligned}
&\Big| \frac{1}{nT} \int_0^{nT} f(u-t)g(t)dt - \frac{1}{T}\int_0^T f(u-t)g(t)dt \Big| \\
&= \Big| \frac{1}{n} \sum_{k=1}^{n} \Big(\frac{1}{T}\int_{(k-1)T}^{kT} f(u-t)g(t)dt - \frac{1}{T}\int_0^T f(u-t)g(t)dt \Big) \Big| \\
&\le \frac{1}{n} \sum_{k=1}^{n} \Big| \frac{1}{T}\int_{(k-1)T}^{kT} f(u-t)g(t)dt - \frac{1}{T}\int_0^T f(u-t)g(t)dt \Big| \\
&\le 4\varepsilon.
\end{aligned}
$$

Since, for $T_1, T_2 \ge T_0$, $T_i^{(m)} = \frac{\lceil mT_i \rceil}{m}$ $(i = 1, 2)$ satisfy that $\frac{T_2^{(m)}}{T_1^{(m)}} = \frac{n_2}{n_1}$ for some n_1, $n_2 \in \mathbb{N}$,

$$
\begin{aligned}
&\Big| \frac{1}{T_1^{(m)}} \int_0^{T_1^{(m)}} f(u-t)g(t)dt - \frac{1}{T_2^{(m)}} \int_0^{T_2^{(m)}} f(u-t)g(t)dt \Big| \\
&= \Big| \frac{1}{T_1^{(m)}} \int_0^{T_1^{(m)}} f(u-t)g(t)dt - \frac{1}{n_2 T_1^{(m)}} \int_0^{n_2 T_1^{(m)}} f(u-t)g(t)dt
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{n_1 T_2^{(m)}} \int_0^{n_1 T_2^{(m)}} f(u-t)g(t)dt - \frac{1}{T_2^{(m)}} \int_0^{T_2^{(m)}} f(u-t)g(t)dt \Big| \\
&\le \Big| \frac{1}{T_1^{(m)}} \int_0^{T_1^{(m)}} f(u-t)g(t)dt - \frac{1}{n_2 T_1^{(m)}} \int_0^{n_2 T_1^{(m)}} f(u-t)g(t)dt \Big| \\
&+ \Big| \frac{1}{n_1 T_2^{(m)}} \int_0^{n_1 T_2^{(m)}} f(u-t)g(t)dt - \frac{1}{T_2^{(m)}} \int_0^{T_2^{(m)}} f(u-t)g(t)dt \Big| \\
&\le 8\varepsilon.
\end{aligned}
$$

Letting $m \to \infty$, we see that

$$
\Big| \frac{1}{T_1} \int_0^{T_1} f(u-t)g(t)dt - \frac{1}{T_2} \int_0^{T_2} f(u-t)g(t)dt \Big| \le 8\varepsilon.
$$

This implies that

$$
\Big| \frac{1}{T} \int_0^{T} f(u-t)g(t)dt - M\big(f(u-\cdot)g(\cdot)\big) \Big| \le 8\varepsilon, \quad {}^\forall T \ge T_0, {}^\forall u \in \mathbb{R}.
$$

Combining this with (1.3), we have

$$
\Big| \frac{1}{T} \int_a^{T+a} f(u-t)g(t)dt - M\big(f(u-\cdot)g(\cdot)\big) \Big| \le 12\varepsilon, \quad {}^\forall T \ge T_0, {}^\forall u, {}^\forall a \in \mathbb{R}. \tag{1.4}
$$

(ii) First, for fixed $T > 0$,

$$
\mathbb{R} \ni u \mapsto \frac{1}{T} \int_0^{T} f(u-t)g(t)dt \in \mathbb{C}
$$

is almost periodic. For it follows from Claim 1.2 that for any real sequence $\{a_n\}_{n=1}^\infty$,

$$
{}^\exists\{n_k\}_{k=1}^\infty\text{: a subsequence, } {}^\exists h \in C(\mathbb{R};\mathbb{C}) \text{ s.t. } f(\cdot + a_{n_k}) \rightrightarrows h(\cdot) \text{ as } k \to \infty.
$$

Then, uniformly in u

$$
\frac{1}{T} \int_0^{T} f(u + a_{n_k} - t)g(t)dt \to \frac{1}{T} \int_0^{T} h(u-t)g(t)dt \quad \text{as } k \to \infty.
$$

By Claim 1.2 again, the function in question is almost periodic.

By (i), uniformly in u

$$
\frac{1}{T} \int_0^{T} f(u-t)g(t)dt \to M\big(f(u-\cdot)g(\cdot)\big) \quad \text{as } T \to \infty.
$$

The assertion of (ii) follows from Claim 1.4 at once. ■

Definition 1.5 We denote $M\big(f(u-\cdot)g(\cdot)\big)$ by $(f \otimes g)(u)$, and call $f \otimes g \in AP(\mathbb{R})$ the *convolution* of f and g.

Claim 1.7 *For $f, g, h \in AP(\mathbb{R})$ and $\xi \in \mathbb{C}$,*
(i) $f \otimes g = g \otimes f$,
(ii) $f \otimes (g + h) = f \otimes g + f \otimes h$, $(\xi f) \otimes g = f \otimes (\xi g) = \xi(f \otimes g)$,
(iii) $f \otimes (g \otimes h) = (f \otimes g) \otimes h$.

Proof. Fix $f, g, h \in AP(\mathbb{R})$ and $\xi \in \mathbb{C}$.
(i) First

$$\begin{aligned}(f \otimes g)(u) &= \lim_{T\to\infty} \frac{1}{T} \int_0^T f(u-t)g(t)dt \\ &= \lim_{T\to\infty} \frac{1}{T} \int_{u-T}^u f(s)g(u-s)ds \quad \big[\because \text{ change of variable: } u-t=s\big].\end{aligned}$$

Since, by (1.4),

$$\Big|\frac{1}{T}\int_{u-T}^u g(u-s)f(s)ds - (g\otimes f)(u)\Big| \le 12\varepsilon, \quad {}^\forall T \ge T_0, {}^\forall u \in \mathbb{R},$$

we see that $(f \otimes g)(u) = (g \otimes f)(u)$.
(ii)

$$\begin{aligned}\big(f \otimes (g+h)\big)(u) &= \lim_{T\to\infty} \int_0^T f(u-t)\big(g(t)+h(t)\big)dt \\ &= \lim_{T\to\infty} \int_0^T f(u-t)g(t)dt + \lim_{T\to\infty} \int_0^T f(u-t)h(t)dt \\ &= (f\otimes g)(u) + (f \otimes h)(u) \\ &= (f\otimes g + f\otimes h)(u),\end{aligned}$$

$$\begin{aligned}\big((\xi f)\otimes g\big)(u) &= \lim_{T\to\infty} \int_0^T \xi f(u-t)g(t)dt \\ &= \xi \lim_{T\to\infty} \int_0^T f(u-t)g(t)dt \\ &= \xi (f\otimes g)(u) \\ &= \big(\xi(f\otimes g)\big)(u).\end{aligned}$$

(iii) First

$$\begin{aligned}\big(f \otimes (g\otimes h)\big)(u) &= \lim_{T\to\infty} \frac{1}{T}\int_0^T f(u-t)(g\otimes h)(t)dt \\ &= \lim_{T\to\infty} \frac{1}{T}\int_0^T f(u-t)dt \lim_{S\to\infty}\frac{1}{S}\int_0^S g(t-r)h(r)dr \\ &= \lim_{T\to\infty}\lim_{S\to\infty} \frac{1}{T}\frac{1}{S}\int_0^T f(u-t)dt \int_0^S g(t-r)h(r)dr \\ &= \lim_{T\to\infty}\lim_{S\to\infty} \frac{1}{T}\frac{1}{S}\int_0^S h(r)dr \int_0^T f(u-t)g(t-r)dt \\ &= \lim_{T\to\infty}\lim_{S\to\infty} \frac{1}{T}\frac{1}{S}\int_0^S h(r)dr \int_{-r}^{T-r} f(u-r-\tau)g(\tau)d\tau \\ &\qquad \big[\because \text{ change of variable: } t-r=\tau\big] \\ &= \lim_{T\to\infty}\lim_{S\to\infty} \frac{1}{S}\int_0^S \Big(\frac{1}{T}\int_{-r}^{T-r} f(u-r-\tau)g(\tau)d\tau\Big)h(r)dr.\end{aligned}$$

By (1.4),

$$\left|\frac{1}{T}\int_{-r}^{T-r} f(u-r-\tau)g(\tau)d\tau - (f\otimes g)(u-r)\right| \le 12\varepsilon, \quad {}^{\forall}T \ge T_0, {}^{\forall}u, {}^{\forall}r \in \mathbb{R},$$

so

$$\begin{aligned}
&\left|\frac{1}{S}\int_0^S \Big(\frac{1}{T}\int_{-r}^{T-r} f(u-r-\tau)g(\tau)d\tau\Big)h(r)dr - \frac{1}{S}\int_0^S (f\otimes g)(u-r)h(r)dr\right| \\
&= \left|\frac{1}{S}\int_0^S \Big(\frac{1}{T}\int_{-r}^{T-r} f(u-r-\tau)g(\tau)d\tau - (f\otimes g)(u-r)\Big)h(r)dr\right| \\
&\le 12\varepsilon \sup_{r\in\mathbb{R}} |h(r)|, \quad {}^{\forall}T \ge T_0, {}^{\forall}S > 0.
\end{aligned}$$

Letting $S \to \infty$, and then $T \to \infty$, we obtain

$$\Big|\big(f\otimes(g\otimes h)\big)(u) - \big((f\otimes g)\otimes h\big)(u)\Big| \le 12\varepsilon \sup_{r\in\mathbb{R}} |h(r)|,$$

which shows that $\big(f\otimes(g\otimes h)\big)(u) = \big((f\otimes g)\otimes h\big)(u)$. ■

1.4 Approximation theorem

Definition 1.6 For $P \in C(\mathbb{R};\mathbb{C})$,

P is a *trigonometric polynomial*

$$\underset{\text{def}}{\Longleftrightarrow} {}^{\exists}k \in \mathbb{N}, {}^{\exists}\xi_1, \dots, {}^{\exists}\xi_k \in \mathbb{C}, {}^{\exists}\lambda_1, \dots, {}^{\exists}\lambda_k \in \mathbb{R} \text{ s.t. } P(t) = \sum_{i=1}^{k} \xi_i e^{\sqrt{-1}\lambda_i t}.$$

Note that by Example 1.1 and Claim 1.3, trigonometric polynomials are almost periodic.

Theorem 1.2 *For* ${}^{\forall}f \in AP(\mathbb{R})$ *and* ${}^{\forall}\varepsilon > 0$, ${}^{\exists}P_\varepsilon$*: a trigonometric polynomial s.t.* $\sup_{t\in\mathbb{R}} |f(t) - P_\varepsilon(t)| \le \varepsilon$.

For the proof, we present two lemmas:

Lemma 1.1 *For* $f \in AP(\mathbb{R})$*, put*

$$g(t) := \sup_{s\in\mathbb{R}} |f(s+t) - f(s)|, \quad t \in \mathbb{R}.$$

(i) $g \in AP(\mathbb{R})$*,* ≥ 0.
(ii) *Put* $F_\varepsilon(t) := \big(1 - \frac{g(t)}{\varepsilon}\big)^+\big(1 - \frac{g(-t)}{\varepsilon}\big)^+$ *for* $\varepsilon > 0$*. Then* $F_\varepsilon \in AP(\mathbb{R})$*,* ≥ 0 *and* $F_\varepsilon(0) = 1$*. Thus*

$$\psi_\varepsilon = \frac{F_\varepsilon}{M(F_\varepsilon)} \in AP(\mathbb{R})$$

is defined, it is an even function and

$$\sup_{t\in\mathbb{R}} \big|f(t) - (f\otimes\psi_\varepsilon\otimes\psi_\varepsilon)(t)\big| \le 2\varepsilon \tag{1.5}$$

holds.

Proof. (i) For $s, t, t' \in \mathbb{R}$,

$$\begin{aligned}|f(s+t) - f(s)| &= |f(s+t') - f(s) + f(s+t) - f(s+t')| \\ &\le |f(s+t') - f(s)| + |f(s+t) - f(s+t')| \\ &\le g(t') + \sup_{r\in\mathbb{R}} |f(r+t'-t) - f(r)|, \\ |f(s+t') - f(s)| &\le |f(s+t) - f(s)| + |f(s+t') - f(s+t)| \\ &\le g(t) + \sup_{r\in\mathbb{R}} |f(r+t'-t) - f(r)|.\end{aligned}$$

Taking the sup in s yields that

$$|g(t') - g(t)| \le \sup_{r\in\mathbb{R}} |f(r+t'-t) - f(r)|, \quad t, t' \in \mathbb{R}.$$

From the uniform continuity of f, it follows that $g \in C(\mathbb{R}; [0,\infty))$. Next, by the almost periodicity of f,

$$^\forall\varepsilon > 0,\ ^\exists l = l(\varepsilon) > 0 \text{ s.t. } \begin{cases} ^\forall\alpha \in \mathbb{R},\ ^\exists\tau \in (\alpha, \alpha + l) \\ \text{s.t. } \sup_{s\in\mathbb{R}} |f(s+\tau) - f(s)| \le \varepsilon. \end{cases}$$

Putting $t' = t + \tau$ in the estimate above yields that

$$|g(t+\tau) - g(t)| \le \sup_{r\in\mathbb{R}} |f(r+\tau) - f(r)| \le \varepsilon, \quad ^\forall t \in \mathbb{R}.$$

This implies that $g \in AP(\mathbb{R})$.
(ii) By Claim 1.3, $g(-\cdot) \in AP(\mathbb{R})$. By the continuity of $\mathbb{R} \ni u \mapsto \left(1 - \frac{u}{\varepsilon}\right)^+ \in [0,\infty)$ and Claim 1.1(iii), $\left(1 - \frac{g(\cdot)}{\varepsilon}\right)^+$, $\left(1 - \frac{g(-\cdot)}{\varepsilon}\right)^+ \in AP(\mathbb{R})$. Thus $F_\varepsilon \in AP(\mathbb{R})$ by Claim 1.3 once more. Clearly $F_\varepsilon \ge 0$ and $F_\varepsilon(0) = 1$. Since $M(F_\varepsilon) > 0$ by Claim 1.5,

$$\psi_\varepsilon = \frac{F_\varepsilon}{M(F_\varepsilon)} \in AP(\mathbb{R})$$

is well-defined. From this definition, it is clear that ψ_ε is even.

It remains to show (1.5). We first note that

$$\left|(f(s-t) - f(s))F_\varepsilon(t)\right| \le \varepsilon F_\varepsilon(t), \quad ^\forall s, ^\forall t \in \mathbb{R}. \tag{1.6}$$

This is seen from the following implications:

$$\begin{aligned}F_\varepsilon(t) > 0 &\Rightarrow \left(1 - \frac{g(-t)}{\varepsilon}\right)^+ > 0 \\ &\Rightarrow g(-t) < \varepsilon \\ &\Rightarrow |f(s-t) - f(s)| < \varepsilon \ (^\forall s \in \mathbb{R}) \\ &\Rightarrow \left|(f(s-t) - f(s))F_\varepsilon(t)\right| = |f(s-t) - f(s)|F_\varepsilon(t) \\ &\qquad\qquad < \varepsilon F_\varepsilon(t) \ (^\forall s \in \mathbb{R}).\end{aligned}$$

Integrating (1.6) in $t \in [0, T]$, we see that

$$\left|\frac{1}{T}\int_0^T f(s-t)F_\varepsilon(t)dt - \frac{f(s)}{T}\int_0^T F_\varepsilon(t)dt\right|$$

$$
\begin{aligned}
&= \left| \frac{1}{T} \int_0^T \big(f(s-t) - f(s)\big) F_\varepsilon(t) dt \right| \\
&\le \frac{1}{T} \int_0^T \big|\big(f(s-t) - f(s)\big) F_\varepsilon(t)\big| dt \\
&\le \frac{\varepsilon}{T} \int_0^T F_\varepsilon(t) dt.
\end{aligned}
$$

Letting $T \to \infty$, and then dividing both sides by $M(F_\varepsilon)$, we have

$$
\big|(f \otimes \psi_\varepsilon)(s) - f(s)\big| \le \varepsilon, \quad {}^\forall s \in \mathbb{R}. \tag{1.7}
$$

Now, since $M(\psi_\varepsilon) = 1$, it follows that

$$
\begin{aligned}
&\big|(f \otimes \psi_\varepsilon)(s-t) - (f \otimes \psi_\varepsilon)(s)\big| \\
&= \lim_{T\to\infty} \left| \frac{1}{T} \int_0^T f(s-t-r)\psi_\varepsilon(r) dr - \frac{1}{T} \int_0^T f(s-r)\psi_\varepsilon(r) dr \right| \\
&= \lim_{T\to\infty} \left| \frac{1}{T} \int_0^T \big(f(s-t-r) - f(s-r)\big)\psi_\varepsilon(r) dr \right| \\
&\le \lim_{T\to\infty} \frac{1}{T} \int_0^T \big|f(s-t-r) - f(s-r)\big| \psi_\varepsilon(r) dr \\
&\le \Big(\sup_{\sigma\in\mathbb{R}} \big|f(\sigma-t) - f(\sigma)\big|\Big) \lim_{T\to\infty} \frac{1}{T} \int_0^T \psi_\varepsilon(r) dr \\
&= \Big(\sup_{\sigma\in\mathbb{R}} \big|f(\sigma-t) - f(\sigma)\big|\Big) M(\psi_\varepsilon) \\
&= \sup_{\sigma\in\mathbb{R}} \big|f(\sigma-t) - f(\sigma)\big|.
\end{aligned}
$$

By this and (1.6),

$$
\Big|\big((f \otimes \psi_\varepsilon)(s-t) - (f \otimes \psi_\varepsilon)(s)\big) F_\varepsilon(t)\Big| \le \varepsilon F_\varepsilon(t), \quad {}^\forall s, {}^\forall t \in \mathbb{R}.
$$

By a similar calculation from (1.6) to (1.7),

$$
\big|(f \otimes \psi_\varepsilon) \otimes \psi_\varepsilon(s) - (f \otimes \psi_\varepsilon)(s)\big| \le \varepsilon, \quad {}^\forall s \in \mathbb{R}.
$$

Finally, combining this estimate with (1.7), we have

$$
\big|f(s) - (f \otimes \psi_\varepsilon) \otimes \psi_\varepsilon(s)\big| \le 2\varepsilon, \quad {}^\forall s \in \mathbb{R},
$$

which is just (1.5). ■

Lemma 1.2 *For $f \in AP(\mathbb{R})$, $\neq 0$, put $f^*(t) := \overline{f(-t)} \in AP(\mathbb{R})$. Then*

$$
{}^\exists\{\lambda_n\}\text{: a real sequence, } {}^\exists\{\gamma_n\}\text{: a positive sequence}
$$

$$
\text{s.t.} \quad \begin{cases} \displaystyle\sum_n \gamma_n < \infty, \\ \displaystyle (f \otimes f^*)(t) = \sum_n \gamma_n e^{\sqrt{-1}\lambda_n t} \quad (t \in \mathbb{R}). \end{cases}
$$

Here $\{\lambda_n\}$ or $\{\gamma_n\}$ may be a finite sequence.

Proof. Fix $f \in AP(\mathbb{R}), \neq 0$. Clearly $\mathbb{R} \ni t \mapsto (f \otimes f^*)(t) \in \mathbb{C}$ is continuous; moreover it is positive definite. Because, for $\xi_1, \dots, \xi_n \in \mathbb{C}$, $t_1, \dots, t_n \in \mathbb{R}$,

$$\begin{aligned}
&\sum_{i,j=1}^{n} \xi_i \overline{\xi_j} (f \otimes f^*)(t_i - t_j) \\
&= \sum_{i,j=1}^{n} \xi_i \overline{\xi_j} \lim_{T\to\infty} \frac{1}{T} \int_0^T f(t_i - t_j - s) f^*(s) ds \\
&= \sum_{i,j=1}^{n} \xi_i \overline{\xi_j} \lim_{T\to\infty} \frac{1}{T} \int_{-t_j}^{T-t_j} f\big(t_i - (t_j + s)\big)\overline{f(-s)} ds \\
&= \sum_{i,j=1}^{n} \xi_i \overline{\xi_j} \lim_{T\to\infty} \frac{1}{T} \int_0^T f(t_i - \sigma)\overline{f(t_j - \sigma)} d\sigma \\
&\qquad \big[\because \text{ change of variable: } \sigma = t_j + s\big] \\
&= \lim_{T\to\infty} \frac{1}{T} \int_0^T \sum_{i,j=1}^{n} \xi_i f(t_i - \sigma)\overline{\xi_j f(t_j - \sigma)} d\sigma \\
&= \lim_{T\to\infty} \frac{1}{T} \int_0^T \Big|\sum_{i=1}^{n} \xi_i f(t_i - \sigma)\Big|^2 d\sigma \\
&\geq 0.
\end{aligned}$$

Particularly, in case $n = 1$ and $\xi_1 = 1$, $(f \otimes f^*)(0) = M\big(|f|^2\big)$. It follows from Claim 1.5 that $(f \otimes f^*)(0) > 0$. Thus Bochner's theorem $\big[$cf. Claim A.4$\big]$ gives that

$$\begin{aligned}
&{}^{\exists}\nu\text{: a probability measure on } (\mathbb{R}, \mathcal{B}(\mathbb{R})) \\
&\text{s.t. } (f \otimes f^*)(t) = (f \otimes f^*)(0) \int_{\mathbb{R}} e^{\sqrt{-1}tx} \nu(dx), \quad t \in \mathbb{R}.
\end{aligned} \tag{1.8}$$

Now, let $D := \big\{\lambda \in \mathbb{R}; \nu(\{\lambda\}) > 0\big\}$. $D_n = \big\{\lambda \in \mathbb{R}; \nu(\{\lambda\}) \geq \frac{1}{n}\big\}$ is a finite set and $D_n \nearrow D$. Put

$$g(t) := \frac{(f \otimes f^*)(t)}{(f \otimes f^*)(0)} - \sum_{\lambda \in D} \nu(\{\lambda\}) e^{\sqrt{-1}\lambda t}, \quad t \in \mathbb{R}.$$

Since $\sum_{\lambda \in D_n} \nu(\{\lambda\}) e^{\sqrt{-1}\lambda t} \in AP(\mathbb{R})$ by Example 1.1, and

$$\sum_{\lambda \in D_n} \nu(\{\lambda\}) e^{\sqrt{-1}\lambda t} \rightrightarrows \sum_{\lambda \in D} \nu(\{\lambda\}) e^{\sqrt{-1}\lambda t} \quad \text{as } n \to \infty,$$

we see from Claim 1.4 that $\sum_{\lambda \in D} \nu(\{\lambda\}) e^{\sqrt{-1}\lambda t} \in AP(\mathbb{R})$. Thus $g \in AP(\mathbb{R})$.

Noting that by (1.8),

$$g(t) = \int_{\mathbb{R}} e^{\sqrt{-1}tx} \nu(dx) - \int_D e^{\sqrt{-1}tx} \nu(dx) = \int_{\mathbb{R}\setminus D} e^{\sqrt{-1}tx} \nu(dx), \quad t \in \mathbb{R},$$

we have

$$
\begin{aligned}
M\big(|g|^2\big) &= \lim_{T\to\infty}\frac{1}{T}\int_0^T |g(t)|^2 dt \\
&= \lim_{T\to\infty}\frac{1}{T}\int_0^T g(t)\overline{g(t)}dt \\
&= \lim_{T\to\infty}\frac{1}{T}\int_0^T dt\int_{\mathbb{R}\setminus D}\int_{\mathbb{R}\setminus D} e^{\sqrt{-1}t(x-y)}\nu(dx)\nu(dy) \\
&= \lim_{T\to\infty}\int_{\mathbb{R}\setminus D}\int_{\mathbb{R}\setminus D}\nu(dx)\nu(dy)\frac{1}{T}\int_0^T e^{\sqrt{-1}t(x-y)}dt \\
&= \int_{\mathbb{R}\setminus D}\int_{\mathbb{R}\setminus D}\nu(dx)\nu(dy)\lim_{T\to\infty}\frac{1}{T}\int_0^T e^{\sqrt{-1}t(x-y)}dt \\
&= \int_{\mathbb{R}\setminus D}\int_{\mathbb{R}\setminus D}\mathbf{1}_{x=y}\nu(dx)\nu(dy) \\
&\quad \left[\because\ \frac{1}{T}\int_0^T e^{\sqrt{-1}t(x-y)}dt = \begin{cases} 1, & x=y, \\ \frac{1}{T}\frac{e^{\sqrt{-1}T(x-y)}-1}{\sqrt{-1}(x-y)}, & x\neq y\end{cases} \to \mathbf{1}_{x=y} \quad \text{as } T\to\infty\right] \\
&= \int_{\mathbb{R}\setminus D}\nu\big((\mathbb{R}\setminus D)\cap\{y\}\big)\nu(dy) \\
&= 0 \\
&\quad \left[\begin{array}{l}\because\ \text{Since } (\mathbb{R}\setminus D)\cap\{y\} = \begin{cases}\emptyset & \text{if } y\in D, \\ \{y\} & \text{if } y\notin D\end{cases}, \\ \nu\big((\mathbb{R}\setminus D)\cap\{y\}\big) = \begin{cases}\nu(\emptyset) & \text{if } y\in D, \\ \nu(\{y\}) & \text{if } y\notin D\end{cases} = 0\end{array}\right].
\end{aligned}
$$

By Claim 1.5, this implies that $g = 0$, i.e.,

$$(f\otimes f^*)(t) = (f\otimes f^*)(0)\sum_{\lambda\in D}\nu(\{\lambda\})e^{\sqrt{-1}\lambda t}, \quad t\in\mathbb{R}. \qquad \blacksquare$$

Proof of Theorem 1.2. Fix $f\in AP(\mathbb{R})$ and $\varepsilon > 0$. Let $\psi_\varepsilon\in AP(\mathbb{R})$ be as in Lemma 1.1(ii). From $\psi_\varepsilon\geq 0$ and $\psi_\varepsilon(-t) = \psi_\varepsilon(t)$, it follows that $\psi_\varepsilon^*(t) = \overline{\psi_\varepsilon(-t)} = \overline{\psi_\varepsilon(t)} = \psi_\varepsilon(t)$. Since $\psi_\varepsilon(0) = \frac{1}{M(F_\varepsilon)} > 0$, we have by Lemma 1.2 that

$$
\begin{aligned}
&{}^\exists\{\lambda_n\}\text{: a real sequence, } {}^\exists\{\gamma_n\}\text{: a positive sequence} \\
&\text{s.t.}\ \begin{cases}\sum_n \gamma_n < \infty, \\ (\psi_\varepsilon\otimes\psi_\varepsilon)(t) = \sum_n \gamma_n e^{\sqrt{-1}\lambda_n t} \quad (t\in\mathbb{R}).\end{cases}
\end{aligned}
$$

We now choose $N\in\mathbb{N}$ so that $\sum_{n>N}\gamma_n < \varepsilon$. Then

$$\left|(f\otimes\psi_\varepsilon\otimes\psi_\varepsilon)(t) - \Big(f\otimes\big(\sum_{n\leq N}\gamma_n e^{\sqrt{-1}\lambda_n\cdot}\big)\Big)(t)\right|$$

$$
\begin{aligned}
&= \left| \Big(f \otimes \big(\psi_\varepsilon \otimes \psi_\varepsilon - \sum_{n \le N} \gamma_n e^{\sqrt{-1}\lambda_n \cdot} \big) \Big)(t) \right| \\
&= \lim_{T\to\infty} \left| \frac{1}{T} \int_0^T f(t-s) \Big(\big(\psi_\varepsilon \otimes \psi_\varepsilon\big)(s) - \sum_{n \le N} \gamma_n e^{\sqrt{-1}\lambda_n s} \Big) ds \right| \\
&= \lim_{T\to\infty} \left| \frac{1}{T} \int_0^T f(t-s) \sum_{n > N} \gamma_n e^{\sqrt{-1}\lambda_n s} ds \right| \\
&\le \lim_{T\to\infty} \frac{1}{T} \int_0^T |f(t-s)| \sum_{n > N} \gamma_n ds \\
&\le \varepsilon \lim_{T\to\infty} \frac{1}{T} \int_{t-T}^t |f(\sigma)| d\sigma \\
&= \varepsilon M(|f|), \quad t \in \mathbb{R}.
\end{aligned}
$$

By this and (1.5),

$$
\left| f(t) - \Big(f \otimes \big(\sum_{n \le N} \gamma_n e^{\sqrt{-1}\lambda_n \cdot} \big) \Big)(t) \right| \le \varepsilon\big(2 + M(|f|)\big), \quad {}^\forall t \in \mathbb{R}.
$$

Noting that

$$
\begin{aligned}
\big(f \otimes e^{\sqrt{-1}\lambda \cdot}\big)(t) &= \lim_{T\to\infty} \frac{1}{T} \int_0^T f(t-s) e^{\sqrt{-1}\lambda s} ds \\
&= e^{\sqrt{-1}\lambda t} \lim_{T\to\infty} \frac{1}{T} \int_0^T f(t-s) e^{\sqrt{-1}\lambda (s-t)} ds \\
&= e^{\sqrt{-1}\lambda t} \lim_{T\to\infty} \frac{1}{T} \int_{t-T}^t f(\sigma) e^{-\sqrt{-1}\lambda \sigma} d\sigma \\
&= M\big(f(\cdot) e^{-\sqrt{-1}\lambda \cdot}\big) e^{\sqrt{-1}\lambda t},
\end{aligned}
$$

we obtain

$$
\sup_{t \in \mathbb{R}} \left| f(t) - \sum_{n \le N} \gamma_n M\big(f(\cdot) e^{-\sqrt{-1}\lambda_n \cdot}\big) e^{\sqrt{-1}\lambda_n t} \right| \le \varepsilon\big(2 + M(|f|)\big). \qquad \blacksquare
$$

1.5 Parseval equality

Theorem 1.3 *For $f \in AP(\mathbb{R})$, $\neq 0$, put $a(\lambda) := M\big(f(\cdot)e^{-\sqrt{-1}\lambda \cdot}\big)$, $\lambda \in \mathbb{R}$. Then* $\mathrm{card}\{\lambda \in \mathbb{R}; a(\lambda) \neq 0\} \le \aleph_0$. *If $\{\lambda_n\} := \{\lambda \in \mathbb{R}; a(\lambda) \neq 0\}$ and $a_n := a(\lambda_n)$, then it holds that*

$$
M\big(|f|^2\big) = \sum_n |a_n|^2.
$$

The numbers a_n are called the *Fourier coefficients* of f, and this identity is called the *Parseval equality*.

Proof. First, note that

$$\begin{aligned}
\overline{a(\lambda)} &= \overline{\lim_{S\to\infty} \frac{1}{S}\int_0^S f(t)e^{-\sqrt{-1}\lambda t}\,dt} \\
&= \overline{\lim_{S\to\infty} \frac{1}{S}\int_{-S}^0 f(t)e^{-\sqrt{-1}\lambda t}\,dt} \\
&= \lim_{S\to\infty} \frac{1}{S}\int_{-S}^0 \overline{f(t)}e^{\sqrt{-1}\lambda t}\,dt \\
&= \lim_{S\to\infty} \frac{1}{S}\int_0^S \overline{f(-r)}e^{-\sqrt{-1}\lambda r}\,dr \quad \left[\because \text{ change of variable: } r = -t\right] \\
&= M\big(\overline{f(-\cdot)}e^{-\sqrt{-1}\lambda\cdot}\big). \qquad (1.9)
\end{aligned}$$

Since, by Lemma 1.2,

$$\begin{aligned}
&{}^\exists\{\lambda_n\}\text{: a real sequence, } {}^\exists\{\gamma_n\}\text{: a positive sequence} \\
&\text{s.t. } \begin{cases} \sum_n \gamma_n < \infty, \\ (f \otimes f^*)(t) = \sum_n \gamma_n e^{\sqrt{-1}\lambda_n t} \quad (t \in \mathbb{R}), \end{cases}
\end{aligned}$$

it follows that for each $\lambda \in \mathbb{R}$,

$$\begin{aligned}
&M\big((f \otimes f^*)(\cdot)e^{-\sqrt{-1}\lambda\cdot}\big) \\
&= \lim_{T\to\infty} \frac{1}{T}\int_0^T (f \otimes f^*)(t)e^{-\sqrt{-1}\lambda t}\,dt \\
&= \lim_{T\to\infty} \frac{1}{T}\int_0^T \Big(\sum_n \gamma_n e^{\sqrt{-1}\lambda_n t}\Big)e^{-\sqrt{-1}\lambda t}\,dt \\
&= \lim_{T\to\infty} \sum_n \gamma_n \frac{1}{T}\int_0^T e^{\sqrt{-1}(\lambda_n-\lambda)t}\,dt \\
&= \sum_n \gamma_n \lim_{T\to\infty} \frac{1}{T}\int_0^T e^{\sqrt{-1}(\lambda_n-\lambda)t}\,dt \\
&\quad \left[\begin{array}{l}\because \text{ Since } \left|\frac{1}{T}\int_0^T e^{\sqrt{-1}(\lambda_n-\lambda)t}\,dt\right| \le \frac{1}{T}\int_0^T |e^{\sqrt{-1}(\lambda_n-\lambda)t}|\,dt = \\ 1 \text{ and } \sum_n \gamma_n < \infty, \text{ we apply Lebesgue's convergence theo-} \\ \text{rem to interchange the limiting procedure}\end{array}\right] \\
&= \sum_n \gamma_n \mathbf{1}_{\lambda_n=\lambda} \\
&\quad \left[\begin{array}{l}\because\ \frac{1}{T}\int_0^T e^{\sqrt{-1}(\lambda_n-\lambda)t}\,dt = \begin{cases} 1 & \text{if } \lambda_n = \lambda, \\ \frac{1}{T}\frac{e^{\sqrt{-1}(\lambda_n-\lambda)T}-1}{\sqrt{-1}(\lambda_n-\lambda)} & \text{if } \lambda_n \ne \lambda \end{cases} \\ \qquad\qquad \to \begin{cases} 1 & \text{if } \lambda_n = \lambda, \\ 0 & \text{if } \lambda_n \ne \lambda \end{cases} \quad \text{as } T \to \infty \end{array}\right] \\
&= \begin{cases} \gamma_n & \text{if } \lambda = \lambda_n \text{ for some } n, \\ 0 & \text{if } \lambda \notin \{\lambda_n\}. \end{cases}
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
&M\big((f \otimes f^*)(\cdot)e^{-\sqrt{-1}\lambda\cdot}\big)\\
&= \lim_{T\to\infty}\frac{1}{T}\int_0^T (f \otimes f^*)(t)e^{-\sqrt{-1}\lambda t}dt\\
&= \lim_{T\to\infty}\frac{1}{T}\int_0^T \Big(\lim_{S\to\infty}\frac{1}{S}\int_0^S f(t-r)f^*(r)dr\Big)e^{-\sqrt{-1}\lambda t}dt\\
&= \lim_{T\to\infty}\lim_{S\to\infty}\frac{1}{T}\frac{1}{S}\int_0^T e^{-\sqrt{-1}\lambda t}dt\int_0^S f(t-r)\overline{f(-r)}dr\\
&= \lim_{T\to\infty}\lim_{S\to\infty}\frac{1}{T}\frac{1}{S}\int_0^S \overline{f(-r)}e^{-\sqrt{-1}\lambda r}dr\int_0^T f(t-r)e^{-\sqrt{-1}\lambda(t-r)}dt\\
&= \lim_{T\to\infty}\lim_{S\to\infty}\frac{1}{S}\int_0^S \overline{f(-r)}e^{-\sqrt{-1}\lambda r}dr\frac{1}{T}\int_{-r}^{T-r} f(\tau)e^{-\sqrt{-1}\lambda\tau}d\tau\\
&\qquad\big[\because \text{ change of variable: } \tau = t-r\big].
\end{aligned}
$$

Recall (1.2). For $T \gg 1$,

$$
\begin{aligned}
&\Big|\frac{1}{S}\int_0^S \overline{f(-r)}e^{-\sqrt{-1}\lambda r}dr\frac{1}{T}\int_{-r}^{T-r} f(\tau)e^{-\sqrt{-1}\lambda\tau}d\tau\\
&\quad -\frac{1}{S}\int_0^S \overline{f(-r)}e^{-\sqrt{-1}\lambda r}dr\frac{1}{T}\int_0^T f(\tau)e^{-\sqrt{-1}\lambda\tau}d\tau\Big|\\
&= \Big|\frac{1}{S}\int_0^S \overline{f(-r)}e^{-\sqrt{-1}\lambda r}dr\Big(\frac{1}{T}\int_{-r}^{T-r} f(\tau)e^{-\sqrt{-1}\lambda\tau}d\tau\\
&\qquad\qquad -\frac{1}{T}\int_0^T f(\tau)e^{-\sqrt{-1}\lambda\tau}d\tau\Big)\Big|\\
&\le \frac{1}{S}\int_0^S |f(-r)|dr\Big|\frac{1}{T}\int_{-r}^{T-r} f(\tau)e^{-\sqrt{-1}\lambda\tau}d\tau\\
&\qquad\qquad -\frac{1}{T}\int_0^T f(\tau)e^{-\sqrt{-1}\lambda\tau}d\tau\Big|\\
&\le \frac{3\varepsilon}{S}\int_0^S |f(-r)|dr\\
&= \frac{3\varepsilon}{S}\int_{-S}^0 |f(r)|dr\\
&\to 3\varepsilon M(|f|) \quad \text{as } S\to\infty.
\end{aligned}
$$

Letting $T\to\infty$, we have

$$\Big|M\big((f \otimes f^*)(\cdot)e^{-\sqrt{-1}\lambda\cdot}\big) - M\big(\overline{f(-\cdot)}e^{-\sqrt{-1}\lambda\cdot}\big)M\big(f(\cdot)e^{-\sqrt{-1}\lambda\cdot}\big)\Big| \le 3\varepsilon M(|f|),$$

so that, letting $\varepsilon\to 0$, we obtain

$$M\big((f \otimes f^*)(\cdot)e^{-\sqrt{-1}\lambda\cdot}\big) = M\big(\overline{f(-\cdot)}e^{-\sqrt{-1}\lambda\cdot}\big)M\big(f(\cdot)e^{-\sqrt{-1}\lambda\cdot}\big)$$

$$
\begin{aligned}
&= \overline{a(\lambda)}a(\lambda) \quad [\because (1.9)] \\
&= |a(\lambda)|^2.
\end{aligned}
$$

Thus

$$
|a(\lambda)|^2 = \begin{cases} \gamma_n & \text{if } \lambda = \lambda_n \text{ for some } n, \\ 0 & \text{if } \lambda \notin \{\lambda_n\}. \end{cases}
$$

This implies that

$$
\begin{aligned}
&\operatorname{card}\{\lambda \in \mathbb{R}; a(\lambda) \neq 0\} = \operatorname{card}\{\lambda_n\} \leq \aleph_0, \\
&M(|f|^2) = (f \otimes f^*)(0) = \sum_n \gamma_n = \sum_n |a(\lambda_n)|^2.
\end{aligned}
$$

■

Corollary 1.1 *Under the same setting as in Theorem 1.3,*

$$
\lim_{N\to\infty} M\Big(\big|f(\cdot) - \sum_{n\leq N} a_n e^{\sqrt{-1}\lambda_n \cdot}\big|^2\Big) = 0.
$$

Proof.

$$
\begin{aligned}
& M\Big(\big|f(\cdot) - \sum_{n\leq N} a_n e^{\sqrt{-1}\lambda_n \cdot}\big|^2\Big) \\
&= \lim_{T\to\infty} \frac{1}{T}\int_0^T \big|f(t) - \sum_{n\leq N} a_n e^{\sqrt{-1}\lambda_n t}\big|^2 dt \\
&= \lim_{T\to\infty} \frac{1}{T}\int_0^T \Big(|f(t)|^2 - f(t)\sum_{n\leq N}\overline{a_n} e^{-\sqrt{-1}\lambda_n t} \\
&\qquad - \sum_{n\leq N} a_n e^{\sqrt{-1}\lambda_n t}\overline{f(t)} + \sum_{m,n\leq N} a_m \overline{a_n} e^{\sqrt{-1}(\lambda_m - \lambda_n)t}\Big) dt \\
&= \lim_{T\to\infty} \Big(\frac{1}{T}\int_0^T |f(t)|^2 dt - \sum_{n\leq N}\overline{a_n}\frac{1}{T}\int_0^T f(t)e^{-\sqrt{-1}\lambda_n t} dt \\
&\qquad - \sum_{n\leq N} a_n \overline{\frac{1}{T}\int_0^T f(t)e^{-\sqrt{-1}\lambda_n t} dt} + \sum_{m,n\leq N} a_m\overline{a_n}\frac{1}{T}\int_0^T e^{\sqrt{-1}(\lambda_m-\lambda_n)t} dt\Big) \\
&= M(|f|^2) - \sum_{n\leq N}\overline{a_n} M\big(f(\cdot)e^{-\sqrt{-1}\lambda_n \cdot}\big) \\
&\qquad - \sum_{n\leq N} a_n \overline{M\big(f(\cdot)e^{-\sqrt{-1}\lambda_n \cdot}\big)} + \sum_{n\leq N} |a_n|^2 \\
&\qquad \Big[\because \text{When } m \neq n,\ \lim_{T\to\infty}\tfrac{1}{T}\textstyle\int_0^T e^{\sqrt{-1}(\lambda_m-\lambda_n)t} dt = 0\Big] \\
&= M(|f|^2) - \sum_{n\leq N} |a_n|^2 \\
&\to 0 \quad \text{as } N \to \infty.
\end{aligned}
$$

■