Chapter 9

Not well-posed results

9.1 Introduction

For the second order differential operator in $\mathbb{R}^2$ with real analytic coefficient $a(x_0, x_1) \geq 0$ defined near the origin

$$P = -D_0^2 + a(x_0, x_1)D_1^2$$

the Cauchy problem is C^∞ well posed near the origin ([40]). Since then it has been conjectured that the Cauchy problem is C^∞ well posed for any second order differential operator of divergence form with real analytic coefficients

$$Pu = -D_0^2 u + \sum_{i,j=1}^{n} D_{x_i}(a_{ij}(x)D_{x_j}u), \quad a_{ij}(x) = a_{ji}(x)$$

where $a_{ij}(x)$ are real analytic and

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq 0, \quad \forall \xi' = (\xi_1, ..., \xi_n) \in \mathbb{R}^n.$$

In Section 8.1 we have shown that the operator P_{mod} is of divergence form and hence this gives a counter example of the conjecture. In this chapter we show somewhat stronger assertion on the well-posedness of the Cauchy problem for P_{mod}, that is the Cauchy problem for $P_{mod} + Q$ is not $\gamma^{(s)}$ well posed for any $s > 6$ whatever the lower order term Q is. Recall that the coefficients of P_{mod} are not only real analytic but also polynomials. This is a quite unexpected fact. On the other hand note that the Cauchy problem for $P_{mod} + Q$ is $\gamma^{(s)}$ well posed for any $1 \leq s \leq 2$ and for any lower order term Q, which is a particular case of the general result proved in [9].

Let us consider again

$$P_{mod}(x, D) = -D_0^2 + 2x_1 D_0 D_2 + D_1^2 + x_1^3 D_2^2 \qquad (9.1.1)$$

in $\mathbb{R}^3$. Then we have

Theorem 9.1.1 ([49]) *The Cauchy problem for*

$$P(x,D) = P_{mod}(x,D) + \sum_{j=0}^{2} b_j D_j$$

is not locally solvable at the origin in $\gamma^{(s)}$ *if* $s > 6$ *for any* b_0, b_1, $b_2 \in \mathbb{C}$. *In particular the Cauchy problem for* P_{mod} *is not* C^∞ *well posed for any lower order term.*

It is easy to modify the proof of Theorem 8.1.1 to get

Proposition 9.1.1 *The Cauchy problem for*

$$P(x,D) = P_{mod}(x,D) + \sum_{j=0}^{1} b_j D_j$$

is not locally solvable in $\gamma^{(s)}$ *if* $s > 5$ *for any* b_0, $b_1 \in \mathbb{C}$.

Thus in order to prove Theorem 9.1.1 we may assume that $b_2 \neq 0$. Moreover, making a change of the coordinate system; $(x_0, x_1, x_2) \to (x_0, x_1, -x_2)$ if necessary, we may assume that $b_2 \in \mathbb{C} \setminus \mathbb{R}^+$.

In Section 10.2, following [18], [23] we construct an asymptotic solution U_λ to $PU_\lambda = 0$ which contradicts the a priori estimates, derived in Section 10.4, when $\lambda \to \infty$ and hence finally we prove Theorem 9.1.1.

9.2 Asymptotic solutions

Let us consider

$$P = -D_0^2 + 2x_1 D_0 D_2 + D_1^2 + x_1^3 D_2^2 + \sum_{j=0}^{2} b_j D_j, \quad b_j \in \mathbb{C}.$$

Make a change of coordinates system

$$x_0 = \lambda^{-1} y_0, \;\; x_1 = \lambda^{-2} y_1, \;\; x_2 = \lambda^{-4} y_2$$

so that we have

$$P_\lambda = -\lambda^{-2} D_0^2 + 2\lambda^{-1} y_1 D_0 D_2 + D_1^2 + \lambda^{-2} y_1^3 D_2^2 + b_2 D_2 + \lambda^{-2} b_1 D_1 + \lambda^{-3} b_0 D_0.$$

We switch the notation to x and set $b_2 = b$ so that we study

$$P_\lambda = -\lambda^{-2} D_0^2 + 2\lambda^{-1} x_1 D_0 D_2 + D_1^2 + \lambda^{-2} x_1^3 D_2^2 + b D_2 + \lambda^{-2} b_1 D_1 + \lambda^{-3} b_0 D_0.$$

Let us denote

$$E_\lambda = \exp\left(i\lambda^2 x_2 + i\lambda\phi(x)\right)$$

and compute $\lambda^{-1}E_\lambda^{-1}P_\lambda E_\lambda$ which yields

$$\begin{aligned}\lambda^{-1}E_\lambda^{-1}P_\lambda E_\lambda = \lambda\big\{2x_1\phi_{x_0} + \phi_{x_1}^2 + x_1^3 + b\big\}\\ +\big\{2x_1D_0 + 2\phi_{x_1}D_1 + 2x_1\phi_{x_0}\phi_{x_2} + b\phi_{x_2} + 2x_1^3\phi_{x_1} - i\phi_{x_1x_1}\big\}\\ +\lambda^{-1}h^{(1)}(x,D) + \lambda^{-2}h^{(2)}(x,D) + \lambda^{-3}h^{(3)}(x,D)\end{aligned}$$

where $h^{(i)}(x,D)$ are differential operators of order 2. We first assume that

$$\mathsf{Im}\, b \neq 0.$$

Take $y_1 \in \mathbb{R}$ small so that

$$\mathsf{Im}\,\frac{b}{2y_1} > 0$$

and work near the point $(x_0,x_1,x_2) = (t,y_1,0) = x^*$. We solve the equation

$$2x_1\phi_{x_0} + \phi_{x_1}^2 + x_1^3 + b = 0 \tag{9.2.1}$$

imposing the condition

$$\phi = (x_1 - y_1) + i(x_1 - y_1)^2 + ix_2^2 \quad \text{on} \quad x_0 = t.$$

Noticing

$$\phi = (x_1 - y_1) + i(x_1 - y_1)^2 + ix_2^2 + \phi_{x_0}(t,x_1,x_2)(x_0 - t) + O((x_0 - t)^2)$$

we conclude

$$\mathsf{Im}\,\phi = (x_1 - y_1)^2 + x_2^2 + \big\{\mathsf{Im}\,\phi_{x_0}(t,y_1,0) + R(x)\big\}(x_0 - t)$$

where $R(x) = O(|x - x^*|)$. Note that

$$\phi_{x_0}(x^*) = \frac{-1-b}{2y_1} - \frac{y_1^2}{2}$$

and hence $\mathsf{Im}\,\phi_{x_0}(x^*) < 0$. Writing $\alpha = \mathsf{Im}\,\phi_{x_0}(x^*)$ we have

$$\begin{aligned}\mathsf{Im}\,\phi = (x_1 - y_1)^2 + x_2^2 + \alpha(x_0 - t) + \frac{1}{2}(\epsilon^{-1}(x_0 - t) + \epsilon R(x))^2\\ -\frac{\epsilon^{-2}}{2}(x_0 - t)^2 - \frac{\epsilon^2}{2}R(x)^2\\ = (x_1 - y_1)^2 + x_2^2 + (x_0 - t)^2 - \frac{\epsilon^2}{2}R(x)^2\\ +\big\{\alpha - (\frac{\epsilon^{-2}}{2} + 1)(x_0 - t)\big\}(x_0 - t) + \frac{1}{2}(\epsilon^{-1}(x_0 - t) + \epsilon R(x))^2\\ = |x - x^*|^2 - \frac{\epsilon^2}{2}R(x)^2 + \frac{1}{2}(\epsilon^{-1}(x_0 - t) + \epsilon R(x))^2\\ +\big\{\alpha - (\frac{\epsilon^{-2}}{2} + 1)(x_0 - t)\big\}(x_0 - t).\end{aligned}$$

Thus $-\mathsf{Im}\,\phi$ attains its strict maximum at x^* in the set $\{x; |x-x^*|<\delta, x_0\le t\}$ if $\delta>0$ is small enough. Let L be a compact set in $\mathbb{R}^3$. For $t\in\mathbb{R}$ recall that we denote $L^t=\{x\in L\mid x_0\le t\}$ and $L_t=\{x\in L\mid x_0\ge t\}$. Then we have

Lemma 9.2.1 *Let K be a small compact neighborhood of x^*. Then we have*

$$\sup_{x\in K^{t+\tau}}\{-\mathsf{Im}\,\phi(x)\}\le 2|\mathsf{Im}\,\phi_{x_0}(x^*)|\tau$$

for any small $\tau>0$. Let $\delta>0$ be small. Then there exist $\nu(\delta)>0$ and $\tau(\delta)>0$ such that

$$\sup_{x\in K^{t+\tau}\cap\{|x-x^*|\ge\delta\}}\{-\mathsf{Im}\,\phi(x)\}\le -\nu(\delta)$$

for any $\tau\le\tau(\delta)$.

Let us denote

$$\lambda^{-1}P_\lambda E_\lambda=E_\lambda Q_\lambda,\quad Q_\lambda=Q_0(x,D)+Q_1(x,\lambda,D)$$

where

$$\begin{cases} Q_0(x,D)=2x_1D_0+2\phi_{x_1}D_1+2x_1\phi_{x_0}\phi_{x_2}+b\phi_{x_2}+2x_1^3\phi_{x_1}-i\phi_{x_1x_1},\\ Q_1(x,\lambda,D)=\lambda^{-1}h^{(1)}(y,D)+\lambda^{-2}h^{(2)}(x,D)+\lambda^{-3}h^{(3)}(x,D).\end{cases}$$

Let us set $V_\lambda=\sum_{n=0}^N v_\lambda^{(n)}$ and determine $v_\lambda^{(n)}$ by solving the Cauchy problem

$$\begin{cases} Q_0(x,D)v_\lambda^{(n)}=-g_\lambda^{(n)}=-Q_1v_\lambda^{(n-1)},\\ v_\lambda^{(0)}(t,x_1,x_2)=1,\\ v_\lambda^{(n)}(t,x_1,x_2)=0,\quad n\ge 1\end{cases}$$

where $v_\lambda^{(-1)}=0$ so that $Q_\lambda V_\lambda=Q_1(x,\lambda,D)v_\lambda^{(N)}$. Hence

$$\lambda^{-1}P_\lambda E_\lambda V_\lambda=E_\lambda Q_1(x,\lambda,D)v_\lambda^{(N)}. \tag{9.2.2}$$

We turn to the case

$$b\in\mathbb{R},\quad b<0.$$

We follow the arguments in [18]. We write $b=-\gamma^2$, $\gamma>0$. We solve the equation (9.2.1) under the condition

$$\phi=-i(x_0-t)+ix_2^2\quad\text{on }\ x_1=0.$$

That is, one solves the equation $\phi_{x_1}=\sqrt{\gamma^2-x_1^3-2x_1\phi_{x_0}}$. It is clear that

$$\phi_{x_1}=\big(\gamma+i\frac{x_1}{\gamma}\big)+O(x_1^2).$$

One can write

$$\phi = -i(x_0 - t) + ix_2^2 + \big(\gamma + i\frac{x_1}{\gamma}\big)x_1 + R(x)$$

where $R(x) = O(x_1^3)$. Note that

$$\begin{aligned} \mathsf{Im}\,\phi &= -(x_0 - t) + x_2^2 + \gamma^{-1}x_1^2 + R(x) \\ &= (x_0 - t)^2 + \gamma^{-1}x_1^2 + x_2^2 + R(x) + \big\{-1 - (x_0 - t)\big\}(x_0 - t) \end{aligned}$$

and hence the same assertion as Lemma 9.2.1 holds. Noting that ϕ_{x_1} is different from zero in an open neighborhood of $x^* = (t, 0, 0)$ we can solve the transport equation in the x_1 direction

$$\begin{cases} Q_0(x, D)v_\lambda^{(n)} = -g_\lambda^{(n)} = -Q_1 v_\lambda^{(n-1)}, \\ v_\lambda^{(0)}(x_0, 0, x_2) = 1, \\ v_\lambda^{(n)}(x_0, 0, x_2) = 0, \quad n \geq 1. \end{cases}$$

9.3 Lemmas

To estimate $E_\lambda V_\lambda$, which is constructed in the previous section, we apply the method of majorant following Ivrii [24]. Consider $Q = \sum_{|\alpha|\leq 1} b_\alpha D^\alpha$ where we assume that the coefficient of D_0 is different from zero near $x = x^*$. We first recall the notion of majorant.

Definition 9.3.1 *Let $\Phi_i(\tau, \eta) = \sum_{j,k\geq 0}^{\infty} C_{ijk}\tau^j\eta^k$, $i = 1, 2$ be two formal power series in (τ, η). Then we write*

$$\Phi_1 \ll \Phi_2$$

if $|C_{1jk}| \leq C_{2jk}$ for any $j, k \geq 0$. We say that Φ_2 is a majorant of Φ_1.

Lemma 9.3.1 *Let $Qv = g$ and let*

$$\Phi(\tau, \eta; v) = \sum_{\alpha=(\alpha_0,\alpha')} \frac{\tau^{\alpha_0}\eta^{|\alpha'|}}{\alpha!}|D^\alpha v(x^*)|.$$

Then we have

$$\frac{\partial}{\partial\tau}\Phi(\tau, \eta; v) \ll C(\tau, \eta)\frac{\partial}{\partial\eta}\Phi(\tau, \eta; v) + C(\tau, \eta)\Phi(\tau, \eta; g)$$

with some holomorphic $C(\tau, \eta)$ at $(0, 0)$ with $C(\tau, \eta) \gg 0$ which depends only on Q.

Proof: Note that

$$\frac{\partial}{\partial\tau}\Phi(\tau, \eta; v) = \sum_{\beta} \frac{\tau^{\beta_0}\eta^{|\beta'|}}{\beta!}|D^\beta(D_0 v)(x^*)| = \Phi(\tau, \eta; D_0 v).$$

On the other hand from $Qv = g$ one sees $D_0 v = \sum_{j=1}^n c_j D_j v + c_0 v$. Since $\Phi(\tau,\eta;fg) \ll \Phi(\tau,\eta;f)\Phi(\tau,\eta;g)$ and hence

$$\frac{\partial}{\partial\tau}\Phi(\tau,\eta;v) \ll C(\tau,\eta)\big(\sum_{j=1}^n \Phi(\tau,\eta;D_j v) + \Phi(\tau,\eta;g)\big).$$

To conclude the assertion it is enough to note

$$\frac{\partial\Phi}{\partial\eta} \gg \sum_{\alpha_j=1} \frac{|\alpha'|\tau^{\alpha_0}\eta^{|\alpha'|-1}}{\alpha!}|D^{\tilde{\alpha}}(D_j v)(x^*)|,$$
$$\frac{|\alpha'|\tau^{\alpha_0}\eta^{|\alpha'|-1}}{\alpha!} = \frac{|\alpha'|\tau^{\tilde{\alpha}_0}\eta^{|\tilde{\alpha}'|}}{\alpha_j\tilde{\alpha}!} \geq \frac{\tau^{\tilde{\alpha}_0}\eta^{|\tilde{\alpha}'|}}{\tilde{\alpha}!}.$$

□

Lemma 9.3.2 *Assume* $Qv = g$ *and*

$$\begin{cases} \dfrac{\partial}{\partial\tau}\Phi^*(\tau,\eta) \gg C(\tau,\eta)\dfrac{\partial}{\partial\eta}\Phi^*(\tau,\eta) + C(\tau,\eta)\Phi(\tau,\eta;g), \\ \Phi^*(0,\eta) \gg \Phi(0,\eta;v). \end{cases}$$

Then we have

$$\Phi(\tau,\eta;v) \ll \Phi^*(\tau,\eta).$$

Proof: Let $\tilde{\Phi}$ be a solution to the Cauchy problem

$$\begin{cases} \dfrac{\partial}{\partial\tau}\tilde{\Phi}(\tau,\eta) = C(\tau,\eta)\dfrac{\partial}{\partial\eta}\tilde{\Phi}(\tau,\eta) + C(\tau,\eta)\Phi(\tau,\eta;g), \\ \tilde{\Phi}(0,\eta) = \Phi^*(0,\eta). \end{cases}$$

Then it is clear that $\Phi(\tau,\eta;v) \ll \tilde{\Phi}(\tau,\eta) \ll \Phi^*(\tau,\eta)$. □

Lemma 9.3.3 *Assume* $0 < a \leq \delta a_1$ *and* $0 < b \leq \delta b_1$ *with some* $0 < \delta < 1$. *Then we have*

(i) $\Big(1 - \dfrac{\eta}{b} - \dfrac{\tau}{a}\Big)^{-1}\Big(1 - \dfrac{\eta}{b_1} - \dfrac{\tau}{a_1}\Big)^{-1} \ll (1-\delta)^{-1}\Big(1 - \dfrac{\eta}{b} - \dfrac{\tau}{a}\Big)^{-1}$,

(ii) $\Big(1 - \dfrac{\eta}{b}\Big)^{-1}\Big(1 - \dfrac{\tau}{a}\Big)^{-1} \ll \Big(1 - \dfrac{\eta}{b} - \dfrac{\tau}{a}\Big)^{-1}$.

Proof: The assertion (i) follows from

$$\Big\{\sum\big(\frac{\eta}{b} + \frac{\tau}{a}\big)^n\Big\}\Big\{\sum\big(\frac{\eta}{b_1} + \frac{\tau}{a_1}\big)^n\Big\}$$
$$= \sum_{n,m}\big(\frac{\eta}{b} + \frac{\tau}{a}\big)^n\big(\frac{\eta}{b_1} + \frac{\tau}{a_1}\big)^m \ll \sum_{n,m}\delta^m\big(\frac{\eta}{b} + \frac{\tau}{a}\big)^{n+m}$$
$$\ll \sum_m \delta^m \sum_n \big(\frac{\eta}{b} + \frac{\tau}{a}\big)^n.$$

To examine the second assertion it is enough to note that the coefficient of $\eta^n\tau^m$ in $\sum(\eta/b)^k\sum(\tau/a)^j$ is $b^{-n}a^{-m}$ while that of $\eta^n\tau^m$ in

$$\sum\big(\frac{\eta}{b}+\frac{\tau}{a}\big)^k$$

is $b^{-n}a^{-m}(n+m)!/(n!m!)$. □

Here we recall that if $\phi(\tau,\eta)$ is holomorphic in a neighborhood of $\{(\tau,\eta)\mid|\eta|\le b,|\tau|\le a\}$ then we have

$$\phi(\tau,\eta)\ll(1-\frac{\tau}{a})^{-1}(1-\frac{\eta}{b})^{-1}\sup_{|\tau|=a,|\eta|=b}|\phi(\tau,\eta)|$$

which follows from the Cauchy's integral formula. Assume that

$$C(\tau,\eta)\ll(1-\frac{\tau}{a_1})^{-1}(1-\frac{\eta}{b_1})^{-1}B\ll\big(1-\frac{\tau}{a_1}-\frac{\eta}{b_1}\big)^{-1}B.$$

Lemma 9.3.4 *Assume that $Qv=g$ and*

$$\Phi(0,\eta;v)\ll\omega^{-1}\big(1-\frac{\eta}{b}\big)^{-n},\ \ \Phi(\tau,\eta;g)\ll L\big(1-\frac{\tau}{a}-\frac{\eta}{b}\big)^{-n}e^{M\tau\omega}.$$

We also assume that $Ba/b\le(1-\delta)$ and $B\le(1-\delta)M$. Then we have

$$\Phi(\tau,\eta;v)\ll L\omega^{-1}\big(1-\frac{\tau}{a}-\frac{\eta}{b}\big)^{-n}e^{M\tau\omega}.$$

Proof: Let us denote $(L\ge1)$

$$\Phi^*=L\omega^{-1}\big(1-\frac{\tau}{a}-\frac{\eta}{b}\big)^{-n}e^{M\tau\omega}.$$

It is easy to see by Lemma 9.3.3 that

$$\frac{\partial\Phi^*}{\partial\tau}\gg C(\tau,\eta)\frac{\partial\Phi^*}{\partial\eta}+C(\tau,\eta)\Phi(\tau,\eta;g).$$

Then the assertion follows from Lemma 9.3.2. □

Let us denote

$$\Phi_\lambda^n=\Phi(\tau,\eta;v_\lambda^{(n)})$$

and hence $\Phi_\lambda^n(0,\eta)=0$ for $n\ge1$ and $\Phi_\lambda^0(0,\eta)=1$. We assume that

$$\Phi_\lambda^n(\tau,\eta)\ll A^{n+1}\lambda^{-n}\sum_{k=0}^{2n}\omega^{n-k}k!\big(1-\frac{\tau}{a}-\frac{\eta}{b}\big)^{-k-1}e^{M\tau\omega}. \tag{9.3.1}$$

For $n=0$ this holds clearly. Suppose that (9.3.1) holds for $\le n-1$. Let

$$g=\Big(\sum_{j=1}^{3}\lambda^{-j}h^{(j)}(x,D)\Big)v_\lambda^{(n-1)}=Q_1(x,\lambda,D)v_\lambda^{(n-1)}$$

and we first show that

$$\Phi(\tau,\eta;g) \ll A^{n+1}\lambda^{-n}\sum_{k=0}^{2n}\omega^{n-k}k!\big(1-\frac{\tau}{a}-\frac{\eta}{b}\big)^{-k-1}e^{M\tau\omega}.$$

As for terms $c(x)D^{\alpha}u$ with $|\alpha|\le 2$ we have

$$\Phi(\tau,\eta;cD^{\alpha}u) \ll C\big(1-\frac{\tau}{a_1}-\frac{\eta}{b_1}\big)^{-1}\Phi(\tau,\eta;D^{\alpha}u)$$
$$\ll C\big(1-\frac{\tau}{a_1}-\frac{\eta}{b_1}\big)^{-1}\Big[\frac{\partial^2}{\partial\tau^2}+\frac{\partial^2}{\partial\tau\partial\eta}+\frac{\partial^2}{\partial\eta^2}\Big]\Phi(\tau,\eta;u).$$

We now estimate

$$\Big[\frac{\partial^2}{\partial\tau^2}+\frac{\partial^2}{\partial\tau\partial\eta}+\frac{\partial^2}{\partial\eta^2}\Big]\sum_{k=0}^{2(n-1)}\omega^{n-1-k}k!\big(1-\frac{\tau}{a}-\frac{\eta}{b}\big)^{-k-1}e^{M\tau\omega}$$

which is bounded by

$$\sum_{k=0}^{2(n-1)} M^2\omega^{n+1-k}k!\big(1-\frac{\tau}{a}-\frac{\eta}{b}\big)^{-k-1}+2M\omega^{n-k}(k+1)!a^{-1}\big(1-\frac{\tau}{a}-\frac{\eta}{b}\big)^{-k-2}$$
$$+\omega^{n-1-k}(k+2)!a^{-2}\big(1-\frac{\tau}{a}-\frac{\eta}{b}\big)^{-k-3}$$
$$+M\omega^{n-k}(k+1)!b^{-1}\big(1-\frac{\tau}{a}-\frac{\eta}{b}\big)^{-k-2}+\omega^{n-1-k}(k+2)!a^{-1}b^{-1}\big(1-\frac{\tau}{a}-\frac{\eta}{b}\big)^{-k-3}$$
$$+\omega^{n-1-k}(k+2)!b^{-2}\big(1-\frac{\tau}{a}-\frac{\eta}{b}\big)^{-k-3}$$
$$\ll \omega\big(M^2+2Ma^{-1}+a^{-2}+Mb^{-1}+a^{-1}b^{-1}+b^{-2}\big)\sum_{k=0}^{2n}\omega^{n-k}k!\big(1-\frac{\tau}{a}-\frac{\eta}{b}\big)^{-k-1}$$

up to the factor $A^n\lambda^{-n+1}e^{M\tau\omega}$. Taking A so that

$$A\ge M^2+2Ma^{-1}+a^{-2}+Mb^{-1}+a^{-1}b^{-1}+b^{-2}$$

we conclude that

$$\Phi(\tau,\eta;g) \ll A^{n+1}\lambda^{-n}\omega\sum_{k=0}^{2n}\omega^{n-k}k!\big(1-\frac{\tau}{a}-\frac{\eta}{b}\big)^{-k-1}e^{M\tau\omega}.$$

Recalling that $\Phi^n_{\lambda}(0,\eta)=0\ll\omega^{-1}\big(1-\frac{\eta}{b}\big)^{-1}$, $n\ge 1$ for any ω and applying Lemma 9.3.4 we see

Lemma 9.3.5 *We have*

$$\Phi^n_{\lambda}(\tau,\eta) \ll A^{n+1}\lambda^{-n}\sum_{k=0}^{2n}\omega^{n-k}k!\big(1-\frac{\tau}{a}-\frac{\eta}{b}\big)^{-k-1}e^{M\tau\omega}$$

for any $\omega\ge 1$.

Lemma 9.3.6 *There are $h > 0$ and $\delta > 0$ such that*

$$\sum_\alpha \frac{h^{|\alpha|}}{\alpha!} \sup_{|x-x^*|\le\delta} |D^\alpha v_\lambda^{(n)}(x)| \le B^{n+1}\lambda^{-n} \sum_{k=0}^{2n} \omega^{n-k} k! e^{M_1\omega}.$$

Proof: Note that

$$\sum_\alpha \frac{\eta^{|\alpha|}}{\alpha!} |D^\alpha v_\lambda^{(n)}(x^*)| \le A^{n+1}\lambda^{-n} \sum_{k=0}^{2n} \omega^{n-k} k! \big(1 - \frac{\eta}{a} - \frac{\eta}{b}\big)^{-k-1} e^{M\eta\omega}$$

and hence for $0 < \eta \le \eta_0$ we have

$$\sum_\alpha \frac{\eta^{|\alpha|}}{\alpha!} |D^\alpha v_\lambda^{(n)}(x^*)| \le B^{n+1}\lambda^{-n} \sum_{k=0}^{2n} \omega^{n-k} k! e^{M\eta_0\omega}.$$

This shows that

$$|v_\lambda^{(n)}(x)| \le \sum_\alpha \frac{|D^\alpha v_\lambda^{(n)}(x^*)|}{\alpha!} |(x-x^*)^\alpha| \le B^{n+1}\lambda^{-n} \sum_{k=0}^{2n} \omega^{n-k} k! e^{M_1\omega}$$

for $|x - x^*| \le \eta_0$. From the Cauchy's inequality it follows that

$$\sup_{|x-x^*|\le\eta_0/2} |D^\alpha v_\lambda^{(n)}(x)| \le (\eta_0/2)^{-|\alpha|} \alpha! B^{n+1}\lambda^{-n} \sum_{k=0}^{2n} \omega^{n-k} k! e^{M_1\omega}$$

and hence we have

$$\sum_\alpha \frac{h^{|\alpha|}}{\alpha!} \sup_{|x-x^*|\le\delta} |D^\alpha v_\lambda^{(n)}(x)| \le B^{n+1}\lambda^{-n} \sum_{k=0}^{2n} \omega^{n-k} k! e^{M_1\omega}$$

for $2h < \eta_0$ and $2\delta < \eta_0$ with a possibly different B. □

Let us define

$$V_\lambda(x) = \sum_{n=0}^{N} v_\lambda^{(n)}(x)$$

where N and ω are chosen so that

$$\omega = 4N, \quad \lambda = \omega B e^L$$

where L will be determined later. Then we have for $n \le N$

$$\sum_{k=0}^{2n} \omega^{n-k} k! e^{M_1\omega} \le \omega^n e^{M_1\omega} \sum_{k=0}^{2n} \Big(\frac{k}{\omega}\Big)^k \le \omega^n e^{M_1\omega} \sum_{k=0}^{2n} \Big(\frac{1}{2}\Big)^k$$

and hence

$$\begin{aligned}\sum_\alpha \frac{h^{|\alpha|}}{\alpha!} \sup_{|x-x^*|\le\delta} |D^\alpha v_\lambda^{(n)}(x)| &\le B^{n+1}\lambda^{-n}\omega^n e^{M_1\omega} \\ &\le B^{n+1}(B^{-1}e^{-L})^n e^{M_1\omega} = Be^{-Ln+M_1\omega}.\end{aligned}$$

In particular one has

$$(9.3.2)\qquad \sum_\alpha \frac{h^{|\alpha|}}{\alpha!}\sup_{|x-x^*|\leq\delta}|D^\alpha v_\lambda^{(N)}(x)|$$
$$\leq Be^{-LN+4M_1N} = Be^{-e^{-L}(L-4M_1)\lambda/4B}.$$

On the other hand, we see

$$(9.3.3)\qquad \sum_\alpha \frac{h^{|\alpha|}}{\alpha!}\sup_{|x-x^*|\leq\delta}|D^\alpha V_\lambda(x)| \leq \sum_{n=0}^N B^{n+1}\lambda^{-n}\omega^n e^{M_1\omega}$$
$$= e^{M_1\omega}B\sum_{n=0}^N\Big(\frac{B\omega}{\lambda}\Big)^n \leq e^{M_1\omega}B = Be^{4M_1N} = Be^{e^{-L}M_1\lambda/B}.$$

9.4 A priori estimates

In this section assuming that the Cauchy problem for $P(x,D)$ is $\gamma^{(s)}$ well posed we derive a priori estimates following [22], [24]. Let L be a compact set in $\mathbb{R}^3$. Recall that

$$\gamma_0^{(s),h}(L) = \{f\in\gamma^{(s)}(\mathbb{R}^3) \mid \operatorname{supp} u\subset L, \exists C>0, h^{|\alpha|}|\partial_x^\alpha f(x)|\leq C(\alpha!)^s\}$$

which is the Banach space equipped with the norm

$$\sup_{x,\alpha}\frac{h^{|\alpha|}|\partial_x^\alpha f(x)|}{(\alpha!)^s}.$$

In the following sections we fix $h>0$ and $\delta>0$ so that Lemma 9.3.6 holds and hence we have (9.3.2) and (9.3.3). Consider

$$P_\lambda = P(\lambda^{-\sigma}x, \lambda^\sigma\xi)$$

where $\lambda^{-\sigma}x = (\lambda^{-\sigma_0}x_0, \lambda^{-\sigma_1}x_1, \lambda^{-\sigma_2}x_2)$ and $\sigma_j\geq 0$. Then we have

Lemma 9.4.1 *Assume that the Cauchy problem for P is $\gamma^{(s)}$ well posed near the origin. Let W be a compact neighborhood of the origin. Then there are $c>0$, $C>0$ such that*

$$|u|_{C^0(W^t)} \leq C\exp\big(c(\lambda^{\sigma_0}/\tau)^{1/(s-\kappa)}\big)\exp\big(\lambda^{\bar\sigma/s'}\big)\sum_\alpha\sup_{x_0\leq t+\tau}\frac{h^{|\alpha|}|\partial_x^\alpha P_\lambda u|}{(\alpha!)^{(s-s')}}$$

for any $u\in\gamma_0^{(s),h}(W_0)$, any $t>0$, $\tau>0$, any $1<s'<s$, any $1<\kappa<s$ where $\bar\sigma = \max_j\{\sigma_j\}$.

Proof: Assume that the Cauchy problem for P is $\gamma^{(s)}$ well posed. Let $h>0$ and K be a compact neighborhood of the origin. From the standard arguments it follows that there exists a neighborhood of the origin D such that for any

$f(x) \in \gamma_0^{(s),h}(K_0)$ there is a $u \in C^2(D)$ satisfying $Pu = f$ in D and $u = 0$ in $x_0 \le 0$ such that for any compact set $L \subset D$ there is $C > 0$ such that

$$|u|_{C^0(L)} \le C \sum_{\alpha} \sup \frac{h^{|\alpha|}|\partial_x^\alpha f(x)|}{(\alpha!)^s}$$

(see for example [39]). We may assume that $K \subset D$. Thus we have

$$|u|_{C^0(L)} \le C \sum_{\alpha} \sup \frac{h^{|\alpha|}|\partial_x^\alpha Pu|}{(\alpha!)^s}, \quad \forall u(x) \in \gamma_0^{(s),h}(K_0).$$

Let $\chi(r) \in \gamma^{(\kappa)}(\mathbb{R})$, $\kappa < s$, such that $\chi(r) = 1$ for $r \le 0$, $\chi(r) = 0$ for $r \ge 1$ and set $\chi_1(x_0) = \chi((x_0 - t)/\tau)$ so that

$$\begin{cases} \chi_1(x_0) = 1 & x_0 \le t, \\ \chi_1(x_0) = 0 & x_0 \ge t + \tau. \end{cases}$$

Let $u \in \gamma_0^{(s),h}(K_0)$ and consider $\chi_1 Pu$. Let $v \in C^2(D)$ be a solution to $Pv = \chi_1 Pu$ with $v = 0$ in $x_0 \le 0$. Since $Pv = Pu$ in $x_0 \le t$ and hence

$$|u|_{C^0(L^t)} = |v|_{C^0(L^t)} \le C \sum_{\alpha} \sup \frac{h^{|\alpha|}|\partial_x^\alpha (\chi_1 Pu)|}{(\alpha!)^s}.$$

Recall that $|\partial_x^\beta \chi_1(x)| \le C^{|\beta|+1}(\beta!)^\kappa \tau^{-|\beta|}$ and hence

$$\begin{aligned} \sum_{\alpha} \sup \frac{h^{|\alpha|}|\partial_x^\alpha (\chi_1 Pu)|}{(\alpha!)^s} &\le \sum \sup \frac{\alpha!}{\alpha_1!\alpha_2!} \frac{h^{|\alpha|}|\partial_x^{\alpha_1}\chi_1||\partial_x^{\alpha_2} Pu|}{(\alpha!)^s} \\ &\le \sum \sup \frac{1}{\alpha_1!\alpha_2!} \frac{h^{|\alpha|}|\partial_x^{\alpha_1}\chi_1||\partial_x^{\alpha_2} Pu|}{(\alpha_1!)^{s-1}(\alpha_2!)^{s-1}} \\ &\le \sum_{\alpha_1} \sup \frac{h^{|\alpha_1|}|\partial_x^{\alpha_1}\chi_1|}{(\alpha_1!)^s} \sum_{\alpha_2} \sup_{x_0 \le t+\tau} \frac{h^{|\alpha_2|}|\partial_x^{\alpha_2} Pu|}{(\alpha_2!)^s}. \end{aligned}$$

Since

$$\begin{aligned} \sum_{\alpha_1} \sup \frac{h^{|\alpha_1|}|\partial_x^{\alpha_1}\chi_1|}{(\alpha_1!)^s} &\le \sum_{\alpha_1} \frac{C^{|\alpha_1|+1}\tau^{-|\alpha_1|}h^{|\alpha_1|}}{(\alpha_1!)^{s-\kappa}} \\ \le C \exp\Big(c\big(\frac{1}{\tau}\big)^{1/(s-\kappa)}\Big) &\sum_{\alpha_1} (Ch)^{|\alpha_1|} \le C_h \exp\Big(c\big(\frac{1}{\tau}\big)^{1/(s-\kappa)}\Big) \end{aligned}$$

we have

$$|u|_{C^0(L^t)} \le C \exp\Big(c\big(\frac{1}{\tau}\big)^{1/(s-\kappa)}\Big) \sum_{\alpha} \sup_{x_0 \le t+\tau} \frac{h^{|\alpha|}|\partial_x^\alpha Pu|}{(\alpha!)^s}. \tag{9.4.1}$$

Let $u \in \gamma_0^{(s),h}(W_0)$. Then it is clear that $u(\lambda^\sigma x) \in \gamma_0^{(s),h}(K_0)$ for large λ. For $v(x) = u(\lambda^\sigma x)$ we apply the inequality (9.4.1) with $t = \lambda^{-\sigma_0}\hat{t}$, $\tau = \lambda^{-\sigma_0}\hat{\tau}$ to get

$$|v|_{C^0(L^t)} \le C \exp\Big(c\big(\frac{\lambda^{\sigma_0}}{\hat{\tau}}\big)^{1/(s-\kappa)}\Big) \sum_\alpha \sup_{x_0 \le t+\tau} \frac{h^{|\alpha|}|\partial_x^\alpha Pv|}{(\alpha!)^s}$$

where $Pv = Pu(\lambda^\sigma x) = (P_\lambda u)(\lambda^\sigma x)$ and hence

$$\partial^\alpha\big[(P_\lambda u)(\lambda^\sigma u)\big] = \lambda^{\langle \sigma,\alpha\rangle}(\partial_x^\alpha P_\lambda u)(\lambda^\sigma x).$$

Thus we have

$$\begin{aligned}
|u|_{C^0(W^{\hat{t}})} &\le Ce^{c\left(\frac{\lambda^{\sigma_0}}{\hat{\tau}}\right)^{1/(s-\kappa)}} \sum_\alpha \sup_{x_0 \le \hat{t}+\hat{\tau}} \frac{h^{|\alpha|}\lambda^{\bar{\sigma}|\alpha|}|\partial_x^\alpha (P_\lambda u)(x)|}{(\alpha!)^s} \\
&= Ce^{c\left(\frac{\lambda^{\sigma_0}}{\hat{\tau}}\right)^{1/(s-\kappa)}} \sum_\alpha \sup_{x_0 \le \hat{t}+\hat{\tau}} \frac{h^{|\alpha|}\lambda^{\bar{\sigma}|\alpha|}|\partial_x^\alpha (P_\lambda u)(x)|}{(\alpha!)^{s'}(\alpha!)^{s-s'}} \\
&\le Ce^{c\left(\frac{\lambda^{\sigma_0}}{\hat{\tau}}\right)^{1/(s-\kappa)}} e^{c\lambda^{\bar{\sigma}/s'}} \sum_\alpha \sup_{x_0 \le \hat{t}+\hat{\tau}} \frac{h^{|\alpha|}|\partial_x^\alpha (P_\lambda u)(x)|}{(\alpha!)^{s-s'}}.
\end{aligned}$$

This proves the assertion. □

9.5 Proof of not well-posed results

Take $\chi(x) \in \gamma_0^{(\kappa)}(W_0)$ such that $\chi(x) = 1$ in a neighborhood of x^* supported in $\{|x - x^*| \le \delta\}$ and $1 < \kappa < s$. Let us set $U_\lambda = E_\lambda V_\lambda \chi \in \gamma_0^{(s),h}(W_0)$ and note $|U_\lambda(x^*)| = 1$. Then we have from (9.2.2)

$$\begin{aligned}
P_\lambda U_\lambda &= (P_\lambda E_\lambda V_\lambda)\chi + \sum_{|\alpha|\le 1, 1\le|\beta|\le 2} c_{\alpha\beta}(x,\lambda)\partial_x^\alpha(E_\lambda V_\lambda)\partial_x^\beta \chi \\
&= E_\lambda Q_1 v_\lambda^{(N)}\chi + \sum_{|\alpha|\le 1, 1\le|\beta|\le 2} c_{\alpha\beta}(x,\lambda)\partial_x^\alpha(E_\lambda V_\lambda)\partial_x^\beta \chi.
\end{aligned}$$

To estimate the right-hand side we note

Lemma 9.5.1 *We have*

$$\sum_\alpha \sup_K \frac{h^{|\alpha|}|\partial_x^\alpha E_\lambda|}{(\alpha!)^s} \le C\exp\big(c\lambda^{2/s} + \lambda \sup_{x\in K}\{-\mathsf{Im}\,\phi(x)\}\big).$$

Proof: Recall that $E_\lambda = \exp\big(i\lambda^2 x_2 + i\lambda\phi(x)\big)$. Since $\phi(x)$ is real analytic in a neighborhood K of x^* then it is not difficult to check that

$$|\partial_x^\alpha e^{i\lambda\phi(x)}| \le C^{|\alpha|+1}(\lambda + |\alpha|)^{|\alpha|}e^{-\lambda \mathsf{Im}\,\phi(x)}, \quad x \in K$$

and hence we have

$$|\partial_x^\alpha E_\lambda| \le C^{|\alpha|+1}(\lambda^2+|\alpha|)^{|\alpha|}e^{-\lambda \mathsf{Im}\,\phi(x)}, \quad x\in K. \tag{9.5.1}$$

Noting that

$$\frac{h^{|\alpha|}(\lambda^2+|\alpha|)^{|\alpha|}}{(\alpha!)^s} \le Ce^{c\lambda^{2/s}}$$

we get the assertion. □

From Lemma 9.2.1 there exist $\nu>0$ and $\bar\tau>0$ such that $-\mathsf{Im}\,\phi(x)\le -\nu$ if $x\in \mathrm{supp}\,[\partial_x^\beta\chi]\cap\{x_0\le t+\tau\}$, $0<\tau\le\bar\tau$, $|\beta|\ge 1$. Then from Lemma 9.5.1 and (9.3.3) it follows that

$$\begin{aligned}\sum_\gamma \sup_{x_0\le t+\tau}&\frac{h^{|\gamma|}|\partial_x^\gamma(\partial_x^\alpha(E_\lambda V_\lambda)\partial_x^\beta\chi)|}{(\gamma!)^s}\\ &\le C\exp\,(c\lambda^{2/s}-\nu\lambda+e^{-L}M_1B^{-1}\lambda).\end{aligned} \tag{9.5.2}$$

We turn to $E_\lambda Q_1 v_\lambda^{(N)}\chi$. Thanks to Lemma 9.2.1 we have $-\mathsf{Im}\,\phi(x)\le 2a\tau$ if $x\in\mathrm{supp}\,[\chi]\cap\{x_0\le t+\tau\}$ where $a=|\mathsf{Im}\,\phi_{x_0}(x^*)|$. Thus from Lemma 9.5.1 and (9.3.2) it follows that

$$\begin{aligned}\sum_\alpha \sup_{x_0\le t+\tau}&\frac{h^{|\alpha|}|\partial_x^\alpha(E_\lambda Q_1 v_\lambda^{(N)}\chi)|}{(\alpha!)^s}\\ &\le C\exp\,(c\lambda^{2/s}+2a\tau\lambda-e^{-L}(L-4M_1)(4B)^{-1}\lambda).\end{aligned} \tag{9.5.3}$$

Let $s-s'>2$. Take L large so that $e^{-L}M_1B^{-1}<\nu$ and $L>4M_1$ and choose $\tau>0$ such that

$$2a\tau-e^{-L}(L-4M_1)(4B)^{-1}<0$$

then it is clear from (9.5.2) and (9.5.3) that

$$\sum_\alpha\sup_{x_0\le t+\tau}\frac{h^{|\alpha|}|\partial_x^\alpha(P_\lambda U_\lambda)|}{(\alpha!)^{s-s'}}\le Ce^{-\nu_1\lambda}$$

with some $\nu_1>0$. We now assume

$$s>6.$$

Recalling $\sigma_0=1$, $\sigma_1=2$, $\sigma_2=4$ and hence $\bar\sigma=4$ then we can choose $s'>4$ such that $s-s'>2$ and $\bar\sigma/s'<1$. Taking $1<\kappa<s$ so that $\sigma_0/(s-\kappa)<1$ we now apply Lemma 9.4.1 to get

$$|U_\lambda|_{C^0(W^t)}\le Ce^{-c\lambda+o(\lambda)}$$

with some $c>0$ as $\lambda\to\infty$. This gives a contradiction because

$$|U_\lambda(x^*)|=1.$$

This completes the proof of Theorem 9.1.1. □