Projective representations and spin characters of complex reflection groups $G(m,p,n)$ and $G(m,p,\infty)$, II

Case of generalized symmetric groups

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Abstract. In this paper we study projective (or spin) irreducible representations and their characters of generalized symmetric groups $G(m,1,n)$, and spin characters of their inductive limit groups $G(m,1,\infty) = \lim_{n\to\infty} G(m,1,n)$. The groups $G(m,1,n)$ form a subcategory of complex reflection groups $G(m,p,n)$, $p|m$, and the present study has a fundamental importance for such studies for general $G(m,p,n)$'s. Schur multipliers $Z = H^2\big(G(m,1,n), \boldsymbol{C}^\times\big)$ are isomorphic to $\boldsymbol{Z}_2^3 = \prod_{1\le i\le 3}\langle z_i\rangle$, $z_i^2 = e$, for $n \ge 4$ and $m \ge 2$ even, and similarly for $n = \infty$. Here, according to the semidirect product structure $G(m,1,n) = D_n \rtimes \mathfrak{S}_n$ with $D_n = \boldsymbol{Z}_m^n$, z_1 corresponds to the double covering group $\widetilde{\mathfrak{S}}_n$ of $\mathfrak{S}_n$, and z_2 to the double covering $\widetilde{D}_n$ of D_n, and z_3 to the twisted action of $\widetilde{\mathfrak{S}}_n$ on $\widetilde{D}_n$. In this case, any such representations and such characters have their own central characters $\chi \in \widehat{Z}$ with $(\beta_1,\beta_2,\beta_3)$, $\beta_i = \chi(z_i) = \pm 1$, called *(spin) type*. Our study here is for two types $(-1,-1,-1)$ and $(-1,-1,\ 1)$, and gives (1) classification and construction of all spin irreducible representations of $G(m,1,n)$, (2) calculation of their characters, (3) calculation of limits of normalized irreducible characters as $n \to \infty$, and (4) explicit determination of all the spin characters of $G(m,1,\infty)$ of these types.[7]

0 Introduction

1. We have nowadays many works on projective (or spin) representations of finite groups (see e.g. [Kar]), in particular for symmetric and alternating groups, after Schur's trilogy [Sch1]–[Sch3], restarting from Morris [Mor] and resulting to a book [HoHu2] by Hoffmann-Humphreys and the one [Kle] by Kleshchev, and so

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on. Also, for the spin theory of the infinite symmetric group $\mathfrak{S}_\infty = \lim_{n\to\infty} \mathfrak{S}_n$, we have a concise and beautiful paper [Naz] by Nazarov.

For the spin theory of the generalized symmetric groups (introduced by Osima [Osi]), we have also several works such as Read [Rea2], Hoffmann-Humphreys [HoHu1], Stembridge [Stem], and Morris and Jones [MoJo].

Starting from [HHH2], we are studying spin representations and spin characters of complex reflection groups $G(m,p,n)$ and of their inductive limits

$$G(m,p,\infty) := \lim_{n\to\infty} G(m,p,n), \qquad p|m.$$

Our methods are quite different from theirs, and elementary and apply fully the semidirect product structure of certain central extensions of $G(m,p,n), 4 \le n \le \infty$, such as $\widetilde{G}_n^{\mathrm{I}}$ and $\widetilde{G}_n^{\mathrm{II}}$, $4 \le n \le \infty$, in (0.1)–(0.2) below. Our methods are explained later in §3 in detail.

As is explained in our previous paper [HHH3], which we quote as [I] in the following, the case of generalized symmetric groups $G(m,1,n)$ and $G(m,1,\infty)$ (case of $p=1$) is decisive, and we call them *mother groups* among $G(m,p,n)$, $n \le \infty$. We treat here these groups in case m is even. (The case of m odd is much simpler, modulo the theory of spin representations for $\mathfrak{S}_n$ and $\mathfrak{A}_n$. Cf. Theorem 2.2 below or [I, Theorem 3.2].)

2. Schur multiplier $Z = H^2\big(G(m,1,n), \boldsymbol{C}^\times\big)$ has been given by Davies-Morris [DaMo]. For $4 \le n < \infty$, it is isomorphic to $\boldsymbol{Z}_2 = \langle z_1\rangle$, $z_1^2 = e$, if $m \ge 1$ odd, and to $\boldsymbol{Z}_2^3 = \prod_{1\le i\le 3}\langle z_i\rangle$, $z_i^2 = e$, if $m \ge 2$ even, and we see that it is also similar for $n = \infty$.

In the case of $G(m,1,n)$, m even, any spin irreducible (or factor) representations and their characters have their own central characters $\chi \in \widehat{Z}$ with $(\beta_1,\beta_2,\beta_3)$, $\beta_i = \chi(z_i) = \pm 1$, called *(spin) type*. Our study here is for a pair of *sister* cases: CASE I, Type $(-1,-1,-1)$, and CASE II, Type $(-1,-1,\ 1)$, and accomplishes

(1) to classify and construct all spin irreducible representations of $G(m,1,n)$,
(2) to calculate their characters and give general spin character formulas,
(3) to calculate limits of normalized such characters as $n \to \infty$, and
(4) to determine explicitly all the spin characters of $G(m,1,\infty)$, of these types.

The simultaneous treatment of CASEs I and II in this paper is very good for comparing the difference and the coincidence between two cases and thus clarifying the situations of spin representations in more details.

3. The representation group $R\big(G(m,1,n)\big)$ is a special central extension of $G(m,1,n)$ by the Schur multiplier $Z = \langle z_1, z_2, z_3\rangle$. Here, according to the semidirect product structure $G(m,1,n) = D_n \rtimes \mathfrak{S}_n$ with $D_n = \boldsymbol{Z}_m^n$, the central element z_1 gives the double covering group $\widetilde{\mathfrak{S}}_n$ of $\mathfrak{S}_n$, and z_2 gives the double covering $\widetilde{D}_n$ of D_n, and z_3 gives a twisted action of $\widetilde{\mathfrak{S}}_n$ on $\widetilde{D}_n$ (cf. Theorem 2.3). As in the previous paper, we choose from two representation groups $\mathfrak{T}_n$ and $\mathfrak{T}'_n$ of $\mathfrak{S}_n$ in [Sch3], the group $\mathfrak{T}'_n$ and denote it by $\widetilde{\mathfrak{S}}_n$.

Every spin representation of $G(m,1,n)$ can be linearized if it is lifted up to

$R\big(G(m,1,n)\big)$. By this, as in the previous paper [I], we divide spin IRs and spin characters of generalized symmetric groups $G(m,1,n)$ and $G(m,1,\infty)$ into 8 cases according to the central character $\chi \in \widehat{Z}$ or Type $(\beta_1, \beta_2, \beta_3)$. Here CASE VIII, Type (1, 1, 1), is the non-spin case of $G(m,1,n)$ and $G(m,1,\infty)$, which has been studied in detail in [HH1]. With these results in non-spin cases as a background, we have studied in Part II of [I] the spin CASE VII, Type $(1,\ 1,-1)$, a *sister* case of CASE VIII.

In the present paper, for the sister CASEs I and II, we introduce the quotient groups of the representation group $R\big(G(m,1,n)\big)$, $4 \le n \le \infty$, as

$$(0.1)\qquad \begin{aligned} \widetilde{G}^{\mathrm{I}}_n &:= R\big(G(m,1,n)\big)/\langle z_2{z_3}^{-1}\rangle && \text{in CASE I ;} \\ \widetilde{G}^{\mathrm{II}}_n &:= R\big(G(m,1,n)\big)/\langle z_3\rangle && \text{in CASE II}. \end{aligned}$$

Then they can be expressed as semidirect products as

$$(0.2)\qquad \widetilde{G}^{\mathrm{Y}}_n = \widetilde{D}_n \overset{\mathrm{Y}}{\rtimes} \widetilde{\mathfrak{S}}_n \qquad (\mathrm{Y{=}I,\ II}),$$

where $\widetilde{D}_n$ denotes the double cover of the canonical normal subgroup D_n, and $\overset{\mathrm{Y}}{\rtimes}$ means that the action of $\widetilde{\mathfrak{S}}_n$ on $\widetilde{D}_n$ is considered in CASE Y. Then, in each CASE Y (Y=I, II), our study on $R\big(G(m,1,n)\big)$ is moved principally to that on $\widetilde{G}^{\mathrm{Y}}_n$. We utilize fully the above semidirect structure of the groups, in particular, we use the *classical induced representation method* (Theorem 4.1 in [E]) (a classic of Mackey type method) to construct all the spin IRs and calculate their characters (spin characters).

4. For the case of inductive limit groups $\widetilde{G}^{\mathrm{Y}}_\infty = \lim_{n\to\infty} \widetilde{G}^{\mathrm{Y}}_n$, we calculate pointwise limits of normalized spin irreducible characters of $\widetilde{G}^{\mathrm{Y}}_n$ as $n \to \infty$ for each of Y = I, II. Then we obtain character formula by this limiting process as follows. Denote by $\mathcal{F}^{\mathrm{Y}}$ the set of all such limit functions on $\widetilde{G}^{\mathrm{Y}}_\infty$ obtained here, then it consists of normalized central spin positive definite functions. Actually they give exactly the set of normalized characters of $\widetilde{G}^{\mathrm{Y}}_\infty$ in CASE Y (see §14 in [I] for general aspects of limiting process and Vershik-Kerov's ergodic method).

We note that Dudko and Nessonov calculated spin characters of $R\big(G(m,1,\infty)\big)$ in [DuNe] by a completely different method (cf. §25).

In CASE I, as is proved in Part I in [I] and is quoted in Table 4.2 below, the criterion (EF) holds for $\widetilde{G}^{\mathrm{I}}_\infty$ (cf. §3) which says

"a normalized central positive definite function f on $\widetilde{G}^{\mathrm{I}}_\infty$ is extremal (i.e., is a character) if and only if it is factorizable."

Using this criterion, we can prove the completeness of the set $\mathcal{F}^{\mathrm{I}}$, that is, $\mathcal{F}^{\mathrm{I}}$ is exactly equal to the set $E^{\mathrm{I}}\big(\widetilde{G}^{\mathrm{I}}_\infty\big)$ of all spin characters of $\widetilde{G}^{\mathrm{I}}_\infty$ of Type $(-1,-1,-1)$. Thus we obtain a general formula and a parametrization of spin characters of $\widetilde{G}^{\mathrm{I}}_\infty$, similarly as in [HH1], for the non-spin case of $G(m,1,\infty)$.

In CASE II, we see, with the explicit form of these limit functions, $f \in \mathcal{F}^{\mathrm{II}}$ is not factorizable in general, and so the criterion (EF) does not hold. For the group $\widetilde{G}^{\mathrm{II}}_{\infty}$, we prove the completeness of $\mathcal{F}^{\mathrm{II}}$ or $\mathcal{F}^{\mathrm{II}} = E^{\mathrm{II}}(\widetilde{G}^{\mathrm{II}}_{\infty})$, by using that of $\mathcal{F}^{\mathrm{I}}$, and give explicitly a surjective (in general 2-1) map from $E^{\mathrm{I}}(\widetilde{G}^{\mathrm{I}}_{\infty})$ onto $E^{\mathrm{II}}(\widetilde{G}^{\mathrm{II}}_{\infty})$. This is similar to the situation studied in Part II of [I], where the completeness for CASE VII, Type $(1, 1, -1)$, is deduced from that for CASE VIII, the non-spin case of $G(m, 1, \infty)$.

The relations of our results on spin characters of $G(m, 1, \infty)$ with those in the work [DuNe] is given in §25 after reviewing it briefly, and the parametrization of spin characters in CASEs I, II and VII is recaptured.

Part I
Preparatory results

1 Generality for projective representations

1.1. Projective representations. Schur [Sch1, 1904] introduced the notion of projective representation of a group under the name "Darstellung durch gebrochene lineare Substitutionen". After him, we define a *projective representation* ρ of a group G, as a map $G \ni g \mapsto \rho(g)$, from G to the set of invertible linear operators on a vector space V satisfying

$$
\begin{aligned}
(1.1) \qquad & \rho(g)\rho(h) = r_{g,h}\, \rho(gh) \quad (g, h \in G),\\
& r_{g,h} \in \boldsymbol{C}^{\times} := \{z \in \boldsymbol{C}; z \neq 0\}.
\end{aligned}
$$

The function $r_{g,h}$ on $G \times G$ is called the *factor set* of ρ.

Replace $\rho(g)$ by $\rho'(g) := \lambda_g\, \rho(g)$ ($\lambda_g \in \boldsymbol{C}^{\times}$), then the factor set changes as

$$
(1.2) \qquad (r_{g,h})_{g,h\in G} \mapsto (r'_{g,h})_{g,h\in G}, \quad \lambda'_{g,h} = \frac{\lambda_g \lambda_h}{\lambda_{gh}} \cdot \lambda_{g,h}.
$$

Defining that $r_{g,h}$ and $r'_{g,h}$ are mutually equivalent, we have the cohomology group $H^2(G, \boldsymbol{C}^{\times})$ of factor sets modulo equivalence. For finite groups G, he proved the following.

(1) Any projective representation ρ can be lifted up to a linear representation of a *representation group* of G.

(2) For any finite group G, there exists a finite number of non-isomorphic representation groups, which are certain central extensions of G.

(3) A representation group of a finite group G is characterized as follows:

Theorem 1.1 (cf. [Sch2, Introduction]). *A group G^* is a representation group of a finite group G if and only if there exists a central subgroup Z of G^* such that*

(i) $Z \subset [G^*, G^*] \cap Z(G^*)$, $Z(G^*) :=$ *the center of* G^*,

(ii) $1 \longrightarrow Z \longrightarrow G^* \longrightarrow G \longrightarrow 1$ (exact).

(iii) $|Z| = |H^2(G, \boldsymbol{C}^\times)|$.

(4) The central subgroup Z is unique: $Z = H^2(G, \boldsymbol{C}^\times)$, which is called *Schur multiplier* of G.

(5) The theory of projective representations is mutually equivalent for any representation group of G.

So we take one of representation groups and denote it by $R(G)$, and even though it is not unique, we can call it as a *universal covering group* of G. In the following we call projective representations also *spin* representations (cf. **1.3** below).

1.2. Representation groups $\mathfrak{T}_n, \mathfrak{T}'_n$ of $\mathfrak{S}_n$. For symmetric groups $\mathfrak{S}_2$ and $\mathfrak{S}_3$, their Schur multipliers are trivial, and so their representation groups are themselves. For n-th symmetric groups $\mathfrak{S}_n$ with $n \geq 4$, Schur [Sch3, 1911] gave two representation groups $\mathfrak{T}_n$ and $\mathfrak{T}'_n$ of n-th symmetric group $\mathfrak{S}_n$ for $n \geq 4$, which are mutually isomorphic only when $n = 6$. The first one $\mathfrak{T}_n$ is used for the study of projective representations in [Sch3], [Mor] and so on, and also $\mathfrak{T}_\infty := \lim_{n\to\infty} \mathfrak{T}_n$ is used in [Naz3]. However we prefer to use the second one $\mathfrak{T}'_n$ and denote it by $R(\mathfrak{S}_n)$ or $\widetilde{\mathfrak{S}}_n$ hereafter. This is given as follows:

Theorem 1.2 ([Sch3]). *For $n \geq 4$, define groups $\mathfrak{T}'_n$ $(=: \widetilde{\mathfrak{S}}_n)$ by giving*

- *generators :* $\{z_1, r_1, r_2, \ldots, r_{n-1}\}$;
- *fundamental relations :*

$$\begin{cases} {z_1}^2 = e\,,\ z_1 r_i = r_i z_1, & 1 \leq i \leq n-1\,; \\ {r_i}^2 = e, & 1 \leq i \leq n-1\,; \\ (r_i r_{i+1})^3 = e, & 1 \leq i \leq n-2\,; \\ r_i r_j = z_1 r_j r_i, & 1 \leq i, j \leq n-1, |i-j| \geq 2\,; \end{cases}$$

$$1 \longrightarrow Z = \langle z_1 \rangle \longrightarrow \mathfrak{T}'_n \xrightarrow{\Phi_{\mathfrak{S}}} \mathfrak{S}_n \longrightarrow 1 \quad \text{(exact)},$$

where e denotes the identity element, and the canonical homomorphism is given by $\Phi_{\mathfrak{S}} : \mathfrak{T}'_n \ni r_i \mapsto s_i = (i\ \ i{+}1) \in \mathfrak{S}_n$. Then, $\mathfrak{T}'_n$ is a representation group of $\mathfrak{S}_n$.

Here the generator of the central subgroup $Z = H^2(\mathfrak{S}_n, \boldsymbol{C}^\times) \cong \boldsymbol{Z}_2$ is denoted by z_1 in accordance with the notation in Theorems 2.2 and 2.3 below.

Schur [Sch3] constructed so called "Hauptdarstellung" Δ_n of $\mathfrak{T}_n$, and used it as the fundamental ingredient to construct all the spin irreducible representations (=IRs) of $\mathfrak{T}_n$. It plays a similar role as the trivial IR $\mathbf{1}_{\mathfrak{S}_n}$ of $\mathfrak{S}_n$ in Frobenius' construction of all IRs of $\mathfrak{S}_n$ in [Frob1, 1900].

Remark 1.1. **(1)** Another representation group $\mathfrak{T}_n$ is given by the set of generators $\{z_1, r'_1, \ldots, r'_{n-1}\}$ and the set of fundamental relations $z_1{}^2 = e$, $z_1 r'_i = r'_i z_1$ $(i \in \boldsymbol{I}_{n-1})$; $r'^2_i = z_1$ $(i \in \boldsymbol{I}_{n-1})$, $(r'_i r'_{i+1})^3 = z_1$ $(i \in \boldsymbol{I}_{n-2})$; $r'_i r'_j = z_1 r'_j r'_i$ $(i, j \in \boldsymbol{I}_{n-1},\ |i-j| \geq 2)$; with $\boldsymbol{I}_n = \{1, 2, \ldots, n\}$ (cf. Theorem 1.1 in [E]).

(2) The reason of our preference of $\mathfrak{T}'_n$ than $\mathfrak{T}_n$ is principally because

(a) $\mathfrak{T}'_n$ is generated by elements of order 2, whereas by elements of order 4 for $\mathfrak{T}_n$, and (b) the braid relation $(r_i r_{i+1})^3 = e$ appears directly, whereas it appears in $\mathfrak{T}_n$ in its *'spin form'*, and further (c) the Schur multiplier of $\mathfrak{S}_3$ is trivial and so its representation group is $\mathfrak{S}_3$ itself, and for any $1 \leq i \leq n-1$, the subgroup $\mathfrak{S}^{(i)} = \langle r_i, r_{i+1} \rangle$ generated by r_i, r_{i+1} is nothing but the representation group $R\big(\mathfrak{S}^{(i)}\big) = \mathfrak{S}^{(i)} \cong \mathfrak{S}_3$ itself.

These facts for $\mathfrak{T}'_n$ facilitate manipulation of fundamental relations in calculations, and give more symmetry or cyclic property in many formulas obtained (for the property of r_i's see Lemmas 1.3$\sim$1.7 below, and cf. also §8 and §15).

(3) Let T be a representation of $\mathfrak{T}'_n$ such that $T(z_1) = -I$, where I denotes the identity operator. Put $T'(r'_j) := \sqrt{-1}\, T(r_j)$ $(j \in \boldsymbol{I}_{n-1})$. Then this gives a representation of $\mathfrak{T}_n$ satisfying $T'(z_1) = -I$, because it satisfies the fundamental relations for $\mathfrak{T}_n$. The correspondence T to T' is bijective.

1.3. Conjugacy relations in $\widetilde{\mathfrak{S}}_n$. The following facts will be necessary in the calculation of characters. First we give a definition as in [I]:

Definition 1.1. Put $r_{i,i+1} := r_i$, and for $i+1 < j$ in $\boldsymbol{I}_n = \{1, 2, \ldots, n\}$,

$$r_{ij} := r_i r_{i+1} \cdots r_{j-2} r_{j-1} r_{j-2} \cdots r_{i+1} r_i, \quad r_{ji} := r_{ij}^{-1} = r_{ij}. \tag{1.3}$$

Then $r_{ij}^2 = e$ and $s_{ij} = \Phi_{\mathfrak{S}}(r_{ij})$ is a transposition $(i\ \ j) \in \mathfrak{S}_n$. We put $r_{ii} = e$ for convenience. For $\sigma' \in \widetilde{\mathfrak{S}}_n$, put $L(\sigma') := L(\sigma)$, $\sigma = \Phi_{\mathfrak{S}}(\sigma')$, the length of σ with respect to simple reflections s_j $(j \in \boldsymbol{I}_{n-1})$. Then $L(r_{ij}) = L(s_{ij}) = 2|j-i| - 1$.

Lemma 1.3. (i) *Suppose* $\mathrm{supp}(r_k) := \{k, k+1\}$ *is disjoint with* $\{i, j\}$. *Then*

$$r_k r_{ij} r_k^{-1} = z_1\, r_{ij}$$

(ii) *For* $i+1 < j$,
$$\begin{cases} r_{i-1} r_{ij} r_{i-1}^{-1} = r_{i-1,j}, & r_i r_{ij} r_i^{-1} = r_{i+1,j}, \\ r_{j-1} r_{ij} r_{j-1}^{-1} = r_{i,j-1}, & r_j r_{ij} r_j^{-1} = r_{i,j+1}. \end{cases}$$

Proof. These are proved by calculations (cf. Proof for [I, Lemma 7.1 (i)]).

(i) If $i+1 < j-1$,
$$\begin{aligned} r_{i+1} r_{ij} r_{i+1}^{-1} &= r_{i+1}(r_i r_{i+1}\ \cdots\ r_{j-1}\ \cdots\ r_{i+1} r_i) r_{i+1} \\ &= (r_{i+1} r_i r_{i+1}) \cdots\ r_{j-1}\ \cdots (r_{i+1} r_i r_{i+1}) \\ &= (r_i r_{i+1} r_i) \cdots\ r_{j-1}\ \cdots (r_i r_{i+1} r_i) =\ \cdots\ = z_1\, r_{ij}\,. \end{aligned}$$

(ii) Using the following identities, the calculations are similar as (i) above:

$$r_{j-1}r_{j-2}r_{j-1}r_{j-2}r_{j-1} = r_{j-2}, \quad r_jr_{j-1}r_j = r_{j-1}r_jr_{j-1}. \qquad \square$$

Lemma 1.4. (i) *Let $i, j, k, \ell \in \boldsymbol{I}_n$ be different. Then* $r_{ij}r_{k\ell} = z_1r_{k\ell}r_{ij}$.
(ii) *Let $j, k, \ell \in \boldsymbol{I}_n$ be mutually different. Then*

$$r_{jk}r_{k\ell}r_{jk}^{-1} = z_1^{k-j-1}r_{j\ell} = z_1^{(L(r_{jk})-1)/2}r_{j\ell}\,.$$

Proof. (i) Suppose $i < j,\ k < \ell$. In case $i < k < j < \ell$, by Lemma 1.3,
$r_{k\ell}^{-1}r_{ij}r_{k\ell} =$
$= (r_kr_{k+1}\cdots r_{\ell-2}r_{\ell-1}r_{\ell-2}\cdots r_{k+1}r_k)\cdot r_{ij}\cdot(r_kr_{k+1}\cdots r_{\ell-2}r_{\ell-1}r_{\ell-2}\cdots r_{k+1}r_k)$
$= (r_kr_{k+1}\cdots r_{\ell-2}r_{\ell-1}r_{\ell-2}\cdots r_jr_{j-1})\cdot z_1^{j-1-k}r_{ij}\cdot(r_{j-1}r_j\cdots r_{\ell-2}r_{\ell-1}r_{\ell-2}\cdots r_{k+1}r_k)$
$= (r_kr_{k+1}\cdots r_{\ell-2}r_{\ell-1}r_{\ell-2}\cdots r_j)\cdot z_1^{j-1-k}r_{i,j-1}\cdot(r_j\cdots r_{\ell-2}r_{\ell-1}r_{\ell-2}\cdots r_{k+1}r_k)$
$= (r_kr_{k+1}\cdots r_{j-2}r_{j-1})\cdot z_1^{j-1-k}z_1r_{i,j-1}\cdot(r_{j-1}r_{j-2}\cdots r_{k+1}r_k)$
$= (r_kr_{k+1}\cdots r_{j-2})\cdot z_1^{j-1-k}z_1r_{ij}\cdot(r_{j-2}\cdots r_{k+1}r_k)$
$= z_1^{j-1-k}z_1z_1^{j-1-k}r_{ij} = z_1r_{ij}.$
For other cases, the calculations are similar and omitted here.

(ii) Suppose $j < k < \ell$. Then
$r_{jk}r_{k\ell}r_{jk}^{-1} =$
$= (r_jr_{j+1}\cdots r_{k-2}r_{k-1}r_{k-2}\cdots r_{j+1}r_j)\cdot r_{k\ell}\cdot(r_jr_{j+1}\cdots r_{k-2}r_{k-1}r_{k-2}\cdots r_{j+1}r_j)$
$= (r_jr_{j+1}\cdots r_{k-2}r_{k-1})\cdot z_1^{k-1-j}r_{k\ell}\cdot(r_{k-1}r_{k-2}\cdots r_{j+1}r_j)$
$= (r_jr_{j+1}\cdots r_{k-2})\cdot z_1^{k-1-j}r_{k-1,\ell}\cdot(r_{k-2}\cdots r_{j+1}r_j) \;=\; \cdots\cdots$
$= z_1^{k-1-j}r_{j\ell}$
The other cases can be reduced to the above case. $\square$

Let $\sigma \in \mathfrak{S}_n$ be a cycle $(k_1\ \ k_2\ \ \ldots\ \ k_\ell)$ and let $\ell(\sigma) := \ell$ be the length of the cycle. Then $L(\sigma) \equiv \ell(\sigma) \pmod 2$. We have two preimages $\sigma' \in \widetilde{\mathfrak{S}}_n$ of σ as $\sigma = \Phi_{\mathfrak{S}}(\sigma')$, one of which is

$$\sigma' := r_{k_1,k_2}\,r_{k_2,k_3}\,\cdots\,r_{k_{\ell-1},k_\ell}. \tag{1.4}$$

To fix a choice of the preimage if necessary, we may assume that k_1 is the smallest among $\{k_1,\ldots,k_\ell\}$. In fact, even though σ can be expressed also as $(k_2\ \ k_3\ \ \ldots\ \ k_\ell\ \ k_1)$ on the level of the base group $\mathfrak{S}_n$, its two preimages $\sigma'' := r_{k_2,k_3}\cdots r_{k_{\ell-1},k_\ell}\,r_{k_\ell,k_1}$ and $\sigma' = r_{k_1,k_2}\,r_{k_2,k_3}\,\cdots\,r_{k_{\ell-1},k_\ell}$ in the covering group $\widetilde{\mathfrak{S}}_n$ are not usually equal to each other, as is shown in the next lemma.

Lemma 1.5. $r_{k_2,k_3}\,\cdots\,r_{k_{\ell-1},k_\ell}\,r_{k_\ell,k_1} = z_1^{X}\,r_{k_1,k_2}\,r_{k_2,k_3}\,\cdots\,r_{k_{\ell-1},k_\ell}$,
with $\quad X = \frac{1}{2}\sum_{2\le p\le \ell-1}\big(L(r_{k_p,k_{p+1}})-1\big).$

Proof. $\sigma''^{-1}\sigma' = (r_{k_\ell,k_1}r_{k_{\ell-1},k_\ell}\cdots r_{k_2,k_3})(r_{k_1,k_2}r_{k_2,k_3}\cdots r_{k_{\ell-1},k_\ell})$
$= r_{k_\ell,k_1}(r_{k_{\ell-1},k_\ell}\cdots r_{k_3,k_4})\,z_1^{(L(r_{k_2,k_3})-1)/2}r_{k_1,k_3}\,(r_{k_3,k_4}\cdots r_{k_{\ell-1},k_\ell})$
$= \cdots = r_{k_\ell,k_1}\,z_1^{X}\,r_{k_1,k_\ell} \;=\; z_1^{X},$

with $X = \frac{1}{2}\big((L(r_{k_2,k_3})-1)+\cdots+(L(r_{k_{\ell-1},k_\ell})-1)\big)$. □

For $\sigma \in \mathfrak{S}_n$, put $\mathrm{supp}(\sigma) := \{p \in \boldsymbol{I}_n \,;\, \sigma(p) \neq p\}$ and $\mathrm{supp}(\sigma') := \mathrm{supp}(\sigma)$ for any preimage σ' of σ. Then, for the cycle σ above, $\mathrm{supp}(\sigma) = \{k_1, k_2, \cdots, k_\ell\}$.

Put $\tau'_\ell = r_1 r_2 \cdots r_{\ell-1} \in \widetilde{\mathfrak{S}}_n$, then $\Phi_{\mathfrak{S}}(\tau'_\ell) = (1\ 2\ 3\ \ldots\ \ell) \in \mathfrak{S}_n$. An arbitrary cycle $\sigma' = r_{k_1,k_2} r_{k_2,k_3} \cdots r_{k_{\ell-1},k_\ell}$ in (1.4) of length ℓ is conjugate to τ'_ℓ under $\widetilde{\mathfrak{S}}_n$ modulo a multiple of powers of z_1, that is, $\tau'\sigma'\tau'^{-1} = z_1^Y$ for some $\tau' \in \widetilde{\mathfrak{S}}_n$. We show in the next lemma that the exponent Y is computable in two steps.

Lemma 1.6. (i) *Let* $\boldsymbol{I}_\ell \backslash \mathrm{supp}(\sigma') = \{u_1, \ldots, u_p\}$, $\mathrm{supp}(\sigma') \backslash \boldsymbol{I}_\ell = \{v_1, \ldots, v_p\}$. *Put* $\tau' = r_{u_1,v_1} \cdots r_{u_p,v_p}$. *Then* $\tau'\sigma'\tau'^{-1}$ *is* z_1^Y*-multiple of the element obtained from* σ' *by replacing* v_i *by* u_i *for* $1 \leq i \leq p$, *with* $Y = p(\ell-2)+\sum_{1\leq i\leq p}(v_i-u_i-1)$, *and* $\mathrm{supp}(\tau'\sigma'\tau'^{-1}) = \boldsymbol{I}_\ell$.

(ii) *Suppose* $\mathrm{supp}(\sigma') = \boldsymbol{I}_\ell$. *Put* $\tau' = r_{k_1,k_2}$ *(resp.* $\tau' = r_{k_{\ell-1},k_\ell}$*). Then* $\tau'\sigma'\tau'^{-1}$ *is* z_1^Y*-times the element obtained from the expression (1.4) of* σ' *by exchanging* k_1 *and* k_2 *(resp.* $k_{\ell-1}$ *and* k_ℓ*), where*

$$Y \equiv k_2 - k_1 + \ell \ \ (\text{resp. } Y \equiv k_p - k_{p-1} + \ell) \pmod 2.$$

Put $\tau' = r_{k_i,k_{i+1}}$ $(2 \leq i \leq \ell-2)$. *Then* $\tau'\sigma'\tau'^{-1}$ *is* z_1^ℓ*-times the element obtained from the expression (1.4) of* σ' *by exchanging* k_i *and* k_{i+1}.

(iii) *For* σ' *in (1.4), take a* $\tau' \in \widetilde{\mathfrak{S}}_n$ *such that* $\tau\sigma\tau^{-1} = s_1 s_2 \cdots s_{\ell-1} = (1\ 2\ \ldots\ \ell)$ *for* $\sigma = \Phi_{\mathfrak{S}}(\sigma')$, $\tau = \Phi_{\mathfrak{S}}(\tau')$. *Then* $\tau'\sigma'\tau'^{-1} = z_1^Y r_1 r_2 \cdots r_{\ell-1}$, *and there exists a process to determine the exponent* Y *by means of* (i) *and* (ii) *above.*

Proof. (i) $r_{u_p,v_p}\sigma' r_{u_p,v_p}$ gives z_1-factor with exponent $(\ell-2)+(v_p-u_p-1)$ by Lemma 1.4, and so on.

(ii) For $i=1$ or $i=\ell-1$, $r_{k_i,k_{i+1}}\sigma' r_{k_i,k_{i+1}}$ gives z_1-factor with exponent $(k_{i+1}-k_i-1)+(\ell-3) \equiv k_{i+1}-k_i+\ell$.

For $2 \leq i \leq \ell-2$, $r_{k_i,k_{i+1}}\sigma' r_{k_i,k_{i+1}}$ gives z_1-factor with exponent $2(k_{i+1}-k_i-1)+(\ell-4) \equiv \ell$. □

Lemma 1.7. *For* $\tau'_\ell = r_1 r_2 \cdots r_{\ell-1} \in \widetilde{\mathfrak{S}}_n$,

$$(\tau'_\ell)^\ell = z_1^Y \quad \text{with} \quad Y = \frac{[\ell/2]\big([\ell/2]-1\big)}{2}. \tag{1.5}$$

If $\ell \equiv 0,1,2,3 \pmod 8$, *then* $(\tau'_\ell)^\ell = e$ *and* τ'_ℓ *is of order* ℓ.
If $\ell \equiv 4,5,6,7 \pmod 8$, *then* $(\tau'_\ell)^\ell = z_1$ *and* τ'_ℓ *is of order* 2ℓ.

Proof. By calculation, we have

$$(\tau'_\ell)^2 = (r_2 \cdots r_{\ell-1})(r_1 r_2 r_3 \cdots r_{\ell-2})$$
$$\big(\because\ (r_1 \cdots r_{\ell-1}) r_i = z_1^{\ell-3} r_{i+1} (r_1 \cdots r_{\ell-1}),\ i \leq \ell-2\big)\ ;$$
$$(\tau'_\ell)^3 = z_1^{\ell-3}(r_2 \cdots r_{\ell-1})^2 (r_1 r_2 r_3 \cdots r_{\ell-3})\ ;$$

$$(\tau'_\ell)^4 = z_1^{(\ell-3)+(\ell-4)\cdot 2}(r_2\cdots r_{\ell-1})^3(r_1r_2r_3\cdots r_{\ell-4})\ ;$$
$$\cdots\quad \cdots\quad \cdots$$
$$(\tau'_\ell)^{\ell-1} = z_1^{(\ell-3)\cdot 1+(\ell-4)\cdot 2+\cdots+1\cdot(\ell-3)}(r_2\cdots r_{\ell-1})^{\ell-2}r_1\ ;$$
$$(\tau'_\ell)^{\ell} = z_1^{(\ell-3)\cdot 1+(\ell-4)\cdot 2+\cdots+1\cdot(\ell-3)}(r_2\cdots r_{\ell-1})^{\ell-1}.$$

Continuing this process, we obtain (1.5). The details are omitted. □

1.4. Terminology. Here we fix our terminology in the following. We call projective representations of a group G also *spin* representations of G similarly as Morris did for the symmetric groups $\mathfrak{S}_n$, and as one did for the rotation group $SO(3)$ etc. classically.

A projective representation of G is a linear representations of a certain covering group $\widetilde{G}$ of G. The relation between the levels of G and $\widetilde{G}$ can be visually expressed as in the following diagram, vertically written.

Diagram 1.1.	covering group level	$\widetilde{G}$	seeing from upper level $\downarrow$
		$\downarrow$	
	base (ground) level	G	seeing from lower level $\uparrow$

We call characters of spin representations as *spin* characters of G although it is a function on $\widetilde{G}$. Our understanding for terminology hereafter is as follows.

The general adjective "**projective**" for representations is used when we see them from the base level G upwards towards the *covering group level* $\widetilde{G}$. In this sense, the use of the adjective "projective" is unilateral.

However, in some cases, we would like to convert the direction of our eyes and see from the level of $\widetilde{G}$ downwards to the *base level* of G (which should be well settled). In those cases, it is convenient to call a linear representation π of $\widetilde{G}$ a "**spin**" representation of $\widetilde{G}$ when it can not be reduced to a linear representation of G. This mode of using adjective "*spin*" is bilateral.

Moreover the adjective use of the word 'spin' goes well for functions on the groups: for instance, if a central positive definite function on $\widetilde{G}$ (e.g. a character) cannot be reduced to G, we call f a 'spin' function on $\widetilde{G}$ (and also a 'spin' function on G). We prefer the word 'spin' character better than 'projective' character.

This kind of terminology is similar, in a sense, to that of Schur in Note 1.1 (1) below for $\widetilde{\mathfrak{S}}_n$ (and $\mathfrak{S}_n$). See also §6, in particular Definition 6.1, and Theorems 13.6 and 13.7.

Note 1.1. (1) Schur called a character χ of $\widetilde{\mathfrak{S}}_n$ "*Charakter erster Art* (of the first kind)" or "*zweiter Art* (of the second kind)" depending on $\chi(z_1\sigma') = \chi(\sigma')$ or $\chi(z_1\sigma') = -\chi(\sigma')$ $(\sigma' \in \widetilde{\mathfrak{S}}_n)$ [Sch3, §13]. He also called a representation ρ of $\widetilde{\mathfrak{S}}_n$ "*Darstellung erster Art*" or "*zweiter Art*" depending on $\rho(z_1\sigma') = \rho(\sigma')$ or $\rho(z_1\sigma') = -\rho(\sigma')$ for $\sigma' \in \widetilde{\mathfrak{S}}_n$ [Sch3, §27].

(2) Projective representations can also be called as **double-valued** representations for $G = \mathfrak{S}_n, SO(n)$ and $SO_0(3,1)$ $\big($with $\widetilde{G} = \widetilde{\mathfrak{S}}_n$, $Spin(n)$ and $SL(2,\boldsymbol{C})\big)$, and in general as **multiple-valued** representations of G.

(3) In the book [HoHu2], Hoffman and Humphreys introduced the terminology **positive** and **negative** for representations (and also for modules and characters) of groups in a certain category which contains covering groups of $\mathfrak{S}_n$.

2 $G(m,p,n)$ and their representation groups

2.1 Definition of complex reflection groups $G(m,p,n)$

2.1.1. Constructive definition. First we give a constructive definition of complex reflection groups $G(m,p,n)$. For a set I, denote by $\mathfrak{S}_I$ the group of finite permutations on I. For $I = \boldsymbol{I}_n = \{1,2,\ldots,n\}$ or $I = \boldsymbol{I}_\infty := \boldsymbol{N}$, the suffices I are usually replaced by n or ∞ respectively: $\mathfrak{S}_{\boldsymbol{I}_n} = \mathfrak{S}_n$, $\mathfrak{S}_{\boldsymbol{N}} = \mathfrak{S}_\infty$. Take a finite abelian group T and define the wreath product groups $\mathfrak{S}_I(T)$ as follows:

$$\mathfrak{S}_I(T) := D_I(T) \rtimes \mathfrak{S}_I, \quad D_I(T) := \prod_{j\in I}{}' T_j, \quad T_j := T \ \ (j \in I), \tag{2.1}$$

where $\prod'$ denotes the restricted direct product if I is infinite, and $\mathfrak{S}_I$ acts on $D_I(T)$ naturally by permuting the components. For a subgroup S of T, we have a canonical normal subgroup of $\mathfrak{S}_I(T)$ given as

$$\mathfrak{S}_I(T)^S := D_I(T)^S \rtimes \mathfrak{S}_I, \quad D_I(T)^S := \{d \in D_I(T)\ ;\ P(d) \in S\}, \tag{2.2}$$

where $P(d) := \prod_{j\in I} t_i$ for $d = (t_j)_{j\in I} \in D_I(T)$. Later on, the index I is replaced by n or ∞ according to $I = \boldsymbol{I}_n$ or $I = \boldsymbol{I}_\infty = \boldsymbol{N}$.

Now let $T = \boldsymbol{Z}_m$, understood as a multiplicative group. Then the groups $\mathfrak{S}_n(\boldsymbol{Z}_m) = D_n(\boldsymbol{Z}_m) \rtimes \mathfrak{S}_n$ were introduced in [Osi] and called *generalized symmetric groups*, and we put $G(m,1,n) := \mathfrak{S}_n(\boldsymbol{Z}_m)$. Any subgroup of $T = \boldsymbol{Z}_m$ is given as

$$S(p) := \{t^p\ ;\ t \in T\} \cong \boldsymbol{Z}_{m/p} \qquad \text{for a divisor } p \text{ of } m, \tag{2.3}$$

and we put $G(m,p,n) := \mathfrak{S}_n(\boldsymbol{Z}_m)^{S(p)}$ for n finite and also for $n = \infty$. Then,

$$G(m,p,\infty) := \lim_{n\to\infty} G(m,p,n) \qquad \text{for } \ p|m. \tag{2.4}$$

In [HH1], we have studied the characters of $\mathfrak{S}_\infty(T)$ and of $\mathfrak{S}_\infty(T)^S$ for any finite abelian group T and its subgroup S. This serves as a basic ingredient in our present study.

2.1.2. Generators and fundamental relations. Let us give a presentation of generalized symmetric groups $G(m,1,n)$, called in [I] *mother groups* among $G(m,p,n)$, by giving a set of generators and a set of fundamental relations. This presentation is convenient to treat their representation groups.

Proposition 2.1. **(i)** *The generalized symmetric group* $G(m,1,n) = \mathfrak{S}_n(\boldsymbol{Z}_m)$, $n \geq 3$, *is presented by*

- *set of generators :* $\{s_1, s_2, \ldots, s_{n-1},\ y_1, y_2, \ldots, y_n\}$,
  where y_j *corresponds to a generator* y *of* $T = \boldsymbol{Z}_m \cong T_j$;
- *set of fundamental relations:*

$$(2.5)\qquad \begin{cases} s_i^{\,2} = e \ \ (1 \leq i \leq n-1), & (s_i s_{i+1})^3 = e \ \ (1 \leq i \leq n-2), \\ s_i s_j = s_j s_i \ \ (|i-j| \geq 2). \end{cases}$$

$$(2.6)\qquad \begin{cases} y_j^{\,m} = e \qquad (1 \leq j \leq n), \\ y_j y_k = y_k y_j \qquad (j \neq k), \\ s_i y_i s_i^{-1} = y_{i+1}, \ \ s_i y_{i+1} s_i^{-1} = y_i \quad (1 \leq i \leq n-1), \\ s_i y_j s_i^{-1} = y_j \qquad (j \neq i, i+1). \end{cases}$$

(ii) *The inductive limit* $G(m,1,\infty)$ *is presented by giving the set of generators* $\big\{s_i,\ y_j\ (i,j \in \boldsymbol{I}_\infty)\big\}$ *and the set of fundamental relations above but without the restrictive conditions containing* n *on the suffices* i *and* j.

2.2 Representation groups of generalized symmetric groups

For a generalized symmetric group $G = G(m,1,n) = \mathfrak{S}_n(\boldsymbol{Z}_m)$, Davies and Morris [DaMo] gave its *Schur multiplier* $H^2(G, \boldsymbol{C}^\times)$ and also one of its representation groups. Thus we have the following depending on the parity of m (cf. Theorems 3.2 and 3.3 in [I]).

Theorem 2.2 (Case m odd). **(i)** *Suppose* $n \geq 4$ *and* m *is odd. For* $G(m,1,n) = \mathfrak{S}_n(\boldsymbol{Z}_m)$, *a representation group* $R\big(G(m,1,n)\big)$ *is given as*

$$\{e\} \longrightarrow Z \longrightarrow R\big(G(m,1,n)\big) \xrightarrow{\ \Phi\ } G(m,1,n) \longrightarrow \{e\} \quad \text{(exact)},$$

- *set of generators :* $\big\{z_1,\ r_i\,(1 \leq i \leq n-1),\ \eta_j\,(1 \leq j \leq n)\big\}$;
  $\Phi(r_i) = s_i\ (1 \leq i \leq n-1),\ \Phi(\eta_j) = y_j\ (1 \leq j \leq n)$;
- *set of fundamental relations :*

$$\text{(i)}\quad z_1^{\,2} = e\,, \quad z_1 \ \text{central element}\,;$$

$$\text{(ii)}\quad \begin{cases} r_i^{\,2} = e \ \ (1 \leq i \leq n-1), & (r_i r_{i+1})^3 = e \ \ (1 \leq i < n-1), \\ r_i r_j = z_1 r_j r_i \ \ (|i-j| \geq 2), \end{cases}$$

$$\text{(iii)}\quad \eta_j^{\,m} = e \qquad (1 \leq j \leq n),$$

$$\text{(iv)}\quad \eta_j \eta_k = \eta_k \eta_j \ \ (j \neq k),$$

$$\text{(v)}\quad \begin{cases} r_i \eta_i r_i^{-1} = \eta_{i+1}, \ \ r_i \eta_{i+1} r_i^{-1} = \eta_i \quad (1 \leq i \leq n-1), \\ r_i \eta_j r_i^{-1} = \eta_j \qquad (j \neq i, i+1)\,; \end{cases}$$

$$Z = H^2\big(G(m,1,n), \boldsymbol{C}^\times\big) = \langle z_1 \rangle \cong \boldsymbol{Z}_2.$$

(ii) *For $n = \infty$, a representation group $R\big(G(m,1,\infty)\big)$ is defined as the inductive limit $\lim_{n\to\infty} R\big(G(m,1,n)\big)$. Then it is presented by a set of generators $\big\{r_i,\ \eta_j\ (i,j \in \boldsymbol{I}_\infty)\big\}$ and a set of fundamental relations above but without the restrictive conditions containing n on the suffices i and j.*

Theorem 2.3 (Case m even)**.** **(i)** *Suppose $n \geq 4$ and m is even. Then, for $G(m,1,n) = \mathfrak{S}_n(\boldsymbol{Z}_m)$, a representation group $R\big(G(m,1,n)\big)$ is given as*

$$\{e\} \longrightarrow Z \longrightarrow R\big(G(m,1,n)\big) \xrightarrow{\Phi} G(m,1,n) \longrightarrow \{e\} \quad \text{(exact)},$$

- *set of generators :* $\big\{z_1, z_2, z_3,\ r_i\,(1 \leq i \leq n-1),\ \eta_j\,(1 \leq j \leq n)\big\}$ *;*
  $\Phi(r_i) = s_i\ (1 \leq i \leq n-1),\ \Phi(\eta_j) = y_j\ (1 \leq j \leq n)$ *;*
- *set of fundamental relations :*

$$\begin{array}{ll}
\text{(i)} & z_i^{\,2} = e \ \ (1 \leq i \leq 3), \quad z_i \ \text{central element}\,; \\
\text{(ii)} & \begin{cases} r_i^{\,2} = e \ \ (1 \leq i \leq n-1), \quad (r_i r_{i+1})^3 = e \ \ (1 \leq i < n-1), \\ r_i r_j = z_1 r_j r_i \quad (|i-j| \geq 2), \end{cases} \\
\text{(iii)} & \eta_j^{\,m} = e \qquad (1 \leq j \leq n), \\
\text{(iv)} & \eta_j \eta_k = z_2 \eta_k \eta_j \ \ (j \neq k), \\
\text{(v)} & \begin{cases} r_i \eta_i r_i^{\,-1} = \eta_{i+1}, \ \ r_i \eta_{i+1} r_i^{\,-1} = \eta_i \quad (1 \leq i \leq n-1), \\ r_i \eta_j r_i^{\,-1} = z_3 \eta_j \qquad (j \neq i, i+1)\,; \end{cases}
\end{array}$$

$$Z = H^2\big(G(m,1,n), \boldsymbol{C}^\times\big) = \langle z_1, z_2, z_3 \rangle \cong \boldsymbol{Z}_2^{\,3}.$$

(ii) *For $n = \infty$, a representation group $R\big(G(m,1,\infty)\big)$ is defined as the inductive limit $\lim_{n\to\infty} R\big(G(m,1,n)\big)$. Then it is presented by a set of generators $\big\{r_i,\ \eta_j\ (i,j \in \boldsymbol{I}_\infty)\big\}$ and a set of fundamental relations above but without the restrictive conditions containing n on the suffices i and j.*

This structure theorem is the starting point of our whole study. We can reconfirm it admitting only th fact that $|H^2\big(G(m,1,n)\big)| = 2^3$. In fact, denote by G^* the group given in Theorem 2.3, then we see by Proposition 2.1 that $G^*/Z \cong G,\ G = G(m,1,n)$, with $Z = \langle z_1, z_2, z_3 \rangle$. Moreover the commutators are given as $[r_1, r_2] = z_1,\ [\eta_1, \eta_2] = z_2,\ [\eta_1, r_2] = z_3$, and so $Z \subset [G^*, G^*] \cap Z(G^*)$. Then, by Theorem 1.1, we see that G^* is a representation group of G.

Remark 2.1. In [I], the groups $G(m,p,n),\ p|m,\ p > 1$, are called *child groups* of the *mother group* $G(m,1,n)$, and their representation groups are given in [I, §3] after [Rea1]. Our present studies on mother groups are decisive for their children (loc.cit.).

Notation 2.1. For a $g = (d,\sigma) \in G(m,1,n) = D_n(\boldsymbol{Z}_m) \rtimes \mathfrak{S}_n$, put $\mathrm{supp}(g) := \mathrm{supp}(d) \cup \mathrm{supp}(\sigma) \subset \boldsymbol{I}_n$, $\mathrm{supp}(d) := \{p \in \boldsymbol{I}_n\,;\, t_p \neq e_T\}$ for $d =$

$(t_p)_{p\in \boldsymbol{I}_n}, t_p \in T_p = T$, and $\mathrm{supp}(\sigma) := \{i \in \boldsymbol{I}_n\,;\, \sigma(i) \neq i\}$, where e_T denotes the identity element of T. For any covering group $\widetilde{G}(m,1,n)$ of $G(m,1,n)$ with the canonical homomorphism $\Phi : \widetilde{G}(m,1,n) \to G(m,1,n)$, we put for $g' \in \widetilde{G}(m,1,n)$, $\mathrm{supp}(g') := \mathrm{supp}(g)$ with $g := \Phi(g') \in G(m,1,n)$.

2.3 CASE I, Type $(-1,-1,-1)$; CASE II, Type $(-1,-1,\ 1)$

Let m be even. In this paper, projective representation of generalized symmetric groups $G(m,1,n)$ for $4 \le n \le \infty$, or linear representations of $R\big(G(m,1,n)\big)$, and their characters are studied. In particular, we study spin irreducible representations (= IRs) and their characters, called also *spin*, for $n < \infty$, and spin characters for $n = \infty$ which correspond to factor representations of finite type. A representation π of such kind has its own (spin) type $\chi \in \widehat{Z}$ defined as $\pi(z) = \chi(z)I$ $(z \in Z)$, where I denotes the identity operator. Since m is assumed to be even, we have $Z = \langle z_1, z_2, z_3\rangle$, and the type χ is given by $\beta = (\beta_1, \beta_2, \beta_3)$, $\beta_i = \chi(z_i) = \pm 1$.

In the previous work [I], we have classified the cases depending on the type β (cf. Tables 8.1 and 9.1 in [I]), and studied fully CASE VII, Type $(1,\ 1,-1)$, in comparison with the non-spin case, CASE VIII, Type $(1,\ 1,\ 1)$. In this paper, we study another pair of *sister cases*, CASE I, Type $(-1,-1,-1)$, and CASE II, Type $(-1,-1,\ 1)$, in parallel.

Since m is even, we see from Theorem 1.3 that representation group $R\big(G(m,1,n)\big), 4 \le n \le \infty$, is a covering group of $G(m,1,n)$ of $2^3\ (= 8)$-fold. To go into detailed study, it is convenient for us to reduce $R\big(G(m,1,n)\big)$ to 4-fold covering groups as

(2.7) $\qquad \widetilde{G}^{\mathrm{I}}(m,1,n) := R\big(G(m,1,n)\big)/\langle z_2 {z_3}^{-1}\rangle \qquad$ in CASE I ;

(2.8) $\qquad \widetilde{G}^{\mathrm{II}}(m,1,n) := R\big(G(m,1,n)\big)/\langle z_3\rangle \qquad$ in CASE II .

The groups $\widetilde{G}^{\mathrm{I}}(m,1,n)$ and $\widetilde{G}^{\mathrm{II}}(m,1,n)$ are presented just as in Theorem 1.3 but replacing z_3 by z_2, and replacing z_3 by e, respectively.

Diagram 2.1. Covering groups

$$
\begin{array}{ll}
R\big(G(m,1,n)\big) & \\
\ \Phi^{Y} \downarrow & \text{double covering} \\
\widetilde{G}^{\mathrm{Y}}(m,1,n) & \text{(Y=I, II)} \\
\ \Phi_{\mathrm{Y}} \downarrow & \text{4-fold covering, Ker} = \widetilde{Z} := \langle z_1, z_2\rangle \\
G(m,1,n) &
\end{array}
$$

We put inside $R\big(G(m,1,n)\big)$, $4 \le n \le \infty$,

(2.9) $\qquad \widetilde{D}(m,n) := \big\langle \eta_j\ (j \in \boldsymbol{I}_n)\big\rangle, \qquad \widetilde{\mathfrak{S}}_n := \big\langle r_i\ (i \in \boldsymbol{I}_{n-1})\big\rangle,$

where we understand $\boldsymbol{I}_{n-1} = \boldsymbol{I}_\infty$ for $n = \infty$. Then they are both canonically imbedded into $\widetilde{G}^{\mathrm{Y}}(m,1,n)$ (Y = I, II) respectively, and $\widetilde{\mathfrak{S}}_n$ acts on $\widetilde{D}(m,n)$ in the following two different ways

$$\text{(I-v)}\quad \begin{cases} r_i\eta_i r_i^{-1} = \eta_{i+1},\ r_i\eta_{i+1}r_i^{-1} = \eta_i \quad (1 \le i \le n-1), \\ r_i\eta_j r_i^{-1} = z_2\eta_j \qquad (j \ne i, i+1)\,; \end{cases} \quad \text{(in CASE I)}\,;$$

$$\text{(II-v)}\quad \begin{cases} r_i\eta_i r_i^{-1} = \eta_{i+1},\ r_i\eta_{i+1}r_i^{-1} = \eta_i \quad (1 \le i \le n-1), \\ r_i\eta_j r_i^{-1} = \eta_j \qquad (j \ne i, i+1)\,; \end{cases} \quad \text{(in CASE II)}\,.$$

Proposition 2.4. *There holds the following semidirect product expression :*

$$\widetilde{G}^{\mathrm{I}}(m,1,n) = \widetilde{D}(m,n) \overset{\mathrm{I}}{\rtimes} \widetilde{\mathfrak{S}}_n \qquad \text{in CASE I}\,; \tag{2.10}$$

$$\widetilde{G}^{\mathrm{II}}(m,1,n) = \widetilde{D}(m,n) \overset{\mathrm{II}}{\rtimes} \widetilde{\mathfrak{S}}_n \qquad \text{in CASE II}\,, \tag{2.11}$$

where the numbers I *and* II *over the symbol '$\rtimes$' means that the action is understood according to* (I-v) *and* (II-v) *respectively, keeping other fundamental relations* (i) (*without* z_3) *and* (ii) – (iv) *in Theorem 1.3 in common.*

Remark 2.2. In §10 in [I], we defined $\widetilde{G}^{\mathrm{Y}}(m,1,\infty) := R\big(G(m,1,\infty)\big)/\mathrm{Ker}(\chi^{\mathrm{Y}})$. This definition differs slightly from the above one. In this paper we prefer the above one because it is more convenient to treat the above sister cases parallel as far as possible.

In CASE I, for the convenience of later calculations, we introduce new generators for $\widetilde{D}(m,n) = \langle \eta_1, \ldots, \eta_n \rangle$ as

$$\eta'_j := z_2^{\,j-1}\eta_j \quad (j \in \boldsymbol{I}_n), \tag{2.12}$$

Then the above relation (I-v) takes the following form:

$$\text{(I-v}'\text{)}\quad \begin{cases} r_i\eta'_i r_i^{-1} = z_2\eta'_{i+1},\ \ r_i\eta'_{i+1}r_i^{-1} = z_2\eta'_i \quad (1 \le i \le n-1), \\ r_i\eta'_j r_i^{-1} = z_2\eta'_j \qquad (j \ne i, i+1)\,; \end{cases} \quad \text{(in CASE I)}.$$

Theorem 2.5. *Suppose $n \ge 4$ and m is even. The covering group $\widetilde{G}^{\mathrm{I}}(m,1,n)$ of CASE I, Type $(-1,-1,-1)$, is given, with respect to the new generators η'_j's and the central subgroup $\widetilde{Z} = \langle z_1, z_2 \rangle$ as*

$$\{e\} \longrightarrow \widetilde{Z} \longrightarrow \widetilde{G}^{\mathrm{I}}(m,1,n) \xrightarrow{\Phi_{\mathrm{I}}} G(m,1,n) \longrightarrow \{e\} \quad \text{(exact)},$$

- *set of generators :* $\big\{z_1, z_2,\ r_i\,(1 \le i \le n-1),\ \eta'_j\,(1 \le j \le n)\big\}$ *;*

  $\Phi_{\mathrm{I}}(r_i) = s_i\ (1 \le i \le n-1),\ \Phi_{\mathrm{I}}(\eta'_j) = y_j\ (1 \le j \le n)$ *;*

- *set of fundamental relations :*

  (i) (*without* z_3), (ii), *and*

(iii′) $\eta_j'^{\,m} = e \qquad (1 \le j \le n)$,
(iv′) $\eta_j'\eta_k' = z_2\eta_k'\eta_j' \;\; (j \ne k)$,
(I-v′) *as above.*

Lemma 2.6. (i) *In the group* $\widetilde{G}^{\rm I}(m,1,n) = \widetilde{D}(m,n) \overset{\rm I}{\rtimes} \widetilde{\mathfrak{S}}_n$ *of CASE I,*

$$r_i\big(\eta_1'\eta_2'\cdots\eta_n'\big)r_i^{-1} = z_2^{\,n}\big(\eta_1'\cdots\eta_{i-1}'\eta_{i+1}'\eta_i'\eta_{i+2}'\cdots\eta_n'\big) =$$
$$= z_2^{n+1}\big(\eta_1'\eta_2'\cdots\eta_n'\big) = \begin{cases} z_2\big(\eta_1'\eta_2'\cdots\eta_n'\big) & \text{for } n = 2n' \text{ even}, \\ \eta_1'\eta_2'\cdots\eta_n' & \text{for } n = 2n'+1 \text{ odd}. \end{cases}$$

(ii) *In the group* $\widetilde{G}^{\rm II}(m,1,n) = \widetilde{D}(m,n) \overset{\rm II}{\rtimes} \widetilde{\mathfrak{S}}_n$ *of CASE II,*

$$r_i\big(\eta_1\eta_2\cdots\eta_n\big)r_i^{-1} = \eta_1\cdots\eta_{i-1}\eta_{i+1}\eta_i\eta_{i+2}\cdots\eta_n = z_2\big(\eta_1\eta_2\cdots\eta_n\big).$$

Note 2.1. For the sake of simplicity, we may denote $\Phi_{\rm Y}$ simply by Φ if there is no danger of misunderstanding.

2.4 Representation group $R(\mathfrak{A}_n)$ of the alternating group

For the alternating group $\mathfrak{A}_n$, its Schur multiplier is given in [Sch3, §4] as

$$H^2(\mathfrak{A}_n, \boldsymbol{C}^\times) = \begin{cases} \boldsymbol{Z}_2 & \text{for } \; n \ge 4, \; n \ne 6,7, \\ \boldsymbol{Z}_6 & \text{for } \; n = 6,7. \end{cases}$$

Its representation group $\mathfrak{B}_n := R(\mathfrak{A}_n)$ is unique and, in the regular cases of $n \ne 6,7$, is realized in $R(\mathfrak{S}_n)$ as the commutator group $\mathfrak{B}_n = \big[R(\mathfrak{S}_n), R(\mathfrak{S}_n)\big]$ with presentation given as follows.

Theorem 2.7 (cf. [Sch3, §5]). *Let $n \ge 4$, and put $v_i' := z_1 r_{i+1} r_1$ $(1 \le i \le n-2)$. Then they generate a double covering group $\widetilde{\mathfrak{A}}_n := \Phi_{\mathfrak{S}}^{-1}\big(\mathfrak{A}_n\big)$ of $\mathfrak{A}_n$, inside $R(\mathfrak{S}_n) = \widetilde{\mathfrak{S}}_n$. Moreover it is defined as an abstract group as follows:*

generators : $\big\{v_i' \,;\; 1 \le i \le n-2\big\}$,

fundamental relations :
$$\begin{cases} v_1'^{\,3} = z_1, \; (v_1'v_2')^3 = z_1, & \\ (v_1'v_i')^2 = z_1 & (3 \le i \le n-2), \\ v_i'^{\,2} = z_1 & (2 \le i \le n-2), \\ (v_i'v_{i+1}')^3 = z_1 & (2 \le i \le n-3), \\ v_i'v_j' = z_1 v_j'v_i' & (2 \le i,j \le n-2, \; |i-j| \ge 2). \end{cases}$$

Here there follows automatically the fact that the element z_1 is central and $z_1^2 = e$.

In case $n \ge 4, \ne 6,7$, the group $\widetilde{\mathfrak{A}}_n$ gives a representation group $R(\mathfrak{A}_n)$ (denoted in [loc. cit.] by $\mathfrak{B}_n$).

3 Method of our study in this paper

Our study in this paper go along the following line.

- The first step, STEP 1, is for constructing spin IRs of generalized symmetric groups $G(m,1,n)$, *mother groups* of all complex reflection groups $G(m,p,n)$, of Types $(-1,-1,-1)$ and $(-1,-1,\ 1)$, and calculating their characters.
- The second step, STEP 2, is for limiting process of irreducible characters as $n \to \infty$ and for studying characters of infinite $G(m,1,\infty)$ of the above (spin) types.

For construction of spin IRs, referring Theorems 3.1 and 4.1 in [E], we explain briefly a generality. Let G be a finite group and a semidirect product of a normal subgroup U and a subgroup S as $G = U \rtimes S$. Take an IR ρ of U and consider its equivalence class $[\rho]$ in the dual $\widehat{U}$ of U. Take a stationary subgroup $S([\rho])$ in S of $[\rho]$, and put $H := U \rtimes S([\rho])$. For each $s \in S([\rho])$ we determine explicitly an intertwining operator $J_\rho(s)$ such that

$$\rho\big(s(u)\big) = J_\rho(s)\,\rho(u)\,J_\rho(s)^{-1} \quad (u \in U), \tag{3.1}$$

where $s(u)$ denotes the action of s on u. Then $J_\rho(s)$ is determined up to a scalar multiple, and the map $S([\rho]) \ni s \mapsto J_\rho(s)$ gives a projective representation of $S([\rho])$. Let $\alpha_{s,t}$ be its factor set given as

$$J_\rho(s)J_\rho(t) = \alpha_{s,t}\,J_\rho(st) \quad \big(s,t \in S([\rho])\big).$$

For $s \mapsto \lambda(s)J_\rho(s),\ \lambda(s) \in \boldsymbol{C}^\times$, its factor set is $\alpha'_{s,t} = \alpha_{s,t}\cdot\lambda(s)\lambda(t)/\lambda(st)$.

Taking an appropriate $\{J_\rho(s)\ ;\ s \in S([\rho])\}$ among them, we have a certain covering group (a central extension) $S([\rho])^\sim$ of $S([\rho])$ such that J_ρ can be lifted up to a linear representation $\widetilde{J}_\rho$ of $S([\rho])^\sim$. Put, for $\widetilde{H} := U \rtimes S([\rho])^\sim$ with $s'(u) := s(u)$, $s' \in S([\rho])^\sim$, $s = \Phi_S(s')$,

$$\pi^0\big((u,s')\big) := \rho(u)\cdot\widetilde{J}_\rho(s') \quad \big(u \in U,\ s' \in S([\rho])^\sim\big), \tag{3.2}$$

then π^0 is an IR of $\widetilde{H}$ which we denote by $\rho\cdot\widetilde{J}_\rho$. Here $\Phi_S : S([\rho])^\sim \to S([\rho])$ is the canonical homomorphism. Take an IR π^1 of $S([\rho])^\sim$ and consider it as a representation of $\widetilde{H}$ through the homomorphism $\widetilde{H} \to S([\rho])^\sim \cong \widetilde{H}/U$, and consider the inner tensor product $\pi := \pi^0 \boxdot \pi^1$ as an IR of $\widetilde{H}$. Let the factor set of π^1, viewed as a spin representation of the base group $S([\rho])$, be $\beta_{s,t}$, then that of π is $\beta_{s,t}\,\alpha_{s,t}$. If J_ρ is replaced by $\lambda(\cdot)J_\rho(\cdot)$, then its effect may be resolved by replacing π^1 by $\chi\cdot\pi^1$ with a character χ of $S([\rho])^\sim$.

Suppose for simplicity that there exists a covering group $\widetilde{S}$ of S such that $S([\rho])^\sim$ is embedded into $\widetilde{S}$ canonically so that $\widetilde{H}$ is embedded into the covering group $\widetilde{G} := U \rtimes \widetilde{S}$ of G with $\widetilde{S}$-action on U through $\widetilde{S} \to S$. Then, for any IR π^1 of $S([\rho])^\sim$, we obtain an IR Π of $\widetilde{G}$ by inducing it up as $\Pi := \mathrm{Ind}_{\widetilde{H}}^{\widetilde{G}}\,\pi$.

If $\beta_{s,t} = \alpha_{s,t}^{-1}$ = the inverse of $\alpha_{s,t}$, then π can be viewed as a representation of $H = U \rtimes S([\rho])$, and accordingly Π is a linear representation of $G = U \rtimes S$, which we denote by $\Pi(\pi^0, \pi^1)$.

STEP1 (Case of n finite). Let $4 \le n < \infty$. We construct all IRs of

$$\widetilde{G}^{\mathrm{Y}}(m,1,n) = \widetilde{D}(m,n) \overset{\mathrm{Y}}{\rtimes} \widetilde{\mathfrak{S}}_n \tag{3.3}$$

for Y = I, II, using their semidirect product structures. So, our task is as follows.

(f-1) Construct all the spin IRs of the normal subgroup $\widetilde{D}(m,n)$.

(f-2) Determine all the orbit in the spin dual (the set of all equivalence classes of spin IRs) of $\widetilde{D}(m,n)$ under the action of $\mathfrak{S}_n \cong \widetilde{\mathfrak{S}}_n/\langle z_1\rangle$, and determine an appropriate complete set $\{\rho\}$ of representatives of these orbits.

(f-3) Calculate the stationary subgroup $\mathfrak{S}_n([\rho])$ in $\mathfrak{S}_n$ of the equivalence class $[\rho]$.

(f-4) Determine intertwining operators $J_\rho(\sigma)$ for $\sigma \in \mathfrak{S}_n([\rho])$ explicitly, and examine (possibly spin) representation $\sigma \mapsto J_\rho(\sigma)$ of $\mathfrak{S}_n([\rho])$.

(f-5) If the representation J_ρ is spin (or double-valued), then it comes from a linear representation $\widetilde{J}_\rho$ of a double covering group $\mathfrak{S}_n([\rho])^\sim$, which may be contained in $\widetilde{\mathfrak{S}}_n$ (this should be checked). If so, we have an IR $\pi^0 := \rho \cdot \widetilde{J}_\rho$ of $\widetilde{H}^{\mathrm{Y}} := \widetilde{D}(m,n) \overset{\mathrm{Y}}{\rtimes} \mathfrak{S}_n([\rho])^\sim$ for Y=I, II respectively.

(f-6) In case J_ρ is spin, we should take a non-spin IR π^1 of $\mathfrak{S}_n([\rho])$ and consider it as an IR of $\widetilde{H}^{\mathrm{Y}}$ through the quotient map $\mathfrak{S}_n([\rho])^\sim \to \mathfrak{S}_n([\rho])$, and take the tensor product $\pi := \pi^0 \boxdot \pi^1$ as an IR of $\widetilde{H}^{\mathrm{Y}}$. Induce it up to $\widetilde{G}_n^{\mathrm{Y}} := \widetilde{G}^{\mathrm{Y}}(m,1,n)$, Y = I, II, then we get a spin IR of $\widetilde{G}_n^{\mathrm{Y}}$ as

$$\Pi(\pi^0, \pi^1) = \mathrm{Ind}_{\widetilde{H}^{\mathrm{Y}}}^{\widetilde{G}_n^{\mathrm{Y}}} \big(\pi^0 \boxdot \pi^1\big). \tag{3.4}$$

(f-5′) + **(f-6′)** If J_ρ is non-spin, then put $\pi^0 := \rho \cdot J_\rho$ as an IR of $H^{\mathrm{Y}} := D(m,n) \overset{\mathrm{Y}}{\rtimes} \mathfrak{S}_n([\rho])$. In this case, we should take a spin IR π^1 of $\mathfrak{S}_n([\rho])$ which is a linear IR of $\mathfrak{S}_n([\rho])^\sim$, not reduced onto $\mathfrak{S}_n([\rho])$. Then $\pi := \pi^0 \boxdot \pi^1$ is again an IR of $\widetilde{H}^{\mathrm{Y}}$, and we get by (3.4) a spin IR $\Pi(\pi^0, \pi^1)$ of $\widetilde{G}_n^{\mathrm{Y}}$.

By these processes, we get all the spin IRs of $\widetilde{G}_n^{\mathrm{Y}}$ for each of Y=I and II, modulo equivalence.

To calculate the characters of IRs Π thus constructed, a difficult task is the next one, since $J_\rho(\sigma)$ can be given for only some generating subset $\{\sigma\}$ of $\mathfrak{S}_n([\rho])$.

(f-7) Calculate the character χ_{π^0} of $\pi^0 = \rho \cdot \widetilde{J}_\rho$ of $\widetilde{H}^{\mathrm{Y}}$ if J_ρ is spin, and that of $\pi^0 = \rho \cdot J_\rho$ of H^{Y} if J_ρ is non-spin.

(f-8) Calculate the character of χ_{π^1} of π^1 too, then the character χ_π is the product $\chi_{\pi^0} \cdot \chi_{\pi^1}$.

(f-9) At last, we calculate the induced character χ_Π from χ_π by the usual method.

This process gives us all the spin irreducible characters of $\widetilde{G}^{\mathrm{Y}}(m,1,n)$ for CASE Y=I, Type $(-1,-1,-1)$, and CASE Y=II, Type $(-1,-1,\ 1)$.

STEP2 (Case of $n=\infty$).

Different from the previous paper [I], we proceed as follows.

(inf-1) For each Y=I, II, examine the limiting process of spin irreducible characters of $\widetilde{G}_n^{\mathrm{Y}}$ as $n\to\infty$, and collect all the 'good' limits.

(inf-2) Let f be a 'good' limit. It is necessarily an central, positive definite function on $\widetilde{G}_\infty^{\mathrm{Y}}=\widetilde{G}^{\mathrm{Y}}(m,1,\infty)$ of the same spin type, and we should check if it is extremal or not in $K_1\big(\widetilde{G}_\infty^{\mathrm{Y}}\big)$.

Here $K_1(G)$ for a topological group G is the set of continuous central (or invariant) positive definite functions f normalized as $f(e)=1$ at the identity element $e\in G$. By definition, a character of G is an extremal element of $K_1(G)$ (cf. §6 in [I]), and denote by $E(G)$ the set of all characters of G.

Definition 3.1. A function f on $G=\widetilde{G}_\infty^{\mathrm{Y}}$ is called *factorizable* if

$$g',g''\in G,\ \mathrm{supp}(g')\cap\mathrm{supp}(g'')=\emptyset\ \Longrightarrow\ f(g'g'')=f(g')f(g''). \tag{3.5}$$

In CASE I, we have the criterion that

(EF) *an* $f\in K_1\big(\widetilde{G}_\infty^{\mathrm{I}}\big)$ *is extremal if and only if it is factorizable.*

But, in CASE II, this criterion does not hold (cf. §11 and Table 13.1 in [I], and Table 4.2 in this paper). However we know from Theorem 3.2 in [HoHH] that any spin character of $\widetilde{G}^{\mathrm{II}}(m,1,\infty)$ is obtained as this kind of pointwise limit as $n\to\infty$ (cf. Theorem 21.2 below), and so our final task will be the following.

(inf-3) Prove that the set of 'good' limits of spin irreducible characters of $\widetilde{G}_n^{\mathrm{Y}}$ as $n\to\infty$, obtained in (**inf-1**) covers all characters of $\widetilde{G}_\infty^{\mathrm{I}}$ of Type $(-1,-1,-1)$ in CASE I, and also of $\widetilde{G}_\infty^{\mathrm{II}}$ of Type $(-1,-1,\ 1)$ in CASE II.

At the same time, thus we will have a unified explicit character formula for such spin characters in each case.

4 Spin characters of $G(m,1,n)$ and $G(m,1,\infty)$

We quote from [I, Part I] some necessary informations on spin characters of $G(m,1,n)$ and $G(m,1,\infty)$, in CASE I, Type $(-1,-1,-1)$, and in CASE II, Type $(-1,-1,\ 1)$.

Let Y = I or II. Take a $g'\in\widetilde{G}^{\mathrm{Y}}(m,1,n)$, then it is expressed as $g'=d'\cdot\sigma'$ with $d'=z_2^{\,b}\prod_{p\in\boldsymbol{I}_n}\eta_p^{\,a_p}\in\widetilde{D}(m,n),\ \sigma'\in\widetilde{\mathfrak{S}}_n$, where $b=0,1;\ 0\le a_p<m\ (p\in\boldsymbol{I}_n)$.

Put $g = \Phi_{\mathrm{Y}}(g') = (d,\sigma) \in G(m,1,n)$ and let its standard decomposition be

$$
(4.1)\quad \begin{cases} g = (d,\sigma) = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r}\, g_1 g_2 \cdots g_s,\ d = \Phi_{\mathrm{Y}}(d'),\ \sigma = \Phi_{\mathrm{Y}}(\sigma'), \\ \xi_q = (t_q, (q)),\ t_q = \Phi_{\mathrm{Y}}\big({\eta_q}^{a_q}\big) = {y_q}^{a_q} \in T_q \cong \boldsymbol{Z}_m\ \ (0 \le a_q \le m-1), \\ g_j = (d_j, \sigma_j),\ \mathrm{supp}(d_j) \subset \mathrm{supp}(\sigma_j) =: K_j, \\ d_j = \Phi_{\mathrm{Y}}\big(\prod_{p\in K_j} {\eta_p}^{a_p}\big) = \prod_{p\in K_j} {y_p}^{a_p}. \end{cases}
$$

Note that the groups $\widetilde{D}(m,n)$ and $\widetilde{\mathfrak{S}}_n$ are both contained canonically in $\widetilde{G}_n^{\mathrm{Y}} := \widetilde{G}^{\mathrm{Y}}(m,1,n)$, $\mathrm{Y} = \mathrm{I}, \mathrm{II}$, and $\Phi_{\mathrm{Y}}(d') = \Phi_D(d')$ $\big(d' \in \widetilde{D}(m,n)\big)$ and $\Phi_{\mathrm{Y}}(\sigma') = \Phi_{\mathfrak{S}}(\sigma')$ $(\sigma' \in \widetilde{\mathfrak{S}}_n)$ both for Y = I and II, where $\Phi_D : \widetilde{D}(m,n) \to D(m,n)$ is the canonical homomorphism (cf. Lemma 5.1).

We quote, as Tables 4.1 and 4.2 below, from Tables 9.1 and 13.1 in [I, Part I] respectively, their parts corresponding to CASEs I and II, together with the parts of CASE VII worked out in [I, Part II], for references.

4.1. Case of finite $\widetilde{G}^{\mathrm{Y}}(m,1,n)$, $4 \le n < \infty$, Y = I, II.

In Table 4.1 below, the support of a character $f(g')$, $g' \in \widetilde{G}^{\mathrm{Y}}(m,1,n)$, Y = I, II (resp. $g' \in R\big(G(m,1,n)\big)$ in CASE VII) of type $(\beta_1,\beta_2,\beta_3)$ ($\chi(z_i) = \beta_i, 1 \le i \le 3$) is evaluated modulo $\widetilde{Z}$ (resp. modulo Z) by using data of $g = \Phi_{\mathrm{Y}}(g') \in G(m,1,n)$ as follows:

Table 4.1. For finite group $\widetilde{G}^{\mathrm{Y}}(m,1,n)$, Y = I, II (and VII), $4 \le n < \infty$, $m = 2m'$:

$1 \longrightarrow \widetilde{Z} = \langle z_1, z_2\rangle \longrightarrow \widetilde{G}^{\mathrm{Y}}(m,1,n) \xrightarrow{\Phi_{\mathrm{Y}}} G(m,1,n) \longrightarrow 1$ (exact):

CASE Y	$(\beta_1,\beta_2,\beta_3)$ (spin) type of projec. represen- tation	$f(g') \neq 0 \implies$ Condition for $g = \Phi_{\mathrm{Y}}(g') = (d,\sigma)$ $= \xi_{q_1}\cdots\xi_{q_r} g_1\cdots g_s,\ \xi_{q_i} = (t_{q_i},(q_i)),\ g_j = (d_j,\sigma_j)$			
		$\mathrm{ord}(d) + L(\sigma) \equiv 0 \pmod 2$		$\mathrm{ord}(d) + L(\sigma) \equiv 1 \pmod 2$	
		$\mathrm{ord}(d) \equiv 0$ $L(\sigma) \equiv 0$	$\mathrm{ord}(d) \equiv 1$ $L(\sigma) \equiv 1$	$\mathrm{ord}(d) \equiv 0$ $L(\sigma) \equiv 1$	$\mathrm{ord}(d) \equiv 1$ $L(\sigma) \equiv 0$
I	(–1,–1,–1) seed repre. in [IhYo], in [DaMo]	$\mathrm{ord}(\xi_{q_i}) \equiv 0$ $(1 \leqslant i \leqslant r)$ $\mathrm{ord}(d_j) + L(\sigma_j) \equiv 0$ $(1 \leqslant j \leqslant s)$		$\lvert\mathrm{supp}(g')\rvert = n$ $\mathrm{ord}(\xi_{q_i}) \equiv 1$ $(1 \leqslant i \leqslant r)$ $\mathrm{ord}(d_j) \equiv 1$ $(1 \leqslant j \leqslant s)$	
II	$(-1,-1,1)$ seed repre. in [DaMo]	$L(\sigma) \equiv 0$ $\mathrm{ord}(\xi_{q_i}) \equiv 0$ $(\forall i)$ $\mathrm{ord}(d_j) + L(\sigma_j)$ $\equiv 0$ $(\forall j)$	$\lvert\mathrm{supp}(g')\rvert = n$ $r+s$ odd $\mathrm{ord}(\xi_{q_i}) \equiv 1$ $(\forall i)$ $\mathrm{ord}(d_j) \equiv 1$ $(\forall j)$ $L(\sigma) \equiv 1$	$\varnothing$	$\lvert\mathrm{supp}(g')\rvert = n$ $r+s$ odd $\mathrm{ord}(\xi_{q_i}) \equiv 1$ $(\forall i)$ $\mathrm{ord}(d_j) \equiv 1$ $(\forall j)$ $L(\sigma_j) \equiv 0$ $(\forall j)$
VII	$(1,\ 1,-1)$ seed repre. in [IhYo]	$\subset \mathfrak{A}_n(T)^{S(2)}$ **No other conditions**	n **even**, $\lvert\mathrm{supp}(g')\rvert = n$ $r = 0,\ s$ odd, $\mathrm{ord}(d_j) \equiv \mathrm{ord}(d)$ $(\forall j)$ $L(\sigma_j) \equiv 1$ $(\forall j)$, $g = g_1 g_2 \cdots g_s$		$\lvert\mathrm{supp}(g')\rvert \geqslant n-1$ $L(\sigma_j) \equiv 0$ $(\forall j)$

Notation : $L(\sigma_j) \equiv \ell(\sigma_j) - 1,\ L(\sigma) \equiv \sum_{1\le j\le s} L(\sigma_j) \pmod 2$,
$\mathrm{ord}(d) := \sum_{p\in \boldsymbol{I}_n} a_p\,,\ \ S(2) = \{t^2; t\in T\} \cong \boldsymbol{Z}_{m/2},\ \ T = \boldsymbol{Z}_m$,
$\mathfrak{A}_n(T)^{S(2)} := \{(d,\sigma)\in \mathfrak{S}_n(T)\ ;\ \sigma\in\mathfrak{A}_n,\ P(d)\in S(2)\}$.

New comments: In CASE I, the condition in the 3rd column and the one in the 4th column correspond respectively to (Conditions I-00) and to (Condition I-11) in §§16 and 19. In CASE II, the conditions in the 3rd column, in the 4th column and in the 6th column correspond respectively to (Conditions II-00), (Condition II-11) in §§17 and 20, and (Condition $\mho$-11s) in §20.3.3.

4.2. Case of infinite $\widetilde{G}^{\mathrm{Y}}(m,1,\infty)$, Y = I, II.

In Table 4.2 below, we give for a character $f(g')$, $g'\in \widetilde{G}^{\mathrm{Y}}(m,1,\infty)$, Y = I, II (resp. $g'\in R\big(G(m,1,\infty)\big)$ in CASE VII) the following:

- in the second column, (spin) type of factor representation or of spin characters, and the information on the basic representations given in [IhYo] or [DaMo] which are called here as *seed representation*;
- in the 3rd column, information on finite-dimensional representations of $\widetilde{G}^{\mathrm{Y}}(m,1,\infty)$;
- in the 4th column, information on the validity of the criterion (EF);
- in the 5th column, (Condition Y) to define $\mathcal{O}(\mathrm{Y})$ for which $\mathrm{supp}(f)\subset \mathcal{O}(\mathrm{Y})$.

Table 4.2. For infinite group $\widetilde{G}^{\mathrm{Y}}(m,1,\infty)$, Y = I, II (and VII) :

$$\widetilde{G}^{\mathrm{Y}}(m,1,\infty)\ni g' \xrightarrow{\Phi_{\mathrm{Y}}} g\in G(m,1,\infty) = \mathfrak{S}_\infty(\boldsymbol{Z}_m) = D(m,\infty)\rtimes\mathfrak{S}_\infty$$

CASE Y	$(\beta_1,\beta_2,\beta_3)$ Type of factor representation	Existence of spin finite-dimensional irred. represen. π	extremal $\Leftrightarrow$ factorizable	$\mathrm{supp}(f)$: $f(g')\neq 0 \implies$ Condition Y : $g=\Phi_{\mathrm{Y}}(g') = (d,\sigma) = \xi_{q_1}\cdots\xi_{q_r} g_1\cdots g_s$, $g_j=(d_j,\sigma_j)$
I	$(-1,-1,-1)$ seed represen. [IhYo], [DaMo]	$\neg\,\exists$ (not exist) π	YES	$\mathrm{ord}(\xi_{q_i})\equiv 0 \pmod 2\ (\forall i)$ i.e., $\xi_{q_i} = (t_{q_i},(q_i)), t_{q_i}\in S(2)$, $\mathrm{ord}(d_j)+L(\sigma_j)\equiv 0\ (\forall j)$
II	$(-1,-1,\ 1)$ seed represen. in [DaMo]	$\neg\,\exists\,\pi$	NO$^{(*)}$	$\subset \mathfrak{A}_\infty(T)^{S(2)}$, i.e., $\mathrm{ord}(d)\equiv L(\sigma)\equiv 0$, and $\mathrm{ord}(\xi_{q_i})\equiv 0\ \ (\forall i)$, $\mathrm{ord}(d_j)+L(\sigma_j)\equiv 0\ (\forall j)$
VII	$(1,\ 1,-1)$ seed represen. in [IhYo]	$\exists$ 2-dimensional irred. represen. π_{2,ζ_k} $(0\leqslant k< m/2)$	NO	$\subset \mathfrak{A}_\infty(T)^{S(2)}$, i.e., $\mathrm{ord}(d)\equiv 0$, $L(\sigma)\equiv 0$

$(*)$ This will be proved in Part V, Corollary 23.8, of the present paper (cf. Lemma 17.3 and Note 17.1 for $n<\infty$)

In this paper we will apply the following notation for simplicity.

Notation 4.1. For $\sigma' \in \widetilde{\mathfrak{S}}_n$ and $d' \in \widetilde{D}(m,n)$, we put, with $\sigma = \Phi(\sigma'),\ d = \Phi(d') \in D(m,n)$, as

$$\begin{cases} \mathrm{sgn}(\sigma') := \mathrm{sgn}(\sigma),\ \ L(\sigma') := L(\sigma),\ \ \sigma'(k) := \sigma(k)\ (k \in \boldsymbol{I}_n)\,; \\ \mathrm{ord}(d') := \mathrm{ord}(d),\ \ \mathrm{supp}(\sigma') := \mathrm{supp}(\sigma)\,; \\ \sigma'\gamma := \sigma\gamma = (\gamma_{\sigma^{-1}(1)}, \gamma_{\sigma^{-1}(2)}, \ldots, \gamma_{\sigma^{-1}(n)})\ \text{ for } \gamma = (\gamma_1, \ldots, \gamma_n),\ \gamma_j \in \boldsymbol{I}_m\,. \end{cases}$$

Part II

Spin irreducible representations of $R\big(G(m,1,n)\big)$, $n<\infty$, of Types $(-1,-1,-1)$ and $(-1,-1,\ 1)$

5 Covering group $\widetilde{D}(m,n)$ and Clifford algebra

5.1 Non-spin and spin IRs of $\widetilde{D}(m,n)$

Let $4 \le n < \infty$. Put $D(m,n) := D_n(\boldsymbol{Z}_m) = \big\langle y_j\ (j \in \boldsymbol{I}_n)\big\rangle$ inside $G(m,1,n)$. Then we have the following.

Lemma 5.1. *The non-abelian group $\widetilde{D}(m,n)$ is a central extension of the abelian group $D(m,n)$ as*

$$\{e\} \longrightarrow Z_2 := \langle z_2\rangle \longrightarrow \widetilde{D}(m,n) \xrightarrow{\Phi_D} D(m,n) \longrightarrow \{e\} \quad \text{(exact)}, \tag{5.1}$$

where $\Phi_D(z_2) = e,\ \Phi_D(\eta_j) = y_j\ \ (j \in \boldsymbol{I}_n)$.

We call an IR ρ of $\widetilde{D}(m,n)$ *spin* or *non-spin* according as $\rho(z_2) = -I$ or $\rho(z_2) = I$. The commutator group of $\widetilde{D}(m,n)$ is nothing but the central subgroup $Z_2 := \langle z_2\rangle \cong \boldsymbol{Z}_2$, and the quotient group $\widetilde{D}(m,n)/Z_2$ is isomorphic to $D(m,n)$. Hence the set of non-spin IRs of $\widetilde{D}(m,n)$ consists of one-dimensional characters of $D(m,n)$.

To look for spin IRs, we take a bigger central normal subgroup

$$\widetilde{D}^0(m,n) := \big\langle \eta_j^2\ (j \in \boldsymbol{I}_n)\big\rangle \subset \widetilde{D}(m,n) = \big\langle z_2,\, \eta_j\ (j \in \boldsymbol{I}_n)\big\rangle, \tag{5.2}$$

and consider the quotient group $\widetilde{D}(m,n)\big/\widetilde{D}^0(m,n)$.

On the other hand, consider a Clifford algebra $\mathcal{C}_n = \langle f_1, f_2, \ldots, f_n\rangle_{\boldsymbol{C}}$ over $\boldsymbol{C}$ associated with a quadratic form $Q(x) = x_1^2 + x_2^2 + \cdots + x_n^2$, i.e.,

$$\begin{cases} \boldsymbol{x} = x_1 f_1 + x_2 f_2 + \cdots + x_n f_n, \quad x = (x_1, x_2, \ldots, x_n), \\ \boldsymbol{x}\cdot\boldsymbol{x} = Q(x) f_0\,, \end{cases} \tag{5.3}$$

where f_0 is the identity element, and take a finite group $\mathcal{F}_n' = \langle f_1, f_2, \ldots, f_n \rangle$ contained in $\mathcal{C}_n$.

Lemma 5.2. *The order $|\mathcal{F}_n'|$ of the group $\mathcal{F}_n'$ is 2^{n+1}, and it is presented as an abstract group $\mathcal{F}_n$ as follows:*

$$\text{set of generators:} \quad \{z_2, f_1, f_2, \ldots, f_n\};$$

$$\text{set of fundamental relations:} \quad \begin{cases} z_2^{\,2} = f_0, \quad z_2 \text{ central element}, \\ f_j^{\,2} = f_0 \quad (j \in \boldsymbol{I}_n), \\ f_j f_k = z_2 f_k f_j \quad (j \neq k). \end{cases}$$

Lemma 5.3. *The quotient group $\widetilde{D}(m,n)/\widetilde{D}^0(m,n)$ is canonically isomorphic to $\mathcal{F}_n$. The group $\widetilde{D}(m,n)$ is a central extension of $\mathcal{F}_n$ as*

$$\{e\} \longrightarrow \widetilde{D}^0(m,n) \longrightarrow \widetilde{D}(m,n) \xrightarrow{\Psi} \mathcal{F}_n \longrightarrow \{e\} \quad (\text{exact}), \tag{5.4}$$

where the canonical homomorphism Ψ is given as $\Psi(\eta_j) = f_j$ $(j \in \boldsymbol{I}_n)$.

5.2 Regular representations $\mathcal{L}$ and $\mathcal{R}$ of $\mathcal{F}_n$

Let us consider the left (resp. right) regular representation $\mathcal{L}$ (resp. $\mathcal{R}$) of $\mathcal{F}_n$, acting on the space $\ell^2(\mathcal{F}_n)$. Moreover we consider 'double' regular representation $\mathcal{L} \cdot \mathcal{R}$ given as

$$(\mathcal{L} \cdot \mathcal{R})(g_0, g_1)\varphi(g) := \varphi(g_0^{-1} g g_1) \quad \big(g, g_0, g_1 \in \mathcal{F}_n,\ \varphi \in \ell^2(\mathcal{F}_n)\big). \tag{5.5}$$

A representation ρ of $\mathcal{F}_n$ is called '*spin*' if $\rho(z_2 g) = -\rho(g)$ $(g \in \mathcal{F}_n)$. Let $V_\pm \subset \ell^2(\mathcal{F}_n)$ be the subspaces of functions φ on $\mathcal{F}_n$ such that $\varphi(z_2 g) = \pm\varphi(g)$. Then V_- carries spin representations of $\mathcal{F}_n$ and

$$\ell^2(\mathcal{F}_n) = V_+ \oplus V_-, \quad \dim V_+ = \dim V_- = 2^n. \tag{5.6}$$

We calculate the trace of the restriction $\pi := (\mathcal{L} \cdot \mathcal{R})|_{V_-}$ of $\mathcal{L} \cdot \mathcal{R}$ onto V_-. The subsets of $\mathcal{F}_n$ consisting of elements of the form $z_2^{\kappa} f_1^{\,a_1} f_2^{\,a_2} \cdots f_n^{\,a_n}$, for $\kappa = 0, 1$, are denoted respectively by $\mathcal{F}_n^{\kappa}$. Then $\mathcal{F}_n = \mathcal{F}_n^0 \sqcup \mathcal{F}_n^1$. An orthogonal basis of V_- is given by the set $\{\delta_h' \,;\, h \in \mathcal{F}_n^0\}$ of equal length $\sqrt{2}$ with

$$\delta_h'(g) = \begin{cases} 1 & \text{for } g = h, \\ -1 & \text{for } g = z_2 h, \\ 0 & \text{for any other } g \in \mathcal{F}_n, \end{cases} \tag{5.7}$$

and π is written with respect to it as

$$\pi(g_0, g_1)\delta_h' = \begin{cases} \delta_{h'}' & \text{if } h' := g_0 h g_1^{-1} \in \mathcal{F}_n^0, \\ -\delta_{h'}' & \text{if } h' := z_2 g_0 h g_1^{-1} \in \mathcal{F}_n^0. \end{cases} \tag{5.8}$$

A basis element δ'_h which contributes to $\mathrm{tr}\big(\pi(g_0,g_1)\big)$ should satisfy $g_0 h g_1^{-1} = z_2^{\kappa} h$, and the contribution from it is ± 1 corresponding to $\kappa = 0,1$.

Note that, for a general element $h = f_{j_1} f_{j_2} \cdots f_{j_p}$, $j_1 < j_2 < \ldots < j_p$, we have $h^{-1} = f_{j_p} \cdots f_{j_2} f_{j_1}$, and so obtain $h g_1 h^{-1} = z_2^{\kappa} g_1$. Hence we see that

$$\mathrm{tr}\big(\pi(g_0,g_1)\big) = 0 \qquad \text{if } g_0 \neq z_2^{\kappa} g_1 \text{ for any } \kappa = 0,1. \tag{5.9}$$

So, let us calculate $\mathrm{tr}\big(\pi(g_0,g_1)\big)$ in case $g_0 = g_1$.

Lemma 5.4. *Let $g_0 = g_1 = f_1 f_2 \cdots f_n$. Then*

$$\mathrm{tr}\big(\pi(g_0,g_1)\big) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 2^n & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let $h = f_1^{a_1} f_2^{a_2} \cdots f_n^{a_n}$. Then

$$\begin{aligned} g_0 h g_1^{-1} &= z_2^{(a_1+a_2+\cdots+a_n)(n-1)} g_0 g_1^{-1} h = z_2^{(a_1+a_2+\cdots+a_n)(n-1)} h \\ &= z_2^{c_h} h \quad \text{with} \quad c_h = \begin{cases} a_1+a_2+\cdots+a_n, & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \\ \therefore\ \mathrm{tr}\big(\pi(g_0,g_1)\big) &= \sum_{h\in\mathcal{F}_n^0} (-1)^{c_h} = \sum_{\substack{a_i=0,1 \\ (i\in \boldsymbol{I}_n)}} (-1)^{c_h} = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 2^n & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

□

Now let $g_0 = g_1 = f_1 f_2 \cdots f_N$ with $1 \leq N < n$, and put $k = n - N$. For

$$h = f_1^{a_1} f_2^{a_2} \cdots f_N^{a_N} \big(f_{N+1}^{b_1} \cdots f_n^{b_k}\big), \quad a_i = 0,1\,; \;\; b_j = 0,1,$$

$$\begin{aligned} g_0 h g_1^{-1} &= z_2^{(a_1+a_2+\cdots+a_N)(N-1)}\, z_2^{(b_1+\cdots+b_k)N} g_0 g_1^{-1} h \\ &= z_2^{(a_1+a_2+\cdots+a_N)(N-1)}\, z_2^{(b_1+\cdots+b_k)N} h = z_2^{c_h} h \\ \text{with} \qquad c_h &= \begin{cases} a_1+a_2+\cdots+a_N, & \text{if } N \text{ is even,} \\ b_1+\cdots+b_k\,, & \text{if } N \text{ is odd.} \end{cases} \\ \therefore \quad \mathrm{tr}\big(\pi(g_0,g_1)\big) &= \sum_{h\in\mathcal{F}_n^0} (-1)^{c_h} = \sum_{\substack{a_i=0,1\,;\,b_j=0,1 \\ (i\in\boldsymbol{I}_N,\, j\in\boldsymbol{I}_k)}} (-1)^{c_h} \;=\; 0. \end{aligned}$$

Lemma 5.5. *The character of $\pi = (\mathcal{L}\cdot\mathcal{R})\big|_{V_-}$ is given as follows:*

$$\begin{aligned} \chi_\pi(g_0,g_1) &= \mathrm{tr}\big(\pi(g_0,g_1)\big) = \\ &= \begin{cases} (-1)^\kappa\, 2^n & \text{if } g_0 = z_2^{\kappa} g_1 \text{ and } g_0 = z_2^{\delta} f_0, \\ (-1)^\kappa\, 2^n & \text{if } g_0 = z_2^{\kappa} g_1 \text{ and } g_0 = z_2^{\delta} f_1 f_2 \cdots f_n,\ n \text{ odd,} \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where "otherwise" *means either $g_0 \neq z_2^{\kappa} g_1$, or*

$$g_0 = z_2^{\kappa} g_1 \ \ \textit{and} \ \ \mathrm{ord}(g_0) \neq \begin{cases} 0 & \textit{if } n \textit{ is even,} \\ 0, n & \textit{if } n \textit{ is odd,} \end{cases}$$

with $\mathrm{ord}(g) := \alpha_1 + \cdots + \alpha_n$ *for* $g = z_2^{\delta} f_1^{\alpha_1} \cdots f_n^{\alpha_n}$, $\alpha_j = 0, 1$, $\delta = 0, 1$.

Theorem 5.6. *Let* $\pi = (\mathcal{L} \cdot \mathcal{R})\big|_{V_-}$ *be the restriction of of* $\mathcal{L} \cdot \mathcal{R}$ *onto* $V_- \subset \ell^2(\mathcal{F}_n)$.

(i) *If* n *is even, then* $\mathcal{F}_n$ *has only one equivalence class of spin IR* ρ_n, *and* π *is irreducible and*

$$\pi \cong \rho_n \otimes \overline{\rho_n}$$

(ii) *If* n *is odd, then* $\mathcal{F}_n$ *has two equivalence classes of spin IRs* $\rho_{n,\pm}$, *and* π *is decomposed as*

$$\pi \cong \big(\rho_{n,+} \otimes \overline{\rho_{n,+}}\big) \bigoplus \big(\rho_{n,-} \otimes \overline{\rho_{n,-}}\big).$$

Proof. Using the above explicit form of the character χ_π, we obtain

$$\frac{1}{|\mathcal{F}_n \times \mathcal{F}_n|} \sum_{g \in \mathcal{F}_n \times \mathcal{F}_n} |\chi_\pi(g)|^2 = \frac{1}{2^{2n+2}} \sum_{g \in \mathcal{F}_n \times \mathcal{F}_n} |\chi_\pi(g)|^2 = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 2 & \text{if } n \text{ is odd.} \end{cases} \qquad \square$$

Lemma 5.7. *The irreducible decomposition of the spin part* $\mathcal{L}\big|_{V_-}$ *of the left regular representation* $\mathcal{L}$ *is given as*

$$\mathcal{L}\big|_{V_-} \cong \begin{cases} \big[2^{[n/2]}\big] \cdot \rho_n & \text{if } n \text{ is even,} \\ \big[2^{[n/2]}\big] \cdot \big(\rho_{n,+} \bigoplus \rho_{n,-}\big) & \text{if } n \text{ is odd;} \end{cases}$$

and the spin part $\widehat{\mathcal{F}_n}^{\mathrm{spin}}$ *of the dual* $\widehat{\mathcal{F}_n}$ *of* $\mathcal{F}_n$ *is given as*

$$(5.10) \qquad \widehat{\mathcal{F}_n}^{\mathrm{spin}} = \begin{cases} \big\{ [\rho_n] \big\}, & \dim \rho_n = 2^{n/2}, & \text{if } n \text{ is even,} \\ \big\{ [\rho_{n,+}], [\rho_{n,-}] \big\}, & \dim \rho_{n,\pm} = 2^{(n-1)/2}, & \text{if } n \text{ is odd,} \end{cases}$$

where $\big[2^{[n/2]}\big] \cdot$ *denotes the multiplicity of IRs, and* $[\rho_n]$ *the equivalence class of* ρ_n.

Proof. If n is even, $\dim \rho_n = \sqrt{\dim V_-} = 2^{n/2}$.
If n is odd, let d be $\dim \rho_{n,+} = \dim \rho_{n,+}$, then,

$$d^2 + d^2 = \dim V_- = 2^n \quad \therefore \quad d = \sqrt{2^{n-1}} = 2^{(n-1)/2}. \qquad \square$$

The following result is more or less known in the theory of Clifford algebras.

Theorem 5.8. *Let* $\mathcal{F}_n = \langle z_2, f_1, f_2, \ldots, f_n \rangle$ *be the finite group in Lemma 5.2, isomorphic to* $\mathcal{F}_n'$ *in the Clifford algebra* $\mathcal{C}_n$. *An IR* ρ *of* $\mathcal{F}_n$ *is called 'spin' IR if* $\rho(z_2 g) = -\rho(g)$ $(g \in \mathcal{F}_n)$.

In the case where n *is even, spin IRs have unique equivalence class* $[\rho_n]$, *and their dimensions are equal to* $2^{n/2}$.

In the case where n *is odd, spin IRs have two equivalence classes* $[\rho_{n,+}], [\rho_{n,-}]$, *and their dimensions are* $2^{(n-1)/2}$.

5.3 Construction of spin IRs of $\mathcal{F}_n$, and their characters

The result in this subsection is essentially contained in [Sch3] (cf. also [Fruc], [IhYo] and [DaMo]). Take a triplet of hermitian matrices of trace zero as

$$a := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad b := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad c := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5.11}$$

which are called Pauli matrices (cf. §1.2.2 in [E]). Then we have

$$\begin{cases} a^2 = b^2 = c^2 = \varepsilon \text{ (identity matrix)}, \quad abc = i\varepsilon\,; \\ ab = -ba = ic, \quad bc = -cb = ia, \quad ca = -ac = ib\,; \\ [a,b] = 2ic, \ \ [b,c] = 2ia, \ \ [c,a] = 2ib. \end{cases} \tag{5.12}$$

Note that, putting $A_1 = ia,\ A_2 = -ib,\ A_3 = ic$ ($i = \sqrt{-1}$), as in [E, (1.7)], we have a canonical basis of the Lie algebra $\mathfrak{su}(2)$ with the commutation relations $[A_i, A_j] = 2A_k$ for cyclically permuted $(i\ j\ k)$ of $(1\ 2\ 3)$.

Let $n = 2n'$ even, or $n = 2n'+1$ odd. Put in $GL(2^{n'}, \boldsymbol{C}) = GL(2^{[n/2]}, \boldsymbol{C})$,

$$\begin{aligned}
Y_1 &= a \otimes \varepsilon \otimes \varepsilon \otimes \cdots \otimes \varepsilon \otimes \varepsilon \otimes \varepsilon = a \otimes \varepsilon^{\otimes(n'-1)} \\
&\qquad\qquad ((n'-1)\text{-times tensor product of } \varepsilon), \\
Y_2 &= b \otimes \varepsilon \otimes \varepsilon \otimes \cdots \otimes \varepsilon \otimes \varepsilon \otimes \varepsilon = b \otimes \varepsilon^{\otimes(n'-1)}, \\
Y_3 &= c \otimes a \otimes \varepsilon \otimes \varepsilon \otimes \cdots \otimes \varepsilon \otimes \varepsilon = c \otimes a \otimes \varepsilon^{\otimes(n'-2)}, \\
Y_4 &= c \otimes b \otimes \varepsilon \otimes \varepsilon \otimes \cdots \otimes \varepsilon \otimes \varepsilon = c \otimes b \otimes \varepsilon^{\otimes(n'-2)}, \\
&\cdots \quad \cdots \quad \cdots \\
Y_{2i-1} &= c \otimes c \otimes \cdots \otimes c \otimes a \otimes \varepsilon \otimes \cdots \otimes \varepsilon = c^{\otimes(i-1)} \otimes a \otimes \varepsilon^{\otimes(n'-i)} \\
&\qquad\qquad (a,\ i\text{-th component}), \\
Y_{2i} &= c \otimes c \otimes \cdots \otimes c \otimes b \otimes \varepsilon \otimes \cdots \otimes \varepsilon = c^{\otimes(i-1)} \otimes b \otimes \varepsilon^{\otimes(n'-i)} \\
&\qquad\qquad (b,\ i\text{-th component}), \\
&\cdots \quad \cdots \quad \cdots \\
Y_{2n'-1} &= c \otimes c \otimes \cdots \otimes c \otimes c \otimes a = c^{\otimes(n'-1)} \otimes a, \\
Y_{2n'} &= c \otimes c \otimes \cdots \otimes c \otimes c \otimes b = c^{\otimes(n'-1)} \otimes b, \\
Y_{2n'+1} &= \underbrace{c \otimes c \otimes \cdots \otimes c \otimes c \otimes c}_{n'\text{-times}} = c^{\otimes(n'-1)} \otimes c\,,
\end{aligned} \tag{5.13}$$

where $\varepsilon^{\otimes i}$ denotes i-times tensor product of ε and so on. Then the set $\{Y_1, Y_2, \ldots, Y_{2n'}\}$, $2n' = 2[n/2]$, generates the algebra $M(2^{n'}, \boldsymbol{C})$ over $\boldsymbol{C}$ and the group $GL(2^{n'}, \boldsymbol{C})$, and gives them special fine structures.

Lemma 5.9. (i) *The set of generators $\{Y_1, Y_2, \ldots, Y_{2n'}\}$ of $M(2^{n'}, \boldsymbol{C})$ satisfies the following set of fundamental relations: with the identity matrix $E = E_{2^{n'}}$,*

$$\begin{cases} Y_j^{\,2} = E \qquad (j \in \boldsymbol{I}_{2n'}), \\ Y_jY_k = -Y_kY_j \quad (j \neq k). \end{cases} \tag{5.14}$$

Conversely, as an abstract algebra over $\boldsymbol{C}$*, take the set of symbols* $\{Y_1, Y_2, \ldots, Y_{2n'}\}$ *as generators, and fundamental relations (5.14) above, where* E *is replaced by the neutral element* e*, then they give an algebra isomorphic to* $M(2^{n'}, \boldsymbol{C})$.

(ii) *If we take the redundant set* $\{Y_1, Y_2, \ldots, Y_{2n'}, Y_{2n'+1}\}$ *as the set of generators, then the set of fundamental relations turns out to be*

$$
\left\{
\begin{array}{rcll}
Y_j^{\,2} & = & E & (j \in \boldsymbol{I}_{2n'+1}),\\
Y_j Y_k & = & -Y_k Y_j & (j \neq k),\\
Y_1 Y_2 \cdots Y_{2n'+1} & = & i^{n'} E, &
\end{array}
\right.
\tag{5.15}
$$

Conversely, as an abstract algebra over $\boldsymbol{C}$*, they give, similarly as above, an algebra isomorphic to* $M(2^{n'}, \boldsymbol{C})$.

Note that the set of fundamental relations (5.14) is symmetric under permutations of Y_j $(j \in \boldsymbol{I}_{2n'})$ under $\mathfrak{S}_{2n'}$, and that the one (5.15) is 'skew-symmetric' under $\mathfrak{S}_{2n'+1}$. These facts will induce spin representations of $\mathfrak{S}_{2n'}$ and $\mathfrak{S}_{2n'+1}$ naturally as is seen in [Sch3] (cf. also §6 in [E]).

Lemma 5.10. (i) *The set of the monomial products* $Y_1^{\,a_1} Y_2^{\,a_2} \cdots Y_{2n'}^{\,a_{2n'}}$, $a_j = 0, 1$ $(1 \leq j \leq 2n')$ *gives a linear basis of the algebra* $M(2^{n'}, \boldsymbol{C})$*, where, in case* $a_j = 0$ *for all* j*, the product means the identity matrix* $E = E_{2^{n'}}$.

(ii) *Any non-trivial monomial product* $Y_{j_1} Y_{j_2} \cdots Y_{j_p}$, $j_1 < j_2 < \ldots < j_p \leq 2n'$*, has trace 0. A monomial product* $Y_{j_1} Y_{j_2} \cdots Y_{j_p} Y_{2n'+1}$, $j_1 < j_2 < \ldots < j_p \leq 2n'$*, containing* $Y_{2n'+1}$*, has non-zero trace only when it is* $Y_1 Y_2 \cdots Y_{2n'} Y_{2n'+1}$*. In other words,*

$$
\mathrm{tr}\big(Y_{j_1} Y_{j_2} \cdots Y_{j_p} Y_{2n'+1}\big) = 0 \quad \text{for } j_1 < j_2 < \ldots < j_p \leq 2n', \tag{5.16}
$$

whenever one of $Y_j, j \leq 2n'$ *is absent.*

Now we give explicit realizations of $\rho_n, \rho_{n,\pm}$ in Theorems 5.6 and 5.8 as follows. Pre-assuming this, we may use the same notation.

Definition 5.1. Put, in case $n = 2n'$ even,

$$
\rho_n(f_j) := Y_j \quad (1 \leq j \leq 2n' = n), \tag{5.17}
$$

and in case $n = 2n' + 1$ odd,

$$
\left\{
\begin{array}{rcll}
\rho_{n,\kappa}(f_j) & := & Y_j & (1 \leq j \leq 2n' = n-1),\\
\rho_{n,\kappa}(f_{2n'+1}) & := & \kappa Y_{2n'+1}, & \kappa = \pm\,.
\end{array}
\right.
\tag{5.18}
$$

We see from the next theorem that the above definition actually gives a realization of IRs $\rho_n, \rho_{n,\pm}$.

Theorem 5.11. (i) Case $n = 2n'$ even: *The set of operators* $\{\rho_n(f_j), j \in \boldsymbol{I}_n\}$ *gives a spin IR* ρ_n *of* $\mathcal{F}_n$*. Every spin IR of* $\mathcal{F}_n$ *is equivalent to* ρ_n.

The character of ρ_n is given by

$$\chi_{\rho_n}(g) = \mathrm{tr}\big(\rho_n(g)\big) = \begin{cases} (-1)^b\, 2^{n/2} & g = z_2^{\,b}\,,\ b = 0,1, \\ 0 & \text{otherwise.} \end{cases} \tag{5.19}$$

(ii) CASE $n = 2n'+1$ ODD: *For each choice of $\kappa = \pm$, the set of operators $\{\rho_{n,\kappa}(f_j), j \in \boldsymbol{I}_n\}$ gives a spin IR $\rho_{n,\kappa}$ of $\mathcal{F}_n$ respectively. Every spin IR of $\mathcal{F}_n$ is equivalent to one of $\rho_{n,+}$ and $\rho_{n,-}$.*

The character of $\rho_{n,\kappa}$, $\kappa = \pm$, is given by

$$\chi_{\rho_{n,\kappa}}(g) = \mathrm{tr}\big(\rho_{n,\kappa}(g)\big) = \begin{cases} (-1)^b\, 2^{[n/2]} & g = z_2^{\,b}\,,\ b = 0,1, \\ \kappa\, i^{[n/2]} \cdot (-1)^b\, 2^{[n/2]} & g = z_2^{\,b} f_1 f_2 \cdots f_n\,, \\ 0 & \text{otherwise.} \end{cases} \tag{5.20}$$

Proof. CASE n EVEN: At first ρ_n is irreducible. Moreover $\dim \rho_n = 2^{[n/2]}$ and so $(\dim \rho_n)^2 = 2^n = |\mathcal{F}_n|/2$. This proves that any spin IR is equivalent to ρ_n.

Another proof for the irreducibility is given by explicit form of characters as

$$\sum_{g \in \mathcal{F}_n} \big|\chi_{\rho_n}(g)\big|^2 = 2 \cdot 2^n = |\mathcal{F}_n| \quad \therefore\ \ \big\|\chi_{\rho_n}\big\|^2 = 1 \ \text{ in } \ \ell^2(\mathcal{F}_n).$$

CASE n ODD: At first $\rho_{n,\pm}$ are irreducible. Since $\rho_{n,\pm}\big|_{\mathcal{F}_{n-1}}$ is already irreducible, there exists no intertwining operator between $\rho_{n,+}$ and $\rho_{n,-}$. Moreover $\dim \rho_{n,\pm} = 2^{[n/2]}$ and so $(\dim \rho_{n,+})^2 + (\dim \rho_{n,-})^2 = 2^n = |\mathcal{F}_n|/2$. This proves that any spin IR is equivalent to one of $\rho_{n,\pm}$.

Another proof for the irreducibility is given by explicit form of characters as

$$\sum_{g \in \mathcal{F}_n} \big|\chi_{\rho_{n,\pm}}(g)\big|^2 = 4 \cdot 2^{2[n/2]} = 2 \cdot 2^n = |\mathcal{F}_n|. \qquad \square$$

5.4 Actions of $\mathfrak{S}_n$ on spin IRs of $\mathcal{F}_n$

From the abstract definition in Lemma 5.2 of the group $\mathcal{F}_n$ by using the sets of generators and of fundamental relations, we see that $\mathcal{F}_n$ admits actions of $\mathfrak{S}_n$ in two different ways as follows: for $\sigma \in \mathfrak{S}_n$,

$$\begin{aligned} \sigma(f_j) &:= f_{\sigma(j)}\,, \\ [\sigma](f_j) &:= \mathrm{sgn}(\sigma) f_{\sigma(j)}\,, \end{aligned} \qquad (1 \le j \le n). \tag{5.21}$$

Accordingly $\mathfrak{S}_n$ acts on representations ρ of $\mathcal{F}_n$ by

$$\begin{aligned} (\sigma\rho)(g) &:= \rho\big(\sigma^{-1}(g)\big), \\ ([\sigma]\rho)(g) &:= \rho\big([\sigma^{-1}](g)\big), \end{aligned} \qquad \text{for } g \in \mathcal{F}_n. \tag{5.22}$$

Theorem 5.12.

$$(5.23)\qquad \begin{cases} \sigma\rho_n \;\cong\; \rho_n & \text{for } \mathcal{F}_n,\ n \text{ even};\\ \sigma\rho_{n,\kappa} \;\cong\; \rho_{n,\mathrm{sgn}(\sigma)\kappa} & \text{for } \mathcal{F}_n,\ n \text{ odd},\ \kappa=\pm; \end{cases}$$

$$(5.24)\qquad [\sigma]\rho_n \;\cong\; \rho_n\ \ (n \text{ even});\qquad [\sigma]\rho_{n,\kappa} \;\cong\; \rho_{n,\kappa}\ \ (n \text{ odd});$$

where the sign $\mathrm{sgn}(\sigma)\kappa$ *in (5.23) is defined by* $\mathrm{sgn}(\sigma)\cdot(\kappa 1) = \big(\mathrm{sgn}(\sigma)\kappa\big)1$.

Proof. This fact can be seen from the explicit form of characters of ρ_n and $\rho_{n,\pm}$ given in the preceding subsection, since they are invariant under $\mathfrak{S}_n$ for n even, and covariant with $\mathrm{sgn}(\sigma)$ for n odd. □

6 Spin IRs of $\widetilde{D}_n := \widetilde{D}(m,n)$

6.1 Induced representations from a central subgroup

Assume $m = 2m'$ be even. We have the central extension

$$(6.1)\qquad \{e\} \longrightarrow Z_2 := \langle z_2\rangle \longrightarrow \widetilde{D}(m,n) \xrightarrow{\Phi_D} D(m,n) \longrightarrow \{e\}\quad \text{(exact)},$$

where the canonical generators η_j ($j \in \boldsymbol{I}_n$) of $\widetilde{D}(m,n)$ are mapped to $y_j = \Phi_D(\eta_j) \in D(m,n)$. Here Φ_D maps each cyclic subgroup $T'_j = \langle\eta_j\rangle$ of $\widetilde{D}(m,n)$ isomorphically onto $T_j = \langle y_j\rangle$ of $D(m,n)$. Let $T = \boldsymbol{Z}_m = \langle y\rangle$ and $T' = \boldsymbol{Z}_m = \langle\eta\rangle$ be the protocols of T_j and T'_j respectively. A character of T is given as $\zeta^{(c)}(y) := \omega^c$ with an integer c considered mod m, and $\omega = e^{2\pi i/m}$, and the corresponding character of T' is denoted by $\chi^{(c)}$: $\chi^{(c)}(\eta) = \omega^c$. Note that $\zeta^{(c+m')}(y) = \omega^{c+m'} = -\zeta^{(c)}(y)$, with $m' = m/2$, and similarly for $\chi^{(c)}$. Denote by $\zeta_{j,\gamma_j} \in \widehat{T_j}$ the copy of $\zeta^{(\gamma_j)}$, i.e., $\zeta_{j,\gamma_j}(y_j^{\,a}) := \omega^{\gamma_j a}$, and similarly for $\chi_{j,\gamma_j} \in \widehat{T'_j}$: $\chi_{j,\gamma_j}(\eta_j^{\,a}) := \omega^{\gamma_j a}$.

For $\gamma = (\gamma_1,\ldots,\gamma_n)$, put $\zeta_\gamma := (\zeta_j)_{j\in\boldsymbol{I}_n}$, $\zeta_j = \zeta_{j,\gamma_j}$, then it covers all one-dimensional characters of $D(m,n) = \prod_{j\in\boldsymbol{I}_n} T_j$.

However, for non-commutative extension $\widetilde{D}(m,n)$, the situation is not so simple.

Definition 6.1. For $\gamma = (\gamma_1,\ldots,\gamma_n)$, define *spin* functions χ_γ and $\mathrm{sgn}_{\widetilde{D}}$ on $\widetilde{D}(m,n)$ as follows: for $d' = z_2^{\,b}\,\eta_1^{\,a_1}\cdots\eta_n^{\,a_n}$,

$$(6.2)\qquad \chi_\gamma(d') := (-1)^b\,\omega^{\gamma_1 a_1}\cdots\omega^{\gamma_n a_n},\quad \mathrm{sgn}_{\widetilde{D}}(d') := (-1)^b,$$

where $b = 0,1$; $0 \le a_j < m = 2m'$ ($j \in \boldsymbol{I}_n$). Define also a (non-spin) character of $\widetilde{D}(m,n)$ by $\zeta_\gamma \circ \Phi_D$ and denote it again by the same symbol ζ_γ.

The character ζ_γ behaves reasonably under the action of $\widetilde{\mathfrak{S}}_n$, but neither $\chi_\gamma = \mathrm{sgn}_{\widetilde{D}}\cdot\zeta_\gamma$ nor $\mathrm{sgn}_{\widetilde{D}}$. However the latters describe well rather complicated behaviors of spin IRs of $\widetilde{D}(m,n)$ under $\widetilde{\mathfrak{S}}_n$ as will be seen below. (In particular,

χ_γ describes globally the character of spin IR P_γ of $\widetilde{D}(m,n)$. Cf. Theorem 6.3 below.) Note that, for d' and $d'' = z_2^{b'}\eta_1^{a'_1}\eta_2^{a'_2}\cdots\eta_n^{a'_n}$,

$$\mathrm{sgn}_{\widetilde{D}}(d'd'') = (-1)^c\,\mathrm{sgn}_{\widetilde{D}}(d')\,\mathrm{sgn}_{\widetilde{D}}(d'') \quad \text{with} \quad c = \textstyle\sum_{j>k} a_j a'_k.$$

The restriction of χ_γ onto $\widetilde{D}^1(m,n) := \langle z_2, \widetilde{D}^0(m,n)\rangle \supset \widetilde{D}^0(m,n)$ is a character, and so is $\chi_\gamma\big|_{T'_j} = \chi_{j,\gamma_j}$ for each j, which corresponds to ζ_{j,γ_j} on $T_j \cong T'_j$.

Actually the above exact sequence (6.1) does not work well to construct spin IRs ρ. We apply another structure of central extension with $\widetilde{D}^0(m,n) = \langle \eta_1^2, \eta_2^2, \ldots, \eta_n^2\rangle$

$$\text{(6.3)} \qquad \{e\} \longrightarrow \widetilde{D}^0(m,n) \longrightarrow \widetilde{D}(m,n) \xrightarrow{\ \Psi\ } \mathcal{F}_n \longrightarrow \{e\} \quad \text{(exact)},$$

that is, $\widetilde{D}(m,n)/\widetilde{D}^0(m,n) \cong \mathcal{F}_n$ under $\Psi : \eta_j \mapsto f_j \in \mathcal{F}_n$ $(j \in \boldsymbol{I}_n)$. Any character of $\widetilde{D}^0(m,n)$ is given as

$$\text{(6.4)} \qquad \chi_\gamma^0 := \chi_\gamma\Big|_{\widetilde{D}^0(m,n)}, \quad \widetilde{D}^0(m,n) = \prod_{j\in \boldsymbol{I}_n} T_j^0\,, \quad T_j^0 := \langle \eta_j^2\rangle.$$

For another $\gamma' = (\gamma'_1, \ldots, \gamma'_n)$, $\chi_\gamma^0 = \chi_{\gamma'}^0$ if and only if $\gamma_j \equiv \gamma'_j \pmod{m'}$ for all $j \in \boldsymbol{I}_n$.

Definition 6.2. Let Γ_n be the set of parameters $\gamma = (\gamma_1, \ldots, \gamma_n)$ satisfying $0 \le \gamma_j < m = 2m'$ $(j \in \boldsymbol{I}_n)$, and define its subset Γ_n^0 as

$$\text{(6.5)} \qquad \Gamma_n^0 := \big\{\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)\ ;\ 0 \le \gamma_j < m' = m/2\ (j \in \boldsymbol{I}_n)\big\} \subset \Gamma_n\,.$$

A character χ_γ^0 of $\widetilde{D}^0(m,n)$ is extended to $\chi_\gamma^1 := \chi_\gamma\big|_{\widetilde{D}^1(m,n)}$ of $\widetilde{D}^1(m,n)$ by putting $\chi_\gamma^1(z_2) = -1$. Now induce it up to $\widetilde{D}(m,n)$ as $\Pi_\gamma := \mathrm{Ind}_{\widetilde{D}^1(m,n)}^{\widetilde{D}(m,n)}\,\chi_\gamma^1$, and decompose it into irreducibles. At first we have the following.

Theorem 6.1. *The trace character χ_{Π_γ} of the induced representation Π_γ of $\widetilde{D}(m,n)$ is given as follows: for $d' = z_2^b\eta_1^{a_1}\eta_2^{a_2}\cdots\eta_n^{a_n} \in \widetilde{D}(m,n)$,*

$$\chi_{\Pi_\gamma}(d') = \begin{cases} (-1)^b\,\chi_\gamma\big(\eta_1^{a_1}\eta_2^{a_2}\cdots\eta_n^{a_n}\big)\cdot 2^n \ =\ (-1)^b\,2^n\cdot\omega^{\gamma_1a_1+\cdots+\gamma_na_n}, \\ \qquad\quad \text{if } \eta_1^{a_1}\eta_2^{a_2}\cdots\eta_n^{a_n} \in \widetilde{D}^0(m,n) \ \text{or}\ a_j \equiv 0 \pmod 2 \ \ (j \in \boldsymbol{I}_n), \\ 0\,, \qquad \text{otherwise.} \end{cases}$$

In the above, $\chi_\gamma\big(\eta_1^{a_1}\eta_2^{a_2}\cdots\eta_n^{a_n}\big) = \zeta_\gamma\big(\eta_1^{a_1}\eta_2^{a_2}\cdots\eta_n^{a_n}\big)$, since it is on $\widetilde{D}^0(m,n)$.

6.2 Explicit form of spin IRs of $\widetilde{D}_n$ and their characters

We prepare a series of matrix IRs of $\widetilde{D}_n := \widetilde{D}(m,n)$. Abbreviate the notation as $\widetilde{D}_n^k := \widetilde{D}^k(m,n)$ for $k = 0, 1$, and let $Y_1, Y_2, \ldots, Y_{2n'}, Y_{2n'+1}$ be $2^{n'} \times 2^{n'}$ type matrices in (5.13). Let $n = 2n'$ or $n = 2n'+1$.

Definition 6.3 (IRs of $\widetilde{D}_n$). Put, for $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Gamma_n$,

$$P_\gamma(\eta_j) := \chi_\gamma(\eta_j) Y_j \quad (j \in \boldsymbol{I}_n), \quad \chi_\gamma(\eta_j) = \zeta_\gamma(\eta_j) = \omega^{\gamma_j}. \tag{6.6}$$

Lemma 6.2. *The system $P_\gamma(\eta_j)$ $(j \in \boldsymbol{I}_n)$ gives an IR of $\widetilde{D}_n$ with $P_\gamma(z_2) = -E$, and for $d' = z_2^{\,b} \eta_{p_1}^{\,b_1} \eta_{p_2}^{\,b_2} \cdots \eta_{p_k}^{\,b_k}$ with any $p_1, p_2, \ldots, p_k \in \boldsymbol{I}_n$ and $b = 0, 1$,*

$$P_\gamma(d') = (-1)^b \zeta_\gamma(d') \cdot Y_{p_1}^{\,b_1} Y_{p_2}^{\,b_2} \cdots Y_{p_k}^{\,b_k}. \tag{6.7}$$

When d' is transcribed as $d' = z_2^{\,b'} \eta_1^{\,a'_1} \eta_2^{\,a'_2} \cdots \eta_n^{\,a'_n}$ by exchanging η_j's, then

$$P_\gamma(d') = \chi_\gamma(d') \cdot Y_1^{\,a'_1} Y_2^{\,a'_2} \cdots Y_n^{\,a'_n}. \tag{6.8}$$

Proof. We can check the fundamental relations (iii)–(iv) in Theorem 2.3 as

$$\begin{cases} P_\gamma(\eta_j)^2 = \zeta_\gamma(\eta_j^{\,2})\, E = \omega^{2\gamma_j}\, E \quad \therefore \;\; P_\gamma(\eta_j)^m = E \quad (j \in \boldsymbol{I}_n), \\ P_\gamma(\eta_j) P_\gamma(\eta_k) = -P_\gamma(\eta_k) P_\gamma(\eta_j) \quad (j \neq k). \end{cases}$$

This proves that P_γ is a representation of $\widetilde{D}_n$. The irreducibility can be seen easily from Lemma 5.9. □

Remark 6.1. The formula (6.7) for *spin* IR P_γ suggests in appearance that the responsibility of non-commutativity among η_j's is shifted through P_γ unilaterally to the side of the product of Y_j's. It is partially true but not completely because of the sign factor. The spin function χ_γ (not a one-dimensional character) in the formula (6.8) represents *spin* property of P_γ and helps to keep in mind the non-commutativity of $\widetilde{D}_n$, and in particular it describe the character χ_{P_γ} of P_γ globally as seen in Theorem 6.3 below. Confer also §§16–17, in particular Remark 17.1, Propositions 17.8 and 17.6, and also Theorem 20.13 etc.

The non-spin character ζ_γ is first defined on $D(m,n)$ and then on $\widetilde{D}(m,n)$ through Φ_D. The spin function χ_γ on $\widetilde{D}(m,n)$ coincides with ζ_γ on $\widetilde{D}^0(m,n)$ and also on each T'_j. A point of danger of confusion exists in

$$\zeta_\gamma(d') = \textstyle\prod_{1 \le i \le k} \chi_\gamma(\eta_{p_i}^{\,b_i}) = \pm(-1)^b \chi_\gamma(d'),$$

where, to get the sign $\pm$ exactly, rearrange $d' = z_2^{\,b} \eta_{p_1}^{\,b_1} \eta_{p_2}^{\,b_2} \cdots \eta_{p_k}^{\,b_k}$ as $d' = z_2^{\,b+b'} \eta_1^{\,a_1} \eta_2^{\,a_2} \cdots \eta_n^{\,a_n}$, then $\chi_\gamma(d') = (-1)^{b+b'} \omega^{\gamma_1 a_1 + \cdots + \gamma_n a_n} = (-1)^{b'} \cdot (-1)^b \zeta_\gamma(d')$.

Theorem 6.3. *The trace character χ_{P_γ} of IR P_γ of $\widetilde{D}_n$ is given as follows.*
(i) *Assume $n = 2n'$. Then,*

$$\chi_{P_\gamma}(d') = \begin{cases} 2^{n'} \cdot \chi_\gamma(d'), & \text{for } d' \in \widetilde{D}_n^1, \\ 0, & \text{otherwise.} \end{cases}$$

(ii) *Assume* $n = 2n' + 1$. *Then,*

$$\chi_{P_\gamma}(d') = \begin{cases} 2^{n'} \cdot \chi_\gamma(d'), & \text{for } d' \in \widetilde{D}_n^1, \\ (2i)^{n'} \cdot \chi_\gamma(d'), & \text{for } d' \in (\eta_1\eta_2\cdots\eta_n)\widetilde{D}_n^1, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We prove only the assertion (ii). We apply Lemma 5.10. The relation $abc = i\varepsilon$ gives us

$$Y_1Y_2\cdots Y_{2n'+1} = (abc)\otimes\cdots\otimes(abc) = (abc)^{\otimes n'} = (i\varepsilon)^{\otimes n'},$$
$$\therefore\quad \mathrm{tr}\big(Y_1Y_2\cdots Y_{2n'+1}\big) = (2i)^{n'}. \qquad \square$$

6.3 Equivalence relations among P_γ's

Definition 6.4. On the space Γ_n, we define an action τ_k for each $k \in \boldsymbol{I}_n$ as

$$\text{(6.9)}\qquad \tau_k\gamma = (\gamma'_1,\ldots,\gamma'_n) \quad\text{with}\quad \begin{cases} \gamma'_k = \gamma_k + m' \pmod m, \\ \gamma'_j = \gamma_j \quad (j \neq k). \end{cases}$$

Lemma 6.4. (i) *Assume* $n = 2n'$. *Then, for* $\gamma' = (\gamma'_1,\ldots,\gamma'_n)$,

$$\text{(6.10)}\qquad P_{\gamma'} \cong P_\gamma \iff \gamma' \approx_n \gamma \quad \textit{under the congruence } \approx_n \textit{ generated by } \{\tau_k\,;\, k \in \boldsymbol{I}_n\}.$$

As parameters for a complete set of representatives of the spin dual of $\widetilde{D}(m,n)$, *we have* $\gamma \in \Gamma_n^0$, *or*

$$\text{(6.11)}\qquad \gamma = (\gamma_1,\ldots,\gamma_n), \quad 0 \le \gamma_j < m' \ (j \in \boldsymbol{I}_n).$$

(ii) *Assume* $n = 2n' + 1$. *Then, for* $\gamma' = (\gamma'_1,\ldots,\gamma'_n)$,

$$\text{(6.12)}\qquad P_{\gamma'} \cong P_\gamma \iff \gamma' \approx_n \gamma \quad \textit{under the congruence } \approx_n \textit{ generated by } \{\tau_k\tau_\ell\,;\, k,\ell \in \boldsymbol{I}_n\}.$$

As parameters for a complete set of representatives of the spin dual of $\widetilde{D}(m,n)$, *we have* γ, $\tau_n\gamma$ $(\gamma \in \Gamma_n^0)$, *or*

$$\text{(6.13)}\qquad \gamma = (\gamma_1,\ldots,\gamma_{n-1},\gamma_n), \quad \begin{cases} 0 \le \gamma_j < m' & (1 \le j \le n-1 = 2n'), \\ 0 \le \gamma_n < m = 2m' & (j = n = 2n'+1). \end{cases}$$

Proof. Using the character formulas in Theorem 6.3, we see that the set of IRs P_γ with parameters in (6.11) or (6.13) respectively are mutually inequivalent.

The completeness of the set of these IRs is proved by dimension calculus such as

$$\sum_{\gamma:\,(6.11)} \big(\dim P_\gamma\big)^2 = (m')^n \times \big(2^{n'}\big)^2 = m^n = \big|\widetilde{D}(m,n)\big|/2, \quad \text{if } n = 2n' \text{ even,}$$

$$\sum_{\gamma:\,(6.13)} \big(\dim P_\gamma\big)^2 = (m')^{2n'} \cdot m \times \big(2^{n'}\big)^2 = m^{2n'+1} = m^n = \big|\widetilde{D}(m,n)\big|/2,$$

if $n = 2n'+1$ odd. □

Theorem 6.5. (i) *In the case where $n = 2n'$ even, a complete set of representatives of spin IRs of $\widetilde{D}(m,n)$ is given by*

$$\text{(6.14)} \qquad \mathrm{Rep}(\widetilde{D}_n) := \big\{ P_\gamma \;;\; \gamma \in \Gamma_n^0 \big\}.$$

(ii) *In the case where $n = 2n'+1$ odd, a complete set of representatives of spin IRs is given by $\big\{P_\gamma\,;\,\gamma$ satisfies (6.13)$\big\}$, and is divided into two subsets as*

$$\text{(6.15)} \qquad \mathrm{Rep}(\widetilde{D}_n) \;=\; \mathrm{Rep}^+(\widetilde{D}_n) \bigsqcup \mathrm{Rep}^-(\widetilde{D}_n)\,,$$

$$\text{(6.16)} \qquad \begin{cases} \mathrm{Rep}^+(\widetilde{D}_n) := \big\{ P_\gamma^+ := P_\gamma \;;\; \gamma \in \Gamma_n^0 \big\}, \\ \mathrm{Rep}^-(\widetilde{D}_n) := \big\{ P_\gamma^- := P_{\tau_n\gamma} \;;\; \gamma \in \Gamma_n^0 \big\}. \end{cases}$$

The reason for dividing into two subsets $\mathrm{Rep}^+(\widetilde{D}_n)$ and $\mathrm{Rep}^-(\widetilde{D}_n)$ is seen from the following. Let $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Gamma_n^0$. Then $\tau_n\gamma = (\gamma_1, \ldots, \gamma_{n-1}, \gamma_n + m')$, $n = 2n'+1$, and for $d' = z_2^{\,b}\,\eta_1^{\,a_1}\eta_2^{\,a_2}\cdots\eta_n^{\,a_n} \in \widetilde{D}_n$, we have $\Psi(d') = f_1^{\,a_1}\cdots f_n^{\,a_n} \in \mathcal{F}_n$, and

$$\text{(6.17)} \qquad P_\gamma^+(d') \;=\; \chi_\gamma(d') Y_1^{a_1}\cdots Y_n^{a_n} = \chi_\gamma(d')\,\rho_{n,+}\big(f_1^{\,a_1}\cdots f_n^{\,a_n}\big).$$

$$\begin{aligned} \text{(6.18)} \qquad P_\gamma^-(d') \;&=\; P_{\tau_n\gamma}(d') = \chi_{\tau_n\gamma}(d') Y_1^{a_1}\cdots Y_n^{a_n} \\ &=\; \chi_\gamma(d') Y_1^{a_1}\cdots Y_{n-1}^{a_{n-1}}(-Y_n)^{a_n} = \chi_\gamma(d')\,\rho_{n,-}\big(f_1^{\,a_1}\cdots f_n^{\,a_n}\big). \end{aligned}$$

The induced representation Π_γ is decomposed into the sum of IRs above as shown below.

Proposition 6.6. *The induced representation $\Pi_\gamma = \mathrm{Ind}_{\widetilde{D}_n^1}^{\widetilde{D}_n}\chi_\gamma^1$ is decomposed as*

$$\begin{aligned} \Pi_\gamma \;&\cong\; \big[2^{n'}\big]\cdot P_\gamma\,, && \text{in case } n = 2n'\,; \\ \Pi_\gamma \;&\cong\; \big[2^{n'}\big]\cdot \big(P_\gamma \oplus P_{\tau_n\gamma}\big), && \text{in case } n = 2n'+1, \end{aligned}$$

where $\big[2^{n'}\big]$ denotes the multiplicity, and $\tau_n\gamma = (\gamma_1, \ldots, \gamma_{n-1}, \gamma_n + m') \pmod m$. In case $n = 2n'+1$ odd, $\Pi_\gamma \;\cong\; \big[2^{n'}\big]\cdot\big(P_\gamma^+ \oplus P_\gamma^-\big)$ if $\gamma \in \Gamma_n^0$.

Proof. By character formulas in Theorem 6.3, we see that, in case $n = 2n'$, $\chi_{\Pi_\gamma} = 2^{n'}\cdot\chi_{P_\gamma}$, which gives the desired result. In case $n = 2n'+1$, we apply

$$\big(\chi_{P_\gamma} + \chi_{P_{\tau_n\gamma}}\big)(d') = \begin{cases} 2\,\chi_{P_\gamma}(d'), & \text{if } d' \in \widetilde{D}_n^1, \\ 0\,, & \text{otherwise,} \end{cases}$$

and an explicit formula for the character Π_γ. □

6.4 Fundamental IRs P_0 for n even, and P_+, P_- for n odd

Put $\gamma^{(0)} := (0, 0, \ldots, 0)$ and $\gamma^{(-)} := \tau_n \gamma^{(0)} = (0, 0, \ldots, 0, m')$, and

$$\text{(6.19)} \qquad \begin{cases} P_0 := P_{\gamma^{(0)}} & \text{if } n = 2n' \text{ even;} \\ P_+ := P_{\gamma^{(0)}},\ P_- := P_{\gamma^{(-)}} & \text{if } n = 2n'+1 \text{ odd.} \end{cases}$$

Then, $P_0(d') = \mathrm{sgn}_{\widetilde{D}}(d')\, Y_1^{a_1} Y_2^{a_2} \cdots Y_n^{a_n}$ for $d' = z_2^b \eta_1^{a_1} \eta_2^{a_2} \cdots \eta_n^{a_n}$. Through $\Psi : \widetilde{D}_n \to \mathcal{F}_n$, P_0 is essentially equal to ρ_n, and $P_\pm$ to $\rho_{n,\pm}$.

Theorem 6.7. *The IRs of* $\widetilde{D}_n$ *are expressed as tensor products of non-spin one-dimensional characters and fundamental spin IRs as follows:*

$$P_\gamma \cong \zeta_\gamma \otimes P_0 \quad (\gamma \in \Gamma_n) \qquad \text{in case } n \text{ even,}$$

$$\begin{cases} P_\gamma^+ \cong \zeta_\gamma \otimes P_+ & (\gamma \in \Gamma_n^0) \\ P_\gamma^- \cong \zeta_\gamma \otimes P_- & (\gamma \in \Gamma_n^0) \end{cases} \qquad \text{in case } n \text{ odd.}$$

7 Actions of $\widetilde{\mathfrak{S}}_n$ on $\widetilde{D}_n$ and stationary subgroups for equivalence classes of IRs

7.1 Actions of $\widetilde{\mathfrak{S}}_n$ on the group $\widetilde{D}_n$, in CASEs I and II

Inside of the base group $G(m,1,n) = D_n \rtimes \mathfrak{S}_n$, $D_n := D(m,n)$, an element $\sigma \in \mathfrak{S}_n$ acts on D_n as

$$\text{(7.1)} \qquad \sigma(y_j) = y_{\sigma(j)} \qquad (j \in \boldsymbol{I}),$$

where y_j is the generator of $T_j \cong T$ corresponding to $y \in T$. However, in the covering groups $\widetilde{G}_n^{\mathrm{Y}} := \widetilde{G}^{\mathrm{Y}}(m,1,n) = \widetilde{D}_n \overset{\mathrm{Y}}{\rtimes} \widetilde{\mathfrak{S}}_n$, $\widetilde{D}_n := \widetilde{D}(m,n)$, of CASE Y = I and II, the covering group $\widetilde{\mathfrak{S}}_n$ acts on $\widetilde{D}_n$ in different ways for Y = I and II. We discuss them independently.

Since the central element z_1 acts trivially, $\widetilde{\mathfrak{S}}_n$ acts through $\Phi_{\mathfrak{S}} : \widetilde{\mathfrak{S}}_n \to \mathfrak{S}_n$. But we prefer to keep the original $r_i \in \widetilde{\mathfrak{S}}_n$ (instead of $s_i \in \mathfrak{S}_n$), and denote the action of r_i on $d' \in \widetilde{D}(m,n)$ as $r_i^{\mathrm{I}}(d')$ $\big($instead of $s_i^{\mathrm{I}}(d')\big)$ in CASE I, and as $r_i^{\mathrm{II}}(d')$ in CASE II, to distinguish them each other.

For conveniences, in CASE I, we use the new generators

$$\text{(7.2)} \qquad \eta'_j = z_2^{j-1} \eta_j \ (j \in \boldsymbol{I}_n)$$

of $\widetilde{D}_n = \widetilde{D}(m,n)$ (cf. §2.3). Then, since $\chi_\gamma(z_2) = -1$ and $P_\gamma(z_2) = -E$, we have

$$\text{(7.3)} \qquad P_\gamma(\eta'_j) = \chi_\gamma(\eta'_j) Y_j = \chi_\gamma(\eta_j) Y'_j \ \ (j \in \boldsymbol{I}_n) \qquad \text{in CASE I},$$

where $Y'_j := (-1)^{j-1} Y_j$ $(j \in \boldsymbol{I}_n)$. In fact, $P_\gamma(\eta'_j) = P_\gamma(z_2^{j-1}\eta_j) = (-1)^{j-1} \chi_\gamma(\eta_j) Y_j = \chi_\gamma(\eta'_j) Y_j = \chi_\gamma(\eta_j) Y'_j$.

From §2.3, we have for $1 \le i \le n-1$,

$$r_i^{\mathrm{I}}(\eta'_j) = z_2 \eta'_{s_i(j)} \quad (j \in \boldsymbol{I}_n), \quad \text{in CASE I ;} \tag{7.4}$$

$$r_i^{\mathrm{II}}(\eta_j) = \eta_{s_i(j)} \quad (j \in \boldsymbol{I}_n), \quad \text{in CASE II .} \tag{7.5}$$

7.2 Action of $\widetilde{\mathfrak{S}}_n$ on IRs and stationary subgroups (CASE I)

Theorem 7.1. *Let $r_i \in \widetilde{\mathfrak{S}}_n$ and $\Phi_{\mathfrak{S}}(r_i) = s_i = (i\ i{+}1) \in \mathfrak{S}_n,\ 1 \leqslant i \leqslant n-1$, a simple transposition. In CASE I, Type $(-1,-1,-1)$, the action on IR P_γ of $\widetilde{D}_n = \widetilde{D}(m,n)$ is as follows: for $\gamma \in \Gamma_n^0$ in (6.5),*

$$\begin{cases} r_i^{\mathrm{I}} P_\gamma \cong P_{s_i\gamma} & (1 \le i \le n-1), \\ \sigma'^{\,\mathrm{I}} P_\gamma \cong P_{\sigma\gamma} & \big(\sigma' \in \widetilde{\mathfrak{S}}_n,\ \sigma = \Phi_{\mathfrak{S}}(\sigma') \in \mathfrak{S}_n\big), \end{cases}$$

where $\sigma'\gamma = \sigma\gamma := \big(\gamma_{\sigma^{-1}(1)}, \gamma_{\sigma^{-1}(2)}, \ldots, \gamma_{\sigma^{-1}(n)}\big)$.

Proof. To prove this, we study characters. Let $n = 2n'$. If $a_j \equiv 0 \pmod 2$ $(j \in \boldsymbol{I}_n)$, then $d' = z_2^{\,b}\, {\eta'_1}^{a_1} \cdots {\eta'_n}^{a_n} = z_2^{\,b}\, \eta_1^{a_1} \cdots \eta_n^{a_n}$, and so $r_i^{\mathrm{I}}(d') = z_2^{\,b}\, s_i\big(\eta_1^{a_1} \cdots \eta_n^{a_n}\big)$. From Theorem 6.3 (i), we see that

$$\chi_{P_\gamma}\big(r_i^{\mathrm{I}}(d')\big) = \chi_{P_{s_i\gamma}}(d') \quad (d' \in \widetilde{D}_n).$$

Let $n = 2n'+1$. If $a_j \equiv 0 \pmod 2$ $(j \in \boldsymbol{I}_n)$, then $\chi_{(r_i^{\mathrm{I}} P_\gamma)}(d') = \chi_{P_{s_i\gamma}}(d')$.
If $a_j \equiv 1 \pmod 2$ $(j \in \boldsymbol{I}_n)$, then

$$\begin{aligned} d' &= z_2^{\,b} {\eta'_1}^{a_1} \cdots {\eta'_n}^{a_n} = z_2^{\,b} z_2^{\,n(n-1)/2}\, \eta_1^{a_1} \cdots \eta_n^{a_n} = z_2^{\,b} z_2^{\,n'}\, \eta_1^{a_1} \cdots \eta_n^{a_n}, \\ r_i^{\mathrm{I}}(d') &= z_2^{\,b} \cdot z_2^{\,n}\, {\eta'_{s_i(1)}}^{a_1} \cdots {\eta'_{s_i(n)}}^{a_n} = z_2^{\,b} z_2^{\,n} \cdot z_2\, {\eta'_1}^{a_{s_i(1)}} \cdots {\eta'_n}^{a_{s_i(n)}} = \\ &= z_2^{\,b} z_2^{\,n'}\, \eta_1^{a_{s_i(1)}} \cdots \eta_n^{a_{s_i(n)}}, \end{aligned}$$

and from Theorem 6.3 (ii), we see that for $\chi_{P_\gamma}\big(r_i^{\mathrm{I}}(d')\big) = \chi_{P_{s_i\gamma}}(d')$ $(d' \in \widetilde{D}_n)$, whence $\chi_{(r_i^{\mathrm{I}} P_\gamma)} = \chi_{P_{s_i\gamma}}$. □

Denote by $[P_\gamma]$ the equivalence class of IR P_γ, and by $\mathcal{S}(P_\gamma)$ the stationary subgroup of $[P_\gamma]$ in $\widetilde{\mathfrak{S}}_n$, and put $S(P_\gamma) := \Phi_{\mathfrak{S}}\big(\mathcal{S}(P_\gamma)\big) \subset \mathfrak{S}_n$. Note that $\Phi_{\mathfrak{S}}\big(\mathcal{S}(P_\gamma)\big) = \Phi_{\mathrm{I}}\big(\mathcal{S}(P_\gamma)\big)$ with $\Phi_{\mathrm{I}} : \widetilde{G}_n^{\mathrm{I}} \to G(m,1,n)$, when $\mathcal{S}(P_\gamma)$ is understood as a subgroup of $\widetilde{G}_n^{\mathrm{I}}$.

Theorem 7.2. *In CASE I, Type $(-1,-1,-1)$, let $\gamma \in \Gamma_n^0$ in (6.5).*
(i) *The stationary subgroup of $[P_\gamma]$ in $\widetilde{\mathfrak{S}}_n$, $n \ge 4$, is given as*

$$\mathcal{S}(P_\gamma) = \big\{\sigma' \in \widetilde{\mathfrak{S}}_n \ ;\ \sigma\gamma = \gamma,\ \sigma = \Phi(\sigma')\big\}.$$

(ii) *Let n be odd. The stationary subgroups of $[P_\gamma^+]$ and $[P_\gamma^-]$ in $\widetilde{\mathfrak{S}}_n$ are given as*

$$\mathcal{S}(P_\gamma^{\pm}) = \big\{\sigma' \in \widetilde{\mathfrak{S}}_n \ ;\ \sigma\gamma = \gamma,\ \sigma = \Phi(\sigma')\big\}.$$

7.3 Action of $\widetilde{\mathfrak{S}}_n$ on IRs and stationary subgroups (CASE II)

In CASE II, Type $\beta = (-1, -1,\ 1)$, in the covering group $\widetilde{G}_n^{\mathrm{II}}$, the group $\widetilde{\mathfrak{S}}_n$ acts on $\widetilde{D}_n$ through $\mathfrak{S}_n \cong \widetilde{\mathfrak{S}}_n/\langle z_1\rangle$ as in (7.5). Then, for $\sigma' \in \widetilde{\mathfrak{S}}_n$, let $\sigma = \Phi(\sigma') \in \mathfrak{S}_n$,

$$\sigma'^{\mathrm{II}}(\eta_j) = \sigma^{\mathrm{II}}(\eta_j) := \eta_{\sigma(j)} \qquad (j \in \boldsymbol{I}_n). \tag{7.6}$$

Hence it acts on a representation π of $\widetilde{D}_n$ as

$$(\sigma'^{\mathrm{II}}\pi)(d') = (\sigma^{\mathrm{II}}\pi)(d') = \pi\big((\sigma^{-1})^{\mathrm{II}}(d')\big) \qquad (d' \in \widetilde{D}_n). \tag{7.7}$$

Theorem 7.3. *Let* $s_i = (i\ \ i{+}1) \in \mathfrak{S}_n$, $1 \le i \le n-1$, *be simple transpositions. In CASE II, Type* $(-1, -1,\ 1)$, *the action on IRs of* $\widetilde{D}_n$ *is given as follows.*

(i) *Let* $n = 2n'$ *be even. Then for* $\gamma \in \Gamma_n^0$,

$$s_i^{\mathrm{II}} P_\gamma \cong P_{s_i\gamma}, \qquad \sigma^{\mathrm{II}} P_\gamma \cong P_{\sigma\gamma} \ \ (\sigma \in \mathfrak{S}_n),$$
$$\sigma\gamma = \big(\gamma_{\sigma^{-1}(1)}, \gamma_{\sigma^{-1}(2)}, \ldots, \gamma_{\sigma^{-1}(n)}\big).$$

(ii) *Let* $n = 2n'+1$ *be odd. Then, for* $\gamma \in \Gamma_n$, $s_i^{\mathrm{II}} P_\gamma \cong P_{\tau_n s_i \gamma}$ $(1 \le i \le n-1)$, *whence for* $\gamma \in \Gamma_n^0$,

$$\begin{cases} \sigma^{\mathrm{II}} P_\gamma^+ \cong P_{\sigma\gamma}^+, & \sigma^{\mathrm{II}} P_\gamma^- \cong P_{\sigma\gamma}^- \qquad \big(\sigma \in \mathfrak{S}_n,\ \mathrm{sgn}(\sigma) = 1\big), \\ \sigma^{\mathrm{II}} P_\gamma^+ \cong P_{\sigma\gamma}^-, & \sigma^{\mathrm{II}} P_\gamma^- \cong P_{\sigma\gamma}^+ \qquad \big(\sigma \in \mathfrak{S}_n,\ \mathrm{sgn}(\sigma) = -1\big). \end{cases}$$

Proof. We apply explicit character formulas for P_γ, $P_\gamma^+ = P_\gamma$ and $P_\gamma^- = P_{\tau_n\gamma}$ in Theorem 6.3. Since the proof for the assertion (i) is similar as that for Theorem 7.1, we treat (ii) here.

For $d' = z_2^b \eta_1^{a_1} \cdots \eta_n^{a_n}$, we have $\chi_{P_\gamma}(d') \ne 0$ if and only if $a_j \equiv 0$ $(\forall j)$ or $a_j \equiv 1$ $(\forall j)$.

In case $a_j \equiv 0$ $(\forall j)$, the components $\eta_k^{a_k}$ commute with each other, and so the transformation $d' \mapsto r_i^{\mathrm{II}}(d')$ is realized by the exchange $a_i \leftrightarrow a_{i+1}$ in $(a_1, a_2, \ldots, a_n)$, whence

$$\chi_{P_\gamma}\big(r_i^{\mathrm{II}}(d')\big) = \chi_{P_{s_i\gamma}}(d') = \chi_{P_{\tau_n s_i\gamma}}(d'). \tag{7.8}$$

In case $a_j \equiv 1$ $(\forall j)$, we have $\ {\eta_{s_i(i)}}^{a_i} {\eta_{s_i(i+1)}}^{a_{i+1}} = z_2\, {\eta_i}^{a_{i+1}} {\eta_{i+1}}^{a_i}$, and

$$\begin{aligned} \chi_{\tau_n\gamma}(d') &= (-1)^b\, \omega^{a_1\gamma_1 + \cdots + a_{n-1}\gamma_{n-1} + a_n(\gamma_n + m')} \\ &= -(-1)^b\, \omega^{a_1\gamma_1 + \cdots + a_{n-1}\gamma_{n-1} + a_n\gamma_n} = -\chi_\gamma(d'), \end{aligned}$$

since $a_n \equiv 1 \pmod 2$. Their effects cancel each other, and so (7.8) above holds in this case too. Altogether $\chi_{P_\gamma}\big(r_i^{\mathrm{II}}(d')\big) = \chi_{P_{\tau_n s_i\gamma}}(d')$ for $d' \in \widetilde{D}_n$ in general.

This proves the assertion (ii), with the help of (6.12). □

In CASE II, the stationary subgroup of the equivalence class $[P_\gamma]$ in $\mathfrak{S}_n$ (resp. in $\widetilde{\mathfrak{S}}_n$) is denoted by $S(P_\gamma)$ $\big($resp. by $\mathcal{S}(P_\gamma)\big)$.

Theorem 7.4. *In CASE II, Type* $(-1,-1,\ 1)$*, the stationary subgroups* $\mathcal{S}(P) \subset \widetilde{\mathfrak{S}}_n$ *and* $S(P) \subset \mathfrak{S}_n$ *for* $P = P_\gamma, P_\gamma^+$ *and* P_γ^- *are given as follows:*

$$\mathcal{S}(P) \ = \ \Phi_{\mathfrak{S}}^{-1}\big(S(P)\big) \qquad \text{for } \ P = P_\gamma,\ P_\gamma^{\pm}, \ \text{with}$$

$$\begin{cases} S(P_\gamma) & = \ \big\{\sigma \in \mathfrak{S}_n \ ; \ \sigma\gamma = \gamma\big\} \subset \mathfrak{S}_n \quad \textit{in case } n = 2n' \textit{ even ;} \\ S(P_\gamma^{\pm}) & = \ \big\{\sigma \in \mathfrak{A}_n \ ; \ \sigma\gamma = \gamma\big\} \subset \mathfrak{A}_n \quad \textit{in case } n = 2n'+1 \textit{ odd.} \end{cases}$$

8 Intertwining operators for IRs of $\widetilde{D}_n$, related spin representations of $\mathfrak{S}_n$ and $\mathfrak{A}_n$

8.1 Operators which intertwine IRs P_γ and $P_{\gamma'}$ of $\widetilde{D}_n$

Assume $4 \le n < \infty$, and let $Y_1, Y_2, \ldots, Y_{2n'}, Y_{2n'+1}$ be as in (5.13), and put

$$Y_j' = (-1)^{j-1} Y_j \quad (j \in \boldsymbol{I}_n). \tag{8.1}$$

Define ∇_n as

$$\nabla_n(r_j) := \frac{(-1)^{j-1}}{\sqrt{2}}\,(Y_j + Y_{j+1}) = \frac{1}{\sqrt{2}}\,(Y_j' - Y_{j+1}') \quad (j \in \boldsymbol{I}_{n-1}). \tag{8.2}$$

Moreover, when $n = 2n'+1$, we put ∇_n^- as

$$\nabla_n^-(r_j) := -Y_{2n'+1}\nabla_n(r_j)\,Y_{2n'+1}^{-1} \qquad (j \in \boldsymbol{I}_{n-1}). \tag{8.3}$$

Then we have

$$\begin{cases} \nabla_n^-(r_j) = \nabla_n(r_j) & (j \in \boldsymbol{I}_{n-2}), \\ \nabla_n^-(r_{n-1}) = \dfrac{(-1)^{n-2}}{\sqrt{2}}\,(Y_{n-1} - Y_n) = \dfrac{1}{\sqrt{2}}\,(Y_{2n'}' + Y_{2n'+1}')\,; \end{cases} \tag{8.4}$$

and also for $n = 2n', 2n'+1$, and $1 \le j \le n-1$,

$$\begin{aligned} &\nabla_n'(r_j) := \frac{1}{\sqrt{2}}\,(Y_j - Y_{j+1})\ ; \\ &\nabla_n''(r_j) := -Y_{2n'+1}\nabla_n'(r_j)\,Y_{2n'+1}^{-1} = -Y_{2n'+1}\nabla_n'(r_j)\,Y_{2n'+1}\,. \end{aligned} \tag{8.5}$$

Then, for $n = 2n'$, $\nabla_n''(r_j) = \nabla_n'(r_j)$ $(j \in \boldsymbol{I}_{n-1})$, and

$$\text{for } n = 2n'+1, \qquad \begin{cases} \nabla_n''(r_j) = \nabla_n'(r_j) & (j \in \boldsymbol{I}_{n-2}), \\ \nabla_n''(r_{n-1}) = \dfrac{1}{\sqrt{2}}\,(Y_{n-1} + Y_n). \end{cases}$$

When $n = 2n'$ even, we put for $1 \le j \le n-1 = 2n'-1$,

$$\nabla_n^{\mathrm{II}}(r_j) := (iY_{2n'+1}) \cdot \nabla'_n(r_j) = \nabla'_n(r_j) \cdot (-iY_{2n'+1}), \tag{8.6}$$

and when $n = 2n'+1$, for $1 \le j \le n-1 = 2n'$,

$$\begin{aligned} \nabla_n^{\mathrm{II}+}(r_j) &:= (iY_{2n'+1}) \cdot \nabla'_n(r_j), \\ \nabla_n^{\mathrm{II}-}(r_j) &:= \nabla'_n(r_j) \cdot (-iY_{2n'+1}). \end{aligned} \tag{8.7}$$

Then $\begin{cases} \nabla_n^{\mathrm{II}\pm}(r_j) = \nabla_n^{\mathrm{II}}(r_j) & (j \in \boldsymbol{I}_{2n'-1}), \\ \nabla_n^{\mathrm{II}\pm}(r_{2n'}) = (iY_{2n'+1}) \cdot \dfrac{1}{\sqrt{2}}(Y_{2n'} \mp Y_{2n'+1}). \end{cases}$

Lemma 8.1. *The following relations hold:*

$$\begin{cases} \nabla_n(r_i)^2 = E & (i \in \boldsymbol{I}_{n-1}), \\ \big(\nabla_n(r_i)\nabla_n(r_{i+1})\big)^3 = E & (i \in \boldsymbol{I}_{n-2}), \\ \nabla_n(r_i)\nabla_n(r_k) = -\nabla_n(r_k)\nabla_n(r_i) & (|i-k| \ge 2)\ ; \end{cases} \tag{8.8}$$

$$\begin{cases} \nabla_n^-(r_i)^2 = E & (i \in \boldsymbol{I}_{n-1}), \\ \big(\nabla_n^-(r_i)\nabla_n^-(r_{i+1})\big)^3 = E & (i \in \boldsymbol{I}_{n-2}), \\ \nabla_n^-(r_i)\nabla_n^-(r_k) = -\nabla_n^-(r_k)\nabla_n^-(r_i) & (|i-k| \ge 2)\ ; \end{cases} \tag{8.9}$$

$$\begin{cases} \nabla'_n(r_i)^2 = E & (i \in \boldsymbol{I}_{n-1}), \\ \big(\nabla'_n(r_i)\nabla'_n(r_{i+1})\big)^3 = E & (i \in \boldsymbol{I}_{n-2}), \\ \nabla'_n(r_i)\nabla'_n(r_k) = -\nabla'_n(r_k)\nabla'_n(r_i) & (|i-k| \ge 2)\ ; \end{cases} \tag{8.10}$$

and similarly for ∇''_n as ∇'_n.
For $n = 2n'$ even,

$$\begin{cases} \nabla_n^{\mathrm{II}}(r_i)^2 = E & (i \in \boldsymbol{I}_{n-1}), \\ \big(\nabla_n^{\mathrm{II}}(r_i)\nabla_n^{\mathrm{II}}(r_{i+1})\big)^3 = E & (i \in \boldsymbol{I}_{n-2}), \\ \nabla_n^{\mathrm{II}}(r_i)\nabla_n^{\mathrm{II}}(r_k) = -\nabla_n^{\mathrm{II}}(r_k)\nabla_n^{\mathrm{II}}(r_i) & (|i-k| \ge 2)\ ; \end{cases} \tag{8.11}$$

Proof. $\nabla_n(r_i)\nabla_n(r_{i+1}) + \nabla_n(r_{i+1})\nabla_n(r_i) + E =$
$= \dfrac{-1}{2}\Big\{(Y_i + Y_{i+1})(Y_{i+1} + Y_{i+2}) + (Y_{i+1} + Y_{i+2})(Y_i + Y_{i+1})\Big\} + E = -E + E = O.$
$\therefore \quad \big(\nabla_n(r_i)\nabla_n(r_{i+1})\big)^3 = E.$
$\nabla_n^-(r_{n-2})\nabla_n^-(r_{n-1}) + \nabla_n^-(r_{n-1})\nabla_n^-(r_{n-2}) + E =$
$= \dfrac{-1}{2}\Big\{(Y_{n-2} + Y_{n-1})(Y_{n-1} - Y_n) + (Y_{n-1} - Y_n)(Y_{n-2} + Y_{n-1})\Big\} + E = -E + E = O.$
$\therefore \quad \big(\nabla_n^-(r_{n-2})\nabla_n^-(r_{n-1})\big)^3 = E.$

Similarly, other relations can be verified by calculations. □

Lemma 8.2. *For $\nabla_n^{\mathrm{II}+}$ and $\nabla_n^{\mathrm{II}-}$, there hold the following relations:*

$$(8.12)\qquad \begin{cases} \nabla^{\mathrm{II}-}(r_i) = \nabla_n^{\mathrm{II}+}(r_i)^{-1} & (i \in \boldsymbol{I}_{n-1})\,, \\ \nabla^{\mathrm{II}-}(r_i) = \nabla_n^{\mathrm{II}+}(r_i) & (i \in \boldsymbol{I}_{n-2})\;; \end{cases}$$

$$(8.13)\qquad \begin{cases} \nabla_n^{\mathrm{II}+}(r_i)^2 = E & (i \in \boldsymbol{I}_{2n'-1}), \\ \big(\nabla_n^{\mathrm{II}+}(r_i)\nabla_n^{\mathrm{II}+}(r_{i+1})\big)^3 = E & (i \in \boldsymbol{I}_{2n'-2}), \\ \nabla_n^{\mathrm{II}+}(r_i)\nabla_n^{\mathrm{II}+}(r_k) = -\nabla_n^{\mathrm{II}+}(r_k)\nabla_n^{\mathrm{II}+}(r_i) & (i,k \in \boldsymbol{I}_{2n'-1}, |i-k|\geq 2); \end{cases}$$

and similarly for $\nabla_n^{\mathrm{II}-}$. Especially for $n = 2n'+1$ odd,

$$(8.14)\qquad \begin{cases} \nabla_n^{\mathrm{II}+}(r_{2n'})^4 \;=\; -E, \quad \big(\nabla_n^{\mathrm{II}+}(r_{2n'-1})\nabla_n^{\mathrm{II}+}(r_{2n'})\big)^3 \;=\; E, \\ \nabla_n^{\mathrm{II}+}(r_i)\nabla_n^{\mathrm{II}+}(r_{2n'}) \neq -\nabla_n^{\mathrm{II}}(r_{2n'})\nabla_n^{\mathrm{II}}(r_i) \qquad (i \in \boldsymbol{I}_{2n'-2}). \end{cases}$$

Moreover, for $n = 2n'+1$,

$$\begin{cases} \nabla_n^{\mathrm{II}+}(r_i)^{-1} = \nabla_n^{\mathrm{II}+}(r_i) & (i \in \boldsymbol{I}_{2n'-1})\;; \\ \nabla_n^{\mathrm{II}+}(r_{2n'})^{-1} = \nabla_n^{\mathrm{II}-}(r_{2n'}) \neq \nabla_n^{\mathrm{II}+}(r_{2n'}) & (i = 2n'). \end{cases}$$

Proof. $\nabla_n^{\mathrm{II}+}(r_{2n'})^2 = -Y_{2n'}Y_{2n'+1} = \varepsilon^{\otimes(n'-1)} \otimes (-bc) = \varepsilon^{\otimes(n'-1)} \otimes (-ia)$,
$\therefore \quad \nabla_n^{\mathrm{II}+}(r_{2n'})^4 = -E.$
$\nabla_n^{\mathrm{II}+}(r_{2n'-1})\nabla^{\mathrm{II}+}(r_{2n'}) = \frac{1}{2}(Y_{2n'-1} - Y_{2n'})(Y_{2n'} - Y_{2n'+1}) = \varepsilon^{\otimes(n'-1)} \otimes A$,
$A = \frac{1}{2}(a-b)(b-c) = \frac{1}{2}(-\varepsilon + ia + ib + ic),\ A^2 = \frac{1}{2}(-\varepsilon - ia - ib - ic),\ A^3 = \varepsilon$,
$\therefore \quad \big(\nabla_n^{\mathrm{II}+}(r_{2n'-1})\nabla^{\mathrm{II}+}(r_{2n'})\big)^3 = E.$
Similarly, other relations can be verified by calculations. □

8.2 Spin representations of $\mathfrak{S}_n$ by intertwining operators

Through the action of $\widetilde{\mathfrak{S}}_n$ on $\widetilde{D}_n$, we have an action of $\widetilde{\mathfrak{S}}_n$ on IRs of $\widetilde{D}_n$. In the preceding subsection, we have prepared operators which will serve to describe intertwining operators between IRs of $\widetilde{D}_n$ for the action of $\widetilde{\mathfrak{S}}_n$. Before going into this subject, we give here a spin representations of $\mathfrak{S}_n$ using these operators. The following is a consequence of Lemmas 8.1 – 8.2.

Theorem 8.3. *Let $n \geq 4$.*
(i) *The map $r_i \mapsto \nabla_n(r_i)$ $\big($resp. $\nabla_n^-(r_i)\big)$, $i \in \boldsymbol{I}_{n-1}$, gives a spin representation of the representation group $\widetilde{\mathfrak{S}}_n = R\big(\mathfrak{S}_n\big) = \langle z_1, r_1, r_2, \ldots, r_{n-1}\rangle$.*
(ii) *Similar assertion holds also for the maps $r_i \mapsto \nabla'_n(r_i)$ and $r_i \mapsto \nabla''_n(r_i)$ $(i \in \boldsymbol{I}_{n-1})$, and for $\sigma' \in \widetilde{\mathfrak{S}}_n$,*

$$\nabla''_n(\sigma') = \mathrm{sgn}(\sigma') \cdot Y_{2n'+1}\nabla'_n(\sigma')\, Y_{2n'+1}^{-1},$$

that is, $\nabla''_n(\sigma') = Y_{2n'+1}\nabla'_n(\sigma')\,Y_{2n'+1}$ *and* $\nabla''_n(\sigma') = -Y_{2n'+1}\nabla'_n(\sigma')\,Y_{2n'+1}$ *according as* $\mathrm{sgn}(\sigma') = 1$ *or* -1.

(iii) *For* $n = 2n'$*, the map* $r_i \mapsto \nabla^{\mathrm{II}}_n(r_i)$, $i \in \boldsymbol{I}_{n-1} = \boldsymbol{I}_{2n'-1}$*, gives a spin representation of the representation group* $\widetilde{\mathfrak{S}}_n$.

(iv) *For* $n = 2n' + 1$*, the map* $r_i \mapsto \nabla^{\mathrm{II}+}_n(r_i) = \nabla^{\mathrm{II}-}_n(r_i)$, $i \in \boldsymbol{I}_{n-2}$*, gives a spin representation of* $\widetilde{\mathfrak{S}}_{n-1} = R(\mathfrak{S}_{n-1})$*, which is equal to* $\nabla^{\mathrm{II}}_{n-1}$*. But the map* $r_i \mapsto \nabla^{\mathrm{II}+}_n(r_i)$ *(resp.* $\nabla^{\mathrm{II}-}_n(r_i)$*),* $i \in \boldsymbol{I}_{n-1}$*, gives no representation of the whole group* $\widetilde{\mathfrak{S}}_n$*.*

Proof. (i) – (iii) follow from Theorem 1.2 on the presentation of the representation group $\widetilde{\mathfrak{S}}_n = \mathfrak{T}'_n$. (iv) follows from (8.14), in particular, $\nabla^{\mathrm{II}\pm}_n(r_{n-1})^4 = -E$. □

Note that the spin representation $\nabla^{\mathrm{II}}_{2n'}$ of $\widetilde{\mathfrak{S}}_{2n'}$ is equal to the restriction of any of matrix-valued functions $\nabla^{\mathrm{II}+}_{2n'+1}$ and $\nabla^{\mathrm{II}-}_{2n'+1}$ of $\widetilde{\mathfrak{S}}_{2n'+1}$ defined in the next subsection.

After the notion of'zweiseitige (=two-sided)' in [Sch3, §14], we introduce a definition as follows.

Definition 8.1. For a spin character χ of $\mathfrak{S}_n$, $4 \le n \le \infty$, the character $\mathrm{sgn}\cdot\chi$ is called *associate character* of χ, and χ is called *self-associate* or *non-self-associate* according as $\mathrm{sgn}\cdot\chi = \chi$ or $\mathrm{sgn}\cdot\chi \neq \chi$. Correspondingly, a spin representation π of $\mathfrak{S}_\infty$ is called *self-associate* or *non-self-associate* according as $\pi \cong \mathrm{sgn}\cdot\pi$ or not. Here sgn is the sign character on $\widetilde{\mathfrak{S}}_n$ given as $\widetilde{\mathfrak{S}}_n \xrightarrow{\Phi_{\mathfrak{S}}} \mathfrak{S}_n \xrightarrow{\mathrm{sgn}} \{\pm 1\}$.

This notion will play important rolls later. See [HHo] and also cf. [Sch3] and the book [HoHu2]. We will see later that Schur's 'Hauptdarstellung' Δ'_n is self-associate or non-self-associate according as n is odd or even (cf. Theorems 15.2 and 15.3). The present spin representation ∇_n is self-associate, and irreducible if n is odd. When n is even, it is equivalent to the direct sum of non-self-associate spin IRs as predicted below. Actually $\nabla_n \cong \Delta'_n \oplus (\mathrm{sgn}\cdot\Delta'_n)$ for $n = 2n'$ (cf. Theorem 15.5).

• **Irreducible decomposition of** $\nabla_{2n'}$. When $n = 2n'$ is even, ∇_n splits into a direct sum of two inequivalent irreducible components as will be seen in Theorem 15.5 by means of characters. At this stage we can show the following. By calculation, for $4 \le n < \infty$,

$$
(8.15)\qquad \begin{cases} \nabla_n(r_j)Y'_j = -Y'_{j+1}\nabla_n(r_j), & \nabla_n(r_j)Y'_{j+1} = -Y'_j\nabla_n(r_j), \\ \nabla_n(r_j)Y'_k = -Y'_k\nabla_n(r_j) & (k \neq j, j+1), \end{cases}
$$

$$
\therefore \quad \nabla_n(r_i)\Theta'_n = -\Theta'_n\nabla_n(r_i) \quad (i \in \boldsymbol{I}_{n-1}), \quad \text{with}
$$

$$
\Theta'_n := \frac{1}{\sqrt{n}}\,(Y'_1 + Y'_2 + \cdots + Y'_n), \quad {\Theta'_n}^2 = E.
$$

Let $n = 2n'$ be even. Then $\nabla_{2n'}(r_i)Y'_{2n'+1} = -Y'_{2n'+1}\nabla_{2n'}(r_i)$ $(i \in \boldsymbol{I}_{2n'-1})$. Put

$$\Theta_n = \Theta_{2n'} := \Theta'_{2n'} \cdot (iY_{2n'+1}), \quad i = \sqrt{-1}, \tag{8.16}$$

then it commutes with $\nabla_{2n'}(\sigma')$, $\sigma' \in \widetilde{\mathfrak{S}}_{2n'}$, and $\Theta_n^2 = E$. Therefore

$$Q_\pm := \frac{1}{2}(E \pm \Theta_{2n'}), \quad Q_+Q_- = Q_-Q_+ = O,\ Q_+ + Q_- = E,$$

gives a decomposition of the space $V = \boldsymbol{C}^{2^{n'}}$ of $\nabla_{2n'}$ into a direct sum of two invariant subspaces $V_\pm := Q_\pm V$. However detailed structure of them are not clear at this stage.

8.3 Extensions $\nabla^{\mathrm{II}+}_{2n'+1}$ and $\nabla^{\mathrm{II}-}_{2n'+1}$ for $\nabla^{\mathrm{II}}_{2n'}$ of $\widetilde{\mathfrak{S}}_{2n'}$

Let $n = 2n' + 1$ odd. The situations for $\nabla^{\mathrm{II}+}_n(r_i)$'s and $\nabla^{\mathrm{II}-}_n(r_i)$'s are rather complicated. First note that each of the matrices $\nabla^{\mathrm{II}\pm}_n(r_i)$ $(1 \le i \le n-1 = 2n')$ are determined as an intertwining operator for the corresponding equivalence relation in (10.1) in Lemma 10.1 (iii) below, or in another notation,

$$\begin{aligned} \nabla^{\mathrm{II}+}_n(r_i)P^+_\gamma(d')\nabla^{\mathrm{II}+}_n(r_i)^{-1} &= P^-_{s_i\gamma}\big(s^{\mathrm{II}}_i(d')\big) \\ \nabla^{\mathrm{II}-}_n(r_i)P^-_\gamma(d')\nabla^{\mathrm{II}-}_n(r_i)^{-1} &= P^+_{s_i\gamma}\big(s^{\mathrm{II}}_i(d')\big) \end{aligned} \quad (\gamma \in \Gamma^0_n\,,\ d' \in \widetilde{D}_n). \tag{8.17}$$

Then each of $\nabla^{\mathrm{II}\pm}_n(r_i)$ is uniquely determined up to a multiplicative constant as a solution of the corresponding equation above.

As is proved, $\nabla^{\mathrm{II}+}_n(r_i) = \nabla^{\mathrm{II}-}_n(r_i) = \nabla^{\mathrm{II}}_{n-1}(r_i)$ $(1 \le i \le n-2 = 2n'-1)$ give a spin representation $\nabla^{\mathrm{II}}_{n-1}$ of $\widetilde{\mathfrak{S}}_{n-1}$. However, if we add $\nabla^{\mathrm{II}+}_n(r_{n-1})$ or $\nabla^{\mathrm{II}-}_n(r_{n-1})$ together with them, we can have no representation of $\widetilde{\mathfrak{S}}_n$, $n = 2n'+1$, at all.

In spite of this, we would like to extend $\nabla^{\mathrm{II}+}_n(r_i)$'s and $\nabla^{\mathrm{II}-}_n(r_i)$'s to reasonable matrix-valued functions on the whole $\widetilde{\mathfrak{S}}_{2n'+1}$ which are two different continuations of IR $\nabla^{\mathrm{II}}_{n-1}$ of $\widetilde{\mathfrak{S}}_{n-1}$. This is to clarify the complicated situation at present and also to use later to express intertwining relations among $P^\pm_\gamma$ $(\gamma \in \Gamma^0_n)$ (see §10).

To give an extension $\nabla^{\mathrm{II}+}_n$ of $\nabla^{\mathrm{II}}_{n-1}$ on $\widetilde{\mathfrak{S}}_{n-1}$, note the following. If $\sigma' \in \widetilde{\mathfrak{S}}_n$ does not belong to $\widetilde{\mathfrak{S}}_{n-1}$, then σ' can be expressed as $\sigma' = \sigma'_2 r_{n-1}\sigma'_1$ with $\sigma'_k \in \widetilde{\mathfrak{S}}_{n-1}$ $(k = 1, 2)$, $\mathrm{sgn}(\sigma'_1) = 1$. Putting $\nabla^{\mathrm{II}\pm}_n(\sigma'') = \nabla^{\mathrm{II}}_n(\sigma'')$ for $\sigma'' \in \widetilde{\mathfrak{S}}_{n-1} \subset \widetilde{\mathfrak{S}}_n$, we define

$$\nabla^{\mathrm{II}+}_n(\sigma') := \nabla^{\mathrm{II}+}_n(\sigma'_2)\nabla^{\mathrm{II}+}_n(r_{n-1})\nabla^{\mathrm{II}+}_n(\sigma'_1). \tag{8.18}$$

Then, since $\nabla^{\mathrm{II}+}_n(\sigma'_1) = \nabla'_n(\sigma'_1)$, $\nabla^{\mathrm{II}+}_n(r_{n-1}) = (iY_{2n'+1})\,\nabla'_n(r_{n-1})$, we have

$$\begin{cases} \nabla^{\mathrm{II}+}_n(\sigma') = \nabla'_n(\sigma') & \text{if } \mathrm{sgn}(\sigma') = 1 \text{ or } \sigma' \in \widetilde{\mathfrak{A}}_n\,, \\ \nabla^{\mathrm{II}+}_n(\sigma') = (iY_{2n'+1})\,\nabla'_n(\sigma') & \text{if } \mathrm{sgn}(\sigma') = -1 \text{ or } \sigma' \notin \widetilde{\mathfrak{A}}_n\,. \end{cases} \tag{8.19}$$

For another extension $\nabla_n^{\mathrm{II}-}$, we take the same expression $\sigma' = \sigma'_2 r_{2n'} \sigma'_1$ as above for $\sigma' \in \widetilde{\mathfrak{S}}_n \setminus \widetilde{\mathfrak{S}}_{n-1}$, and put

$$\nabla_n^{\mathrm{II}-}(\sigma') := \nabla_n^{\mathrm{II}-}(\sigma'_2)\nabla_n^{\mathrm{II}-}(r_{n-1})\nabla_n^{\mathrm{II}-}(\sigma'_1). \tag{8.20}$$

Then, since $\nabla_n^{\mathrm{II}-}(\sigma'_1) = \nabla'_n(\sigma'_1)$, $\nabla_n^{\mathrm{II}-}(r_{n-1}) = \nabla'_n(r_{n-1})\,(-iY_{2n'+1})$, we have

$$\begin{cases} \nabla_n^{\mathrm{II}-}(\sigma') = Y_{2n'+1}\nabla'_n(\sigma')\,Y_{2n'+1} = \nabla''_n(\sigma') & \text{if } \mathrm{sgn}(\sigma') = 1\,, \\ \nabla_n^{\mathrm{II}-}(\sigma') = \nabla'_n(\sigma')\,(-iY_{2n'+1}) & \text{if } \mathrm{sgn}(\sigma') = -1\,. \end{cases} \tag{8.21}$$

8.4 Spin representations of $\mathfrak{A}_n$ by intertwining operators

Let $n = 2n'+1$ odd. From the spin representations ∇'_n and ∇''_n of $\widetilde{\mathfrak{S}}_n$, we get those of the double covering group $\widetilde{\mathfrak{A}}_n$ by restrictions. Note that, as remarked in §2.4, the representation group $\mathfrak{B}_n = R(\mathfrak{A}_n)$ of $\mathfrak{A}_n$ is special for $n = 6, 7$, since it is 6-fold covering whereas it is double covering for $n \geq 4, \neq 6, 7$. In this paper, we treat only this double covering $\widetilde{\mathfrak{A}}_n$, generated by $v'_i = z_1 r_{i+1} r_1 \in \widetilde{\mathfrak{S}}_n$ $(1 \leq i \leq n-2)$ (cf. Theorem 2.7). We put for $v' \in \widetilde{\mathfrak{A}}_n$

$$\begin{aligned} \mho_n^+(v') &:= \nabla'_n(v'), \\ \mho_n^-(v') &:= \nabla''_n(v'), \end{aligned} \tag{8.22}$$

and in particular $\mho_n^{\pm}(z_1) = -E$. Then, $\mho_n^+(v') = \mho_n^-(v')$ for $v' \in \widetilde{\mathfrak{A}}_{n-1}$, and we get from the results in the preceding subsection the following: for $v' \in \widetilde{\mathfrak{A}}_n$,

$$\begin{aligned} \mho_n^+(v') &= \nabla_n^{\mathrm{II}+}(v'), \\ \mho_n^-(v') &= \nabla_n^{\mathrm{II}-}(v') = Y_{2n'+1}\mho_n^+(v')\,Y_{2n'+1}. \end{aligned} \tag{8.23}$$

Theorem 8.4. *Let $n \geq 4$. The maps $v'_i \mapsto \mho_n^+(v'_i)$ and $v'_i \mapsto \mho_n^-(v'_i)$, $1 \leq i \leq n-2$, give respectively spin representations of the double covering group $\widetilde{\mathfrak{A}}_n$, which is the representation group $\mathfrak{B}_n = R(\mathfrak{A}_n)$ if $n \neq 6, 7$.*

Note 8.1. It will be seen in Theorem 15.5 (ii) that $\mho_n^+ \cong \mho_n^-$ is a direct sum of two inequivalent spin IRs of $\widetilde{\mathfrak{A}}_n$.

8.5 Conjugations by $Y_{2n'+1}, \nabla_n(r_i), \nabla_n^-(r_i), \nabla'_n(r_i)$ and $\nabla_n^{\mathrm{II}\pm}(r_i)$

Denote by $\iota(A)$ the conjugation $B \mapsto ABA^{-1}$ on $GL(2^{n'}, \boldsymbol{C})$.

Lemma 8.5. (i) *The conjugation $\iota(Y_{2n'+1})$ yields the transformation*

$$Y_j \mapsto -Y_j \ (1 \leq j \leq 2n'), \quad Y_{2n'+1} \mapsto Y_{2n'+1}.$$

(ii) *For $1 \leq i \leq n-1$, the conjugation $\iota\big(\nabla_n(r_i)\big)$ yields the transformation:*

$$Y'_j \mapsto -Y'_{s_i(j)} \ \ (j \in \boldsymbol{I}_n) \quad \text{with} \ \ Y'_j = (-1)^{j-1} Y_j \ (j \in \boldsymbol{I}_n)$$

(iii) *In case $n=2n'+1$ odd, the conjugation $\iota\big(\nabla_n^-(r_{n-1})\big)$ yields the transformation:*

$$Y_j' \mapsto -Y_j' \ \ (j \in \boldsymbol{I}_{n-2}), \quad Y_{n-1}' \mapsto Y_n', \ \ Y_n' \mapsto Y_{n-1}'.$$

(iv) *For $1 \le i \le n-1$, the conjugation $\iota\big(\nabla'_n(r_i)\big)$ yields the transformation:*

$$Y_j \mapsto -Y_{s_i(j)} \quad (j \in \boldsymbol{I}_n).$$

(v) *For $1 \le i \le 2n'-1$, the conjugation $\iota\big(\nabla_n^{\mathrm{II}+}(r_i)\big)$ yields the transformation:*

$$Y_j \mapsto Y_{s_i(j)} \ \ (j \in \boldsymbol{I}_{2n'}), \quad Y_{2n'+1} \mapsto -Y_{2n'+1}\,.$$

(vi) *The conjugation $\iota\big(\nabla_n^{\mathrm{II}+}(r_{2n'})\big)$ $\big($resp. $\iota\big(\nabla_n^{\mathrm{II}-}(r_{2n'})\big)\big)$ yields the transformation:*

$$\begin{cases} Y_j \mapsto Y_j \quad (j \le 2n'-1), \\ Y_{2n'} \mapsto -Y_{2n'+1}, \\ Y_{2n'+1} \mapsto Y_{2n'}. \end{cases} \qquad \text{resp.} \qquad \begin{cases} Y_j \mapsto Y_j \quad (j \le 2n'-1), \\ Y_{2n'} \mapsto Y_{2n'+1}, \\ Y_{2n'+1} \mapsto -Y_{2n'}. \end{cases}$$

Proof. These are proved by calculations. □

The conjugation $\iota(Y_{2n'+1})$ yields the transformation of IRs P_γ of $\widetilde{D}_n$, through Lemma 8.5 (i), as follows. We will discuss later for other types of conjugations.

Lemma 8.6. *The conjugation $\iota(Y_{2n'+1})$ yields the transformation*

$$\iota(Y_{2n'+1})P_\gamma(d') = P_{(\tau_1\tau_2\cdots\tau_{2n'})\gamma}(d') \quad (d' \in \widetilde{D}_n),$$

or $\iota(Y_{2n'+1})P_\gamma = P_{(\tau_1\tau_2\cdots\tau_{2n'})\gamma}$. *Moreover* $(\tau_1\cdots\tau_{2n'})\gamma \approx_n \gamma$.

Proof. For $1 \le j \le 2n'$ and $j=2n'+1$ respectively,

$$Y_{2n'+1}P_\gamma(\eta_j)Y_{2n'+1}^{-1} = Y_{2n'+1}\omega^{\gamma_j}Y_jY_{2n'+1}^{-1} = -\omega^{\gamma_j}\,Y_j = \omega^{\gamma_j+m'}\,Y_j,$$
$$Y_{2n'+1}P_\gamma(\eta_{2n'+1})Y_{2n'+1}^{-1} = Y_{2n'+1}\omega^{\gamma_{2n'+1}}Y_{2n'+1}Y_{2n'+1}^{-1} = \omega^{\gamma_{2n'+1}}\,Y_{2n'+1}.$$ □

9 Intertwining relations among P_γ's under $\widetilde{\mathfrak{S}}_n$ in CASE I

Lemma 9.1. *In CASE I, Type $(-1,-1,-1)$, the following intertwining relations hold.*

(i) *For $1 \le i \le n-1$, the conjugation $\iota\big(\nabla_n(r_i)\big)$ yields for $\gamma \in \Gamma_n$,*

$$\iota\big(\nabla_n(r_i)\big)P_\gamma(d') = P_{s_i\gamma}\big(r_i^{\mathrm{I}}(d')\big) \quad (d' \in \widetilde{D}_n).$$

(ii) *For $n = 2n'+1$ odd, $\gamma \in \Gamma_n^0$ in (6.5), and for $1 \le i \le n-1$,*

$$\iota\big(\nabla_n^-(r_i)\big)P_\gamma^-(d') = P_{s_i\gamma}^-\big(r_i^{\mathrm{I}}(d')\big) \quad (d' \in \widetilde{D}_n).$$

(iii) *For $1 \le i \le n-1$, the conjugation $\iota\big(\nabla'_n(r_i)\big)$ yields*

$$\iota\big(\nabla'_n(r_i)\big)P_\gamma(d') = P_{(\tau_i\tau_{i+1})s_i\gamma}\big(r_i^{\mathrm{I}}(d')\big) \quad (d' \in \widetilde{D}_n),$$

and $(\tau_i\tau_{i+1})s_i\gamma = s_i(\tau_i\tau_{i+1})\gamma \approx_n s_i\gamma$.

(iv) *For $1 \le i \le 2n'-1$, the conjugation $\iota\big(\nabla_n^{\mathrm{II}+}(r_i)\big)$ yields*

$$\iota\big(\nabla_n^{\mathrm{II}+}(r_i)\big)P_\gamma(d') = P_{s_i\gamma'}\big(r_i^{\mathrm{I}}(d')\big) \quad (d' \in \widetilde{D}_n),$$

where $\gamma' = \big(\prod_{j\ne i,i+1,2n'+1}\tau_j\big)\gamma$, and $\gamma' \approx_n \gamma$.

(v) *In case $n = 2n'+1$ odd, the conjugation $\iota\big(\nabla_n^{\mathrm{II}+}(r_{2n'})\big)$ yields*

$$\iota\big(\nabla_n^{\mathrm{II}+}(r_{2n'})\big)P_\gamma(d') = P_{s_{2n'}(\tau_1\tau_2\cdots\tau_{2n'})\gamma}\big(r_{2n'}^{\mathrm{I}}(d')\big) \quad (d' \in \widetilde{D}_n),$$

and $(\tau_1\tau_2\cdots\tau_{2n'})\gamma \approx_n \gamma$.

Proof. **(i)** For $i \in \boldsymbol{I}_{n-1}$,

$$\iota\big(\nabla_n(r_i)\big)P_\gamma(\eta'_i) = \iota\big(\nabla_n(r_i)\big)\big(\omega^{\gamma_i+(i-1)m'}Y_i\big) = \omega^{\gamma_i+(i-1)m'}Y_{i+1}$$
$$= \omega^{\gamma_i+(i-1)m'}Y_{i+1} = P_{s_i\gamma}\big(r_i^{\mathrm{I}}(\eta'_i)\big),$$
$$\iota\big(\nabla_n(r_i)\big)P_\gamma(\eta'_{i+1}) = \iota\big(\nabla_n(r_i)\big)\big(\omega^{\gamma_{i+1}+im'}Y_{i+1}\big) = \omega^{\gamma_{i+1}+im'}Y_i = P_{s_i\gamma}\big(r_i^{\mathrm{I}}(\eta'_{i+1})\big);$$

and for $j \ne i, i+1$,

$$\iota\big(\nabla_n(r_i)\big)P_\gamma(\eta'_j) = \iota\big(\nabla_n(r_i)\big)\big(\omega^{\gamma_j+(j-1)m'}Y_j\big) = -\omega^{\gamma_j+(j-1)m'}Y_j$$
$$= \omega^{\gamma_j+jm'}Y_j = P_{s_i\gamma}\big(r_i^{\mathrm{I}}(\eta'_j)\big).$$

(ii) For $n = 2n'+1$ odd,

$$\iota\big(\nabla_n^-(r_{n-1})\big)P_\gamma^-(\eta'_{n-1}) = \iota\big(\nabla_n^-(r_{n-1})\big)\big(\chi_{\tau_n\gamma}(\eta'_{n-1})Y_{n-1}\big) = (-1)^{n-2}\,\omega^{\gamma_{n-1}}\,(-Y_n)$$
$$= (-1)^{n-1}\,\omega^{\gamma_{n-1}}Y_n = P_{s_{n-1}\gamma}^-\big(r_{n-1}^{\mathrm{I}}(\eta'_{n-1})\big),$$
$$\iota\big(\nabla_n^-(r_{n-1})\big)P_\gamma^-(\eta'_n) = \iota\big(\nabla_n^-(r_{n-1})\big)\big(\chi_{\tau_n\gamma}(\eta'_n)Y_n\big) = (-1)^{n-1}\,\omega^{\gamma_n+m'}\,(-Y_{n-1})$$
$$= (-1)^{n-1}\,\omega^{\gamma_n}Y_{n-1} = P_{s_{n-1}\gamma}^-\big(r_{n-1}^{\mathrm{I}}(\eta'_n)\big).$$

(iii) For $i \in \boldsymbol{I}_{n-1}$, let $j \ne i, i+1$,

$$\iota\big(\nabla'_n(r_i)\big)P_\gamma(\eta'_i) = \iota\big(\nabla'_n(r_i)\big)\big(\omega^{\gamma_i+(i-1)m'}Y_i\big) = -\omega^{\gamma_i+(i-1)m'}Y_{i+1}$$
$$= \omega^{\gamma_i+im'}Y_{i+1} = P_{(\tau_i\tau_{i+1})s_i\gamma}\big(r_i^{\mathrm{I}}(\eta'_i)\big),$$
$$\iota\big(\nabla'_n(r_i)\big)P_\gamma(\eta'_{i+1}) = \iota\big(\nabla'_n(r_i)\big)\big(\omega^{\gamma_{i+1}+im'}Y_{i+1}\big) = -\omega^{\gamma_{i+1}+im'}Y_i$$
$$= \omega^{\gamma_{i+1}+(i+1)m'}Y_i = P_{(\tau_i\tau_{i+1})s_i\gamma}\big(r_i^{\mathrm{I}}(\eta'_{i+1})\big);$$
$$\iota\big(\nabla'_n(r_i)\big)P_\gamma(\eta'_j) = \iota\big(\nabla'_n(r_i)\big)\big(\omega^{\gamma_j+(j-1)m'}Y_j\big) = -\omega^{\gamma_j+(j-1)m'}Y_j$$

$= \omega^{\gamma_j + jm'} Y_j = P_{(\tau_i\tau_{i+1})s_i\gamma}\big(r_i^{\mathrm{I}}(\eta'_j)\big).$

(iv) For $i \in \boldsymbol{I}_{2n'-1}$, let $\gamma' = \big(\prod_{j\neq i,i+1,2n'+1} \tau_j\big)\gamma$,

$$\iota\big(\nabla_n^{\mathrm{II}+}(r_i)\big)P_\gamma(\eta'_i) = \iota\big(\nabla_n^{\mathrm{II}+}(r_i)\big)\big(\omega^{\gamma_i+(i-1)m'}Y_i\big) = \omega^{\gamma_i+(i-1)m'}Y_{i+1} = P_{s_i\gamma'}\big(r_i^{\mathrm{I}}(\eta'_i)\big),$$
$$\iota\big(\nabla_n^{\mathrm{II}+}(r_i)\big)P_\gamma(\eta'_{i+1}) = \iota\big(\nabla_n^{\mathrm{II}+}(r_i)\big)\big(\omega^{\gamma_{i+1}+im'}Y_{i+1}\big) = \omega^{\gamma_{i+1}+im'}Y_i = P_{s_i\gamma'}\big(r_i^{\mathrm{I}}(\eta'_{i+1})\big);$$

and for $j \in \boldsymbol{I}_{2n'},\ \neq i, i+1$,

$$\iota\big(\nabla_n^{\mathrm{II}+}(r_i)\big)P_\gamma(\eta'_j) = \iota\big(\nabla_n^{\mathrm{II}+}(r_i)\big)\big(\omega^{\gamma_j+(j-1)m'}Y_j\big) = \omega^{\gamma_j+(j-1)m'}Y_j = P_{s_i\gamma'}\big(r_i^{\mathrm{I}}(\eta'_j)\big);$$
$$\iota\big(\nabla_n^{\mathrm{II}+}(r_i)\big)P_\gamma(\eta'_{2n'+1}) = \iota\big(\nabla_n^{\mathrm{II}+}(r_i)\big)\big(\omega^{\gamma_{2n'+1}+2n'm'}Y_{2n'+1}\big)$$
$$= -\omega^{\gamma_{2n'+1}+2n'm'}Y_{2n'+1} = P_{s_i\gamma'}\big(r_i^{\mathrm{I}}(\eta'_{2n'+1})\big).$$

(v) For $i = 2n'$ $(n = 2n'+1)$, and $j \neq 2n', 2n'+1$,

$$\iota\big(\nabla_n^{\mathrm{II}+}(r_{2n'})\big)P_\gamma(\eta'_j) = \iota\big(\nabla_n^{\mathrm{II}+}(r_{2n'})\big)\big(\omega^{\gamma_j+(j-1)m'}Y_j\big) = \omega^{\gamma_j+(j-1)m'}Y_j$$
$$= P_{s_{2n'}\gamma''}\big(r_{2n'}^{\mathrm{I}}(\eta'_j)\big);$$

and for $j = 2n', 2n'+1$,

$$\iota\big(\nabla_n^{\mathrm{II}+}(r_{2n'})\big)P_\gamma(\eta'_{2n'}) = \iota\big(\nabla_n^{\mathrm{II}+}(r_{2n'})\big)\big(\omega^{\gamma_{2n'}+(2n'-1)m'}Y_{2n'}\big) =$$
$$= -\omega^{\gamma_{2n'}+(2n'-1)m'}\,Y_{2n'+1} = P_{s_{2n'}\gamma''}\big(r_{2n'}^{\mathrm{I}}(\eta'_{2n'})\big);$$
$$\iota\big(\nabla_n^{\mathrm{II}+}(r_{2n'})\big)P_\gamma(\eta'_{2n'+1}) = \iota\big(\nabla_n^{\mathrm{II}+}(r_{2n'})\big)\big(\omega^{\gamma_{2n'+1}+2n'm'}Y_{2n'+1}\big)$$
$$= \omega^{\gamma_{2n'+1}+2n'm'}Y_{2n'} = P_{s_{2n'}\gamma''}\big(r_{2n'}^{\mathrm{I}}(\eta'_{2n'+1})\big). \qquad \square$$

From the assertions (i) and (ii), we get the following intertwining relations.

Theorem 9.2. *Suppose we are in CASE I, Type* $(-1,-1,-1)$.

(i) *For* $\sigma' \in \widetilde{\mathfrak{S}}_n$, *let* $\sigma = \Phi(\sigma')$, *then for* $\gamma \in \Gamma_n$,

$$\iota\big(\nabla_n(\sigma')\big)P_\gamma(d') = P_{\sigma\gamma}\big({\sigma'}^{\mathrm{I}}(d')\big) = P_{\sigma\gamma}\big(\sigma^{\mathrm{I}}(d')\big) \qquad (d' \in \widetilde{D}_n),$$

or in another notation, ${\sigma'}^{\mathrm{I}}P_\gamma = \iota\big(\nabla_n(\sigma')\big)^{-1}\big(P_{\sigma\gamma}\big)$.

(ii) *In the case where* $n = 2n'+1$ *odd, there holds for* $\gamma \in \Gamma_n^0$,

$$\iota\big(\nabla_n^-(\sigma')\big)P_\gamma^-(d') = P_{\sigma\gamma}^-\big({\sigma'}^{\mathrm{I}}(d')\big) = P_{\sigma\gamma}^-\big(\sigma^{\mathrm{I}}(d')\big) \quad (d' \in \widetilde{D}_n),$$

or in another notation, ${\sigma'}^{\mathrm{I}}P_\gamma^- = \iota\big(\nabla_n^-(\sigma')\big)^{-1}\big(P_{\sigma\gamma}^-\big)$.

10 Intertwining relations among P_γ's under $\widetilde{\mathfrak{S}}_n$ in CASE II

Recall that $r_i^{\mathrm{II}}(d') = s_i^{\mathrm{II}}(d')$ and ${\sigma'}^{\mathrm{II}}(d') = \sigma^{\mathrm{II}}(d')$ with $\sigma = \Phi(\sigma') \in \mathfrak{S}_n$ for $\sigma' \in \widetilde{\mathfrak{S}}_n$, from (7.6). This action of $\widetilde{\mathfrak{S}}_n$ in CASE II on $\widetilde{D}(m,n)$ is the non-twisted natural one through $\widetilde{\mathfrak{S}}_n \to \mathfrak{S}_n$, compared with the twisted one in CASE I.

Lemma 10.1. *In CASE II, Type $(-1,-1,\ 1)$, the following intertwining relations hold.*

(i) *For $i \in \boldsymbol{I}_{n-1}$, the conjugation $\iota\big(\nabla'_n(r_i)\big)$ yields, for $d' \in \widetilde{D}_n$,*

$$\iota\big(\nabla'_n(r_i)\big)P_\gamma(d') = P_{(\tau_1\tau_2\cdots\tau_n)s_i\gamma}\big(r_i^{\mathrm{II}}(d')\big) = P_{(\tau_1\tau_2\cdots\tau_n)s_i\gamma}\big(s_i^{\mathrm{II}}(d')\big),$$

and $(\tau_1\tau_2\cdots\tau_n)s_i\gamma = s_i(\tau_1\tau_2\cdots\tau_n)\gamma$. For $\gamma' := (\tau_1\tau_2\cdots\tau_n)\gamma$, we have $\gamma' \approx_n \gamma$ in case $n = 2n'$, and $\gamma' \approx_n \tau_n\gamma$ in case $n = 2n'+1$.

(ii) *In case $n = 2n'$ even, the conjugation $\iota\big(\nabla^{\mathrm{II}}_n(r_i)\big)$ yields, for $i \in \boldsymbol{I}_{n-1}$ and $\gamma \in \Gamma_n$,*

$$\iota\big(\nabla^{\mathrm{II}}_n(r_i)\big)P_\gamma(d') = P_{s_i\gamma}\big(s_i^{\mathrm{II}}(d')\big) \quad (d' \in \widetilde{D}_n).$$

(iii) *Let $n = 2n'+1$ be odd. For $i \in \boldsymbol{I}_{n-1}$, $\gamma \in \Gamma_n$ and $d' \in \widetilde{D}_n$,*

$$\iota\big(\nabla^{\mathrm{II}+}_n(r_i)\big)P_\gamma(d') = P_{\tau_n s_i\gamma}\big(s_i^{\mathrm{II}}(d')\big),$$
$$\iota\big(\nabla^{\mathrm{II}-}_n(r_i)\big)P_\gamma(d') = P_{s_i\tau_n\gamma}\big(s_i^{\mathrm{II}}(d')\big).$$

In other notations, for $\gamma \in \Gamma^0_n$, $d' \in \widetilde{D}_n$, and $1 \le i \le 2n' = n-1$,

$$\begin{aligned}\iota\big(\nabla^{\mathrm{II}+}_n(r_i)\big)P^+_\gamma(d') &= P^-_{s_i\gamma}\big(s_i^{\mathrm{II}}(d')\big),\\ \iota\big(\nabla^{\mathrm{II}-}_n(r_i)\big)P^-_\gamma(d') &= P^+_{s_i\gamma}\big(s_i^{\mathrm{II}}(d')\big).\end{aligned} \tag{10.1}$$

Proof. **(i)** For $i \in \boldsymbol{I}_{n-1}$ let $j \ne i, i+1$. Then, with $\gamma' = (\tau_1\tau_2\cdots\tau_n)\gamma$,

$$\iota\big(\nabla'_n(r_i)\big)P_\gamma(\eta_i) = \iota\big(\nabla'_n(r_i)\big)\big(\omega^{\gamma_i}Y_i\big) = -\omega^{\gamma_i}Y_{i+1} = \omega^{\gamma_i+m'}Y_{i+1} = P_{s_i\gamma'}\big(s_i^{\mathrm{II}}(\eta_i)\big),$$
$$\begin{aligned}\iota\big(\nabla'_n(r_i)\big)P_\gamma(\eta_{i+1}) = \iota\big(\nabla'_n(r_i)\big)\big(\omega^{\gamma_{i+1}}Y_{i+1}\big) &= -\omega^{\gamma_{i+1}}Y_i = \omega^{\gamma_{i+1}+m'}Y_i\\ &= P_{s_i\gamma'}\big(s_i^{\mathrm{II}}(\eta_{i+1})\big),\end{aligned}$$
$$\iota\big(\nabla'_n(r_i)\big)P_\gamma(\eta_j) = \iota\big(\nabla'_n(r_i)\big)\big(\omega^{\gamma_j}Y_j\big) = -\omega^{\gamma_j}Y_j = \omega^{\gamma_j+m'}Y_j = P_{s_i\gamma'}\big(s_i^{\mathrm{II}}(\eta_j)\big).$$

(ii) For $i \in \boldsymbol{I}_{2n'-1}$,

$$\iota\big(\nabla^{\mathrm{II}}_n(r_i)\big)P_\gamma(\eta_i) = \iota\big(\nabla^{\mathrm{II}}_n(r_i)\big)\big(\omega^{\gamma_i}Y_i\big) = \omega^{\gamma_i}Y_{i+1} = P_{s_i\gamma}\big(s_i^{\mathrm{II}}(\eta_i)\big),$$
$$\iota\big(\nabla^{\mathrm{II}}_n(r_i)\big)P_\gamma(\eta_{i+1}) = \iota\big(\nabla^{\mathrm{II}}_n(r_i)\big)\big(\omega^{\gamma_{i+1}}Y_{i+1}\big) = \omega^{\gamma_{i+1}}Y_i = P_{s_i\gamma}\big(s_i^{\mathrm{II}}(\eta_{i+1})\big),$$
$$\iota\big(\nabla^{\mathrm{II}}_n(r_i)\big)P_\gamma(\eta_j) = \iota\big(\nabla^{\mathrm{II}}_n(r_i)\big)\big(\omega^{\gamma_j}Y_j\big) = \omega^{\gamma_j}Y_j = P_{s_i\gamma}\big(s_i^{\mathrm{II}}(\eta_j)\big).$$

(iii) Note that $\iota\big(\nabla^{\mathrm{II}+}_n(r_i)\big) = \iota\big(iY_{2n'+1}\big)\iota\big(\nabla'_n(r_i)\big)$, then by (i) above and Lemma 8.6,

$$\begin{aligned}\iota\big(\nabla^{\mathrm{II}+}_n(r_i)\big)P_\gamma(d') &= \iota\big(iY_{2n'+1}\big)\iota\big(\nabla'_n(r_i)\big)P_\gamma(d') = \iota\big(iY_{2n'+1}\big)P_{(\tau_1\tau_2\cdots\tau_n)s_i\gamma}\big(s_i^{\mathrm{II}}(d')\big)\\ &= P_{(\tau_1\cdots\tau_{2n'})(\tau_1\tau_2\cdots\tau_n)s_i\gamma}\big(s_i^{\mathrm{II}}(d')\big) = P_{\tau_n s_i\gamma}\big(s_i^{\mathrm{II}}(d')\big).\end{aligned}$$

Similarly for $\nabla^{\mathrm{II}-}_n(r_i)$. □

From Lemma 10.1, we have the following.

Theorem 10.2. *In CASE II, Type* $(-1,-1,\ 1)$:
(i) *For* $\sigma' \in \widetilde{\mathfrak{S}}_n$, *put* $\sigma = \Phi(\sigma')$, *and let* $\gamma \in \Gamma_n$ *and* $d' \in \widetilde{D}_n$.
If $\mathrm{sgn}(\sigma) = 1$, *then* $\nabla'_n(\sigma') = \mho^+_n(\sigma')$ *and*

$$\iota\big(\nabla'_n(\sigma')\big)P_\gamma(d') = \iota\big(\mho^+_n(\sigma')\big)P_\gamma(d') = P_{\sigma\gamma}\big(\sigma^{\mathrm{II}}(d')\big).$$

If $\mathrm{sgn}(\sigma) = -1$, *then*

$$\iota\big(\nabla'_n(\sigma')\big)P_\gamma(d') = P_{(\tau_1\tau_2\cdots\tau_n)\sigma\gamma}\big(\sigma^{\mathrm{II}}(d')\big).$$

In another notation, these are expressed respectively as

$$\sigma^{\mathrm{II}}P_\gamma = \iota\big(\nabla'_n(\sigma')\big)^{-1}\big(P_{\sigma\gamma}\big) = \iota\big(\mho^+_n(\sigma')\big)^{-1}\big(P_{\sigma\gamma}\big) \quad \text{if } \mathrm{sgn}(\sigma) = 1\ ;$$
$$\sigma^{\mathrm{II}}P_\gamma = \iota\big(\nabla'_n(\sigma')\big)^{-1}\big(P_{(\tau_1\tau_2\cdots\tau_n)\sigma\gamma}\big) \quad \text{if } \mathrm{sgn}(\sigma) = -1.$$

(ii) *Let* $n = 2n'$ *be even, then for* $\gamma \in \Gamma_n$ *and* $d' \in \widetilde{D}_n$,

$$\iota\big(\nabla^{\mathrm{II}}_n(\sigma')\big)P_\gamma(d') = P_{\sigma\gamma}\big(\sigma^{\mathrm{II}}(d')\big)\ ,$$
$$\text{or} \qquad \sigma^{\mathrm{II}}P_\gamma = \iota\big(\nabla^{\mathrm{II}}_n(\sigma')\big)^{-1}\big(P_{\sigma\gamma}\big).$$

(iii) *Let* $n = 2n'+1$ *be odd. The two extensions of* $\nabla^{\mathrm{II}}_{n-1}$ *on* $\widetilde{\mathfrak{S}}_{n-1}$ *as matrix-valued functions* $\nabla^{\mathrm{II}+}_n$ *and* $\nabla^{\mathrm{II}-}_n$ *are given respectively in (8.19) and (8.21).*
Suppose $\mathrm{sgn}(\sigma') = \mathrm{sgn}(\sigma) = 1$ *for* $\sigma' \in \widetilde{\mathfrak{S}}_n$, $\sigma = \Phi(\sigma')$. *Then*

$$\begin{aligned} \nabla^{\mathrm{II}+}_n(\sigma') &= \nabla'_n(\sigma') = \mho^+_n(\sigma'), \\ \nabla^{\mathrm{II}-}_n(\sigma') &= \nabla''_n(\sigma') = \mho^-_n(\sigma')\ ; \end{aligned} \tag{10.2}$$

and, for $\gamma \in \Gamma^0_n$ *and* $d' \in \widetilde{D}_n$,

$$\begin{aligned} \iota\big(\mho^+_n(\sigma')\big)P^+_\gamma(d') &= P^+_{\sigma\gamma}\big(\sigma^{\mathrm{II}}(d')\big), \\ \iota\big(\mho^-_n(\sigma')\big)P^-_\gamma(d') &= P^-_{\sigma\gamma}\big(\sigma^{\mathrm{II}}(d')\big), \end{aligned} \tag{10.3}$$

which are expressed in another notation as

$$\begin{aligned} \sigma^{\mathrm{II}}P^+_\gamma &= \iota\big(\mho^+_n(\sigma')\big)^{-1}\big(P^+_{\sigma\gamma}\big), \\ \sigma^{\mathrm{II}}P^-_\gamma &= \iota\big(\mho^-_n(\sigma')\big)^{-1}\big(P^-_{\sigma\gamma}\big). \end{aligned} \tag{10.4}$$

(iv) *Let* $n = 2n' + 1$ *be odd. Suppose* $\mathrm{sgn}(\sigma') = \mathrm{sgn}(\sigma) = -1$ *for* $\sigma' \in \widetilde{\mathfrak{S}}_n$. *Then*

$$\begin{aligned} \nabla^{\mathrm{II}+}_n(\sigma') &= (iY_{2n'+1}) \cdot \nabla'(\sigma'), \\ \nabla^{\mathrm{II}-}_n(\sigma') &= \nabla'(\sigma') \cdot (-iY_{2n'+1}) = (-iY_{2n'+1}) \cdot \nabla''(\sigma')\ ; \end{aligned} \tag{10.5}$$

and, for $\gamma \in \Gamma^0_n$ *and* $d' \in \widetilde{D}_n$,

$$\begin{aligned} \iota\big(\nabla^{\mathrm{II}+}_n(\sigma')\big)P^+_\gamma(d') &= P^-_{\sigma\gamma}\big(\sigma^{\mathrm{II}}(d')\big), \\ \iota\big(\nabla^{\mathrm{II}-}_n(\sigma')\big)P^-_\gamma(d') &= P^+_{\sigma\gamma}\big(\sigma^{\mathrm{II}}(d')\big), \end{aligned} \tag{10.6}$$

which are expressed in another notation as

$$\begin{aligned} \sigma^{\mathrm{II}}P^+_\gamma &= \iota\big(\nabla^{\mathrm{II}+}_n(\sigma')\big)^{-1}\big(P^-_{\sigma\gamma}\big), \\ \sigma^{\mathrm{II}}P^-_\gamma &= \iota\big(\nabla^{\mathrm{II}-}_n(\sigma')\big)^{-1}\big(P^+_{\sigma\gamma}\big). \end{aligned} \tag{10.7}$$

11 Classification of spin IRs of $G(m,1,n)$, CASE I, Type $(-1,-1,-1)$

We have already several classifications of spin IRs of generalized symmetric groups $G(m,1,n)$ such as [Rea2] and [HoHu4]. Here we give a classification by means of (say) a constructive method, which serves directly to explicit calculations of their characters.

11.1 Irreducible spin representations $\Pi^{\mathrm{I}}_{c,\Lambda}$ and $\Pi^{\mathrm{I}\pm}_{c,\Lambda}$ of $\widetilde{G}^{\mathrm{I}}_n$

Let m be even. Take a spin representation Π of

$$(11.1)\qquad \widetilde{G}^{\mathrm{I}}_n = \widetilde{G}^{\mathrm{I}}(m,1,n) = R\big(G(m,1,n)\big)/\langle z_2{z_3}^{-1}\rangle = \widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \widetilde{\mathfrak{S}}_n$$

such that $\Pi(z_1)=\Pi(z_2)=-I$ or of Type $(-1,-1,-1)$, where $\widetilde{D}_n=\widetilde{D}(m,n)$.

Theorem 11.1. *Suppose* $n\geq 4$ *and* m *is even. Let* Π *be a representation of the covering group* $\widetilde{G}^{\mathrm{I}}_n = R\big(G(m,1,n)\big)/\langle z_2{z_3}^{-1}\rangle$ *such that*

(i) $\qquad \Pi(z_1)=-I,\quad \Pi(z_2)=-I.$

Then it is actually a spin representation of $G(m,1,n)$*, of CASE I, Type* $(-1,-1,-1)$*. The operators* $\Pi(r_i)$*'s and* $\Pi(\eta'_j)$*'s satisfy the following:*

$$\text{(ii)}\qquad \begin{cases} \Pi(r_i)^2=I \ \ (i\in \boldsymbol{I}_{n-1}), \quad \big(\Pi(r_i)\Pi(r_{i+1})\big)^3 = I \ \ (i\in\boldsymbol{I}_{n-2}), \\ \Pi(r_i)\Pi(r_j) = -\Pi(r_j)\Pi(r_i) \quad (|i-j|\geq 2), \end{cases}$$

$$\text{(iii)}\qquad \Pi(\eta'_j)^m = I \qquad (j\in\boldsymbol{I}_n),$$

$$\text{(iv)}\qquad \Pi(\eta'_j)\Pi(\eta'_k) = -\Pi(\eta'_k)\Pi(\eta'_j) \qquad (j\neq k),$$

$$\text{(v)}\qquad \Pi(r_i)\Pi(\eta'_j)\Pi({r_i}^{-1}) = -\Pi\big(\eta'_{s_i(j)}\big) \qquad (j\in\boldsymbol{I}_n).$$

Now consider spin representations ∇_n and ∇^-_n of $\widetilde{\mathfrak{S}}_n$ in Theorem 8.3 and the IR P_γ of $\widetilde{D}_n$ in (6.6):

$$(11.2)\qquad \nabla_n(r_i) = \frac{(-1)^{j-1}}{\sqrt{2}}\,(Y_j+Y_{j+1}) \qquad (i\in\boldsymbol{I}_{n-1}) \qquad \text{of } \widetilde{\mathfrak{S}}_n\,;$$

$$(11.3)\qquad \begin{cases} \nabla^-_n(r_i) \ = \ \nabla_n(r_i) & (i\in\boldsymbol{I}_{n-2}) \\ \nabla^-_n(r_{n-1}) \ = \ \dfrac{(-1)^{n-2}}{\sqrt{2}}\,(Y_{n-1}-Y_n) & (i=n-1) \end{cases} \qquad \text{of } \widetilde{\mathfrak{S}}_n\,;$$

$$(11.4)\qquad P_\gamma(\eta'_j) = (-1)^{j-1}\omega^{\gamma_j}\rho(\eta_j) = \chi_\gamma(\eta'_j)\,Y_j = \zeta_\gamma(\eta'_j)\,Y'_j \ \ (j\in\boldsymbol{I}_n) \quad \text{of } \widetilde{D}_n\,.$$

11.2 Stationary subgroups of $[P_\gamma]$ and $[P_\gamma^{\pm}]$

Take the complete sets of representatives of the set of equivalence classes $\widehat{\widetilde{D}_n}^{\rm spin}$ of spin IRs of $\widetilde{D}_n$ in Theorem 6.5 as

$$\text{(11.5)} \qquad \text{Case } n = 2n' \text{ even:} \qquad \mathrm{Rep}(\widetilde{D}_n) = \big\{[P_\gamma]\ ;\ \gamma \in \Gamma_n^0\big\}\ ;$$

$$\text{(11.6)} \qquad \text{Case } n = 2n'+1 \text{ odd:} \qquad \mathrm{Rep}(\widetilde{D}_n) = \mathrm{Rep}^+(\widetilde{D}_n) \bigsqcup \mathrm{Rep}^-(\widetilde{D}_n),$$

$$\text{with} \qquad \begin{cases} \mathrm{Rep}^+(\widetilde{D}_n) = \big\{[P_\gamma^+] = [P_\gamma]\ \ ;\ \gamma \in \Gamma_n^0\big\}, \\ \mathrm{Rep}^-(\widetilde{D}_n) = \big\{[P_\gamma^-] = [P_{\tau_n\gamma}]\ ;\ \gamma \in \Gamma_n^0\big\}. \end{cases}$$

Here the subset Γ_n^0 of Γ_n is defined by the condition

$$\text{(11.7)} \qquad \gamma = (\gamma_1, \ldots, \gamma_n), \quad 0 \le \gamma_j < m' \ \ (j \in \boldsymbol{I}_n).$$

Lemma 11.2. *For representatives P_γ (resp. $P_\gamma^+ = P_\gamma$ and $P_\gamma^- = P_{\tau_n\gamma}$ if n is odd) with $\gamma \in \Gamma_n^0$, of equivalence classes of spin IRs of $\widetilde{D}_n$, there holds*

$$\text{(11.8)} \qquad \sigma^{\rm I}\big(P_\gamma\big) \cong P_{\sigma\gamma}\,, \quad \sigma^{\rm I}\big(P_\gamma^{\pm}\big) \cong P_{\sigma\gamma}^{\pm} \quad (\sigma \in \mathfrak{S}_n),$$

and the stationary subgroups in $\widetilde{\mathfrak{S}}_n$ of their equivalence classes are given as

$$\text{(11.9)} \qquad \begin{aligned} &\mathcal{S}(P_\gamma) := \mathcal{S}([P_\gamma]) = \big\{\sigma' \in \widetilde{\mathfrak{S}}_n\ ;\ \sigma'\gamma = \gamma\big\}\ ; \\ &\mathcal{S}(P_\gamma^{\pm}) := \mathcal{S}([P_\gamma^{\pm}]) = \big\{\sigma' \in \widetilde{\mathfrak{S}}_n\ ;\ \sigma'\gamma = \gamma\big\} \quad \textit{in case } n \textit{ is odd.} \end{aligned}$$

Proof. The first relation (11.8) follows from Theorem 9.2.

For the second relations, we know from Theorem 6.4 that $P_{\gamma'} \cong P_\gamma$ if and only if $\gamma' \approx_n \gamma$, since γ is taken from Γ_n^0. Then (11.9) for $P_\gamma^+ = P_\gamma$ follows from Theorem 9.2. For $P_\gamma^- = P_{\tau_n\gamma}$, letting $\sigma^{-1}(k) = n$ for $\sigma' \in \widetilde{\mathfrak{S}}_n$, $\sigma = \Phi(\sigma')$, we have

$$\begin{aligned} &\sigma^{\rm I}\big(P_\gamma^-\big) = P_{\sigma(\tau_n\gamma)} = P_{\tau_k(\sigma\gamma)} \cong P_{\tau_n(\sigma\gamma)} = P_{(\sigma\gamma)}^-, \\ \because\ \ &\sigma(\tau_n\gamma) = \big(\gamma_{\sigma^{-1}(1)}, \ldots, \gamma_{\sigma^{-1}(k)} + m', \gamma_{\sigma^{-1}(k+1)}, \ldots, \gamma_{\sigma^{-1}(n)}\big) \\ &\qquad\quad = \tau_k(\sigma\gamma) \approx_n \tau_n(\sigma\gamma). \end{aligned}$$

□

In CASE I (contrary to CASE II), it is not so much necessary to use the notation P_γ^+ in case $n = 2n'+1$. However we prefer hereafter to use it principally in place of P_γ in accordance with CASE II, and we denote ∇_n also by ∇_n^+. We put for the subgroups $\widetilde{D}_n \overset{\rm I}{\rtimes} \mathcal{S}(P_\gamma)$ and $\widetilde{D}_n \overset{\rm I}{\rtimes} \mathcal{S}(P_\gamma^{\pm})$ of $\widetilde{G}_n^{\rm I}$ as

$$\text{(11.10)} \qquad \pi_\gamma^{\rm I}(d', \sigma') \ :=\ P_\gamma(d') \cdot \nabla_n(\sigma') \qquad \big((d', \sigma') \in \widetilde{D}_n \overset{\rm I}{\rtimes} \mathcal{S}(P_\gamma)\big),$$

$$\text{(11.11)} \qquad \pi_\gamma^{\rm I\pm}(d', \sigma') \ :=\ P_\gamma^{\pm}(d') \cdot \nabla_n^{\pm}(\sigma') \qquad \big((d', \sigma') \in \widetilde{D}_n \overset{\rm I}{\rtimes} \mathcal{S}(P_\gamma^{\pm})\big).$$

Then they are spin IRs of $\widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P_\gamma)$ and of $\widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P_\gamma^{\pm})$ by Theorem 9.2, which are also denoted as $P_\gamma \cdot \nabla_n$ and $P_\gamma^{\pm} \cdot \nabla_n^{\pm}$ respectively.

On the other hand, we see from Theorem 9.2 that, under $\widetilde{\mathfrak{S}}_n$, P_γ (resp. $P_\gamma^{\pm}$ if n is odd) is conjugate to $P_{\sigma\gamma}$ (resp. to $P_{\sigma\gamma}^{\pm}$) for any $\sigma \in \mathfrak{S}_n$. Therefore, as a representative of their conjugacy classes under $\widetilde{\mathfrak{S}}_n$, we can choose P_γ (resp. $P_\gamma^{\pm}$ if n is odd) with a parameter $\gamma \in \Gamma_n^0$ of the following normal form.

Definition 11.1. A parameter $\gamma \in \Gamma_n$ is said to be *normalized*, if there exist a series of integers

$$c = (c_1, \ldots, c_K), \quad 0 \le c_1 < c_2 < \cdots < c_K < m', \tag{11.12}$$

and a partition of $\boldsymbol{I}_n = \{1, 2, \ldots, n\}$ into subsets as

$$\boldsymbol{I}_n = \bigsqcup_{k \in \mathcal{K}} I_k\,, \quad \mathcal{K} := \{1, 2, \ldots, K\}, \tag{11.13}$$

such that $\gamma_j = c_k \ (j \in I_k)$ for $k \in \mathcal{K}$. In a visual form, when I_k's are consecutive intervals,

$$\gamma = (\underbrace{c_1, \ldots, c_1}_{j\in I_1}, \underbrace{c_2, \ldots, c_2}_{j \in I_2}, c_3, \ldots \ \ldots, c_{K-1}, \underbrace{c_K, \ldots, c_K}_{j \in I_K}). \tag{11.14}$$

Lemma 11.3. *Let γ be as in (11.12) – (11.14). Then stationary subgroups $\mathcal{S}(P_\gamma)$ in $\widetilde{\mathfrak{S}}_n$ are of the following form:*

$$\mathcal{S}(P_\gamma) := \mathcal{S}([P_\gamma]) = \Phi_{\mathfrak{S}}^{-1}\Big(\prod_{k\in\mathcal{K}} \mathfrak{S}_{I_k}\Big) \cong \Phi_{\mathfrak{S}}^{-1}\Big(\prod_{k\in\mathcal{K}} \mathfrak{S}_{n_k}\Big), \tag{11.15}$$

where $n_k := |I_k|$, $n_1 + \cdots + n_K = n$, and $\Phi_{\mathfrak{S}}$ is the canonical homomorphism $\widetilde{\mathfrak{S}}_n \to \mathfrak{S}_n$.

Similarly for $\mathcal{S}(P_\gamma^{\pm})$ in case n is odd.

11.3 IRs of stationary subgroups $\mathcal{S}(P_\gamma)$, $\mathcal{S}(P_\gamma^{\pm})$ (CASE I)

Thus $\mathcal{S}(P_\gamma)$ (and $\mathcal{S}(P_\gamma^{\pm})$ if n is odd) is a double covering group, with the central subgroup $\langle z_1 \rangle \cong \boldsymbol{Z}_2$, of the direct product $\prod_{k\in\mathcal{K}} \mathfrak{S}_{I_k}$ of smaller symmetric groups. Its IRs are of two kinds:

(1) spin IRs $\pi_{\mathcal{S}}$ with $\pi_{\mathcal{S}}(z_1) = -I$, and

(2) non-spin IRs $\pi_{\mathcal{S}}$ with $\pi_{\mathcal{S}}(z_1) = I$.

Our basic representation ∇_n (resp. $\nabla_n^{\pm}$ if n is odd) is spin one and π_γ^{I} (resp. $\pi_\gamma^{\mathrm{I}\pm}$ if n is odd) is already of (spin) Type $(-1,-1,-1)$. Therefore, to get spin representations of Type $(-1,-1,-1)$ of $G(m,1,n)$ itself by the inducing-up process, we are forced to pick up, at this stage, non-spin linear IRs of $\prod_{k\in\mathcal{K}} \mathfrak{S}_{I_k}$ as

$$\pi_{\mathcal{S}} = \pi_\Lambda := \bigotimes_{k\in\mathcal{K}} \pi_{\Lambda_k}^{(k)}, \quad \Lambda := (\Lambda_1, \ldots, \Lambda_K), \tag{11.16}$$

where $\pi^{(k)}_{\Lambda_k}$ denotes an IR of $\mathfrak{S}_{I_k} \cong \mathfrak{S}_{n_k}$ corresponding to a (Frobenius-) Young diagram Λ_k of size n_k. For $n_k = 1$, we put $\Lambda_k = \varnothing$.

Consider $\pi_{\mathcal{S}}$ as an IR of $\widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P_\gamma)$ (resp. of $\widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P^{\pm}_\gamma)$ if n is odd) through the quotient map, and take an inner tensor product with $\pi^{\mathrm{I}}_\gamma = P_\gamma \cdot \nabla_n$ in (11.10) (resp. $\pi^{\mathrm{I}\pm}_\gamma = P^{\pm}_\gamma \cdot \nabla^{\pm}_n$ in (11.11)) as

$$\pi^{\mathrm{I}}_\gamma \boxdot \pi_{\mathcal{S}} = \pi^{\mathrm{I}}_\gamma \boxdot \pi_\Lambda \quad \big(\text{resp. } \pi^{\mathrm{I}\pm}_\gamma \boxdot \pi_{\mathcal{S}} = \pi^{\mathrm{I}\pm}_\gamma \boxdot \pi_\Lambda \ \text{ if } n \text{ is odd}\big). \tag{11.17}$$

As a natural parameter for this IR of $\widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P_\gamma)$ (resp. $\widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P^{\pm}_\gamma)$ if n is odd), we take (c, Λ) and denote it as $\pi^{\mathrm{I}}_{c,\Lambda}$ (resp. $\pi^{\mathrm{I}\pm}_{c,\Lambda}$ if n is odd), where $c = (c_1, \ldots, c_K)$.

11.4 IRs of $\widetilde{G}^{\mathrm{I}}(m,1,n)$ in CASE I, Type $(-1,-1,-1)$

By inducing it up to $\widetilde{G}^{\mathrm{I}}_n$ we obtain spin IRs of CASE I, Type $(-1,-1,-1)$, as

$$\begin{aligned} \Pi^{\mathrm{I}}_{c,\Lambda} &:= \operatorname{Ind}^{\widetilde{G}^{\mathrm{I}}_n}_{\widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P_\gamma)} \pi^{\mathrm{I}}_{c,\Lambda} \quad \text{in case } n \text{ is even,} \\ \Pi^{\mathrm{I}\pm}_{c,\Lambda} &:= \operatorname{Ind}^{\widetilde{G}^{\mathrm{I}}_n}_{\widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P^{\pm}_\gamma)} \pi^{\mathrm{I}\pm}_{c,\Lambda} \quad \text{in case } n \text{ is odd.} \end{aligned} \tag{11.18}$$

Then we obtain the following result as a consequence of the standard inducing-up method for semidirect product groups.

Theorem 11.4. *Let m be even and $n \geq 4$. A complete set* $\mathrm{spinIR}^{\mathrm{I}}\big(G(m,1,n)\big)$ *of representatives of equivalence classes of spin IRs of CASE I, (spin) Type $(-1,-1,-1)$, of $G(m,1,n)$ is given by the set of the following IRs :*

$$\begin{aligned} &\text{Case } n = 2n' \text{ even :} && \Pi^{\mathrm{I}}_{c,\Lambda}\ ; \\ &\text{Case } n = 2n'+1 \text{ odd :} && \Pi^{\mathrm{I}+}_{c,\Lambda} \ \text{ and } \ \Pi^{\mathrm{I}-}_{c,\Lambda}\ ; \end{aligned}$$

where, for parameter (c, Λ), $c = (c_1, \ldots, c_K)$ is related to γ through (11.12)–(11.14), and $\Lambda = (\Lambda_1, \ldots, \Lambda_K)$ runs over K-tuples of Young diagrams of size $n_1, n_2, \ldots, n_K$ in (11.16).

Example 11.1. Most simple spin IRs of $\widetilde{G}^{\mathrm{I}}_n$ are given in special cases where $\mathcal{S}(P_\gamma) = \widetilde{\mathfrak{S}}_n$ or $\mathcal{S}(P^{\pm}_\gamma) = \widetilde{\mathfrak{S}}_n$. For $0 \leq k < m' = m/2$, put $\gamma^{(k)} := (k, k, \ldots, k)$, then it is $\mathfrak{S}_n$-invariant and so $\mathcal{S}(P_{\gamma^{(k)}}) = \widetilde{\mathfrak{S}}_n$ and $\widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P_{\gamma^{(k)}}) = \widetilde{G}^{\mathrm{I}}_n$. Similarly also for $P^{\pm}_{\gamma^{(k)}}$ in case $n = 2n'+1$.

Take $\Lambda^{(0)} = (n)$ a Young diagram with one row of length n, then $\pi_{\Lambda^{(0)}}$ is the trivial representation $\mathbf{1}_{\widetilde{\mathfrak{S}}_n}$ of $\widetilde{\mathfrak{S}}_n$. Then the parameters (c, Λ) of the corresponding spin IRs of $\widetilde{G}^{\mathrm{I}}_n$ in these cases are given by $c = (k)$ with $K = 1$ and $\Lambda = \Lambda^{(0)}$:

$$\begin{aligned} \Pi^{\mathrm{I}}_{k,\Lambda^{(0)}} &= \pi^{\mathrm{I}}_{k,\Lambda^{(0)}} = P_{\gamma^{(k)}} \cdot \nabla_n \quad && \text{in case } n \text{ is even ;} \\ \Pi^{\mathrm{I}\pm}_{k,\Lambda^{(0)}} &= \pi^{\mathrm{I}\pm}_{k,\Lambda^{(0)}} = P^{\pm}_{\gamma^{(k)}} \cdot \nabla^{\pm}_n \quad && \text{in case } n \text{ is odd.} \end{aligned}$$

Especially, take $k=0$, then $P_{\gamma^{(0)}}$, $P^+_{\gamma^{(0)}}$ and $P^-_{\gamma^{(0)}}$ are denoted in (6.19) respectively by P_0, P_+ and P_- (cf. Theorem 6.7). The special spin IRs

$$\Pi^{\rm I}_0 := P_0 \cdot \nabla_n, \quad \Pi^{\rm I}_+ := P_+ \cdot \nabla^+_n, \quad \Pi^{\rm I}_- := P_- \cdot \nabla^-_n \,. \tag{11.19}$$

will play very important roles in the theory of spin irreducible characters of $\widetilde{G}^{\rm I}_n$, and also in that of spin characters of $\widetilde{G}^{\rm I}_\infty$ under taking limits as $n \to \infty$.

On the other hand, take $\Lambda^{(1)} = (1,1,\ldots,1)$, then $\pi_{\Lambda^{(1)}} = \mathrm{sgn}$, and

$$\Pi^{\rm I}_{k,\Lambda^{(1)}} = \pi^{\rm I}_{k,\Lambda^{(1)}} = P_{\gamma^{(k)}} \cdot (\nabla_n \,\mathrm{sgn}).$$

11.5 Another parametrization of IRs of $\widetilde{G}^{\rm I}(m,1,n)$

Parametrization by (c,Λ), using normalized γ, is good for describing equivalence classes of IRs of $\widetilde{G}^{\rm I}(m,1,n)$, similarly as in §17 in [I]. However, for giving characters of IRs $\widetilde{G}^{\rm I}(m,1,n)$ explicitly in later sections and also for studying their limits in the next part, another parameterization is better.

Denote by $\boldsymbol{Y}$ the set of all Young diagrams Λ, in which the empty set $\varnothing$ with zero box is contained by definition. For $T = \boldsymbol{Z}_m$, put as in [HoHH, §1]

$$\boldsymbol{Y}_n(T) := \{\Lambda = (\lambda^\zeta)_{\zeta\in\widehat{T}} \;;\; \lambda^\zeta \in \boldsymbol{Y},\ \sum_{\zeta\in\widehat{T}} |\lambda^\zeta| = n\}, \tag{11.20}$$

where the size of $\lambda \in \boldsymbol{Y}$ is denoted by $|\lambda|$. Moreover we define a representative space of a quotient of $\boldsymbol{Y}_n(T)$ under certain equivalence relation as

$$\boldsymbol{Y}_n(T)^0 := \{\Lambda = (\lambda^\zeta)_{\zeta\in\widehat{T}^0} \;;\; \lambda^\zeta \in \boldsymbol{Y},\ \sum_{\zeta\in\widehat{T}^0} |\lambda^\zeta| = n\}, \tag{11.21}$$

where $\widehat{T}^0$ is the half of $\widehat{T}$ given by

$$\widehat{T}^0 := \big\{\zeta \in \widehat{T} \;;\; \zeta(\eta) = \omega^a,\ 0 \le a < m' = m/2\big\}. \tag{11.22}$$

Then a parameter (c,Λ) in Theorem 11.4 above corresponds bijectively to an element in $\boldsymbol{Y}_n(T)^0$, as is explained below.

We give in §6.1 two abelian subgroups of $\widetilde{D}(m,n)$ as $\widetilde{D}^0(m,n) \subset \widetilde{D}^1(m,n) = \big\langle z_2, \widetilde{D}^0(m,n)\big\rangle$. A spin character of $\widetilde{D}^1(m,n)$ is given by restricting a spin function χ_γ in (6.2), whereas a character ζ_γ of the base group $D(m,n)$ and a non-spin character $\zeta_\gamma \circ \Phi_D$ of $\widetilde{D}(m,n)$ are defined as $\zeta_\gamma = (\zeta_1,\ldots,\zeta_n)$ with $\zeta_j = \zeta_{j,\gamma_j} = \zeta_\gamma\big|_{T_j}$ for j-th copy $T_j = \langle y_j\rangle$ of the protocol $T = \langle y\rangle$. We identify ζ_j with $\zeta^{(\gamma_j)} \in \widehat{T}$ through $T_j \cong T$.

The decomposition of $\boldsymbol{I}_n$ in Definition 11.1 is a special case of the following one.

Definition 11.2. For $\gamma \in \Gamma_n$, we define a partition of $\boldsymbol{I}_n$ by $\zeta_\gamma = (\zeta_j)_{j \in \boldsymbol{I}_n}$ as

$$\boldsymbol{I}_n = \bigsqcup_{\zeta \in \widehat{T}} I_{n,\zeta}\,, \quad I_{n,\zeta} := \{j \in \boldsymbol{I}_n\ ;\ \zeta_j = \zeta\}.$$

If $\gamma \in \Gamma_n^0$ corresponds to $c = (c_1, \ldots, c_K)$ and $\boldsymbol{I}_n = \bigsqcup_{k \in \mathcal{K}} I_k$, then the set of Young diagrams $\Lambda^n = (\lambda^{n,\zeta})_{\zeta \in \widehat{T}^0}$ with $|\lambda^{n,\zeta}| = |I_{n,\zeta}|$ given by (c, Λ) just belongs to $\boldsymbol{Y}_n(T)^0$. Then Theorem 11.4 can be restated with this new parameter as follows.

Theorem 11.5. *Let m be even and $n \geq 4$. A complete set* $\mathrm{spinIR}^{\mathrm{I}}\big(G(m,1,n)\big)$ *of representatives of equivalence classes of spin IRs of CASE I, Type* $(-1,-1,-1)$, *of* $G(m,1,n)$ *is given by the set of the following IRs :*

$$\begin{array}{lll} \text{Case } n = 2n' \text{ even :} & \Pi^{\mathrm{I}}_{\Lambda^n}\,, & \Lambda^n \in \boldsymbol{Y}_n(T)^0\ ; \\ \text{Case } n = 2n'+1 \text{ odd :} & \Pi^{\mathrm{I}+}_{\Lambda^n}\,,\ \Pi^{\mathrm{I}-}_{\Lambda^n}\,, & \Lambda^n \in \boldsymbol{Y}_n(T)^0. \end{array}$$

12 Classification of spin IRs of $G(m,1,n)$, CASE II, Type $(-1,-1,\ 1)$

12.1 Irreducible spin representations of $\widetilde{G}^{\,\mathrm{II}}_n := \widetilde{G}^{\,\mathrm{II}}(m,1,n)$

Let Π be a spin representation of

$$\widetilde{G}^{\,\mathrm{II}}_n := \widetilde{G}^{\,\mathrm{II}}(m,1,n) = R\big(G(m,1,n)\big)/\langle z_3\rangle = \widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \widetilde{\mathfrak{S}}_n\,,\ \ \widetilde{D}_n = \widetilde{D}(m,n),$$

such that $\Pi(z_1) = \Pi(z_2) = -I$ or of (spin) Type $(-1,-1,\ 1)$.

Theorem 12.1. *Suppose $n \geq 4$ and m is even. Let Π be a representation of the covering group $\widetilde{G}^{\,\mathrm{II}}(m,1,n) = R\big(G(m,1,n)\big)/\langle z_3\rangle$ such that*

(i) $\qquad \Pi(z_1) = -I, \quad \Pi(z_2) = -I.$

Then the operators $\Pi(r_i)$'s and $\Pi(\eta_j)$'s satisfy the following:

(ii) $$\begin{cases} \Pi(r_i)^2 = I \ \ (i \in \boldsymbol{I}_{n-1}), & \big(\Pi(r_i)\Pi(r_{i+1})\big)^3 = I \ \ (i \in \boldsymbol{I}_{n-2}), \\ \Pi(r_i)\Pi(r_j) = -\Pi(r_j)\Pi(r_i) & (|i-j| \geq 2), \end{cases}$$

(iii) $$\Pi(\eta_j)^m = I \qquad (j \in \boldsymbol{I}_n),$$

(iv) $$\Pi(\eta_j)\Pi(\eta_k) = -\Pi(\eta_k)\Pi(\eta_j) \qquad (j \neq k),$$

(v) $$\Pi(r_i)\Pi(\eta_j)\Pi(r_i^{-1}) = \Pi\big(\eta_{s_i(j)}\big) \qquad (j \in \boldsymbol{I}_n).$$

Now we prepare several spin representations as follows. For $\widetilde{\mathfrak{S}}_n$ with $n = 2n', 2n'+1$, two representations ∇'_n and ∇''_n in Theorem 8.3 as

$$(12.1) \qquad \begin{cases} \nabla'_n(r_j) &= \dfrac{1}{\sqrt{2}}\,(Y_j - Y_{j+1}) \\ \nabla''_n(r_j) &= -Y_{2n'+1}\nabla'_n(r_j)\,Y_{2n'+1}^{-1} \end{cases} \qquad (j \in \boldsymbol{I}_{n-1});$$

and one more spin representation $\nabla_n^{\rm II}$, in case $n = 2n'$ even, for $\widetilde{\mathfrak{S}}_{2n'}$ as

$$\nabla_{2n'}^{\rm II}(r_j) = (iY_{2n'+1}) \cdot \nabla'_{2n'}(r_j) = \nabla'_{2n'}(r_j) \cdot (-iY_{2n'+1}) \quad (j \in \boldsymbol{I}_{n-1}); \tag{12.2}$$

and, in case $n = 2n'+1$, two spin representations $\mho_n^+$ and $\mho_n^-$ of $\widetilde{\mathfrak{A}}_n$ in (8.22) as

$$\begin{cases} \mho_n^+(v') &= \nabla'_n(v') \\ \mho_n^-(v') &= \nabla''_n(v') = Y_{2n'+1}\nabla'_n(v')\,Y_{2n'+1}^{-1} \end{cases} \qquad (v' \in \widetilde{\mathfrak{A}}_n). \tag{12.3}$$

As spin IRs of $\widetilde{D}_n$, we have in (6.6) and in (6.16) respectively

$$\begin{aligned} P_\gamma(\eta_j) \;&=\; \omega^{\gamma_j}\,\rho(\eta_j) = \chi_\gamma(\eta_j)\,Y_j \quad\; (j \in \boldsymbol{I}_n,\ \gamma \in \Gamma_n)\ ; \\ P_\gamma^+ = P_\gamma,\;\; P_\gamma^- &= P_{\tau_n\gamma} \quad (\gamma \in \Gamma_n^0) \quad \text{ if } n \text{ is odd,} \end{aligned} \tag{12.4}$$

and so $P_\gamma^+(\eta_j) = \chi_\gamma(\eta_j)\,Y_j \ \ (j \in \boldsymbol{I}_{n-1}),\ \ P_\gamma^-(\eta_n) = -\chi_\gamma(\eta_n)\,Y_n$.

12.2 Stationary subgroups of $[P_\gamma]$ and $[P_\gamma^\pm]$

Take the complete sets of representatives of equivalence classes of spin IRs $\widehat{\widetilde{D}_n}^{\rm spin}$ in Theorem 6.5 quoted above in (11.5) – (11.6).

Lemma 12.2. *For representatives P_γ in case n even (resp. P_γ^+ and P_γ^- in case n odd), with $\gamma \in \Gamma_n^0$, of equivalence classes of spin IRs of $\widetilde{D}_n$, the stationary subgroups in $\widetilde{\mathfrak{S}}_n$ of their equivalence classes, and their $\mathfrak{S}_n$-orbits are given as follows.*

(i) *Let $n = 2n'$ even. For representatives P_γ of $\widehat{\widetilde{D}_n}^{\rm spin}$,*

$$\mathcal{S}(P_\gamma) := \mathcal{S}([P_\gamma]) = \big\{\sigma' \in \widetilde{\mathfrak{S}}_n\ ;\ \sigma\gamma = \gamma\big\} \quad \big(\sigma = \Phi(\sigma')\big), \tag{12.5}$$

and $\sigma^{\rm II}P_\gamma \cong P_{\sigma\gamma}$ for $\sigma \in \mathfrak{S}_n$.

(ii) *Let $n = 2n'+1$ odd. For representatives P_γ^+ and P_γ^- of $\widehat{\widetilde{D}_n}^{\rm spin}$,*

$$\begin{cases} \mathcal{S}(P_\gamma^+) := \mathcal{S}([P_\gamma^+]) = \big\{\sigma' \in \widetilde{\mathfrak{A}}_n\ ;\ \sigma\gamma = \gamma\big\}, \\ \mathcal{S}(P_\gamma^-) := \mathcal{S}([P_\gamma^-]) = \big\{\sigma' \in \widetilde{\mathfrak{A}}_n\ ;\ \sigma\gamma = \gamma\big\}. \end{cases} \tag{12.6}$$

According as ${\rm sgn}(\sigma) = 1$ *or* ${\rm sgn}(\sigma) = -1$ *for $\sigma \in \mathfrak{S}_n$, there holds respectively*

$$\sigma^{\rm II}P_\gamma^\pm \cong P_{\sigma\gamma}^\pm \qquad \text{i.e.,} \qquad \sigma^{\rm II}P_\gamma^+ \cong P_{\sigma\gamma}^-\,,\ \ \sigma^{\rm II}P_\gamma^- \cong P_{\sigma\gamma}^+\,.$$

Proof. Recall that $P_{\gamma'} \cong P_\gamma$ if and only if $\gamma' \approx_n \gamma$. Then (i) follows from Theorem 10.2 (ii). Similarly (ii) follows from Theorem 10.2 (iii) and (iv). □

We put for the subgroup $\widetilde{D}_n \overset{\rm II}{\rtimes} \mathcal{S}(P_\gamma)$ $\big($resp. $\widetilde{D}_n \overset{\rm II}{\rtimes} \mathcal{S}(P_\gamma^\pm)\big)$ of $\widetilde{G}^{\rm II}(m,1,n)$ as follows.

- In case $n = 2n'$ even, for $(d', \sigma') \in \widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \mathcal{S}(P_\gamma)$,

$$\pi_\gamma^{\mathrm{II}}(d', \sigma') \;:=\; P_\gamma(d') \cdot \nabla_n^{\mathrm{II}}(\sigma') . \tag{12.7}$$

- In case $n = 2n' + 1$ odd, we put, for $(d', \sigma') \in \widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \mathcal{S}(P_\gamma^+) = \widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \mathcal{S}(P_\gamma^-)$,

$$\begin{aligned} \pi_\gamma^{\mho+}(d', \sigma') &:= P_\gamma^+(d') \cdot \mho_n^+(\sigma') \,; \\ \pi_\gamma^{\mho-}(d', \sigma') &:= P_\gamma^-(d') \cdot \mho_n^-(\sigma') . \end{aligned} \tag{12.8}$$

Then they are spin IRs of the corresponding subgroups by Theorem 10.2.

As a complete set of representative of their conjugacy classes in $\widehat{\widetilde{D}_n}^{\mathrm{spin}}$ under the action of $\widetilde{\mathfrak{S}}_n$, we can choose from the following

- in case $n = 2n'$ even, $\{P_\gamma \,;\, \gamma \in \Gamma_n^0\}$, under the action of $\widetilde{\mathfrak{S}}_n$;
- in case $n = 2n' + 1$ odd, $\{P_\gamma^+ \,;\, \gamma \in \Gamma_n^0\}$, under the action of $\widetilde{\mathfrak{A}}_n$.

In fact, in case n is odd, we have, by Theorem 10.2(iv), $\sigma^{\mathrm{II}} P_\gamma^- = \iota\big(\nabla_n^{\mathrm{II}-}(\sigma')\big)^{-1}\big(P_{\sigma\gamma}^+\big)$, $\sigma = \Phi(\sigma')$, for $\sigma' \in \widetilde{\mathfrak{S}}_n,\ \mathrm{sgn}(\sigma') = -1$. We can choose those γ's normalized in the sense of Definition 11.1 (slightly modified in case n is odd), and arrive to a parametrization, by (c, Λ)'s, of equivalence classes of IRs of $\widetilde{G}_n^{\mathrm{II}}$. However, here we prefer to follow another parametrization as in §11.5.

For $\gamma \in \Gamma_n^0$, we have a non-spin character $\zeta_\gamma = (\zeta_1, \ldots, \zeta_n),\ \zeta_j = \zeta_{j,\gamma_j} \in \widehat{T}^0$, of $\widetilde{D}_n$ and a partition $\boldsymbol{I}_n = \bigsqcup_{\zeta \in \widehat{T}^0} I_{n,\zeta}$ in Definition 11.2.

Lemma 12.3. *Let $\gamma \in \Gamma_n^0$. Then stationary subgroup $\mathcal{S}(P_\gamma)$ and $\mathcal{S}(P_\gamma^+)$ in $\widetilde{\mathfrak{S}}_n$ are respectively of the following form.*

(i) *In the case where $n = 2n'$ even,*

$$\mathcal{S}(P_\gamma) = \Phi_{\mathfrak{S}}^{-1}\Big(\prod_{\zeta \in \widehat{T}^0} \mathfrak{S}_{I_{n,\zeta}} \Big), \quad \mathfrak{S}_{I_{n,\zeta}} \cong \mathfrak{S}_{|I_{n,\zeta}|}. \tag{12.9}$$

(ii) *In the case where $n = 2n' + 1$ odd,*

$$\mathcal{S}(P_\gamma^+) = \Phi_{\mathfrak{S}}^{-1}\Big(\prod_{\zeta \in \widehat{T}^0} \mathfrak{S}_{I_{n,\zeta}} \bigcap \mathfrak{A}_n \Big), \tag{12.10}$$

and the index $\big[\prod_{\zeta \in \widehat{T}^0} \mathfrak{S}_{I_{n,\zeta}} : \prod_{\zeta \in \widehat{T}^0} \mathfrak{S}_{I_{n,\zeta}} \bigcap \mathfrak{A}_n\big] = 2$, *except the cases where*

$$|I_{n,\zeta}| = 1 \ (\zeta \in \widehat{T}^0) \quad (\therefore n \le m' = m/2), \quad \prod_{\zeta \in \widehat{T}^0} \mathfrak{S}_{I_{n,\zeta}} = \{e\}. \tag{12.11}$$

Notation 12.1. A subgroup of $\mathfrak{S}_n$ of the form $\prod_{\zeta \in \widehat{T}^0} \mathfrak{S}_{I_{n,\zeta}} \bigcap \mathfrak{A}_n$ is denoted as

$$\mathfrak{A}\big(\textstyle\prod_{\zeta \in \widehat{T}^0} \mathfrak{S}_{I_{n,\zeta}} \big) := \prod_{\zeta \in \widehat{T}^0} \mathfrak{S}_{I_{n,\zeta}} \bigcap \mathfrak{A}_n$$

in [I] of this series of papers. In accordance with this notation, the subgroup of $\widetilde{\mathfrak{S}}_n$ in the right hand side of (12.10) will be denoted as

$$\widetilde{\mathfrak{A}}\Big(\prod\nolimits_{\zeta\in\widehat{T}^0}\mathfrak{S}_{I_{n,\zeta}}\Big) := \Phi_{\mathfrak{S}}^{-1}\Big(\prod\nolimits_{\zeta\in\widehat{T}^0}\mathfrak{S}_{I_{n,\zeta}}\bigcap\mathfrak{A}_n\Big). \tag{12.12}$$

Example 12.1. In case $J=\{1,2,3\},\ \{2,3\},\{3\}$, we have respectively

$$\Phi_{\mathfrak{S}}^{-1}(\mathfrak{S}_J) \ = \ \langle z_1,r_1,r_2\rangle,\ \ \langle z_1,r_2\rangle,\ \ \langle z_1\rangle.$$

12.3 IRs of stationary subgroups $\mathcal{S}(P_\gamma)$, $\mathcal{S}(P_\gamma^+)$ (CASE II)

• CASE $n=2n'$ EVEN :

The subgroup $\mathcal{S}(P_\gamma)$ is a double covering group (with the central subgroup $\langle z_1\rangle\cong \boldsymbol{Z}_2$) of the direct product $\prod_{\zeta\in\widehat{T}^0}\mathfrak{S}_{I_{n,\zeta}}$. Its IRs are of two kinds: (1) spin IRs $\pi_{\mathcal{S}}$ with $\pi_{\mathcal{S}}(z_1)=-I$, and (2) non-spin IRs $\pi_{\mathcal{S}}$ with $\pi_{\mathcal{S}}(z_1)=I$.

Since the representation $\nabla_n^{\rm II}$ is spin one and $\pi_\gamma^{\rm II}$ is of Type $(-1,-1,\ 1)$ already, to get spin representations of Type $(-1,-1,\ 1)$ of $G(m,1,n)$, we are forced to pick up non-spin, linear IRs of $\prod_{\zeta\in\widehat{T}^0}\mathfrak{S}_{I_{n,\zeta}}$ as

$$\pi_{\mathcal{S}}=\pi_{\Lambda^n} := \bigotimes_{\zeta\in\widehat{T}^0}\pi^{(\zeta)}_{\lambda^{n,\zeta}},\quad \Lambda^n := (\lambda^{n,\zeta})_{\zeta\in\widehat{T}^0}, \tag{12.13}$$

where $\pi^{(\zeta)}_{\lambda^{n,\zeta}}$ denotes an IR of $\mathfrak{S}_{I_{n,\zeta}}\cong\mathfrak{S}_{|I_{n,\zeta}|}$ corresponding to a Young diagram $\lambda^{n,\zeta}$ of size $|I_{n,\zeta}|$.

• CASE $n=2n'+1$ ODD :

The subgroup $\mathcal{S}(P_\gamma^+)$ is a double covering group (with the central subgroup $\langle z_1\rangle$) of $\mathfrak{A}\big(\prod_{\zeta\in\widehat{T}^0}\mathfrak{S}_{I_{n,\zeta}}\big)$. Its IRs are of two kinds: (1) spin IRs $\pi_{\mathcal{S}}$ with $\pi_{\mathcal{S}}(z_1)=-I$, and (2) non-spin IRs $\pi_{\mathcal{S}}$ with $\pi_{\mathcal{S}}(z_1)=I$.

Since the representations $\mho_n^+$ of $\widetilde{\mathfrak{A}}_n$ are spin one, and $\pi_\gamma^{\mho+}$ are already of Type $(-1,-1,\ 1)$, to get a spin representation of Type $(-1,-1,\ 1)$ of $G(m,1,n)$, we are forced to pick up a non-spin, linear IR of $\widetilde{\mathfrak{A}}\big(\prod_{\zeta\in\widehat{T}^0}\mathfrak{S}_{I_{n,\zeta}}\big)$, or an IR of $\mathfrak{A}\big(\prod_{\zeta\in\widehat{T}^0}\mathfrak{S}_{I_{n,\zeta}}\big)$.

To describe them, put for a moment $G=\prod_{\zeta\in\widehat{T}^0}\mathfrak{S}_{I_{n,\zeta}}$ and $H=\mathfrak{A}\big(\prod_{\zeta\in\widehat{T}^0}\mathfrak{S}_{I_{n,\zeta}}\big)$, then H is normal and of index two in G. As an IR of G, Take π_{Λ^n} above, and put $\rho_{\Lambda^n}=\pi_{\Lambda^n}\big|_H$. Then we know the following (cf. Lemmas 17.5 and 17.6 in [I]).

Lemma 12.4. *Let $s\in G\setminus H$, and assume that the partition $\boldsymbol{I}_n=\bigsqcup_{\zeta\in\widehat{T}^0}I_{n,\zeta}$ is not in the exceptional cases in (12.11).*

(CASE TΛ-1). *Assume ${}^t\Lambda^n\neq\Lambda^n$, where ${}^t\Lambda^n := ({}^t\lambda^{n,\zeta})_{\zeta\in\widehat{T}^0}$. Then $\pi_{{}^t\Lambda^n}\cong$ $\mathrm{sgn}\cdot\pi_\Lambda$, and*

$$\begin{cases} \rho_{{}^t\Lambda^n}\cong\rho_{\Lambda^n} \quad \text{irreducible}, \\ \big(\rho_{\Lambda^n}\big)^s \ \cong\ \rho_{\Lambda^n}, \qquad \big(\rho_{\Lambda^n}\big)^s(h) := \rho_{\Lambda^n}(shs^{-1})\ (h\in H), \\ \mathrm{Ind}_H^G\,\rho_{\Lambda^n}\cong\pi_{\Lambda^n}\oplus\pi_{{}^t\Lambda^n}. \end{cases}$$

(Case TΛ-2). *Assume* ${}^t\Lambda^n = \Lambda^n$. *Then* $\pi_{\Lambda^n} \cong \mathrm{sgn}\cdot\pi_{\Lambda^n}$, *and*

$$\begin{cases} \rho_{\Lambda^n} \cong \rho^{(0)}_{\Lambda^n} \oplus \rho^{(1)}_{\Lambda^n}, \quad \rho^{(0)}_{\Lambda^n} \not\cong \rho^{(1)}_{\Lambda^n} \ \text{ irreducible}, \\ \big(\rho^{(0)}_{\Lambda^n}\big)^s \cong \rho^{(1)}_{\Lambda^n}, \\ \mathrm{Ind}^G_H\, \rho^{(\alpha)}_{\Lambda^n} \cong \pi_{\Lambda^n} \quad (\alpha = 0, 1). \end{cases}$$

A complete set of representatives of equivalence classes of spin IRs for $\mathfrak{A}\big(\prod_{\zeta\in\widehat{T}^0}\mathfrak{S}_{I_{n,\zeta}}\big)$ *is given by the set of*

$$\begin{cases} \rho^n_\Lambda = \pi_{\Lambda^n}\big|_{\mathfrak{A}\left(\prod_{\zeta\in\widehat{T}^0}\mathfrak{S}_{I_{n,\zeta}}\right)}, & \text{in case } {}^t\Lambda^n \neq \Lambda^n, \text{ where } \rho_{{}^t\Lambda^n} \cong \rho_{\Lambda^n}\ ; \\ \rho^{(0)}_{\Lambda^n}, \quad \rho^{(1)}_{\Lambda^n}, & \text{in case } {}^t\Lambda^n = \Lambda^n, \\ & \text{where } \pi_{\Lambda^n}\big|_{\mathfrak{A}\left(\prod_{\zeta\in\widehat{T}^0}\mathfrak{S}_{I_{n,\zeta}}\right)} = \rho^{(0)}_{\Lambda^n} \oplus \rho^{(1)}_{\Lambda^n}. \end{cases}$$

Remark 12.1. For the difference between irreducible components $\rho^{(0)}_{\Lambda^n}$ and $\rho^{(1)}_{\Lambda^n}$ of $\pi_{\Lambda^n}\big|_{\mathfrak{A}\left(\prod_{\zeta\in\widehat{T}^0}\mathfrak{S}_{I_{n,\zeta}}\right)}$, see [I, §17.5].

On the basis of this lemma, we prepare a parameter space $\boldsymbol{Y}^{\mathfrak{A}}_n(T)^0$, $T = \boldsymbol{Z}_m$, for equivalence classes of IRs of $\mathfrak{A}\big(\prod_{\zeta\in\widehat{T}^0}\mathfrak{S}_{I_{n,\zeta}}\big)$ as follows: put

$$\begin{aligned} &\boldsymbol{Y}^{\mathfrak{A}}_n(T)^{0,1} := \big\{\{\Lambda^n, {}^t\Lambda^n\}\,;\, \Lambda^n = (\lambda^{n,\zeta})_{\zeta\in\widehat{T}^0} \in \boldsymbol{Y}_n(T)^0,\ {}^t\Lambda^n \neq \Lambda^n\big\}, \\ &\boldsymbol{Y}^{\mathfrak{A}}_n(T)^{0,2} := \big\{(\Lambda^n, \kappa)\,;\, \Lambda^n \in \boldsymbol{Y}_n(T)^0,\ {}^t\Lambda^n = \Lambda^n, \\ &\qquad\qquad\qquad\qquad |\lambda^{n,\zeta}| \geq 2\ (\exists\zeta),\ \kappa = 0, 1\big\}, \\ &\boldsymbol{Y}^{\mathfrak{A}}_n(T)^{0,3} := \big\{\Lambda^n = (\lambda^{n,\zeta})_{\zeta\in\widehat{T}^0}\ ;\ |\lambda^{n,\zeta}| \leq 1\ (\forall\zeta)\big\}. \end{aligned} \tag{12.14}$$

Then $\boldsymbol{Y}^{\mathfrak{A}}_n(T)^{0,3} \neq \emptyset$ if and only if $n \leq m' = m/2$, and $\boldsymbol{Y}^{\mathfrak{A}}_n(T)^{0,3}$ corresponds to the exceptional case in (12.11). Put

$$\boldsymbol{Y}^{\mathfrak{A}}_n(T)^0 := \boldsymbol{Y}^{\mathfrak{A}}_n(T)^{0,1} \sqcup \boldsymbol{Y}^{\mathfrak{A}}_n(T)^{0,2} \sqcup \boldsymbol{Y}^{\mathfrak{A}}_n(T)^{0,3}, \tag{12.15}$$

Then, in case $n > m' = m/2$, $\quad \boldsymbol{Y}^{\mathfrak{A}}_n(T)^0 = \boldsymbol{Y}^{\mathfrak{A}}_n(T)^{0,1} \sqcup \boldsymbol{Y}^{\mathfrak{A}}_n(T)^{0,2}$.

12.4 IRs of $\widetilde{G}^{\mathrm{II}}(m,1,n)$ in CASE II, Type $(-1,-1,\ 1)$

Let $\widetilde{G}^{\mathrm{II}}_n = \widetilde{G}^{\mathrm{II}}(m,1,n) = R\big(G(m,1,n)\big)/\langle z_3\rangle$ be as before. Take a non-spin IR $\pi_{\mathcal{S}}$ of $\mathcal{S}(P_\gamma)$ $\big($resp. of $\mathcal{S}(P^+_\gamma)\big)$, and consider it as an IR of $\widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \mathcal{S}(P_\gamma)$ (resp. $\widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \mathcal{S}(P^+_\gamma)$) through the quotient map, and take an inner tensor product as follows.

Case $n = 2n'$ even :

$$\pi^{\mathrm{II}}_\gamma \boxdot \pi_{\mathcal{S}} = \pi^{\mathrm{II}}_\gamma \boxdot \pi_{\Lambda^n}\,. \tag{12.16}$$

As a natural parameter for this IR of $\widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \mathcal{S}(P_\gamma)$, we take $\Lambda^n = (\lambda^{n,\zeta})_{\zeta\in\widehat{T}^0} \in \boldsymbol{Y}_n(T)^0$ and denote it as $\pi^{\mathrm{II}}_{\Lambda^n}$.

CASE $n = 2n'+1$ ODD :

$$(12.17)\qquad \pi_\gamma^{\mho+} \boxdot \pi_{\mathcal{S}} = \begin{cases} \pi_\gamma^{\mho+} \boxdot \rho_{\Lambda^n} & \text{for } \{\Lambda^n, {}^t\Lambda^n\} \in \boldsymbol{Y}_n^{\mathfrak{A}}(T)^{0,1}, \\ \pi_\gamma^{\mho+} \boxdot \rho_{\Lambda^n}^{(\kappa)},\ \kappa = 0,1 & \text{for } (\Lambda^n, \kappa) \in \boldsymbol{Y}_n^{\mathfrak{A}}(T)^{0,2}, \\ \pi_\gamma^{\mho+} \boxdot \rho_{\Lambda^n} & \text{for } \Lambda^n \in \boldsymbol{Y}_n^{\mathfrak{A}}(T)^{0,3}. \end{cases}$$

Here the 3rd row on the left hand side is the exceptional case (12.11), and $\Lambda^n = (\lambda^{n,\zeta})_{\zeta\in\widehat{T}^0}$, $\lambda^{n,\zeta} = \varnothing$ or $\square$ (one box) for $\zeta \in \widehat{T}^0$, and $\pi_{\mathcal{S}} = \rho_{\Lambda^n} = \mathbf{1}$ the trivial representation (cf. Example 12.3 below). We denote these representations as $\pi_{\Lambda^n}^{\mho+}$ and $\pi_{\Lambda^n,\kappa}^{\mho+}$ respectively.

We have picked up only P_γ^+'s, discarding P_γ^-'s. Here we remark a simple conjugacy between $\pi_\gamma^{\mho+}$ and $\pi_\gamma^{\mho-}$ as follows.

Lemma 12.5. *Let* $s'_0 \in \Phi_{\mathfrak{S}}^{-1}\big(\prod_{\zeta\in\widehat{T}^0} \mathfrak{S}_{I_{n,\zeta}}\big) \setminus \mathcal{S}(P_\gamma^\pm)$. *Then* $\mathrm{sgn}(s'_0) = \mathrm{sgn}(s_0) = -1$ *and* $s_0\gamma = \gamma$ *with* $s_0 = \Phi(s'_0)$. *Then, for* $g' \in \widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \mathcal{S}(P_\gamma^+)$,

$$\pi_\gamma^{\mho-}(s'_0 g' {s'_0}^{-1}) = \big((iY_n)\nabla'_n(s'_0)\big) \cdot \pi_\gamma^{\mho+}(g') \cdot \big((iY_n)\nabla'_n(s'_0)\big)^{-1}.$$

Proof. First, by Theorem 10.2 (i), $\iota\big(\nabla'_n(\sigma')\big)P_\gamma(d') = P_{(\tau_1\tau_2\cdots\tau_n)\sigma\gamma}\big({\sigma'}^{\mathrm{II}}(d')\big)$ for $\sigma' \in \widetilde{\mathfrak{S}}_n$ if $\mathrm{sgn}(\sigma') = -1$, where $\sigma = \Phi(\sigma')$. Take $\sigma' = {s'_0}^{-1}$, then ${s_0}^{-1}\gamma = \gamma$, and so we have

$$\nabla'_n(s'_0)^{-1} P_\gamma\big({s'_0}^{\mathrm{II}}(d')\big)\nabla'_n(s'_0) = P_{(\tau_1\tau_2\cdots\tau_n)\gamma}(d') \quad (d' \in \widetilde{D}_n).$$

Note that $s'_0 g' {s'_0}^{-1} = \big({s'_0}^{\mathrm{II}}(d'), s'_0\sigma'{s'_0}^{-1}\big)$ for $g' = (d', \sigma')$, then

$$\begin{aligned} \pi_\gamma^{\mho-}(s'_0 g' {s'_0}^{-1}) &= P_\gamma^-\big({s'_0}^{\mathrm{II}}(d')\big) \cdot \mho_n^-\big(s'_0\sigma'{s'_0}^{-1}\big) \\ &= P_{\tau_n\gamma}\big({s'_0}^{\mathrm{II}}(d')\big) \cdot (iY_n)\nabla'_n\big(s'_0\sigma'{s'_0}^{-1}\big)(iY_n)^{-1} \\ &= (iY_n) \cdot P_{(\tau_1\cdots\tau_n)\gamma}\big({s'_0}^{\mathrm{II}}(d')\big)\, \nabla'_n\big(s'_0\sigma'{s'_0}^{-1}\big) \cdot (iY_n)^{-1} \\ &= (iY_n)\nabla'_n(s'_0) \cdot \big(P_\gamma(d')\,\nabla'_n(\sigma')\big) \cdot \nabla'_n(s'_0)^{-1}(iY_n)^{-1} \\ &= (iY_n)\nabla'_n(s'_0) \cdot \pi_\gamma^{\mho+}(g') \cdot \nabla'_n(s'_0)^{-1}(iY_n)^{-1}. \end{aligned}$$

□

Finally, by inducing up to $\widetilde{G}_n^{\mathrm{II}}$, we obtain IRs of CASE II, Type $(-1,-1,\ 1)$.

CASE $n = 2n'$ EVEN :

$$(12.18)\qquad \Pi_{\Lambda^n}^{\mathrm{II}} := \mathrm{Ind}_{\widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \mathcal{S}(P_\gamma)}^{\widetilde{G}_n^{\mathrm{II}}} \pi_{\Lambda^n}^{\mathrm{II}}\ ;$$

CASE $n = 2n'+1$ ODD :

$$(12.19)\qquad \Pi_{\Lambda^n}^{\mho+} := \mathrm{Ind}_{\widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \mathcal{S}(P_\gamma^+)}^{\widetilde{G}_n^{\mathrm{II}}} \pi_{\Lambda^n}^{\mho+}, \qquad \Pi_{\Lambda^n,\kappa}^{\mho+} := \mathrm{Ind}_{\widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \mathcal{S}(P_\gamma^+)}^{\widetilde{G}_n^{\mathrm{II}}} \pi_{\Lambda^n,\kappa}^{\mho+} \ (\kappa = 0,1).$$

Then we obtain the following result in CASE II.

Theorem 12.6. *Let m be even and $n \geq 4$. A complete set* $\mathrm{spinIR}^{\mathrm{II}}\big(G(m,1,n)\big)$ *of representatives of equivalence classes of spin IRs of CASE II, (spin) Type* $(-1,-1,\ 1)$*, of $G(m,1,n)$ is given as follows.*

(i) Case $n = 2n'$ even :
$\mathrm{spinIR}^{\mathrm{II}}\big(G(m,1,n)\big)$ *consists of* $\Pi^{\mathrm{II}}_{\Lambda^n}$, $\Lambda^n \in \boldsymbol{Y}_n(T)^0$.

(ii) Case $n = 2n'+1$ odd, and $n > m' = m/2$:
$\mathrm{spinIR}^{\mathrm{II}}\big(G(m,1,n)\big)$ *consists of IRs*

$$\begin{cases} \Pi^{\mho+}_{\Lambda^n}, & \{\Lambda^n, {}^t\Lambda^n\} \in \boldsymbol{Y}^{\mathfrak{A}}_n(T)^{0,1}, \\ \Pi^{\mho+}_{\Lambda^n,\kappa}, & (\Lambda^n,\kappa) \in \boldsymbol{Y}^{\mathfrak{A}}_n(T)^{0,2},\ \kappa = 0,1. \end{cases} \tag{12.20}$$

(iii) Case $n = 2n'+1$ odd, and $n \leq m' = m/2$:
$\mathrm{spinIR}^{\mathrm{II}}\big(G(m,1,n)\big)$ *consists of IRs:*

$$\begin{cases} \Pi^{\mho+}_{\Lambda^n}, \quad \Pi^{\mho+}_{\Lambda^n,\kappa}\ (\kappa=0,1), \quad \text{in (12.20)}, \\ \Pi^{\mho+}_{\Lambda^n}, \qquad \Lambda^n \in \boldsymbol{Y}^{\mathfrak{A}}_n(T)^{0,3}. \end{cases} \tag{12.21}$$

Example 12.2. Simple spin IRs of $\widetilde{G}^{\mathrm{II}}_n$ are given in the cases where $\mathcal{S}(P_\gamma) = \widetilde{\mathfrak{S}}_n$ for n even, and $\mathcal{S}(P^+_\gamma) = \widetilde{\mathfrak{A}}_n$ for n odd. For $0 \leq k < m'$, put $\gamma = \gamma^{(k)} := (k,k,\ldots,k) \in \Gamma^0_n$.

In case n is even, $\mathcal{S}(P_{\gamma^{(k)}}) = \widetilde{\mathfrak{S}}_n$, and $\widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \mathcal{S}(P_{\gamma^{(k)}}) = \widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \widetilde{\mathfrak{S}}_n = \widetilde{G}^{\mathrm{II}}_n$. Take $\Lambda^n = (\lambda^{n,\zeta})_{\zeta\in\widehat{T}^0}$. Then $\Pi^{\mathrm{II}}_{\Lambda^n} = \big(P_{\gamma^{(k)}} \cdot \nabla^{\mathrm{II}}_n\big) \boxdot \pi_{\Lambda^n}$. As its special case for $k=0$ and $\Lambda^n = (n)$, we have a simple spin IR of $\widetilde{G}^{\mathrm{II}}_n$

$$\Pi^{\mathrm{II}}_0 := P_0 \cdot \nabla^{\mathrm{II}}_n, \quad P_0 = P_{\gamma^{(0)}}. \tag{12.22}$$

In case n is odd, $\mathcal{S}(P^+_{\gamma^{(k)}}) = \widetilde{\mathfrak{A}}_n$, and $\widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \mathcal{S}(P^+_{\gamma^{(k)}}) = \widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \widetilde{\mathfrak{A}}_n =: \widetilde{H}^{\mathrm{II}}_n$, which is a normal subgroup of $\widetilde{G}^{\mathrm{II}}_n$ of index 2. Take $\Lambda^n = (\lambda^{n,\zeta})_{\zeta\in\widehat{T}^0}$ such that ${}^t\Lambda^n \neq \Lambda^n$ and put $\rho_{\Lambda^n} = \pi_{\Lambda^n}\big|_{\widetilde{\mathfrak{A}}_n}$. Then

$$\Pi^{\mho+}_{\Lambda^n} = \mathrm{Ind}^{\widetilde{G}^{\mathrm{II}}_n}_{\widetilde{H}^{\mathrm{II}}_n} \pi^{\mho+}_{\Lambda^n} = \mathrm{Ind}^{\widetilde{G}^{\mathrm{II}}_n}_{\widetilde{H}^{\mathrm{II}}_n} \big(P^+_{\gamma^{(k)}} \cdot \mho^+_n\big) \boxdot \rho_{\Lambda^n}.$$

As its special case for $k=0$ and $\Lambda^n = (n)$, we have simple spin IRs as

$$\begin{aligned} &\Pi^{\mathrm{II}}_+ := \mathrm{Ind}^{\widetilde{G}^{\mathrm{II}}_n}_{\widetilde{H}^{\mathrm{II}}_n} \Pi^{\mathrm{II},\widetilde{H}}_+ && \text{of } \widetilde{G}^{\mathrm{II}}_n, \\ &\Pi^{\mathrm{II},\widetilde{H}}_+ := P_+ \cdot \mho^+_n \ \text{ with } P_+ = P^+_{\gamma^{(0)}} && \text{of } \widetilde{H}^{\mathrm{II}}_n = \widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \widetilde{\mathfrak{A}}_n. \end{aligned} \tag{12.23}$$

Spin IRs in (12.22)–(12.23) will play important roles in studying spin irreducible characters.

Example 12.3. Let n be odd. For $\Lambda^n \in \boldsymbol{Y}^{\mathfrak{A}}_n(T)^{0,3}$, we start with $\gamma = (\gamma_1,\ldots,\gamma_n)$ such that γ_j's are all different ($\therefore\ n \leq m' = m/2$), and so $\mathcal{S} =$

$\mathcal{S}(P_\gamma^+) = Z_1 = \langle z_1\rangle$. Then $\mho_n^+$ is the spin character $\chi_1(z_1) = -1$ of Z_1, and ρ_{Λ^n} is the trivial representation $\mathbf{1}$, and

$$\begin{aligned}&\big(P_\gamma^+ \cdot \mho_n^+\big)\big((d', {z_1}^a)\big) = P_\gamma^+(d') \cdot (-1)^a \qquad (a = 0, 1)\ ;\\ &\Pi_{\Lambda^n}^{\mho+} = \mathrm{Ind}_{\widetilde{D}_n \times Z_1}^{\widetilde{G}^{\mathrm{II}}}\big(P_\gamma^+ \cdot \chi_1\big) \qquad \big(\text{IR of } \widetilde{G}_n^{\mathrm{II}}\big).\end{aligned}$$

Part III

Spin irreducible characters of $R\big(G(m,1,n)\big)$ of Types $(-1,-1,\pm 1)$

13 Conjugacies in $\mathfrak{S}_n$ and $\mathfrak{A}_n$, and in $\widetilde{\mathfrak{S}}_n$ and $\widetilde{\mathfrak{A}}_n$

13.1 Standard elements in $\mathfrak{S}_n$ and in $\widetilde{\mathfrak{S}}_n$

To give characters of spin representations of permutation groups, we should study conjugacy classes in $\widetilde{\mathfrak{S}}_n$ and $\widetilde{\mathfrak{A}}_n$, together with those in the base groups $\mathfrak{S}_n$ and $\mathfrak{A}_n$.

Conjugacy class of $\sigma \in \mathfrak{S}_n$ is denoted by $[\sigma]_{\mathfrak{S}}$ and that under conjugations of $\mathfrak{A}_n$ is denoted by $[\sigma]_{\mathfrak{A}}$. Conjugacy class $[\sigma]_{\mathfrak{S}}$ is determined by the type of its decomposition into disjoint cycles. This means the following. Let $\sigma = \sigma_1 \cdots \sigma_s$ be a cycle decomposition such that $\ell_1 \geq \ell_2 \geq \ldots \geq \ell_s > 1$ with $\ell_j = \ell(\sigma_j)$. Add $\ell_{s+1} = \ldots = \ell_t = 1$ if necessary, so that we have a partition of n as

$$\sum_{1 \leq j \leq t} \ell_j = n, \quad \ell_1 \geq \ell_2 \geq \ldots \geq \ell_t \geq 1. \tag{13.1}$$

Put $m_k := \sharp\{j\,;\,\ell_j = k\}$, then $1^{m_1}2^{m_2}\cdots n^{m_n}$ is called the type of σ or of the partition above.

As a representative of the conjugacy class of type $1^{m_1}2^{m_2}\cdots n^{m_n}$, we define a *standard* element (or permutation) in $\mathfrak{S}_n$, and also a corresponding *standard* element in the covering group $\widetilde{\mathfrak{S}}_n = \mathfrak{T}'_n$ as follows.

Definition 13.1. Put $N_0 = 0,\ N_1 = \ell_1,\ N_k = \ell_1 + \cdots + \ell_k\ (1 \leq k \leq t)$, and

$$\begin{aligned}&\sigma_1 = s_1 s_2 \cdots s_{N_1 - 1},\ \ \sigma_k = s_{N_{k-1}+1} \cdots s_{N_k - 1}\ (1 \leq k \leq t);\\ &\sigma'_1 = r_1 r_2 \cdots r_{N_1 - 1},\ \ \sigma'_k = r_{N_{k-1}+1} \cdots r_{N_k - 1}\ (1 \leq k \leq t),\end{aligned} \tag{13.2}$$

where the elements σ_k and σ'_k for $k > s$ mean the identity elements in $\mathfrak{S}_n$ and $\widetilde{\mathfrak{S}}_n$ respectively. We call $\sigma = \sigma_1\sigma_2\cdots\sigma_t$ and $\sigma' = \sigma'_1\sigma'_2\cdots\sigma'_t$ *standard* elements of type $1^{m_1}2^{m_2}\cdots n^{m_n}$ in $\mathfrak{S}_n$ and $\widetilde{\mathfrak{S}}_n$ respectively.

Note that in the expression of σ' the order of product is essential, since $\sigma'_j\sigma'_k = z_1^{L_jL_k}\sigma'_k\sigma'_j$, $L_j = \ell_j - 1$, for $j \neq k$.

13.2 First, second and third kind of elements

In this and the next subsection, the main reference is [Sch3, Abschnitt II]. After Schur, we define the first, the second and the third kind of elements (or permutations) in $\mathfrak{S}_n$, and transfer this definition to the covering group $\widetilde{\mathfrak{S}}_n$.

Definition 13.2. For $\sigma \in \mathfrak{S}_n$, let $\sigma = \sigma_1\sigma_2\cdots\sigma_t$ be a cycle decomposition of σ, and $\ell_i = \ell(\sigma_i)$ length of σ_i, and $1^{m_1}2^{m_2}\cdots n^{m_n}$ type of σ.

(1) σ is called *of first kind*, if either σ is even and contains at least one odd cycle σ_i (or ℓ_i even), or σ is odd and contains at least one pair of cycles of the same lengths;

(2) σ is called *of second kind*, if it is not of the first kind, that is, if either σ is even and contains only even cycles (or all ℓ_i odd), or σ is odd and lengths of cycles are all different;

(3) σ is called *of third kind*, if σ is even and lengths of cycles are all different, so that the number of the trivial cycle (= cycle of length 1) should be ≤ 1.

An element $\sigma' \in \widetilde{\mathfrak{S}}_n$ is called *of first kind*, *of second kind*, and *of third kind*, if so is the image $\sigma = \Phi_{\mathfrak{S}}(\sigma') \in \mathfrak{S}_n$ respectively.

Lemma 13.1. *Let $n \geq 4$. For $\sigma \in \mathfrak{A}_n$, let $\sigma = \sigma_1\sigma_2\cdots\sigma_t$ be its cycle decomposition. Then its conjugacy class $[\sigma]_{\mathfrak{S}}$ in $\mathfrak{S}_n$ splits into two conjugacy classes in $\mathfrak{A}_n$ if and only if σ is of 2nd kind and 3rd kind at the same time, that is, all σ_i's are even and with different lengths. In that case, for σ standard,*

$$[\sigma]_{\mathfrak{S}} \;=\; [\sigma]_{\mathfrak{A}} \sqcup [s_1\sigma s_1^{-1}]_{\mathfrak{A}}\,. \tag{13.3}$$

Note that this has been given in the study of characters of $\mathfrak{A}_n$ by Frobenius [Frob2].

13.3 Conjugacies in $\widetilde{\mathfrak{S}}_n$ and in $\widetilde{\mathfrak{A}}_n$ and central functions

For $\sigma' \in \widetilde{\mathfrak{S}}_n$, denote by $[\sigma']_{\widetilde{\mathfrak{S}}}$ its conjugacy class under $\widetilde{\mathfrak{S}}_n$, and by $[\sigma']_{\widetilde{\mathfrak{A}}}$ its conjugacy class under $\widetilde{\mathfrak{A}}_n$. For $\sigma' \in \widetilde{\mathfrak{A}}_n$ standard, $[\sigma']_{\widetilde{\mathfrak{S}}} = [\sigma']_{\widetilde{\mathfrak{A}}} \cup [r_1\sigma' r_1^{-1}]_{\widetilde{\mathfrak{A}}}$, and we ask relations among 4 conjugacy classes $[\sigma']_{\widetilde{\mathfrak{A}}}$, $[r_1\sigma' r_1^{-1}]_{\widetilde{\mathfrak{A}}}$, $[z_1\sigma']_{\widetilde{\mathfrak{A}}}$ and $[z_1r_1\sigma' r_1^{-1}]_{\widetilde{\mathfrak{A}}}$.

Theorem 13.2. *Let $n \geq 4$, and $\sigma' \in \widetilde{\mathfrak{S}}_n$. Under conjugations of $\widetilde{\mathfrak{S}}_n$, σ' is conjugate to $z_1\sigma'$ or not, according as σ' is of 1st kind or of 2nd kind respectively.*

Theorem 13.3. *Let $n \geq 4$, and $\sigma' \in \widetilde{\mathfrak{A}}_n$ be standard.*

(i) *If σ' is of 1st kind and not of 3rd kind, then σ' is conjugate to $z_1\sigma'$ under $\widetilde{\mathfrak{A}}_n$:*

$$[\sigma]_{\mathfrak{A}} = [s_1\sigma s_1^{-1}]_{\mathfrak{A}}\,;$$
$$[\sigma']_{\widetilde{\mathfrak{S}}} = [z_1\sigma']_{\widetilde{\mathfrak{S}}}\,,$$
$$[\sigma']_{\widetilde{\mathfrak{A}}} = [z_1\sigma']_{\widetilde{\mathfrak{A}}} = [r_1\sigma' r_1^{-1}]_{\widetilde{\mathfrak{A}}} = [z_1r_1\sigma' r_1^{-1}]_{\widetilde{\mathfrak{A}}}\,.$$

(ii) *If σ' is of 1st kind and also of 3rd kind, then σ' is conjugate to $z_1\sigma'$ under $\widetilde{\mathfrak{S}}_n$, but not under $\widetilde{\mathfrak{A}}_n$, and $[\sigma']_{\widetilde{\mathfrak{S}}}$ does not split under $\widetilde{\mathfrak{A}}_n$:*

$$[\sigma]_{\mathfrak{A}} = [s_1\sigma s_1^{-1}]_{\mathfrak{A}}\,;$$
$$[\sigma']_{\widetilde{\mathfrak{S}}} = [z_1\sigma']_{\widetilde{\mathfrak{S}}}\,,$$
$$[\sigma']_{\widetilde{\mathfrak{A}}} \neq [z_1\sigma']_{\widetilde{\mathfrak{A}}}\,, \quad [r_1\sigma' r_1^{-1}]_{\widetilde{\mathfrak{A}}} \neq [z_1r_1\sigma' r_1^{-1}]_{\widetilde{\mathfrak{A}}}\,, \quad [\sigma']_{\widetilde{\mathfrak{A}}} = [z_1r_1\sigma' r_1^{-1}]_{\widetilde{\mathfrak{A}}}\,.$$

(iii) *If σ' is of 2nd kind and not of 3rd kind, then σ' is not conjugate to $z_1\sigma'$ under $\widetilde{\mathfrak{S}}_n$, and $[\sigma']_{\widetilde{\mathfrak{S}}}$ does not split under $\widetilde{\mathfrak{A}}_n$:*

$$[\sigma]_{\mathfrak{A}} = [s_1\sigma s_1^{-1}]_{\mathfrak{A}}\,;$$
$$[\sigma']_{\widetilde{\mathfrak{S}}} \neq [z_1\sigma']_{\widetilde{\mathfrak{S}}}\,,$$
$$[\sigma']_{\widetilde{\mathfrak{A}}} = [r_1\sigma' r_1^{-1}]_{\widetilde{\mathfrak{A}}} \neq [z_1\sigma']_{\widetilde{\mathfrak{A}}} = [z_1r_1\sigma' r_1^{-1}]_{\widetilde{\mathfrak{A}}}\,.$$

(iv) *If σ' is of 2nd kind and also of 3rd kind, then σ' is not conjugate to $z_1\sigma'$ under $\widetilde{\mathfrak{S}}_n$ and $[\sigma']_{\widetilde{\mathfrak{S}}}$ splits into two conjugacy classes under $\widetilde{\mathfrak{A}}_n$:*

$$[\sigma]_{\mathfrak{A}} \neq [s_1\sigma s_1^{-1}]_{\mathfrak{A}}\,;$$
$$[\sigma']_{\widetilde{\mathfrak{S}}} \neq [z_1\sigma']_{\widetilde{\mathfrak{S}}}\,,$$
$$[\sigma']_{\widetilde{\mathfrak{S}}} = [\sigma']_{\widetilde{\mathfrak{A}}} \sqcup [r_1\sigma' r_1^{-1}]_{\widetilde{\mathfrak{A}}}\,,$$
$$[\sigma']_{\widetilde{\mathfrak{A}}}\,, \ [z_1\sigma']_{\widetilde{\mathfrak{A}}}\,, \ [r_1\sigma' r_1^{-1}]_{\widetilde{\mathfrak{A}}}\,, \ [z_1r_1\sigma' r_1^{-1}]_{\widetilde{\mathfrak{A}}} \ \textit{(4 are different).}$$

Theorem 13.4. *A complete set of representatives of conjugacy classes in $\widetilde{\mathfrak{S}}_n$ is given as follows: denote a standard element of types $1^{m_1}2^{m_2}\cdots n^{m_n}$ simply by σ', then*

$$\begin{cases} \sigma' & \text{of 1st kind,} \\ \sigma',\ z_1\sigma' & \text{of 2nd kind.} \end{cases}$$

Theorem 13.5. *A complete set of representatives of conjugacy classes in $\widetilde{\mathfrak{A}}_n$ is given as follows: denote a standard element of types $1^{m_1}2^{m_2}\cdots n^{m_n}$ simply by σ', then*

$$\begin{cases} \sigma' & \text{of 1st kind and not of 3rd kind,} \\ \sigma',\ z_1\sigma' \ (\text{or } \ r_1\sigma' r_1^{-1}) & \text{of 1st kind and also of 3rd kind,} \\ \sigma',\ z_1\sigma' \ (\text{or } \ z_1r_1\sigma' r_1^{-1}) & \text{of 2nd kind and not of 3rd kind,} \\ \sigma',\ z_1\sigma',\ r_1\sigma' r_1^{-1},\ z_1r_1\sigma' r_1^{-1} & \text{of 2nd kind and also of 3rd kind.} \end{cases}$$

A function f on $\widetilde{\mathfrak{S}}_n$ or on $\widetilde{\mathfrak{A}}_n$ is called *spin* if $f(z_1\sigma') = -f(\sigma')$.

Theorem 13.6. *Let f be a spin central function on $\widetilde{\mathfrak{S}}_n$. Then $f(\sigma')=0$ if σ' is of the 1st kind. Moreover f is completely determined by the values $f(\sigma')$ for every standard elements σ' of the 2nd kind.*

Theorem 13.7. *Let f be a spin central function on $\widetilde{\mathfrak{A}}_n$. Then $f(\sigma')=0$ if σ' is of the 1st kind and not of the 3rd kind. Moreover f is completely determined by*

(1) the values $f(\sigma')$ for every standard elements σ' either of the 1st and the 3rd kind or of the 2nd and not the 3rd kind, and

(2) the values $f(\sigma')$, $f(r_1\sigma' r_1^{-1})$ for every standard elements σ' of the 2nd and the 3rd kind.

14 Characters of spin representations of $\mathfrak{S}_n$ and $\mathfrak{A}_n$

14.1 Traces of spin representations ∇_n and ∇_n^- of $\mathfrak{S}_n$

Let $4 \le n < \infty$. We defined in (8.2) and (8.3), spin representations ∇_n and ∇_n^- as follows: put $Y_j' = (-1)^{j-1}Y_j$ $(j \in \boldsymbol{I}_n)$, then

$$(14.1) \qquad \nabla_n(r_i) \;:=\; \frac{(-1)^{j-1}}{\sqrt{2}}\,(Y_i+Y_{i+1}) = \frac{1}{\sqrt{2}}\,(Y_i'-Y_{i+1}') \quad (i \in \boldsymbol{I}_{n-1})\;;$$

and for $n=2n'+1$ odd,

$$(14.2) \qquad \begin{cases} \nabla_n^-(r_i) := \nabla_n(r_i) & (i \in \boldsymbol{I}_{n-2}), \\ \nabla_n^-(r_{n-1}) := -Y_n\nabla_n(r_{n-1})Y_n^{-1} = \dfrac{(-1)^{n-1}}{\sqrt{2}}\,(Y_{n-1}-Y_n). \end{cases}$$

Since the characters χ_{∇_n} and $\chi_{\nabla_n^-}$ are invariant under $\widetilde{\mathfrak{S}}_n$, by Theorem 13.6, it is sufficient for us to calculate the trace $\mathrm{tr}\big(\nabla_n(\sigma')\big), \mathrm{tr}\big(\nabla_n^-(\sigma')\big)$ for standard σ' of 2nd kind. Let $\sigma' \in \widetilde{\mathfrak{S}}_n$ and $\sigma = \Phi(\sigma') \in \mathfrak{S}_n$ be standard elements in Definition 13.1, but hereafter we omit the trivial factors $\sigma_{s+1},\ldots,\sigma_t$ and $\sigma'_{s+1},\ldots,\sigma'_t$ added in (13.2) in case $s<t$ as

$$(14.3) \qquad \begin{cases} \sigma = \sigma_1\sigma_2\cdots\sigma_s\,, \\ \sigma' = \sigma_1'\sigma_2'\cdots\sigma_s'\,, \end{cases} \qquad \sigma_j = \Phi(\sigma_j'), \quad \ell_1 \ge \ell_2 \ge \ldots \ge \ell_s > 1,$$

$$(14.4) \qquad \ell_k := \ell(\sigma_k), \;\; \ell(\sigma_k) = N_k - N_{k-1} \ge 2 \;\; (k \in \boldsymbol{I}_s).$$

- **Trace of spin representation ∇_n of $\mathfrak{S}_n$:**

  The matrix corresponding to σ' is expressed as

$$(14.5) \qquad \nabla_n(\sigma') \;=\; \nabla_n(\sigma_1')\cdots\nabla_n(\sigma_s'),$$

$$\begin{aligned}\nabla_n(\sigma'_k) &= \nabla_n\big(r_{N_{k-1}}\big)\cdots\nabla_n\big(r_{N_k-2}\big)\\ &= \frac{1}{2^{(\ell_k-1)/2}}\prod_{N_{k-1}+1\le i\le N_k-1}(-1)^{i-1}(Y_i+Y_{i+1}),\end{aligned} \tag{14.6}$$

where the product is taken in the natural order of the index i.

In addition to Lemmas 5.9 – 5.10, the following lemma will be used repeatedly in calculating traces.

Lemma 14.1. (i) *A monomial product* $Y_{k_1}Y_{k_2}\cdots Y_{k_{2n'}}Y_{k_{2n'+1}}$ *such that* $\{k_1,k_2,\dots,k_{2n'+1}\}=\boldsymbol{I}_{2n'+1}$ *has non-zero trace given as*

$$\mathrm{tr}\big(Y_{k_1}Y_{k_2}\cdots Y_{k_{2n'}}Y_{k_{2n'+1}}\big)=(2i)^{n'}\mathrm{sgn}(\sigma), \tag{14.7}$$

where $\sigma\in\mathfrak{S}_{2n'+1}$ *is such that* $\sigma(i)=k_i$ $(i\in\boldsymbol{I}_{2n'+1})$.

(ii) *For a formal monomial* $F=Y_1^{a_1}Y_2^{a_2}\cdots Y_n^{a_n}$, *put* $\mathrm{supp}(F):=\{j\in\boldsymbol{I}_n\,;\,a_j\not\equiv 0\ (\mathrm{mod}\ 2)\}$. *For two such monomials* F_1 *and* F_2, *suppose* $\mathrm{supp}(F_1)\cap\mathrm{supp}(F_2)=\emptyset$, *and* $\mathrm{supp}(F_1)\cup\mathrm{supp}(F_2)\subset\boldsymbol{I}_{2n'}\subsetneq\boldsymbol{I}_{2n'+1}$, *then*

$$\mathrm{tr}(F_1F_2)=2^{n'}\cdot\frac{\mathrm{tr}(F_1)}{2^{n'}}\cdot\frac{\mathrm{tr}(F_2)}{2^{n'}} \tag{14.8}$$

Proof. (ii) Let $F_1=Y_1^{a_1}Y_2^{a_2}\cdots Y_n^{a_n}$ and $F_2=Y_1^{b_1}Y_2^{b_2}\cdots Y_n^{b_n}$, then $a_jb_j=0$ for all j, and $F_1F_2=\pm Y_1^{c_1}Y_2^{c_2}\cdots Y_n^{c_n}$ with $c_j=a_j+b_j$. If $\mathrm{tr}(F_1F_2)\neq 0$, then $c_j\equiv 0\ (\mathrm{mod}\ 2)$ for all $j\in\boldsymbol{I}_n$, since $\mathrm{supp}(F_1F_2)\neq\boldsymbol{I}_{2n'+1}$. This means that $a_j\equiv 0\ (\mathrm{mod}\ 2)$ and $b_j\equiv 0\ (\mathrm{mod}\ 2)$ for $j\in\boldsymbol{I}_n$. Then, actually as matrices, $F_1=E_{2^{n'}}$, $F_2=E_{2^{n'}}$.

If $\mathrm{tr}(F_1F_2)=0$, then $c_j\equiv 1\ (\mathrm{mod}\ 2)$ for at least one of $j\in\boldsymbol{I}_n$. For that j, a_j or b_j is odd, and accordingly $\mathrm{tr}(F_1)=0$ or $\mathrm{tr}(F_2)=0$. □

CASE 1: when $Y_{2n'+1}$ does not appear in (14.5) – (14.6) :

Note that when $\nabla_n(\sigma'_j)$'s are expanded into linear combinations of monomials of the form $\prod_{p\in K_j}Y_p^{c_p}$, $K_j:=\mathrm{supp}(\sigma'_j)$, their supports are mutually disjoint for $j\in\boldsymbol{I}_s$, since so are K_j's, and that, when $\mathrm{tr}\big(\nabla_n(\sigma'_j)\big)\neq 0$, its monomial term with non-zero trace is of the form $\prod_{p\in K_j}Y_p^{2c'_p}$ and is unique among its monomials (to see this, we appeal to the explicit form of $\nabla(r_i)$ in (14.1)). Then we see from Lemmas 5.9 – 5.10 and 14.1 the following:

(1-1) $\qquad \mathrm{tr}\big(\nabla_n(\sigma')\big)=2^{n'}\cdot\dfrac{\mathrm{tr}\big(\nabla_n(\sigma'_1)\big)}{2^{n'}}\cdots\dfrac{\mathrm{tr}\big(\nabla_n(\sigma'_s)\big)}{2^{n'}}\ ;$

(1-2) $\qquad \mathrm{tr}\big(\nabla_n(\sigma'_k)\big)\neq 0\implies L(\sigma'_k)\equiv 0\ (\mathrm{mod}\ 2).$

Lemma 14.2. *Suppose that* $Y_{2n'+1}$ *does not appear in (14.5) – (14.6). Then, for a standard* $\sigma'=\sigma'_1\cdots\sigma'_s$ *in (14.3),* $L(\sigma')=\sum_{k\in\boldsymbol{I}_s}L(\sigma'_k)$, *and*

$$\mathrm{tr}\big(\nabla_n(\sigma')\big)=\begin{cases}2^{n'}(-2)^{-\sum_{k\in\boldsymbol{I}_s}(\ell_k-1)/2}, & \text{if } L(\sigma'_k)\equiv\ell_k-1\equiv 0\ (\text{mod } 2,\ \forall k\in\boldsymbol{I}_s),\\ 0\,, & \text{otherwise.}\end{cases}$$

CASE 2: when $Y_{2n'+1}$ appears in (14.5) – (14.6) :

From the setting, we have $|\mathrm{supp}(\sigma)| = n = 2n'+1$ in this case if $\mathrm{tr}\big(\nabla_n(\sigma')\big) \neq 0$. By Lemmas 5.9 – 5.10 and 14.1, as monomials with non-zero traces, there are two cases:

Case (2-1): $Y_{k_1}Y_{k_2}\cdots Y_{k_{2n'}}Y_{k_{2n'+1}}, \ \{k_1, k_2, \ldots, k_{2n'+1}\} = \boldsymbol{I}_{2n'+1}$;

Case (2-2): $Y_{q_1}^2 Y_{q_2}^2 \cdots Y_{q_u}^2$.

For Case (2-1), from the degree of the monomial, we see that if $\mathrm{tr}\big(\nabla_n(\sigma')\big) \neq 0$, then $\sum_k(\ell_k - 1)$ should be equal to $n = 2n'+1$. But this is impossible.

For Case (2-2), similarly as in CASE 1, all σ_k should be even, and we have the following.

Lemma 14.3. *Suppose that $Y_{2n'+1}$ appears in (14.5) – (14.6). Then we have $n = 2n'+1$ odd and $|\mathrm{supp}(\sigma)| = n = 2n'+1$. For a standard $\sigma' = \sigma'_1\cdots\sigma'_s$ in (14.3),*

$$\mathrm{tr}\big(\nabla_n(\sigma')\big) = \begin{cases} 2^{n'}\cdot(-2)^{-\sum_{k\in\boldsymbol{I}_s}(\ell_k-1)/2}, & \text{if } L(\sigma'_k) \equiv 0 \ (\mathrm{mod}\ 2,\ \forall k \in \boldsymbol{I}_s), \\ 0\,, & \text{otherwise.} \end{cases}$$

Thus, from Lemmas 14.2 and 14.3, we obtain for the trace of ∇_n the following:

Theorem 14.4. *Let $n \geq 4$ and put $n' = [n/2]$. For a standard element $\sigma' = \sigma'_1\cdots\sigma'_s \in \widetilde{\mathfrak{S}}_n$ in (13.2),*

$$\mathrm{tr}\big(\nabla_n(\sigma')\big) = \begin{cases} 2^{[n/2]}\cdot(-2)^{-\sum_{k\in\boldsymbol{I}_s}(\ell_k-1)/2} = 2^{n'}\cdot\prod_{k\in\boldsymbol{I}_s}(-2)^{-(\ell_k-1)/2}, \\ \qquad\qquad \text{if } L(\sigma'_k) \equiv 0 \ (\mathrm{mod}\ 2) \text{ for all } k \in \boldsymbol{I}_s, \\ 0\,, \qquad\qquad \text{otherwise.} \end{cases}$$

As a corollary, we have the following. According to Definition 8.1, for a spin character χ of $\mathfrak{S}_n$ the *associate character* is $\mathrm{sgn}\cdot\chi$, and χ is called *self-associate* if $\mathrm{sgn}\cdot\chi = \chi$. Then the character χ_{∇_n} is self-associate, or $\mathrm{sgn}\cdot\chi_{\nabla_n} = \chi_{\nabla_n}$.

Corollary 14.5. *The product of the representation ∇_n and the one-dimensional character* sgn *is again equivalent to ∇_n, or $\mathrm{sgn}\cdot\nabla_n \cong \nabla_n$, that is, the spin representation ∇_n is self-associate.*

- **Trace of spin representation ∇_n^- of $\mathfrak{S}_n$:**

From the formula (14.1) – (14.2), we see that

$$-\,Y_n\,\nabla_n(r_i)\,Y_n^{-1} = \nabla_n^-(r_i) \qquad (1 \leq i \leq n-1). \tag{14.9}$$

This gives us the following result.

Theorem 14.6. *The spin representation ∇_n^- is equivalent to $\mathrm{sgn}\cdot\nabla_n$. Moreover it is also equivalent to ∇_n since $\mathrm{sgn}\cdot\nabla_n \cong \nabla_n$, so that*

$$\nabla_n^- \cong \nabla_n \quad \text{and} \quad \chi_{\nabla_n^-} = \chi_{\nabla_n}\,.$$

14.2 Traces of spin representations ∇'_n and ∇^{II}_n

14.2.1 Trace of spin representation ∇'_n of $\mathfrak{S}_n$

Let the notation be as in Definition 13.1 and (14.3) – (14.4). Then, under the spin representation ∇'_n, the matrix corresponding to a standard σ' is expressed as

$$\begin{aligned} \nabla'_n(\sigma') &= \nabla'_n(\sigma'_1)\cdots\nabla'_n(\sigma'_s), \\ \nabla'_n(\sigma'_k) &= \frac{1}{2^{(\ell_k-1)/2}} \prod_{N_{k-1}+1\le i\le N_k-1} (Y_i - Y_{i+1}), \end{aligned} \tag{14.10}$$

where the product is taken in the natural order.

CASE 1: when $Y_{2n'+1}$ does not appear in (14.10):
Then we see that

(1) $\qquad \mathrm{tr}\big(\nabla'_n(\sigma')\big) = 2^{n'}\cdot\dfrac{\mathrm{tr}\big(\nabla'_n(\sigma'_1)\big)}{2^{n'}}\cdots\dfrac{\mathrm{tr}\big(\nabla'_n(\sigma'_s)\big)}{2^{n'}}\;;$

(2) $\qquad \mathrm{tr}\big(\nabla'_n(\sigma'_k)\big)\neq 0 \implies L(\sigma_k)\equiv \ell_k-1\equiv 0 \pmod 2.$

Lemma 14.7. *Suppose that $Y_{2n'+1}$ does not appear in (14.10), or equivalently that we are either in the case of even $n=2n'$, or in the case of odd $n=2n'+1$ and $|\mathrm{supp}(\sigma)|<n=2n'+1$. Then, for a standard $\sigma'=\sigma'_1\cdots\sigma'_s$ in (14.3),*

$$\mathrm{tr}\big(\nabla'_n(\sigma')\big) = \begin{cases} 2^{n'}\cdot(-2)^{-\sum_{k\in \boldsymbol{I}_s}(\ell_k-1)/2}, & \text{if } L(\sigma'_k)\equiv 0 \ (\mathrm{mod}\ 2,\ \forall k\in \boldsymbol{I}_s), \\ 0\,, & \text{otherwise.} \end{cases}$$

CASE 2: when $Y_{2n'+1}$ appears in (14.10):
We can discuss similarly as for ∇_n, and obtain the following.

Lemma 14.8. *Suppose that $Y_{2n'+1}$ appears in (14.10). If $\mathrm{tr}\big(\nabla'_n(\sigma')\big)\neq 0$, then $n=2n'+1$ and $|\mathrm{supp}(\sigma)|=n=2n'+1$. For a standard $\sigma'=\sigma'_1\cdots\sigma'_s$ in (14.3),*

$$\mathrm{tr}\big(\nabla'_n(\sigma')\big) = \begin{cases} 2^{n'}\cdot(-2)^{-\sum_{k\in \boldsymbol{I}_s}(\ell_k-1)/2}, & \text{if } L(\sigma'_k)\equiv 0 \ (\mathrm{mod}\ 2,\ \forall k\in \boldsymbol{I}_s), \\ 0\,, & \text{otherwise.} \end{cases}$$

Thus, from Lemmas 14.7 and 14.8, we obtain for the trace of ∇'_n the following:

Theorem 14.9. *Let $n\ge r$ and put $n'=[n/2]$.*
(i) *For a standard element $\sigma'=\sigma'_1\cdots\sigma'_s\in\widetilde{\mathfrak{S}}_n$ in (14.3),*

$$\mathrm{tr}\big(\nabla'_n(\sigma')\big) = \begin{cases} 2^{[n/2]}\cdot(-2)^{-\sum_{k\in \boldsymbol{I}_s}(\ell_k-1)/2} = 2^{n'}\cdot\prod_{k\in\boldsymbol{I}_s}(-2)^{-(\ell_k-1)/2}, \\ \qquad\qquad \text{if } L(\sigma'_k)\equiv 0 \ (\mathrm{mod}\ 2) \text{ for all } k\in \boldsymbol{I}_s, \\ 0\,, \qquad\qquad \text{otherwise.} \end{cases}$$

(ii) *There hold equivalencies $\nabla'_n\cong\nabla_n\cong\nabla^-_n$, and ∇'_n is self-associate.*

14.2.2 Trace of spin representation ∇_n^{II} of $\mathfrak{S}_n$, $n=2n'$

Recall that, for $n=2n'$, we defined in (8.6) operators $\nabla_n^{\mathrm{II}}(r_j)$ for $j \in \boldsymbol{I}_{n-1}$, and proved that they give a spin representation of $\mathfrak{S}_n$:

$$\nabla_n^{\mathrm{II}}(r_j) = (iY_{2n'+1}) \cdot \nabla'_n(r_j) = \nabla'_n(r_j) \cdot (-iY_{2n'+1}). \tag{14.11}$$

For $\sigma' = r_{j_1} r_{j_2} \cdots r_{j_L}$, $j_1 < j_2 < \ldots < j_L < 2n'$,

$$\begin{aligned} \nabla_n^{\mathrm{II}}(\sigma') &= \nabla'_n(\sigma') = \nabla'_n(r_{j_1})\nabla'_n(r_{j_2}) \cdots \nabla'_n(r_{j_L}) & \text{if } L(\sigma') \equiv 0 \pmod 2, \\ \nabla_n^{\mathrm{II}}(\sigma') &= \nabla'_n(\sigma')\,(-iY_{2n'+1}) & \text{if } L(\sigma') \equiv 1 \pmod 2. \end{aligned}$$

Theorem 14.10. *For $\widetilde{\mathfrak{S}}_n$, $n=2n'$, the characters $\chi_{\nabla_n^{\mathrm{II}}}$ and $\chi_{\nabla'_n}$ of ∇_n^{II} and ∇'_n coincide with each other, and so these spin representations are mutually equivalent.*

Proof. Suppose $L(\sigma') \equiv 0 \pmod 2$. Then $\nabla_n^{\mathrm{II}}(\sigma') = \nabla'_n(\sigma')$.

Suppose $L(\sigma') \equiv 1 \pmod 2$. Then $\nabla_n^{\mathrm{II}}(\sigma') = (iY_{2n'+1}) \cdot \nabla'_n(\sigma')$, and $\nabla'_n(\sigma')$ does not contain $Y_{2n'+1}$ and is a linear combination of monomial terms $Y_{k_1}Y_{k_2}\cdots$ of order $\leq 2n'-1$. Therefore $(iY_{2n'+1})\,\nabla'_n(\sigma')$ cannot contain $Y_{2n'+1}Y_{q_1}Y_{q_2}\cdots$ of degree $2n'+1$, whence every homogeneous terms in it are of trace 0, and in total the trace of $\nabla_n^{\mathrm{II}}(\sigma')$ is zero. □

Note 14.1. As will be seen in Theorem 15.5 (i), when n even, $\nabla'_n \cong \mathrm{sgn}\cdot\nabla'_n$ is a direct sum of inequivalent spin IRs Δ'_n and $\mathrm{sgn}\cdot\Delta'_n$: $\nabla'_n \cong \Delta'_n \oplus \mathrm{sgn}\cdot\Delta'_n$.

14.3 Traces of spin representations $\mho_n^{\pm}$ of $\mathfrak{A}_n$

Let $n = 2n'+1$ odd. Spin representations $\mho_n^{\pm}$ of $\mathfrak{A}_n$ are defined by the formula (12.3). Then, for $v' = z_1{}^b\sigma'$, $\sigma' = r_{i_1} r_{i_2} \cdots r_{i_L} \in \widetilde{\mathfrak{A}}_n$, $L(\sigma') \equiv L$ even,

$$\begin{aligned} \mho_n^{+}(v') &= \nabla'_n(v') = (-1)^b \cdot \nabla'_n(\sigma')\,; \\ \mho_n^{-}(v') &= (iY_{2n'+1}) \cdot \nabla'_n(v') \cdot (iY_{2n'+1})^{-1}. \end{aligned} \tag{14.12}$$

Thus $\mho_n^{\pm}$ are restrictions of spin representations, equivalent to ∇'_n, of the group $\widetilde{\mathfrak{S}}_n$ to the subgroup $\widetilde{\mathfrak{A}}_n$. So their characters are invariant under $\widetilde{\mathfrak{S}}_n$. Therefore, in spite of Theorems 13.5 and 13.7 on a complete set of representatives of conjugacy classes in $\widetilde{\mathfrak{S}}_n$ and on central functions on it respectively, the characters $\chi_{\mho_n^{\pm}}$ are completely determined by their values on standard elements in $\widetilde{\mathfrak{A}}_n$.

Theorem 14.11. *The traces of the spin representations $\mho_n^{+}$ and $\mho_n^{-}$ are restrictions of the character $\chi_{\nabla'_n}$ of $\widetilde{\mathfrak{S}}_n$ onto $\widetilde{\mathfrak{A}}_n$.*

For $v' = z_1{}^b\sigma' \in \widetilde{\mathfrak{A}}_n$ *with a standard* σ'*,*

$$\mathrm{tr}\big(\mho_n^{\pm}(v')\big) = \begin{cases} (-1)^b \cdot 2^{n'} \cdot (-2)^{-\sum_{k\in \boldsymbol{I}_s}(\ell_k-1)/2}, & \\ \qquad \text{if } \sigma = \Phi_{\mathfrak{S}}(\sigma') \text{ has a cycle decomposition} \\ \qquad \sigma = \sigma_1 \cdots \sigma_s,\ L(\sigma_k) \equiv 0 \pmod 2\ \ (k \in \boldsymbol{I}_s); \\ 0\,, \qquad \text{otherwise.} \end{cases}$$

Note 14.2. As will be seen in Theorem 15.5 (ii), when $n = 2n' + 1$ odd, $\mho_n^+ \cong \mho_n^-$ is a direct sum of two inequivalent spin IRs $\mho_n^{(\kappa)},\ \kappa = 0, 1$, of $\widetilde{\mathfrak{A}}_n$, where $\mho_n^{(1)} = \big(\mho_n^{(0)}\big)^{r_1}$.

14.4 Behaviors of $\nabla_n^{\mathrm{II}+}$ and $\nabla_n^{\mathrm{II}-}$ on $\widetilde{\mathfrak{S}}_n,\ n = 2n' + 1$

For $n = 2n' + 1$, the matrix-valued functions $\nabla_n^{\mathrm{II}+}$ and $\nabla_n^{\mathrm{II}-}$ on $\widetilde{\mathfrak{S}}_n$, defined in (8.7), coincide with the spin representation $\nabla_{n-1}^{\mathrm{II}}$ on the subgroup $\widetilde{\mathfrak{S}}_{n-1} = \langle z_1, r_1, \ldots, r_{n-2}\rangle$, and also give equivalent spin representations $\mho_n^+$ and $\mho_n^-$ on the subgroup $\widetilde{\mathfrak{A}}_n$ by restriction. Consider them on the remainder subset $\widetilde{\mathfrak{S}}_n \setminus \big(\widetilde{\mathfrak{S}}_{n-1} \bigcup \widetilde{\mathfrak{A}}_n\big)$, then, since $n = 2n' + 1$, it is a union of $\widetilde{\mathfrak{A}}_{n-1}$-cosets as

$$\bigsqcup_{j\in \boldsymbol{I}_{n-1},\mathrm{even}} \widetilde{\mathfrak{A}}_{n-1} r_{n-1} r_{n-2} \cdots r_j \ \bigsqcup \ \bigsqcup_{j\in \boldsymbol{I}_{n-1},\mathrm{odd}} \widetilde{\mathfrak{A}}_{n-1} r_1 r_{n-1} r_{n-2} \cdots r_j\,.$$

Proposition 14.12. *For* $\sigma' \in \widetilde{\mathfrak{S}}_n \backslash \big(\widetilde{\mathfrak{S}}_{n-1} \bigcup \widetilde{\mathfrak{A}}\big)$*, there holds* $\mathrm{tr}\big(\nabla_n^{\mathrm{II}-}(\sigma')\big) = -\mathrm{tr}\big(\nabla_n^{\mathrm{II}+}(\sigma')\big)$*. In particular, for* $\sigma' = \sigma'' r_{n-1}$ *with a standard* $\sigma'' \in \widetilde{\mathfrak{A}}_{n-1}$*,*

$$\mathrm{tr}\big(\nabla_n^{\mathrm{II}\pm}(\sigma')\big) = \mathrm{tr}\big(\nabla_{n-1}^{\mathrm{II}}(\sigma'')\big) \cdot (\mp i/\sqrt{2}\,).$$

Proof. Under the conjugation of $\widetilde{\mathfrak{A}}_{n-1}$, σ' is conjugate, modulo a multiple of powers of z_1, to an element τ' such that $\tau' = \tau_1'\tau_2' \cdots \tau_s'$ with disjoint cycles, $\tau_k' = r_{N_{k-1}} r_{N_{k-1}+1} \cdots r_{N_k - 1}$ $(k < s)$ and $\tau_s' = r_p r_{p+1} \cdots r_{n-1}$. Put $\tau'' = \tau_1' \cdots \tau_{s-1}'(r_p r_{p+1} \cdots r_{n-2})$, then $\tau' = \tau'' r_{n-1}$ with $\tau'' \in \widetilde{\mathfrak{A}}_{n-1}$. Note that $\nabla_n^{\mathrm{II}\pm}(\sigma') = \nabla_n^{\mathrm{II}\pm}(\tau_1') \cdots \nabla_n^{\mathrm{II}\pm}(\tau_s')$, and

$$\nabla_n^{\mathrm{II}\pm}(\tau_s') = \nabla_{n-1}^{\mathrm{II}}(r_p) \cdots \nabla_{n-1}^{\mathrm{II}}(r_{n-2}) \cdot \tfrac{i}{\sqrt{2}}\,(Y_{2n'+1}Y_{2n'} \mp E_{2^{n'}}).$$

Expand the right hand side of $\nabla_n^{\mathrm{II}\pm}(\sigma')$ into a linear combination of monomials $Y_1{}^{a_1} \cdots Y_n{}^{a_n}$ as before, then we see from Lemmas 5.9 – 5.10 that, as the contribution from the last term of $\nabla_n^{\mathrm{II}\pm}(r_{n-1})$, we should take $\mp E_{2^{n'}}$ (not $Y_{2n'+1}Y_{2n'}$). This gives us the first assertion.

The proof of the second assertion is similar. □

15 Relations to Hauptdarstellung of Schur

15.1 Rewriting of Hauptdarstellung Δ_n for $\widetilde{\mathfrak{S}}_n$

Schur's Hauptdarstellung Δ_n is given for the representation group $\mathfrak{T}_n$ [Sch3, §22]. So we rewrite it for another representation group $\widetilde{\mathfrak{S}}_n = \mathfrak{T}_n'$ of our present choice, and denote it by Δ'_n. Then we compare it with ∇_n, ∇^-_n and $\nabla'_n \cong \nabla''_n$. Put

$$N = [(n-1)/2] \qquad (n = 2N+1 \text{ or } 2N+2) \tag{15.1}$$

and define (only in this section)

$$\begin{cases} X'_{2k-1} & := \; c^{\otimes(k-1)} \otimes a \otimes \varepsilon^{\otimes(N-k)} \quad (1 \le k \le N), \\ X'_{2k} & := \; c^{\otimes(k-1)} \otimes b \otimes \varepsilon^{\otimes(N-k)} \quad (1 \le k \le N), \\ X'_{2N+1} & := \; c^{\otimes N}. \end{cases} \tag{15.2}$$

Note that we have defined in (5.13) matrices Y_j $(j \in \boldsymbol{I}_n)$ of type $2^{n'} \times 2^{n'}$ with $n' = [n/2]$, and here X'_j $(j \in \boldsymbol{I}_n)$ are similar but of type $2^N \times 2^N$.

Put for $1 \le j \le 2N+1$,

$$T'_j := a_{j-1} X'_{j-1} + b_j X'_j \qquad (X'_0 := O), \tag{15.3}$$

in such a way that they satisfy the relations

$$\begin{cases} (T'_j)^2 = E & (1 \le j \le 2N+1), \\ T'_j T'_{j+1} + T'_{j+1} T'_j + E = O & (1 \le j \le 2N), \end{cases}$$

where $E = E_{2^N}$. Then the equations for a_j's and b_j's are

$$\begin{cases} a_0 = 0, \quad b_1^{\,2} = 1, \\ a_{j-1}^{\;2} + b_j^{\,2} = 1 \quad (1 \le j \le 2N+1). \\ 2a_j b_j = -1 \qquad (1 \le j \le 2N). \end{cases} \tag{15.4}$$

A set of solutions is immediate from [Sch3, Abschnitt VI] as follows.

Lemma 15.1.

$$\begin{cases} a_0 = 0, \quad b_1 = 1, \\ a_{2\nu} = -\dfrac{\sqrt{\nu}}{\sqrt{2\nu+1}}\,, \qquad b_{2\nu+1} = \dfrac{\sqrt{\nu+1}}{\sqrt{2\nu+1}} \quad (1 \le \nu \le N), \\ a_{2\nu+1} = -\dfrac{\sqrt{2\nu+1}}{2\sqrt{\nu+1}}\,, \qquad b_{2\nu+2} = \dfrac{\sqrt{2\nu+3}}{2\sqrt{\nu+1}} \quad (1 \le \nu \le N-1). \end{cases}$$

Thus we get a spin representation $r_j \mapsto T'_j$ $(1 \le j \le n-1)$ of $\widetilde{\mathfrak{S}}_n$, which is denoted by Δ'_n and called 'Hauptdarstellung' of $\widetilde{\mathfrak{S}}_n = \mathfrak{T}'_n$, for $n = 2N+1$ or $n = 2N+2$.

15.2 Character of 'Hauptdarstellung' Δ'_n of $\widetilde{\mathfrak{S}}_n = \mathfrak{T}'_n$

Theorem 15.2 (cf. [Sch3, §23]). *Let $n = 2N+2$ even. The character of Δ'_n is given as follows: for a standard $\sigma' = \sigma'_1 \sigma'_2 \cdots \sigma'_s \in \widetilde{\mathfrak{S}}_n$, $\sigma_k = \Phi(\sigma'_k)$ disjoint cycles, with $\ell_k = \ell(\sigma'_k)$,*

$$\chi_{\Delta'_n}(\sigma') = \begin{cases} 2^N \cdot (-2)^{-\sum_{k \in \boldsymbol{I}_s} (\ell_k - 1)/2} = 2^{n/2-1}\,(-2)^{-\sum_{k \in \boldsymbol{I}_s} (\ell_k - 1)/2}, \\ \qquad\qquad\qquad\qquad\qquad \text{if } L(\sigma'_k) \equiv 0 \pmod 2 \;\; (k \in \boldsymbol{I}_s), \\ i^N \sqrt{N+1} = i^{n/2-1}\sqrt{n/2}\,, \quad \text{if } \sigma' = r_1 r_2 \cdots r_{n-1} \text{ (odd)}, \end{cases}$$

and $\chi_{\Delta'_n}(\sigma') = 0$ *if* σ' *is not conjugate to an element above or to* z_1*-times of it.*

Proof. For $\sigma' = r_1 r_2 \cdots r_{n-1}$, we express the following product as a linear combination of monomial terms ${X'_1}^{a_1} {X'_2}^{a_2} \cdots {X'_n}^{a_n}$:

$$T'_1 T'_2 \cdots T'_{n-1} = X'_1 \prod_{2 \le j \le 2N+1} \big(a_{j-1} X'_{j-1} + b_j X'_j\big), \quad 2N+1 = n-1.$$

Then, since the degrees of each monomials are all $n-1 = 2N+1$ odd, the unique monomial term with non-zero trace is

$$X'_1 \cdot b_2 X'_2 \cdot \cdots \cdot b_{2N+1} X'_{2N+1} = b_2 \cdots b_{2N+1} \cdot (abc)^{\otimes N} = b_2 \cdots b_{2N+1} \cdot (i\varepsilon)^{\otimes N}.$$

On the other hand,

$$\prod_{0 \le \nu \le N-1} b_{2\nu+1} b_{2\nu+2} \times b_{2N+1} = \prod_{0 \le \nu \le N-1} \frac{\sqrt{2\nu+3}}{2\sqrt{2\nu+1}} \times \frac{\sqrt{N+1}}{\sqrt{2N+1}} = \frac{\sqrt{N+1}}{2^N}$$

Hence we obtain

$$\chi_{\Delta'_n}(\sigma') = \frac{\sqrt{N+1}}{2^N} \cdot (2i)^N = i^N \sqrt{N+1} = i^{n/2-1} \sqrt{n/2}\,. \qquad \square$$

Theorem 15.3 (cf. [Sch3, §23]). *Let* $n = 2N+1$ *odd. The character of* Δ'_n *is given as follows: for a standard* $\sigma' = \sigma'_1 \sigma'_2 \cdots \sigma'_s$, $\sigma_k = \Phi_{\mathfrak{S}}(\sigma'_k)$ *disjoint cycles,*

$$\chi_{\Delta'_n}(\sigma') = \begin{cases} 2^N \cdot (-2)^{-\sum_{k \in \boldsymbol{I}_s} (\ell_k - 1)/2} = 2^{(n-1)/2} \, (-2)^{-\sum_{k \in \boldsymbol{I}_s} (\ell_k - 1)/2}, \\ \qquad\qquad \text{if } L(\sigma'_k) \equiv 0 \pmod 2 \ \ (k \in \boldsymbol{I}_s), \\ 0, \qquad\qquad \text{otherwise.} \end{cases}$$

15.3 Relations to 'Hauptdarstellung' Δ'_n

Lemma 15.4. *In the case where* $n = 2N+2$ *is even,*

$$\chi_{\Delta'_n} + \chi_{\mathrm{sgn}\cdot\Delta'_n} = \begin{cases} 2 \cdot 2^N \cdot (-2)^{-\sum_{k \in \boldsymbol{I}_s} (\ell_k - 1)/2} = 2^{n/2} \cdot (-2)^{-\sum_{k \in \boldsymbol{I}_s} (\ell_k - 1)/2}, \\ \qquad\qquad \text{if } L(\sigma'_k) \equiv 0 \pmod 2 \ \ (\forall k), \\ 0, \qquad\qquad \text{otherwise.} \end{cases}$$

Theorem 15.5. (i) *The spin representations* $\nabla_n \cong \nabla_n^- \cong \nabla'_n$ *of* $\widetilde{\mathfrak{S}}_n \cong \mathfrak{T}'_n$ *is related to Schur's 'Hauptdarstellung'* Δ'_n *as*

$$\begin{aligned} \nabla_n &\cong \Delta'_n \oplus (\mathrm{sgn} \cdot \Delta'_n), && \text{if } n = 2n' \text{ even } (N = n'-1), \\ \nabla_n &\cong \Delta'_n \ \ \text{(irreducible)}, && \text{if } n = 2n'+1 \text{ odd } (N = n'). \end{aligned}$$

(ii) *The restriction of* Δ'_n, $n \ge 4$, *onto the subgroup* $\widetilde{\mathfrak{A}}_n$ *are as follows:*

- *When* n *is even,* Δ'_n *is not self-associate, and* $\Delta'_n\big|_{\widetilde{\mathfrak{A}}_n}$ *is irreducible.*

• *When n is odd, Δ'_n is self-associate, and $\Delta'_n|_{\widetilde{\mathfrak{A}}_n} \cong \mho_n^+ \cong \mho_n^-$, and it is a direct sum of two inequivalent spin IRs $\mho_n^{(\kappa)}$, $\kappa = 0, 1$, of $\widetilde{\mathfrak{A}}_n$, where $\mho_n^{(1)} = \big(\mho_n^{(0)}\big)^{r_1}$.*

Proof. (i) follows from Theorem 15.3 and Lemma 15.4 above, together with the results on characters in §14.

(ii) is proved by applying Lemma 17.4 in [I] (cf. also Lemma 12.4), and character formulas in §§14–15. □

In case n is odd, our representation ∇_n is another realization of 'Hauptdarstellung' Δ'_n of Schur. On the contrary, in case n is even, ∇_n is decomposed into the direct sum of Δ'_n (cf. §8.2).

16 Characters of IRs of $\widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P_\gamma)$, $\widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P_\gamma^\pm)$

This section for CASE I, Type $(-1, -1, -1)$, and the next one for CASE II, Type $(-1, -1, 1)$, correspond to the step **(f-7)** in §3, and are very important steps on the way of getting character formulas for spin IRs of $R\big(G(m, 1, n)\big)$ of CASEs I and II.

16.1 Formulas for calculating trace of $\pi_\gamma^{\mathrm{I}}(g')$, $\pi_\gamma^{\mathrm{I}\pm}(g')$

16.1.1 Preparation for calculating characters

As in §11, consider spin representation ∇_n and ∇_n^- of $\widetilde{\mathfrak{S}}_n$ and the IR P_γ of $\widetilde{D}_n$:

$$\nabla_n(r_i) = \frac{(-1)^{i-1}}{\sqrt{2}}\,(Y_i + Y_{i+1}) = \frac{1}{\sqrt{2}}\,(Y'_i - Y'_{i+1}) \quad (i \in \boldsymbol{I}_{n-1})\,; \tag{16.1}$$

$$\begin{cases} \nabla_n^-(r_i) \ = \nabla_n(r_i) & (i \in \boldsymbol{I}_{n-2}), \\ \nabla_n^-(r_{n-1}) \ = \dfrac{(-1)^{n-2}}{\sqrt{2}}\,(Y_{n-1} - Y_n) = \dfrac{1}{\sqrt{2}}\,(Y'_{n-1} + Y'_n) & (i = n-1)\,; \end{cases} \tag{16.2}$$

$$P_\gamma(\eta'_j) = (-1)^{j-1}\omega^{\gamma_j}\rho(\eta_j) = \chi_\gamma(\eta'_j)\,Y_j = \zeta_\gamma(\eta'_j)Y'_j \quad (j \in \boldsymbol{I}_n)\,; \tag{16.3}$$

where $Y'_j = (-1)^{j-1}Y_j$ $(j \in \boldsymbol{I}_n)$.

The complete sets of representatives of the set of equivalence classes $\widehat{\widetilde{D}_n}^{\mathrm{spin}}$ are

$$\begin{array}{ll} \text{Case } n = 2n' \text{ even :} & \mathrm{Rep}(\widetilde{D}_n) = \big\{[P_\gamma]\,;\ \gamma \in \Gamma_n^0\,\big\}\,; \\ \text{Case } n = 2n'+1 \text{ odd :} & \mathrm{Rep}(\widetilde{D}_n) = \mathrm{Rep}^+(\widetilde{D}_n)\bigsqcup\mathrm{Rep}^-(\widetilde{D}_n), \end{array} \tag{16.4}$$

where $\mathrm{Rep}^\pm(\widetilde{D}_n) = \big\{[P_\gamma^\pm]\,;\, \gamma \in \Gamma_n^0\big\}$, $P_\gamma^+ = P_\gamma$, $P_\gamma^- = P_{\tau_n\gamma}$. The set Γ_n^0 consists of $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ satisfying $0 \le \gamma_k < m' = m/2$ $(k \in \boldsymbol{I}_n)$ as in (11.7), and $\tau_n\gamma = (\gamma_1, \ldots, \gamma_{n-1}, \gamma_n + m')$.

For representatives P_γ (resp. P_γ^+ and P_γ^- if n is odd) with $\gamma \in \Gamma_n^0$, the stationary subgroups in $\widetilde{\mathfrak{S}}_n$ of their equivalence classes are given as

$$(16.5)\qquad \begin{array}{ll} \mathcal{S}(P_\gamma) = \{\sigma' \in \widetilde{\mathfrak{S}}_n \,;\, \sigma'\gamma = \gamma\} & \text{in case } n \text{ is even ;} \\ \mathcal{S}(P_\gamma^{\pm}) = \{\sigma' \in \widetilde{\mathfrak{S}}_n \,;\, \sigma'\gamma = \gamma\} & \text{in case } n \text{ is odd.} \end{array}$$

Recall IRs of the subgroup $\widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P_\gamma)$ of $\widetilde{G}_n^{\mathrm{I}}$ given as

$$(16.6)\qquad \begin{array}{lll} \pi_\gamma^{\mathrm{I}}(d',\sigma') &:=& P_\gamma(d')\cdot\nabla_n(\sigma') \qquad \big((d',\sigma')\in \widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P_\gamma)\big)\ ; \\ \pi_\gamma^{\mathrm{I}\pm}(d',\sigma') &:=& P_\gamma^{\pm}(d')\cdot\nabla_n^{\pm}(\sigma') \qquad \big((d',\sigma')\in \widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P_\gamma^{\pm})\big), \end{array}$$

or $\pi_\gamma^{\mathrm{I}} = P_\gamma\cdot\nabla_n$ and $\pi_\gamma^{\mathrm{I}\pm} = P_\gamma^{\pm}\cdot\nabla_n^{\pm}$ with $\nabla_n^{+} = \nabla_n$. Put their characters as

$$(16.7)\qquad \begin{array}{lll} f_\gamma^{\mathrm{I}}(g') &:=& \mathrm{tr}\big(\pi_\gamma^{\mathrm{I}}(g')\big) \qquad \big(g'\in \widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P_\gamma)\big) \quad \text{if } n \text{ is even ;} \\ f_\gamma^{\mathrm{I}\pm}(g') &:=& \mathrm{tr}\big(\pi_\gamma^{\mathrm{I}\pm}(g')\big) \qquad \big(g'\in \widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P_\gamma^{\pm})\big) \quad \text{if } n \text{ is odd.} \end{array}$$

Before starting calculation of them, we refer Table 4.1 on the general property of supports of characters. We keep to the notations in §6.1 and Definition 11.2, which we recall briefly. We have two abelian subgroups of $\widetilde{D}_n = \widetilde{D}(m,n)$ defined as

$$(16.8)\qquad \widetilde{D}^0(m,n) = \langle {\eta_1}^2, {\eta_2}^2, \ldots, {\eta_n}^2\rangle \subset \widetilde{D}^1(m,n) = \big\langle z_2, \widetilde{D}^0(m,n)\big\rangle.$$

A spin character of $\widetilde{D}_n^1 = \widetilde{D}^1(m,n)$ is given as $\chi_\gamma^1 = \chi_\gamma\big|_{\widetilde{D}_n^1}$ with a spin function χ_γ in (6.2) on $\widetilde{D}_n$. A character $\zeta_\gamma = (\zeta_1,\ldots,\zeta_n)$ on $D(m,n)$ and a non-spin character $\zeta_\gamma\circ\Phi_D$ (denoted again by ζ_γ) on $\widetilde{D}(m,n)$ are defined for $\gamma\in\Gamma_n$. Here $\zeta_j = \zeta_{j,\gamma_j} = \zeta_\gamma\big|_{T_j}$ for j-th copy T_j of the protocol $T \cong \boldsymbol{Z}_m$. Define a partition of $\boldsymbol{I}_n$ by $\zeta_\gamma = (\zeta_j)_{j\in\boldsymbol{I}_n}$, $\gamma\in\Gamma_n^0$, as

$$(16.9)\qquad \boldsymbol{I}_n = \bigsqcup\nolimits_{\zeta\in\widehat{T}^0} I_{n,\zeta}\,, \quad I_{n,\zeta} := \{j\in\boldsymbol{I}_n \,;\, \zeta_j = \zeta\}.$$

We normalize $\gamma = (\gamma_1,\ldots,\gamma_n)\in\Gamma_n^0$ under conjugation of $\mathfrak{S}_n$ so that (11.12) – (11.14) hold. Then the above partition is nothing but (11.13) with $I_k = I_{n,\zeta}$ an interval for $\zeta = \zeta^{(c_k)}$.

16.1.2 Normalization of representatives of conjugacy classes in CASE I

Any element $g''\in \widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P_\gamma)$ $\big($resp. $\widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P_\gamma^{\pm})$ if n is odd$\big)$ has a standard decomposition as

$$(16.10)\qquad \left\{ \begin{array}{l} g'' = {z_1}^{b_1}{z_2}^{b_2}\,g', \quad g' = (d',\sigma') = \xi'_{q_1}\cdots\xi'_{q_r}\,g'_1\cdots g'_s, \\ \xi'_q = t_q = {\eta'_q}^{a_q}, \quad \sigma' = \sigma'_1\cdots\sigma'_s\,, \ d' = \xi'_{q_1}\cdots\xi'_{q_r}\cdot {z_2}^X d'_1\cdots d'_s, \\ g'_j = (d'_j,\sigma'_j), \ \sigma_j = \Phi_{\mathfrak{S}}(\sigma'_j) \text{ a cycle, } K_j := \mathrm{supp}(\sigma'_j) \supset \mathrm{supp}(d'_j), \\ d'_j = \prod_{p\in K_j} {\eta'_p}^{a_p} = d_j^0 h_j, \ d_j^0 \in \widetilde{D}_n^0 = \widetilde{D}^0(m,n), \ h_j = \prod_{p\in K_j} {\eta'_p}^{\beta_p}, \ \beta_p = 0,1, \end{array} \right.$$

$$\text{(16.11)} \qquad \text{with} \quad X=\sum_{2\le j\le s}\operatorname{ord}(d'_j)\big(L(\sigma'_1)+\cdots+L(\sigma'_{j-1})\big),$$

since $r_i^{\mathrm{I}}(\eta'_j)=z_2\eta'_{s_i(j)}$. Here the product of ${\eta'_p}^{a_p}$ for d'_j is in the natural order except otherwise indicated. Note that the top term $z_1^{b_1}z_2^{b_2}$ has usually some ambiguity, because $\xi'_{q_i}\xi'_{q_k}=z_2\xi'_{q_k}\xi'_{q_i}$ $(q_i\neq q_k)$ if $\operatorname{ord}(\xi'_{q_i})\equiv\operatorname{ord}(\xi'_{q_k})\equiv 1 \pmod 2$ and $\sigma'_j\sigma'_k=z_1\sigma'_k\sigma'_j$ if $L(\sigma'_j)\equiv L(\sigma'_k)\equiv 1 \pmod 2$ etc. When we study spin characters of Type $(-1,-1,-1)$, it is sufficient to determine their values for conjugacy classes modulo the center $\widetilde{Z}=\langle z_1,z_2\rangle$, so that we consider principally elements of the form g' above.

We normalize representatives $g'=\xi'_{q_1}\cdots\xi'_{q_r}g'_1\cdots g'_s$, $g'_j=(d'_j,\sigma'_j)$, under conjugacy of $\widetilde{D}_n\overset{\mathrm{I}}{\rtimes}\Phi_{\mathfrak{S}}\big(\prod_{\zeta\in\widehat{T}}\mathfrak{S}_{I_{n,\zeta}}\big)$. To do so, recall Lemmas 1.3 – 1.6. Let $\sigma_j=\Phi(\sigma'_j)$ be a cycle $(k_1\ k_2\ \ldots\ k_{\ell_j})$, $\ell_j=\ell(\sigma'_j)$. To fix the choice of the preimage σ'_j of σ_j, let k_1 be the smallest among $\{k_1,\ldots,k_{\ell_j}\}$, and put

$$\text{(16.12)} \qquad \sigma'_j:=r_{k_1,k_2}r_{k_2,k_3}\cdots r_{k_{\ell_j-1},k_{\ell_j}}.$$

The support $K_j=\operatorname{supp}(\sigma'_j)=\{k_1,k_2,\ldots,k_{\ell_j}\}$ is contained in some $I_{n,\zeta}$, and suppose here for simplicity that $I_{n,\zeta}$ is equal to $[1,M]\subset\boldsymbol{I}_n$.

Lemma 16.1. (i) *Put* $\sigma_j^{0\prime}:=r_1r_2\cdots r_{\ell_j-1}$, *then* $\sigma_j^0=\Phi(\sigma_j^{0\prime})=(1\ 2\ \ldots\ \ell_j)$. *Take a* $\tau'\in\widetilde{\mathfrak{S}}_M$ *such that* $\tau\sigma_j\tau^{-1}=\sigma_j^0$ *with* $\tau=\Phi_{\mathfrak{S}}(\tau')$. *Then* $\tau'\sigma'_j{\tau'}^{-1}=z_1^Y\sigma_j^{0\prime}$, *where the exponent* Y *is computable.*

(ii) *Let* $\sigma'_j=\sigma_j^{0\prime}$. *Then* $K_j=[1,N]$, $N=\ell_j$, *and there exists a* $\tilde d\in\widetilde{D}_N$ *such that*

$$\text{(16.13)} \qquad \tilde d\,g'_j\,\tilde d^{-1}=(d''_j,\sigma'_j) \quad \text{with} \quad d''_j=z_2^{\operatorname{ord}(\sigma'_j)\operatorname{ord}(\tilde d)+X}\,{\eta'_1}^{\operatorname{ord}(d'_j)},$$

where, express d'_j *with new generators* η'_k *as* $d'_j={\eta'_1}^{a_1}{\eta'_2}^{a_2}\cdots{\eta'_N}^{a_N}$, *then*

$$X\equiv\sum_{2\le p<q\le N-1}b_pb_q \pmod 2 \quad \text{with} \quad b_p:=a_2+\cdots+a_p\ (p\ge 2).$$

(iii) *Let* $g'_j=(d'_j,\sigma'_j)=({\eta'_k}^b,\sigma_j^{0\prime})$. *Then* ${\sigma'_j}^{-1}g'_j\sigma'_j=(d''_j,\sigma_j^{0\prime})$ *with*

$$d''_j=\begin{cases}(z_2\eta'_{k-1})^b & \text{if } \operatorname{ord}(d'_j)\equiv L(\sigma'_j)\equiv 1 \pmod 2,\\ (\eta'_{k-1})^b & \text{otherwise.}\end{cases}$$

Proof. Note that $\operatorname{ord}(d'_j)\equiv a_1+\cdots+a_N \pmod m$. Let $\tilde d={\eta'_1}^{b'_1}{\eta'_2}^{b'_2}\cdots{\eta'_N}^{b'_N}$. Then $\tilde d\,g'_j\,\tilde d^{-1}=\big(\tilde d\,d'_j\,{\sigma'_j}^{\mathrm{I}}(\tilde d)^{-1},\ \sigma'_j\big)$ and

$$\begin{aligned}{\sigma'_j}^{\mathrm{I}}(\tilde d)=\sigma'_j\,\tilde d\,{\sigma'_j}^{-1}&=z_2^{\operatorname{ord}(\tilde d)L(\sigma'_j)}\,{\eta'_{\sigma_j(1)}}^{b'_1}{\eta'_{\sigma_j(2)}}^{b'_2}\cdots{\eta'_{\sigma_j(N)}}^{b'_N}=\\ &=z_2^{\operatorname{ord}(\tilde d)L(\sigma'_j)}\,{\eta'_2}^{b'_1}{\eta'_3}^{b'_2}\cdots{\eta'_1}^{b'_N},\end{aligned}$$

$$\begin{aligned}\tilde{d}\, d'_j\, \sigma'^{\mathrm{I}}_j(\tilde{d}\,)^{-1} &= \big(\eta'^{\,b'_1}_1 \eta'^{\,b'_2}_2 \cdots \eta'^{\,b'_N}_N\big)\big(\eta'^{\,a_1}_1 \eta'^{\,a_2}_2 \cdots \eta'^{\,a_N}_N\big) \times \\ &\quad \times \big(z_2^{\mathrm{ord}(\tilde{d})L(\sigma'_j)}\, \eta'^{\,-b'_N}_1 \eta'^{\,-b'_{N-1}}_N \cdots \eta'^{\,-b'_2}_3 \eta'^{\,-b'_1}_2\big) \\ &= \eta'^{\,b'_1+a_1-b'_N}_1 \eta'^{\,b'_2+a_2-b'_1}_2 \cdots \eta'^{\,b'_N+a_N-b'_{N-1}}_N \times z_2^{\,X}.\end{aligned}$$

Put $b'_{k+1} + a_{k+1} - b'_k = 0$ $(1 \le k \le N-1)$. Then a solution is given by

$$b'_1 = 0,\; b'_k = -(a_2 + a_3 + \cdots + a_k) = -b_k \quad (2 \le k \le N).$$

Moreover X is computed to be equal to $\sum_{2 \le p < q \le N-1} b_p b_q$ modulo 2. □

By this lemma, we see that each g'_j is conjugate under $\widetilde{\mathfrak{S}}_{I_{n,\zeta}}$, modulo powers of z_1 and z_2, to the following normal form $g^{0\prime}_j$:

- *Let* $K_j \subset I_{n,\zeta}$ *be an interval* $[n_j, n_j + \ell_j - 1] \subset \boldsymbol{I}_n$, *then for some* $k_j \in K_j$,

(16.14) $$g^{0\prime}_j = (\eta'^{\,\mathrm{ord}(d'_j)}_{k_j}, \sigma^{0\prime}_j) \quad \text{with} \quad \sigma^{0\prime}_j = r_{n_j} r_{n_j+1} \cdots r_{n_j+\ell_j-2}.$$

For such a normalization, we can take k_j the smallest in K_j, but here we take k_j an arbitrary element in K_j to prepare for calculation of characters of induced representations. In this connection, see also §22.2, Part V, below.

Example 16.1 (Representatives in $\mathfrak{S}_n(T)$ under conjugacy). In the case of a wreath product group $\mathfrak{S}_n(T) := D_n(T) \rtimes \mathfrak{S}_n$, $D_n(T) := \prod_{k \in \boldsymbol{I}_n} T_k$, $T_k = T$, with a non-abelian T, a general element is expressed as

$$g = \xi_{q_1} \cdots \xi_{q_r} g_1 \cdots g_s,\; \xi_q = (t_q),\; g_j = (d_j, \sigma_j),\; d_j = (t_p)_{p \in K_j},\; K_j = \mathrm{supp}(\sigma_j),$$

with σ_j disjoint cycles, $t_k \in T_k$. The datum determining its conjugacy class (cf. Theorem 1.2 in [HHH1, p.6]) is the set of $\{[t_{q_1}], \ldots, [t_{q_r}]\}$ and $\{(P_{\sigma_j}(d_j), \ell(\sigma_j));\ j \in \boldsymbol{I}_s\}$, where $[t_k]$ denotes the conjugacy class of t_k in $T_k = T$, and for $\sigma_j = (k_1\ k_2\ \ldots\ k_{\ell_j})$,

$$P_{\sigma_j}(d_j) := [t_{k_{\ell_j}} \cdots t_{k_2} t_{k_1}].$$

Example 16.2 (Representatives in $\widetilde{G}^{\mathrm{I}}_n$ under conjugacy). For simplicity, let

$$g' = \xi'_1 g'_1,\; \xi'_1 = \eta'_1,\; g'_1 = (d'_1, \sigma'_1),\; \sigma'_1 = r_2 r_3 r_4,\; d'_1 = \eta'^{\,a_2}_2 \eta'^{\,a_3}_3 \eta'^{\,a_4}_4 \eta'^{\,a_5}_5.$$

Then $g = \Phi(g') = \xi_1 g_1$, $\xi_1 = y_1$, $g_1 = (d_1, \sigma_1)$, $\sigma_1 = s_2 s_3 s_4 = (2\ 3\ 4\ 5)$, $d_1 = y_2^{a_2} y_3^{a_3} y_4^{a_4} y_5^{a_5}$. Take $d' = \eta'^{\,c_1}_1 \eta'^{\,c_2}_2 \eta'^{\,c_3}_3 \eta'^{\,c_4}_4$ with $c_4 = a_5$, $c_3 = a_4 + a_5$, $c_2 = a_3 + a_4 + a_5$, and consider the conjugate $d' g' d'^{-1}$. Then

$$\begin{aligned} d' g' d'^{-1} &= \xi''_1 g''_1,\; \xi''_1 = d' \xi'_1 d'^{-1},\; g''_1 = d' g'_1 d'^{-1}, \\ \xi''_1 &= z_2^{\,c_2+c_3+c_4} \xi'_1,\;\; g''_1 = z_2^{\,x} \eta'^{\,\mathrm{ord}(d'_1)}_2, \end{aligned}$$

with $x = (c_1 + a_3 + a_5) + c_1(a_2 + a_3 + a_4 + a_5) + (a_2 + a_3)a_5 + a_2(a_4 + a_5)$.

This simple example shows that the set of data

$$\mathrm{ord}(\xi'_{q_i})\ (i \in \boldsymbol{I}_r), \quad \big(\mathrm{ord}(d'_j), \ell(\sigma'_j)\big)\ (j \in \boldsymbol{I}_s)$$

usually does not determine uniquely the conjugacy class of g' in (16.10), whereas it determines the conjugacy class modulo the central group $\widetilde{Z} = \langle z_1, z_2 \rangle$.

16.1.3 Product formulas for traces of $\pi^{\mathrm{I}}_\gamma(g')$ and $\pi^{\mathrm{I}\pm}_\gamma(g')$

For $g' = \xi'_{q_1}\cdots\xi'_{q_r}g'_1\cdots g'_s,\ \ g'_j = (d'_j, \sigma'_j) \in \widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P_\gamma)$, put

$$Q := \{q_1, q_2, \ldots, q_r\}, \quad J := \boldsymbol{I}_s = \{1, 2, \ldots, s\}. \tag{16.15}$$

CASE $n = 2n'$ EVEN :

$$\pi^{\mathrm{I}}_\gamma(g') \ = \ \prod_{q\in Q} P_\gamma(\xi'_{q_i}) \times \prod_{j\in J}\Big(P_\gamma(d'_j)\nabla_n(\sigma'_j)\Big), \tag{16.16}$$

$$P_\gamma(\xi'_q) \ = \ \chi_\gamma({\eta'_q}^{a_q})\, Y_q^{a_q} = \zeta_\gamma({\eta'_q}^{a_q})\, {Y'_q}^{a_q} \qquad (q \in Q)\,, \tag{16.17}$$

$$P_\gamma(d'_j) \ = \ \prod_{p\in K_j} \chi_\gamma\big({\eta'_p}^{a_p}\big) Y_p^{a_p} = \zeta_\gamma(d'_j)\cdot \prod_{p\in K_j} {Y'_p}^{\beta_p}, \tag{16.18}$$

where $a_p \equiv \beta_p$ (mod 2), and the product is taken in the natural order. After normalization (16.14), $a_p \neq 0$ at most for one $p \in K_j$, and $\nabla_n(\sigma'_j)$ is given by a product of $\nabla_n(r_i)$ in (16.1) correspondingly.

CASE OF $n = 2n'+1$ ODD :

For $\nabla^+_n = \nabla_n,\ P^+_\gamma = P_\gamma$ and $\pi^{\mathrm{I}+}_\gamma = \pi^{\mathrm{I}}_\gamma$, the same formulas as (16.1) and (16.16) – (16.18) hold automatically. Also recall that

$$\begin{aligned} &\pi^{\mathrm{I}-}_\gamma(g') = P^-_\gamma(d')\cdot\nabla^-_n(\sigma') \quad \big(g' = (d', \sigma') \in \widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P^-_\gamma)\big) \\ &P^-_\gamma = P_{\tau_n\gamma}, \quad \nabla^-_n(r_j) := -Y_n\,\nabla_n(r_j)\,Y_n^{-1} \qquad (j \in \boldsymbol{I}_{n-1}). \end{aligned}$$

Calculations for $\pi^{\mathrm{I}+}_\gamma$ for n odd are carried out together with π^{I}_γ for n even, and distinction depending on the parity of n will be remarked at the place.

In the expression of $\pi^{\mathrm{I}}_\gamma(g'_j) = P_\gamma(d'_j)\nabla_n(\sigma'_j)$ for $g'_j = (d'_j, \sigma'_j)$ in (16.16), we expand the right hand side into a linear combination of monomial terms such as $Y_{k_1}^{c_1}Y_{k_2}^{c_2}\cdots Y_{k_n}^{c_n}$. We know that the terms with non-zero traces are those such that

$$Y_{k_1}^{c_1}Y_{k_2}^{c_2}\cdots Y_{k_n}^{c_n}, \qquad c_1 \equiv c_2 \equiv \ldots \equiv c_n \equiv 0 \pmod 2), \ \text{ or} \tag{16.19}$$

$$Y_{k_1}Y_{k_2}\cdots Y_{k_{2n'+1}}, \quad \text{with } \{k_1, k_2, \ldots, k_{2n'+1}\} = \boldsymbol{I}_{2n'+1}. \tag{16.20}$$

Similar assertions hold also for $\pi^{\mathrm{I}\pm}_\gamma$. Therefore, since the supports of ξ'_{q_i}'s and g'_j's are mutually disjoint, we have the following.

Lemma 16.2. (i) CASE OF n EVEN : $\quad f^{\mathrm{I}}_\gamma(g') \neq 0 \implies$ (Condition I-00) :

(Condition I-00) $\quad \begin{cases} \mathrm{ord}(d') + L(\sigma') \equiv 0 \ (\mathrm{mod}\ 2)\ ; \\ \mathrm{ord}(\xi'_{q_i}) \equiv 0\ (\forall i),\ \mathrm{ord}(d'_j) + L(\sigma'_j) \equiv 0 \ (\mathrm{mod}\ 2) \ \ (\forall j). \end{cases}$

(ii) CASE OF n ODD :

if $|\mathrm{supp}(g')| < n$, $f_\gamma^{\mathrm{I}\pm}(g') \neq 0 \implies$ (Condition I-00) ;
if $|\mathrm{supp}(g')| = n$, $f_\gamma^{\mathrm{I}\pm}(g') \neq 0 \implies$ (Condition I-00) or (Condition I-11) :

(Condition I-11) $\begin{cases} |\mathrm{supp}(g')| = n \text{ odd}, \ \ \mathrm{ord}(d') + L(\sigma') \equiv 1 \ (\mathrm{mod}\ 2)\ ; \\ \mathrm{ord}(\xi'_{q_i}) \equiv 1 \ (\forall i), \ \ \mathrm{ord}(d'_j) \equiv 1 \ (\mathrm{mod}\ 2) \ \ (\forall j). \end{cases}$

Proof. The assertion (i) comes from the statement above on (16.19). The first half of (ii) is similar as (i), and the second half should come from Lemma 14.1 on $Y_{k_1} Y_{k_2} \cdots Y_{k_{2n'+1}}$ in (16.20) with $n = 2n' + 1$, and this will be confirmed from the detailed discussion in §16.3 below. However here we quote it from Table 4.1. □

For $g' = (d', \sigma') \in \widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P_\gamma)$, we separate g' into two cases as

CASE 1 : $\mathrm{ord}(d') + L(\sigma') \equiv 0 \ (\mathrm{mod}\ 2)$;
CASE 2 : $\mathrm{ord}(d') + L(\sigma') \equiv 1 \ (\mathrm{mod}\ 2)$.

Note that, in $\widetilde{G}_n^{\mathrm{I}}$, since their supports are mutually disjoint (cf. (I-v) and (I-v′) in §2.3),

$$(16.21) \qquad \begin{cases} d'_i g'_j \ = \ z_2^{\,\mathrm{ord}(d'_i)(\mathrm{ord}(d'_j) + L(\sigma'_j))} g'_j d'_i \,, \\ g'_j g'_k \ = \ z_2^{\,\mathrm{ord}(d'_j)\,\mathrm{ord}(d'_k) + \mathrm{ord}(d'_j)\,L(\sigma'_k) + L(\sigma'_j)\,\mathrm{ord}(d'_k)} \, z_1^{\,L(\sigma'_j)\,L(\sigma'_k)} \, g'_k g'_j \,. \end{cases}$$

Then we obtain the following lemma.

Lemma 16.3. *Under (Condition I-00), the operators* $\pi_\gamma^{\mathrm{I}}(\xi'_{q_i})$*'s and* $\pi_\gamma^{\mathrm{I}}(g'_j)$*'s all commute with each other.*

Lemma 16.4. (i) CASE OF $n = 2n'$ EVEN : *The following product formula holds*

$$f_\gamma^{\mathrm{I}}(g') = 2^{n'} \cdot \prod_{q \in Q} \frac{f_\gamma^{\mathrm{I}}(\xi'_q)}{2^{n'}} \cdot \prod_{j \in J} \frac{f_\gamma^{\mathrm{I}}(g'_j)}{2^{n'}} \,.$$

(ii) CASE OF n ODD : *Except the case of (Condition I-11), the analogous product formulas hold for* $f_\gamma^{\mathrm{I}+}$ *and* $f_\gamma^{\mathrm{I}-}$ *respectively.*

Proof. (i) In the expansion of the operator $\pi_\gamma^{\mathrm{I}}(g'_j)$ into a linear combination of monomial terms, those with non-zero traces are of the form $Y_{j_1}^{2c'_1} Y_{j_2}^{2c'_2} \cdots Y_{j_p}^{2c'_p}$ $\big(= E_{2^{n'}}\big)$. Moreover, for each of $f_\gamma^{\mathrm{I}}\big(\xi'_{q_i}\big)$'s and $f_\gamma^{\mathrm{I}}\big(g'_j\big)$'s, the supports in $\boldsymbol{I}_n$ of their monomials with non-zero traces are mutually disjoint since so are $\mathrm{supp}\big(\xi'_{q_i}\big)$'s and $\mathrm{supp}(g'_j)$'s. Here the support of a monomial $Y_{j_1}^{c_1} Y_{j_2}^{c_2} \cdots Y_{j_p}^{c_p}$ means the collection of suffices $j_k \in \boldsymbol{I}_n$ with $c_k \neq 0$. This proves the assertion (i).

The proof for (ii) is similar. □

(NM) Normalization of the situation:

(NM1) Since we work with a representative of $\widehat{\widetilde{D}_n}^{\,\mathrm{spin}}$ of an $\mathfrak{S}_n$-orbit, we can assume that every $I_{n,\zeta} \subset \boldsymbol{I}_n$ is an interval in $\boldsymbol{I}_n$.

(NM2) Assume the normalization (NM1). Then, inside each $I_{n,\zeta}$, the character of π^{I}_γ and $\pi^{\mathrm{I}\pm}_\gamma$ are invariant at least under $\widetilde{\mathfrak{S}}_{I_{n,\zeta}} \subset \mathcal{S}(P_\gamma)$ $\big($resp. $\subset \mathcal{S}(P^{\pm}_\gamma)\big)$. So, to calculate their characters, we can assume that every $K_j = \mathrm{supp}(\sigma_j)$, contained in some $I_{n,\zeta}$, is itself an interval.

16.2 Characters of IRs π^{I}_γ of $\widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P_\gamma)$, CASE 1

To start with, we assume only the condition $\mathrm{ord}(d') + L(\sigma') \equiv 0 \pmod 2$ for $g' = (d', \sigma')$. Let us calculate $f^{\mathrm{I}}_\gamma(g') = \mathrm{tr}\big(\pi^{\mathrm{I}}_\gamma(g')\big)$. This is automatically common to $f^{\mathrm{I}+}_\gamma(g') = \mathrm{tr}\big(\pi^{\mathrm{I}+}_\gamma(g')\big)$ in case n odd. By Lemma 16.4, it is enough for us to calculate $\mathrm{tr}\big(\pi^{\mathrm{I}}_\gamma(\xi'_q)\big)$ and $\mathrm{tr}\big(\pi^{\mathrm{I}}_\gamma(g'_j)\big)$ independently.

(1-1) Case of ξ'_q $(q \in Q)$:

$$f^{\mathrm{I}}_\gamma\big(\xi'_q\big) = \mathrm{tr}\big(\pi^{\mathrm{I}}_\gamma(\xi'_q)\big) = \begin{cases} 2^{n'}\,\chi_\gamma\big(\xi'_q\big), & \text{if } \mathrm{ord}\big(\xi'_q\big) \equiv 0 \pmod 2, \\ \ 0\,, & \text{if } \mathrm{ord}\big(\xi'_q\big) \equiv 1 \pmod 2\,. \end{cases} \tag{16.22}$$

(1-2) Case of $g'_j = (d'_j, \sigma'_j),\ j \in J = \boldsymbol{I}_s$:

$$d'_j = \prod_{p \in K_j} {\eta'_p}^{a_p} = d^0_j h_j,\ \ d^0_j \in \widetilde{D}^0_n,\ \ h_j = \prod_{p \in K_j} {\eta'_p}^{\beta_p},\ \beta_p = 0, 1\,; \tag{16.23}$$

$$\begin{aligned} \pi^{\mathrm{I}}_\gamma(g'_j) &= P_\gamma(d'_j)\nabla_n(\sigma'_j) \\ &= \chi_\gamma(d'_j) \cdot \prod_{p \in K_j} {Y_p}^{\beta_p} \times \prod_{p,\,p+1 \in K_j} \frac{(-1)^{p-1}}{\sqrt{2}}\big(Y_p + Y_{p+1}\big), \end{aligned} \tag{16.24}$$

where the products on $p \in K_j$ are in the natural order of p. In fact, $\zeta_\gamma(d'_j) \cdot \prod_{p \in K_j} {Y'_p}^{\beta_p} = \chi_\gamma(d'_j) \cdot \prod_{p \in K_j} {Y_p}^{\beta_p}$. In the rightmost hand side of (16.24), we see from Lemmas 5.9 – 5.10 and 14.1 that, to have a non-zero trace, it is necessary that the degrees of monomials in its expansion should be even, i.e.,

$$\mathrm{ord}(d'_j) + L(\sigma'_j) \equiv 0 \pmod 2. \tag{16.25}$$

Note that $f^{\mathrm{I}}_\gamma(g'_j)$ is invariant under $\widetilde{D}_{K_j}$ and $\widetilde{\mathfrak{S}}_{K_j} \subset \widetilde{\mathfrak{S}}_{I_{n,\zeta}}$, then we may assume by Lemma 16.1 to simplify complicated suffices that g'_j is normalized modulo $z_1^a z_2^b$ as

$$\begin{cases} K_j = \mathrm{supp}(\sigma'_j) = \{n_j, n_j{+}1, \ldots, n_j{+}\ell_j{-}1\}, \text{ an interval in } \boldsymbol{I}_n, \\ d'_j = {\eta'_{k_j}}^{\mathrm{ord}(d'_j)} \ \text{ for some } k_j \in K_j, \\ \sigma'_j = r_{n_j} r_{n_j+1} \cdots r_{n_j+\ell_j-2},\ \sigma_j = \Phi(\sigma'_j) = (n_j\ \ n_j{+}1\ \ldots\ n_j{+}\ell_j{-}1). \end{cases} \tag{16.26}$$

For simplicity of calculations, we take $n_j = 1$ and put $N = n_j + \ell_j - 1 = \ell_j$ so that $K_j = [1, N] \subset \boldsymbol{I}_n$. The expression $d'_j = d_j^0 h_j$ is given as

$$(16.27) \qquad d_j^0 = {\eta'_{k_j}}^{2[\mathrm{ord}(d'_j)/2]}, \quad h_j = \begin{cases} \eta'_{k_j} & \text{if } \mathrm{ord}(d'_j) \equiv 1 \pmod 2, \\ e_T & \text{if } \mathrm{ord}(d'_j) \equiv 0 \pmod 2. \end{cases}$$

Put $\kappa_p := (-1)^{p-1}/\sqrt{2}$ for simplicity, then

$$(16.28) \qquad \prod_{p,\, p+1 \in K_j} \kappa_p = (-1)^{[(\ell_j-1)/2]} \cdot 2^{-(\ell_j-1)/2}.$$

In fact, $\prod_{p,\,p+1\in K_j} (-1)^{p-1} = (-1)^{(\ell_j-1)(\ell_j-2)/2} = (-1)^{[(\ell_j-1)/2]}$. Thus, modulo constant coefficient $(-1)^{[(\ell_j-1)/2]} \cdot 2^{-(\ell_j-1)/2}$, we come to calculate the trace of

$$(16.29) \qquad \begin{cases} Y_{k_j}\,(Y_1+Y_2)(Y_2+Y_3)\cdots(Y_{N-1}+Y_N) & \text{if } \mathrm{ord}(d'_j) \equiv 1 \pmod 2, \\ (Y_1+Y_2)(Y_2+Y_3)\cdots(Y_{N-1}+Y_N) & \text{if } \mathrm{ord}(d'_j) \equiv 0 \pmod 2. \end{cases}$$

To get a monomial term of the form (16.19) with non-zero trace from expansion of (16.29), we put on each Y_p ($p \in \boldsymbol{I}_N$) a color *black* or *white* depending on if it comes in or not to form this monomial term. When Y_{k_j} corresponds to $\eta'_{k_j} = h_j$, we express it as $Y_{k_j} \in h_j$. First we start with putting color *black* to $Y_{k_j} \in h_j$, the front multiplicative factor in (16.29).

Then, we proceed successively from $\nabla_n(r_1) = \kappa_1(Y_1 + Y_2)$ until $\nabla_n(r_{N-1}) = \kappa_{N-1}(Y_{N-1} + Y_N)$ in (16.24), putting colors on Y_p and Y_{p+1} in $\nabla_n(r_p) = \kappa_p(Y_p + Y_{p+1})$ successively. At the point for $\nabla_n(r_q) = \kappa_q(Y_q + Y_{q+1})$, if we pick up the first component Y_q to get Y_q^2, then we color it *black* and the second component Y_{q+1} *white*, and similarly in the reversed case. The mode of coloring of two terms Y_p, Y_{p+1} of $\nabla_n(r_p)$ is called the *parity* of $\nabla_n(r_p)$, for which we denote (*black, white*) simply by (*b, w*), and (*white, black*) by (*w, b*).

Step 1. First start with $\nabla_n(r_1) = \kappa_1(Y_1 + Y_2)$.

Case (1): If $Y_1 \in h_j$, then Y_1 in $\nabla_n(r_1)$ should be *black* to get Y_1^2, and the parity of $\nabla_n(r_1)$ is (*black, white*) = (*b, w*).

Case (2): If $Y_1 \notin h_j$, then Y_1 in $\nabla_n(r_1)$ should be *white* and automatically Y_2 in $\nabla_n(r_1)$ should be *black*, and the parity is (*w, b*).

Step 2. Next take $\nabla_n(r_2) = \kappa_2(Y_2 + Y_3)$.

- Suppose the parity of $\nabla_n(r_1)$ is (*b, w*).

Case (bw): Since $h_j = \eta'_1$, Y_2 in $\nabla_n(r_2)$ should be *white*, and the parity of $\nabla_n(r_2)$ is (*w, b*) expecting Y_3^2, by supposing to get *black* Y_3 in the next step.

- Suppose the parity of $\nabla_n(r_1)$ is (*w, b*).

Case (wb-1): If $Y_2 \in h_j$, then Y_2 in $\nabla_n(r_2)$ should be *white* to have twice (not triple) of *black* Y_2 in total. The parity of $\nabla_n(r_2)$ is (*w, b*).

Case (wb-2): Otherwise, Y_2 in $\nabla_n(r_2)$ should be *black*, and the parity of $\nabla_n(r_2)$ is (*b, w*).

Step $q+1$**.** Assume that the parity of $\nabla_n(r_p)$ has been decided for $p = 1, \ldots, q$.

• Firstly suppose the parity of $\nabla_n(r_q)$ is $(b,\ w)$.

Case (bw-1): If $Y_{q+1} \in h_j$, then Y_{q+1} in $\nabla_n(r_{q+1}) = \kappa_{q+1}(Y_{q+1} + Y_{q+2})$ should be *black*, and the parity of $\nabla_n(r_{q+1})$ is $(b,\ w)$.

Case (bw-2): If $Y_{q+1} \notin h_j$, then Y_{q+1} in $\nabla_n(r_{q+1}) = \kappa_{q+1}(Y_{q+1} + Y_{q+2})$ should be *white*, and the parity of $\nabla_n(r_{q+1})$ is $(w,\ b)$.

• Secondly suppose the parity of $\nabla_n(r_q)$ is $(w,\ b)$.

Case (wb-1): If $Y_{q+1} \in h_j$, then Y_{q+1} in $\nabla_n(r_{q+1}) = \kappa_{q+1}(Y_{q+1} + Y_{q+2})$ should be *white*, not to get $Y_{q+1}^{\,3}$ (triple of Y_{q+1}), and the parity of $\nabla_n(r_{q+1})$ is $(w,\ b)$.

Case (wb-2): If $Y_{q+1} \notin h_j$, then Y_{q+1} in $\nabla_n(r_{q+1}) = \kappa_{q+1}(Y_{q+1} + Y_{q+2})$ should be *black*, and the parity of $\nabla_n(r_{q+1})$ is $(b,\ w)$.

Step $N-1$**.** At the end,

Case (1): if $Y_N \in h_j$, then the parity of $\nabla_n(r_{N-1})$ should be $(w,\ b)$.

Case (2): If $Y_N \notin h_j$, then the parity should be $(b,\ w)$.

This last step is guaranteed if and only if $\mathrm{ord}(d'_j) + L(\sigma'_j) \equiv 0$. In fact, the front factor in (16.29) coming from $h_j = \eta'_{k_j}$ controls the change of parities of $\nabla_n(r_p)$ according to p as follows.

> **Parity Rule I-00.** *The parity of* $\nabla_n(r_p)$ *remains unchanged between* $p = k_j - 1,\ k_j$*, and at* $p = k_j + 1$ *it changes. If we put aside of consideration* $\nabla_n(r_{k_j})$ *for* $h_j = \eta'_{k_j}$*, then for the rest of* $\nabla_n(r_p)$*'s their parities change alternatively, from* (w, b) *to* (b, w)*, or from* (b, w) *to* (w, b)*.*

In this situation, we should have even number of *blacks*, and so $L(\sigma'_j) - |\mathrm{supp}(h_j)| = L(\sigma'_j) - \mathrm{ord}(h_j)$ should be even. This is equivalent to $\mathrm{ord}(d'_j) + L(\sigma'_j) \equiv 0$.

The above process to determine parities of $\nabla_n(r_p)$ is illustrated in the table below.

Table 16.1. Parity of $\nabla_n(r_p)$ $(p = 1, 2, \ldots, N-1)$

In each small unit table, on the 1st row, columns after the separation '||' are for $Y_{k_j} \in h_j$. On the 2nd row, sections are for parities $(b,\ w)$ or $(w,\ b)$ of $\nabla_n(r_p)$, $p = 1, 3, \ldots$, and on the 3rd row, sections are for $\nabla_n(r_p)$, $p = 2, 4, \ldots$

CASE : $N = \ell(\sigma'_j) = 8$ $\big(L(\sigma'_j) \equiv 7\big)$:

$Y_{k_j} \in h_j = \eta_1$	b							
$\nabla_n(r_p)$ $(p = 1, 3, 5, 7)$	b	w	b	w	b	w	b	w
$\nabla_n(r_p)$ $(p = 2, 4, 6)$	$\times$	w	b	w	b	w	b	$\times$

$Y_{k_j} \in h_j = \eta_2$		b						
$\nabla_n(r_p)$ $(p = 1, 3, 5, 7)$	w	b	b	w	b	w	b	w
$\nabla_n(r_p)$ $(p = 2, 4, 6)$	$\times$	w	b	w	b	w	b	$\times$

$Y_{k_j} \in h_j = \eta_7$							b	
$\nabla_n(r_p)$ $(p = 1,3,5,7)$	w	b	w	b	w	b	b	w
$\nabla_n(r_p)$ $(p = 2,4,6)$	$\times$	b	w	b	w	b	w	$\times$

$Y_{k_j} \in h_j = \eta_8$								b
$\nabla_n(r_p)$ $(p = 1,3,5,7)$	w	b	w	b	w	b	w	b
$\nabla_n(r_p)$ $(p = 2,4,6)$	$\times$	b	w	b	w	b	w	$\times$

$h_j = e$								
$\nabla_n(r_p)$ $(p = 1,3,5,7)$	w	b	w	b	w	b	?	?
$\nabla_n(r_p)$ $(p = 2,4,6)$	$\times$	b	w	b	w	b	w	$\times$

NOT OK

In this way, we obtain the only one monomial term with non-zero trace, apart from its coefficient $(-1)^{[(\ell_j-1)/2]} \cdot 2^{-(\ell_j-1)/2}\chi_\gamma(d'_j)$, as

(16.30)
$$\prod_{k \in \mathrm{supp}(h_j)} Y_k \times \prod_{1 \le p \le \ell_j - 1} X_p = E_{2^{n'}},$$

(16.31)
$$\text{with} \qquad X_p := \begin{cases} Y_p & \text{if } Y_p \text{ is } \textit{black} \text{ for } \nabla_n(r_p), \\ Y_{p+1} & \text{if } Y_{p+1} \text{ is } \textit{black} \text{ for } \nabla_n(r_p). \end{cases}$$

Lemma 16.5. *For $g'_j = (d'_j, \sigma'_j)$, assume it is normalized as in (16.26). If the condition (16.25) holds, then*

(16.32)
$$\begin{aligned} \mathrm{tr}\big(\pi^{\mathrm{I}}_\gamma(g'_j)\big) &= \mathrm{tr}\big(P_\gamma(d'_j)\nabla_n(\sigma'_j)\big) \\ &= \chi_\gamma(d'_j) \cdot 2^{n'} \cdot (-1)^{[(\ell_j-1)/2]} \cdot 2^{-(\ell_j-1)/2}. \end{aligned}$$

Otherwise $\mathrm{tr}\big(\pi^{\mathrm{I}}_\gamma(g'_j)\big) = 0$.

Moreover, when $\pi^{\mathrm{I}}_\gamma(g'_j) = P_\gamma(d'_j)\nabla_n(\sigma'_j)$ is expanded into a linear combination of monomial terms $Y_1^{c_1} Y_2^{c_2} \cdots Y_N^{c_N}$, $N = \ell_j$, there exists at most one term with trace non-zero, which is of the form $Y_1^{2c'_1} Y_2^{2c'_2} \cdots Y_N^{2c'_N}$ $(= E_{2^{n'}})$.

Proof. Let $h_j = \eta'_{k_j}$. Then the factor Y_{k_j} is put in front of $\nabla_n(r_1)$ as

(16.33)
$$Y_{k_j} \cdot \kappa_1(Y_1 + Y_2) \cdot \kappa_2(Y_2 + Y_3) \cdots \kappa_{N-1}(Y_{N-1} + Y_N).$$

We send Y_{k_j} step by step by exchanging with $\nabla_n(r_1) = \kappa_1(Y_1 + Y_2), \ldots, \nabla_n(r_{k_j - 1}) = \kappa_{k_j-1}(Y_{k_j-1} + Y_{k_j})$. Since we are only interested in the unique monomial with non-zero trace in (16.30) – (16.31) obtained from (16.33) by expansion, the sign changes to send Y_{k_j} from the front of $\nabla_n(r_1)$ until the back of $\nabla_n(r_{k_j-1})$ accumulate in total to even number of times, since Y_{k_j} permutes elements of the following form

$$\begin{cases} Y_{j_1}^{2c'_1} \cdots Y_{j_u}^{2c'_u}, & j_1 < \ldots < j_u < k_j, \qquad \text{or} \\ Y_{j_1}^{2c'_1} \cdots Y_{j_u}^{2c'_u} Y_{k_j}, & j_1 < \ldots < j_u < k_j. \end{cases}$$ □

Summarizing the results until now, we have the following.

Proposition 16.6. *Assume that* $g' = (d', \sigma') \in \widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P_\gamma)$, *expressed as in (16.10), satisfies the condition* $\mathrm{ord}(d') + L(\sigma') \equiv 0 \pmod 2$.

(i) *For* $f_\gamma^{\mathrm{I}}(g') = \mathrm{tr}\big(\pi_\gamma^{\mathrm{I}}(g')\big) \neq 0$, *it is necessary and sufficient that (Condition I-00) holds for* g'. *In that case the following product formula holds*

$$f_\gamma^{\mathrm{I}}(g') = 2^{n'} \cdot \prod_{q \in Q} \frac{f_\gamma^{\mathrm{I}}(\xi'_q)}{2^{n'}} \cdot \prod_{j \in J} \frac{f_\gamma^{\mathrm{I}}(g'_j)}{2^{n'}}\,, \tag{16.34}$$

where $f_\gamma^{\mathrm{I}}(\xi'_q)$ *is given in (16.22), and* $f_\gamma^{\mathrm{I}}(g'_j)$ *is in Lemma 16.5.*

(ii) *Suppose* $f_\gamma^{\mathrm{I}}(g') = \mathrm{tr}\big(\pi_\gamma^{\mathrm{I}}(g')\big) \neq 0$. *When* $\pi_\gamma^{\mathrm{I}}(g')$ *is expanded into a linear combination of monomial terms* $Y_1^{c_1} Y_2^{c_2} \cdots Y_n^{c_n}$, *there exists only one term with trace non-zero, which is of the form* $Y_1^{2c'_1} Y_2^{2c'_2} \cdots Y_n^{2c'_n}$ $(= E_{2^{n'}})$.

Remark 16.1. Note that $\chi_\gamma(\eta'_p) = (-1)^{p-1}\zeta_\gamma(\eta_p)$ and by definition $\zeta_\gamma(\eta'_p) = \zeta_\gamma(\eta_p)$. The difference between χ_γ and ζ_γ will have an important meaning in the following (cf. Part III). So, to make the difference much clearer, we introduce a sign function on $\widetilde{D}_n$: for $d'' \in \widetilde{D}_n$, express it as $d'' = z_2^{\,b} \eta_1^{\,b_1} \eta_2^{\,b_2} \cdots \eta_n^{\,b_n}$, and put

$$\mathrm{sgn}_{\widetilde{D}}(d'') := (-1)^b = \chi_\gamma(d'')/\zeta_\gamma(d''). \tag{16.35}$$

The point is that $\mathrm{sgn}_{\widetilde{D}}(d'')$ and $\chi_\gamma(d'')$ do not behave well under the action of $\widetilde{\mathfrak{S}}_n$. In fact, for r_p $(p \in \boldsymbol{I}_{n-1})$, in CASE I,

$$\mathrm{sgn}_{\widetilde{D}}\big(r_p^{\mathrm{I}}(d'')\big) = (-1)^{\mathrm{ord}(d'') + b_p b_{p+1} + b_p + b_{p+1}}\, \mathrm{sgn}_{\widetilde{D}}(d''), \tag{16.36}$$

with $\mathrm{ord}(d'') = b_1 + \cdots + b_n$.

However, when d'' has a normalized form as $d'' = {\eta'_{k_j}}^{b_{k_j}}$ or $d'' = \eta_{k_j}^{\,b_{k_j}}$, $b_{k_j} = \mathrm{ord}(d'')$, the sign function $\mathrm{sgn}_{\widetilde{D}}(d'')$ behave much simple way as

$$\mathrm{sgn}_{\widetilde{D}}\big(r_p^{\mathrm{I}}(d'')\big) = \begin{cases} (-1)^{\mathrm{ord}(d'')}\mathrm{sgn}_{\widetilde{D}}(d'') & \text{for } p \neq k_j, k_j+1\,; \\ \mathrm{sgn}_{\widetilde{D}}(d'') & \text{for } p = k_j, k_j+1. \end{cases} \tag{16.37}$$

In particular, if $\mathrm{ord}(d'')$ is even, then $\mathrm{sgn}_{\widetilde{D}}(d'')$ is invariant under $\widetilde{\mathfrak{S}}_n$ in the sense that $\mathrm{sgn}_{\widetilde{D}}\big({\sigma'}^{\mathrm{I}}(d'')\big) = \mathrm{sgn}_{\widetilde{D}}(d'')$ $(\sigma' \in \widetilde{\mathfrak{S}}_n)$. This is one of the reasons for the choice of a representative of the conjugacy class of $g'_j = (d'_j, \sigma'_j)$ modulo $\widetilde{Z}$ in the form of $d'_j = {\eta'_{k_j}}^{b_j}$, $b_j = \mathrm{ord}(d'_j)$ $(\exists k_j \in K_j)$.

Here we add a remark for CASE II too. Recall that $r_p^{\mathrm{II}}(\eta_k) = \eta_{s_p(k)}$, $s_p = \Phi_{\mathfrak{S}}(r_p)$, then we have $\mathrm{sgn}_{\widetilde{D}}\big(r_p^{\mathrm{II}}(d'')\big) = (-1)^{b_p b_{p+1}}\, \mathrm{sgn}_{\widetilde{D}}(d'')$. Therefore, if d'' has a normalized form as $d'' = \eta_k^{\,b_k}$, $b_k = \mathrm{ord}(d'')$, then $\mathrm{sgn}_{\widetilde{D}}(d'')$ is invariant under $\widetilde{\mathfrak{S}}_n$ in the sense that $\mathrm{sgn}_{\widetilde{D}}\big({\sigma'}^{\mathrm{II}}(d'')\big) = \mathrm{sgn}_{\widetilde{D}}(d'')$ $(\sigma' \in \widetilde{\mathfrak{S}}_n)$.

16.3 Characters of $\pi_\gamma^{\mathrm{I}+}$ of $\widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P_\gamma)$, CASE 2

To start with, we assume only the condition $\mathrm{ord}(d') + L(\sigma') \equiv 1 \pmod 2$ for $g' = (d', \sigma')$. Here also the formulas (16.23) – (16.24) are valid.

In the right hand side of (16.24), when expanded into a linear combination of monomials in Y_k's, the only one which has non-zero trace is, multiplicatively modulo $\prod_{k \in \boldsymbol{I}_n} Y_k^{2c'_k}$,

$$Y_1 Y_2 \cdots Y_{2n'} Y_{2n'+1} \left(= i^{n'} E_{2^{n'}} \right), \tag{16.38}$$

and so there should be

$$n = 2n' + 1 \quad \text{odd}, \quad \text{and} \quad |\mathrm{supp}(g')| = n = 2n' + 1. \tag{16.39}$$

This turns out to be a case only for $\pi_\gamma^{\mathrm{I}+}$ ($= \pi_\gamma^{\mathrm{I}}$ for n odd). We prefer to change here the notations to those with superfices + (for instance as $P_\gamma^+ = P_\gamma$, $\nabla_n^+ = \nabla_n$, $\pi_\gamma^{\mathrm{I}+} = \pi_\gamma^{\mathrm{I}}$ etc.), in comparison to the case of $\pi_\gamma^{\mathrm{I}-}$ which will be treated later.

Let us determine contributions to the monomial $Y_1 Y_2 \cdots Y_{2n'+1}$ from each of $\pi_\gamma^{\mathrm{I}+}(\xi'_q)$ and $\pi_\gamma^{\mathrm{I}+}(g'_j)$.

(2-1) Case of $\pi_\gamma^{\mathrm{I}+}(\xi'_q)$: For each $\pi_\gamma^{\mathrm{I}+}(\xi'_q) = P_\gamma^+(\xi'_q)$, we should have the term Y_q so that $\mathrm{ord}(\xi'_q) \equiv 1$ ($q \in Q$).

(2-2) Case of $\pi_\gamma^{\mathrm{I}+}(g'_j)$: For each $\pi_\gamma^{\mathrm{I}+}(g'_j) = P_\gamma^+(d'_j)\nabla_n^+(\sigma'_j)$, we should have $\prod_{k \in K_j} Y_k$ multiplicatively modulo $\prod_{k \in K_j} Y_k^{2c'_k}$, and accordingly $\mathrm{ord}(d'_j) + L(\sigma'_j) \equiv |K_j|$. On the other hand, $|K_j| \equiv L(\sigma'_j) + 1 \pmod 2$, and accordingly $\mathrm{ord}(d'_j) \equiv 1$ ($j \in \boldsymbol{I}_s$). Hence, if $f_\gamma^{\mathrm{I}+}(g') \neq 0$, then g' satisfies (Condition I-11) in Lemma 16.2.

As in CASE 1, we may assume that g'_j is normalized as in (16.26), and put here for simplicity of calculations as $n_j = 1$ and $N = n_j + \ell_j - 1 = \ell_j$ so that $K_j = [1, N] \subset \boldsymbol{I}_n$. Note that $d'_j = d_j^0 h_j$, $d_j^0 = {\eta'_{k_j}}^{\mathrm{ord}(d'_j)-1}$, $h_j = \eta'_{k_j}$ ($k_j \in K_j$). Then we have

$$\begin{aligned} P_\gamma^+(d'_j)\, \nabla_n^+(\sigma'_j) = (-1)^{[(\ell_j-1)/2]}\, 2^{-(\ell_j-1)/2}\, \chi_\gamma(d'_j) \times \\ \times Y_{k_j}\, (Y_1 + Y_2)(Y_2 + Y_3) \cdots (Y_{N-1} + Y_N). \end{aligned} \tag{16.40}$$

To get in total the monomial term (16.38), we should pick up from (16.40) a monomial term $Y_1 Y_2 \cdots Y_N$ in modulo. On picking up from Y_1 until Y_N successively, the parity of each $\nabla_n^+(r_p)$ will be uniquely determined one after another as is seen below. As in CASE 1, $Y_{k_j} \in h_j$ should be *black* from the beginning.

Step 1. We have two cases as follows.

Case (1): $Y_1 \in h_j$. In this case, Y_1 in $\nabla_n^+(r_1)$ should be *white* and so Y_2 in it is forced to be *black*, and the parity of $\nabla_n^+(r_1)$ should be (w, b).

Case (2): $Y_1 \notin h_j$. In this case, Y_1 in $\nabla_n^+(r_1)$ should be *black*, and so the parity of $\nabla_n^+(r_1)$ should be (b, w).

Now suppose that, the parities of $\nabla_n^+(r_p),\ p=1,\ldots,q$, have been determined.

Step $q+1$**.**

• Suppose the parity of $\nabla_n^+(r_q)$ is $(w,\ b)$. Then $h_j=\eta'_{k_j}$ with $k_j\le q$.

Case (wb): Here $Y_{q+1}\notin h_j$, and Y_{q+1} in $\nabla_n^+(r_{q+1})$ should be *white*, to keep Y_{q+1} once, already taken from $\nabla_n^+(r_q)$ as *black*, and so the parity of $\nabla_n^+(r_{q+1})$ should be $(w,\ b)$.

• Suppose the parity of $\nabla_n^+(r_q)$ is $(b,\ w)$.

Case (bw-1): If $Y_{q+1}\in h_j$, then Y_{q+1} in $\nabla_n^+(r_{q+1})=\kappa_{q+1}(Y_{q+1}+Y_{q+2})$ should be *white*, and so the parity should be $(w,\ b)$.

Case (bw-2): If $Y_{q+1}\notin h_j$, then Y_{q+1} in $\nabla_n^+(r_{q+1})$ should be *black*, and so the parity should be $(b,\ w)$.

Parity Rule I-11. *As a general rule, at the step for* $\nabla_n^+(r_{q+1})$*,*

(1) *if* $Y_{q+1}\in h_j$*, then the parity of* $\nabla_n^+(r_{q+1})$ *is reversed from that of* $\nabla_n^+(r_q)$*;*

(2) *if* $Y_{q+1}\notin h_j$*, then the parity of* $\nabla_n^+(r_{q+1})$ *is unchanged from that of* $\nabla_n^+(r_q)$*.*

Step $N-1$**.** At the end,

Case (1): if $Y_N\in h_j$, then Y_N in $\nabla_n^+(r_{N-1})=\kappa_{N-1}(Y_{N-1}+Y_N)$ should be *white*, to have Y_N only one time (from h_j). So the parity should be $(b,\ w)$.

Case (2): If $Y_N\notin h_j$, then the parity should be $(w,\ b)$.

Let check the consistency of Step $N-1$ above with **Parity Rule I-11**. Note that, in the above, the factor $Y_{k_j}\in h_j$ in the front multiplicative factor in (16.40) changes the parity of $\nabla_n^+(r_{k_j})=\kappa_{k_j}(Y_{k_j}+Y_{k_j+1})$. Consider three cases depending on

(a) whether $Y_1\in h_j$ or not, and

(b) whether $Y_N\in h_j$ or not.

We see below each case is possible under $\mathrm{ord}(d'_j)\equiv L(\sigma'_j)\equiv 1$.

Case (Yes, No): The parities of $\nabla_n^+(r_p)$ starts from $(w,\ b)$ for $p=1$ and should end at $(w,\ b)$ for $p=N$.

Case (No, Yes): The parities of $\nabla_n^+(r_p)$ starts from $(b,\ w)$ for $p=1$ and should end at $(b,\ w)$ for $p=N$.

Case (No, No): The parities of $\nabla_n^+(r_p)$ starts from $(b,\ w)$ for $p=1$ and should end at $(w,\ b)$ for $p=N$.

Altogether we see the following.

Lemma 16.7. *Assume a* $g'=(d',\sigma')\in\widetilde{D}_n\overset{\mathrm{I}}{\rtimes}\mathcal{S}(P_\gamma)$ *is in CASE 2:* $\mathrm{ord}(d')+L(\sigma')\equiv 1\ (\mathrm{mod}\ 2)$*. Then, to have* $\mathrm{tr}\big(\pi_\gamma^{\mathrm{I}}(g')\big)\neq 0$*, n should be odd and* $|\mathrm{supp}(g')|=n$*. In this case,* $f_\gamma^{\mathrm{I}+}(g')=\mathrm{tr}\big(\pi_\gamma^{\mathrm{I}+}(g')\big)\neq 0$ *if and only if (Condition I-11) holds for* g'*.*

Moreover, in the expansion of $\pi_\gamma^{\mathrm{I}+}(g')$ *into a linear combination of monomials terms* $Y_1^{c_1}Y_2^{c_2}\ldots Y_n^{c_n}$*, there exists one and only one term with non-zero trace, which is* $Y_1\cdots Y_{2n'}Y_{2n'+1}$ *(multiplicatively modulo* $Y_1^{2c'_1}Y_2^{2c'_2}\ldots Y_n^{2c'_n}$*) in*

(16.38).

The way of determining parities of $\nabla_n^+(r_p)$'s is illustrated in the following table.

Table 16.2. Parities of $\nabla_n^+(r_p) = \kappa_p(Y_p + Y_{p+1})$ $(p+1 \in K_j)$ in $\pi_\gamma^{\mathrm{I}+}(g'_j)$.

CASE: $N = \ell(\sigma'_j) = 10$ $\big(L(\sigma'_j) \equiv 9\big)$: $K_j = \{1, 2, \ldots, N\}$,

$Y_{k_j} \in h_j = \eta_1$	b									
$\nabla_n^+(r_p)$ $(p = 1,3,5,7,9)$	w	b	w	b	w	b	w	b	w	b
$\nabla_n^+(r_p)$ $(p = 2,4,6,8)$	$\times$	w	b	w	b	w	b	w	b	$\times$

$Y_{k_j} \in h_j = \eta_2$		b								
$\nabla_n^+(r_p)$ $(p = 1,3,5,7,9)$	b	w	w	b	w	b	w	b	w	b
$\nabla_n^+(r_p)$ $(p = 2,4,6,8)$	$\times$	w	b	w	b	w	b	w	b	$\times$

$Y_{k_j} \in h_j = \eta_9$									b	
$\nabla_n^+(r_p)$ $(p = 1,3,5,7,9)$	b	w	b	w	b	w	b	w	w	b
$\nabla_n^+(r_p)$ $(p = 2,4,6,8)$	$\times$	b	w	b	w	b	w	b	w	$\times$

$Y_{k_j} \in h_j = \eta_{10}$										b
$\nabla_n^+(r_p)$ $(p = 1,3,5,7,9)$	b	w	b	w	b	w	b	w	b	w
$\nabla_n^+(r_p)$ $(p = 2,4,6,8)$	$\times$	b	w	b	w	b	w	b	w	$\times$

$h_j = e_T$											
$\nabla_n^+(r_p)$ $(p = 1,3,5,7,9)$	b	w	b	w	b	w	b	w	?	?	NOT OK
$\nabla_n^+(r_p)$ $(p = 2,4,6,8)$	$\times$	b	w	b	w	b	w	b	w	$\times$	

Thus, under (Condition I-11), we have a unique monomial term inside $\pi_\gamma^{\mathrm{I}+}(g'_j)$ for each $j \in \boldsymbol{I}_s$, to get $Y_1 Y_2 \cdots Y_{2n'+1}$ in total, together with $\pi_\gamma^{\mathrm{I}+}(\xi'_q)$ $(q \in Q)$. The specified monomial term from $\pi_\gamma^{\mathrm{I}+}(g'_j)$, apart from its coefficient $(-1)^{[(\ell_j-1)/2]} \cdot 2^{-(\ell_j-1)/2} \cdot \chi_\gamma(d'_j)$, is in case $h_j = \eta'_{k_j}$ $(\exists k_j \in K_j)$

$$Y_{k_j} \cdot \prod_{1 \le p \le N-1} X_p = \varepsilon^{\mathrm{I}+}(g'_j) \prod_{p \in K_j} Y_p \,, \tag{16.41}$$

$$\text{with} \quad X_p = \begin{cases} Y_p & \text{if } Y_p \text{ is } \textit{black} \text{ for } \nabla_n^+(r_p), \\ Y_{p+1} & \text{if } Y_{p+1} \text{ is } \textit{black} \text{ for } \nabla_n^+(r_p), \end{cases} \tag{16.42}$$

where $\varepsilon^{\mathrm{I}+}(g'_j) := (-1)^{k_j - 1}$, in the above setting that $K_j = [1, N]$.

For $\pi_\gamma^{\mathrm{I}+}(g')$ in total, by taking product over $q \in Q := \{q_i \;;\; i \in \boldsymbol{I}_r\}$ and $j \in J := \boldsymbol{I}_s$, we get the following.

Proposition 16.8. *Let n be odd, and $g' = (d', \sigma') \in \widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P_\gamma^+)$ satisfies (Condition I-11). Assume that each g'_j $(j \in J)$ is normalized as in (16.26). Then $f_\gamma^{\mathrm{I}+}(g') = \mathrm{tr}\big(\pi_\gamma^{\mathrm{I}+}(g')\big)$ is given by*

$$f_\gamma^{\mathrm{I}+}(g') \;=\; \prod_{q \in Q} \chi_\gamma(\xi'_q) \cdot \prod_{j \in J} \varepsilon^{\mathrm{I}+}(g'_j) \chi_\gamma(d'_j)\, (-1)^{[(\ell_j-1)/2]}\, 2^{-(\ell_j-1)/2} \times$$

$$\times\ \mathrm{tr}\Big(\prod_{q\in Q} Y_q \cdot \prod_{j\in J}\prod_{p\in K_j} Y_p\Big)$$

$$= \ \varepsilon^{\mathrm{I}}(g')\cdot \prod_{q\in Q}\chi_\gamma(\xi'_q)\cdot\prod_{j\in J}\chi_\gamma(d'_j)\cdot(2i)^{n'}\prod_{j\in J}(-1)^{[(\ell_j-1)/2]}\,2^{-(\ell_j-1)/2},$$

where the sign $\varepsilon^{\mathrm{I}+}(g'_j)$ *is given by (16.41)–(16.42), and the one* $\varepsilon^{\mathrm{I}}(g')$ *comes from rearrangement to have* $Y_1Y_2\cdots Y_{2n'+1}$ *as*

$$\prod_{q\in Q} Y_q \times \prod_{j\in J}\Big\{Y_{k_j}\prod_{\substack{p\in K_j\setminus\{k_j\}:\\ Y_p\ \text{black}}} Y_p\Big\} \ = \ \varepsilon^{\mathrm{I}}(g')\cdot Y_1Y_2\cdots Y_n\,. \tag{16.43}$$

Moreover, since $\mathrm{ord}(d'_j)\equiv 1$ $(j\in J)$, $d' = z_2^{\,X}\xi'_{q_1}\cdots\xi'_{q_r}d'_1\cdots d'_s$ *with* X *in (16.11) which is given here as* $X=\sum_{2\le j\le s}\big(L(\sigma'_1)+\cdots+L(\sigma'_{j-1})\big)$.

16.4 Characters of IRs $\pi_\gamma^{\mathrm{I}-} = P_\gamma^-\cdot\nabla_n^-$ of $\widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P_\gamma^-)$

Assume that $n=2n'+1$ odd. Then, for $\pi_\gamma^{\mathrm{I}-}=P_\gamma^-\cdot\nabla_n^-$, $P_\gamma^-=P_{\tau_n\gamma}$ with $\gamma\in\Gamma_n^0$, we have the following formulas by (16.1) – (16.3),

$$\begin{cases} \nabla_n^-(r_i)=\nabla_n^+(r_i)=\dfrac{(-1)^{i-1}}{\sqrt2}\,(Y_i+Y_{i+1}) & (i\in \boldsymbol{I}_{n-2}),\\ \nabla_n^-(r_{n-1})=\dfrac{(-1)^{n-2}}{\sqrt2}\,(Y_{n-1}-Y_n) & (i=n-1)\ ; \end{cases} \tag{16.44}$$

$$\begin{cases} P_\gamma^-(\eta'_j)=P_{\tau_n\gamma}(\eta'_j)=\omega^{\gamma_j}\rho(\eta'_j)=\chi_\gamma(\eta'_j)\,Y_j & (j\in\boldsymbol{I}_{n-1}=\boldsymbol{I}_{2n'}),\\ P_\gamma^-(\eta'_n)=P_{\tau_n\gamma}(\eta'_n)=\omega^{\gamma_n+m'}\rho(\eta'_n)=-\chi_\gamma(\eta'_n)\,Y_n & (j=n=2n'+1). \end{cases} \tag{16.45}$$

Let $g'=(d',\sigma')\in\widetilde{D}_n\overset{\mathrm{I}}{\rtimes}\mathcal{S}(P_\gamma^-)$. We can discuss just as in §§16.2 – 16.3 for $\pi_\gamma^{\mathrm{I}+}$, and only the difference from there is that,

(A) $\chi_\gamma(\eta'_n)Y_n, n=2n'+1$, is replaced by $\chi_{\tau_n\gamma}(\eta'_n)Y_n=-\chi_\gamma(\eta'_n)Y_n$,
(from the contribution of d'-side);

(B) Y_n in $\nabla_n^{\mathrm{I}+}(r_{n-1})=\kappa_{n-1}(Y_{n-1}+Y_n)$ is replaced by $-Y_n$ in $\nabla_n^{\mathrm{I}-}(r_{n-1})=\kappa_{n-1}(Y_{n-1}-Y_n)$, where $\kappa_p=(-1)^{p-1}/\sqrt2$,
(from the contribution of σ'-side).

Altogether, we see that, for $\pi_\gamma^{\mathrm{I}-}$, it is enough to replace Y_n by $-Y_n$ everywhere on the way of calculating characters of $\pi_\gamma^{\mathrm{I}+}$. In more detail, we see the following.

CASE 1 : $\mathrm{ord}(d')+L(\sigma')\equiv 0 \pmod 2$:

In the case where $Y_n, n=2n'+1$, does not contribute to the unique monomial term with non-zero trace $Y_1^{2c'_1}Y_2^{2c'_2}\cdots Y_n^{2c'_n}$, we have $\mathrm{tr}\big(\pi_\gamma^{\mathrm{I}+}(g')\big)=\mathrm{tr}\big(\pi_\gamma^{\mathrm{I}-}(g')\big)$.

On the contrary, when $Y_n, n=2n'+1$, actually contributes to the unique monomial term with non-zero term $Y_1^{2c'_1}Y_2^{2c'_2}\cdots Y_n^{2c'_n}$, there are two cases:

Case (1-1) The case where Y_n appears only as a contribution from d'. In this case, there appear $\big(\chi_\gamma(\eta'_n)Y_n\big)^{2c'_n}$ for $\pi_\gamma^{\mathrm{I}+}(g')$ in §16.2, and $\big(-\chi_\gamma(\eta'_n)Y_n\big)^{2c'_n}$ for $\pi_\gamma^{\mathrm{I}-}(g')$ here. Thus we have $\mathrm{tr}\big(\pi_\gamma^{\mathrm{I}+}(g')\big) = \mathrm{tr}\big(\pi_\gamma^{\mathrm{I}-}(g')\big)$.

Case (1-2) The case where Y_n appears as contributions both from d' and from σ'. In this case, for $\pi_\gamma^{\mathrm{I}+}(g')$ in §16.2, there appear as contributions from the side of d' and from the side of σ' respectively:

$$\big(\chi_\gamma(\eta'_n)Y_n\big)^{2c'_n-1} \quad \text{and} \quad Y_n \ \text{in} \ \nabla_n^+(r_{n-1}) = \kappa_{n-1}(Y_{n-1}+Y_n).$$

On the other hand, for $\pi_\gamma^{\mathrm{I}-}(g')$ here, there appear as contributions from the side of d' and from the side of σ' respectively:

$$\big(-\chi_\gamma(\eta'_n)Y_n\big)^{2c'_n-1} \quad \text{and} \quad -Y_n \ \text{in} \ \nabla_n^-(r_{n-1}) = \kappa_{n-1}\,(Y_{n-1}-Y_n).$$

Both for $\pi_\gamma^{\mathrm{I}+}$ and $\pi_\gamma^{\mathrm{I}-}$, these two elements contribute as their product to the monomial with non-zero trace, and so the difference of the signs cancels out. Hence $\mathrm{tr}\big(\pi_\gamma^{\mathrm{I}+}(g')\big) = \mathrm{tr}\big(\pi_\gamma^{\mathrm{I}-}(g')\big)$.

Lemma 16.9. *Suppose* $n = 2n'+1$ *odd, and* $\mathrm{ord}(d') + L(\sigma') \equiv 0 \pmod 2$ *for* $g' = (d',\sigma') \in \widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P_\gamma^-)$. *Then*

$$\mathrm{tr}\big(\pi_\gamma^{\mathrm{I}-}(g')\big) = \mathrm{tr}\big(\pi_\gamma^{\mathrm{I}+}(g')\big),$$

and $\mathrm{tr}\big(\pi_\gamma^{\mathrm{I}-}(g')\big) \neq 0$ *if and only if (Condition I-00) holds for* g'. *In that case, when* $\pi_\gamma^{\mathrm{I}-}(g')$ *is expanded into a linear combination of monomial terms such as* $Y_1^{c_1}Y_2^{c_2}\cdots Y_n^{c_n}$, *the term with non-zero trace is unique and of the form* $Y_1^{2c'_1}Y_2^{2c'_2}\cdots Y_n^{2c'_n}\ \big(= E_{2^{n'}}\big)$.

CASE 2 : $\mathrm{ord}(d') + L(\sigma') \equiv 1 \pmod 2$:

In this case, we can discuss as in §16.3 for $\pi_\gamma^{\mathrm{I}+}$. The unique non-zero trace comes from $Y_1Y_2\cdots Y_{2n'+1}$ multiplicatively modulo $\prod_{j\in \boldsymbol{I}_n} Y_j^{2c'_j}$ as in (16.38), and to have such a term, g' should satisfy the condition (16.39).

Moreover, in a similar discussion here for $\pi_\gamma^{\mathrm{I}-}$ as that for $\pi_\gamma^{\mathrm{I}+}$ in §16.3, the only difference between them is (A) and (B) above. There follows from this the following.

Lemma 16.10. *Suppose* $n = 2n'+1$ *odd, and* $\mathrm{ord}(d') + L(\sigma') \equiv 1 \pmod 2$ *for* $g' = (d',\sigma') \in \widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P_\gamma^-)$. *Then*

$$\mathrm{tr}\big(\pi_\gamma^{\mathrm{I}-}(g')\big) = -\mathrm{tr}\big(\pi_\gamma^{\mathrm{I}+}(g')\big),$$

and $\mathrm{tr}\big(\pi_\gamma^{\mathrm{I}-}(g')\big) \neq 0$ *if and only if (Condition I-11) holds for* g'. *In that case, when* $\pi_\gamma^{\mathrm{I}-}(g')$ *is expanded into a linear combination of monomial terms such as* $Y_1^{c_1}Y_2^{c_2}\cdots Y_n^{c_n}$, *the term with non-zero trace is unique and of the form* $Y_1Y_2\cdots Y_n\ \big(= i^{n'}E_{2^{n'}}\big)$.

Remark 16.2 (Sum of IRs $\pi_\gamma^{\mathrm{I}+} \oplus \pi_\gamma^{\mathrm{I}-}$ for n odd). From the above lemma, the character of the direct sum $\pi_\gamma^{\mathrm{I,odd}} := \pi_\gamma^{\mathrm{I}+} \oplus \pi_\gamma^{\mathrm{I}-}$ for n odd is given as follows: for $g' = (d', \sigma') \in \widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P_\gamma^+) = \widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P_\gamma^-)$,

$$(16.46) \quad \mathrm{tr}\big(\pi_\gamma^{\mathrm{I,odd}}(g')\big) = \begin{cases} 2 \cdot \mathrm{tr}\big(\pi_\gamma^{\mathrm{I}+}(g')\big) & \text{if } \mathrm{ord}(d') + L(\sigma') \equiv 0 \pmod 2 \,; \\ 0 & \text{if } \mathrm{ord}(d') + L(\sigma') \equiv 1 \pmod 2 \,. \end{cases}$$

By the results in §5, we see that $\pi_\gamma^{\mathrm{I,odd}}$ is naturally constructed using the representation $\rho_{n,+} \oplus \rho_{n,-}$ of $\mathcal{F}_n$.

16.5 Character formulas for π_γ^{I} (n even), and $\pi_\gamma^{\mathrm{I}\pm}$ (n odd)

Summarizing the results until here we have character formulas for π_γ^{I} of $\widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P_\gamma)$, and $\pi_\gamma^{\mathrm{I}\pm}$ of $\widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P_\gamma^\pm)$ as follows.

Theorem 16.11. *Let $\gamma \in \Gamma_n^0$. Take a $g' = (d', \sigma')$ from $\widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P_\gamma)$ for $n = 2n'$ even, or from $\widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P_\gamma^\pm)$ for $n = 2n' + 1$ odd respectively, and express it as $g' = (d', \sigma') = \xi'_{q_1} \cdots \xi'_{q_r} g'_1 \cdots g'_s$ in (16.10). Assume that g' is normalized modulo $\widetilde{Z} = \langle z_1, z_2 \rangle$ in such a way that each g'_j ($j \in J = \boldsymbol{I}_s$) satisfies (16.26).*

(i) Case $n = 2n'$ even: *The normalized character $\widetilde{f}_\gamma^{\mathrm{I}} = \mathrm{tr}\big(\pi_\gamma^{\mathrm{I}}\big)/2^{n'}$ is factorizable in the sense that the product formula (16.47) below holds in general. If $\mathrm{ord}(d') + L(\sigma') \equiv 0 \pmod 2$, then $f_\gamma^{\mathrm{I}}(g') = \mathrm{tr}\big(\pi_\gamma^{\mathrm{I}}(g')\big) \neq 0$ if and only if (Condition I-00) in Lemma 16.2 holds for g'. In that case,*

$$(16.47) \qquad \begin{aligned} \widetilde{f}_\gamma^{\mathrm{I}}(g') &= \prod_{q \in Q} \widetilde{f}_\gamma^{\mathrm{I}}(\xi'_q) \cdot \prod_{j \in J} \widetilde{f}_\gamma^{\mathrm{I}}(g'_j)\,, \\ \widetilde{f}_\gamma^{\mathrm{I}}(\xi'_q) &= \chi_\gamma(\xi'_q) \qquad\qquad (q \in Q)\,, \\ \widetilde{f}_\gamma^{\mathrm{I}}(g'_j) &= \chi_\gamma(d'_j) \cdot (-1)^{[(\ell_j - 1)/2]}\, 2^{-(\ell_j - 1)/2} \quad (j \in J). \end{aligned}$$

If $\mathrm{ord}(d') + L(\sigma') \equiv 1 \pmod 2$, then $f_\gamma^{\mathrm{I}}(g') = 0$ identically.

(ii) Case $n = 2n' + 1$ odd: *The normalized characters $\widetilde{f}_\gamma^{\mathrm{I}\pm} = \mathrm{tr}\big(\pi_\gamma^{\mathrm{I}\pm}\big)/2^{n'}$ are not factorizable. If $\mathrm{ord}(d') + L(\sigma') \equiv 0 \pmod 2$, then $f_\gamma^{\mathrm{I}+}(g') = \mathrm{tr}\big(\pi_\gamma^{\mathrm{I}+}(g')\big)$ is given by the same formula as for $f_\gamma^{\mathrm{I}}(g') = \mathrm{tr}\big(\pi_\gamma^{\mathrm{I}}(g')\big)$ above, and*

$$f_\gamma^{\mathrm{I}-}(g') \;=\; f_\gamma^{\mathrm{I}+}(g').$$

If $\mathrm{ord}(d') + L(\sigma') \equiv 1 \pmod 2$, then $f_\gamma^{\mathrm{I}+}(g') \neq 0$ if and only if (Condition I-11) holds for g'. In that case,

$$f_\gamma^{\mathrm{I}+}(g') \;=\; \varepsilon^{\mathrm{I}}(g')\, \chi_\gamma(d') \cdot (2i)^{n'} \prod_{j \in J} (-1)^{[(\ell_j - 1)/2]}\, 2^{-(\ell_j - 1)/2},$$

$$f_\gamma^{\rm I-}(g') \;=\; -\,f_\gamma^{\rm I+}(g')\,,$$

where the sign $\varepsilon^{\rm I}(g')$ *comes from rearrangement to have* $Y_1Y_2\cdots Y_n$ *as*

$$\prod_{q\in Q} Y_q \times \prod_{j\in J}\Big\{Y_{k_j}\prod_{\substack{p\in K_j\setminus\{k_j\}:\\ Y_p\ \text{black}}} Y_p\Big\} \;=\; \varepsilon^{\rm I}(g')\cdot Y_1Y_2\cdots Y_n\,.$$

(iii) *Suppose* $f_\gamma^{\rm I}(g')\neq 0$ (*resp.* $f_\gamma^{\rm I\pm}(g')\neq 0$) *for* $g'=(d',\sigma')\in \widetilde{D}_n \overset{\rm I}{\rtimes} \mathcal{S}(P_\gamma)$ $\big($*resp.* $\widetilde{D}_n \overset{\rm I}{\rtimes} \mathcal{S}(P_\gamma^\pm)\big)$. *Then, when* $\pi_\gamma^{\rm I}(g') = P_\gamma(d')\nabla_n(\sigma')$ $\big($*resp.* $\pi_\gamma^{\rm I\pm}(g') = P_\gamma^\pm(d')\nabla_n^\pm(\sigma')\big)$ *is expanded into a linear combination of monomial terms such as* $Y_1^{c_1}Y_2^{c_2}\cdots Y_n^{c_n}$, *there exists only one monomial term having non-zero trace, which is of the form* $Y_1^{2c_1'}Y_2^{2c_2'}\cdots Y_n^{2c_n'}$ $\big(=E_{2^{n'}}\big)$ *if* $\mathrm{ord}(d')+L(\sigma')\equiv 0 \pmod 2$, *or of the form* $Y_1Y_2\cdots Y_n$ $\big(= i^{n'}E_{2^{n'}}\big)$ *if* n *is odd and* $\mathrm{ord}(d')+L(\sigma')\equiv 1 \pmod 2$.

Suppose $f_\gamma^{\rm I}(g')=0$ (*resp.* $f_\gamma^{\rm I\pm}(g')=0$), *then all the monomial terms have trace zero.*

17 Characters of IRs of $\widetilde{D}_n \overset{\rm II}{\rtimes} \mathcal{S}(P_\gamma)$, $\widetilde{D}_n \overset{\rm II}{\rtimes} \mathcal{S}(P_\gamma^\pm)$

17.1 Formulas for calculating trace of $\pi_\gamma^{\rm II}(g')$, $n=2n'$

Recall the definitions of spin representations $\nabla_n^{\rm II}$ in §8.1 of $\widetilde{\mathfrak{S}}_n$, and $\pi_\gamma^{\rm II}$ in §12.2:

(17.1) $\qquad \nabla_n'(r_p) = \dfrac{1}{\sqrt{2}}\,(Y_p - Y_{p+1}) \quad (p\in \boldsymbol{I}_{n-1}) \qquad$ of $\widetilde{\mathfrak{S}}_n$;

(17.2) $\qquad \nabla_n^{\rm II}(r_p) = (iY_{2n'+1})\,\nabla_n'(r_p) = \nabla_n'(r_p)\,(-i\,Y_{2n'+1}) \;\; (p\in \boldsymbol{I}_{2n'-1}) \quad$ of $\widetilde{\mathfrak{S}}_{2n'}$;

(17.3) $\qquad \pi_\gamma^{\rm II}(d',\sigma') = P_\gamma(d')\cdot\nabla_n^{\rm II}(\sigma') \qquad \big((d',\sigma')\in \widetilde{D}_n \overset{\rm II}{\rtimes} \mathcal{S}(P_\gamma)\big) \quad$ for $n=2n'$.

Let $n=2n'$ even, and put the character of $\pi_\gamma^{\rm II} = P_\gamma\cdot\nabla_n^{\rm II}$ as

(17.4) $\qquad f_\gamma^{\rm II}(g') := \mathrm{tr}\big(\pi_\gamma^{\rm II}(g')\big) \qquad \big(g'\in \widetilde{D}_n \overset{\rm II}{\rtimes} \mathcal{S}(P_\gamma)\big).$

Then $f_\gamma^{\rm II}(z_1g') = f_\gamma^{\rm II}(z_2g') = -f_\gamma^{\rm II}(g')$, since $\pi_\gamma^{\rm II}(z_1)=\pi_\gamma^{\rm II}(z_2) = -E$.

Here we refer to Table 4.1 for general information on the supports of spin characters of CASE II. We keep to the notation in the preceding section.

Any element $g''\in \widetilde{D}_n \overset{\rm II}{\rtimes} \mathcal{S}(P_\gamma) \subset \widetilde{G}_n^{\rm II} = \widetilde{D}_n \overset{\rm II}{\rtimes} \widetilde{\mathfrak{S}}_n$ has a standard decomposition as

(17.5)
$$\begin{cases} g'' = z_1^{b_1}z_2^{b_2}\,g', \;\; g'=(d',\sigma') = \xi_{q_1}'\cdots\xi_{q_r}'\,g_1'\cdots g_s', \\ \xi_q' = t_q = \eta_q^{a_q}, \;\; \sigma' = \sigma_1'\cdots\sigma_s'\,, \; d' = \xi_{q_1}'\cdots\xi_{q_r}'\cdot d_1'd_2'\cdots d_s', \\ g_j' = (d_j',\sigma_j'), \; \sigma_j = \Phi_{\mathfrak{S}}(\sigma_j') \text{ a cycle}, \; K_j := \mathrm{supp}(\sigma_j') \supset \mathrm{supp}(d_j'), \\ d_j' = \prod_{p\in K_j}\eta_p^{a_p} = d_j^0h_j, \; d_j^0\in \widetilde{D}_n^0 = \widetilde{D}^0(m,n), \; h_j = \prod_{p\in K_j}\eta_p^{\beta_p}, \; \beta_p = 0,1\,, \end{cases}$$

where the product over $p \in K_j$ is in the natural order of p. In appearance, this decomposition is similar as (16.10), but we use here the generators $\eta_1, \dots, \eta_n$ of $\widetilde{D}_n$ instead of $\eta'_1, \dots, \eta'_n$ there, and the multiplication rule here should be understood in CASE II. For instance, the factor ${z_2}^X$ in (16.10) in front of the expression of d' does not appear here in (17.5) because $r_i^{\mathrm{II}}(\eta_k) = \eta_{s_i(k)}$ $(k \in \boldsymbol{I}_n)$.

We normalize $\gamma = (\gamma_1, \dots, \gamma_n) \in \Gamma_n^0$ under $\mathfrak{S}_n$ so that (11.12) – (11.14) hold, in particular, every $I_{n,\zeta}$ is an interval of $\boldsymbol{I}_n$.

Lemma 17.1. *For $g'_j = (d'_j, \sigma'_j)$, assume that $K_j = \mathrm{supp}(\sigma'_j) \subset I_{n,\zeta} = [1, M]$.*

(i) *Put $\sigma_j^{0\prime} := r_1 r_2 \cdots r_{\ell_j - 1}$, and take a $\tau' \in \widetilde{\mathfrak{S}}_{I_{n,\zeta}}$ such that $\tau \sigma_j \tau^{-1} = \sigma_j^0 = \Phi_{\mathfrak{S}}(\sigma_j^{0\prime})$ in $\mathfrak{S}_n$ with $\sigma^0 = (1\ 2\ \dots\ \ell_j)$ and $\tau = \Phi_{\mathfrak{S}}(\tau')$. Then $\tau' \sigma'_j \tau'^{-1}$ is equal to $\sigma_j^{0\prime}$ modulo a power of z_1.*

(ii) *In case $\sigma'_j = \sigma_j^{0\prime}$, $K_j = \mathrm{supp}(\sigma'_j) = \boldsymbol{I}_{\ell_j}$, there exists a $\tilde{d} \in \widetilde{D}_{K_j}$ such that*

$$\tilde{d}\, g'_j\, \tilde{d}^{-1} = {z_2}^x \cdot (d''_j, \sigma'_j) \quad \text{with} \quad d''_j = {\eta_k}^{\mathrm{ord}(d'_j)},$$

for some $k \in K_j$ with a certain exponent x.

(iii) *Let $g'_j = (d'_j, \sigma'_j)$ be such that $d'_j = \eta_k^b$, $\sigma'_j = \sigma_j^{0\prime}$. Then, with suffix $k-1$ modulo ℓ_j,*

$${\sigma'_j}^{-1} g'_j \sigma'_j = ({\eta_{k-1}}^b, \sigma_j^{0\prime}).$$

For each $j \in J = \boldsymbol{I}_s$, we have $K_j \subset I_{n,\zeta}$ for some $\zeta \in \widehat{T}^0$, since $g' = (d', \sigma') \in \widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \mathcal{S}(P_\gamma)$ with $\mathcal{S}(P_\gamma) = \Phi_{\mathfrak{S}}^{-1}\big(\prod_{\zeta \in \widehat{T}^0} \mathfrak{S}_{I_{n,\zeta}}\big)$ in case $n = 2n'$ even. We have for $n = 2n'$ even, with $Q = \{q_1, q_2, \dots, q_r\}$,

$$\pi_\gamma^{\mathrm{II}}(g') = \prod_{q \in Q} P_\gamma(\xi'_q) \times \prod_{j \in J} \Big(P_\gamma(d'_j) \nabla_n^{\mathrm{II}}(\sigma'_j) \Big), \tag{17.6}$$

$$P_\gamma(\xi'_q) = \chi_\gamma({\eta_q}^{a_q})\, {Y_q}^{a_q} = \zeta_\gamma({\eta_q}^{a_q})\, {Y_q}^{a_q} \qquad (q \in Q)\,, \tag{17.7}$$

$$P_\gamma(d'_j) = \prod_{p \in K_j} \chi_\gamma\big({\eta_p}^{a_p}\big) {Y_p}^{a_p} = \zeta_\gamma(d'_j) \cdot \prod_{p \in K_j} {Y_p}^{\beta_p} \quad (j \in J)\,, \tag{17.8}$$

where $\nabla_n^{\mathrm{II}}(\sigma'_j)$ is given by a product of $\nabla_n^{\mathrm{II}}(r_p)$ in (17.2), and the product over $p \in K_j$ should be in the natural order of p.

To calculate characters, we can assume as in §16.1.3 the normalization of the situation according to **(NM) Normalization of the situation:** (NM1)+(NM2).

We work dividing the situation into two cases as

CASE 1sig: $L(\sigma') \equiv \sum_{j \in J} L(\sigma'_j) \equiv 0 \pmod 2$**,**

CASE 2sig: $L(\sigma') \equiv \sum_{j \in J} L(\sigma'_j) \equiv 1 \pmod 2$**.**

17.2 Characters of $\pi_\gamma^{\mathrm{II}} = P_\gamma \cdot \nabla_n^{\mathrm{II}}$ of $\widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \mathcal{S}(P_\gamma)$, CASE 1sig

17.2.1 Reduction of calculations in CASE 1sig: $L(\sigma') \equiv 0$

In the expression of $\pi_\gamma^{\mathrm{II}}(d'_j, \sigma'_j) = P_\gamma(d'_j)\nabla_n^{\mathrm{II}}(\sigma'_j)$ in terms of Y_k's, there appears the term $Y_{2n'+1}$ if $L(\sigma'_j) \equiv 1 \pmod 2$. However, in the present case of $L(\sigma') \equiv 0$, $Y_{2n'+1}$ appears even number of times, and so it disappears in the total product expression of $\pi_\gamma^{\mathrm{II}}(g')$ in (17.6), since ${Y_{2n'+1}}^2 = E_{2^{n'}}$.

Moreover, in case $L(\sigma') \equiv 0$, to express $\pi_\gamma^{\mathrm{II}}(d', \sigma')$, we may use only the operators $\nabla'_n(r_j) = \frac{1}{\sqrt{2}}(Y_j - Y_{j+1})$ instead of $\nabla_n^{\mathrm{II}}(r_j) = \nabla'_n(r_j) \cdot (-iY_{2n'+1})$. Expanding the right hand side of (17.6) into a linear combination of monomial terms such as ${Y_1}^{c_1}{Y_2}^{c_2}\cdots{Y_n}^{c_n}$, $n = 2n' < 2n'+1$, we see that the terms with non-zero traces are only those such that

$$
{Y_1}^{2c'_1}{Y_2}^{2c'_2}\cdots{Y_n}^{2c'_n} \quad (= E_{2^{n'}}). \tag{17.9}
$$

Since the supports of ξ'_{q_i}'s and g'_j's are mutually disjoint, $f_\gamma^{\mathrm{I}}(g') \neq 0$ implies

(Condition II-00) $\begin{cases} \mathrm{ord}(d') \equiv 0, \quad L(\sigma') \equiv 0 \pmod 2\,; \\ \mathrm{ord}(\xi'_{q_i}) \equiv 0\ (\forall i),\ \mathrm{ord}(d'_j) + L(\sigma'_j) \equiv 0 \pmod 2 \quad (\forall j), \end{cases}$

because $L(\sigma') \equiv 0$ is assumed. Note that, in $\widetilde{G}_n^{\mathrm{II}} = \widetilde{G}^{\mathrm{II}}(m,1,n)$,

$$
g'_j g'_l = {z_2}^{\mathrm{ord}(d'_j)\,\mathrm{ord}(d'_l)} \cdot {z_1}^{L(\sigma'_j)\,L(\sigma'_l)}\, g'_l g'_j \qquad (j \neq l)\,, \tag{17.10}
$$

then we see that, if $\mathrm{ord}(d'_j) \equiv L(\sigma'_j) \equiv 1$, $\mathrm{ord}(d'_l) \equiv L(\sigma'_l) \equiv 1$,

$$
\begin{aligned}
& g'_j g'_l = z_1 z_2\, g'_l g'_j \qquad \big(\text{in } \widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \mathcal{S}(P_\gamma)\big), \\
\therefore\quad & \pi_\gamma^{\mathrm{II}}(g'_j)\pi_\gamma^{\mathrm{II}}(g'_l) = \pi_\gamma^{\mathrm{II}}(g'_l)\pi_\gamma^{\mathrm{II}}(g'_j).
\end{aligned}
$$

Lemma 17.2. *Assume* $L(\sigma') \equiv 0 \pmod 2$. *Under (Condition II-00), the operators* $\pi_\gamma^{\mathrm{II}}(\xi'_{q_i})$*'s and* $\pi_\gamma^{\mathrm{II}}(g'_j)$*'s commute with each other. In particular, if* $\mathrm{ord}(d'_j) \equiv L(\sigma'_j) \equiv \mathrm{ord}(d'_l) \equiv L(\sigma'_l) \equiv 1$, *then* $\pi_\gamma^{\mathrm{II}}(g'_j g'_l) = \pi_\gamma^{\mathrm{II}}(g'_l g'_j)$.

Lemma 17.3. *Let* $n = 2n'$ *be even. Assume* $L(\sigma') \equiv 0 \pmod 2$ *and (Condition II-00) for* $g' = (d', \sigma') \in \widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \mathcal{S}(P_\gamma)$.

(i) *Divide the index set* $J = \{1, 2, \ldots, s\}$ *as*

$$
J = J_+ \bigsqcup J_-, \quad J_\pm := \{j \in J\,;\, \mathrm{sgn}(\sigma'_j) = \pm 1\}, \tag{17.11}
$$

and divide J_- *into pairs as* $J_- = \bigsqcup\{j_1, j_2\}$ *and denote this as* $\{j_1, j_2\} \sqsubset J_-$. *Then*

$$
f_\gamma^{\mathrm{II}}(g') = 2^{n'} \cdot \prod_{q \in Q} \frac{f_\gamma^{\mathrm{II}}(\xi'_q)}{2^{n'}} \cdot \prod_{j \in J_+} \frac{f_\gamma^{\mathrm{II}}(g'_j)}{2^{n'}} \cdot \prod_{\{j_1, j_2\} \sqsubset J_-} \frac{f_\gamma^{\mathrm{II}}(g'_{j_1} g'_{j_2})}{2^{n'}}\,. \tag{17.12}
$$

(ii) *For* $\{j_1, j_2\} \sqsubset J_-$ *or for a pair* $\{j_1, j_2\}$ *such that* $L(\sigma'_{j_1}) \equiv L(\sigma'_{j_2}) \equiv 1$,

$$f_\gamma^{\rm II}(g'_{j_1} g'_{j_2}) = f_\gamma^{\rm II}(g'_{j_2} g'_{j_1}).$$

Proof. (i) Since $L(\sigma') \equiv 0$, the operator $Y_{2n'+1}$ actually appears neither in $\pi_\gamma^{\rm II}(g'_j)$ for $j \in J_+$ nor in $\pi_\gamma^{\rm II}(g'_{j_1} g'_{j_2})$ for $\{j_1, j_2\} \sqsubset J_-$, because $\nabla^{\rm II}(\sigma'') = \nabla'(\sigma'')$ if $\mathrm{sgn}(\sigma'') = 1$ for $\sigma'' \in \widetilde{\mathfrak{S}}_{2n'}$. Monomial terms with non-zero traces in these cases are of the form $Y_{j_1}^{2c'_1} Y_{j_2}^{2c'_2} \cdots Y_{j_p}^{2c'_p}$. Moreover, for $\pi_\gamma^{\rm II}(\xi'_{q_i})$, $\pi_\gamma^{\rm II}(g'_j)$ and $\pi_\gamma^{\rm II}(g'_{j_1} g'_{j_2})$, supports in $\boldsymbol{I}_n$ of their monomials in Y_j's are mutually disjoint. Hence the assertion follows from Lemma 14.1 (ii).

(ii) This assertion follows from Lemma 17.2. □

Note 17.1. For g'_j with $L(\sigma'_j) \equiv 1$ or equivalently $j \in J_-$, we have $f_\gamma^{\rm II}(g'_j) = 0$ except when $|\mathrm{supp}(g'_j)| = n$ as will be seen in CASE 2sig below. This is the reason why an exact product formula (not a quasi-product-formula such as (17.12)) does not hold for the character $f_\gamma^{\rm II}$.

Actually for $\{j_1, j_2\} \sqsubset J_-$, $f_\gamma^{\rm II}(g'_{j_1} g'_{j_2}) \neq 0 = f_\gamma^{\rm II}(g'_{j_1}) f_\gamma^{\rm II}(g'_{j_2})$.

Example 17.1. Let (d'_j, σ'_j) be with $d'_j = \eta_k^b$, $\sigma'_j = r_1 r_2 \cdots r_{N-1}$, $N \leq n = 2n'$. Then $L(\sigma_j) = N-1$, $\mathrm{ord}(d_j) = b$, $\therefore$ $\mathrm{ord}(d_j) + L(\sigma_j) = b + (N-1)$, and

$$\begin{aligned} &P_\gamma(d'_j) = \chi_\gamma(\eta_k^b) Y_k^b = \omega^{b\gamma_k} Y_k^b, \\ &\pi_\gamma^{\rm II}(g'_j) = \omega^{b\gamma_k}\, 2^{(N-1)/2}\, Y_k^b \times \\ &\quad \times \begin{cases} (Y_1 - Y_2)(Y_2 - Y_3) \cdots (Y_{N-1} - Y_N)\, (-iY_{2n'+1}) & \text{if } N \text{ is even ;} \\ (Y_1 - Y_2)(Y_2 - Y_3) \cdots (Y_{N-1} - Y_N) & \text{if } N \text{ is odd.} \end{cases} \end{aligned}$$

If b is even and N is odd, then the unique monomial term with non-zero trace in the right hand side is $(-Y_2{}^2) \cdots (-Y_{N-1}{}^2)$ and

$$\mathrm{tr}\big(\pi_\gamma^{\rm II}(g'_j)\big) = \omega^{b\gamma_k}\, (-1)^{(N-1)/2}\, 2^{n'-(N-1)/2}.$$

If b is odd and $N = 2n'$ even, then the unique monomial term with non-zero trace is

$$\begin{cases} Y_1(-Y_2) \cdots (-Y_{2n'})(-iY_{2n'+1}) = i\, Y_1 Y_2 \cdots Y_{2n'+1} & \text{if } k = 1, \\ Y_k \cdot Y_1 \cdots Y_{k-1}(-Y_{k+1}) \cdots (-Y_{2n'})(-iY_{2n'+1}) = i\, Y_1 Y_2 \cdots Y_{2n'+1} & \text{otherwise,} \end{cases}$$

$$\therefore \quad \mathrm{tr}\big(\pi_\gamma^{\rm II}(g'_j)\big) = \omega^{b\gamma_k}\, 2^{-(N-1)/2} \cdot i(2i)^{n'} = \omega^{b\gamma_k}\, i^{n'+1}\, 2^{1/2}.$$

In all other cases, we have $\mathrm{tr}\big(\pi_\gamma^{\rm II}(g'_j)\big) = 0$.

17.2.2 Calculation of $f_\gamma^{\rm II}(\cdot) = \mathrm{tr}(\pi_\gamma^{\rm II}(\cdot))$, CASE 1sig-1 and CASE 1sig-2

(CASE 1sig-1) Case of $q \in Q$:

We have $f_\gamma^{\rm I}(\xi'_{q_i}) = \begin{cases} 2^{n'}\, \chi_\gamma(\xi'_q), & \text{if } \mathrm{ord}(\xi'_q) \equiv 0 \pmod 2, \\ 0\,, & \text{if } \mathrm{ord}(\xi'_q) \equiv 1 \pmod 2\,. \end{cases}$

Now we divide g'_j into two subcases, (CASE 1sig-2) or (CASE 1sig-3), depending on $j \in J_+$ or $j \in J_-$.

(CASE 1sig-2) CASE OF $g'_j,\ j \in J_+$:
Suppose $j \in J_\zeta$. Since $\mathrm{sgn}(\sigma'_j) = 1$ or $L(\sigma'_j) \equiv 0 \pmod 2$ in this case, we have $\mathrm{ord}(d'_j) \equiv 0 \pmod 2$ under (Condition II-00), and $\nabla^{\mathrm{II}}(\sigma'_j) = \nabla'(\sigma'_j)$, and

$$\begin{aligned}(17.13)\qquad \pi_\gamma^{\mathrm{II}}(g'_j) &= P_\gamma(d'_j)\nabla'_n(\sigma'_j)\\ &= \chi_\gamma(d'_j)\cdot \prod_{p\in K_j} Y_p^{\beta_p} \times \prod_{p,\,p+1\in K_j} \frac{1}{\sqrt{2}}\big(Y_p - Y_{p+1}\big).\end{aligned}$$

Note that $f_\gamma^{\mathrm{II}}(\cdot)$ is invariant under $\widetilde{\mathfrak{S}}_{I_{n,\zeta}}$ if $K_j \subset I_{n,\zeta}$, then we may assume, to simplify complicated suffices, that modulo $\widetilde{Z}$ (cf. Lemma 17.1)

$$(17.14)\qquad \begin{cases} K_j = \mathrm{supp}(\sigma'_j) = [n_j, n_j+\ell_j-1], \text{ an interval in } \boldsymbol{I}_n,\\ d'_j = {\eta_{k_j}}^{b_j},\ b_j = \mathrm{ord}(d'_j),\ \text{ for some } k_j \in K_j,\\ \sigma'_j = r_{n_j} r_{n_j+1}\cdots r_{n_j+\ell_j-2},\ \sigma_j = \Phi(\sigma'_j) = (n_j\ \ n_j{+}1\ \ldots\ n_j{+}\ell_j{-}1).\end{cases}$$

For calculations at present, we put $n_j = 1$ and $N = n_j + \ell_j - 1 = \ell_j$ odd, then

$$(17.15)\qquad \sigma'_j = r_1 r_2 \cdots r_{N-1}, \quad \sigma_j = \Phi(\sigma'_j) = (1\ 2\ \ldots\ N),$$

Since $\mathrm{ord}(d'_j) \equiv 0 \pmod 2$, the decomposition $d'_j = d^0_j h_j$ is trivial as $h_j = e_T$. Thus we come to calculate the trace of

$$(17.16)\qquad 2^{-(\ell_j-1)/2}\,(Y_1 - Y_2)(Y_2 - Y_3)\cdots(Y_{N-1} - Y_N).$$

When this is expanded as a linear combination of monomial terms in $Y_1, \ldots, Y_N$, the unique term with non-zero trace, up to $2^{-(\ell_j-1)/2}$, is

$$(-{Y_2}^2)\cdots(-{Y_{N-1}}^2) = (-1)^{(\ell_j-1)/2}{Y_2}^2\cdots{Y_{N-1}}^2.$$

Lemma 17.4. *In CASE 1sig-2, under the above normalization of* $g'_j = (d'_j, \sigma'_j)$,

$$f_\gamma^{\mathrm{II}}(g'_j) = \mathrm{tr}\big(P_\gamma(d'_j)\nabla'_n(\sigma'_j)\big) = \chi_\gamma(d'_j)\,(-1)^{(\ell_j-1)/2}\,2^{n'-(\ell_j-1)/2}.$$

17.2.3 Calculation of $f_\gamma^{\mathrm{II}}(\cdot) = \mathrm{tr}\big(\pi_\gamma^{\mathrm{II}}(\cdot)\big)$, CASE 1sig-3

(CASE 1sig-3) CASE OF A PRODUCT $g'_{j_1} g'_{j_2},\ \{j_1, j_2\} \sqsubset J_-$:
Here $\mathrm{ord}(d'_{j_1}) \equiv \mathrm{ord}(d'_{j_2}) \equiv L(\sigma'_{j_1}) \equiv L(\sigma'_{j_2}) \equiv 1 \pmod 2$, and so $\mathrm{ord}(d'_{j_1} d'_{j_2}) \equiv L(\sigma'_{j_1}\sigma'_{j_2}) \equiv 0 \pmod 2$. Hence, by $\sigma'_{j_1} d'_{j_2} = d'_{j_2}\sigma'_{j_1}$ in CASE II,

$$(17.17)\qquad \pi_\gamma^{\mathrm{II}}(g'_{j_1} g'_{j_1}) = P_\gamma(d'_{j_1} d'_{j_2})\nabla^{\mathrm{II}}(\sigma'_{j_1}\sigma'_{j_2}) = P_\gamma(d'_{j_1} d'_{j_2})\nabla'_n(\sigma'_{j_1}\sigma'_{j_2}).$$

$$\therefore \quad f_\gamma^{\rm II}(g'_{j_1}g'_{j_1}) = {\rm tr}\big(\pi_\gamma^{\rm II}(g'_{j_1}g'_{j_1})\big) = {\rm tr}\big(P_\gamma(d'_{j_1}d'_{j_2})\nabla'_n(\sigma'_{j_1}\sigma'_{j_2})\big).$$

Remark 17.1. For $n \geq 4$ general, we have from Lemma 10.1 (i)

$$\iota\big(\nabla'_n(\sigma')\big)P_\gamma(d') = \begin{cases} P_{(\tau_1\tau_2\cdots\tau_n)\gamma}\big({\sigma'}^{\rm II}(d')\big) & \text{if } {\rm sgn}(\sigma') = -1\ ; \\ P_\gamma\big({\sigma'}^{\rm II}(d')\big) & \text{if } {\rm sgn}(\sigma') = 1, \end{cases} \tag{17.18}$$

for $g' = (d', \sigma') \in \widetilde{D}_n \overset{\rm II}{\rtimes} \mathcal{S}(P_\gamma)$. Moreover, for $\{j_1, j_2\} \sqsubset J_-$,

$$\begin{aligned} P_\gamma(d'_{j_1}d'_{j_2})\nabla'_n(\sigma'_{j_1}\sigma'_{j_2}) &= P_\gamma(d'_{j_1})P_\gamma(d'_{j_2})\nabla'_n(\sigma'_{j_1})\nabla'_n(\sigma'_{j_2}) \\ &= -\,P_\gamma(d'_{j_1})\nabla'_n(\sigma'_{j_1}) \cdot P_\gamma(d'_{j_2})\nabla'_n(\sigma'_{j_2}). \end{aligned} \tag{17.19}$$

Therefore, the map $\widetilde{D}_n \rtimes \mathcal{S}(P_\gamma) \ni g' \mapsto P_\gamma(d')\nabla'_n(\sigma')$ does not give a representation of this group, but it does if it is restricted on the subgroup $\widetilde{D}_n \rtimes \big(\mathcal{S}(P_\gamma)\cap\widetilde{\mathfrak{A}}_n\big)$. In this sense, when ${\rm sgn}(\sigma') = -1$, we understand the symbol $P_\gamma(d')\nabla'_n(\sigma')$ simply as an operator (or a matrix), and do not use a representation-like notation such as $\pi'_\gamma(g')$ for $P_\gamma(d')\nabla'_n(\sigma')$.

As in (17.14), g'_j $(j = j_1, j_2)$ are supposed to be normalized as follows: with ℓ_j even,

$$\begin{cases} \sigma'_j = r_{n_j}r_{n_j+1}\cdots r_{n_j+\ell_j-2}, \quad K_j = [n_j, n_j+\ell_j-1] \subset \boldsymbol{I}_n, \\ d'_j = d^0_j h_j, \quad d^0_j = \eta_{k_j}^{b_j-1},\ h_j = \eta_{k_j},\ b_j = {\rm ord}(d'_j)\ \ (\exists k_j \in K_j). \end{cases} \tag{17.20}$$

Then, taking into account (17.19), we calculate the trace of the following

$$\begin{aligned} & P_\gamma(d'_{j_1}d'_{j_2})\nabla'_n(\sigma'_{j_1}\sigma'_{j_2}) = \chi_\gamma(d'_{j_1})\chi_\gamma(d'_{j_2}) \cdot (-1) \times \\ & \times 2^{-(\ell_{j_1}-1)/2}\, Y_{k'_1}(Y_{n'_1} - Y_{n'_1+1})(Y_{n'_1+1} - Y_{n'_1+2}) \cdots (Y_{n'_1+N_1-2} - Y_{n'_1+N_1-1}) \\ & \times 2^{-(\ell_{j_2}-1)/2}\, Y_{k'_2}(Y_{n'_2} - Y_{n'_2+1})(Y_{n'_2+1} - Y_{n'_2+2}) \cdots (Y_{n'_2+N_2-2} - Y_{n'_2+N_2-1}), \end{aligned} \tag{17.21}$$

where $n'_t = n_{j_t},\ N_t = \ell_{j_t}\ (t = 1, 2)$.

To get a monomial term of the form (17.9) from the right hand side, we put on each Y_p a color *black* or *white* depending on if it comes in or not, to form a monomial term such as (17.9). At the point of starting, we first color *black* all $Y_{k_j} \in h_j$ $(j = j_1, j_2)$.

The process of coloring is independent for each of the 2nd line (for j_1) and the 3rd line (for j_2) of (17.21). It is similar to that in CASE 1sig-2, and we have the following rule which is illustrated in Table 17.1 below.

Parity Rule II-00. *For $Y_{k_j} \cdot \nabla'(r_{n'_j})\nabla'(r_{n'_j+1}) \cdots \nabla'(r_{n'_j+N_j-2})$, the parity of $\nabla'(r_i)$ begins with (b, w) or (w, b) depending on $k_j = n_j$ or not. The parity changes alternatively until $i = k_j - 1$. From $i = k_j - 1$ to $i = k_j$ it remains unchanged, and it again changes alternatively starting from $i = k_j$.*

Table 17.1. Parities of $\nabla'_n(r_p) = \frac{1}{\sqrt{2}}(Y_p - Y_{p+1})$ $(p, p+1 \in K_j)$ in $\pi^{\mathrm{II}}_\gamma(g'_j)$.
CASE: $N_j = \ell_j = 10$ $\big(L(\sigma'_j) \equiv 9\big)$: $K_j = [n_j, n_j+\ell_j-1] = [1, N_j]$,

$Y_{k_j} \in h_j = \eta_1$	b									
$\nabla'_n(r_p)$ $(p = 1,3,5,7,9)$	b	w	b	w	b	w	b	w	b	w
$\nabla'_n(r_p)$ $(p = 2,4,6,8)$	$\times$	w	b	w	b	w	b	w	b	$\times$

accumulated sign $= (-1)^4$

$Y_{k_j} \in h_j = \eta_2$		b								
$\nabla'_n(r_p)$ $(p = 1,3,5,7,9)$	w	b	b	w	b	w	b	w	b	w
$\nabla^+_n(r_p)$ $(p = 2,4,6,8)$	$\times$	w	b	w	b	w	b	w	b	$\times$

accumulated sign $= (-1)^5$

$Y_{k_j} \in h_j = \eta_9$									b	
$\nabla^+_n(r_p)$ $(p = 1,3,5,7,9)$	w	b	w	b	w	b	w	b	b	w
$\nabla^+_n(r_p)$ $(p = 2,4,6,8)$	$\times$	b	w	b	w	b	w	b	w	$\times$

accumulated sign $= (-1)^4$

$Y_{k_j} \in h_j = \eta_{10}$										b
$\nabla^+_n(r_p)$ $(p = 1,3,5,7,9)$	w	b	w	b	w	b	w	b	w	b
$\nabla^+_n(r_p)$ $(p = 2,4,6,8)$	$\times$	b	w	b	w	b	w	b	w	$\times$

accumulated sign $= (-1)^5$

Thus we get the unique monomial term with non-zero trace as

$$\prod_{t=1,2} 2^{-(\ell_{j_t}-1)/2}\Big(Y_{k_t} \times \prod_{p,\, p+1 \in K_{j_t}} X_p\Big), \tag{17.22}$$

$$\text{with} \quad X_p := \begin{cases} Y_p & \text{if the parity of } \nabla'_n(r_p) \text{ is } (b,\ w), \\ -Y_{p+1} & \text{if the parity of } \nabla'_n(r_p) \text{ is } (w,\ b). \end{cases} \tag{17.23}$$

In our circumstances where $\mathrm{ord}(d'_{j_1}) \equiv \mathrm{ord}(d'_{j_2}) \equiv 1 \pmod 2$, we have $d'_{j_1}d'_{j_2} = z_2 d'_{j_2}d'_{j_1}$, and so $\chi_\gamma(d'_{j_1}d'_{j_2}) = -\chi_\gamma(d'_{j_2}d'_{j_1})$ and $\chi_\gamma(d'_{j_1})\chi_\gamma(d'_{j_2}) = \chi_\gamma(d'_{j_1}d'_{j_2})$ if K_{j_1} is on the left of K_{j_2}.

On the other hand, $\zeta_\gamma(d'_{j_1})\zeta_\gamma(d'_{j_2}) = \zeta_\gamma(d'_{j_1}d'_{j_2}) = \zeta_\gamma(d'_{j_2}d'_{j_1})$ in general, and if $d'_j = \prod_{p \in K_j} {\eta_p}^{a_p}$, a product in the natural order, without z_2-factor, then $\chi_\gamma(d'_j) = \zeta_\gamma(d_j)$, and when this is the case for $j = j_1, j_2$, we have $\chi_\gamma(d'_{j_1})\chi_\gamma(d'_{j_2}) = \zeta_\gamma(d'_{j_1})\zeta_\gamma(d'_{j_2}) = \zeta_\gamma(d'_{j_1}d'_{j_2})$. So, we may use ζ_γ instead of χ_γ with careful attention on their differences.

Lemma 17.5. *In CASE 1sig-3, for $\{j_1, j_2\} \sqsubset J_-$, under the normalization (17.20),*

$$\begin{aligned} \mathrm{tr}(\pi^{\mathrm{II}}_\gamma(g'_{j_1}g'_{j_2})) &= \mathrm{tr}\big(P_\gamma(d'_{j_1}d'_{j_2})\nabla'_n(\sigma'_{j_1}\sigma'_{j_2})\big) \\ &= \varepsilon(g'_{j_1}g'_{j_2}) \cdot 2^{n'} \cdot (-1) \prod_{j=j_1,\, j_2} \chi_\gamma(d'_j)\,(-1)^{\ell_j/2}\, 2^{-(\ell_j-1)/2}, \end{aligned}$$

where $\chi_\gamma(d'_{j_1})\chi_\gamma(d'_{j_2}) = \zeta_\gamma(d'_{j_1}d'_{j_2})$, and the sign $\varepsilon(g'_{j_1}g'_{j_2}) = \pm 1$ is determined so that, in (17.22),

$$\prod_{t=1,2} 2^{-(\ell_{j_t}-1)/2}\Big(Y_{k_t} \times \prod_{p,\, p+1 \in K_{j_t}} X_p\Big) =$$

$$(17.24) \qquad = \varepsilon(g'_{j_1} g'_{j_2}) \cdot (-1) \prod_{j=j_1, j_2} (-1)^{\ell_j/2}\, 2^{-(\ell_j-1)/2} \cdot E_{2n'}\,.$$

Practically speaking, suppose, for $j = j_1, j_2$, that $k_j \in K_j$ is the c_j-th number from the smallest one in the increasing order in K_j, then

$$(17.25) \qquad \varepsilon(g'_{j_1} g'_{j_2}) = \prod_{j=j_1, j_2} (-1)^{c_j}$$

Proof. In the middle line ($j_1 = 1$) in (17.21), we pick up a monomial as follows:

$$(-Y_2{}^2)\cdots(-Y_{2u}{}^2)\cdot(Y_{2u+1}{}^2)\cdot(-Y_{2u+3}{}^2)\cdots(-Y_{\ell_1-1}{}^2) = (-1)^{\ell_1/2-1} E_{2n'} \quad \text{if } k_1 = 2u+1;$$

$$(-Y_2{}^2)\cdots(-Y_{2u-2}{}^2)\cdot(-Y_{2u}{}^2)\cdot(-Y_{2u+1}{}^2)\cdots(-Y_{\ell_1-1}{}^2) = (-1)^{\ell_1/2} E_{2n'} \quad \text{if } k_1 = 2u.$$

Similarly for the last line in (17.21). □

17.3 Characters of $\pi_\gamma^{\mathrm{II}} = P_\gamma \cdot \nabla_n^{\mathrm{II}}$ of $\widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \mathcal{S}(P_\gamma)$, CASE 2sig

Recall that CASE 2sig for $n = 2n'$ even is defined by $L(\sigma') \equiv \sum_{1 \le j \le s} L(\sigma'_j) \equiv 1 \pmod 2$, and in this case, $\nabla_n^{\mathrm{II}}(\sigma') = \nabla'_n(\sigma')\,(-iY_{2n'+1})$. Moreover $\nabla_n^{\mathrm{II}}(\sigma'_j) = \nabla'(\sigma'_j)$ or $\nabla_n^{\mathrm{II}}(\sigma'_j) = \nabla'_n(\sigma'_j)\,(-iY_{2n'+1})$ according as $L(\sigma'_j) \equiv 0$ or $\equiv 1$, and we have in total

$$(17.26) \qquad \pi_\gamma^{\mathrm{II}}(g') = \kappa \prod_{q \in Q} P_\gamma(\xi'_q) \cdot \prod_{j \in J} \Big(P_\gamma(d'_j)\, \nabla'_n(\sigma'_j) \Big) \times (-iY_{2n'+1}), \quad \kappa = \pm 1.$$

Here the orders of products on $q \in Q$ and on $j \in J$ should follow the order in the expression $g' = \xi'_{q_1} \cdots \xi'_{q_r}\, g'_1 \cdots g'_s$. Even so, there remains some ambiguity for the sign κ. In fact, in general

$$(17.27) \qquad g'_j g'_{j'} = z_1^{\,L(\sigma'_j)L(\sigma'_{j'})}\, z_2^{\,\mathrm{ord}(d'_j)\mathrm{ord}(d'_{j'})}\, g'_{j'} g'_j.$$

Suppose $\big(\mathrm{ord}(d'_j), L(\sigma'_j)\big) \equiv (1,0)$ and $\big(\mathrm{ord}(d'_{j'}), L(\sigma'_{j'})\big) \equiv (0,1)$, then g'_j and $g'_{j'}$ are commutative as $g'_j g'_{j'} = g'_{j'} g'_j$. However $P_\gamma(d'_j)\,\nabla'_n(\sigma'_{j'}) = -\nabla'_n(\sigma'_{j'})\,P_\gamma(d'_j)$, and so

$$\Big(P_\gamma(d'_j)\,\nabla'_n(\sigma'_j)\Big)\Big(P_\gamma(d'_{j'})\,\nabla'_n(\sigma'_{j'})\Big) = -\Big(P_\gamma(d'_{j'})\,\nabla'_n(\sigma'_{j'})\Big)\Big(P_\gamma(d'_j)\,\nabla'_n(\sigma'_j)\Big).$$

This ambiguity of κ in (17.26) is introduced by using ∇'_n instead of ∇_n^{II}. To avoid this, we take a representative of g' in (17.5) of conjugacy class modulo $\widetilde{Z}$ expressed as

$$(17.28) \qquad g'_j\ (j \in J_+) \text{ are placed before } g'_j\ (j \in J_-).$$

Then $\kappa = 1$ in (17.26). Here $|J_-|$ is odd since $L(\sigma') \equiv 1 \pmod 2$.

In the right hand side of (17.26), when expanded into a linear combination of monomials in Y_k's, the only one with non-zero trace is, multiplicatively modulo $\prod_{i\in \boldsymbol{I}_n} Y_i^{2c_i'}$,

$$Y_1 Y_2 \cdots Y_{2n'} Y_{2n'+1}\,, \tag{17.29}$$

since $n = 2n'$, and so there should be $|\mathrm{supp}(g')| = n$. Since the degree of monomials $Y_1^{c_1} \cdots Y_{2n'}^{c_{2n'}}$ coming out of $\prod_{j\in J} \nabla'_n(\sigma'_j)$, $J = \boldsymbol{I}_s$, is odd, there should be $\mathrm{ord}(d') \equiv 1$ to get together $Y_1 Y_2 \cdots Y_{2n'}$ multiplicatively modulo $\prod_{i\in\boldsymbol{I}_n} Y_i^{2c_i'}$.

For each $P_\gamma(\xi'_q)$, we should have the term Y_q multiplicatively modulo Y_q^{2c} so that $\mathrm{ord}(\xi_q) \equiv 1$ for all $q \in Q$. Moreover, for each $P_\gamma(d'_j)\nabla'_n(\sigma'_j)$, we should have $\prod_{k\in K_j} Y_k$ multiplicatively modulo $\prod_{k\in K_j} Y_k^{2c_k'}$, and accordingly $\mathrm{ord}(d'_j) + L(\sigma'_j) \equiv |K_j|$. On the other hand, $|K_j| \equiv L(\sigma'_j) + 1$, and accordingly $\mathrm{ord}(d'_j) \equiv 1$ $(j \in J)$. Therefore $\mathrm{ord}(d') = \sum_i \mathrm{ord}(\xi'_{q_i}) + \sum_j \mathrm{ord}(d'_j) \equiv r + s \equiv 1$, so $r+s$ odd and

(Condition II-11) $$\begin{cases} |\mathrm{supp}(g')| = n = 2n', \ \ \mathrm{ord}(d') \equiv L(\sigma') \equiv 1, \ \ r+s \equiv 1, \\ \mathrm{ord}(\xi'_{q_i}) \equiv 1 \ (i \in \boldsymbol{I}_r), \ \ \mathrm{ord}(d'_j) \equiv 1 \ (j \in \boldsymbol{I}_s) \quad (\mathrm{mod}\ 2). \end{cases}$$

We normalize $g'_j = (d'_j, \sigma'_j)$ as in (17.20), with $d'_j = \eta_{k_j}^{b_j}$, $b_j = \mathrm{ord}(d'_j)$ odd, and $\sigma'_j = r_{n_j} r_{n_j+1} \cdots r_{n_j+\ell_j-2}$, then

$$\begin{aligned} P_\gamma(d'_j)\nabla'_n(\sigma'_j) = \zeta_\gamma(d'_j) \cdot 2^{-(\ell_j-1)/2} \times \\ \times Y_{k_j}(Y_{n_j} - Y_{n_j+1})(Y_{n_j+1} - Y_{n_i+2}) \cdots (Y_{n_j+\ell_j-2} - Y_{n_j+\ell_j-1}). \end{aligned} \tag{17.30}$$

Expanding into a linear combination of monomials of Y_p's, we are forced to pick up the term

$$\begin{aligned} &\zeta_\gamma(d'_j)\, 2^{-(\ell_j-1)/2} \cdot Y_{k_j} \cdot Y_{n_j} Y_{n_j+1} \cdots Y_{k_j-1} \cdot (-Y_{k_j+1}) \cdots (-Y_{n_j+\ell_j-1}) \\ &= \ \zeta_\gamma(d'_j)\, 2^{-(\ell_j-1)/2}\, (-1)^{\ell_j-1} \cdot Y_{n_j} Y_{n_j+1} \cdots Y_{k_j} \cdots Y_{n_j+\ell_j-1}, \end{aligned} \tag{17.31}$$

where the first line should be appropriately understood in the extremal cases: $k_j = n_j$ or $k_j = n_j + \ell_j - 1$. Note that $\prod_{j\in J}(-1)^{\ell_j-1} = \mathrm{sgn}(\sigma') = -1$ in this case.

Proposition 17.6. *Let* $g' = (d', \sigma') \in \widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \mathcal{S}(P_\gamma)$ *be in CASE 2sig, that is,* $L(\sigma') \equiv 1 \ (\mathrm{mod}\ 2)$. *Then* $f_\gamma^{\mathrm{II}}(g') = \mathrm{tr}\big(\pi_\gamma^{\mathrm{II}}(g')\big) \neq 0$ *if and only if (Condition II-11) holds for* $g' = \xi'_{q_1} \cdots \xi'_{q_r}\, g'_1 \cdots g'_s$.

Assume that g' *is normalized, modulo* $\widetilde{Z}$, *as in (17.14). Then*

$$\begin{aligned} f_\gamma^{\mathrm{II}}(g') \ &= \ \prod_{q\in Q} \chi_\gamma(\xi'_q) \cdot \prod_{j\in J} \chi_\gamma(d'_j)\, (-1)^{\ell_j-1}\, 2^{-(\ell_j-1)/2} \times 2^{n'} \times \\ &\quad \times \mathrm{tr}\Big(\prod_{q\in Q} Y_q \cdot \prod_{j\in J} \prod_{p\in K_j} Y_p \cdot (-iY_{2n'+1}) \Big) \end{aligned}$$

$$= \quad -\varepsilon^{\mathrm{II}}(g') \cdot \zeta_\gamma(d') \cdot i^{n'-1}\, 2^{n'} \cdot \prod_{j\in J} 2^{-(\ell_j-1)/2},$$

where the product $\prod_{p\in K_j} Y_p$ *is along the natural order of* $p \in K_j$*, and the sign* $\varepsilon^{\mathrm{II}}(g') = \pm 1$ *is given by rearranging the product of* Y_p*'s as*

$$\textstyle\prod_{i\in Q} Y_q \cdot \prod_{j\in J} \big(\prod_{p\in K_j} Y_p\big) = \varepsilon^{\mathrm{II}}(g') \cdot Y_1 Y_2 \cdots Y_{2n'}\,.$$

Note that the normalization (17.28) is not asked here, instead of it we introduce the sign $\varepsilon^{\mathrm{II}}(g')$.

17.4 Character formula for $\pi^{\mathrm{II}}_\gamma = P_\gamma \cdot \nabla^{\mathrm{II}}_n$ of $\widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \mathcal{S}(P_\gamma), n = 2n'$

Define normalized character $\widetilde{f}^{\,\mathrm{II}}_\gamma(g') := f^{\mathrm{II}}_\gamma(g')/\dim \pi^{\mathrm{II}}_\gamma = \mathrm{tr}\big(\pi^{\mathrm{II}}_\gamma(g')\big)/2^{n'}$.

Theorem 17.7 (Case $n = 2n'$ even). *Assume that* $g' = (d', \sigma') \in \widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \mathcal{S}(P_\gamma)$ *is expressed as* $g' = \xi'_{q_1} \cdots \xi'_{q_r}\, g'_1 \cdots g'_s,\ g'_j = (d'_j, \sigma'_j)$*. Put* $Q = \{q_1, \ldots, q_r\}$ *and*

$$(17.32) \qquad J = \boldsymbol{I}_s, \quad J = J_+ \sqcup J_-, \quad J_\pm = \{j \in J\,;\ \mathrm{sgn}(\sigma'_j) = \pm 1\}.$$

(i) Case $L(\sigma') \equiv 0 \pmod 2$ or $\mathrm{sgn}(\sigma') = 1$: *Here* $\widetilde{f}^{\,\mathrm{II}}_\gamma(g') \neq 0$ *if and only if (Condition II-00) holds for* g'*. In that case, there holds a product formula as*

$$\widetilde{f}^{\,\mathrm{II}}_\gamma(g') \;=\; \prod_{q\in Q} \widetilde{f}^{\,\mathrm{II}}_\gamma(\xi'_q) \times \prod_{j\in J_+} \widetilde{f}^{\,\mathrm{II}}_\gamma(g'_j) \times \prod_{\{j_1, j_2\}\sqsubset J_-} \widetilde{f}^{\,\mathrm{II}}_\gamma\big(g'_{j_1} g'_{j_2}\big).$$

If g' *is normalized modulo* $\widetilde{Z}$ *as in (17.14), then*

$$\begin{aligned}
\widetilde{f}^{\,\mathrm{II}}_\gamma(\xi'_q) &= \zeta_\gamma(\xi'_q), \\
\widetilde{f}^{\,\mathrm{II}}_\gamma(g'_j) &= \zeta_\gamma(d'_j)\,(-1)^{(\ell_j-1)/2}\, 2^{-(\ell_j-1)/2}, && \text{for } j \in J_+, \\
\widetilde{f}^{\,\mathrm{II}}_\gamma\big(g'_{j_1} g'_{j_2}\big) &= \varepsilon(g'_{j_1} g'_{j_2}) \cdot (-1) \prod_{j=j_1, j_2} \zeta_\gamma(d'_j)\,(-1)^{\ell_j/2}\, 2^{-(\ell_j-1)/2}, && \text{for } \{j_1, j_2\} \sqsubset J_-\,,
\end{aligned}$$

where the sign $\varepsilon(g'_{j_1} g'_{j_2})$ *is defined by (17.22)–(17.24), and suppose, for* $j = j_1, j_2$, $k_j \in K_j$ *is the* c_j*-th number from the smallest one in* K_j*, then*

$$(17.33) \qquad \varepsilon(g'_{j_1} g'_{j_2}) = \prod_{j=j_1, j_2} (-1)^{c_j}$$

(ii) Case $L(\sigma') \equiv 1 \pmod 2$ or $\mathrm{sgn}(\sigma') = -1$: *Here* $\widetilde{f}^{\,\mathrm{II}}_\gamma(g') \neq 0$ *if and only if (Condition II-11) holds for* g'*. In that case, if* g' *is taken in a normalized form as in (17.14), then*

$$\widetilde{f}^{\,\mathrm{II}}_\gamma(g') = -\varepsilon^{\mathrm{II}}(g')\, i^{n'-1} \cdot \zeta_\gamma(d') \cdot \prod_{j\in J} 2^{-(\ell_j-1)/2},$$

where $\varepsilon^{\mathrm{II}}(g') = \pm 1$ *is given by*

$$\prod_{q\in Q} Y_q \cdot \prod_{j\in J} \prod_{p\in K_j} Y_p = \varepsilon^{\mathrm{II}}(g') \cdot Y_1 Y_2 \cdots Y_{2n'} \,. \tag{17.34}$$

17.5 Character formula for $\pi_\gamma^{\mho\pm} = P_\gamma^\pm \cdot \mho_n^\pm$, $n = 2n'+1$

Recall that $\pi_\gamma^{\mho\pm} = P_\gamma^\pm \cdot \mho_n^\pm$ for $n = 2n'+1$, where two spin representations $\mho_n^+$ and $\mho_n^-$ of $\widetilde{\mathfrak{A}}_n$ are given as

$$\begin{cases} \mho_n^+(\sigma') = \nabla'_n(\sigma') \\ \mho_n^-(\sigma') = \nabla''_n(\sigma') = Y_{2n'+1} \cdot \nabla'_n(\sigma') \cdot Y_{2n'+1}^{-1} \end{cases} \qquad (\sigma' \in \widetilde{\mathfrak{A}}_n). \tag{17.35}$$

with two spin representations for $\widetilde{\mathfrak{S}}_n$:

$$\begin{cases} \nabla'_n(r_i) &= \dfrac{1}{\sqrt{2}}(Y_i - Y_{i+1}) \\ \nabla''_n(r_i) &= -Y_{2n'+1} \cdot \nabla'_n(r_i) \cdot Y_{2n'+1}^{-1} \end{cases} \qquad (i \in \boldsymbol{I}_{n-1}).$$

Assume $n = 2n'+1$. For $\gamma \in \Gamma_n^0$, we prepare two spin IRs of $\widetilde{D}_n$,

$$P_\gamma^+(\eta_j) = \omega^{\gamma_j}\,\rho(\eta_j) = \chi_\gamma(\eta_j)\,Y_j \qquad (j \in \boldsymbol{I}_n)\ ; \tag{17.36}$$

$$P_\gamma^-(\eta_j) = \chi_{\tau_n\gamma}(\eta_j)\,Y_j = \begin{cases} \chi_\gamma(\eta_j) Y_j & (j \in \boldsymbol{I}_{n-1}), \\ -\chi_\gamma(\eta_n) Y_n & (j = n). \end{cases} \tag{17.37}$$

Stationary subgroups of the equivalence classes $[P_\gamma^+]$ and $[P_\gamma^-]$ are

$$\mathcal{S}(P_\gamma^\pm) = \{\sigma' \in \widetilde{\mathfrak{A}}_n \,;\, \sigma(\gamma) = \gamma\} = \widetilde{\mathfrak{A}}_n \bigcap \Phi_{\mathfrak{S}}^{-1}\Big(\prod\nolimits_{\zeta\in\widehat{T}^0} \mathfrak{S}_{I_{n,\zeta}}\Big). \tag{17.38}$$

In the following, we denote by $\widetilde{\mathfrak{A}}\big(\prod_{\zeta\in\widehat{T}^0} \mathfrak{S}_{I_\zeta}\big)$ the group at the right hand side.

Lemma 17.8. *For* $g' = (d', \sigma') \in \widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \mathcal{S}(P_\gamma^+) = \widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \mathcal{S}(P_\gamma^-)$,

$$\begin{aligned} \mathrm{tr}\big(\pi_\gamma^{\mho+}(g')\big) &= \mathrm{tr}\big(P_\gamma(d')\,\nabla'_n(\sigma')\big)\ ; \\ \mathrm{tr}\big(\pi_\gamma^{\mho-}(g')\big) &= \mathrm{tr}\big(P_{(\tau_1\tau_2\cdots\tau_n)\gamma}(d')\,\nabla'_n(\sigma')\big) \;=\; (-1)^{\mathrm{ord}(d')}\,\mathrm{tr}\big(\pi_\gamma^{\mho+}(g')\big). \end{aligned}$$

Proof. For the second equality, we have

$$\begin{aligned} &\pi_\gamma^{\mho-}(g') = P_\gamma^-(d')\,\mho_n^-(\sigma') = P_{\tau_n\gamma}(d') \cdot Y_n\,\nabla'_n(\sigma')\,Y_n^{-1}, \\ &Y_n^{-1} P_{\tau_n\gamma}(d')\,Y_n = P_{(\tau_1\tau_2\cdots\tau_n)\gamma}(d') = (-1)^{\mathrm{ord}(d')} P_\gamma(d')\,. \end{aligned}$$ □

By this lemma, our task is reduced to calculate the trace of the operators $P_\gamma(d')\,\nabla'_n(\sigma')$. Put $f_\gamma^{\mho\pm}(g') := \mathrm{tr}\big(\pi_\gamma^{\mho\pm}(g')\big)$, Then, for $g' = (d', \sigma') \in \widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \mathcal{S}(P_\gamma^\pm)$,

$$f_\gamma^{\mho-}(g') = (-1)^{\mathrm{ord}(d')}\, f_\gamma^{\mho+}(g'). \tag{17.39}$$

Define normalized character as

$$\widetilde{f}_\gamma^{\mho\pm}(g') := f_\gamma^{\mho\pm}(g')/\dim \pi_\gamma^{\mho\pm}, \quad \dim \pi_\gamma^{\mho\pm} = 2^{n'},\ n' = [n/2].$$

Take $g' = (d', \sigma') \in \widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \mathcal{S}(P_\gamma^\pm)$ normalized modulo $\widetilde{Z}$ as

$$g' = \prod_{q \in Q} \xi'_q \cdot \prod_{j \in J_+} g'_j \cdot \prod_{\{j_1, j_2\} \sqsubset J_-} g'_{j_1} g'_{j_2}, \qquad g'_j = (d'_j, \sigma'_j), \tag{17.40}$$

where $Q,\ J$ and $J_\pm$ are as in (17.32). Note that, since $\mathcal{S}(P_\gamma^\pm) \subset \widetilde{\mathfrak{A}}_n,\ g'_j\ (j \in J_-)$ should be considered always as products $g'_{j_1} g'_{j_2}$ of pairs of two elements.

•• **CASE** $\mathrm{ord}(d') \equiv 0 \pmod 2$ **:**

Since $L(\sigma') \equiv 0 \pmod 2$ a priori, the calculations are quite similar as those in §17.2 for π_γ^{II} in CASE 1sig. At first, it is proved that $f_\gamma^{\mho\pm}(g') \neq 0$ if and only if (Condition II-00) below holds for g' :

(Condition II-00) $\begin{cases} \mathrm{ord}(d') \equiv 0, \quad L(\sigma') \equiv 0 \pmod 2\ ; \\ \mathrm{ord}(\xi'_q) \equiv 0\ (q \in Q),\ \mathrm{ord}(d'_j) + L(\sigma'_j) \equiv 0 \ \ (j \in J). \end{cases}$

In that case, we get the explicit form of $f_\gamma^{\mho\pm}(g')$ as in Theorem 17.9 (i) below.

•• **CASE** $\mathrm{ord}(d') \equiv 1 \pmod 2$ **:**

Note that, in CASE II, we have $(d'_j, \sigma'_j)(d'_k, \sigma'_k) = (d'_k d'_j, \sigma'_k \sigma'_j)\ (j \neq k)$, and so, $\pi_\gamma^{\mho\pm}(g'_j g'_k) = \pi_\gamma^{\mho\pm}\big((d'_j d'_k, \sigma'_j \sigma'_k)\big)$. Since $L(\sigma') \equiv 0$ a priori, we have $\mathrm{ord}(d') + L(\sigma') \equiv 1 \pmod 2$. Hence when we expand the operator

$$\pi_\gamma^{\mho\pm}(g') = \prod_{q \in Q} P_\gamma^\pm(\xi_q) \cdot \prod_{j \in J_+} P_\gamma^\pm(d'_j) \mho_n^\pm(\sigma'_j) \cdot \prod_{\{j_1, j_2\} \sqsubset J_-} \pi_\gamma^{\mho\pm}\big((d'_{j_1} d'_{j_2}, \sigma'_{j_1} \sigma'_{j_2})\big)$$

into a linear combination of monomial terms in Y_j's such as ${Y_1}^{c_1} {Y_2}^{c_2} \cdots {Y_{2n'+1}}^{c_n}$, only one monomial term with trace non-zero is

$$Y_1 Y_2 \cdots Y_{2n'+1}\ (= i^{n'} E_{2^{n'}}),$$

multiplicatively modulo ${Y_1}^{2c'_1} {Y_2}^{2c'_2} \cdots {Y_{2n'+1}}^{2c'_n}$. Then we see that $f_\gamma^{\mho\pm}(g') \neq 0$ if and only if (Condition $\mho$-11) below holds for g' :

(Condition $\mho$-11) $\begin{cases} |\mathrm{supp}(g')| = n = 2n'+1,\ \mathrm{ord}(d') \equiv 1,\ L(\sigma') \equiv 0 \ \ (\mathrm{mod}\ 2), \\ \mathrm{ord}(\xi'_q) \equiv 1\ (q \in Q),\ \mathrm{ord}(d'_j) \equiv 1\ (j \in J) \quad (\therefore\ r+s \equiv 1). \end{cases}$

Case of $j \in J_+$: Suppose that g'_j is normalized modulo $\widetilde{Z}$ as in (17.14). Then, we pick up from the expansion of

$$Y_{k_j} \cdot 2^{-(\ell_j - 1)/2} \cdot \prod_{n_j \leq p < n_j + \ell_j - 1} (Y_p - Y_{p+1}), \tag{17.41}$$

a monomial term

$$(17.42)\qquad \begin{aligned} &Y_{k_j}\cdot 2^{-(\ell_j-1)/2}\cdot Y_{n_j}\cdots Y_{k_j-1}(-Y_{k_j+1})\cdots(-Y_{n_j+\ell_j-1}) \\ &\quad = 2^{-(\ell_j-1)/2}\,(-1)^{\ell_j-1}\cdot Y_{n_j}\cdots Y_{k_j-1}Y_{k_j}Y_{k_j+1}\cdots Y_{n_j+\ell-1}. \end{aligned}$$

Case of $\{j_1,j_2\}\sqsubset J_-$: Note first that $g'_{j_1}g'_{j_2}=z_1z_2g'_{j_2}g'_{j_1}$, and so $\pi_\gamma^{\mho+}(g'_{j_1}g'_{j_2}) = \pi_\gamma^{\mho+}(g'_{j_2}g'_{j_1})$. Suppose that, for $j=j_1$ and j_2, g'_j's are normalized modulo $\widetilde{Z}$ as above. Then we should pick up from

$$Y_{k_{j_1}}Y_{k_{j_2}}\cdot\prod_{j=j_1,j_2}2^{-(\ell_j-1)/2}\cdot(Y_{n_j}-Y_{n_j+1})(Y_{n_j+1}-Y_{n_j+2})\cdots(Y_{n_j+\ell_j-2}-Y_{n_j+\ell_j-1}),$$

a monomial term

$$-\prod_{j=j_1,j_2}2^{-(\ell_j-1)/2}\,(-1)^{\ell_j-1}\cdot Y_{n_j}\cdots Y_{k_j-1}Y_{k_j}Y_{k_j+1}\cdots Y_{n_j+\ell_j-1}\,.$$

Thus, in total, we have

$$\pi_\gamma^{\mho+}(g')=\varepsilon^{\mho}(g')\cdot\zeta_\gamma(d')\cdot(-1)^{|J_-|/2}\cdot\prod_{j\in J}2^{-(\ell_j-1)/2}\,(-1)^{\ell_j-1}\times Y_1Y_2\cdots Y_n\,,$$

where the sign $\varepsilon^{\mho}(g')=\pm1$ is determined by

$$(17.43)\qquad \prod_{q\in Q}Y_q\times\prod_{j\in J}\Big(\prod_{p\in K_j}Y_p\Big)\ =\ \varepsilon^{\mho}(g')\cdot Y_1Y_2\cdots Y_n\,.$$

Its trace is $f_\gamma^{\mho+}(g')=\varepsilon^{\mho}(g')\cdot\zeta_\gamma(d')\cdot(-1)^{|J_-|/2}\cdot(2i)^{n'}\cdot\prod_{j\in J}2^{-(\ell_j-1)/2}$, since $\prod_{j\in J}(-1)^{\ell_j-1}=\mathrm{sgn}(\sigma')=1$.

Theorem 17.9 (Case $n=2n'+1$ odd). *Let* $g'=(d',\sigma')=\xi'_{q_1}\cdots\xi'_{q_r}\,g'_1\cdots g'_s$, $g'_j=(d'_j,\sigma'_j)$, *be an element of* $\widetilde{D}_n\overset{\mathrm{II}}{\rtimes}\mathcal{S}(P_\gamma^\pm)$, *where* $\mathcal{S}(P_\gamma^\pm)=\widetilde{\mathfrak{A}}\big(\prod_{\zeta\in\widehat{T}^0}\mathfrak{S}_{I_\zeta}\big)$.

(i) *Suppose* $\mathrm{ord}(d')\equiv 0 \pmod 2$. *Then* $f_\gamma^{\mho+}(g')=f_\gamma^{\mho-}(g')$, *and* $f_\gamma^{\mho+}(g')\neq 0$ *if and only if (Condition II-00) holds for* g'. *Moreover there holds a product formula as*

$$\widetilde{f}_\gamma^{\mho\pm}(g')=\prod_{q\in Q}\widetilde{f}_\gamma^{\mho\pm}(\xi'_q)\times\prod_{j\in J_+}\widetilde{f}_\gamma^{\mho\pm}(g'_j)\times\prod_{\{j_1,j_2\}\sqsubset J_-}\widetilde{f}_\gamma^{\mho\pm}(g'_{j_1}g'_{j_2}).$$

If g' *is normalized as in (17.14), then*

$$\widetilde{f}_\gamma^{\mathrm{II}}(\xi'_q)=\zeta_\gamma(\xi'_q),$$

$$\widetilde{f}_\gamma^{\mathrm{II}}(g'_j)=\zeta_\gamma(d'_j)\,(-1)^{(\ell_j-1)/2}\,2^{-(\ell_j-1)/2},\qquad \text{for } j\in J_+,$$

$$\widetilde{f}_\gamma^{\mathrm{II}}(g'_{j_1}g'_{j_2})=\varepsilon(g'_{j_1}g'_{j_2})\cdot(-1)\prod_{j=j_1,j_2}\zeta_\gamma(d'_j)\,(-1)^{\ell_j/2}\,2^{-(\ell_j-1)/2},\ \ \text{for } \{j_1,j_2\}\sqsubset J_-\,,$$

where the sign $\varepsilon(g'_{j_1}g'_{j_2})$ *is defined by (17.22) – (17.24), and also by (17.25).*

(ii) *Suppose* $\mathrm{ord}(d') \equiv 1 \pmod 2$. *Then* $f_\gamma^{\mho+}(g') = -f_\gamma^{\mho-}(g')$ *and* $f_\gamma^{\mho+}(g') \neq 0$ *if and only if (Condition* $\mho$*-11) holds for* g'. *If* g' *is normalized as in (17.14), then*

$$f_\gamma^{\mho+}(g') \;=\; \varepsilon^{\mho}(g')\cdot\zeta_\gamma(d')\cdot(-1)^{|J_-|/2}\cdot(2i)^{n'}\cdot\prod_{j\in J}2^{-(\ell_j-1)/2},$$

where the sign $\varepsilon^{\mho}(g') = \pm 1$ *is determined by (17.43).*

Remark 17.2. In the special case where $|I_{n,\zeta}| \le 1$ $(\forall\zeta\in\widehat{T})$ or $\mathcal{S}(P_\gamma^\pm) = Z_1 = \langle z_1\rangle$, we see in Example 12.3 that $\pi_\gamma^{\mho+} = P_\gamma^+\cdot\chi_1$, so its character is easy to get.

17.6 A covariance property of characters $f_\gamma^{\mho\pm} = \mathrm{tr}\big(\pi_\gamma^{\mho\pm}\big)$

Recall that $\pi_\gamma^{\mho\pm}$ is no more a group representation when we go out from

$$\widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \mathcal{S}(P_\gamma^\pm) = \widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \widetilde{\mathfrak{A}}\big(\textstyle\prod_{\zeta\in\widehat{T}^0}\mathfrak{S}_{I_{n,\zeta}}\big) \quad\text{to}\quad \widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \Phi_{\mathfrak{S}}^{-1}\big(\textstyle\prod_{\zeta\in\widehat{T}^0}\mathfrak{S}_{I_{n,\zeta}}\big).$$

Hence the following covariance property of $f_\gamma^{\mho\pm} = \mathrm{tr}\big(\pi_\gamma^{\mho\pm}\big)$ is very interesting, and it plays an important role in calculating characters of spin IRs of $\widetilde{G}_n^{\mathrm{II}}$ in §20.3.

Let $s'_0 \in \Phi_{\mathfrak{S}}^{-1}\big(\prod_{\zeta\in\widehat{T}^0}\mathfrak{S}_{I_{n,\zeta}}\big)$. If $\mathrm{sgn}(s'_0) = 1$, then $f_\gamma^{\mho\pm}(s'_0 g' {s'_0}^{-1}) = f_\gamma^{\mho\pm}(g')$, since $s'_0 \in \mathcal{S}(P_\gamma^\pm)$.

Lemma 17.10. *Let* $\mathrm{sgn}(s'_0) = -1$ *or* $s'_0 \notin \widetilde{\mathfrak{A}}\big(\prod_{\zeta\in\widehat{T}^0}\mathfrak{S}_{I_{n,\zeta}}\big)$. *Take a* $g' \in \widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \mathcal{S}(P_\gamma^\pm)$ *normalized as* $g' = \xi'_{q_1}\cdots\xi'_{q_r}\,g'_1\cdots g'_s$ *with* $d'_j = \eta_{k_j}^{b_j}$ $(\exists k_j\in K_j,\ j\in J)$. *Then*

$$f_\gamma^{\mho\pm}(s'_0 g' {s'_0}^{-1}) = (-1)^{\mathrm{ord}(d')}\, f_\gamma^{\mho\pm}(g').$$

Proof. First, by Theorem 10.2 (i), $\iota\big(\nabla'_n(\sigma')\big)P_\gamma(d') = P_{(\tau_1\tau_2\cdots\tau_n)\sigma\gamma}\big({\sigma'}^{\mathrm{II}}(d')\big)$ for $\sigma'\in\widetilde{\mathfrak{S}}_n$ if $\mathrm{sgn}(\sigma') = -1$, where $\sigma = \Phi(\sigma')$. Take $\sigma' = {s'_0}^{-1}$, then $s_0^{-1}\gamma = \gamma$, and so we have

$$\nabla'_n(s'_0)^{-1}P_\gamma\big({s'_0}^{\mathrm{II}}(d')\big)\nabla'_n(s'_0) = P_{(\tau_1\tau_2\cdots\tau_n)\gamma}(d') = (-1)^{\mathrm{ord}(d')}\,P_\gamma(d'), \tag{17.44}$$

for $d'\in\widetilde{D}_n$. Note that $s'_0 g' {s'_0}^{-1} = \big({s'_0}^{\mathrm{II}}(d'), s'_0\sigma'{s'_0}^{-1}\big)$, then

$$\begin{aligned}
\pi_\gamma^{\mho+}(s'_0 g' {s'_0}^{-1}) &= P_\gamma^+\big({s'_0}^{\mathrm{II}}(d')\big)\cdot\mho_n^+\big(s'_0\sigma'{s'_0}^{-1}\big) = P_\gamma\big({s'_0}^{\mathrm{II}}(d')\big)\cdot\nabla'_n\big(s'_0\sigma'{s'_0}^{-1}\big)\\
&= \nabla'_n(s'_0)\Big(\big(\nabla'_n(s'_0)^{-1}P_\gamma\big({s'_0}^{\mathrm{II}}(d')\big)\nabla'_n(s'_0)\big)\cdot\nabla'_n(\sigma')\Big)\nabla'_n({s'_0}^{-1})\\
&= \nabla'_n(s'_0)\cdot\big(P_{(\tau_1\tau_2\cdots\tau_n)\gamma}(d')\,\nabla'_n(\sigma')\big)\cdot\nabla'_n({s'_0}^{-1})
\end{aligned}$$

$$= \nabla'_n(s'_0)\cdot(-1)^{\mathrm{ord}(d')}\,P_\gamma(d')\nabla'_n(\sigma')\cdot\nabla'_n({s'_0}^{-1})$$
$$= (-1)^{\mathrm{ord}(d')}\cdot\nabla'_n(s'_0)\,\pi_\gamma^{\mho+}(g')\,\nabla'_n(s'_0)^{-1}.$$

Hence we get $f_\gamma^{\mho+}(s'_0 g' {s'_0}^{-1}) = (-1)^{\mathrm{ord}(d')} f_\gamma^{\mho+}(g')$. □

Another proof. We appeal to detailed calculations in the preceding subsection. Note that

$$s'_0 g' {s'_0}^{-1} = \big({s'_0}^{\mathrm{II}}(d'), s'_0\sigma'{s'_0}^{-1}\big) = {s'_0}^{\mathrm{II}}(\xi'_{q_1})\cdots {s'_0}^{\mathrm{II}}(\xi'_{q_r})\,(s'_0 g'_1 {s'_0}^{-1})\,\cdots\,(s'_0 g'_s {s'_0}^{-1}),$$

where $s'_0 g'_j {s'_0}^{-1} = \big({s'_0}^{\mathrm{II}}(d'_j), s'_0\sigma'_j{s'_0}^{-1}\big)$. On the one hand, for $\xi'_q = {\eta_q}^{a_q}$ and $d'_j = \prod_{p\in K_j}{\eta_p}^{a_p}$,

$${s'_0}^{\mathrm{II}}(\xi'_q) = {\eta_{s_0(q)}}^{a_q}, \quad {s'_0}^{\mathrm{II}}(d'_j) = \prod_{p\in K_j}{\eta_{s_0(p)}}^{a_p},$$

whence $P_\gamma^+\big({s'_0}^{\mathrm{II}}(\xi'_q)\big) = \big(\omega^{\gamma_{s_0(q)}}Y_{s_0(q)}\big)^{a_q}$, $P_\gamma^+\big({s'_0}^{\mathrm{II}}(d'_j)\big) = \prod_{p\in K_j}\big(\omega^{\gamma_{s_0(p)}}Y_{s_0(p)}\big)^{a_p}$. On the other hand, note that, for $j\in J_+$ and $\{j_1,j_2\}\sqsubset J_-$ respectively

$$\nabla'_n\big(s'_0\sigma'_j{s'_0}^{-1}\big) = \nabla'_n(s'_0)\nabla'_n(\sigma'_j)\nabla'_n(s'_0)^{-1}\ ;$$
$$\nabla'_n\big(s'_0\sigma'_{j_1}\sigma'_{j_2}{s'_0}^{-1}\big) = \nabla'_n(s'_0)\nabla'_n(\sigma'_{j_1}\sigma'_{j_2})\nabla'_n(s'_0)^{-1}\,,$$

and that $\nabla'_n(s'_0)Y_p\nabla'_n(s'_0)^{-1} = -Y_{s_0(p)}$ ($p\in\boldsymbol{I}_n$).

From these two facts, the one for P_γ^+ and the other for ∇'_n, and also from $L(\sigma'_j)\equiv 0$, $L(\sigma'_{j_1}\sigma'_{j_2})\equiv 0 \pmod 2$, we see that the effect of $\pi_\gamma^{\mho+}(g')\mapsto \pi_\gamma^{\mho+}(s'_0 g'{s'_0}^{-1})$, on their expressions by means of Y_p's, is just the replacement $Y_p\mapsto Y_{s_0(p)}$ ($p\in\boldsymbol{I}_n$), and on their coefficients is $\chi_\gamma(d') = \omega^{a_1\gamma_1+\cdots+a_n\gamma_n}\mapsto \omega^{a_1\gamma_{s_0(1)}+\cdots+a_n\gamma_{s_0(n)}} = \chi_\gamma(d')$ since ${s_0}^{-1}\gamma=\gamma$.

Under this replacement $Y_p\mapsto Y_{s_0(p)}$, every monomial ${Y_1}^{c_1}{Y_2}^{c_2}\cdots{Y_n}^{c_n}$ with trace zero (resp. trace non-zero) is sent to such a one with trace zero (resp. trace non-zero).

In case $\mathrm{ord}(d')\equiv 0 \pmod 2$, the traces themselves remain unchanged.

In case $\mathrm{ord}(d')\equiv 1 \pmod 2$, $f_\gamma^{\mho+}(g')\neq 0$ if and only if g' satisfies (Condition $\mho$-11), and the sign $\varepsilon^{\mho}(g')$ changes as $\varepsilon^{\mho}(s'_0 g'{s'_0}^{-1}) = -\varepsilon^{\mho}(g')$, since

$$\prod_{q\in Q}Y_{s_0(q)}\times\prod_{j\in J}\prod_{p\in K_j}Y_{s_0(p)} = \mathrm{sgn}(s_0)\cdot\prod_{q\in Q}Y_q\times\prod_{j\in J}\prod_{p\in K_j}Y_p\,.$$

This proves the assertions for $f_\gamma^{\mho+}$. □

18 Factorisability of characters and covariance of trace functions

18.1 Origin of factorizability of normalized characters

We explain here the origin of the factorizability of the normalized character $\widetilde{f}_\gamma^{\mathrm{I}}(g') = \mathrm{tr}\big(\pi_\gamma^{\mathrm{I}}(g')\big)/\dim\pi_\gamma^{\mathrm{I}}$ such as $\widetilde{f}_\gamma^{\mathrm{I}}(g'_jg'_l) = \widetilde{f}_\gamma^{\mathrm{I}}(g'_j)\,\widetilde{f}_\gamma^{\mathrm{I}}(g'_l)$. This will be important in studying induced characters of π_γ^{I}. Similar for $\widetilde{f}_\gamma^{\mathrm{II}}$ etc.

Put $n' = [n/2]$ as before. Among monomial terms in $Y_1, \ldots, Y_{2n'+1}$ such as $Y_1^{c_1} \cdots Y_{2n'+1}^{c_{2n'+1}}$, those who have non-zero trace are of the form $Y_1^{2c'_1} \cdots Y_n^{2c'_n}$ $(= E_{2^{n'}})$ or $Y_1 Y_2 \cdots Y_{2n'+1}$ $(= i^{n'} E_{2^{n'}})$ (multiplicatively modulo $Y_1^{2c'_1} \cdots Y_{2n'+1}^{2c'_{2n'+1}}$). Any of other monomials, which has trace zero, will be called an *odd monomial*.

To calculate $f_\gamma^{\mathrm{I}}(g'_j)$ (resp. $f_\gamma^{\mathrm{II}}(g'_j)$), we expand $\pi_\gamma^{\mathrm{I}}(g'_j)$ (resp. $\pi_\gamma^{\mathrm{II}}(g'_j)$) into a linear combination of monomial terms in $Y_1, \ldots, Y_{2n'+1}$ such as $Y_1^{c_1} \cdots Y_{2n'+1}^{c_{2n'+1}}$. Then we appeal to the following lemma.

Lemma 18.1. *When $\pi_\gamma^{\mathrm{I}}(g'_j)$ is expanded into a linear combination of monomial terms in $Y_1, \ldots, Y_{2n'+1}$ such as $Y_1^{c_1} \cdots Y_{2n'+1}^{c_{2n'+1}}$, there exists only one term with trace non-zero and all others are* **odd** *monomials, or*

$$\pi_\gamma^{\mathrm{I}}(g'_j) = \lambda_j E_{2^{n'}} + \sum_{\mathrm{odd}} \lambda^{(j)}_{c_1,\ldots,c_n} Y_1^{c_1} \cdots Y_n^{c_n}, \tag{18.1}$$

where $\lambda_j = \mathrm{tr}\big(\pi_\gamma(g'_j)\big)/2^{n'} = \widetilde{f}_\gamma^{\mathrm{I}}(g'_j)$, and $\lambda^{(j)}_{c_1,\ldots,c_n}$ are constants such that if $\lambda^{(j)}_{c_1,\ldots,c_n} \neq 0$, then $\mathrm{supp}\big(Y_1^{c_1} \cdots Y_n^{c_n}\big) := \{p \in \boldsymbol{I}_n \,;\, c_p \neq 0\} \subset K_j$. Similar assertion holds for $\pi_\gamma^{\mathrm{II}}(g'_j)$ and $\widetilde{f}_\gamma^{\mathrm{II}}(g'_j)$.

A proof for the factorizability of $\widetilde{f}_\gamma^{\mathrm{I}}$ under (Condition I-00) :

Let us prove $\widetilde{f}_\gamma^{\mathrm{I}}(g'_j g'_l) = \widetilde{f}_\gamma^{\mathrm{I}}(g'_j)\widetilde{f}_\gamma^{\mathrm{I}}(g'_l)$ for $j \neq l$. Expand $\pi_\gamma^{\mathrm{I}}(g'_l)$ also as

$$\pi_\gamma^{\mathrm{I}}(g'_l) = \lambda_l E_{2^{n'}} + \sum_{\mathrm{odd}} \lambda^{(l)}_{c_1,\ldots,c_n} Y_1^{c_1} \cdots Y_n^{c_n}, \tag{18.2}$$

where $\lambda^{(l)}_{c_1,\ldots,c_n} \neq 0$ implies that $\mathrm{supp}\big(Y_1^{c_1} \cdots Y_n^{c_n}\big) \subset K_l$. Multiply (18.1) with (18.2), and note that supports of monomials in (18.1) and (18.2) are mutually disjoint because $K_j \cap K_l = \emptyset$, we have

$$\pi_\gamma^{\mathrm{I}}(g'_j g'_l) = \pi_\gamma^{\mathrm{I}}(g'_j)\pi_\gamma^{\mathrm{I}}(g'_l) = \lambda_j \lambda_l E_{2^{n'}} + \sum_{\mathrm{odd}} \lambda^{(jl)}_{c_1,\ldots,c_n} Y_1^{c_1} \cdots Y_n^{c_n}.$$

Taking the traces of both sides, we obtain $\widetilde{f}_\gamma^{\mathrm{I}}(g'_j g'_l) = \lambda_j \lambda_l = \widetilde{f}_\gamma^{\mathrm{I}}(g'_j)\widetilde{f}_\gamma^{\mathrm{I}}(g'_l)$. □

18.2 Covariance of certain trace functions (CASE I)

Let us study the behavior of characters of stationary subgroups under the conjugation of $\widetilde{\mathfrak{S}}_n$. To calculate induced characters, e.g., for π_γ^{I}, we should calculate $f_\gamma^{\mathrm{I}}(s' g' s'^{-1})$ for $s' \in \widetilde{\mathfrak{S}}_n$ in case $s' g' s'^{-1} \in \widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P_\gamma)$, and similarly for $\pi_\gamma^{\mathrm{I}\pm}$ etc.

More generally we define several *(spin) trace functions* on $\widetilde{G}_n^{\mathrm{I}}$ as follows: for $\gamma \in \Gamma_n$ and $g' = (d', \sigma') \in \widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \widetilde{\mathfrak{S}}_n = \widetilde{G}_n^{\mathrm{I}}$,

$$\mathcal{T}_\gamma^{\mathrm{I}}(g') := \mathrm{tr}\big(P_\gamma(d')\nabla_n(\sigma')\big), \quad \mathcal{T}_\gamma^{\mathrm{I}\pm}(g') := \mathrm{tr}\big(P_\gamma^{\pm}(d')\nabla_n^{\mathrm{I}\pm}(\sigma')\big), \tag{18.3}$$

and study their covariance under conjugation of $s' \in \widetilde{\mathfrak{S}}_n$.

Suppose $g' = (d', \sigma')$ is in the stationary subgroup $\widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P_\gamma)$ (resp. $\widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P_\gamma^{\pm})$), and express it (modulo $z_1{}^a z_2{}^b$) as

$$\text{(18.4)} \qquad \begin{cases} g' = \xi'_{q_1} \cdots \xi'_{q_r}\, g'_1 g'_2 \cdots g'_s, \ \ \xi'_q = \eta'_q{}^{a_q} \ (q \in Q = \{q_1, \ldots, q_r\}), \\ g'_j = (d'_j, \sigma'_j), \ \ d'_j = \prod_{p \in K_j} \eta'_p{}^{a_p}, \ \ K_j = \mathrm{supp}(g'_j) \ \ (j \in J = \boldsymbol{I}_s). \end{cases}$$

For any $s' \in \widetilde{\mathfrak{S}}_n$, put $s = \Phi_{\mathfrak{S}}(s') \in \mathfrak{S}_n$, then

$$\text{(18.5)} \qquad \begin{cases} s'g's'^{-1} = \big(s'^{\mathrm{I}}(d'), s'\sigma's'^{-1}\big), \ \ s'^{\mathrm{I}}(\xi'_q) = \big(z_2{}^{L(s)}\eta'_{s(q)}\big)^{a_q}, \\ s'g'_js'^{-1} = \big(s'^{\mathrm{I}}(d'_j), s'\sigma'_js'^{-1}\big), \ \ s'^{\mathrm{I}}(d'_j) = \prod_{p \in K_j}(z_2{}^{L(s)}\eta'_{s(p)})^{a_p}. \end{cases}$$

Lemma 18.2. *Under conjugation of $s' \in \widetilde{\mathfrak{S}}_n$, we have a covariance as*

$$\begin{aligned} P_\gamma\big(s'^{\mathrm{I}}(\xi'_q)\big) &= \zeta_{s^{-1}\gamma}(\xi'_q)\big(\mathrm{sgn}(s)Y'_{s(q)}\big)^{a_q} \\ &= \begin{cases} \zeta_{s^{-1}\gamma}(\xi'_q)\, E_{2^{n'}} & \text{if } a_q \equiv 0 \pmod 2\ , \\ \zeta_{s^{-1}\gamma}(\xi'_q)\,\mathrm{sgn}(s)Y'_{s(q)} & \text{if } a_q \equiv 1 \pmod 2). \end{cases} \\ \frac{\mathrm{tr}\big(P_\gamma(s'\xi'_qs'^{-1})\big)}{\zeta_{s^{-1}\gamma}(\xi'_q)} &= \frac{\mathrm{tr}\big(P_\gamma(\xi'_q)\big)}{\zeta_\gamma(\xi'_q)} \qquad (q \in Q,\ s' \in \widetilde{\mathfrak{S}}_n). \end{aligned}$$

Proof. $P_\gamma\big(s'^{\mathrm{I}}(\xi'_q)\big) = \big(\zeta_\gamma(\eta'_{s(q)})\mathrm{sgn}(s)Y'_{s(q)}\big)^{a_q} = \zeta_{s^{-1}\gamma}(\xi'_q)\big(\mathrm{sgn}(s)Y'_{s(q)}\big)^{a_q}$. □

Lemma 18.3. *The operator $P_\gamma(d'_j)\nabla_n(\sigma'_j)$ is transformed by conjugation of $s' \in \widetilde{\mathfrak{S}}_n$ as*

$$\text{(18.6)} \qquad P_\gamma\big(s'^{\mathrm{I}}(d'_j)\big) = \zeta_{s^{-1}\gamma}(d'_j) \cdot \prod_{p \in K_j} \big(\mathrm{sgn}(s)Y'_{s(p)}\big)^{a_p}.$$

Express $\nabla_n(\sigma')$ as a product of $\nabla_n(r_p)$'s and then expand it into a linear combination of monomial terms $\prod_{p \in K_j} Y_p^{c_p}$ as

$$\text{(18.7)} \qquad \nabla_n(\sigma'_j) = \sum_c \lambda_c \cdot \prod_{p \in K_j} Y'_p{}^{c_p}, \quad c = (c_p)_{p \in K_j}, \ \ \lambda_c \in \boldsymbol{C},$$

then its conjugation by $s' \in \widetilde{\mathfrak{S}}_n$ is

$$\text{(18.8)} \qquad \nabla_n(s'\sigma'_js'^{-1}) = \sum_c \lambda_c \cdot \prod_{p \in K_j}\big(\mathrm{sgn}(s)Y'_{s(p)}\big)^{c_p},$$

where the products are along with the natural order of $p \in K_j$. Moreover

$$\text{(18.9)} \qquad \frac{\mathrm{tr}\Big(P_\gamma\big(s'^{\mathrm{I}}(d'_j)\nabla_n(s'\sigma'_js'^{-1})\big)\Big)}{\zeta_{s^{-1}\gamma}(d'_j)} = \frac{\mathrm{tr}\big(P_\gamma(d'_j)\nabla_n(\sigma')\big)}{\zeta_\gamma(d'_j)} \quad (j \in J, s' \in \widetilde{\mathfrak{S}}_n).$$

Proof. (i) Note that $\nabla_n(s'\sigma'_j s'^{-1}) = \nabla_n(s')\nabla_n(\sigma'_j)\nabla_n(s')^{-1}$ and

$$\begin{aligned} \nabla_n(r_i)Y'_p\nabla_n(r_i)^{-1} &= -Y'_{s_i(p)} \quad (p \in \boldsymbol{I}_n) \\ \therefore \quad \nabla_n(s')Y'_p\nabla_n(s')^{-1} &= \mathrm{sgn}(s)Y'_{s(p)} \quad (p \in \boldsymbol{I}_n). \end{aligned}$$

Compare the expansion of $P_\gamma(d'_j)\nabla(\sigma'_j)$ into a linear combination of monomials in Y'_p's obtained from (18.7) with that of $P_\gamma\big({s'}^{\mathrm{I}}(d'_j)\big)\nabla_n(s'\sigma'_j s'^{-1})$ obtained from (18.6) and (18.8). Then, apart from the proportional constants $\zeta_\gamma(d'_j)$ and $\zeta_{s^{-1}\gamma}(d'_j)$, the latter is obtained from the former by replacing each Y'_p by $\mathrm{sgn}(s)Y'_{s(p)}$.

On the other hand, as is remarked in Lemma 16.5 and in Theorem 16.11, in the expansion of $P_\gamma(d'_j)\nabla(\sigma'_j)$, there exists at most one monomial term with trace non-zero and all others are **odd** monomials (cf. also Lemma 18.1). Since the monomial term with trace non-zero is of the form $\prod_{p\in K_j} {Y'_p}^{2c'_p}$ $(= E_{2^{n'}})$, we have the following rule.

> **Consequence of Replacement I-00:** *Under replacement I-00:* $Y'_p \mapsto \mathrm{sgn}(s)Y'_{s(p)}$ $(p \in K_j)$, *the unique monomial term with trace non-zero* $\prod_{p\in K_j} {Y'_p}^{2c'_p} = E_{2^{n'}}$ *for* g'_j *remains to be equal to* $E_{2^{n'}}$, *and all the other odd monomials are mapped to odd monomials with supports inside* $s(K_j)$.

Thus we get the covariance relation (18.9). □

Proposition 18.4. *Let* $n = 2n'$ *even, and* $s' \in \widetilde{\mathfrak{S}}_n$. *For* $g' = (d', \sigma') \in \widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P_\gamma)$, *expressed as in (18.4), the value* $\mathcal{T}^{\mathrm{I}}_\gamma(s'g's'^{-1})$ *is obtained as follows. According to the decomposition* $\mathcal{T}^{\mathrm{I}}_\gamma(g') = \prod_{q\in Q}\mathcal{T}^{\mathrm{I}}_\gamma(\xi'_q)\cdot\prod_{j\in J}\mathcal{T}^{\mathrm{I}}_\gamma(g'_j)$,

$$\mathcal{T}^{\mathrm{I}}_\gamma(s'g's'^{-1}) = \prod_{q\in Q}\mathcal{T}^{\mathrm{I}}_\gamma(s'\xi'_q s'^{-1})\cdot\prod_{j\in J}\mathcal{T}^{\mathrm{I}}_\gamma(s'g'_j s'^{-1}), \tag{18.10}$$

and $\mathcal{T}^{\mathrm{I}}_\gamma(s'\xi'_q s'^{-1})$ *(resp.* $\mathcal{T}^{\mathrm{I}}_\gamma(s'g'_j s'^{-1})$*) is obtained from the formula in Theorem 16.11 (i) for* $\mathcal{T}^{\mathrm{I}}_\gamma(\xi'_q)$ *(resp.* $\mathcal{T}^{\mathrm{I}}_\gamma(g'_j)$*) by replacing* γ *with* $s^{-1}\gamma$. *In short,*

> $\mathcal{T}^{\mathrm{I}}_\gamma(s'g's'^{-1})$ *is obtained from the formula of* $\mathcal{T}^{\mathrm{I}}_\gamma(g')$ *by replacing* γ *with* $s^{-1}\gamma$.

Proof. By Theorem 9.2 (i), for any $s' \in \widetilde{\mathfrak{S}}_n$ and $d' \in \widetilde{D}_n$, $\nabla_n(s')P_\gamma(d')\nabla_n(s')^{-1} = P_{s\gamma}\big({s'}^{\mathrm{I}}(d')\big)$. Then we have

$$\begin{aligned} P_\gamma\big({s'}^{\mathrm{I}}(d')\big)\nabla_n(s'\sigma's'^{-1}) &= \nabla_n(s')P_{s^{-1}\gamma}(d')\nabla_n(s')^{-1}\cdot\nabla_n(s')\nabla_n(\sigma')\nabla_n(s')^{-1} \\ &= \nabla_n(s')\big(P_{s^{-1}\gamma}(d')\nabla_n(\sigma')\big)\nabla_n(s')^{-1}. \end{aligned}$$

Taking traces of both extremities, we see that $\mathcal{T}^{\mathrm{I}}_\gamma(s'g's'^{-1})$ is obtained from $\mathcal{T}^{\mathrm{I}}_\gamma(g')$ by replacing γ with $s^{-1}\gamma$. □

Lemma 18.5. *Let $n = 2n'+1$ odd. Then, for $g' = (d', \sigma') \in \widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P_\gamma^+)$ and $s' \in \widetilde{\mathfrak{S}}_n$. Express g' as in (18.4).*

(i) *If* $\mathrm{ord}(d') + L(\sigma') \equiv 0 \pmod 2$, *then there holds for $\mathcal{T}_\gamma^{\mathrm{I}+}$ the similar assertion as in Lemma 18.3.*

(ii) *Assume* $\mathrm{ord}(d') + L(\sigma') \equiv 1 \pmod 2$. *Then* $\varepsilon^{\mathrm{I}}(s'g's'^{-1}) = \varepsilon^{\mathrm{I}}(g')$ *and*

$$\frac{\mathcal{T}_\gamma^{\mathrm{I}+}(s'g's'^{-1})}{\zeta_{s^{-1}\gamma}(d')} = \frac{\mathcal{T}_\gamma^{\mathrm{I}+}(g')}{\zeta_\gamma(d')}.$$

Proof. The proof is similar as for Lemma 18.3 except the proof for $\varepsilon^{\mathrm{I}}(s'g's'^{-1}) = \varepsilon^{\mathrm{I}}(g')$ in (ii). For this, when $\pi_\gamma^{\mathrm{I}+}(g')$ is expanded into a linear combination of monomial terms in Y_p''s, there exists only one with trace non-zero given as

$$\prod_{q\in Q} Y_q' \cdot \prod_{j\in J} \prod_{\substack{p\in K_j:\\ Y_p' \text{ black}}} Y_p' = \varepsilon^{\mathrm{I}}(g') \cdot Y_1'Y_2' \cdots Y_n' \ \bigl(= \varepsilon^{\mathrm{I}}(g') \cdot (-i)^{n'} E_{2^{n'}}\bigr),$$

and all others are odd monomials. Under the replacement $Y_p' \mapsto \mathrm{sgn}(s) Y'_{s(p)}$ ($p \in \boldsymbol{I}_n$), it is replaced by

$$\prod_{q\in Q} \bigl(\mathrm{sgn}(s)Y'_{s(q)}\bigr) \cdot \prod_{j\in J} \prod_{\substack{p\in K_j:\\ Y_p' \text{ black}}} \bigl(\mathrm{sgn}(s)Y'_{s(p)}\bigr) = \mathrm{sgn}(s)^{n+1} \prod_{q\in Q} Y_q' \cdot \prod_{j\in J} \prod_{\substack{p\in K_j:\\ Y_p' \text{ black}}} Y_p'. \quad \square$$

Similarly as Proposition 18.4, we can prove the following.

Proposition 18.6. *Let $n = 2n'+1$, $s' \in \widetilde{\mathfrak{S}}_n$ and $g' = (d', \sigma') \in \widetilde{G}_n^{\mathrm{I}}$. Then,*

> *$\mathcal{T}_\gamma^{\mathrm{I}+}(s'g's'^{-1})$ (resp. $\mathcal{T}_\gamma^{\mathrm{I}-}(s'g's'^{-1})$) is obtained from the formula of $\mathcal{T}_\gamma^{\mathrm{I}+}(g')$ (resp. $\mathcal{T}_\gamma^{\mathrm{I}-}(g')$) by replacing γ with $s^{-1}\gamma$.*

Proof. For $\mathcal{T}_\gamma^{\mathrm{I}+}$, the proof for $\mathcal{T}_\gamma^{\mathrm{I}}$ in Proposition 18.4 is also valid, since $P_\gamma^+ = P_\gamma$ and $\nabla_n^+ = \nabla_n$. For $\mathcal{T}_\gamma^{\mathrm{I}-}$, by Theorem 9.2 (ii), we have, for any $s' \in \widetilde{\mathfrak{S}}_n$ and $d' \in \widetilde{D}_n$, $\nabla_n^-(s') P_\gamma^-(d') \nabla_n^-(s')^{-1} = P_{s\gamma}^-\bigl({s'}^{\mathrm{I}}(d')\bigr)$. Then

$$P_\gamma^-\bigl({s'}^{\mathrm{I}}(d')\bigr)\nabla_n^-(s'\sigma's'^{-1}) = \nabla_n^-(s')\bigl(P_{s^{-1}\gamma}^-(d')\nabla_n^-(\sigma')\bigr)\nabla_n^-(s')^{-1}.$$

Taking traces of both sides, we obtain the assertion. □

18.3 Covariance of certain trace functions (CASE II)

We define *(spin) trace functions* on $\widetilde{G}_n^{\mathrm{II}}$ as follows: for $\gamma \in \Gamma_n$ and $g' = (d', \sigma') \in \widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \widetilde{\mathfrak{S}}_n = \widetilde{G}_n^{\mathrm{II}}$,

(18.11) $$\mathcal{T}_\gamma^{\mathrm{II}}(g') := \mathrm{tr}\bigl(P_\gamma(d')\nabla_n^{\mathrm{II}}(\sigma')\bigr), \quad \mathcal{T}_\gamma^{\mho\pm}(g') := \mathrm{tr}\bigl(P_\gamma^\pm(d')\mho_n^\pm(\sigma')\bigr).$$

Proposition 18.7. *Let $n = 2n' \geq 4$. For $s' \in \widetilde{\mathfrak{S}}_n$ and $g' = (d', \sigma') \in \widetilde{G}^{\mathrm{II}}_n$, $\mathcal{T}^{\mathrm{II}}_\gamma(s'g's'^{-1})$ is obtained from the formula of $\mathcal{T}^{\mathrm{II}}_\gamma(g')$ by replacing γ with $s^{-1}\gamma$.*

Proof. By Theorem 10.2 (ii), for any $s' \in \widetilde{\mathfrak{S}}_n$ and $d' \in \widetilde{D}_n$, $\nabla^{\mathrm{II}}_n(s') P_\gamma(d') \nabla^{\mathrm{II}}_n(s')^{-1} = P_{s\gamma}\big(s'^{\,\mathrm{II}}(d')\big)$. Then we have

$$P_\gamma\big(s'^{\,\mathrm{II}}(d')\big)\nabla^{\mathrm{II}}_n(s'\sigma's'^{-1}) = \nabla^{\mathrm{II}}_n(s')\big(P_{s^{-1}\gamma}(d')\nabla^{\mathrm{II}}_n(\sigma')\big)\nabla_n(s')^{-1}.$$

Taking traces of both sides, we obtain the assertion. □

Proposition 18.8. *Let $n = 2n'+1 \geq 5$. For $s' \in \widetilde{\mathfrak{A}}_n$ and $g' = (d', \sigma') \in \widetilde{G}^{\mathrm{II}}_n$ with $\sigma' \in \widetilde{\mathfrak{A}}_n$,*

$\mathcal{T}^{\mho+}_\gamma(s'g's'^{-1})$ (resp. $\mathcal{T}^{\mho-}_\gamma(s'g's'^{-1})$) is obtained from the formula of $\mathcal{T}^{\mho+}_\gamma(g')$ (resp. $\mathcal{T}^{\mho-}_\gamma(g')$) by replacing γ with $s^{-1}\gamma$.

Proof. By Theorem 10.2 (iii), for any $s' \in \widetilde{\mathfrak{A}}_n$ and $d' \in \widetilde{D}_n$, $\mho^\pm_n(s') P^\pm_\gamma(d') \mho^\pm_n(s')^{-1} = P^\pm_{s\gamma}\big(s'^{\,\mathrm{II}}(d')\big)$. Then we have for $\sigma' \in \widetilde{\mathfrak{A}}_n$

$$P^\pm_\gamma\big(s'^{\,\mathrm{II}}(d')\big)\mho^\pm_n(s'\sigma's'^{-1}) = \mho^\pm_n(s')\big(P^\pm_{s^{-1}\gamma}(d')\mho^\pm_n(\sigma')\big)\mho^\pm_n(s')^{-1}.$$

Taking traces of both sides, we obtain the assertion. □

19 Characters of spin IRs of $\widetilde{G}^{\mathrm{I}}_n$ (CASE I)

19.1 Formula for calculating characters of spin IRs

First we prepare formulas for characters of induced representations. For $\gamma \in \Gamma^0_n$, we have $\zeta_\gamma = (\zeta_j)_{j \in \boldsymbol{I}_n}$ with $\zeta_j = \zeta_{j,\gamma_j}$ identified with $\zeta^{(\gamma_j)} \in \widehat{T}^0 \subset \widehat{T}$, and a partition in Definition 11.2 as

$$\boldsymbol{I}_n = \bigsqcup\nolimits_{\zeta \in \widehat{T}^0} I_{n,\zeta}, \quad I_{n,\zeta} = \{j \in \boldsymbol{I}_n\,;\, \zeta_j = \zeta\}. \tag{19.1}$$

Here $\widehat{T}^0 = \{\eta^{(a)} \in \widehat{T}\,;\, \eta^{(a)}(\eta) = \omega^a,\, 0 \leq a < m' = m/2\}$. We assume that γ is normalized (under the action of $\mathfrak{S}_n$) so that all $I_{n,\zeta}$'s are intervals in $\boldsymbol{I}_n$. IRs of $\prod_{\zeta \in \widehat{T}^0} \mathfrak{S}_{I_{n,\zeta}}$ are parametrized by a set of Young diagrams $\Lambda^n = \big(\lambda^{n,\zeta}\big)_{\zeta \in \widehat{T}^0}$ as

$$\pi_{\Lambda^n} = \mathop{\boxtimes}_{\zeta \in \widehat{T}^0} \pi(\lambda^{n,\zeta})\,, \tag{19.2}$$

where $\lambda^{n,\zeta}$ is of size $|I_{n,\zeta}|$ and $\pi(\lambda^{n,\zeta})$ is an IR of $\mathfrak{S}_{I_{n,\zeta}}$ corresponding to $\lambda^{n,\zeta}$. Consider IRs of $\widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P_\gamma)$, and those of $\widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P^\pm_\gamma)$ given as

$$\begin{aligned} &\pi^{\mathrm{I}}_{\gamma,\Lambda^n} := \pi^{\mathrm{I}}_\gamma \boxdot \pi_{\Lambda^n} \quad \text{if } n = 2n' \text{ even}\,; \\ &\pi^{\mathrm{I}\pm}_{\gamma,\Lambda^n} := \pi^{\mathrm{I}\pm}_\gamma \boxdot \pi_{\Lambda^n} \quad \text{if } n = 2n'+1 \text{ odd}. \end{aligned} \tag{19.3}$$

By inducing them up to $\widetilde{G}^{\rm I}_n = \widetilde{G}^{\rm I}(m,1,n)$, we obtain spin IRs of CASE I, Type $(-1,-1,-1)$, as

$$(19.4)\qquad \begin{array}{ll} \Pi^{\rm I}_{\Lambda^n} = {\rm Ind}^{\widetilde{G}^{\rm I}_n}_{\widetilde{D}_n \overset{\rm I}{\rtimes} \mathcal{S}(P_\gamma)} \pi^{\rm I}_{\gamma,\Lambda^n} & \text{in case } n \text{ is even ;} \\ \Pi^{\rm I\pm}_{\Lambda^n} = {\rm Ind}^{\widetilde{G}^{\rm I}_n}_{\widetilde{D}_n \overset{\rm I}{\rtimes} \mathcal{S}(P^\pm_\gamma)} \pi^{\rm I\pm}_{\gamma,\Lambda^n} & \text{in case } n \text{ is odd.} \end{array}$$

Remark 19.1. In the parametrization above such as $\Pi^{\rm I}_{\Lambda^n}$ and $\Pi^{\rm I\pm}_{\Lambda^n}$, γ is implicit but, as is shown by the parametrization in §11.5 and §12, it is subordinate to Λ^n. In fact, the information of (the equivalence class of) γ is fully contained in $\Lambda^n = (\lambda^{n,\zeta})_{\zeta\in\widehat{T}^0}$, since $|\lambda^{n,\zeta}|$ is equal to the multiple $|I_{n,\zeta}|$ of ζ in $\zeta_\gamma = (\zeta_1,\dots,\zeta_n)$.

Put $f_{\Lambda^n}(\sigma') := {\rm tr}\big(\pi_{\Lambda^n}(\sigma')\big)$, then $f_{\Lambda^n}(\sigma') = f_{\Lambda^n}(\sigma) = {\rm tr}\big(\pi_{\Lambda^n}(\sigma)\big)$, $\sigma = \Phi(\sigma')$, and

$$(19.5)\qquad \begin{array}{lll} f^{\rm I}_{\gamma,\Lambda^n} = {\rm tr}\big(\pi^{\rm I}_{\gamma,\Lambda^n}\big), & F^{\rm I}_{\Lambda^n} = {\rm tr}\big(\Pi^{\rm I}_{\Lambda^n}\big) & \text{in case } n \text{ is even ;} \\ f^{\rm I\pm}_{\gamma,\Lambda^n} = {\rm tr}\big(\pi^{\rm I\pm}_{\gamma,\Lambda^n}\big), & F^{\rm I\pm}_{\Lambda^n} = {\rm tr}\big(\Pi^{\rm I\pm}_{\Lambda^n}\big) & \text{in case } n \text{ is odd.} \end{array}$$

Then we have at first

$$(19.6)\qquad \begin{array}{l} f^{\rm I}_{\gamma,\Lambda^n}\big((d',\sigma')\big) = f^{\rm I}_\gamma\big((d',\sigma')\big)\cdot f_{\Lambda^n}(\sigma), \\ f^{\rm I\pm}_{\gamma,\Lambda^n}\big((d',\sigma')\big) = f^{\rm I\pm}_\gamma\big((d',\sigma')\big)\cdot f_{\Lambda^n}(\sigma). \end{array}$$

Let $G' = \widetilde{G}^{\rm I}_n$, and $H' = \widetilde{D}_n \overset{\rm I}{\rtimes} \mathcal{S}$ be one of $\widetilde{D}_n \overset{\rm I}{\rtimes} \mathcal{S}(P_\gamma)$ and $\widetilde{D}_n \overset{\rm I}{\rtimes} \mathcal{S}(P^\pm_\gamma)$, where $\mathcal{S}$ is one of $\mathcal{S}(P_\gamma)$ and $\mathcal{S}(P^\pm_\gamma)$ correspondingly. Also let π be one of IRs of H' in (19.3), and put $\Pi = {\rm Ind}^{G'}_{H'}\pi$. Denote by f_π and F_Π their characters. Extend f_π from H' to G' by putting identically zero outside H'. Then we have

$$(19.7)\qquad F_\Pi(g') \;=\; \frac{1}{|H'|}\sum_{g'_0\in G'} f_\pi(g'_0 g' {g'_0}^{-1}).$$

From this, we see that $F_\Pi(g') = 0$ if g' is not conjugate under G' to an element of H', and so F_Π is completely determined by its values on H'.

Lemma 19.1. *Let $g' = (d',\sigma') \in H' = \widetilde{D}_n \overset{\rm I}{\rtimes} \mathcal{S}$, then*

$$F_\Pi(g') \;=\; \frac{1}{|\mathcal{S}|}\sum_{\substack{s'\in\widetilde{\mathfrak{S}}_n:\\ s'\sigma's'^{-1}\in\mathcal{S}}} f_\pi\big(({s'}^{\rm I}(d'), s'\sigma's'^{-1})\big).$$

Proof. This follows from the facts that f_π is invariant under $\widetilde{D}_n \subset H'$ and that $s'g's'^{-1} = ({s'}^{\rm I}(d'), s'\sigma's'^{-1})$. □

Lemma 19.2. *Let $g' = (d',\sigma') \in H' = \widetilde{D}_n \overset{\rm I}{\rtimes} \mathcal{S}$, and $f^{\rm I}_\gamma = {\rm tr}\big(\pi^{\rm I}_\gamma\big)$, $f^{\rm I\pm}_\gamma = {\rm tr}\big(\pi^{\rm I\pm}_\gamma\big)$.*

(i) *In case* $n = 2n'$ *even*, $\mathcal{S} = \mathcal{S}(P_\gamma) = \Phi_{\mathfrak{S}}^{-1}\big(\prod_{\zeta\in\widehat{T}^0} \mathfrak{S}_{I_{n,\zeta}}\big)$ *and with* $s = \Phi(s')$,

$$F^{\mathrm{I}}_{\Lambda^n}(g') \;=\; \sum_{\substack{s'\in\mathcal{S}\backslash\widetilde{\mathfrak{S}}_n:\\ s'\sigma's'^{-1}\in\mathcal{S}}} f^{\mathrm{I}}_\gamma\big((s'^{\mathrm{I}}(d'), s'\sigma's'^{-1})\big)\cdot f_{\Lambda^n}(s\sigma s^{-1}).$$

(ii) *In case* $n = 2n'+1$ *odd*, $\mathcal{S} = \mathcal{S}(P^{\pm}_\gamma) = \Phi_{\mathfrak{S}}^{-1}\big(\prod_{\zeta\in\widehat{T}^0} \mathfrak{S}_{I_{n,\zeta}}\big)$ *and*

$$F^{\mathrm{I}\pm}_{\Lambda^n}(g') \;=\; \sum_{s':\ \text{as above}} f^{\mathrm{I}\pm}_\gamma\big((s'^{\mathrm{I}}(d'), s'\sigma's'^{-1})\big)\cdot f_{\Lambda^n}(s\sigma s^{-1}).$$

Now we prepare here, as an important ingredient, explicit formulas for characters of the special spin IRs of $\widetilde{G}^{\mathrm{I}}_n$ given in Example 11.1:

$$\Pi^{\mathrm{I}}_0 = P_0\cdot\nabla_n \quad \text{with } P_0 = P_{\gamma^{(0)}} \qquad \text{for } \widetilde{G}^{\mathrm{I}}_n,\ n\ge 4 \text{ even},$$
$$\Pi^{\mathrm{I}}_\pm = P_\pm\cdot\nabla^\pm_n \quad \text{with } P_\pm = P^\pm_{\gamma^{(0)}} \qquad \text{for } \widetilde{G}^{\mathrm{I}}_n,\ n\ge 5 \text{ odd}.$$

Their characters are calculated in §16 as special cases, which we denote by $F^{\mathrm{I}}_{0,n}$, $F^{\mathrm{I}}_{+,n}$ and $F^{\mathrm{I}}_{-,n}$ respectively. Their intimate relations with the general case of γ facilitate to understand the covariance in §18.2 above, and also the calculation below.

An element of $\widetilde{G}^{\mathrm{I}}_n$ is expressed as $g'' = z_1{}^a z_2{}^b g'$, $g' = (d',\sigma') = \xi'_{q_1}\cdots\xi'_{q_r} g'_1\cdots g'_s$ with $g'_j = (d'_j,\sigma'_j)$ ($j\in J = \boldsymbol{I}_s$). Normalize g' as in (16.26) modulo $z_1{}^a z_2{}^b$.

Theorem 19.3. (i) Case $n = 2n'$ even: *If* $\mathrm{ord}(d') + L(\sigma') \equiv 0 \pmod 2$, *then* $F^{\mathrm{I}}_{0,n}(g') = \mathrm{tr}\big(\Pi^{\mathrm{I}}_0(g')\big) \ne 0$ *if and only if (Condition I-00) in Lemma 16.2 holds for* g'. *In that case, with* $\ell_j = \ell(\sigma'_j)$,

$$F^{\mathrm{I}}_{0,n}(g') = 2^{n'}\cdot\prod_{j\in J}(-1)^{[(\ell_j-1)/2]}\,2^{-(\ell_j-1)/2}$$

If $\mathrm{ord}(d') + L(\sigma') \equiv 1 \pmod 2$, *then* $F^{\mathrm{I}}_{0,n}(g') = 0$ *identically.*

(ii) Case $n = 2n'+1$ odd: *Suppose* $\mathrm{ord}(d') + L(\sigma') \equiv 0 \pmod 2$. *Then,* $F^{\mathrm{I}}_{-,n}(g') = F^{\mathrm{I}}_{+,n}(g') = F^{\mathrm{I}}_{0,n}(g')$, *and is given by the above formula.*

Suppose $\mathrm{ord}(d') + L(\sigma') \equiv 1 \pmod 2$. *Then* $F^{\mathrm{I}}_{-,n}(g') = -F^{\mathrm{I}}_{+,n}(g')$, *and* $F^{\mathrm{I}}_{+,n}(g') \ne 0$ *if and only if (Condition I-11) in Lemma 16.2 holds for* g'. *In that case,*

$$F^{\mathrm{I}}_{+,n}(g') = \varepsilon^{\mathrm{I}}(g')\,(2i)^{n'}\cdot\prod_{j\in J}(-1)^{[(\ell_j-1)/2]}\,2^{-(\ell_j-1)/2},$$

where the sign $\varepsilon^{\mathrm{I}}(g')$ *comes from rearrangement to have* $Y_1Y_2\cdots Y_n$ *as*

$$\prod_{q\in Q} Y_q \times \prod_{j\in J}\Big\{Y_{k_j}\prod_{\substack{p\in K_j\backslash\{k_j\}:\\ Y_p\ \text{black}}} Y_p\Big\} \;=\; \varepsilon^{\mathrm{I}}(g')\cdot Y_1Y_2\cdots Y_n\,. \tag{19.8}$$

19.2 Reduction of the summation over $s' \in \mathcal{S}\backslash\widetilde{\mathfrak{S}}_n$

Let $\gamma \in \Gamma_n^0$. Take a $g' = (d', \sigma')$ from $\widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P_\gamma)$ for $n = 2n'$ even, or from $\widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P_\gamma^\pm)$ for $n = 2n'+1$ odd respectively, and express it as $g' = (d', \sigma') = \xi'_{q_1} \cdots \xi'_{q_r} g'_1 \cdots g'_s$. Normalize g' as in (16.26) modulo $z_1{}^a z_2{}^b$. From Theorem 16.11, we get the following.

Lemma 19.4. (i) CASE $n = 2n'$ EVEN: *If* $\mathrm{ord}(d') + L(\sigma') \equiv 0 \pmod 2$, *then* $f_\gamma^{\mathrm{I}}(g') = \mathrm{tr}\big(\pi_\gamma^{\mathrm{I}}(g')\big) \neq 0$ *if and only if (Condition I-00) in Lemma 16.2 holds for* g'. *If* $\mathrm{ord}(d') + L(\sigma') \equiv 1 \pmod 2$, *then* $f_\gamma^{\mathrm{I}}(g') = 0$ *identically. In general, for* $g' = (d', \sigma') \in \widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P_\gamma)$,

$$f_\gamma^{\mathrm{I}}(g') = \zeta_\gamma(d') \cdot F_{0,n}^{\mathrm{I}}(g').$$

(ii) CASE $n = 2n'+1$ ODD: *If* $\mathrm{ord}(d') + L(\sigma') \equiv 0 \pmod 2$, *then* $f_\gamma^{\mathrm{I}-}(g') = f_\gamma^{\mathrm{I}+}(g')$. *If* $\mathrm{ord}(d') + L(\sigma') \equiv 1 \pmod 2$, *then* $f_\gamma^{\mathrm{I}-}(g') = -f_\gamma^{\mathrm{I}+}(g')$, *and* $f_\gamma^{\mathrm{I}+}(g') \neq 0$ *if and only if (Condition I-11) in Lemma 16.2 holds for* g'. *In general, for* $g' = (d', \sigma') \in \widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P_\gamma^+)$,

$$f_\gamma^{\mathrm{I}+}(g') = \zeta_\gamma(d') \cdot F_{+,n}^{\mathrm{I}}(g').$$

Using Propositions 18.4 and 18.6 together with Lemma 19.2, we obtain the following summation formulas for spin irreducible characters $F_{\Lambda^n}^{\mathrm{I}}$, $F_{\Lambda^n}^{\mathrm{I}+}$ and $F_{\Lambda^n}^{\mathrm{I}-}$.

Proposition 19.5. *Let* $\gamma \in \Gamma_n^0$ *and* $\Lambda^n = (\lambda^{n,\zeta})_{\zeta \in \widehat{T}^0}$ *be as in (19.2) – (19.4). Take* $g' = (d', \sigma') \in \widetilde{G}_n^{\mathrm{I}}$ *and let*

$$g = \Phi(g') = (d, \sigma) \in G_n = D_n \rtimes \mathfrak{S}_n = G(m, 1, n),$$

with $d = \Phi(d') \in D_n$, $\sigma = \Phi(\sigma') \in \mathfrak{S}_n$. *For* $s' \in \widetilde{\mathfrak{S}}_n$, *put* $s = \Phi(s')$, *then* $\Phi(s'g's'^{-1}) = sgs^{-1} = \big(s(d), s\sigma s^{-1}\big)$, *and* $\zeta_{s^{-1}\gamma}(d') = \zeta_{s^{-1}\gamma}(d) = \zeta_\gamma\big(s(d)\big)$.

(i) CASE $n = 2n'$ EVEN: *If* $\mathrm{ord}(d') + L(\sigma') \equiv 0 \pmod 2$, *then* $F_{\Lambda^n}^{\mathrm{I}}(g') = \mathrm{tr}\big(\Pi_{\Lambda^n}^{\mathrm{I}}(g')\big) \neq 0$ *only if (Condition I-00) in Lemma 16.2 holds for* g'. *If* $\mathrm{ord}(d') + L(\sigma') \equiv 1 \pmod 2$, *then* $F_{\Lambda^n}^{\mathrm{I}}(g') = 0$ *identically. In general, for* $g' = (d', \sigma') \in \widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P_\gamma)$, *with* $S_\gamma := \Phi\big(\mathcal{S}(P_\gamma)\big) = \{s \in \mathfrak{S}_n \; ; \; s\gamma = \gamma\}$,

$$F_{\Lambda^n}^{\mathrm{I}}(g') = F_{0,n}^{\mathrm{I}}(g') \cdot \sum_{\substack{s \in S_\gamma \backslash \mathfrak{S}_n : \\ s\sigma s^{-1} \in S_\gamma}} \zeta_\gamma\big(s(d)\big) \, f_{\Lambda^n}(s\sigma s^{-1}).$$

(ii) CASE $n = 2n'+1$ ODD: *If* $\mathrm{ord}(d') + L(\sigma') \equiv 0 \pmod 2$, *then* $F_{\Lambda^n}^{\mathrm{I}-}(g') = F_{\Lambda^n}^{\mathrm{I}+}(g') = F_{\Lambda^n}^{\mathrm{I}}(g')$. *If* $\mathrm{ord}(d') + L(\sigma') \equiv 1 \pmod 2$, *then* $F_{\Lambda^n}^{\mathrm{I}-}(g') = -F_{\Lambda^n}^{\mathrm{I}+}(g')$, *and* $F_{\Lambda^n}^{\mathrm{I}+}(g') \neq 0$ *only if (Condition I-11) in Lemma 16.2 holds for* g'. *In general, for* $g' = (d', \sigma') \in \widetilde{D}_n \overset{\mathrm{I}}{\rtimes} \mathcal{S}(P_\gamma^+)$,

$$F_{\Lambda^n}^{\mathrm{I}+}(g') = F_{+,n}^{\mathrm{I}}(g') \cdot \sum_{s:\text{ as above}} \zeta_\gamma\big(s(d)\big) \, f_{\Lambda^n}(s\sigma s^{-1}).$$

19.3 Relations to non-spin irreducible characters of $G(m,1,n)$

The group $\widetilde{G}^{\mathrm{I}}_n$ is a quadruple covering group of the base group $G_n = D_n \rtimes \mathfrak{S}_n = G(m,1,n)$, and the sum

$$\sum_{s\in S_\gamma\backslash\mathfrak{S}_n\,:\,s\sigma s^{-1}\in S_\gamma} \zeta_\gamma\big(s(d)\big)\, f_{\Lambda^n}(s\sigma s^{-1}) \tag{19.9}$$

is the value of a (non-spin) irreducible character of G_n at the point $g = \Phi(g')$.

Let us explain this a little more in detail. Take $\gamma \in \Gamma_n$ and a character $\zeta_\gamma = (\zeta_1, \zeta_2, \ldots, \zeta_n)$ of D_n, and define a partition of $\boldsymbol{I}_n$ as

$$\mathcal{I}_n := \big(I_{n,\zeta}\big)_{\zeta\in\widehat{T}}\,, \quad \boldsymbol{I}_n = \bigsqcup_{\zeta\in\widehat{T}} I_{n,\zeta}\,, \quad I_{n,\zeta} = \{j \in \boldsymbol{I}_n\ ;\ \zeta_j = \zeta\}. \tag{19.10}$$

Then the stationary subgroup S_γ in $\mathfrak{S}_n$ is given as $S_\gamma = \prod_{\zeta\in\widehat{T}} \mathfrak{S}_{I_{n,\zeta}}$. Take an IR $\pi_{\Lambda^n} := \boxtimes_{\zeta\in\widehat{T}}\, \pi(\lambda^{n,\zeta})$ of S_γ with $\Lambda^n = (\lambda^{n,\zeta})_{\zeta\in\widehat{T}}$, $|\lambda^{n,\zeta}| = |I_{n,\zeta}|$. Then we have an IR $\breve{\pi}_{\gamma,\Lambda^n}(g) = \zeta_\gamma(d)\, \pi_{\Lambda^n}(\sigma)$ of $H_n := D_n \rtimes S_\gamma$, and inducing it up, an IR of $G_n = D_n \rtimes \mathfrak{S}_n$ as

$$\breve{\Pi}_{\gamma,\Lambda^n} := \mathrm{Ind}_{H_n}^{G_n} \breve{\pi}_{\gamma,\Lambda^n}\,. \tag{19.11}$$

Lemma 19.6. *The characters $\breve{f}_{\gamma,\Lambda^n}$ of $\breve{\pi}_{\gamma,\Lambda^n}$ and $\breve{F}_{\gamma,\Lambda^n}$ of $\breve{\Pi}_{\gamma,\Lambda^n}$ are respectively given by*

$$\breve{f}_{\gamma,\Lambda^n}(h) = \zeta_\gamma(d)\, f_{\Lambda^n}(\sigma) \qquad \big(h = (d,\sigma) \in D_n \rtimes S_\gamma\big)\ ;$$

$$\breve{F}_{\gamma,\Lambda^n}(g) = \sum_{\substack{s\in S_\gamma\backslash\mathfrak{S}_n\,:\\ s\sigma s^{-1}\in S_\gamma}} \breve{f}_{\gamma,\Lambda^n}(sgs^{-1}) \quad \big(g = (d,\sigma) \in D_n \rtimes \mathfrak{S}_n\big).$$

Note that the above sum is nothing but the sum in (19.9). On the other hand, the irreducible character $\breve{F}_{\gamma,\Lambda^n}$ is explicitly expressed as follows.

Denote by $\chi(\lambda^{n,\zeta};\sigma)$ the character value of $\pi(\lambda^{n,\zeta})$ at $\sigma \in \mathfrak{S}_{I_{n,\zeta}}$. If $\sigma = \sigma_1\sigma_2\cdots\sigma_t$ is a cycle decomposition of σ, then the value $\chi(\lambda^{n,\zeta};\sigma)$ is determined by the set $\{\ell_p = \ell(\sigma_p); 1 \le p \le t\}$ of lengths, and so it is also denoted by $\chi(\lambda^{n,\zeta};(\ell_p)_{1\le p\le t})$, that is,

$$\chi\big(\lambda^{n,\zeta};(\ell_p)_{1\le p\le t}\big) := \chi(\lambda^{n,\zeta};\sigma) \quad \text{if } \sigma = \sigma_1\sigma_2\cdots\sigma_t,\ \ell_p = \ell(\sigma_p). \tag{19.12}$$

Recall that the redundancy $\ell_j = 1$ $(s < j \le t)$ is admitted, and $g = (d,\sigma) \in G_n$ is expressed by its decomposition into basic elements as

$$\begin{cases} g = (d,\sigma) = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r} g_1 g_2 \cdots g_s, \quad \xi_q = {y_q}^{a_q}\ (q \in Q), \\ g_j = (d_j,\sigma_j),\ \mathrm{supp}(d_j) \subset \mathrm{supp}(\sigma_j)\ \ (j \in J), \end{cases} \tag{19.13}$$

with $Q = \{q_1, q_2, \ldots, q_r\}$ and $J = \{1, 2, \ldots, s\}$. Consider partitions of Q and J as

$$\mathcal{Q} = (Q_\zeta)_{\zeta\in\widehat{T}},\ Q = \bigsqcup\nolimits_{\zeta\in\widehat{T}} Q_\zeta\,, \ \text{ and } \ \mathcal{J} = (J_\zeta)_{\zeta\in\widehat{T}},\ J = \bigsqcup\nolimits_{\zeta\in\widehat{T}} J_\zeta\,. \tag{19.14}$$

For a pair of partitions $(\mathcal{Q},\mathcal{J})$ satisfying the condition

(Condition QJ) $\qquad |Q_\zeta| + \sum_{j\in J_\zeta} \ell_j \;\le\; |I_{n,\zeta}| \qquad (\zeta \in \widehat{T}),$

we define a function of g given as

$$(19.15)\qquad X\big(\Lambda^n\,;\, \mathcal{Q},\, \mathcal{J}\,;\, g\big) := \prod_{\zeta\in\widehat{T}} \Big(\zeta\big(\textstyle\prod_{q\in Q_\zeta}\xi_q\cdot\prod_{j\in J_\zeta} d_j\big)\times\chi\big(\lambda^{n,\zeta};(\ell_j)_{j\in J_\zeta}\big)\Big).$$

Note that, if g is conjugate under $\mathfrak{S}_n$ to an element of $D_n \rtimes S_\gamma$, then there exists at least one $(\mathcal{Q},\, \mathcal{J})$ which satisfies (Condition QJ). We have

$$(19.16)\qquad n - |\mathrm{supp}(g)| = \sum_{\zeta\in\widehat{T}} \Big(|I_{n,\zeta}| - |Q_\zeta| - \sum_{j\in J_\zeta} \ell_j\Big).$$

Define an important number $N(\mathcal{I}_n;\mathcal{Q},\mathcal{J};g)$ as

$$(19.17)\qquad N(\mathcal{I}_n\,;\, \mathcal{Q},\, \mathcal{J}\,;\, g) \;:=\; (n-|\mathrm{supp}(g)|)!\,\times$$
$$\times\prod_{\zeta\in\widehat{T}} |I_{n,\zeta}|\big(|I_{n,\zeta}|-1\big)\cdots\Big(|I_{n,\zeta}| - |Q_\zeta| - \sum_{j\in J_\zeta}\ell_j + 1\Big).$$

Then $\sum_{(\mathcal{Q},\mathcal{J})} N(\mathcal{I}_n;\mathcal{Q},\mathcal{J};g) = n!$, where $(\mathcal{Q},\mathcal{J})$ runs over those satisfying (Condition QJ). Furthermore we put

$$(19.18)\qquad b(\mathcal{I}_n;\mathcal{Q},\mathcal{J};g) := \frac{N(\mathcal{I}_n;\mathcal{Q},\mathcal{J};g)}{|S_\gamma|} = \frac{(n-|\mathrm{supp}(g)|)!}{\prod_{\zeta\in\widetilde{T}}\Big(|I_{n,\zeta}| - |Q_\zeta| - \sum_{j\in J_\zeta}\ell_j\Big)!}.$$

Here, even for a pair $(\mathcal{Q},\mathcal{J})$ which does not satisfy (Condition QJ), the above formulas have meaning if we understand as $N(\mathcal{I}_n;\mathcal{Q},\mathcal{J};g) = 0$.

By a simpler discussions than those in [HHH1, §§4.3 – 4.4], we obtain the character of an IR of the wreath product $G_n = \mathfrak{S}_n(T) = D_n \rtimes \mathfrak{S}_n$, $D_n = D_n(T)$, of the cyclic group $T = \boldsymbol{Z}_m$ with the n-th symmetric group $\mathfrak{S}_n$ as follows.

Theorem 19.7 (Non-spin Case)**.**

(i) *Let* $\Lambda^n = (\lambda^{n,\zeta})_{\zeta\in\widehat{T}} \in \boldsymbol{Y}_n(T)$ *be a set of Young diagrams such that* $\lambda^{n,\zeta}$ *determines an IR* $\pi(\lambda^{n,\zeta})$ *of* $\mathfrak{S}_{I_{n,\zeta}} \cong \mathfrak{S}_{|I_{n,\zeta}|}$, $|I_{n,\zeta}| = |\lambda^{n,\zeta}|$. *Put* $\breve{\pi}_{\gamma,\Lambda^n} = \zeta_\gamma \boxdot \pi_{\Lambda^n}$. *Then* $\breve{\pi}_{\gamma,\Lambda^n}$ *is an IR of* $D_n \rtimes S_\gamma$ *with* $S_\gamma = \prod_{\zeta\in\widehat{T}} \mathfrak{S}_{I_{n,\zeta}}$, *and the induced representation* $\breve{\Pi}_{\gamma,\Lambda^n}$ *is irreducible. Every IR of* G_n *is equivalent to an induced representation of this type.*

(ii) *Take a* $g = (d,\sigma) \in D_n \rtimes S_\gamma \subset G_n$, *and let its standard decomposition be as in (19.13), and put* $Q = \{q_1, q_2, \ldots, q_r\}$, $J = \{1, 2, \ldots, s\}$. *Then the character value* $\breve{F}_{\gamma,\Lambda^n}(g)$ *of the IR* $\breve{\Pi}_{\gamma,\Lambda^n}$ *of* G_n *is given by*

$$(19.19)\qquad \breve{F}_{\gamma,\Lambda^n}(g) = \sum_{(\mathcal{Q},\mathcal{J})} b(\mathcal{I}_n;\mathcal{Q},\mathcal{J};g)\, X\big(\Lambda^n\,;\, \mathcal{Q},\, \mathcal{J}\,;\, g\big),$$

where the summation runs over all pairs of partitions $(\mathcal{Q}, \mathcal{J})$ *for which (Condition QJ) holds, and* $b(\mathcal{I}_n; \mathcal{Q}, \mathcal{J}; g)$ *is given in (19.18), and* $X\big(\Lambda^n\,;\, \mathcal{Q},\, \mathcal{J}\,;\, g\big)$ *in (19.15).*

(iii) *For a* $g = (d, \sigma) \in G_n$ *which is not conjugate to any element in* $D_n \rtimes S_\gamma$*, the character vanishes. The above character formula (19.19) is also valid in this case, in the sense that there is no pair* $(\mathcal{Q}, \mathcal{J})$ *satisfying (Condition QJ).*

Lemma 19.8. *For* $g = (d, \sigma) \in G_n = G(m, 1, n)$*,*

$$\begin{aligned} \breve{F}_{\gamma, {}^t\Lambda^n}(g) &= \mathrm{sgn}(\sigma) \cdot \breve{F}_{\gamma, \Lambda^n}(g)\ ; \\ \breve{F}_{(\tau_1\tau_2\cdots\tau_n)\gamma, \Lambda^n}(g) &= (-1)^{\mathrm{ord}(d)} \cdot \breve{F}_{\gamma, \Lambda^n}(g). \end{aligned}$$

Remark 19.2. One dimensional characters of $G_n = D_n \rtimes \mathfrak{S}_n$ are given as follows: for (k, ε), $0 \le k < m$, $\varepsilon = 0,\, 1$, and for $g = (d, \sigma) \in G_n$, $d = (t_i)_{i \in \boldsymbol{I}_n}$,

$$\chi_{k,\varepsilon}(g) := \zeta_k\big(P(d)\big) \cdot \mathrm{sgn}(\sigma)^\varepsilon = \omega^{k \cdot \mathrm{ord}(d)} \cdot \mathrm{sgn}(\sigma)^\varepsilon,$$

where $P(d) := t_n t_{n-1} \cdots t_1$ and $\omega = e^{2\pi i/m}$.

19.4 Explicit formula for spin irreducible characters of $\widetilde{G}^{\mathrm{I}}_n$

Summarizing the above results we obtain the following formulas for spin irreducible characters $F^{\mathrm{I}}_{\Lambda^n}$, $F^{\mathrm{I}+}_{\Lambda^n}$ and $F^{\mathrm{I}-}_{\Lambda^n}$.

Theorem 19.9. *Let* $\gamma \in \Gamma^0_n$ *and take* $\Lambda^n = \big(\lambda^{n,\zeta}\big)_{\zeta \in \widehat{T}^0} \in \boldsymbol{Y}_n(T)^0$ *correspondingly. Take* $g' = (d', \sigma') \in \widetilde{G}^{\mathrm{I}}_n$ *expressed as* $g' = \xi'_{q_1} \cdots \xi'_{q_r}\, g'_1 g'_2 \cdots g'_s$ *with* $\xi'_q = {\eta'_q}^{a_q}$ $(q \in Q = \{q_1, \ldots, q_r\})$ *and* $g'_j = (d'_j, \sigma'_j)$ $(j \in J = \boldsymbol{I}_s)$*. Let* $g = \Phi(g') = (d, \sigma) \in G_n = D_n \rtimes \mathfrak{S}_n = G(m, 1, n)$*.*

(i) Case $n = 2n'$ even: *If* $\mathrm{ord}(d') + L(\sigma') \equiv 0 \pmod 2$*, then* $F^{\mathrm{I}}_{\Lambda^n}(g') = \mathrm{tr}\big(\Pi^{\mathrm{I}}_{\Lambda^n}(g')\big) \ne 0$ *only if (Condition I-00) in Lemma 16.2 holds for* g'*. If* $\mathrm{ord}(d') + L(\sigma') \equiv 1 \pmod 2$*, then* $F^{\mathrm{I}}_{\Lambda^n}(g') = 0$ *identically. In general,*

$$F^{\mathrm{I}}_{\Lambda^n}(g') = F^{\mathrm{I}}_{0,n}(g') \times \breve{F}_{\gamma, \Lambda^n}(g),$$

where $\breve{F}_{\gamma, \Lambda^n}(g)$ *is given in (19.19).*

(ii) Case $n = 2n'+1$ odd: *If* $\mathrm{ord}(d') + L(\sigma') \equiv 0 \pmod 2$*, then* $F^{\mathrm{I}+}_{\Lambda^n}(g') \ne 0$ *only if (Condition I-00) in Lemma 16.2 holds for* g'*. In that case,* $F^{\mathrm{I}-}_{\Lambda^n}(g') = F^{\mathrm{I}+}_{\Lambda^n}(g') = F^{\mathrm{I}}_{\Lambda^n}(g')$*. If* $\mathrm{ord}(d') + L(\sigma') \equiv 1 \pmod 2$*, then* $F^{\mathrm{I}-}_{\Lambda^n}(g') = -\,F^{\mathrm{I}+}_{\Lambda^n}(g')$*, and* $F^{\mathrm{I}+}_{\Lambda^n}(g') \ne 0$ *only if (Condition I-11) in Lemma 16.2 holds for* g'*. In general,*

$$F^{\mathrm{I}+}_{\Lambda^n}(g') = F^{\mathrm{I}}_{+,n}(g') \times \breve{F}_{\gamma, \Lambda^n}(g).$$

(iii) *With* $\mathrm{sgn}(\sigma') = \mathrm{sgn}(\sigma)$*,*

$$F^{\rm I}_{{}^t\Lambda^n}(g') = \mathrm{sgn}(\sigma')\, F^{\rm I}_{\Lambda^n}(g'), \quad F^{\rm I\pm}_{{}^t\Lambda^n}(g') = \mathrm{sgn}(\sigma')\, F^{\rm I\pm}_{\Lambda^n}(g').$$

Corollary 19.10. *Spin IRs in CASE I, Type* $(-1,-1,-1)$*, are expressed as tensor products of special spin IRs with non-spin IRs as follows:*

$$\Pi^{\rm I}_{\Lambda^n} \cong \Pi^{\rm I}_0 \otimes \breve{\Pi}_{\gamma,\Lambda^n} \quad \text{for } \Lambda^n \in \boldsymbol{Y}_n(T)^0,$$
$$\Pi^{\rm I+}_{\Lambda^n} \cong \Pi^{\rm I}_+ \otimes \breve{\Pi}_{\gamma,\Lambda^n} \quad \text{for } \Lambda^n \in \boldsymbol{Y}_n(T)^0.$$

20 Characters of spin IRs of $\widetilde{G}_n^{\rm II}$ (CASE II)

20.1 Formulas for calculating spin IRs of $\widetilde{G}_n^{\rm II}$

For $\gamma \in \Gamma_n^0$, let $\zeta_\gamma = (\zeta_j)_{j\in \boldsymbol{I}_n}$ be as before, and denote by $\mathcal{I}_n$ the partition $\boldsymbol{I}_n = \bigsqcup_{\zeta\in\widehat{T}^0} I_{n,\zeta}$, $I_{n,\zeta} = \{j \in \boldsymbol{I}_n\,;\, \zeta_j = \zeta\}$ in Definition 11.2 or (19.1). Here we assume that γ is normalized so that every $I_{n,\zeta}$ is an interval in $\boldsymbol{I}_n$.

As seen in Theorem 12.6, equivalence classes of IRs of $\widetilde{G}_n^{\rm II} = \widetilde{G}^{\rm II}(m,1,n)$ in CASE II, Type $(-1,-1,\ 1)$, are realized as induced representations as follows. Recall that, in CASE II, the stationary subgroup $\mathcal{S}$ is $\mathcal{S}(P_\gamma) = \Phi_{\mathfrak{S}}^{-1}\big(\prod_{\zeta\in\widehat{T}^0} \mathfrak{S}_{I_{n,\zeta}}\big)$ or $\mathcal{S}(P_\gamma^\pm) = \widetilde{\mathfrak{A}}\big(\prod_{\zeta\in\widehat{T}^0} \mathfrak{S}_{I_{n,\zeta}}\big)$ according as n is even or odd.

Case $n = 2n'$ even : For $\Lambda^n = \big(\lambda^{n,\zeta}\big)_{\zeta\in\widehat{T}^0} \in \boldsymbol{Y}_n(T)^0$,

$$\text{(20.1)}\qquad \begin{aligned} &\pi^{\rm II}_{\gamma,\Lambda^n} = \pi^{\rm II}_\gamma \boxdot \pi_{\Lambda^n}\,, && \text{IR of } \widetilde{D}_n \overset{\rm II}{\rtimes} \mathcal{S}(P_\gamma)\,,\\ &\Pi^{\rm II}_{\Lambda^n} = \mathrm{Ind}^{\widetilde{G}_n^{\rm II}}_{\widetilde{D}_n \overset{\rm II}{\rtimes} \mathcal{S}(P_\gamma)} \pi^{\rm II}_{\gamma,\Lambda^n}\,, && \text{IR of } \widetilde{G}_n^{\rm II}\,. \end{aligned}$$

Put $f_{\Lambda^n}(\sigma') := \mathrm{tr}\big(\pi_{\Lambda^n}(\sigma')\big)$, then $f_{\Lambda^n}(\sigma') = f_{\Lambda^n}(\sigma) := \mathrm{tr}\big(\pi_{\Lambda^n}(\sigma)\big), \sigma = \Phi(\sigma')$, and

$$\text{(20.2)}\qquad \begin{aligned} &f^{\rm II}_{\gamma,\Lambda^n} := \mathrm{tr}\big(\pi^{\rm II}_{\gamma,\Lambda^n}\big), \quad F^{\rm II}_{\Lambda^n} := \mathrm{tr}\big(\Pi^{\rm II}_{\Lambda^n}\big)\ ;\\ &f^{\rm II}_{\gamma,\Lambda^n}\big((d',\sigma')\big) = f^{\rm II}_\gamma\big((d',\sigma')\big)\cdot f_{\Lambda^n}(\sigma),\\ &f^{\rm II}_\gamma\big((d',\sigma')\big) := \mathrm{tr}\big(\pi^{\rm II}_\gamma\big((d',\sigma')\big)\big). \end{aligned}$$

Case $n = 2n'+1$ odd : For $\{\Lambda^n, {}^t\Lambda^n\} \in \boldsymbol{Y}_n^{\mathfrak{A}}(T)^{0,1}$ and $(\Lambda^n,\kappa) \in \boldsymbol{Y}_n^{\mathfrak{A}}(T)^{0,2}$ respectively, we have, with two spin representations $\mho_n^+$, $\mho_n^-$ of $\widetilde{\mathfrak{A}}_n$ in (17.35),

$$\text{(20.3)}\qquad \left\{\begin{aligned} &\pi^{\mho\pm}_{\gamma,\Lambda^n} = \pi^{\mho\pm}_\gamma \boxdot \rho_{\Lambda^n} \text{ with } \pi^{\mho\pm}_\gamma = P^\pm_\gamma \cdot \mho^\pm_n\,,\\ &\pi^{\mho\pm}_{\gamma,\Lambda^n,\kappa} = \pi^{\mho\pm}_\gamma \boxdot \rho^{(\kappa)}_{\Lambda^n}\,, \end{aligned}\right. \quad \text{IR of } \widetilde{D}_n \overset{\rm II}{\rtimes} \mathcal{S}\big(P^\pm_\gamma\big)\ ;$$

$$\text{(20.4)}\qquad \left\{\begin{aligned} &\Pi^{\mho\pm}_{\Lambda^n} = \mathrm{Ind}^{\widetilde{G}_n^{\rm II}}_{\widetilde{D}_n \overset{\rm II}{\rtimes} \mathcal{S}(P^\pm_\gamma)} \pi^{\mho\pm}_{\gamma,\Lambda^n}\,,\\ &\Pi^{\mho\pm}_{\Lambda^n,\kappa} = \mathrm{Ind}^{\widetilde{G}_n^{\rm II}}_{\widetilde{D}_n \overset{\rm II}{\rtimes} \mathcal{S}(P^\pm_\gamma)} \pi^{\mho\pm}_{\gamma,\Lambda^n,\kappa}\,, \end{aligned}\right. \quad \text{IR of } \widetilde{G}_n^{\rm II}\ ;$$

$$(20.5)\quad \begin{cases} f^{\mho\pm}_{\gamma,\Lambda^n} := \mathrm{tr}\big(\pi^{\mho\pm}_{\gamma,\Lambda^n}\big), & F^{\mho\pm}_{\Lambda^n} := \mathrm{tr}\big(\Pi^{\mho\pm}_{\Lambda^n}\big), \\ f^{\mho\pm}_{\gamma,\Lambda^n,\kappa} := \mathrm{tr}\big(\pi^{\mho\pm}_{\gamma,\Lambda^n,\kappa}\big), & F^{\mho\pm}_{\Lambda^n,\kappa} := \mathrm{tr}\big(\Pi^{\mho\pm}_{\Lambda^n,\kappa}\big)\ ; \end{cases}$$

$$(20.6)\quad \begin{cases} f^{\mho\pm}_{\gamma,\Lambda^n}\big((d',\sigma')\big) = f^{\mho\pm}_{\gamma}\big((d',\sigma')\big)\cdot f^{\rho}_{\Lambda^n}(\sigma), \\ \qquad f^{\mho\pm}_{\gamma} := \mathrm{tr}\big(\pi^{\mho\pm}_{\gamma}\big), \quad f^{\rho}_{\Lambda^n}(\sigma) := \mathrm{tr}\big(\rho_{\Lambda^n}(\sigma)\big), \\ f^{\mho\pm}_{\gamma,\Lambda^n,\kappa}\big((d',\sigma')\big) = f^{\mho\pm}_{\gamma}\big((d',\sigma')\big)\cdot f^{\rho}_{\Lambda^n,\kappa}(\sigma), \\ \qquad\qquad f^{\rho}_{\Lambda^n,\kappa}(\sigma) := \mathrm{tr}\big(\rho^{(\kappa)}_{\Lambda^n}(\sigma)\big). \end{cases}$$

Lemma 20.1. *Let $\gamma \in \Gamma^0_n$, and take $g' = (d',\sigma') \in \widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \mathcal{S}$.*
(i) *In case $n = 2n'$ even, $\mathcal{S} = \mathcal{S}(P_\gamma) = \Phi^{-1}_{\mathfrak{S}}\big(\prod_{\zeta\in\widehat{T}^0}\mathfrak{S}_{I_{n,\zeta}}\big)$ and with $s = \Phi(s')$,*

$$f^{\mathrm{II}}_{\Lambda^n}(g') \;=\; \sum_{\substack{s'\in\mathcal{S}\backslash\widetilde{\mathfrak{S}}_n:\\ s'\sigma's'^{-1}\in\mathcal{S}}} f^{\mathrm{II}}_{\gamma}\big((s'^{\,\mathrm{II}}(d'), s'\sigma's'^{-1})\big)\cdot f_{\Lambda^n}(s\sigma s^{-1}).$$

(ii) *In case $n = 2n'+1$ odd, $\mathcal{S} = \mathcal{S}(P^{\pm}_\gamma) = \widetilde{\mathfrak{A}}\big(\prod_{\zeta\in\widehat{T}^0}\mathfrak{S}_{I_{n,\zeta}}\big)$ and*

$$F^{\mho\pm}_{\Lambda^n}(g') = \sum_{s'\,:\ \text{as above}} f^{\mho\pm}_{\gamma}\big((s'^{\,\mathrm{II}}(d'), s'\sigma's'^{-1})\big)\cdot f^{\rho}_{\Lambda^n}(s\sigma s^{-1})\ ;$$

$$F^{\mho\pm}_{\Lambda^n,\kappa}(g') = \sum_{s'\,:\ \text{as above}} f^{\mho\pm}_{\gamma}\big((s'^{\,\mathrm{II}}(d'), s'\sigma's'^{-1})\big)\cdot f^{\rho}_{\Lambda^n,\kappa}(s\sigma s^{-1}).$$

On the other hand, special spin IRs are given in Example 12.2:

$\Pi^{\mathrm{II}}_0 = P_0\cdot\nabla^{\mathrm{II}}_n$ with $P_0 = P_{\gamma^{(0)}}$ for $\widetilde{G}^{\mathrm{II}}_n$, $n \geq 4$ even,

$\Pi^{\mathrm{II},\widetilde{H}}_+ = P_+\cdot\mho^+_n$ with $P_+ = P^+_{\gamma^{(0)}}$ for $\widetilde{H}^{\mathrm{II}}_n = \widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \widetilde{\mathfrak{A}}_n$, $n \geq 5$ odd,

$\Pi^{\mathrm{II}}_+ = \mathrm{Ind}^{\widetilde{G}^{\mathrm{II}}_n}_{\widetilde{H}^{\mathrm{II}}_n}\Pi^{\mathrm{II},\widetilde{H}}_+$ for $\widetilde{G}^{\mathrm{II}}_n$.

Their characters are calculated in §17 as special cases, and we denote them by $F^{\mathrm{II}}_{0,n}$, $F^{\mathrm{II},\widetilde{H}}_{+,n}$ and $F^{\mathrm{II}}_{+,n}$ respectively. Their intimate relations with the general case of γ facilitate to understand the covariance in §18.3 above, and also the calculations below.

Take a $g' = (d',\sigma')$ from $\widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \mathcal{S}(P_\gamma)$ for $n = 2n'$ even, or from $\widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \mathcal{S}(P^{\pm}_\gamma)$ for $n = 2n'+1$ odd respectively, expressed as $g' = \xi'_{q_1}\cdots\xi'_{q_r}\,g'_1g'_2\cdots g'_s$ and normalized as in (17.14), or, $\xi'_q = \eta_q^{\,a_q}$ ($q \in Q = \{q_1,\ldots,q_r\}$) and $g'_j = (d'_j,\sigma'_j)$ ($j \in J = \boldsymbol{I}_n$) with $\ell_j = \ell(\sigma'_j)$ and

$$(20.7)\quad \begin{cases} d'_j = \eta_{k_j}^{\;b_j}\ \ \big(\exists k_j \in K_j = [n_j, n_j+\ell_j-1]\big), \\ \sigma'_j = r_{n_j}r_{n_j+1}\cdots r_{n_j+\ell_j-2}\,,\ \sigma = \Phi(\sigma') = (n_j\ \ n_j{+}1\ \ldots\ n_j{+}\ell_j{-}1). \end{cases}$$

Theorem 20.2. (i) Case $n = 2n'$ even: *If $L(\sigma') \equiv 0 \pmod 2$, then $F^{\mathrm{II}}_{0,n}(g') = \mathrm{tr}\big(\Pi^{\mathrm{II}}_0(g')\big) \neq 0$ if and only if (Condition II-00) in §17.2.1 holds for*

g'. *In that case,*

$$F_{0,n}^{\rm II}(g') = 2^{n'} \cdot \varepsilon\big(\textstyle\prod_{j\in J_-} g'_j\big) \cdot \displaystyle\prod_{j\in J}(-1)^{(\ell_j-1)/2}\, 2^{-(\ell_j-1)/2},$$

where $\varepsilon\big(\prod_{j\in J_-} g'_j\big) := \prod_{\{j_1,j_2\}\sqsubset J_-} \varepsilon(g'_{j_1} g'_{j_2})$, *with the sign* $\varepsilon(g'_{j_1} g'_{j_2}) = \pm 1$ *defined by (17.22)–(17.24), and also by (17.25), and for* $j \in J_-$, ℓ_j *are even and we use the rule* $(-1)^{1/2}(-1)^{1/2} = -1$.

If $L(\sigma') \equiv 1 \pmod 2$, *then* $F_{0,n}^{\rm II}(g') \neq 0$ *if and only if (Condition II-11) in §17.3 holds for* g'. *In that case,*

$$\begin{aligned} F_{0,n}^{\rm II}(g') &= \varepsilon^{\rm II}(g') \cdot 2^{n'} i^{n'-1} \cdot \prod_{j\in J}(-1)^{\ell_j-1}\, 2^{-(\ell_j-1)/2} \\ &= -\varepsilon^{\rm II}(g') \cdot 2^{n'} i^{n'-1} \cdot \prod_{j\in J} 2^{-(\ell_j-1)/2}, \end{aligned}$$

where $\varepsilon^{\rm II}(g') = \pm 1$ *is given by (17.34).*

(ii) CASE $n = 2n'+1$ ODD: *Let* $g' = (g', \sigma') \in \widetilde{H}_n^{\rm II} = \widetilde{D}_n \overset{\rm II}{\rtimes} \widetilde{\mathfrak{A}}_n \subsetneq \widetilde{G}_n^{\rm II}$.

If $\mathrm{ord}(d') \equiv 0 \pmod 2$, *then* $F_{+,n}^{{\rm II},\widetilde{H}}(g') \neq 0$ *if and only if (Condition II-00) in §17.5 holds for* g'. *In that case,*

$$F_{+,n}^{{\rm II},\widetilde{H}}(g') = 2^{n'} \cdot \varepsilon\big(\textstyle\prod_{j\in J_-} g'_j\big) \cdot \displaystyle\prod_{j\in J}(-1)^{(\ell_j-1)/2}\, 2^{-(\ell_j-1)/2}. \tag{20.8}$$

If $\mathrm{ord}(d') \equiv 1 \pmod 2$, *then* $F_{+,n}^{{\rm II},\widetilde{H}}(g') \neq 0$ *if and only if (Condition* $\mho$*-11) in §17.5 holds for* g'. *In that case,*

$$F_{+,n}^{{\rm II},\widetilde{H}}(g') = \varepsilon^{\mho}(g') \cdot (-1)^{|J_-|/2} \cdot (2i)^{n'} \cdot \prod_{j\in J} 2^{-(\ell_j-1)/2}, \tag{20.9}$$

where the sign $\varepsilon^{\mho}(g') = \pm 1$ *is determined by (17.43).*

Let $\tau' \in \widetilde{\mathfrak{S}}_n \setminus \widetilde{\mathfrak{A}}_n$. *If* $\mathrm{ord}(d') \equiv 0 \pmod 2$, *then* $F_{+,n}^{{\rm II},\widetilde{H}}(\tau' g' \tau'^{-1}) = F_{+,n}^{{\rm II},\widetilde{H}}(g')$. *If* $\mathrm{ord}(d') \equiv 1 \pmod 2$, *then* $F_{+,n}^{{\rm II},\widetilde{H}}(\tau' g' \tau'^{-1}) = -F_{+,n}^{{\rm II},\widetilde{H}}(g')$.

(iii) CASE $n = 2n'+1$ ODD: *Let* $g' = (d', \sigma') \in \widetilde{H}_n^{\rm II}$ *and* $\tau' \in \widetilde{\mathfrak{S}}_n \setminus \widetilde{\mathfrak{A}}_n$, *then*

$$F_{+,n}^{\rm II}(g') = \begin{cases} 2\,F_{+,n}^{{\rm II},\widetilde{H}}(g') & \text{if } \mathrm{ord}(d') \equiv 0 \pmod 2, \\ 0 & \text{if } \mathrm{ord}(d') \equiv 1 \pmod 2, \end{cases} \tag{20.10}$$

and $F_{+,n}^{\rm II}(\tau' g') = 0$ *identically on* $\tau' \widetilde{H}_n^{\rm II}$.

From Theorems 17.7 and 17.9, we get the following. In particular, for the assertion (ii) below, we should note that $\zeta_{(\tau_1\tau_2\cdots\tau_n)\gamma}(d') = (-1)^{\mathrm{ord}(d')}\,\zeta_\gamma(d')$.

Lemma 20.3. *Let* $\gamma \in \Gamma_n^0$ *and* $g' = (d', \sigma') \in \widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \mathcal{S}$ *be as above.*

(i) Case $n = 2n'$ even: $\mathcal{S} = \mathcal{S}(P_\gamma) = \Phi^{-1}\big(\prod_{\zeta\in\widehat{T}^0} \mathfrak{S}_{I_{n,\zeta}}\big)$.

If $L(\sigma') \equiv 0 \pmod 2$, *then* $f_\gamma^{\mathrm{II}}(g') = \mathrm{tr}\big(\pi_\gamma^{\mathrm{II}}(g')\big) \neq 0$ *if and only if (Condition II-00) in §17.2.1 holds for* g'. *If* $L(\sigma') \equiv 1 \pmod 2$, *then* $f_\gamma^{\mathrm{II}}(g') \neq 0$ *if and only if (Condition II-11) in §17.3 holds for* g'. *In general,*

$$f_\gamma^{\mathrm{II}}(g') = \zeta_\gamma(d') \cdot F_{0,n}^{\mathrm{II}}(g').$$

(ii) Case $n = 2n'+1$ odd: $\mathcal{S} = \mathcal{S}(P_\gamma^+) = \widetilde{\mathfrak{A}}\big(\prod_{\zeta\in\widehat{T}^0} \mathfrak{S}_{I_{n,\zeta}}\big)$.

If $\mathrm{ord}(d') \equiv 0 \pmod 2$, *then* $f_\gamma^{\mho+}(g') \neq 0$ *if and only if (Condition II-00) in §17.5 holds for* g'. *If* $\mathrm{ord}(d') \equiv 1 \pmod 2$, *then* $f_\gamma^{\mho+}(g') \neq 0$ *if and only if (Condition* $\mho$*-11) in §17.5 holds for* g'. *In general,*

$$f_\gamma^{\mho+}(g') = \zeta_\gamma(d') \cdot F_{+,n}^{\mathrm{II},\widetilde{H}}(g').$$

20.2 Spin irreducible characters of $\widetilde{G}_n^{\mathrm{II}}$ (Case n even)

Using Proposition 18.7 together with Lemma 20.1, we obtain the following formula for spin irreducible characters $F_{\Lambda^n}^{\mathrm{II}}$, n even, for $\Lambda^n = (\lambda^{n,\zeta})_{\zeta\in\widehat{T}^0} \in \boldsymbol{Y}_n(T)^0$.

Take $g' = (d', \sigma') \in \widetilde{G}_n^{\mathrm{II}}$, and let

$$g = \Phi(g') = (d, \sigma) \in G_n = D_n \rtimes \mathfrak{S}_n = G(m, 1, n), \tag{20.11}$$

with $d = \Phi(d') \in D_n$, $\sigma = \Phi(\sigma') \in \mathfrak{S}_n$. The non-spin character $\zeta_\gamma(d')$ on $\widetilde{D}_n$ is considered also as a character of D_n which is denoted by the same symbol as $\zeta_\gamma(d)$.

For $s' \in \widetilde{\mathfrak{S}}_n$, put $s = \Phi(s')$, then $\Phi(s'g's'^{-1}) = sgs^{-1} = \big(s(d), s\sigma s^{-1}\big)$, and $\zeta_{s^{-1}\gamma}(d') = \zeta_{s^{-1}\gamma}(d) = \zeta_\gamma\big(s(d)\big)$. Let $\breve{F}_{\gamma,\Lambda^n}(g)$ be the character of an IR of the base group $G_n = G(m, 1, n)$ given in (19.15) – (19.19).

By Proposition 18.7 and Lemma 20.3 (i), we obtain the following, through similar discussions as for Theorem 19.9.

Theorem 20.4 (Case $n = 2n' \geq 4$ even)**.** *Let* $\Lambda^n = (\lambda^{n,\zeta})_{\zeta\in\widehat{T}^0} \in \boldsymbol{Y}_n(T)^0$.

(i) *Take* $g' = (d', \sigma') \in \widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \mathcal{S}(P_\gamma)$, $\mathcal{S}(P_\gamma) = \Phi_{\mathfrak{S}}^{-1}\big(\prod_{\zeta\in\widehat{T}^0} \mathfrak{S}_{I_{n,\zeta}}\big)$. *Suppose* $L(\sigma') \equiv 0 \pmod 2$. *Then* $F_{\Lambda^n}^{\mathrm{II}}(g') = \mathrm{tr}\big(\Pi_{\Lambda^n}^{\mathrm{II}}(g')\big) \neq 0$ *only if (Condition II-00) in §17.2.1 holds for* g'. *In that case,*

$$F_{{}^t\Lambda^n}^{\mathrm{II}}(g') = F_{\Lambda^n}^{\mathrm{II}}(g'),$$
$$F_{\Lambda^n}^{\mathrm{II}}(g') = F_{0,n}^{\mathrm{II}}(g') \times \breve{F}_{\gamma,\Lambda^n}(g).$$

Suppose $L(\sigma') \equiv 1 \pmod 2$. *Then* $F_{\Lambda^n}^{\mathrm{II}}(g') = \mathrm{tr}\big(\Pi_{\Lambda^n}^{\mathrm{II}}(g')\big) \neq 0$ *only if (Condition II-11) in §17.3 holds for* g'. *In that case,*

$$F_{{}^t\Lambda^n}^{\mathrm{II}}(g') = -F_{\Lambda^n}^{\mathrm{II}}(g'),$$

$$F_{\Lambda^n}^{\mathrm{II}}(g') = F_{0,n}^{\mathrm{II}}(g') \times \breve{F}_{\gamma,\Lambda^n}(g).$$

(ii) *If* $g' \in \widetilde{G}_n^{\mathrm{II}}$ *is not conjugate to an element of* $\widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \mathcal{S}(P_\gamma)$, *then* $F_{\Lambda^n}^{\mathrm{II}}(g') = 0$.

(iii) *In total,* $F_{\Lambda^n}^{\mathrm{II}} = F_{0,n}^{\mathrm{II}} \times \breve{F}_{\gamma,\Lambda^n}$, *and so* $\Pi_{\Lambda^n}^{\mathrm{II}} \cong \Pi_0^{\mathrm{II}} \otimes \breve{\Pi}_{\gamma,\Lambda^n}$.

20.3 Spin irreducible characters of $\widetilde{G}_n^{\mathrm{II}}$ (Case n odd, Case of $\{\Lambda^n, {}^t\Lambda^n\} \in \boldsymbol{Y}_n^{\mathfrak{A}}(T)^{0,1}$)

20.3.1 Preparatory formulas in Case n odd

Let $n = 2n'+1 \geq 5$ be odd, and $\gamma \in \Gamma_n^0$. Note that, as is remarked in §14.3.1, the character of $\mho_n^+$ is invariant under $\widetilde{\mathfrak{S}}_n$, since $\mho_n^+$ is the restriction to the subgroup $\widetilde{\mathfrak{A}}_n$ of a spin representation of the group $\widetilde{\mathfrak{S}}_n$ which is equivalent to ∇'_n.

Take s'_0 from $\Phi_{\mathfrak{S}}^{-1}\big(\prod_{\zeta\in\widehat{T}^0} \mathfrak{S}_{I_{n,\zeta}}\big) \setminus \widetilde{\mathfrak{A}}\big(\prod_{\zeta\in\widehat{T}^0} \mathfrak{S}_{I_{n,\zeta}}\big)$, then $s_0\gamma = \gamma$ and

$$s_0 = \Phi(s'_0) \in S_\gamma \setminus A_\gamma, \text{ with } A_\gamma := \mathfrak{A}\Big(\prod_{\zeta\in\widehat{T}^0} \mathfrak{S}_{I_{n,\zeta}}\Big),\ S_\gamma = \prod_{\zeta\in\widehat{T}^0} \mathfrak{S}_{I_{n,\zeta}}\,, \tag{20.12}$$

and $S_\gamma = A_\gamma \sqcup s_0 A_\gamma$. Moreover we have by Lemmas 12.4 and 17.10 respectively

$$\begin{cases} \big(\rho_{\Lambda^n}\big)^{s_0} \cong \rho_{\Lambda^n}\,, & \text{for } \{\Lambda^n, {}^t\Lambda^n\} \in \boldsymbol{Y}_n^{\mathfrak{A}}(T)^{0,1}\ ; \\ \big(\rho_{\Lambda^n}^{(\kappa)}\big)^{s_0} \cong \rho_{\Lambda^n}^{(\kappa+1)}, & \text{for } (\Lambda^n, \kappa) \in \boldsymbol{Y}_n^{\mathfrak{A}}(T)^{0,2}\,, \end{cases} \tag{20.13}$$

$$f_\gamma^{\mho+}(s'_0 g' {s'_0}^{-1}) = (-1)^{\mathrm{ord}(d')}\, f_\gamma^{\mho+}(g'). \tag{20.14}$$

where the superfix $\kappa+1$ is understood modulo 2. For $d' \in \widetilde{D}_n$ and $s' \in \widetilde{\mathfrak{S}}_n$, put $d = \Phi(d'),\ s = \Phi(s')$, then

$$\begin{aligned} &\zeta_\gamma\big((s'_0 s')^{\mathrm{II}}(d')\big) = \zeta_\gamma\big(s_0 s(d)\big) = \zeta_\gamma\big(s(d)\big) = \zeta_\gamma\big({s'}^{\mathrm{II}}(d')\big)\ ; \\ &\zeta_{(\tau_1\tau_2\cdots\tau_n)s^{-1}\gamma}(d') = (-1)^{\mathrm{ord}(d)}\,\zeta_{s^{-1}\gamma}(d) = (-1)^{\mathrm{ord}(d)}\,\zeta_\gamma\big(s(d)\big). \end{aligned} \tag{20.15}$$

By Proposition 18.8, we have, for $g' = (d', \sigma') \in \widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \widetilde{\mathfrak{A}}_n \subset \widetilde{G}_n^{\mathrm{II}}$ and $s' \in \widetilde{\mathfrak{A}}_n$,

$$\mathrm{tr}\Big(P_\gamma^+\big({s'}^{\mathrm{II}}(d')\big)\mho_n^+(s'\sigma'{s'}^{-1})\Big) = \mathrm{tr}\big(P_\gamma^+(d')\mho_n^+(\sigma')\big)\cdot\frac{\zeta_{s^{-1}\gamma}(d')}{\zeta_\gamma(d')}, \tag{20.16}$$

and in particular, for $g' = (d', \sigma') \in \widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \mathcal{S}(P_\gamma^+)$ and $s' \in \widetilde{\mathfrak{A}}_n$,

$$f_\gamma^{\mho+}(s'g'{s'}^{-1}) = f_\gamma^{\mho+}(g')\cdot\frac{\zeta_{s^{-1}\gamma}(d')}{\zeta_\gamma(d')}. \tag{20.17}$$

Therefore, noting that $s'_0 g' {s'_0}^{-1} = \big({s'_0}^{\mathrm{II}}(d'), s'_0\sigma'{s'_0}^{-1}\big)$, we have by Lemma 20.3 (ii) the following.

Lemma 20.5. *Let* $g' = (d', \sigma') \in \widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \mathcal{S}(P_\gamma^\pm)$ *and* $s' \in \widetilde{\mathfrak{A}}_n$ *.*
(i) *Suppose* $\mathrm{ord}(d') \equiv 0 \pmod 2$*. Then, under (Condition II-00) on* g'*,*

$$f_\gamma^{\mho+}(s'g's'^{-1}) = F_{+,n}^{\mathrm{II},\widetilde{H}}(g') \times \zeta_\gamma\big(s(d)\big)\ ;$$
$$f_\gamma^{\mho+}(s's_0'g's_0'^{-1}s'^{-1}) = F_{+,n}^{\mathrm{II},\widetilde{H}}(g') \times \zeta_\gamma\big(ss_0(d)\big).$$

(ii) *Suppose* $\mathrm{ord}(d') \equiv 1 \pmod 2$*. Then, under (Condition* $\mho$*-11) on* g'*,*

$$f_\gamma^{\mho+}(s'g's'^{-1}) = F_{+,n}^{\mathrm{II},\widetilde{H}}(g') \times \zeta_\gamma\big(s(d)\big)\ ;$$
$$f_\gamma^{\mho+}(s's_0'g's_0'^{-1}s'^{-1}) = -F_{+,n}^{\mathrm{II},\widetilde{H}}(g') \times \zeta_\gamma\big(ss_0(d)\big).$$

20.3.2 Case of $\Lambda^n = (\lambda^{n,\zeta})_{\zeta\in\widehat{T}^0}$, $\{\Lambda^n, {}^t\Lambda^n\} \in \boldsymbol{Y}_n^{\mathfrak{A}}(T)^{0,1}$ (Part 1)

Using the above results, we discuss spin irreducible characters $F_{\Lambda^n}^{\mho+}$, depending on the cases whether $\mathrm{ord}(d') \equiv 0 \pmod 2$ or $\mathrm{ord}(d') \equiv 1 \pmod 2$. Here we treat the case of $\mathrm{ord}(d') \equiv 0 \pmod 2$.

We obtain from Lemmas 19.8 and 20.1 (ii) the following result.

Lemma 20.6. *Let* $\{\Lambda^n, {}^t\Lambda^n\} \in \boldsymbol{Y}_n^{\mathfrak{A}}(T)^{0,1}$*,* $n = 2n'+1$*. For* $g' = (d', \sigma') \in \widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \mathcal{S}(P_\gamma^+)$*,* $\mathcal{S}(P_\gamma^+) = \widetilde{\mathfrak{A}}\big(\prod_{\zeta\in\widehat{T}^0} \mathfrak{S}_{I_{n,\zeta}}\big)$*, assume that* $\mathrm{ord}(d') \equiv 0 \pmod 2$*. Then, under (Condition II-00) on* g'*,*

$$\begin{aligned} F_{\Lambda^n}^{\mho+}(g') &= F_{+,n}^{\mathrm{II},\widetilde{H}}(g') \times \\ &\times \sum_{\substack{s\in A_\gamma\backslash\mathfrak{A}_n:\\ s\sigma s^{-1}\in A_\gamma}} \Big(\zeta_\gamma\big(s(d)\big)\, f_{\Lambda^n}^\rho(s\sigma s^{-1}) + \zeta_\gamma\big(ss_0(d)\big)\, f_{\Lambda^n}^\rho\big(ss_0\sigma(ss_0)^{-1}\big)\Big). \end{aligned} \tag{20.18}$$

For $g = \Phi(g') \in D_n \rtimes A_\gamma$, the character of an IR $\zeta_\gamma\cdot\rho_{\Lambda^n}$ of $D_n \rtimes A_\gamma$ is given by

$$X_{\gamma,\Lambda^n}(g) := \zeta_\gamma(d)\cdot f_{\Lambda^n}^\rho(\sigma).$$

The sum in (20.18) gives the character of the induced representation

$$\mathrm{Ind}_{D_n\rtimes A_\gamma}^{D_n\rtimes\mathfrak{S}_n}\big(\zeta_\gamma\cdot\rho_{\Lambda^n}\big) \cong \mathrm{Ind}_{D_n\rtimes S_\gamma}^{D_n\rtimes\mathfrak{S}_n}\Big(\mathrm{Ind}_{D_n\rtimes A_\gamma}^{D_n\rtimes S_\gamma}\big(\zeta_\gamma\cdot\rho_{\Lambda^n}\big)\Big). \tag{20.19}$$

With an element $s_0 \in S_\gamma \setminus A_\gamma$, the character of $\mathrm{Ind}_{D_n\rtimes A_\gamma}^{D_n\rtimes S_\gamma}(\zeta_\gamma\cdot\rho_{\Lambda^n})$ is given as

$$X_{\gamma,\Lambda^n}(g) + X_{\gamma,\Lambda^n}(s_0 g s_0{}^{-1}) = \zeta_\gamma(d)\big(f_{\Lambda^n}^\rho(\sigma) + f_{\Lambda^n}^\rho\big(s_0\sigma s_0{}^{-1}\big)\big),$$

because $s_0{}^{-1}\gamma = \gamma$ and so $\zeta_\gamma\big(s_0(d)\big) = \zeta_\gamma(d)$. Moreover the above sum is the character of $\mathrm{Ind}_{A_\gamma}^{S_\gamma}\rho_{\Lambda^n} \cong \pi_{\Lambda^n} \oplus \pi_{{}^t\Lambda^n}$ by Lemma 12.4, which is zero outside of A_γ. Thus we see that the sum in (20.18) is equal to $\breve{F}_{\gamma,\Lambda^n}(g) + \breve{F}_{\gamma,{}^t\Lambda^n}(g) = \big(1+\mathrm{sgn}(\sigma)\big)\breve{F}_{\gamma,\Lambda^n}(g) = 2\breve{F}_{\gamma,\Lambda^n}(g)$.

Proposition 20.7. *For $g' = (d', \sigma') \in \widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \mathcal{S}(P_\gamma^{\pm})$ with* $\mathrm{ord}(d') \equiv 0 \pmod 2$, *$F_{\Lambda^n}^{\mho\pm}(g') \neq 0$ only if g' satisfies (Condition II-00). In that case,*

$$F_{\Lambda^n}^{\mho+}(g') = F_{+,n}^{\mathrm{II},\widetilde{H}}(g') \cdot 2 \cdot \breve{F}_{\gamma,\Lambda^n}(g) = F_{+,n}^{\mathrm{II}}(g') \times \breve{F}_{\gamma,\Lambda^n}(g).$$

20.3.3 Case of $\Lambda^n = (\lambda^{n,\zeta})_{\zeta \in \widehat{T}^0}$, $\{\Lambda^n, {}^t\Lambda^n\} \in \mathbf{Y}_n^{\mathfrak{A}}(T)^{0,1}$ (Part 2)

Here we treat the case of $\mathrm{ord}(d') \equiv 1 \pmod 2$. In this case, as is seen in Lemma 17.9 (ii), $f_\gamma^{\mho\pm}(g') \neq 0$ if and only if the following condition holds for g' :

$$\textbf{(Condition } \mho\textbf{-11)} \quad \begin{cases} |\mathrm{supp}(g')| = n = 2n'+1,\ \mathrm{ord}(d') \equiv 1,\ L(\sigma') \equiv 0 \pmod 2, \\ \mathrm{ord}(\xi_q') \equiv 1\ (q \in Q),\ \mathrm{ord}(d_j') \equiv 1\ (j \in J) \quad (\therefore\ r+s \equiv 1). \end{cases}$$

Apply this, and use Lemma 20.5 (ii) and $\big(\rho_{\Lambda^n}\big)^{s_0} \cong \rho_{\Lambda^n}$, then we obtain the following.

Lemma 20.8. *Let $g' = (d', \sigma') \in \widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \mathcal{S}(P_\gamma^{\pm})$ be as in (20.7). Assume* $\mathrm{ord}(d') \equiv 1 \pmod 2$. *Then, possibly $F_{\Lambda^n}^{\mho\pm}(g') \neq 0$ only if g' satisfies (Condition $\mho$-11). In that case,*

$$F_{\Lambda^n}^{\mho+}(g') = F_{+,n}^{\mathrm{II},\widetilde{H}}(g') \times \\ \times \sum_{s \in A_\gamma \backslash \mathfrak{A}_n,\ s\sigma s^{-1} \in A_\gamma} \Big(\zeta_\gamma\big(s(d)\big)\, f_{\Lambda^n}^{\rho}(s\sigma s^{-1}) - \zeta_\gamma\big(ss_0(d)\big)\, f_{\Lambda^n}^{\rho}(ss_0\sigma {s_0}^{-1} s^{-1}) \Big).$$

Now, on the base group level, we consider induced representations of $\zeta_\gamma \boxdot \rho_{\Lambda^n}$ from $D_n \rtimes A_\gamma = \Phi\big(\widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \mathcal{S}(P_\gamma^{\pm})\big)$ up to $H_n := D_n \rtimes \mathfrak{A}_n$ and also up to $G_n = D_n \rtimes \mathfrak{S}_n$ as

$$(20.20) \qquad \begin{array}{rcll} \breve{P}_{\gamma,\Lambda^n}^{H_n} & := & \mathrm{Ind}_{D_n \rtimes A_\gamma}^{H_n} \big(\zeta_\gamma \boxdot \rho_{\Lambda^n}\big) & \text{of } H_n\,; \\ \breve{P}_{\gamma,\Lambda^n}^{G_n} & := & \mathrm{Ind}_{D_n \rtimes A_\gamma}^{G_n} \big(\zeta_\gamma \boxdot \rho_{\Lambda^n}\big) & \text{of } G_n\,. \end{array}$$

Since $\mathrm{Ind}_{A_\gamma}^{S_\gamma} \rho_{\Lambda^n} \cong \pi_{\Lambda^n} \oplus \pi_{{}^t\Lambda^n}$, we have $\breve{P}_{\gamma,\Lambda^n}^{G_n} \cong \breve{\Pi}_{\gamma,\Lambda^n} \oplus \breve{\Pi}_{\gamma,{}^t\Lambda^n}$.

Denote by $\breve{F}_{\gamma,\Lambda^n}^{H_n}$ the character of $\breve{P}_{\gamma,\Lambda^n}^{H_n}$. Then the sum in the above lemma gives us $\breve{F}_{\gamma,\Lambda^n}^{H_n}(g) - \breve{F}_{\gamma,\Lambda^n}^{H_n}\big(s_0 g {s_0}^{-1}\big)$. On the other hand, since $\Lambda^n \neq {}^t\Lambda^n$, we have ${}^{{s_0}^{-1}}(\rho_{\Lambda^n}) \cong \rho_{\Lambda^n}$ and ${}^{{s_0}^{-1}}(\zeta_\gamma \cdot \rho_{\Lambda^n}) \cong \zeta_\gamma \cdot \rho_{\Lambda^n}$. This gives us $\breve{F}_{\gamma,\Lambda^n}^{H_n}(g) = \breve{F}_{\gamma,\Lambda^n}^{H_n}\big(s_0 g {s_0}^{-1}\big)$.

Theorem 20.9 (CASE $n = 2n'+1 \geq 5$ ODD). *Let $\{\Lambda^n, {}^t\Lambda^n\} \in \mathbf{Y}_n^{\mathfrak{A}}(T)^{0,1}$.*

(i) *Let $g' = (d', \sigma') \in \widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \mathcal{S}(P_\gamma^{\pm})$ be as in (20.7).*

Assume $\mathrm{ord}(d') \equiv 0 \pmod 2$. *Then $F_{\Lambda^n}^{\mho+}(g') \neq 0$ only if g' satisfies (Condition II-00). In that case,*

$$F_{\Lambda^n}^{\mho+}(g') = F_{+,n}^{\mathrm{II},\widetilde{H}}(g') \cdot 2 \cdot \breve{F}_{\gamma,\Lambda^n}(g) = F_{+,n}^{\mathrm{II}}(g') \times \breve{F}_{\gamma,\Lambda^n}(g).$$

where $\breve{F}_{\gamma,\Lambda^n}(g)$ *denotes the character of non-spin IR of the base group* $G_n = D_n \rtimes \mathfrak{S}_n = G(m,1,n)$ *given in (19.19) in Theorem 19.7.*

Assume $\mathrm{ord}(d') \equiv 1 \pmod 2$. *Then* $F^{\mho+}_{\Lambda^n}(g') = 0$, $F^{\mathrm{II}}_{+,n}(g') = 0$, *and* $F^{\mho+}_{\Lambda^n}(g') = F^{\mathrm{II}}_{+,n}(g') \times \breve{F}_{\gamma,\Lambda^n}(g)$.

(ii) *If* $g' \in \widetilde{G}^{\mathrm{II}}_n$ *is not conjugate to an element in* $\widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \mathcal{S}(P^{\pm}_{\gamma})$, *then* $F^{\mho+}_{\Lambda^n}(g') = 0$, $\breve{F}_{\gamma,\Lambda^n}(g) = 0$, *and* $F^{\mho+}_{\Lambda^n}(g') = F^{\mathrm{II}}_{+,n}(g') \times \breve{F}_{\gamma,\Lambda^n}(g)$.

(iii) *Let* $\breve{\Pi}_{\gamma,\Lambda^n} = \mathrm{Ind}^{G_n}_{H_n} \breve{\pi}_{\gamma,\Lambda^n}$ *be as in (19.11). Then*

$$F^{\mho+}_{\Lambda^n} = F^{\mathrm{II}}_{+,n} \times \breve{F}_{\gamma,\Lambda^n} \quad \textit{and} \quad \Pi^{\mho+}_{\Lambda^n} \cong \Pi^{\mathrm{II}}_{+} \otimes \breve{\Pi}_{\gamma,\Lambda^n}.$$

Remark 20.1. Let $n \geq 5$ be odd, and $\{\Lambda^n, {}^t\Lambda^n\} \in \boldsymbol{Y}^{\mathfrak{A}}_n(T)^{0,1}$. Then,

$$\Pi^{\mho-}_{\Lambda^n} \cong \Pi^{\mho+}_{\Lambda^n}, \qquad \text{as IRs of } \widetilde{G}^{\mathrm{II}}_n.$$

20.4 Spin irreducible characters of $\widetilde{G}^{\mathrm{II}}_n$ (Case n odd, Case of $(\Lambda^n, \kappa) \in \boldsymbol{Y}^{\mathfrak{A}}_n(T)^{0,2}$, $\kappa = 0, 1$)

20.4.1 Irreducible characters of $H_n = D_n \rtimes \mathfrak{A}_n$

First we discuss on the base group level. Recall Definition 13.2 for $\mathfrak{S}_n$ and refer Theorems 13.3, 13.5 and 13.7 for $\widetilde{\mathfrak{A}}_n$.

Lemma 20.10. *Let* $g = (d, \sigma) = \xi_{q_1} \cdots \xi_{q_r} g_1 \cdots g_s$ *be an element of* $H_n = D_n \rtimes \mathfrak{A}_n$, *with* $\xi_q = y_q^{\,a_q}$, $g_j = (d_j, \sigma_j)$, *for* $n \geq 4$. *Let* $\tau \in \mathfrak{S}_n \setminus \mathfrak{A}_n$. *Then the conjugacy class of* g *under* $G_n = D_n \rtimes \mathfrak{S}_n$ *splits into two conjugacy classes under* H_n *if and only if* $r \leq 1$ *and* $\sigma = \sigma_1 \cdots \sigma_s \in \mathfrak{A}_n$ *is of the 2nd kind and of the 3rd kind (in the sense of Schur) at the same time, that is,* $\ell(\sigma_j)$ $(j \in J)$ *are all odd and mutually different.*

Let $\Lambda^n = (\lambda^{n,\zeta})_{\zeta \in \widehat{T}} \in \boldsymbol{Y}_n(T)$ be such that ${}^t\Lambda^n = \Lambda^n$ and $|\lambda^{n,\zeta}| \geq 2$ $(\exists \zeta)$, and let $\gamma \in \Gamma_n$ is subordinate to Λ^n. On the base group level, for the subgroup $D_n \rtimes A_\gamma = \Phi\big(\widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \mathcal{S}(P^{\pm}_\gamma)\big)$ of $G_n = D_n \rtimes \mathfrak{S}_n$, consider its IRs $\zeta_\gamma \boxdot \rho^{(\kappa)}_{\Lambda^n}$ $(\kappa = 0, 1)$, and take their induced representations up to $H_n := D_n \rtimes \mathfrak{A}_n$ and also up to $G_n = D_n \rtimes \mathfrak{S}_n$ as

$$\begin{aligned} \breve{P}^{H_n}_{\gamma,\Lambda^n,\kappa} &:= \mathrm{Ind}^{H_n}_{D_n \rtimes A_\gamma} \big(\zeta_\gamma \boxdot \rho^{(\kappa)}_{\Lambda^n}\big) \qquad \text{of } H_n\,; \\ \breve{P}^{G_n}_{\gamma,\Lambda^n,\kappa} &:= \mathrm{Ind}^{G_n}_{D_n \rtimes A_\gamma} \big(\zeta_\gamma \boxdot \rho^{(\kappa)}_{\Lambda^n}\big) \qquad \text{of } G_n\,. \end{aligned} \tag{20.21}$$

Take an $s_0 \in S_\gamma \setminus A_\gamma$. Then $\zeta_\gamma(s_0 d s_0^{\,-1}) = \zeta_\gamma(d)$ $(d \in D_n)$, ${}^{s_0}\big(\rho^{(\kappa)}_{\Lambda^n}\big) \cong \rho^{(\kappa+1)}_{\Lambda^n}$, and $\mathrm{Ind}^{S_\gamma}_{A_\gamma} \rho^{(\kappa)}_{\Lambda^n} \cong \mathrm{Ind}^{S_\gamma}_{A_\gamma} \rho^{(\kappa+1)}_{\Lambda^n} \cong \pi_{\Lambda^n}$, and accordingly

$$\breve{P}^{G_n}_{\gamma,\Lambda^n,\kappa} \cong \breve{\Pi}_{\gamma,\Lambda^n} \qquad (\kappa = 0, 1). \tag{20.22}$$

Denote by $\breve{F}^{H_n}_{\gamma,\Lambda^n,\kappa}$ the characters of $\breve{P}^{H_n}_{\gamma,\Lambda^n,\kappa}$. Then, for $h=(d,\sigma)\in H_n$,

$$\text{(20.23)}\qquad \begin{cases} \breve{F}^{H_n}_{\gamma,\Lambda^n,\kappa}(s_0hs_0{}^{-1}) = \breve{F}^{H_n}_{\gamma,\Lambda^n,\kappa+1}(h)\ ; \\ \breve{F}^{H_n}_{\gamma,\Lambda^n,0}(h) + \breve{F}^{H_n}_{\gamma,\Lambda^n,1}(h) = \breve{F}_{\gamma,\Lambda^n}(h)\ ; \\ \breve{F}^{H_n}_{(\tau_1\tau_2\cdots\tau_n)\gamma,\Lambda^n,\kappa}(h) = (-1)^{\mathrm{ord}(d)}\,\breve{F}^{H_n}_{\gamma,\Lambda^n,\kappa}(h). \end{cases}$$

Since $\breve{F}_{\gamma,\Lambda^n}(g)=0$ outside H_n, the second equality above can be expressed as $\breve{F}_{\gamma,\Lambda^n} = \breve{F}^{H_n}_{\gamma,\Lambda^n,0} + \breve{F}^{H_n}_{\gamma,\Lambda^n,1}$ if each $\breve{F}^{H_n}_{\gamma,\Lambda^n,\kappa}$ is extended identically zero outside H_n.

Proposition 20.11 (non-spin irreducible characters of H_n)**.** *Let Λ^n be such that ${}^t\Lambda^n=\Lambda^n$, and $\gamma\in\Gamma$ is subordinate to Λ^n. For a $g=(d,\sigma)\in D_n\rtimes A_\gamma\subset H_n=D_n\rtimes\mathfrak{A}_n$, let its standard decomposition be as in (19.13), and put $Q=\{q_1,q_2,\ldots,q_r\}$, $J=\{1,2,\ldots,s\}$. Then the character $\breve{F}^{H_n}_{\gamma,\Lambda^n,\kappa}(g)$ of IR $\breve{P}^{H_n}_{\gamma,\Lambda^n,\kappa}$ of H_n is given by*

$$\text{(20.24)}\qquad \breve{F}^{H_n}_{\gamma,\Lambda^n,\kappa}(g) = \frac{1}{2}\sum_{(\mathcal{Q},\mathcal{J})} b(\mathcal{I}_n;\mathcal{Q},\mathcal{J};g)\,X\big(\Lambda^n,\,\kappa\,;\,\mathcal{Q},\,\mathcal{J}\,;\,g\big),$$

where the summation runs over all pairs of partitions $(\mathcal{Q},\mathcal{J})$ for which (Condition QJ) in §19.3 holds, and the coefficient $b(\mathcal{I}_n;\mathcal{Q},\mathcal{J};g)$ is given in (19.18), and

$$\text{(20.25)}\qquad X\big(\Lambda^n,\,\kappa;\,\mathcal{Q},\,\mathcal{J}\,;\,g\big) := \prod_{\zeta\in\widehat{T}}\zeta\Big(\textstyle\prod_{q\in Q_\zeta}\xi_q\cdot\prod_{j\in J_\zeta}d_j\Big)\times f^{\rho}_{\Lambda^n,\kappa}\big(\sigma_{\mathcal{J}}\big).$$

Here $\sigma_{\mathcal{J}}$ denotes an $\mathfrak{A}_n$-conjugate of σ such that $\sigma_{\mathcal{J}}=\prod_{\zeta\in\widehat{T}}\sigma_\zeta$ with $\sigma_\zeta\in\mathfrak{S}_{I_{n,\zeta}}$ a product of cycles of lengths ℓ_j ($j\in J_\zeta$).

For any $g=(d,\sigma)\in H_n$ which is not conjugate to any element in $D_n\rtimes A_\gamma$, the character vanishes.

To calculate $\breve{F}^{H_n}_{\gamma,\Lambda^n,\kappa}(g) - \breve{F}^{H_n}_{\gamma,\Lambda^n,\kappa}(s_0gs_0{}^{-1})$, the evaluation of the difference $f^{\rho}_{\Lambda^n,\kappa}\big(\sigma_{\mathcal{J}}\big) - f^{\rho}_{\Lambda^n,\kappa}\big(s_0\sigma_{\mathcal{J}}s_0{}^{-1}\big)$ is essential. The latter has been already studied for irreducible characters of $\mathfrak{A}_n$ by Frobenius [Frob2], and here we remark only the following.

Lemma 20.12. *The difference $f^{\rho}_{\Lambda^n,\kappa}\big(\sigma_{\mathcal{J}}\big) - f^{\rho}_{\Lambda^n,\kappa}\big(s_0\sigma_{\mathcal{J}}s_0{}^{-1}\big)$ of character values is non-zero only when, for each $\zeta\in\widehat{T}$,*

$$|I_{n,\zeta}| - \textstyle\sum_{j\in J_\zeta}\ell_j \le 1\,, \ \ \text{and}\ \ \ell_j\ (j\in J_\zeta)\ \text{are all different, odd integers.}$$

20.4.2 Characters of spin IRs $\Pi^{\mho+}_{\Lambda^n,\kappa}$ of $\widetilde{G}_n^{\rm II}$ for $(\Lambda^n,\kappa)\in\boldsymbol{Y}_n^{\mathfrak{A}}(T)^{0,2}$

Now we study characters of spin IRs $\Pi^{\mho+}_{\Lambda^n,\kappa}$ of $\widetilde{G}_n^{\rm II}$ for $(\Lambda^n,\kappa)\in\boldsymbol{Y}_n^{\mathfrak{A}}(T)^{0,2}$, and apply Lemma 20.12 and Proposition 20.11. Introduce a condition on g' as

(Condition Ʊ-11s) $\begin{cases} |\mathrm{supp}(g')| = n = 2n'+1, \ \mathrm{ord}(d') \equiv 1, \ L(\sigma') \equiv 0 \ (\mathrm{mod}\ 2), \\ \mathrm{ord}(\xi'_q) \equiv 1 \ (q \in Q), \ \mathrm{ord}(d'_j) \equiv 1 \ (j \in J), \ r \leq 1 \ (|Q| \leq 1), \\ L(\sigma'_j) \equiv 0 \ (j \in J), \ \text{and} \ \ \ell(\sigma'_j) \ (j \in J) \ \text{all different.} \end{cases}$

Theorem 20.13 (CASE $n = 2n' + 1 \geq 5$ ODD)**.**
Let $(\Lambda^n, \kappa) \in \boldsymbol{Y}_n^{\mathfrak{A}}(T)^{0,2}$, $\kappa = 0, 1$.

(i) *Let* $g' = (d', \sigma') \in \widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \mathcal{S}(P_\gamma^{\pm})$ *be as in (20.7).*

Assume $\mathrm{ord}(d') \equiv 0 \ (\mathrm{mod}\ 2)$*. Then* $F^{Ʊ\pm}_{\Lambda^n,\kappa}(g') \neq 0$ *only if (Condition II-00) holds for* g'*. In that case,*

$$F^{Ʊ+}_{\Lambda^n,0}(g') = F^{Ʊ+}_{\Lambda^n,1}(g') = F^{\mathrm{II},\widetilde{H}}_{+,n}(g') \times \breve{F}_{\gamma,\Lambda^n}(g),$$

where $\breve{F}_{\gamma,\Lambda^n}(g)$ *denotes the character of non-spin IR of the base group* $G_n = D_n \rtimes \mathfrak{S}_n = G(m,1,n)$ *given in (19.19), and* $\breve{F}_{\gamma,\Lambda^n}(g) = 0$ *if* $\mathrm{sgn}(\sigma) = -1$.

Assume $\mathrm{ord}(d') \equiv 1 \ (\mathrm{mod}\ 2)$*. Then* $F^{Ʊ\pm}_{\Lambda^n,\kappa}(g') \neq 0$ *only if (Condition Ʊ-11s) holds for* g'*, and* $F^{Ʊ+}_{\Lambda^n,1}(g') = -F^{Ʊ+}_{\Lambda^n,0}(g')$*, and*

$$F^{Ʊ+}_{\Lambda^n,\kappa}(g') = F^{\mathrm{II},\widetilde{H}}_{+,n}(g') \times \big(\breve{F}^{H_n}_{\gamma,\Lambda^n,\kappa}(g) - \breve{F}^{H_n}_{\gamma,\Lambda^n,\kappa+1}(g)\big) \qquad (\kappa = 0, 1).$$

(ii) *If* $g' \in \widetilde{G}_n^{\mathrm{II}}$ *is not conjugate to an element in* $\widetilde{D}_n \overset{\mathrm{II}}{\rtimes} \mathcal{S}(P_\gamma^{\pm})$*, then* $F^{Ʊ\pm}_{\Lambda^n,\kappa}(g') = 0$.

Proof. We start with the summation formula in Lemma 20.1. Take an $s'_0 \in \Phi^{-1}\big(\prod_{\zeta \in \widehat{T}^0} \mathfrak{S}_{I_{n,\zeta}}\big) \setminus \mathcal{S}$. Then the main sum in the formula is rewritten as

$$\sum_{s' \in \mathcal{S} \setminus \widetilde{\mathfrak{A}}_n \,:\, s'\sigma' s'^{-1} \in \mathcal{S}} \Big\{ f_\gamma^{Ʊ+}\big((s'^{\,\mathrm{II}}(d'), s'\sigma' s'^{-1})\big) \cdot f^{\rho}_{\Lambda^n,\kappa}(s\sigma s^{-1}) + f_\gamma^{Ʊ+}\big((s'^{\,\mathrm{II}}(s_0'^{\,\mathrm{II}}(d')), s'(s'_0\sigma' s_0'^{-1})s'^{-1})\big) \cdot f^{\rho}_{\Lambda^n,\kappa}(s(s_0\sigma s_0^{-1})s^{-1}) \Big\}.$$

By Proposition 18.8, for $g' = (d', \sigma')$, this is equal to

$$\sum_{s' \,:\, \text{as above}} \Big\{ f^{Ʊ+}_{s^{-1}\gamma}(g') \cdot f^{\rho}_{\Lambda^n,\kappa}(s\sigma s^{-1}) + f^{Ʊ+}_{s^{-1}\gamma}(s'_0 g' s_0'^{-1}) \cdot f^{\rho}_{\Lambda^n,\kappa}(s(s_0\sigma s_0^{-1})s^{-1}) \Big\}.$$

By Theorem 17.9 (ii), we have $f_\gamma^{Ʊ+}(g') = f_\gamma^{Ʊ+}(g') = \zeta_\gamma(d') \cdot F^{\mathrm{II},\widetilde{H}}_{+,n}(g')$.

On the other hand, by Lemma 17.10, $f_\gamma^{Ʊ+}(s'_0 g' s_0'^{-1}) = (-1)^{\mathrm{ord}(d')} f_\gamma^{Ʊ+}(g')$. Therefore, under the condition $\mathrm{sgn}(\sigma') = 1$, the sum above is equal to

$$\sum_{s' \,:\, \text{as above}} f^{Ʊ+}_{s^{-1}\gamma}(g') \cdot \big(f^{\rho}_{\Lambda^n,\kappa}(s\sigma s^{-1}) \pm f^{\rho}_{\Lambda^n,\kappa+1}(s\sigma s^{-1})\big),$$

according as $(-1)^{\mathrm{ord}(d')} = \pm 1$. This gives us the formulas for $F^{Ʊ+}_{\Lambda^n,\kappa}$.

Remark that $\breve{F}^{H_n}_{(\tau_1\tau_2\cdots\tau_n)\gamma,\Lambda^n,\kappa}(g) = (-1)^{{\rm ord}(d)}\,\breve{F}^{H_n}_{\gamma,\Lambda^n,\kappa}(g)$, then the results for $\Pi^{\mho-}_{\Lambda^n,\kappa}$ are obtained from those for $\Pi^{\mho+}_{\Lambda^n,\kappa}$. □

Remark 20.2. Let $n \geq 5$ odd, and $(\Lambda^n,\kappa) \in \boldsymbol{Y}_n^{\mathfrak{A}}(T)^{0,2}$, $\kappa = 0,1$. Then,

$$\Pi^{\mho-}_{\Lambda^n,0} \cong \Pi^{\mho+}_{\Lambda^n,1}\,, \quad \Pi^{\mho-}_{\Lambda^n,1} \cong \Pi^{\mho+}_{\Lambda^n,0}\,, \quad \text{as IRs of } \widetilde{G}_n^{\rm II}\,.$$

20.5 Spin irreducible characters of $\widetilde{G}_n^{\rm II}$ (Case n odd, Case of $\Lambda^n \in \boldsymbol{Y}_n^{\mathfrak{A}}(T)^{0,3}$)

This is an exceptional case in the case of n odd, and we say that the corresponding IRs $\Pi^{\mho+}_{\Lambda^n}$ are in Case (Exc). As seen in Example 12.3, $P_\gamma^+\cdot\mho_n^+ = P_\gamma^+\cdot\chi_1$ with a spin character χ_1 of $Z_1 = \langle z_1\rangle \subset \widetilde{\mathfrak{A}}_n : \chi(z_1) = -1$, and $\rho_{\Lambda^n} = \mathbf{1}$ for $\Lambda^n \in \boldsymbol{Y}_n^{\mathfrak{A}}(T)^{0,3}$. Here $\widetilde{D}_n \overset{\rm II}{\rtimes} \mathcal{S}(P_\gamma^+) = \widetilde{D}_n \times Z_1$, and $\Pi^{\mho+}_{\Lambda^n} = {\rm Ind}^{\widetilde{G}^{\rm II}}_{\widetilde{D}_n\times Z_1}\big(P_\gamma^+\cdot\chi_1\big)$.

Theorem 20.14. *Let $n = 2n'+1 \leq m' = m/2$, and $\Lambda^n = (\lambda^{n,\zeta})_{\zeta\in\widehat{T}^0} \in \boldsymbol{Y}_n^{\mathfrak{A}}(T)^{0,3}$. The character $F^{\mho+}_{\Lambda^n}$ of $\Pi^{\mho+}_{\Lambda^n}$ is given as follows. Let $\gamma = (\gamma_1,\ldots,\gamma_n) \in \Gamma_n^0$ in (11.7) be subordinate to Λ^n, then γ_j's are all different.*

Let $g'' = (d'',\sigma') \in \widetilde{G}^{\rm II} = \widetilde{D}_n \overset{\rm II}{\rtimes} \widetilde{\mathfrak{S}}_n$ with $d'' = {z_2}^b d', d' = {\eta_1}^{a_1}{\eta_2}^{a_2}\cdots{\eta_n}^{a_n}$. Then $F^{\mho+}_{\Lambda^n}(g'') = 0$ except when $\sigma' = {z_1}^a$. For $g'' = (d'',{z_1}^a)$,

$$(*)\qquad F^{\mho+}_{\Lambda^n}(g'') = \begin{cases} (-1)^a(-1)^b\,2^{n'}\sum_{\sigma\in\mathfrak{S}_n}\zeta_{\sigma\gamma}(d'), \\ \qquad\qquad \text{if } d' \in \widetilde{D}_n^0 \text{ or } a_q \equiv 0 \pmod 2\ \ (q\in Q)\,; \\ (-1)^a(-1)^b\,(2i)^{n'}\big\{\sum_{\sigma\in\mathfrak{A}_n}\zeta_{\sigma\gamma}(d') - \sum_{\sigma\in\mathfrak{A}_n}\zeta_{\sigma s_1\gamma}(d')\big\}, \\ \qquad\qquad \text{if } a_q \equiv 1 \pmod 2\ \ (q\in Q)\,; \\ 0, \qquad\qquad \text{otherwise.} \end{cases}$$

where χ_γ is given in Definition 6.1, and $\zeta_\gamma(d') = \omega^{\gamma_1a_1+\cdots+\gamma_na_n}$, $\omega = e^{2\pi i/m}$, and $s_1 = (1\ 2) \in \mathfrak{S}_n$.

Note 20.1. In this exceptional case, $\Lambda^n \in \boldsymbol{Y}_n^{\mathfrak{A}}(T)^{0,3}$, all $0 \leq \gamma_j \leq m' = m/2$ $(j \in \boldsymbol{I}_n)$ are mutually different. So $\zeta_{\sigma\gamma}$, $\sigma \in \mathfrak{S}_n$, are all different non-spin characters of D_n. Moreover, in the middle line in $(*)$, put $a_j = 2b_j+1$, then

$$\zeta_{\sigma^{-1}\gamma}(d') = \omega^{\gamma_1+\cdots+\gamma_n}\,\big(\omega^2\big)^{\gamma_{\sigma(1)}b_1+\cdots+\gamma_{\sigma(n)}b_n}.$$

This shows that we have there a non-zero function on $(\eta_1\cdots\eta_n)\widetilde{D}_n^0$.

Part IV
Spin characters of infinite group $R\big(G(m,1,\infty)\big)$ of Types $(-1,-1,\pm1)$

For a finite dimensional representation π of a group H, its normalized character is defined as $\widetilde{\chi}_\pi(h) := \mathrm{tr}\big(\pi(h)\big)/\dim\pi$ $(h \in H)$. In this part, we study pointwise limits of normalized spin irreducible characters of $R\big(G(m,1,n)\big)$ as $n \to \infty$, in CASEs I and II. We prove that the set of all limits gives exactly the set of all characters of the infinite group $R\big(G(m,1,\infty)\big)$ of each spin type.

21 Towards limits of spin irreducible characters

21.1. Summary on spin IRs and spin irreducible characters.
To start with, we summarize the fundamental results on spin IRs and their characters of $R\big(G(m,1,n)\big)$ in CASEs I and II as follows. For CASE I, Type $(-1,-1,-1)$, spin IRs are classified in Theorem 11.5 with their parameter space in (11.21), and their characters are given in Theorem 19.9. For CASE II, Type $(-1,-1,\ 1)$, spin IRs are classified in Theorem 12.6 with their parameter spaces in (12.14), and their characters are given in Theorems 20.4, 20.9, 20.13 and 20.14. The parameter spaces for them are given as follows: let $\boldsymbol{Y}$ be the set of all Young diagrams, and $\widehat{T}^0 = \{\zeta \in \widehat{T}\,;\ \zeta(\eta) = \omega^a,\ 0 \le a < m' = m/2\}$, then

$$\boldsymbol{Y}_n(T)^0 = \big\{\Lambda = (\lambda^\zeta)_{\zeta\in\widehat{T}^0}\ ;\ \lambda^\zeta \in \boldsymbol{Y},\ \textstyle\sum_{\zeta\in\widehat{T}^0}|\lambda^\zeta| = n\big\}\ ;$$

$$\boldsymbol{Y}^{\mathfrak{A}}_n(T)^0 = \boldsymbol{Y}^{\mathfrak{A}}_n(T)^{0,1} \sqcup \boldsymbol{Y}^{\mathfrak{A}}_n(T)^{0,2} \sqcup \boldsymbol{Y}^{\mathfrak{A}}_n(T)^{0,3}, \qquad \text{with}$$

$$\begin{cases} \boldsymbol{Y}^{\mathfrak{A}}_n(T)^{0,1} = \big\{\{\Lambda^n, {}^t\Lambda^n\}\,;\ \Lambda^n = (\lambda^{n,\zeta})_{\zeta\in\widehat{T}^0} \in \boldsymbol{Y}_n(T)^0,\ {}^t\Lambda^n \ne \Lambda^n\big\}, \\ \boldsymbol{Y}^{\mathfrak{A}}_n(T)^{0,2} = \big\{(\Lambda^n,\kappa)\,;\ \Lambda^n \in \boldsymbol{Y}_n(T)^0,\ {}^t\Lambda^n = \Lambda^n,\ |\lambda^{n,\zeta}| \ge 2\ (\exists\zeta),\ \kappa = 0,1\big\}, \\ \boldsymbol{Y}^{\mathfrak{A}}_n(T)^{0,3} = \big\{\Lambda^n = (\lambda^{n,\zeta})_{\zeta\in\widehat{T}^0}\ ;\ |\lambda^{n,\zeta}| \le 1\ (\forall\zeta)\big\}. \end{cases}$$

Table 21.1. Spin IRs and spin irreducible characters for $G(m,1,n)$.

CASE *Type*	Parity of n	Symbol of IRs	irreducible characters	given in Theorem
I $(-1,-1,-1)$	n even	$\Pi^{\mathrm{I}}_{\Lambda^n}$ $(\Lambda^n \in \boldsymbol{Y}_n(T)^0)$	$F^{\mathrm{I}}_{\Lambda^n}$	19.9 (i)
	n odd	$\Pi^{\mathrm{I}+}_{\Lambda^n},\ \Pi^{\mathrm{I}-}_{\Lambda^n}$ $(\Lambda^n \in \boldsymbol{Y}_n(T)^0)$	$F^{\mathrm{I}+}_{\Lambda^n},\ F^{\mathrm{I}-}_{\Lambda^n}$	19.9 (ii)
II $(-1,-1,\ 1)$	n even	$\Pi^{\mathrm{II}}_{\Lambda^n}$ $(\Lambda^n \in \boldsymbol{Y}^{\mathfrak{A}}_n(T)^0)$	$F^{\mathrm{II}}_{\Lambda^n}$	20.4
	n odd	$\Pi^{\mho+}_{\Lambda^n}$ $(\{\Lambda^n, {}^t\Lambda^n\} \in \boldsymbol{Y}^{\mathfrak{A}}_n(T)^{0,1})$	$F^{\mho+}_{\Lambda^n}$	20.9
		$\Pi^{\mho+}_{\Lambda^n,\kappa}$ $((\Lambda^n,\kappa) \in \boldsymbol{Y}^{\mathfrak{A}}_n(T)^{0,2})$	$F^{\mho+}_{\Lambda^n,\kappa}$	20.13
		$\Pi^{\mho+}_{\Lambda^n}$ $(\Lambda^n \in \boldsymbol{Y}^{\mathfrak{A}}_n(T)^{0,3})$	$F^{\mho+}_{\Lambda^n}$	20.14

21.2. Evaluation of supports of spin irreducible characters.

We evaluate the supports of each irreducible characters and obtain the following Table 21.2, which is a refinement of Table 4.1.

Table 21.2. Supports of spin irreducible characters for $\widetilde{G}^{\mathrm{Y}}(m,1,n)$, Y=I, II, $4 \le n < \infty,\ m = 2m'$:

CASE Y (*spin*) *Type* $(\beta_1,\beta_2,\beta_3)$ parity of n	$f(g') \neq 0 \Rightarrow$ Condition for $g' = d'\sigma' = \xi'_{q_1}\cdots\xi'_{q_r}g'_1\cdots g'_s,\ g'_j = (d'_j,\sigma'_j)$			
	$\mathrm{ord}(d') + L(\sigma') \equiv 0 \pmod 2$		$\mathrm{ord}(d') + L(\sigma') \equiv 1 \pmod 2$	
	$\mathrm{ord}(d') \equiv 0$ $L(\sigma') \equiv 0$	$\mathrm{ord}(d') \equiv 1$ $L(\sigma') \equiv 1$	$\mathrm{ord}(d') \equiv 0$ $L(\sigma') \equiv 1$	$\mathrm{ord}(d') \equiv 1$ $L(\sigma') \equiv 0$
I (n even) $(-1,-1,-1)$	Condition I-00		$\varnothing$	
I (n odd) $(-1,-1,-1)$	Condition I-00		Condition I-11	
II (n even) $(-1,-1,\ 1)$	Condition II-00	Condition II-11	$\varnothing$	
II (n odd) $(-1,-1,\ 1)$	Condition II-00	$\varnothing$		except Case (Exc): Condition $\mho$-11s in Case (Exc): Condition $\mho$-Exc

Here the symbol $\varnothing$ means that characters are identically zero there.

(Condition I-00) and (Condition I-11) are given in Lemma 16.2, (Condition II-00) in §17.2.1, (Condition II-11) in §17.3, (Condition $\mho$-11) in §17.5, and (Condition $\mho$-11s) in §20.4.2. For the convenience of readers, we list up them here, together with (Condition $\mho$-Exc) below appearing in Case (Exc) in §20.5:

(Condition I-00)
$$\begin{cases} \mathrm{ord}(d') + L(\sigma') \equiv 0 \pmod 2, \\ \mathrm{ord}(\xi'_q) \equiv 0\ (q \in Q),\ \mathrm{ord}(d'_j) + L(\sigma'_j) \equiv 0 \pmod 2\ \ (j \in J)\,; \end{cases}$$

(Condition I-11)
$$\begin{cases} |\mathrm{supp}(g')| = n \text{ odd},\ \ \mathrm{ord}(d') + L(\sigma') \equiv 1 \pmod 2, \\ \mathrm{ord}(\xi'_q) \equiv 1\ (q \in Q),\ \ \mathrm{ord}(d'_j) \equiv 1 \pmod 2\ \ (j \in J)\,; \end{cases}$$

(Condition II-00)
$$\begin{cases} \mathrm{ord}(d') \equiv 0, \quad L(\sigma') \equiv 0 \pmod 2, \\ \mathrm{ord}(\xi'_q) \equiv 0\ (q \in Q),\ \mathrm{ord}(d'_j) + L(\sigma'_j) \equiv 0 \pmod 2\ \ (j \in J)\,; \end{cases}$$

(Condition II-11)
$$\begin{cases} |\mathrm{supp}(g')| = n = 2n',\ \ \mathrm{ord}(d') \equiv L(\sigma') \equiv 1,\ \ r+s \equiv 1, \\ \mathrm{ord}(\xi'_q) \equiv 1\ (q \in Q),\ \ \mathrm{ord}(d'_j) \equiv 1 \pmod 2\ \ (j \in J)\,; \end{cases}$$

(Condition $\mho$-11)
$$\begin{cases} |\mathrm{supp}(g')| = n \text{ odd},\ \mathrm{ord}(d') \equiv 1,\ L(\sigma') \equiv 0 \pmod 2, \\ \mathrm{ord}(\xi'_q) \equiv 1\ (q \in Q),\ \mathrm{ord}(d'_j) \equiv 1\ (j \in J)\ \ (\therefore\ r+s \equiv 1)\,; \end{cases}$$

(Condition $\mho$-11s)
$$\begin{cases} |\mathrm{supp}(g')| = n \text{ odd},\ \mathrm{ord}(d') \equiv 1,\ L(\sigma') \equiv 0 \pmod 2, \\ \mathrm{ord}(\xi'_q) \equiv 1\ (q \in Q),\ \mathrm{ord}(d'_j) \equiv 1\ (j \in J),\ r \le 1\ (|Q| \le 1), \\ L(\sigma'_j) \equiv 0\ (j \in J),\ \text{and}\ \ \ell(\sigma'_j)\ (j \in J)\ \text{ all different}\,; \end{cases}$$

(Condition $\mho$-Exc)
$$\begin{cases} |\mathrm{supp}(g')| = n\ (\le m')\ \text{odd},\ \mathrm{ord}(d') \equiv 1 \pmod 2, \\ \sigma' = z_1{}^a\ (a = 0,1)\,;\ \ \mathrm{ord}(\xi'_q) \equiv 1\ (\forall q \in Q = \boldsymbol{I}_n). \end{cases}$$

When we consider limits of characters of spin IRs of $\widetilde{G}_n^{\mathrm{Y}}$, $\mathrm{Y}=\mathrm{I},\mathrm{II}$ as $n\to\infty$, as soon as $n>|\mathrm{supp}(g')|$, there does not hold any of (Condition I-11), (Condition II-11), (Condition Ʊ-11) and (Condition Ʊ-11s) for g', no more. So we are naturally led to consider only the cases of g' under (Condition I-00) or (Condition II-00). The case of $\Lambda^n\in \boldsymbol{Y}_n(T)^{0,3}$ is out of our present consideration since $n\le m'=m/2$.

The inductive limits $\widetilde{G}_\infty^{\mathrm{Y}}:=\lim_{n\to\infty}\widetilde{G}_n^{\mathrm{Y}}$ (Y=I, II) are covering groups of $G_\infty := G(m,1,\infty)=\lim_{n\to\infty}G(m,1,n)$ and are quotients of the representation group

$$R\big(G(m,1,\infty)\big):=\lim_{n\to\infty}R\big(G(m,1,n)\big) \tag{21.1}$$

by a central subgroup $Z^{\mathrm{I}}:=\langle z_2{z_3}^{-1}\rangle$ or $Z^{\mathrm{II}}:=\langle z_3\rangle$ corresponding respectively to Y=I or Y=II as

$$\widetilde{G}_\infty^{\mathrm{Y}}=R\big(G(m,1,\infty)\big)/Z^{\mathrm{Y}}\qquad (\mathrm{Y}=\mathrm{I},\ \mathrm{II}). \tag{21.2}$$

We may consider as $\widetilde{G}_\infty^{\mathrm{Y}}=\bigcup_{n\gg 0}\widetilde{G}_n^{\mathrm{Y}}$, and as topological groups with the inductive limit topologies, they are discrete and countable. (Condition Y-00) for each $\widetilde{G}_n^{\mathrm{Y}}$ can be extended to (Condition Y-00) for $\widetilde{G}_\infty^{\mathrm{Y}}$. Denote by $\widetilde{F}_{\Lambda^n}^{\mathrm{I}}$ the normalized character $F_{\Lambda^n}^{\mathrm{I}}/F_{\Lambda^n}^{\mathrm{I}}(e)$, and similarly for other cases.

Lemma 21.1. (i) *In CASE I, Type $(-1,-1,-1)$, the limits*

$$\lim_{n=2n'\to\infty}\widetilde{F}_{\Lambda^n}^{\mathrm{I}}(g'),\qquad \lim_{n=2n'+1\to\infty}\widetilde{F}_{\Lambda^n}^{\mathrm{I}+}(g')=\lim_{n=2n'+1\to\infty}\widetilde{F}_{\Lambda^n}^{\mathrm{I}-}(g')$$

are all zero if $g'\in\widetilde{G}_\infty^{\mathrm{I}}$ does not satisfy (Condition I-00).

(ii) *In CASE II, Type $(-1,-1,\ 1)$, the limits*

$$\lim_{n=2n'\to\infty}\widetilde{F}_{\Lambda^n}^{\mathrm{II}}(g'),\qquad \lim_{n=2n'+1\to\infty}\widetilde{F}_{\Lambda^n}^{\mho+}(g'),\qquad \lim_{n=2n'+1\to\infty}\widetilde{F}_{\Lambda^n,\kappa}^{\mho+}(g')$$

are all zero if $g'\in\widetilde{G}_\infty^{\mathrm{II}}$ does not satisfy (Condition II-00).

21.3. General theory on limiting process as $n\to\infty$.

In §§2–3 of [HoHH], we studied harmonic analysis on a general branching graph, allowing infinite valencies of the graph (see Theorem 3.2, loc. cit. in particular). Generalizing it, we give Theorem 14.3 in our first paper [I], for the case of an increasing sequence of compact groups. As its corollary in the case of increasing sequence of finite groups, we have the following.

Theorem 21.2. *Let $\cdots\subset H_n\subset H_{n+1}\subset\cdots$ be a sequence of finite groups, and $H_\infty=\lim_{n\to\infty}H_n$ be its inductive limit. Let f be a character of H_∞ (i.e., an extremal, normalized central positive definite function on H_∞). Then there exists at least a sequence of IRs π_n of H_n such that f is the pointwise limit of normalized irreducible characters $\widetilde{\chi}_{\pi_n}=\chi_{\pi_n}/\dim\pi_n$ as $n\to\infty$.*

With this as our background, to obtain the set $E(G)$ of all characters, for $G=H_\infty$, by the limiting process, there are the following two points to be checked:

(1) To pick up all the *good* limit functions on G,

(2) Not to pick up any *bad* limit functions on G.

In our previous papers [HHH1] and [HoHH], we have studied the case of wreath products $H_n = \mathfrak{S}_n(T)$ and $G = H_\infty = \mathfrak{S}_\infty(T)$, with T any compact group. In [HHH1], concerning the point (2) above, it is proved that, if T is infinite, then there exists always *non-continuous* limit functions (of course, positive definite), which are certainly out of $K_1(G)$ and are called as *bad limits.* Also a necessary and sufficient condition is given for a path $\{\pi_n;\, n \in \boldsymbol{N}\}$ to have a *good limit* (cf. Theorems 6.1 and 7.1, loc. cit.). Moreover, in Example 6.1, loc. cit., for $G = \mathfrak{S}_\infty(T)$ with certain infinite T, interesting bad limits are given explicitly.

21.4. Limits of the special spin irreducible characters.
In CASE I, we gave in Example 11.1 (§11.4) special spin IRs

$\Pi^{\rm I}_0$ of $\widetilde{G}^{\rm I}_n$, in case $n = 2n'$ is even,

$\Pi^{\rm I}_+$ of $\widetilde{G}^{\rm I}_n$, in case $n = 2n' + 1$ is odd,

and their characters $F^{\rm I}_{0,n}$ and $F^{\rm I}_{+,n}$ are given in Theorem 19.3 (§19.1). Note that if $F^{\rm I}_{0,n}(g') \neq 0$ $\big($resp. $F^{\rm I}_{+,n}(g') \neq 0\big)$, then $g' \in \widetilde{G}^{\rm I}_n$ satisfies (Condition I-00), and that (Condition I-00) does not contain any condition explicitly referring n. Then we see easily that, when $n' \to \infty$, these characters, after normalization, have naturally pointwise limits on $\widetilde{G}^{\rm I}_\infty = \lim_{n'\to\infty} \widetilde{G}^{\rm I}_{2n'} = \lim_{n'\to\infty} \widetilde{G}^{\rm I}_{2n'+1}$. We denote them by $\widetilde{F}^{\rm I}_{0,\infty}$ and $\widetilde{F}^{\rm I}_{+,\infty}$ respectively.

In CASE II, we gave special spin IRs in Example 12.2 (§12.4) as follows:

$\Pi^{\rm II}_0$ of $\widetilde{G}^{\rm II}_n = \widetilde{D}_n \overset{\rm II}{\rtimes} \widetilde{\mathfrak{S}}_n$, in case $n = 2n'$;

$\Pi^{{\rm II},\widetilde{H}}_+$ of $\widetilde{H}^{\rm II} = \widetilde{D}_n \overset{\rm II}{\rtimes} \widetilde{\mathfrak{A}}_n$, and $\Pi^{\rm II}_+ = {\rm Ind}^{\widetilde{G}^{\rm II}_n}_{\widetilde{H}^{\rm II}_n} \Pi^{{\rm II},\widetilde{H}}_+$ of $\widetilde{G}^{\rm II}_n$, in case $n=2n'+1$.

Their characters are denoted by $F^{\rm II}_{0,n}$, $F^{{\rm II},\widetilde{H}}_{+,n}$ and $F^{\rm II}_{+,n}$ respectively and are given in Theorem 20.2 (§20.1). Note that if $F^{\rm II}_{0,n}(g') \neq 0$ $\big($resp. $F^{{\rm II},\widetilde{H}}_{+,n}(g') \neq 0$ or $F^{\rm II}_{+,n}(g') \neq 0\big)$, then $g' \in \widetilde{G}^{\rm II}_n$ satisfies (Condition II-00), and that (Condition II-00) does not contain any condition explicitly referring n, then we see that, when $n' \to \infty$, these characters, after normalization, have naturally pointwise limits on $\widetilde{G}^{\rm II}_\infty = \lim_{n'\to\infty} \widetilde{G}^{\rm II}_{2n'} = \lim_{n'\to\infty} \widetilde{G}^{\rm II}_{2n'+1}$. We denote these limits by $\widetilde{F}^{\rm II}_{0,\infty}$, $\widetilde{F}^{{\rm II},\widetilde{H}}_{+,\infty}$ and $\widetilde{F}^{\rm II}_{+,\infty}$ respectively.

For ${\rm Y} = {\rm I}, {\rm II}$, denote by $\mathcal{O}^{\rm Y}$ the subset of $\widetilde{G}^{\rm Y}_\infty$ consisting of all g' which satisfy (Condition Y-00). Then, as we see above, the supports of limiting functions above are contained in $\mathcal{O}^{\rm Y}$ in CASE Y.

22 Limits of special spin irreducible characters

For Y=I, II, denote by $\mathcal{F}^{\rm Y}$ the set of all the limits of normalized spin irreducible characters of $\widetilde{G}^{Y}_n$ as $n \to \infty$. Then $\mathcal{F}^{\rm Y}$ consists of central positive definite functions on $\widetilde{G}^{Y}_\infty = \lim_{n\to\infty} \widetilde{G}^{\rm Y}_n$, and we see from Theorem 21.2 that $\mathcal{F}^{\rm Y}$ contains the set $E^{\rm Y}\big(\widetilde{G}^{\rm Y}_\infty\big)$ of all the characters of $\widetilde{G}^{\rm Y}_\infty$ of (spin) Type Y:

$$\mathcal{F}^{\rm Y} \supset E^{\rm Y}\big(\widetilde{G}^{\rm Y}_\infty\big) \qquad ({\rm Y{=}I,\ II}). \tag{22.1}$$

To obtain these sets $\mathcal{F}^{\mathrm{Y}}$, the following calculations of limits of normalized characters of special spin IRs play important roles as is explained in the first subsection below and will be seen in the next section. Our goal is to give explicit form of each limit functions $f \in \mathcal{F}^{\mathrm{Y}}$, and also to prove the equality $\mathcal{F}^{\mathrm{Y}} = E^{\mathrm{Y}}\big(\widetilde{G}^{\mathrm{Y}}_\infty\big)$, and thus to get character formulas.

22.1 General situation before taking limits

CASE I. For general spin IRs, we have in Theorem 19.9 tensor product structures as follows: let $\Lambda^n = \big(\lambda^{n,\zeta}\big)_{\zeta\in\widehat{T}^0} \in \boldsymbol{Y}_n(T)^0$ and $\gamma \in \Gamma^0_n$ subordinate to Λ^n, then

$$(22.2)\quad \begin{cases} \Pi^{\mathrm{I}}_{\Lambda^n} \cong \Pi^{\mathrm{I}}_0 \otimes \breve{\Pi}_{\gamma,\Lambda^n}, \quad F^{\mathrm{I}}_{\Lambda^n}(g') = F^{\mathrm{I}}_{0,n}(g') \times \breve{F}_{\gamma,\Lambda^n}(g), \quad \text{in case } n \geq 4 \text{ even}, \\ \Pi^{\mathrm{I}+}_{\Lambda^n} \cong \Pi^{\mathrm{I}}_+ \otimes \breve{\Pi}_{\gamma,\Lambda^n}, \quad F^{\mathrm{I}+}_{\Lambda^n}(g') = F^{\mathrm{I}}_{+,n}(g') \times \breve{F}_{\gamma,\Lambda^n}(g), \quad \text{in case } n \geq 5 \text{ odd}, \end{cases}$$

where $g' \in \widetilde{G}^{\mathrm{I}}_n$, $g = \Phi(g')$, and $\breve{F}_{\gamma,\Lambda^n}(g)$ denotes the character given in (19.19) of non-spin IR $\breve{\Pi}_{\gamma,\Lambda^n} = \mathrm{Ind}^{G_n}_{H_n} \breve{\pi}_{\gamma,\Lambda^n}$ of $G_n = \Phi\big(\widetilde{G}^{\mathrm{I}}_n\big)$ in (19.11).

Denote by $\breve{F}^{\sim}_{\gamma,\Lambda^n}$ the normalized character $\breve{F}_{\gamma,\Lambda^n}/\breve{F}_{\gamma,\Lambda^n}(e)$. Then we have the following product formula:

$$(22.3)\quad \begin{cases} \widetilde{F}^{\mathrm{I}}_{\Lambda^n}(g') = \widetilde{F}^{\mathrm{I}}_{0,n}(g') \times \breve{F}^{\sim}_{\gamma,\Lambda^n}(g), \quad \text{in case } n \geq 4 \text{ even}, \\ \widetilde{F}^{\mathrm{I}+}_{\Lambda^n}(g') = \widetilde{F}^{\mathrm{I}}_{+,n}(g') \times \breve{F}^{\sim}_{\gamma,\Lambda^n}(g), \quad \text{in case } n \geq 5 \text{ odd}. \end{cases}$$

CASE II. For general spin IRs, we have in Theorems 20.4, 20.9 and 20.13 tensor product structures as follows:

Case $n = 2n' \geq 4$ even:

For $\Lambda^n \in \boldsymbol{Y}_n(T)^0$, $\quad \Pi^{\mathrm{II}}_{\Lambda^n} \cong \Pi^{\mathrm{II}}_0 \otimes \breve{\Pi}_{\gamma,\Lambda^n}$ and $F^{\mathrm{II}}_{\Lambda^n} = F^{\mathrm{II}}_{0,n} \times \breve{F}_{\gamma,\Lambda^n}$.

Case $n = 2n'+1 \geq 5$ odd:

For $\{\Lambda^n, {}^t\Lambda^n\} \in \boldsymbol{Y}^{\mathfrak{A}}_n(T)^{0,1}$, $\quad \Pi^{\mho+}_{\Lambda^n} \cong \Pi^{\mathrm{II}}_+ \otimes \breve{\Pi}_{\gamma,\Lambda^n}$ and $F^{\mho+}_{\Lambda^n} = F^{\mathrm{II}}_{+,n} \times \breve{F}_{\gamma,\Lambda^n}$.

For $(\Lambda^n, \kappa) \in \boldsymbol{Y}^{\mathfrak{A}}_n(T)^{0,2}$, $\kappa = 0, 1$, for $g' = (d', \sigma') \in \widetilde{G}^{\mathrm{II}}_n$,

$$\text{if } \mathrm{ord}(d') \equiv 0 \pmod 2, \quad F^{\mho+}_{\Lambda^n,0}(g') = F^{\mho+}_{\Lambda^n,1}(g') = F^{\mathrm{II},\widetilde{H}}_{+,n}(g') \times \breve{F}_{\gamma,\Lambda^n}(g),$$

$$\text{if } \mathrm{ord}(d') \equiv 1 \pmod 2, \quad F^{\mho+}_{\Lambda^n,\kappa}(g') = F^{\mathrm{II},\widetilde{H}}_{+,n}(g') \times \big(\breve{F}^{H_n}_{\gamma,\Lambda^n,\kappa}(g) - \breve{F}^{H_n}_{\gamma,\Lambda^n,\kappa+1}(g)\big),$$

From the above list, together with the detailed data from Theorem 20.13, we obtain the following product formula:

In case $n \geq 4$ is even, $\quad \widetilde{F}^{\mathrm{II}}_{\Lambda^n}(g') = \widetilde{F}^{\mathrm{II}}_{0,n}(g') \times \breve{F}^{\sim}_{\gamma,\Lambda^n}(g), \quad \Lambda^n \in \boldsymbol{Y}_n(T)^0$;

In case $n \geq 5$ odd, $\quad \widetilde{F}^{\mathrm{II}}_{\Lambda^n} = \widetilde{F}^{\mathrm{II}}_{0,n} \times \breve{F}^{\sim}_{\gamma,\Lambda^n}, \quad \{\Lambda^n, {}^t\Lambda^n\} \in \boldsymbol{Y}^{\mathfrak{A}}_n(T)^{0,1}$, and

$$\widetilde{F}^{\mho+}_{\Lambda^n,\kappa}(g') = \widetilde{F}^{\mathrm{II},\widetilde{H}}_{+,n}(g') \times \breve{F}^{\sim}_{\gamma,\Lambda^n}(g), \quad (\Lambda^n, \kappa) \in \boldsymbol{Y}^{\mathfrak{A}}_n(T)^{0,2}, \text{ if } |\mathrm{supp}(g')| < n.$$

From the above product formulas, we see for each of CASEs I and II, the following.

On the subset $\mathcal{O}^{\mathrm{I}}$, for which $\mathcal{O}^{\mathrm{I}} \cap \widetilde{G}^{\mathrm{I}}_n$ contains the supports of $F^{\mathrm{I}}_{0,n}$ and $F^{\mathrm{I}}_{+,n}$, the study of limits of $\widetilde{F}^{\mathrm{I}}_{\Lambda^n}(g')$ and $\widetilde{F}^{\mathrm{I}+}_{\Lambda^n}(g')$ as $n = 2n' \to \infty$ or $n = 2n'+1 \to \infty$, is equivalent to that on limits of non-spin normalized irreducible characters $\breve{F}^{\sim}_{\gamma,\Lambda^n}(g)$ on the subsets $O^{\mathrm{I}} = \Phi_{\mathrm{I}}\big(\mathcal{O}^{\mathrm{I}}\big)$ of G_∞.

Similarly, on the subset $\mathcal{O}^{\mathrm{II}}$ containing the supports of $F^{\mathrm{II}}_{0,n}$ $(n = 2n')$ and $F^{\mathrm{II},\widetilde{H}}_{+,n}$ $(n = 2n' + 1)$ if $|\mathrm{supp}(g')| < n$, the study of limits of $\widetilde{F}^{\mathrm{II}}_{\Lambda^n}(g')$ and $\widetilde{F}^{\mho+}_{\Lambda^n,\kappa}(g')$, $\kappa = 0, 1$, as $n' \to \infty$, is equivalent to that on limits of non-spin normalized irreducible characters $\breve{F}_{\gamma,\Lambda^n}^{\sim}(g)$ on the subsets $O^{\mathrm{II}} = \Phi_{\mathrm{II}}(\mathcal{O}^{\mathrm{II}})$ of G_∞.

22.2 Explicit formulas for special limit characters

As indicated above, special limit functions $\widetilde{F}^{\mathrm{I}}_{0,\infty}$, $\widetilde{F}^{\mathrm{I}}_{+,\infty}$ etc. are important and also will be known to be characters on $\widetilde{G}^{\mathrm{I}}_\infty$ etc. Their explicit formulas follow from Theorems 19.3 and 20.2 respectively.

CASE I, Type $(-1, -1, -1)$:

Take $g' = (d', \sigma') \in \widetilde{G}^{\mathrm{I}}_\infty = \widetilde{D}_\infty \overset{\mathrm{I}}{\rtimes} \widetilde{\mathfrak{S}}_\infty$ normalized as before as

$$(22.4)\qquad \begin{cases} g' = \xi'_{q_1} \cdots \xi'_{q_r}\, g'_1 g'_2 \cdots g'_s, \;\; \xi'_q = {\eta'_q}^{a_q} \;(q \in Q = \{q_1, \ldots, q_r\}), \\ g'_j = (d'_j, \sigma'_j), \;\; d'_j = {\eta'_{k_j}}^{b_j} \;\big(\exists k_j \in K_j = \mathrm{supp}(g'_j),\; j \in J = \boldsymbol{I}_s\big)\,; \\ \sigma'_j = r_{n_j} r_{n_j+1} \ldots r_{n_j+\ell_j-2}, \;\; K_j = [n_j, n_j + \ell_j - 1]. \end{cases}$$

Lemma 22.3. (i) *If* $\mathrm{ord}(d') + L(\sigma') \equiv 0 \pmod 2$, *then* $\widetilde{F}^{\mathrm{I}}_{0,\infty}(g') \neq 0$ *if and only if (Condition I-00) holds for* g'. *In that case, with* $\ell_j = \ell(\sigma'_j)$,

$$(22.5)\qquad \widetilde{F}^{\mathrm{I}}_{0,\infty}(g') = \prod_{j \in J} (-1)^{[(\ell_j - 1)/2]}\, 2^{-(\ell_j - 1)/2}.$$

If $\mathrm{ord}(d') + L(\sigma') \equiv 1 \pmod 2$, *then* $\widetilde{F}^{\mathrm{I}}_{0,\infty}(g') = 0$ *identically.*

(ii) *There holds* $\widetilde{F}^{\mathrm{I}}_{0,\infty} = \widetilde{F}^{\mathrm{I}}_{+,\infty}$ *on the whole group* $\widetilde{G}^{\mathrm{I}}_\infty$.

CASE II, Type $(-1, -1, 1)$:

Let $g' = (d', \sigma') \in \widetilde{G}^{\mathrm{II}}_\infty = \widetilde{D}_\infty \overset{\mathrm{II}}{\rtimes} \widetilde{\mathfrak{S}}_\infty$ be normalized as

$$(22.6)\qquad \begin{cases} g' = \xi'_{q_1} \cdots \xi'_{q_r}\, g'_1 g'_2 \cdots g'_s, \;\; \xi'_q = \eta_q^{a_q} \;(q \in Q = \{q_1, \ldots, q_r\}), \\ g'_j = (d'_j, \sigma'_j), \;\; d'_j = \eta_{k_j}^{b_j} \;(\exists k_j \in K_j = \mathrm{supp}(g'_j),\; j \in J = \boldsymbol{I}_s)\,; \\ \sigma'_j = r_{n_j} r_{n_j+1} \ldots r_{n_j+\ell_j-2}, \;\; K_j = [n_j, n_j + \ell_j - 1]. \end{cases}$$

Lemma 22.4. (i) *If* $L(\sigma') \equiv 0 \pmod 2$, *then* $\widetilde{F}^{\mathrm{II}}_{0,\infty}(g') \neq 0$ *if and only if (Condition II-00) holds for* g'. *In that case,*

$$(22.7)\qquad \widetilde{F}^{\mathrm{II}}_{0,\infty}(g') = \varepsilon\big(\textstyle\prod_{j \in J_-} g'_j\big) \cdot \displaystyle\prod_{j \in J} (-1)^{(\ell_j - 1)/2}\, 2^{-(\ell_j - 1)/2},$$

where $\varepsilon\big(\prod_{j \in J_-} g'_j\big) := \prod_{\{j_1, j_2\} \sqsubset J_-} \varepsilon(g'_{j_1} g'_{j_2})$, *with the sign* $\varepsilon(g'_{j_1} g'_{j_2}) = \pm 1$ *defined by (17.22) – (17.24), and also by (17.25).*

If $L(\sigma') \equiv 1 \pmod 2$, *then* $\widetilde{F}^{\mathrm{II}}_{0,\infty}(g') = 0$ *identically.*

(ii) *Let* $g' = (g', \sigma') \in \widetilde{H}^{\mathrm{II}}_\infty = \widetilde{D}_\infty \overset{\mathrm{II}}{\rtimes} \widetilde{\mathfrak{A}}_\infty \subsetneq \widetilde{G}^{\mathrm{II}}_\infty$. *If* $\mathrm{ord}(d') \equiv 0 \pmod 2$, *then* $\widetilde{F}^{\mathrm{II},\widetilde{H}}_{+,\infty}(g') \neq 0$ *if and only if (Condition II-00) holds for* g'. *In that case,*

$$(22.8)\qquad \widetilde{F}^{\mathrm{II},\widetilde{H}}_{+,\infty}(g') = \varepsilon\big(\textstyle\prod_{j \in J_-} g'_j\big) \cdot \displaystyle\prod_{j \in J} (-1)^{(\ell_j - 1)/2}\, 2^{-(\ell_j - 1)/2}.$$

If $\mathrm{ord}(d') \equiv 1 \pmod 2$, *then* $\widetilde{F}^{\mathrm{II},\widetilde{H}}_{+,\infty}(g') = 0$ *identically.*

(iii) *Let* $g' = (d', \sigma') \in \widetilde{H}^{\mathrm{II}}_\infty$ *and* $\tau' \in \widetilde{\mathfrak{S}}_\infty \setminus \widetilde{\mathfrak{A}}_\infty$, *then* $\widetilde{F}^{\mathrm{II}}_{+,\infty}(g') = \widetilde{F}^{\mathrm{II},\widetilde{H}}_{+,\infty}(g')$, *and* $\widetilde{F}^{\mathrm{II}}_{+,\infty}(\tau' g') = 0$ *identically.*

22.3 Notes on representative elements of conjugacy classes

For the formula in Lemma 22.3, we take $g' = (d', \sigma') \in \widetilde{G}^{\mathrm{I}}_\infty = \widetilde{D}_\infty \overset{\mathrm{I}}{\rtimes} \widetilde{\mathfrak{S}}_\infty$ normalized as in (22.4) above.

Let us give a note on the normalization of d'_j in $g'_j = (d'_j, \sigma'_j)$ as $d'_j = {\eta'_{k_j}}^{b_j}$. As is seen above, it is enough to consider $\lim_{n\to\infty} \widetilde{F}^{\mathrm{I}}_{\Lambda^n}(g')$ under (Condition I-00). Now take a general $g' \in \widetilde{G}^{\mathrm{I}}_\infty$ for which d'_j's are not yet normalized. For any $h' = (d'', \sigma'') \in \widetilde{D}_{K_j} \overset{\mathrm{I}}{\rtimes} \widetilde{\mathfrak{S}}_{K_j}$, we have $h' \xi'_q h'^{-1} = \xi'_q$ and for $k \neq j$,

$$(22.9) \qquad h' g'_k h'^{-1} = z_1^{L(\sigma'_k)L(\sigma'')} z_2^{\mathrm{ord}(d'_k)L(\sigma'')} g'_k = \begin{cases} g'_k & \text{if } k \in J_+, \\ (z_1 z_2)^{L(\sigma'')} g'_k & \text{if } k \in J_-. \end{cases}$$

Moreover, for the transform $g'_j \mapsto h' g'_j h'^{-1}$, suppose $\sigma_j = (1\ 2\ \ldots\ \ell_j)$ and so $K_j = \boldsymbol{I}_{\ell_j}$. Then, for $g'_j = (d'_j, \sigma'_j)$ and $i \in K_j$,

$$(22.10) \qquad \eta'_i g'_j \eta'^{-1}_i = z_2^{L(\sigma'_j)} (\eta'_i d'_j \eta'_{i+1},\, \sigma'_j) = \begin{cases} (\eta'_i d'_j \eta'_{i+1},\, \sigma'_j) & \text{for } j \in J_+\,; \\ z_2 (\eta'_i d'_j \eta'_{i+1},\, \sigma'_j) & \text{for } j \in J_-\,. \end{cases}$$

Case of $j \in J_+ = \{j \in J\,;\, \mathrm{sgn}(\sigma'_j) = 1\}$**:** In this case, $L(\sigma'_j) \equiv 0$, $\mathrm{ord}(d'_j) \equiv 0 \pmod 2$. For example, for $d'_j = {\eta'_1}^{a_1} {\eta'_2}^{a_2} {\eta'_3}^{a_3}$, we have $\mathrm{ord}(d'_j) = a_1 + a_2 + a_3 \equiv 0$ and

$$\begin{aligned} {\eta'_1}^{-a_1} g'_j\, {\eta'_1}^{a_1} &= ({\eta'_2}^{a_2} {\eta'_3}^{a_3} {\eta'_2}^{a_1},\, \sigma'_j) = z_2^{a_1 a_3} ({\eta'_2}^{a_1+a_2} {\eta'_3}^{a_3},\, \sigma'_j), \\ {\eta'_2}^{-a_1-a_2} \big({\eta'_1}^{-a_1} g'_j\, {\eta'_1}^{a_1}\big) {\eta'_2}^{a_1+a_2} &= z_2^{a_1 a_3} ({\eta'_3}^{a_1+a_2+a_3},\, \sigma'_j), \end{aligned}$$

and $z_2^{a_1 a_3}$ can be equal to z_2 or to e under the fixed even $a_1 + a_2 + a_3$.

This explains the necessity to take d'_j in the normal form as ${\eta'_{k_j}}^{b_j}$, $b_j = \mathrm{ord}(d'_j)$.

Moreover note that $\sigma'_j {\eta'_{k_j}}^{b_j} {\sigma'_j}^{-1} = {\eta'_{k_j+1}}^{b_j}$, then we see that the choice of $k_j \in K_j$ may be arbitrary.

Case of $j \in J_- = \{j \in J\,;\, \mathrm{sgn}(\sigma'_j) = -1\}$**:** In this case, $L(\sigma'_j) \equiv 1$, $\mathrm{ord}(d'_j) \equiv 1 \pmod 2$. Note that, for $d'_j = {\eta'_1}^{a_1} {\eta'_2}^{a_2} {\eta'_3}^{a_3}$, we have $\mathrm{ord}(d'_j) = a_1 + a_2 + a_3 \equiv 1$ and

$$\begin{aligned} {\eta'_1}^{-a_1} g'_j\, {\eta'_1}^{a_1} &= z_2^{a_1} ({\eta'_2}^{a_2} {\eta'_3}^{a_3} {\eta'_2}^{a_1},\, \sigma'_j) = z_2^{a_1 + a_1 a_3} ({\eta'_2}^{a_1+a_2} {\eta'_3}^{a_3},\, \sigma'_j), \\ {\eta'_2}^{-a_1-a_2} \big({\eta'_1}^{-a_1} g'_j\, {\eta'_1}^{a_1}\big) {\eta'_2}^{a_1+a_2} &= z_2^{a_1 + a_1 a_3} z_2^{a_1+a_2} ({\eta'_3}^{a_1+a_2+a_3},\, \sigma'_j), \end{aligned}$$

and $z_2^{a_1 + a_1 a_3} z_2^{a_1+a_2} = z_2^{a_1 a_3 + a_2}$ can be equal to z_2 or to e under the fixed odd $a_1 + a_2 + a_3$. This explains the necessity to take d'_j in the normal form as ${\eta'_{k_j}}^{b_j}$.

Moreover we have $\sigma'_j {\eta'_{k_j}}^{b_j} {\sigma'_j}^{-1} = z_2 {\eta'_{k_j+1}}^{b_j}$ or $\sigma'_j {\eta_{k_j}}^{b_j} {\sigma'_j}^{-1} = {\eta_{k_j+1}}^{b_j}$. If $\gamma_{k_j} = \gamma_{k_j+1}$, then since $b_j \equiv 1 \pmod 2$,

$$\chi_\gamma\big({\eta'_{k_j}}^{b_j}\big) = (-1)^{(k_j-1)b_j} \omega^{\gamma_{k_j}} = -(-1)^{(k_j+1-1)b_j} \omega^{\gamma_{k_j+1}} = \chi_\gamma\big(z_2 {\eta'_{k_j+1}}^{b_j}\big).$$

Note that this equality has an important meaning with respect to the formula of $\mathrm{tr}\big(\pi^{\mathrm{I}}_\gamma(g'_j)\big)$ in Lemma 16.5, for which we should pay attention on the factor $\chi_\gamma(d'_j)$ at the rightmost hand side of the formula (16.32).

On the other hand, we have $\chi_\gamma\big({\eta'_{k_j}}^{b_j}\big) = -\chi_\gamma\big({\eta'_{k_j+1}}^{b_j}\big)$. This is consistent with the property of a central positive definite function f on $\widetilde{G}^{\mathrm{I}}_\infty$ of type $(-1,-1,-1)$ such that $f(z_2 g') = -f(g')$. In more detail, for the conjugacy class $[g'_j]_{\widetilde{Z}}$ of g'_j modulo $\widetilde{Z}$ under $\widetilde{D}_{K_j} \overset{\mathrm{I}}{\rtimes} \widetilde{\mathfrak{S}}_{K_j}$, we can take, as its representative, $g^{(1)}_j = \big({\eta'_{k_j}}^{b_j},\, \sigma'_j\big)$ and also $g^{(2)}_j = \big({\eta'_{k_j+1}}^{b_j},\, \sigma'_j\big)$, which are conjugate to each other modulo $\langle z_2\rangle$ under conjugation by $\widetilde{\mathfrak{S}}_{K_j}$ as $\sigma'_j g^{(1)}_j {\sigma'_j}^{-1} = z_2 g^{(2)}_j$.

23 Limits of spin irreducible characters

23.1 Limits of non-spin irreducible characters $\breve{F}_{\gamma,\Lambda^n}$ of G_n

For the completeness, we refer here briefly §§5–6 in [HHH1] in the simpler case of $G_n = D_n(\boldsymbol{Z}_m) \rtimes \mathfrak{S}_n$ and $G_\infty = D_\infty(\boldsymbol{Z}_m) \rtimes \mathfrak{S}_\infty$. Put, for a non-spin IR $\pi_{\lambda^{n,\zeta}}$ of $\mathfrak{S}_{I_{n,\zeta}}$,

$$\widetilde{\chi}(\lambda^{n,\zeta}\,;\sigma) := \frac{\mathrm{tr}\big(\pi_{\lambda^{n,\zeta}}(\sigma)\big)}{\dim\,\pi_{\lambda^{n,\zeta}}} \qquad \big(\sigma \in \mathfrak{S}_{I_{n,\zeta}}\big),$$

and when σ is a product of disjoint cycles of lengths ℓ_j $(j \in J_\zeta)$, it is denoted also by $\widetilde{\chi}\big(\lambda^{n,\zeta}\,;\, (\ell_j)_{j\in J_\zeta}\big)$. Then, for non-spin IRs Π_{γ,Λ^n} of G_n, with $\Lambda^n = \big(\lambda^{n,\zeta}\big)_{\zeta\in\widehat{T}^0} \in \boldsymbol{Y}_n(T)^0$, its normalized character is given as follows (cf. §19.3).

With a partition $\mathcal{I}_n = (I_{n,\zeta})_{\zeta\in\widehat{T}^0}$ of $\boldsymbol{I}_n$, and partitions $\mathcal{Q} = (Q_\zeta)_{\zeta\in\widehat{T}^0}$ of $Q = \{q_1,\ldots,q_r\}$, and $\mathcal{J} = (J_\zeta)_{\zeta\in\widehat{T}^0}$ of $J = \{1,\ldots,s\}$ satisfying the condition (Condition QJ) in §19.3, we define a function of $g \in G_n$ as

$$\begin{aligned} c(\mathcal{I}_n\,;\,\mathcal{Q},\,\mathcal{J}\,;\,g) \;&:=\; \frac{n^{|\mathrm{supp}(g)|}}{n(n-1)\cdots(n-|\mathrm{supp}(g)|+1)} \times \\ &\times \prod_{\zeta\in\widehat{T}^0} \frac{|I_{n,\zeta}|}{n}\cdot\frac{|I_{n,\zeta}|-1}{n}\cdot\cdots\cdot\frac{|I_{n,\zeta}|-|Q_\zeta|-\sum_{j\in J_\zeta}\ell_j+1}{n}. \\ \widetilde{X}\big(\Lambda^n\,;\,\mathcal{Q},\,\mathcal{J}\,;\,g\big) &:= \prod_{\zeta\in\widehat{T}^0}\Big(\zeta\big(\textstyle\prod_{q\in Q_\zeta}\xi_q\cdot\prod_{j\in J_\zeta} d_j\big)\times\widetilde{\chi}\big(\lambda^{n,\zeta};(\ell_j)_{j\in J_\zeta}\big)\Big). \end{aligned}$$

$$\text{(23.1)} \qquad \breve{F}^{\sim}_{\gamma,\Lambda^n}(g) = \frac{\breve{F}_{\gamma,\Lambda^n}(g)}{\breve{F}_{\gamma,\Lambda^n}(e)} = \sum_{(\mathcal{Q},\mathcal{J})} c(\mathcal{I}_n;\mathcal{Q},\mathcal{J};g)\,\widetilde{X}\big(\Lambda^n\,;\,\mathcal{Q},\,\mathcal{J}\,;\,g\big).$$

First consider $\breve{F}^{\sim}_{\gamma,\Lambda^n}$ on the subgroup $D_n \subset G_n$ through the above formula.

Lemma 23.1. *To have a pointwise limit* $\lim_{n\to\infty}\breve{F}^{\sim}_{\gamma,\Lambda^n}(d)$ *on* D_n, *it is necessary and sufficient that the following limits exist : for every* $\zeta\in\widehat{T}^0$,

$$\text{(23.2)} \qquad \exists\ \lim_{n\to\infty}\frac{|I_{n,\zeta}|}{n}\ \ \big(=: B_\zeta\ \ (\text{put})\big).$$

In that case, $\sum_{\zeta\in\widehat{T}^0} B_\zeta = 1$.

On the other hand, we know the following about the asymptotic behavior of irreducible characters of $\mathfrak{S}_n$ (cf. [VK] or [HHH1, Theorem 5.4]). For a Young diagram λ^n of size n, let lengths of its rows and columns be $r_i(\lambda^n),\ c_i(\lambda^n)$ $(i\in \boldsymbol{I}_n)$ respectively, and put

$$a_i(\lambda^n) := r_i(\lambda^n) - i + 1/2,\ \ b_i(\lambda^n) := c_i(\lambda^n) - i + 1/2 \quad (i\in \boldsymbol{I}_n). \tag{23.3}$$

Lemma 23.2. *Suppose that the following limits exist:*

$$\alpha'_i := \lim_{n\to\infty} \frac{a_i(\lambda^n)}{n},\quad \beta'_i := \lim_{n\to\infty} \frac{b_i(\lambda^n)}{n} \quad (i\in \boldsymbol{N}). \tag{23.4}$$

Then there exists the pointwise limit of $\widetilde{\chi}\big(\lambda^n;(\ell_j)_{j\in J}\big)$ *as*

$$\lim_{n\to\infty} \widetilde{\chi}\big(\lambda^n;(\ell_j)_{j\in J}\big) = \prod_{j\in J}\Big(\sum_{i\in \boldsymbol{N}} {\alpha'_i}^{\ell_j} + (-1)^{\ell_j-1}\sum_{i\in \boldsymbol{N}} {\beta'_i}^{\ell_j}\Big). \tag{23.5}$$

Lemma 23.3. *Assume that* $B_\zeta = \lim_{n\to\infty} |I_{n,\zeta}|/n$ *exists for any* $\zeta\in\widehat{T}^0$*, and that, for every* $\zeta\in\widehat{T}^0$ *with* $B_\zeta > 0$*, the following limits exist:*

$$\alpha'_{\zeta,i} := \lim_{n\to\infty} \frac{a_i(\lambda^{n,\zeta})}{|I_{n,\zeta}|},\ \ \beta'_{\zeta,i} := \lim_{n\to\infty} \frac{b_i(\lambda^{n,\zeta})}{|I_{n,\zeta}|} \quad (i\in \boldsymbol{N}). \tag{23.6}$$

Then the limit of $\breve{F}^{\sim}_{\gamma,\Lambda^n}(g)$ *as* $n\to\infty$ *exists and*

$$\lim_{n\to\infty} \breve{F}^{\sim}_{\gamma,\Lambda^n}(g) = \sum_{\mathcal{Q},\mathcal{J}} \prod_{\zeta\in\widehat{T}^0} \Big\{\zeta\Big(\prod_{q\in Q_\zeta}\xi_q\cdot\prod_{j\in J_\zeta} d_j\Big)\cdot \cdot B_\zeta^{|Q_\zeta|}\cdot \prod_{j\in J_\zeta} B_\zeta^{\ell_j}\cdot\Big(\sum_{i\in\boldsymbol{N}} {\alpha'_{\zeta,i}}^{\ell_j} + (-1)^{\ell_j-1}\sum_{i\in\boldsymbol{N}} {\beta'_{\zeta,i}}^{\ell_j}\Big)\Big\}. \tag{23.7}$$

Put

$$\begin{cases} \alpha_{\zeta,i} := B_\zeta\cdot\alpha'_{\zeta,i}\,,\ \ \beta_{\zeta,i} := B_\zeta\cdot\beta'_{\zeta,i} & (i\in\boldsymbol{N}),\\ \mu_\zeta := B_\zeta - \sum_{i\in\boldsymbol{N}}\alpha_{\zeta,i} - \sum_{i\in\boldsymbol{N}}\beta_{\zeta,i} & (\zeta\in\widehat{T}^0), \end{cases} \tag{23.8}$$

and reorder $(\alpha_{\zeta,i})_{i\in\boldsymbol{N}}$, $(\beta_{\zeta,i})_{i\in\boldsymbol{N}}$ in the descending order, and then introduce a parameter A as

$$A = (\alpha,\beta\,;\mu),\quad \alpha = \big(\alpha_\zeta\big)_{\zeta\in\widehat{T}^0},\ \beta = \big(\beta_\zeta\big)_{\zeta\in\widehat{T}^0},\ \mu = \big(\mu_\zeta\big)_{\zeta\in\widehat{T}^0}\,; \tag{23.9}$$

$$\begin{cases} \alpha_\zeta = (\alpha_{\zeta,i})_{i\in\boldsymbol{N}},\ \ \alpha_{\zeta,1}\ge\alpha_{\zeta,2}\ge\alpha_{\zeta,3}\ge\cdots\ge 0\,,\\ \beta_\zeta = (\beta_{\zeta,i})_{i\in\boldsymbol{N}},\ \ \beta_{\zeta,1}\ge\beta_{\zeta,2}\ge\beta_{\zeta,3}\ge\cdots\ge 0\,, \end{cases} \qquad \mu_\zeta \ge 0\,, \tag{23.10}$$

$$\sum_{\zeta\in\widehat{T}^0}\Big(\|\alpha_\zeta\| + \|\beta_\zeta\| + \mu_\zeta\Big) = 1,\quad \|\alpha_\zeta\| := \sum_{i\in\boldsymbol{N}}\alpha_{\zeta,i}\,. \tag{23.11}$$

Then we have the following result on as a special case of Theorem 7.1 in [HHH1].

Theorem 23.4. (i) *Assume that* $B_\zeta = \lim_{n\to\infty} |I_{n,\zeta}|/n$ *exists for any* $\zeta \in \widehat{T}^0$, *and that, for any* $\zeta \in \widehat{T}^0$ *with* $B_\zeta > 0$, *the limits in Lemma 23.3 exist. Then, for* $g = \prod_{q\in Q} \xi_q \cdot \prod_{j\in J} g_j$, $g_j = (d_j, \sigma_j)$, $\ell_j = \ell(\sigma_j)$,

$$\lim_{n\to\infty} \breve{F}^{\sim}_{\gamma,\Lambda^n}(g) = \sum_{\mathcal{Q},\mathcal{J}} \prod_{\zeta\in\widehat{T}^0} \Big\{ \prod_{q\in Q_\zeta} \Big(\sum_{i\in\boldsymbol{N}} \alpha_{\zeta,i} + \sum_{i\in\boldsymbol{N}} \beta_{\zeta,i} + \mu_\zeta \Big) \zeta(\xi_q) \times \times \prod_{j\in J_\zeta} \Big(\sum_{i\in\boldsymbol{N}} \alpha_{\zeta,i}^{\ell_j} + (-1)^{\ell_j-1} \sum_{i\in\boldsymbol{N}} \beta_{\zeta,i}^{\ell_j} \Big) \zeta(d_j) \Big\}, \tag{23.12}$$

where $(\mathcal{Q},\mathcal{J})$ *runs over all pairs of partitions* $\mathcal{Q} = (Q_\zeta)_{\zeta\in\widehat{T}^0}$ *of* $Q = \{q_1,\dots,q_r\}$ *and* $\mathcal{J} = (J_\zeta)_{\zeta\in\widehat{T}^0}$ *of* $J = \{1,\dots,s\}$. *The right hand side is a function* f_A *with the parameter* $A = (\alpha,\beta\,;\mu)$ *given as*

$$f_A(g) = \prod_{q\in Q} f_A(\xi_q) \cdot \prod_{j\in J} f_A(g_j)\,, \tag{23.13}$$

$$f_A(\xi_q) = \sum_{\zeta\in\widehat{T}^0} \Big(\|\alpha_\zeta\| + \|\beta_\zeta\| + \mu_\zeta \Big)\, \zeta(\xi_q), \tag{23.14}$$

$$f_A(g_j) = \sum_{\zeta\in\widehat{T}^0} \Big(\sum_{i\in\boldsymbol{N}} \alpha_{\zeta,i}^{\ell_j} + (-1)^{\ell_j-1} \sum_{i\in\boldsymbol{N}} \beta_{\zeta,i}^{\ell_j} \Big)\, \zeta(d_j). \tag{23.15}$$

(ii) *Put*

$$\begin{aligned} &\mathcal{A}(\widehat{T}^0) := \{A = (\alpha,\beta\,;\mu)\,;\ \text{parameter satisfying (23.9)–(23.11)}\}, \\ &\mathcal{F}(\widehat{T}^0) := \{f_A\,;\ A \in \mathcal{A}(\widehat{T}^0)\}. \end{aligned} \tag{23.16}$$

Then $\mathcal{F}(\widehat{T}^0)$ *is the set of all pointwise limits of normalized irreducible characters* $\breve{F}^{\sim}_{\gamma,\Lambda^n}$ *as* $n \to \infty$.

Put ${}^tA := (\beta,\alpha\,;\mu)$, then $\tau :\ A \mapsto {}^tA$ is an involutive action on $\mathcal{A}(\widehat{T}^0)$

Lemma 23.5. (i) *For* $A \in \mathcal{A}(\widehat{T}^0)$, *the restriction of* f_A *onto the subset* $O^{\mathrm{I}} \subset G_\infty$ *determines its parameter* A *completely. In other words, let* $A' = (\alpha',\beta';\mu') \in \mathcal{A}(\widehat{T}^0)$ *be another parameter such that* $f_{A'}\big|_{O^{\mathrm{I}}} = f_A\big|_{O^{\mathrm{I}}}$, *then* $A' = A$.

(ii) *For* $A, A' \in \mathcal{A}(\widehat{T}^0)$, *their restrictions on the subset* $O^{\mathrm{II}} \subset G_\infty$ *coincide with each other, or* $f_{A'}\big|_{O^{\mathrm{II}}} = f_A\big|_{O^{\mathrm{II}}}$, *if and only if* $A' = A$ *or* $A' = {}^tA = \tau(A)$.

Proof. (i) is proved by using the following equality: for $z \in \boldsymbol{C}$,

$$\exp\Big\{ \sum_{\ell\ge 2} \sum_{i\in\boldsymbol{N}} \big(\alpha_i^\ell + (-1)^{\ell-1}\beta_i^\ell\big) \frac{z^\ell}{\ell} \Big\} = \exp\Big\{ -z \sum_{i\in\boldsymbol{N}} (\alpha_i + \beta_i) \Big\} \prod_{i\in\boldsymbol{N}} \frac{1+\beta_i z}{1-\alpha_i z}\,.$$

(ii) The equality $f_{A'}\big|_{O^{\mathrm{II}}} = f_A\big|_{O^{\mathrm{II}}}$ holds if and only if the following equalities hold. For $\zeta \in \widehat{T}^0$, $\ell \ge 1$, with

$$C(\zeta,\ell) := \sum_{i\in\boldsymbol{N}} \alpha_{\zeta,i}^\ell + (-1)^{\ell-1} \sum_{i\in\boldsymbol{N}} \beta_{\zeta,i}^\ell,\ \ C'(\zeta,\ell) := \sum_{i\in\boldsymbol{N}} {\alpha'}_{\zeta,i}^{\ell} + (-1)^{\ell-1} \sum_{i\in\boldsymbol{N}} {\beta'}_{\zeta,i}^{\ell},$$

$$\begin{cases} C(\zeta,1)+\mu_\zeta = C'(\zeta,1)+\mu'_\zeta & (\ell=1)\ ; \\ C(\zeta,\ell) = C'(\zeta,\ell) & (\ell \geq 3\ \text{ odd})\ ; \\ C(\zeta,\ell)\,C(\zeta,k) = C'(\zeta,\ell)\,C'(\zeta,k) & (\ell,k \geq 2\ \text{ even}). \end{cases}$$

From the last equation we have either $C(\zeta,\ell) = C'(\zeta,\ell)$ or $C(\zeta,\ell) = -C'(\zeta,\ell)$ for any $\zeta \in \widehat{T}^0$, $\ell \geq 3$ odd. Actually the former gives us $A' = A$ and the latter gives $A' = {}^tA$. We omit the details. □

23.2 Limits of spin irreducible characters for (CASE I)

Under the condition $|\mathrm{supp}(g')| < n$, the characters $F^{\mathrm{I}}_{\Lambda^n}$ and $F^{\mathrm{I}+}_{\Lambda^n}$, with $\Lambda^n \in \boldsymbol{Y}_n(T)^0$, have the same form. If $g' \in \widetilde{G}^{\mathrm{I}}_n$ satisfies (Condition I-00), then the normalized characters corresponding to them are expressed as

$$\widetilde{F}^{\mathrm{I}}_{0,n}(g') \times \breve{F}^{\sim}_{\gamma,\Lambda^n}(g), \tag{23.17}$$

where $g = \Phi(g')$ and $\breve{F}^{\sim}_{\gamma,\Lambda^n}(g)$ is given in (23.1). Then we get the following result on the limit of normalized spin irreducible characters of $\widetilde{G}^{\mathrm{I}}_n$ as $n \to \infty$. Note that this limit concerns essentially on the subsets $\mathcal{O}^{\mathrm{I}} \cap \widetilde{G}^{\mathrm{I}}_n$ of $\widetilde{G}^{\mathrm{I}}_n$ and $\mathcal{O}^{\mathrm{I}}$ of $\widetilde{G}^{\mathrm{I}}_\infty$ both defined by (Condition I-00).

Theorem 23.6 (CASE I). *Assume that $B_\zeta = \lim_{n\to\infty} |I_{n,\zeta}|/n$ exists for any $\zeta \in \widehat{T}^0$, and that, for any $\zeta \in \widehat{T}^0$ with $B_\zeta > 0$, the limits in (23.4) exist.*

(i) *The normalized spin irreducible characters $\widetilde{F}^{\mathrm{I}}_{\Lambda^n}(g')$ have a limit $f^{\mathrm{I}}_A(g') := \lim_{n\to\infty} \widetilde{F}^{\mathrm{I}}_{\Lambda^n}(g')$ with a parameter $A \in \mathcal{A}(\widehat{T}^0)$ as*

$$f^{\mathrm{I}}_A(g') = \widetilde{F}^{\mathrm{I}}_{0,\infty}(g') \times f_A(g). \tag{23.18}$$

(ii) *In another expression, the limit function f^{I}_A is factorizable in the sense that, for $g' = \xi'_{q_1}\xi'_{q_2}\cdots\xi'_{q_r}\,g'_1 g'_2\cdots g'_s$ in (22.4),*

$$f^{\mathrm{I}}_A(g') = \prod_{q\in Q} f^{\mathrm{I}}_A(\xi'_q)\cdot\prod_{j\in J} f^{\mathrm{I}}_A(g'_j). \tag{23.19}$$

And $f^{\mathrm{I}}_A(g') \neq 0$ only if g' satisfies (Condition I-00) or $g' \in \mathcal{O}^{\mathrm{I}}$, and for $\xi'_q = {\eta'_q}^{a_q}$ $\big(\mathrm{ord}(\xi'_q) \equiv 0 \ (\mathrm{mod}\ 2)\big)$, $g'_j = (d'_j,\sigma'_j)$ $\big(\mathrm{ord}(d'_j) + L(\sigma'_j) \equiv 0\ (\mathrm{mod}\ 2)\big)$ with $d'_j = {\eta'_{k_j}}^{b_j}$ $(\exists k_j \in K_j)$,

$$f^{\mathrm{I}}_A(\xi'_q) = \sum_{\zeta\in\widehat{T}^0} \Big(\|\alpha_\zeta\| + \|\beta_\zeta\| + \mu_\zeta\Big)\,\zeta(\xi'_q), \tag{23.20}$$

$$f^{\mathrm{I}}_A(g'_j) = \sum_{\zeta\in\widehat{T}^0} (-1)^{[(\ell_j-1)/2]}\,2^{-(\ell_j-1)/2}\Big(\sum_{i\in\boldsymbol{N}} {\alpha_{\zeta,i}}^{\ell_j} + (-1)^{\ell_j-1}\sum_{i\in\boldsymbol{N}} {\beta_{\zeta,i}}^{\ell_j}\Big)\,\zeta(d'_j). \tag{23.21}$$

If g' does not satisfy (Condition I-00), then $f^{\mathrm{I}}_A(g') = 0$.

(iii) *Let $\mathcal{F}^{\mathrm{I}}$ be the set of all the limits of normalized spin irreducible characters on $\widetilde{G}^{\mathrm{I}}_n$ of Type $(-1,-1,-1)$ as $n \to \infty$. Then*

$$\mathcal{F}^{\mathrm{I}} = \big\{f^{\mathrm{I}}_A\ ;\ A \in \mathcal{A}(\widehat{T}^0)\big\} = \big\{\widetilde{F}^{\mathrm{I}}_{0,\infty}\cdot f_A\ ;\ A \in \mathcal{A}(\widehat{T}^0)\big\}. \tag{23.22}$$

23.3 Limits of spin irreducible characters (CASE II)

Recall that (Condition II-00) in §21.1 is used both in Case $n = 2n'$ even and in Case $n = 2n' + 1$ odd in common.

23.3.1. Case $n = 2n' \geq 4$ even.

By Theorem 20.4, irreducible spin characters are given by $F^{\mathrm{II}}_{\Lambda^n} = \mathrm{tr}\big(\Pi^{\mathrm{II}}_{\Lambda^n}\big)$ with $\Lambda^n = (\lambda^{n,\zeta})_{\zeta\in\widehat{T}^0} \in \boldsymbol{Y}_n(T)^0$, and that $F^{\mathrm{II}}_{\Lambda^n} = F^{\mathrm{II}}_{0,n} \times \breve{F}_{\gamma,\Lambda^n}$ on $\widetilde{G}^{\mathrm{II}}_n$, and moreover that, if $|\mathrm{supp}(g')| < n$, then $F^{\mathrm{II}}_{\Lambda^n}(g') \neq 0$ only if (Condition II-00) holds for g'. The normalized character is given as

$$\widetilde{F}^{\mathrm{II}}_{\Lambda^n}(g') = \widetilde{F}^{\mathrm{II}}_{0,n}(g') \times \breve{F}^{\sim}_{\gamma,\Lambda^n}(g). \tag{23.23}$$

23.3.2. Case $n = 2n' + 1 \geq 5$ odd.

(1) Let $\{\Lambda^n, {}^t\Lambda^n\} \in \boldsymbol{Y}^{\mathfrak{A}}_n(T)^{0,1}$. Then, from Theorem 20.9, the corresponding irreducible spin characters are given as $F^{\mho+}_{\Lambda^n} = F^{\mho+}_{{}^t\Lambda^n}$ and

$$F^{\mho+}_{\Lambda^n}(g') = F^{\mathrm{II}}_{0,n}(g') \times \breve{F}_{\gamma,\Lambda^n}(g) \quad (g' \in \widetilde{G}^{\mathrm{II}}_n,\ g = \Phi(g') \in G_n)$$

and the normalized ones are

$$\widetilde{F}^{\mho+}_{\Lambda^n}(g') = \widetilde{F}^{\mathrm{II}}_{0,n}(g') \times \breve{F}^{\sim}_{\gamma,\Lambda^n}(g).$$

Moreover, if $|\mathrm{supp}(g')| < n$, then $F^{\mho+}_{\Lambda^n}(g') \neq 0$ only if g' satisfies (Condition II-00).

(2) Let $(\Lambda^n, \kappa) \in \boldsymbol{Y}^{\mathfrak{A}}_n(T)^{0,2}$ with ${}^t\Lambda^n = \Lambda^n$, $\kappa = 0, 1$. Then, by Theorem 20.2 and 20.13, under $\mathrm{ord}(d') \equiv 0 \pmod 2$, $F^{\mho+}_{\Lambda^n,\kappa}(g') \neq 0$ only if (Condition II-00) holds for g', and $F^{\mho+}_{\Lambda^n,0}(g') = F^{\mho+}_{\Lambda^n,1}(g')$,

$$F^{\mho+}_{\Lambda^n,0}(g') = F^{\mathrm{II},\widetilde{H}}_{+,n}(g') \times \breve{F}_{\gamma,\Lambda^n}(g),$$

and, under $\mathrm{ord}(d') \equiv 0 \pmod 2$, if $|\mathrm{supp}(g')| < n$, then $F^{\mho+}_{\Lambda^n,\kappa}(g') = 0$. Therefore, if $|\mathrm{supp}(g')| < n$, the normalized ones is

$$\widetilde{F}^{\mho+}_{\Lambda^n,0}(g') = \widetilde{F}^{\mathrm{II}}_{+,n}(g') \times \breve{F}^{\sim}_{\gamma,\Lambda^n}(g) \quad (g' \in \widetilde{G}^{\mathrm{II}}_n).$$

We see from Theorem 20.2 that, if $|\mathrm{supp}(g')| < n$, then $\widetilde{F}^{\mathrm{II}}_{+,n}(g') = \widetilde{F}^{\mathrm{II}}_{0,n}(g')$, and also by Lemma 22.4 that, for the limits as $n \to \infty$, $\widetilde{F}^{\mathrm{II}}_{0,\infty} = \widetilde{F}^{\mathrm{II}}_{+,\infty}$ on $\widetilde{G}^{\mathrm{II}}_\infty$.

23.3.3. Limits of normalized spin characters (CASE II).

As is seen above, the normalized spin irreducible characters $\widetilde{F}^{\mathrm{II}}_{\Lambda^n}(g')$, $\widetilde{F}^{\mho+}_{\Lambda^n}(g')$, and $\widetilde{F}^{\mho+}_{\Lambda^n,0}(g')$, under $|\mathrm{supp}(g')| < n$, are all of the same form: $\widetilde{F}^{\mathrm{II}}_{0,n}(g') \times \breve{F}^{\sim}_{\gamma,\Lambda^n}(g)$, which is not zero only when g' satisfies (Condition II-00). Therefore, on the subset $O^{\mathrm{II}} = \bigcup_{4\leq n<\infty} O^{\mathrm{II}}_n$ of $\widetilde{G}^{\mathrm{II}}_\infty$, the existence of the limit as $n \to \infty$ depends completely on that of the series of non-spin irreducible characters $\breve{F}^{\sim}_{\gamma,\Lambda^n}(g)$.

The latter is discussed in Theorem 23.4, and we get the following.

Theorem 23.7 (CASE II). *Assume that $B_\zeta = \lim_{n\to\infty} |I_{n,\zeta}|/n$ exists for any $\zeta \in \widehat{T}^0$, and that, for any $\zeta \in \widehat{T}^0$ with $B_\zeta > 0$, the limits in (23.4) exist.*

(i) *The series of normalized spin irreducible characters* $\widetilde{F}^{\rm II}_{\Lambda^n}(g')$, $\widetilde{F}^{\mho+}_{\Lambda^n}(g')$, *and* $\widetilde{F}^{\mho+}_{\Lambda^n,0}(g')$ *have the same limit* $f^{\rm II}_A(g')$ *with a parameter* $A \in \mathcal{A}(\widehat{T}^0)$ *written as*

$$f^{\rm II}_A(g') = \widetilde{F}^{\rm II}_{0,\infty}(g') \times f_A(g). \tag{23.24}$$

(ii) *In another expression, the limit function* $f^{\rm II}_A$ *is **weaklyfactorizable** in the following sense. For* $g' = \xi'_{q_1}\xi'_{q_2}\cdots\xi'_{q_r}\, g'_1 g'_2 \cdots g'_s \in \widetilde{G}^{\rm II}_\infty = \bigcup_{4\leqslant n<\infty} \widetilde{G}^{\rm II}_n$, *put* $Q = \{q_1, \ldots, q_r\}$, $J = \boldsymbol{I}_s$, $J_\pm = \{j \in J\,;\, \mathrm{sgn}(\sigma'_j) = \pm 1\}$. *Suppose* g' *satisfies (Condition II-00) or* $g' \in \mathcal{O}^{\rm II}$, *then* $|J_-|$ *is even, and decompose* J_- *into disjoint pairs as* $J_- = \bigsqcup\{j_1, j_2\}$, *expressed as* $\{j_1, j_2\} \sqsubset J_-$, *and*

$$f^{\rm II}_A(g') = \prod_{q\in Q} f^{\rm II}_A(\xi'_q)\cdot \prod_{j\in J_+} f^{\rm II}_A(g'_j)\cdot \prod_{\{j_1,j_2\}\sqsubset J_-} f^{\rm II}_A(g'_{j_1}g'_{j_2}), \tag{23.25}$$

where each factors are given as, after assuming that the expression of g' *is normalized, modulo* $z_1{}^a z_2{}^b$, *as in (22.6),*

$$f^{\rm II}_A(\xi'_q) = \sum_{\zeta\in\widehat{T}^0} \Big(\|\alpha_\zeta\| + \|\beta_\zeta\| + \mu_\zeta\Big)\,\zeta(\xi'_q) \qquad \text{for } q \in Q,$$

$$f^{\rm II}_A(g'_j) = \sum_{\zeta\in\widehat{T}^0} (-1)^{(\ell_j-1)/2}\, 2^{(\ell_j-1)/2}\Big(\sum_{i\in\boldsymbol{N}} \alpha_{\zeta,i}{}^{\ell_j} + \sum_{i\in\boldsymbol{N}} \beta_{\zeta,i}{}^{\ell_j}\Big)\,\zeta(d'_j) \quad \text{for } j \in J_+,$$

and for $\{j_1, j_2\} \sqsubset J_-$, *with the sign* $\varepsilon(g_{j_1}g_{j_2})$ *in (17.22)–(17.24), or in (17.25),*

$$f^{\rm II}_A(g'_{j_1}g'_{j_2}) = \varepsilon(g_{j_1}g_{j_2})\cdot \prod_{j=j_1,j_2}\Big\{\sum_{\zeta\in\widehat{T}^0} (-1)^{\ell_j/2-1}\, 2^{(\ell_j-1)/2}\Big(\sum_{i\in\boldsymbol{N}} \alpha_{\zeta,i}{}^{\ell_j} - \sum_{i\in\boldsymbol{N}} \beta_{\zeta,i}{}^{\ell_j}\Big)\,\zeta(d'_j)\Big\}.$$

Suppose g' *does not satisfy (Condition II-00), then* $f^{\rm II}_A(g') = 0$ *identically.*

(iii) *Let* $\mathcal{F}^{\rm II}$ *be the set of all the limits of normalized spin irreducible characters of* $\widetilde{G}^{\rm I}_n$ *of Type* $(-1,-1,-1)$ *as* $n \to \infty$. *Then*

$$\mathcal{F}^{\rm II} = \big\{f^{\rm II}_A\,;\ A \in \mathcal{A}(\widehat{T}^0)\big\} = \big\{\widetilde{F}^{\rm II}_{0,\infty}\cdot f_A\,;\ A \in \mathcal{A}(\widehat{T}^0)\big\}. \tag{23.26}$$

Corollary 23.8. *Suppose* $\alpha_{\zeta,i} \neq \beta_{\zeta,i}$ *for at least one of* $(\zeta, i) \in \widehat{T}^0 \times \boldsymbol{N}$. *Then the limit function* $f^{\rm II}_A$ *is **not** factorizable because, for* $g' = \xi'_{q_1}\xi'_{q_2}\cdots\xi'_{q_r}\, g'_1 g'_2\cdots g'_s$ *in (22.6) satisfying (Condition II-00),* $f^{\rm II}_A(g'_j) = 0$ *for any* g'_j, $j \in J_-$, *whereas*

$$f^{\rm II}_A(g'_{j_1}g'_{j_2}) \neq 0 \ \text{ for an appropriate choice of } g'_{j_1},\ g'_{j_2} \text{ with } \{j_1, j_2\} \subset J_-.$$

The limit function $f^{\rm II}_A$ *is factorizable if and only if* $\alpha_\zeta = \beta_\zeta$ *for all* $\zeta \in \widehat{T}^0$ *or* $\alpha = \beta$.

Proof. We have $f^{\rm II}_A(g'_{j_1}g'_{j_2}) = 0$ always, if and only if, for any $(\zeta, i) \in \widehat{T}^0 \times \boldsymbol{N}$,

$$\sum_{i\in\boldsymbol{N}} \alpha_{\zeta,i}{}^{\ell_j} = \sum_{i\in\boldsymbol{N}} \beta_{\zeta,i}{}^{\ell_j} \quad \text{for all even } \ell_j \geq 2.$$

On the other hand, note that

$$\exp\Big\{\sum_{2\le k<\infty}\ \sum_{i\in\boldsymbol{N}} x_i{}^k\,\frac{z^k}{k}\Big\} = \exp\Big\{-z\sum_{1\le i<\infty} x_i\Big\}\prod_{1\le i<\infty}\frac{1}{1-x_i z}\quad (z\in\boldsymbol{C}). \tag{23.27}$$

Then we have $\alpha_{\zeta,i}{}^2=\beta_{\zeta,i}{}^2$ and so $\alpha_{\zeta,i}=\beta_{\zeta,i}$ $(i\in\boldsymbol{N})$. □

24 Determination of spin characters of $G(m,1,\infty)$

24.1 Preliminaries

Recall that (cf. §3), for a topological group G', $K(G')$ denotes the set of all central, continuous positive definite functions on G', $K_1(G')$ the set of $f\in K(G')$ normalized as $f(e)=1$ at the identity element $e\in G'$, and $E(G')$ the set of extremal points (called characters) in the convex set $K_1(G')$. Let Z' be a central subgroup of G' and χ a character of G'. Denote by $K_1^\chi(G')$ $\big($resp. $E^\chi(G')\big)$ the elements $f\in K_1(G')$ $\big($resp. $E(G')\big)$ satisfying $f(z'g')=\chi(z')f(g')$ $(z'\in Z',\ g'\in G')$. Then the set of all extremal points of $K_1^\chi(G')$ is equal to $E^\chi(G')$ and $E(G')=\bigsqcup_{\chi\in\widehat{Z'}}E^\chi(G')$.

Furthermore let N' be a normal subgroup of G', and define a restriction map for $f\in K_1(G')$ by $R^{N'}_{G'}f:=f\big|_{N'}$. Then, as is seen in §6 in [I], we have the following.

Lemma 24.1 (cf. [I, §6]). *Suppose G' is discrete.*

(i) *The restriction map $R^{N'}_{G'}$ maps $K_1(G')$ into $K_1(N')$, and $E(G')$ into $E(N')$.*

(ii) *Suppose that a function φ on N' is invariant under G': $\varphi(g'h'g'^{-1})=\varphi(h')$ $(g'\in G', h'\in N')$, if it is invariant (under N'). Then $R^{N'}_{G'}$ maps $K_1(G')$ onto $K_1(N')$.*

(iii) *Suppose moreover that a central subgroup Z' of G' is contained in N', and that every $f\in K_1^\chi(G')$ vanishes outside N', then $R^{N'}_{G'}$ maps $E^\chi(G')$ onto $E^\chi(N')$.*

Now put $\widetilde{G}_\infty:=R(G_\infty)$, $G_\infty=G(m,1,\infty)$, and let $\widetilde{G}^{\rm Y}_\infty$, Y=I, II, be the quotient groups of $\widetilde{G}_\infty$ given as

$$\widetilde{G}^{\rm Y}_\infty:=\widetilde{G}_\infty/Z^{\rm Y}\cong\lim_{n\to\infty}\widetilde{G}^{\rm Y}_n,\quad Z^{\rm Y}=\begin{cases}\langle z_2{z_3}^{-1}\rangle & \text{for Y}=\text{I}\,;\\ \langle z_3\rangle & \text{for Y}=\text{II}.\end{cases} \tag{24.1}$$

Denote by Φ (resp. $\Phi^{\rm Y}$ and $\Phi_{\rm Y}$) the canonical homomorphism $\widetilde{G}_\infty\to G_\infty$ $\big($resp. $\widetilde{G}_\infty\to\widetilde{G}^{\rm Y}_\infty$ and $\widetilde{G}^{\rm Y}_\infty\to G_\infty$, similar to those in Diagram 2.1.

Define subsets $\mathcal{O}(\mathrm{I})$ and $\mathcal{O}(\mathrm{II})$ in $\widetilde{G}_\infty$ respectively by (Condition I-00) and (Condition II-00) as follows:

$$\begin{aligned}\mathcal{O}(\mathrm{I})&:=\big\{g''\in\widetilde{G}_\infty\,;\ g'=\Phi^{\rm I}(g'')\in\widetilde{G}^{\rm I}_\infty\ \text{satisfies (Condition I-00)}\big\},\\ \mathcal{O}(\mathrm{II})&:=\big\{g''\in\widetilde{G}_\infty\,;\ g'=\Phi^{\rm II}(g'')\in\widetilde{G}^{\rm II}_\infty\ \text{satisfies (Condition II-00)}\big\}.\end{aligned} \tag{24.2}$$

Then $\Phi^{\rm Y}\big(\mathcal{O}(\mathrm{Y})\big)=\mathcal{O}^{\rm Y}\subset\widetilde{G}^{\rm Y}_\infty$, $\Phi\big(\mathcal{O}(\mathrm{Y})\big)=O^{\rm Y}\subset G_\infty$, for Y=I, II. Consider a normal subgroup $\widetilde{N}:=\Phi^{-1}(N)$ of $\widetilde{G}_\infty$ of finite index, where $N:=\mathfrak{A}_\infty(T)^{S(2)}$ a normal subgroup of $G_\infty = \mathfrak{S}_\infty(T)$:

$$
(24.3) \qquad \begin{cases} \widetilde{N} = \Phi^{-1}\big(\mathfrak{A}_\infty(T)^{S(2)}\big),\ T = \boldsymbol{Z}_m,\ S(2) = \{t^2\,;\, t \in T\} \subset T = \boldsymbol{Z}_m\,, \\ N = \mathfrak{A}_\infty(T)^{S(2)} = \big\{(d,\sigma) \in G_\infty\ ;\ P(d) \in S(2),\ \sigma \in \mathfrak{A}_\infty\big\} \\ \quad = \big\{(d,\sigma) \in G_\infty\ ;\ \mathrm{ord}(d) \equiv 0,\ L(\sigma) \equiv 0 \ (\mathrm{mod}\ 2)\big\}. \end{cases}
$$

Then $\widetilde{N}$ contains the subset $\mathcal{O}(\mathrm{II})$ but not $\mathcal{O}(\mathrm{I})$.

As in [I, §10], for the mother group $\widetilde{G}_\infty = R\big(G(m,1,\infty)\big)$, let χ^{Y} ($\mathrm{Y} = \mathrm{I} \sim \mathrm{VIII}$) be the character of $Z = \langle z_1, z_2, z_3\rangle \subset \widetilde{G}_\infty$ in CASE Y, Type $(\beta_1,\beta_2,\beta_3)$, that is,

$$
(24.4) \qquad \chi^{\mathrm{Y}}(z_i) = \beta_i \quad (i = 1,2,3),
$$

and put $K_1^{\mathrm{Y}}(\widetilde{G}_\infty) := K_1^{\chi^{\mathrm{Y}}}(\widetilde{G}_\infty)$, $E^{\mathrm{Y}}\big(\widetilde{G}_\infty\big) := E^{\chi^{\mathrm{Y}}}\big(\widetilde{G}_\infty\big)$, and call $f \in E^{\mathrm{Y}}\big(\widetilde{G}_\infty\big)$ a character of $\widetilde{G}_\infty$ of CASE Y, Type $(\beta_1,\beta_2,\beta_3)$. We have a decomposition of the set of characters as

$$
(24.5) \qquad E\big(\widetilde{G}_\infty)\big) = \bigsqcup_{\mathrm{Y}=\mathrm{I}\sim\mathrm{VIII}} E^{\mathrm{Y}}\big(\widetilde{G}_\infty\big).
$$

From the studies in [I, §§9–10] on the supports of central functions, we have the following (cf. the summary in Table 10.1 in [I]).

Lemma 24.2. *For Y=I, II, every $f \in K_1^{\mathrm{Y}}(\widetilde{G}_\infty)$ vanishes outside the subset $\mathcal{O}(\mathrm{Y})$.*

We have studied in [I, §§10–11] the properties of $K_1^{\mathrm{Y}}(\widetilde{G}_\infty)$ and the subsets $\mathcal{O}(\mathrm{Y})$, and summarized the results in Tables 10.1 and 13.1, loc. cit. In particular, by the detailed study of structure of $\mathcal{O}(\mathrm{I})$, we obtain in [I, Theorem 11.1 (ii)] the following important result.

Lemma 24.3. *For $f \in K_1^{\mathrm{Y}}\big(\widetilde{G}_\infty\big)$ of CASE* Y*, the criterion (EF) to be a character holds for* $\mathrm{Y} = \mathrm{I}$.

For the level of quotient groups $G' = \widetilde{G}_\infty^{\mathrm{Y}}$, Y = I, II, we set similarly as above: let χ' be the character of $\widetilde{Z} = \langle z_1, z_2\rangle$ given by $\chi'(z_i) = -1$ $(i = 1,2)$, and put $E^{\mathrm{Y}}(G') = E^{\chi'}(G')$ for $G' = \widetilde{G}_\infty^{\mathrm{Y}}$. An element $f \in E^{\mathrm{Y}}\big(\widetilde{G}_\infty^{\mathrm{Y}}\big)$ is a character of a factor representation of $\widetilde{G}_\infty^{\mathrm{Y}}$ and the natural correspondence

$$
(24.6) \qquad E^{\mathrm{Y}}\big(\widetilde{G}_\infty^{\mathrm{Y}}\big) \ni f \longrightarrow f \circ \Phi^{\mathrm{Y}} \in E^{\mathrm{Y}}\big(\widetilde{G}_\infty\big)
$$

is a bijection.

Problem setting: *Prove the completeness of the set of limit functions $\mathcal{F}^{\mathrm{Y}}$ in CASE* Y *for CASE* Y, Y=I *and* II, *that is, $\mathcal{F}^{\mathrm{Y}} = E^{\mathrm{Y}}\big(\widetilde{G}_\infty^{\mathrm{Y}}\big)$.*

The inclusion $\mathcal{F}^{\mathrm{Y}} \supset E^{\mathrm{Y}}\big(\widetilde{G}_\infty^{\mathrm{Y}}\big)$ is already known in (22.1), so that the problem is to see the converse inclusion $\mathcal{F}^{\mathrm{Y}} \subset E^{\mathrm{Y}}\big(\widetilde{G}_\infty^{\mathrm{Y}}\big)$.

24.2 $\mathcal{F}^{\rm I} = E^{\rm I}\big(\widetilde{G}^{\rm I}_\infty\big)$ and parametrization of spin characters

Every limit function f in $\mathcal{F}^{\rm I}$ (on $\widetilde{G}^{\rm I}_\infty$) is factorizable as proved in Theorem 23.6 (ii), and so, it is extremal by Lemma 24.3, whence $\mathcal{F}^{\rm I} \subset E^{\rm I}\big(\widetilde{G}^{\rm I}_\infty\big)$. Thus we obtain the following theorem.

Theorem 24.4. (i) *Every limit as* $n \to \infty$ *of normalized spin irreducible characters of* $\widetilde{G}^{\rm I}_n = \widetilde{G}^{\rm I}(m,1,n)$, *in CASE I, Type* $(-1,-1,-1)$, *is extremal or is a character of* $\widetilde{G}^{\rm I}_\infty = \widetilde{G}^{\rm I}(m,1,\infty)$, *so that* $\mathcal{F}^{\rm I} = E^{\rm I}\big(\widetilde{G}^{\rm I}_\infty\big)$, *and this gives a formula of spin characters as*

$$E^{\rm I}\big(\widetilde{G}^{\rm I}_\infty\big) = \{f^{\rm I}_A\,;\, A \in \mathcal{A}(\widehat{T}^0)\} = \{\widetilde{F}^{\rm I}_{0,\infty}\cdot f_A\,;\, A \in \mathcal{A}(\widehat{T}^0)\},$$

where $\mathcal{A}(\widehat{T}^0)$ *denotes the set of all parameters* $A = (\alpha,\beta\,;\mu)$ *satisfying (23.9)–(23.11). The above equality can be rewritten as* $E^{\rm I}\big(\widetilde{G}_\infty\big) = \mathcal{F}^{\rm I}\circ\Phi^{\rm I}$.

(ii) *The space* $E^{\rm I}\big(\widetilde{G}^{\rm I}_\infty\big) \subset K_1\big(\widetilde{G}^{\rm I}_\infty\big)$ *of spin characters of* $G(m,1,\infty)$ *in CASE I is parametrized by* $\mathcal{A}(\widehat{T}^0)$ *and is compact as a topological space, and the map* $\mathcal{A}(\widehat{T}^0) \ni A \mapsto f^{\rm I}_A \in E^{\rm I}\big(\widetilde{G}^{\rm I}_\infty\big)$ *is homeomorphic.*

24.3 $\mathcal{F}^{\rm II} = E^{\rm II}\big(\widetilde{G}^{\rm II}_\infty\big)$ and parametrization of spin characters

24.3.1. Restriction map from $\widetilde{G}_\infty$ to $\widetilde{N} = \Phi^{-1}(N),\ N = \mathfrak{A}_\infty(T)^{S(2)}$.

Denote by $K^{\rm Y}_1\big(\widetilde{N}\big)$, ${\rm Y} = {\rm I}$, II, the set of all normalized central positive definite function f on $\widetilde{N}$ satisfying

$$f(zh') = \chi^{\rm Y}(z)\,f(h') \quad (z \in Z = \langle z_1, z_2, z_3\rangle,\ h' \in \widetilde{N}), \tag{24.7}$$

where $\chi^{\rm I}(z_i) = -1$ $(i = 1,2,3)$, and $\chi^{\rm II}(z_i) = -1$ $(i = 1,2)$, $\chi^{\rm II}(z_3) = 1$, respectively. Moreover, let $E^{\rm Y}\big(\widetilde{N}\big)$ be the set of extremal elements of the convex set $K^{\rm Y}_1\big(\widetilde{N}\big)$, that is, the set of characters of $\widetilde{N}$ of Type $(-1,-1,-1)$ or $(-1,-1,\ 1)$ respectively.

First we have the following fact, similar to Theorem 6.2 in [I]. Define a restriction map ${\rm R}^{\widetilde{N}}_{\widetilde{G}}$ from $K_1\big(\widetilde{G}_\infty\big)$ as

$${\rm R}^{\widetilde{N}}_{\widetilde{G}}:\ K_1\big(\widetilde{G}_\infty\big) \ni F \mapsto f = F|_{\widetilde{N}} \in K_1\big(\widetilde{N}\big) \tag{24.8}$$

Proposition 24.5. (i) *A function* f *on the normal subgroup* $\widetilde{N}$ *is* $\widetilde{G}_\infty$*-invariant if and only if it is invariant (under* $\widetilde{N}$*).*

(ii) *The restriction map* ${\rm R}^{\widetilde{N}}_{\widetilde{G}}$ *gives a surjection from* $E\big(\widetilde{G}_\infty\big)$ *onto* $E\big(\widetilde{N}\big)$, *and also a surjection from* $E^{\rm Y}\big(\widetilde{G}_\infty\big)$ *onto* $E^{\rm Y}\big(\widetilde{N}\big)$ *for* ${\rm Y} = {\rm I}$, II.

The proof is similar as that for Theorem 6.2 (ii) and (iii), loc. cit., and omitted.

Now put, for ${\rm Y} = {\rm I}$, II,

$$\mathcal{R}^{\rm Y}(f) := {\rm R}^{\widetilde{N}}_{\widetilde{G}}\big(f\circ\Phi^{\rm Y}\big) \qquad \big(f \in K_1\big(\widetilde{G}^{\rm Y}_\infty\big)\big). \tag{24.9}$$

By the proposition above, we obtain from Theorem 24.4 the following.

Proposition 24.6. *The set of characters $E^{\mathrm{I}}(\widetilde{N})$ of the normal subgroup $\widetilde{N} = \Phi^{-1}(N)$ in CASE I, Type $(-1,-1,-1)$, is equal to the set of $\mathcal{R}^{\mathrm{I}}(f_A^{\mathrm{I}})$:*

$$E^{\mathrm{I}}(\widetilde{N}) = \{\mathcal{R}^{\mathrm{I}}(f_A^{\mathrm{I}})\ ;\ A = (\alpha, \beta\,; \mu) \in \mathcal{A}(\widehat{T}^0)\}.$$

As a property of the restriction maps $\mathcal{R}^{\mathrm{Y}}, \mathrm{Y} = \mathrm{I}, \mathrm{II}$, we remark the following.

Proposition 24.7. (i) *For* $\mathrm{Y} = \mathrm{I}$, *we have* $\mathcal{R}^{\mathrm{I}}(f_A^{\mathrm{I}}) = \mathcal{R}^{\mathrm{I}}(f_{{}^tA}^{\mathrm{I}})$ *for* $A \in \mathcal{A}(\widehat{T}^0)$, *and the restriction map $\mathcal{R}^{\mathrm{I}}$ is 2-to-1 if ${}^tA \neq A$, and 1-to-1 if ${}^tA = A$.*

(ii) *For* $\mathrm{Y} = \mathrm{II}$, *the restriction map $\mathcal{R}^{\mathrm{II}}$ is bijective.*

Proof. We see easily that $\mathcal{R}^{\mathrm{I}}(f_A^{\mathrm{I}}) = \mathcal{R}^{\mathrm{I}}(f_{{}^tA}^{\mathrm{I}})$ from the explicit form of f_A^{I} given in Theorem 23.6, or more directly from Lemma 23.5 (ii). On the contrary, suppose that, for $A = (\alpha, \beta\,; \mu)$ and $A' = (\alpha', \beta', \mu')$, there holds $(f_A^{\mathrm{I}} \circ \Phi^{\mathrm{I}})|_{\widetilde{N}} = (f_{A'}^{\mathrm{I}} \circ \Phi^{\mathrm{I}})|_{\widetilde{N}}$. Then, we can discuss just as in the proof of Lemma 23.5 (ii).

The assertion (ii) is easy to prove. □

24.3.2. Operators $\mathcal{M}$ and $\mathcal{N}$ of multiplying the character $\widetilde{\chi}_{\pi_{2,\zeta_0}}$.

In CASE VII, Type $(1,\ 1,-1)$, there exist 2-dimensional IRs π_{2,ζ_k} of $\widetilde{G}_\infty = R(G_\infty)$, given in [I, §12]. Here $\zeta_k \in \widehat{T}$, $\zeta_k(\eta) := \omega^k$, with the generator η of $T = \boldsymbol{Z}_m$, and $\pi_{2,\zeta_k}(z_1^a z_2^b z_3^c) = (-1)^c E_2$, and

$$\pi_{2,\zeta_k}(\eta'_j) = \begin{pmatrix} \zeta_k(\eta) & 0 \\ 0 & -\zeta_k(\eta) \end{pmatrix}, \quad \pi_{2,\zeta_k}(r_i) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (i, j \in \boldsymbol{N}), \tag{24.10}$$

where $\eta'_j = z_2^{j-1}\eta_j$ ($\eta_j \in T_j \cong T$, $j \in \boldsymbol{N}$) form another set of generators of $\widetilde{D}_\infty$.

The trace character of π_{2,ζ_k} is given as follows. For $g'' = z_1^a z_2^b z_3^c\, d'\sigma' \in R(G)$, $d' = \prod_{j\in\boldsymbol{N}} {\eta'_j}^{a_j}$, $\sigma' \in \mathfrak{S}_\infty$, put $\mathrm{ord}(d') := \sum_{j\in\boldsymbol{N}} a_j \pmod m$, then

$$\chi_{\pi_{2,\zeta_k}}(g'') = \begin{cases} 2\cdot(-1)^c \zeta_k(\eta)^{\mathrm{ord}(d')} & \text{if } L(\sigma') \equiv 0,\ \mathrm{ord}(d') \equiv 0 \pmod 2, \\ 0 & \text{otherwise.} \end{cases} \tag{24.11}$$

So the support of the character $\chi_{\pi_{2,\zeta_k}}$ is equal to $\widetilde{N} = \Phi^{-1}(N)$, $N = \mathfrak{A}_\infty(T)^{S(2)}$.

Let $\widetilde{\chi}_{\pi_{2,\zeta_k}}$ denote the normalized character $\chi_{\pi_{2,\zeta_k}}/2$, then the inner tensor product $\pi_{2,\zeta_k} \boxtimes \pi_{2,\zeta_\ell}$ has normalized character given as

$$\widetilde{\chi}_{\pi_{2,\zeta_k}}(g'') \cdot \widetilde{\chi}_{\pi_{2,\zeta_\ell}}(g'') = \big(\zeta_k(\eta)\,\zeta_\ell(\eta)\big)^{\mathrm{ord}(d')} \cdot X_{\widetilde{N}}(g''),$$

where $X_{\widetilde{N}}$ denotes the indicator function of $\widetilde{N}$. In particular, for $k = \ell = 0$, we have

$$\big(\widetilde{\chi}_{\pi_{2,\zeta_0}}\big)^2 = X_{\widetilde{N}}. \tag{24.12}$$

Noting that $\widetilde{\chi}_{2,\zeta_0}$ is a character of $\widetilde{G}_\infty = R(G_\infty)$ of Type $(1,\ 1,-1)$, we define two maps $\mathcal{M}$ and $\mathcal{N}$ similarly as [I, Definition 16.1]:

Definition 24.1. Between the set of normalized central positive definite functions $K_1^{\rm I}(\widetilde{N})$ in CASE I and the set of such functions $K_1^{\rm II}(\widetilde{N})$ in CASE II, we define two maps $\mathcal{M}$ and $\mathcal{N}$ as follows: for $F \in K_1^{\rm I}(\widetilde{N})$ and $f \in K_1^{\rm II}(\widetilde{N})$, put

$$(24.13) \qquad \left\{ \begin{array}{lcl} \mathcal{M}(F)(g'') & := & \widetilde{\chi}_{\pi_{2,\zeta_0}}(g'') \cdot F(g''), \\ \mathcal{N}(f)(g'') & := & \widetilde{\chi}_{\pi_{2,\zeta_0}}(g'') \cdot f(g''), \end{array} \right. \qquad (g'' \in \widetilde{N}).$$

We see from [I, §10, Table 10.1] that for any $f \in K_1^{\rm II}(\widetilde{G}_\infty)$, its support $\mathrm{supp}(f)$ is contained in the subset $\mathcal{O}(\mathrm{II}) \subset \widetilde{N}$. Therefore the restriction map

$$(24.14) \qquad K_1^{\rm II}(\widetilde{G}_\infty) \ni f \longmapsto f\big|_{\widetilde{N}} \in K_1^{\rm II}(\widetilde{N})$$

is bijective, and maps $E^{\rm II}(\widetilde{G}_\infty)$ onto $E^{\rm II}(\widetilde{N})$ bijectively.

Moreover note that $(\widetilde{\chi}_{\pi_{2,\zeta_0}})^2$ is just the indicator function of $N = \mathfrak{A}_\infty(T)^{S(2)} \subset G_\infty$ if considered as a function of $g = \Phi(g'')$ through modulo Z, and is the indicator function of $\widetilde{N} = \Phi^{-1}(N)$ as a function in $g'' \in \widetilde{G}_\infty$.

These facts guarantee that $\mathcal{M}$ and $\mathcal{N}$ are mutually the inverse of the other and so both are bijective. Moreover since they are both linear, they map the sets of extremal points $E^{\rm I}(\widetilde{N})$ and $E^{\rm II}(\widetilde{N})$ mutually each other.

24.3.3. Spin characters in CASE II, Type $(-1,-1,\ 1)$.

Proposition 24.8. (i) *The map* $\mathcal{M}$ *and* $\mathcal{N}$ *between* $K_1^{\rm I}(\widetilde{N})$ *and* $K_1^{\rm II}(\widetilde{N})$ *are mutually the inverse of the other, and both preserve convex combinations. They induce bijective maps* $\mathcal{M}'$ *and* $\mathcal{N}'$ *between the sets of extremal points* $E^{\rm I}(\widetilde{N})$ *and* $E^{\rm II}(\widetilde{N})$.

(ii) *With restriction maps from* $E^{\rm Y}(\widetilde{G}_\infty)$, $\mathrm{Y}=\mathrm{I}, \mathrm{II}$, *there gives rise to the following diagram, with* $\widetilde{G}_\infty = R\big(G(m,1,\infty)\big)$ *:*

$$\begin{array}{ccccc} E^{\rm I}(\widetilde{G}^{\rm I}_\infty) \ \cong \ E^{\rm I}(\widetilde{G}_\infty) & & & & E^{\rm II}(\widetilde{G}_\infty) \ \cong \ E^{\rm II}(\widetilde{G}^{\rm II}_\infty) \\ \mathrm{R}^{\widetilde{N}}_{\widetilde{G}} \downarrow \text{surj.} & & & & \text{bijec.} \downarrow \mathrm{R}^{\widetilde{N}}_{\widetilde{G}} \\ E^{\rm I}(\widetilde{N}) & & \begin{array}{c} \xrightarrow{\mathcal{M}'} \\ \xleftarrow[\mathcal{N}']{} \end{array} & & E^{\rm II}(\widetilde{N}) \end{array}$$

(iii) *Under these maps, the correspondence of functions are as follows: for* $A \in \mathcal{A}(\widehat{T}^0)$, $f_A^{\rm I} = \widetilde{F}^{\rm I}_{0,\infty} \cdot f_A$, $f_A^{\rm II} = \widetilde{F}^{\rm II}_{0,\infty} \cdot f_A$,

$$\begin{array}{ccccc} E^{\rm I}(\widetilde{G}^{\rm I}_\infty) \ni f_A^{\rm I} & \leftrightarrow & f_A^{\rm I} \circ \Phi^{\rm I} & f_A^{\rm II} \circ \Phi^{\rm II} & \leftrightarrow \ f_A^{\rm II} \in E^{\rm II}(\widetilde{G}^{\rm II}_\infty) \\ & & \text{surj.} \downarrow \mathcal{R}^{\rm I} & \mathcal{R}^{\rm II} \downarrow \text{bijec.} & \\ & & (f_A^{\rm I} \circ \Phi^{\rm I})\big|_{\widetilde{N}} \ \rightleftarrows & (f_A^{\rm II} \circ \Phi^{\rm II})\big|_{\widetilde{N}} & \end{array}$$

Note 24.1. As a spacial case of the proposition above, we have for the special characters,

$$\widetilde{\chi}_{\pi_{\pi_2,\zeta_0}} \times \big(\widetilde{F}^{\rm I}_{0,\infty} \circ \Phi^{\rm I}\big) = \big(\widetilde{F}^{\rm II}_{0,\infty} \circ \Phi^{\rm II}\big).$$

Now, appealing to Theorem 24.4 in CASE I, we arrive at our final theorem as follows. As a parameter space for $E^{\mathrm{II}}(\widetilde{G}_\infty)$, we have the quotient space $\mathcal{A}(\widehat{T}^0)/\langle\tau\rangle$ of the set $\mathcal{A}(\widehat{T}^0)$ of $A = (\alpha, \beta\,; \mu)$ satisfying (23.9)–(23.11) by the action $\tau :\ A \mapsto {}^tA$.

Theorem 24.9. (i) *Every limit as $n \to \infty$ of normalized spin irreducible characters of $\widetilde{G}^{\mathrm{II}}_n = \widetilde{G}^{\mathrm{II}}(m,1,n)$, in CASE II, Type $(-1,-1,\,1)$, is extremal or a character of $\widetilde{G}^{\mathrm{II}}_\infty = \widetilde{G}^{\mathrm{II}}(m,1,\infty)$, so that $\mathcal{F}^{\mathrm{II}} = E^{\mathrm{II}}(\widetilde{G}^{\mathrm{II}}_\infty)$. This gives a formula of spin characters in CASE II, Type $(-1,-1,\,1)$, as*

$$E^{\mathrm{II}}(\widetilde{G}^{\mathrm{II}}_\infty) = \{f^{\mathrm{II}}_A\,;\ A \in \mathcal{A}(\widehat{T}^0)\} = \{\widetilde{F}^{\mathrm{II}}_{0,\infty}\cdot f_A\,;\, A \in \mathcal{A}(\widehat{T}^0)\}.$$

This equality can be rewritten as $E^{\mathrm{II}}(\widetilde{G}_\infty) = \mathcal{F}^{\mathrm{II}} \circ \Phi^{\mathrm{II}}$.

(ii) *The map $\mathcal{A}(\widehat{T}^0) \ni A \mapsto f^{\mathrm{II}}_A \in E^{\mathrm{II}}(\widetilde{G}^{\mathrm{II}}_\infty) \subset K_1(\widetilde{G}^{\mathrm{II}}_\infty)$ is continuous and, as its image, the space $E^{\mathrm{II}}(\widetilde{G}^{\mathrm{II}}_\infty)$ is homeomorphic to the quotient space $\mathcal{A}(\widehat{T}^0)/\langle\tau\rangle$ and is compact as a topological space.*

25 Structure of the space of spin characters of $G(m,1,\infty)$

25.1. Multiplicative structure of spin characters of $G(m,1,\infty)$.

Here we quote the main result of the paper [DuNe] by Dudko and Nessonov on spin characters of $G(m,1,\infty)$. The notations ibid. are as follows:

$$B_m := \mathfrak{S}_\infty \ltimes \mathbb{Z}_m^\infty, \quad \mathbb{Z}_m^\infty := \textstyle\bigcup_{n\geqslant 1} \mathbb{Z}_m^n$$

$\big(B_m \cong G(m,1,\infty)\big)$, and $\widetilde{B}_m$ a representation group of B_m isomorphic to $R\big(G(m,1,\infty)\big)$ (cf. Theorems 3.2 and 3.3 in the present paper). The canonical homomorphism $\widetilde{B}_m \to B_m$ is denoted by $\mathfrak{pr}$.

The *cocycles* $\theta = [t_\vartheta, t_\mu, t_\nu]$ in Definition 9, p.1428, of a factor representation π of $\widetilde{B}_m$ is equal to our *(spin) Type* $\beta = (\beta_1, \beta_2, \beta_3)$ modulo ordering as $[t_\vartheta, t_\mu, t_\nu] = [\beta_1, \beta_3, \beta_2]$. There are 8 cocycles named as

$$\begin{aligned}&\theta_0 = [1,\,1,\,1], \quad \theta_1 = [-1,\,1,\,1], \quad \theta_2 = [1,-1,\,1], \quad \theta_3 = [-1,-1,\,1],\\ &\theta_4 = [1,\,1,-1],\ \theta_5 = [-1,\,1,-1],\ \theta_6 = [1,-1,-1],\ \theta_7 = [-1,-1,-1].\end{aligned}$$

For each cocycle θ, except $\theta = \theta_0$, a special factor representation π^b_θ, called *basis representation* of finite type of $\widetilde{B}_m$, is constructed, in such a way that, in its representation space $V(\pi^b_\theta)$, there exists a unit cyclic vector ξ^b_θ such that $(\pi^b_\theta(g)\xi^b_\theta, \xi^b_\theta) = \chi^b_\theta(g)$ $(g \in \widetilde{B}_m)$, where χ^b_θ denotes the character of π^b_θ. (This representation, uniquely determined modulo equivalence by positive definite function $f = \chi^b_\theta$, can be called as Gelfand-Raikov representation associated to f, cf. [GeRa].). First, for three cocycles $\theta = \theta_1, \theta_2$ and θ_7, the representation π^b_θ is defined by specifying the operator $\pi^b_\theta(g)$ for each standard generators g of $\widetilde{B}_m$. Second, for a cycle θ other than the above, write θ as a product

$$\theta = \theta_1^{\,i}\theta_2^{\,j}\theta_7^{\,k}, \qquad i,j,k \in \{0,1\},$$

and consider the tensor product

$$\pi'_\theta := (\pi^b_1)^{\otimes i} \otimes (\pi^b_2)^{\otimes j} \otimes (\pi^b_7)^{\otimes k}.$$

Put $\xi^b_\theta := (\xi^b_{\theta_1})^{\otimes i} \otimes (\xi^b_{\theta_2})^{\otimes j} \otimes (\xi^b_{\theta_7})^{\otimes k} \in V(\pi'_\theta)$ and take the closed subspace H^b_θ spanned by $\pi'_\theta(\widetilde{B}_m)\xi^b_\theta$, then $\pi^b_\theta := \pi'_\theta|_{H^b_\theta}$.

It is claimed that Theorem 10 on p.1429 is the main result of the paper:

Theorem 10. *Let χ be an arbitrary indecomposable character of $\widetilde{B}_m$ corresponding to a cocycle θ_i. Then there exists an indecomposable character χ' on B_m such that*

$$\chi(g) = \chi^b_{\theta_i}(g)\,\chi'\big(\mathfrak{pr}(g)\big) \tag{15}$$

holds for each $g \in \widetilde{B}_m$. Conversely, every function χ of the form (15) is an indecomposable character on $\widetilde{B}_m$ corresponding to the cocycle θ_i.

25.2. Parameter spaces of spin characters of $\mathfrak{S}_\infty$ and $G(m,1,\infty)$.

As the parameter space of (non-spin) characters of the infinite symmetric group $\mathfrak{S}_\infty$, the set of Thoma parameters is given (Satz 3 in [Tho2]) as

$$\mathcal{A}^1 := \big\{(\alpha,\beta)\,;\ \alpha = (\alpha_i)_{i\in\boldsymbol{N}},\ \beta = (\beta_i)_{i\in\boldsymbol{N}}\ \text{ satisfying (25.1)}\big\},$$

$$\text{(25.1)}\qquad \begin{cases} \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_i \geq \ldots \geq 0,\ \ \beta_1 \geq \beta_2 \geq \ldots \geq \beta_i \geq \ldots \geq 0, \\ \|\alpha\| + \|\beta\| \leq 1, \quad \|\alpha\| := \sum_{i\in\boldsymbol{N}} \alpha_i, \quad \|\beta\| := \sum_{i\in\boldsymbol{N}} \beta_i. \end{cases}$$

Define a subset $\mathcal{C}^1 \subset \mathcal{A}^1$ of " nearly of half a dimension " of $\mathcal{A}^1$ (even though both of infinite dimensions) as

$$\mathcal{C}^1 := \big\{\gamma\,;\ \gamma = (\gamma_i)_{i\in\boldsymbol{N}}\ \text{ satisfying (25.2)}\big\},$$

$$\text{(25.2)}\qquad \gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_i \geq \ldots \geq 0, \quad \|\gamma\| \leq 1.$$

Moreover, for a subset K of the dual $\widehat{T}$ of the group $T = \boldsymbol{Z}_m$, put

$$\mathcal{A}(K) := \big\{(\alpha,\beta;\mu)\,;\ \alpha = (\alpha_\zeta)_{\zeta\in K},\ \beta = (\beta_\zeta)_{\zeta\in K},\ \mu = (\mu_\zeta)_{\zeta\in K}\ \text{ satisfying (25.3)–(25.4)}\big\},$$

$$\text{(25.3)}\qquad \begin{cases} \alpha_\zeta = (\alpha_{\zeta,i})_{i\in\boldsymbol{N}}, \quad \beta_\zeta = (\beta_{\zeta,i})_{i\in\boldsymbol{N}}, \\ \alpha_{\zeta,1} \geq \alpha_{\zeta,2} \geq \ldots \geq 0,\ \beta_{\zeta,1} \geq \beta_{\zeta,2} \geq \ldots \geq 0\,;\ \mu_\zeta \geq 0, \end{cases}$$

$$\text{(25.4)}\qquad \|\alpha\| + \|\beta\| + \|\mu\| = 1,$$

where $\|\alpha\| := \sum_{\zeta\in K} \|\alpha_\zeta\|$, $\|\beta\| := \sum_{\zeta\in K} \|\beta_\zeta\|$, $\|\mu\| := \sum_{\zeta\in K} \mu_\zeta$. Then, $\mathcal{A}(\widehat{T})$ for $K = \widehat{T}$ is the parameter spaces for non-spin characters, and the subset $\mathcal{A}(\widehat{T}^0)$ for $K = \widehat{T}^0$ ("of nearly half a dimension") is used to prepare parameter spaces for spin characters of $G(m,1,\infty)$.

Let $\widetilde{f}_\infty$ be the pointwise limit (on $\widetilde{\mathfrak{S}}_\infty$) of the normalized character $\widetilde{\chi}_{\Delta'_n}$ as $n \to \infty$. The exact form of $\widetilde{f}_\infty$ is obtained directly from Theorems 15.2 and 15.3, and see that the support of $\widetilde{f}_\infty$ is contained in the subset $\mathcal{B} \subset \widetilde{\mathfrak{A}}_\infty := \Phi_{\mathfrak{S}}^{-1}(\mathfrak{A}_\infty)$ consisting of elements $\sigma' \in \widetilde{\mathfrak{A}}_\infty$ of the following form (cf. Definition 13.2 (2)):

$$\sigma' = \sigma'_1\sigma'_2\cdots\sigma'_s,\ \sigma_k = \Phi_{\mathfrak{S}}(\sigma'_k) \text{ disjoint cycles, } \ell_k = \ell(\sigma'_k) \text{ all odd.}$$

Nazarov proved that any spin character of $\widetilde{\mathfrak{S}}_\infty$ is given as a product $\widetilde{f}_\infty \cdot f$, of $\widetilde{f}_\infty$ and a non-spin normalized character f of $\mathfrak{S}_\infty$ (Theorem 3.3 in [Naz], cf. also Ivanov [Iva]). The latter is given by a parameter $(\alpha,\beta) \in \mathcal{A}^1$ as $f = f_{\alpha,\beta}$. Every $f \in E(\mathfrak{S}_\infty)$ is *factorizable* in the sense that, if $\sigma = \sigma_1\sigma_2$ in $\mathfrak{S}_\infty$ is a decomposition such that $\mathrm{supp}(\sigma_k)$ $(k=1,2)$ are mutually disjoint, then $f(\sigma) = f(\sigma_1)\,f(\sigma_2)$. So, $f_{\alpha,\beta}$ is uniquely determined by the following formula: for a cycle τ_ℓ of length ℓ,

$$f_{\alpha,\beta}(\tau_\ell) = \sum\nolimits_{i\in\boldsymbol{N}} \alpha_i^{\,\ell} + (-1)^{\ell-1}\sum\nolimits_{i\in\boldsymbol{N}} \beta_i^{\,\ell}, \tag{25.5}$$

where $(-1)^{\ell-1} = \mathrm{sgn}(\tau_\ell)$. Since $\mathrm{supp}(\widetilde{f}_\infty) \subset \mathcal{B}$, we have $\widetilde{f}_\infty \cdot f_{\alpha,\beta} = \widetilde{f}_\infty \cdot f_{\gamma,0}$, where $\gamma = \alpha \sqcup \beta$ is given from (α,β) by rearranging the union of $\{\alpha_i\ (i \in \boldsymbol{N})\}$ and $\{\beta_i\ (i \in \boldsymbol{N})\}$ in the descending order.

Table 25.1. Spaces of non-spin and spin characters of $\mathfrak{S}_\infty$ and $G(m,1,\infty)$.

		parametrization through	parameter space	subset $\supset \mathrm{supp}(f)$
symmetric group	$\mathfrak{S}_\infty$	$(\alpha,\beta) \mapsto f_{\alpha,\beta}$	$(\alpha,\beta) \in \mathcal{A}^1$	$\mathfrak{S}_\infty$
covering group	$\widetilde{\mathfrak{S}}_\infty$	$\gamma \mapsto \widetilde{f}_\infty \cdot f_{\gamma,0}$	$\gamma \in \mathcal{C}^1 \subset \mathcal{A}^1$	$\mathcal{B} \subset \widetilde{\mathfrak{A}}_\infty$

For $G(m,1,\infty) = \mathfrak{S}_\infty(\boldsymbol{Z}_m)$ and its covering groups:

CASE Y	(spin) type	cocy-cle	parametriza-tion through	parameter space	subset $\supset \mathrm{supp}(f)$
I	(–1,–1,–1)	θ_7	$A \mapsto \widetilde{F}^{\mathrm{I}}_\infty \cdot f_A$	$A \in \mathcal{A}(\widehat{T}^0)$	$\mathcal{O}(\mathrm{I})$
II	(–1,–1, 1)	θ_5	$A \mapsto \widetilde{F}^{\mathrm{II}}_\infty \cdot f_A$	$\{A, {}^tA\} \in \mathcal{A}(\widehat{T}^0)/\langle\tau\rangle$	$\mathcal{O}(\mathrm{II})$
VII	(1, 1,–1)	θ_2	$A \mapsto \widetilde{\chi}_{\pi_{2,\zeta_0}} \cdot f_A$	$\{A, {}^tA\} \in \mathcal{A}(\widehat{T}^0)/\langle\tau\rangle$	$\mathcal{O}(\mathrm{VII})$
VIII	(1, 1, 1)	θ_0	$A \mapsto f_A$	$A \in \mathcal{A}(\widehat{T})$	$\mathfrak{S}_\infty(\boldsymbol{Z}_m)$

The subsets $\mathcal{O}(\mathrm{I})$, $\mathcal{O}(\mathrm{II})$ and $\mathcal{O}(\mathrm{VII})$ are given in Table 10.1 in [I], and $\mathcal{O}(\mathrm{I})$, $\mathcal{O}(\mathrm{II})$ are redefined in (24.2). Moreover, $\mathcal{O}(\mathrm{VII}) = \Phi^{-1}(N)$, $N = \mathfrak{A}_\infty(\boldsymbol{Z}_m)^{S(2)}$.

Note 25.1. The factor representation $\pi_2^b = \pi_{\theta_2}^b$ in [DuNe] is equal to 2-times multiple of our 2-dimensional spin representation π_{2,ζ_0} in CASE VII (Theorem 12.1 in [I]), and that $\chi_{\theta_2}^b$ is equal to $\widetilde{\chi}_{2,\zeta_0}$. The one $\pi_7^b = \pi_{\theta_7}^b$ loc. cit. can be constructed from IRs Π_0^{I}, Π_+^{I} of $\widetilde{G}_n^{\mathrm{I}}$ (depending on n even or odd) in Example 11.1 as $n \to \infty$, and $\chi_{\theta_7}^b$ is equal to $\widetilde{F}_\infty^{\mathrm{I}}$. Similarly the one $\pi_5^b = \pi_{\theta_5}^b$ loc. cit. can be constructed from Π_0^{II}, Π_+^{II} in Example 12.2 as $n \to \infty$, and $\chi_{\theta_5}^b$ is equal to $\widetilde{F}_\infty^{\mathrm{II}}$.

The problem of transcribing the parametrization of spin characters of $G(m, 1, \infty)$, in Table 25.1, in terms of parameters in [MoJo] for spin IRs (cf. [Iva]), is an interesting problem but left to be open here.

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List of symbols and definitions for [II]:

Definitions:

(Condition I-00) : Lem. 16.2
(Condition I-11) : Lem. 16.2
(Condition II-00) : **17.2.1**
(Condition II-11) : **17.3**
(Condition $\mho$-11) : **17.5**
Criterion (EF) : Def. 3.1
factorizable for f : Def. 3.1
Parity Rule I-00 : **16.2**
Parity Rule I-11 : **15.3**
Parity Rule II-00 : **17.2.3**
Parity Rule II-11 : **17.3**
representation group of G : Th. 1.1
(spin) type of projective IR : **2.3**, Table 4.1
standard decomposition of g : (4.1)
standard decomposition of g'' : **16.1.2**, **17.1**
standard decomposition of σ' : Def. 13.1
type $\beta = (\beta_1, \beta_2, \beta_3)$: **2.3**, Table 4.1

Symbols:

$A = (\alpha, \beta; \mu)$: (23.9)
$\mathcal{A}(\widehat{T}^0)$: (23.16)
$a,\ b,\ c$ triplet : (5.11)
$\widetilde{\mathfrak{A}}_n = \Phi_{\mathfrak{S}}^{-1}(\mathfrak{A}_n), \mathfrak{B}_n = R(\mathfrak{A}_n)$: Th. 2.7
CASE I, CASE II : **2.3**
$\mathcal{C}_n$ Clifford algebra : **5.1**
$\gamma,\ \Gamma_n,\ \Gamma_n^0$: Def. 6.2
$\widetilde{D}(m,n)$: **2.3**
$\widetilde{D}_n = \widetilde{D}(m,n)$: **6.1**
$\widetilde{D}^0(m,n)$: (5.2)
Δ'_n : **15.1**
ε 2×2 unit matrix : **5.3**
$f_\gamma^{\mathrm{I}} = \mathrm{tr}(\pi_\gamma)$: **16.2**
$\widetilde{f}_\gamma^{\mathrm{I}}$ normalized character : **16.5**
$f_\gamma^{\mathrm{II}} = \mathrm{tr}(\pi_\gamma^{\mathrm{II}})$: **17.1**
$\mathcal{F}_n,\ \mathcal{F}'_n$: **5.1**
$\mathcal{F}^{\mathrm{Y}}$ (Y=I,II) : top of §**22**
$\zeta_\gamma,\ \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$: **6.1**
$g_j = (d_j, \sigma_j)$: (4.1)
$G(m,p,n) = \mathfrak{S}_n(\boldsymbol{Z}_m)^{S(p)}$: **2.1.1**
$G(m,p,\infty)$: **2.1.1**
$\widetilde{G}^{\mathrm{I}}(m,1,n),\ \widetilde{G}^{\mathrm{II}}(m,1,n)$: **2.3**
$\eta_j\ (j \in \boldsymbol{I}_n)$: Th's 2.2–2.3
$\eta'_j\ (j \in \boldsymbol{I}_n)$: (2.12), (7.2)
$\boldsymbol{I}_n = \{1, 2, \ldots, n\}$: Rem. 1.1
$I_{n,\zeta}$: Def. 11.2, (19.1)
$\iota(A)B = ABA^{-1}$: **8.5**
$\ell(\sigma)$ length of a cycle : **1.3**
$\ell_j = \ell(\sigma_j)$: **13.1**
$L(\sigma),\ L(\sigma')$: Def. 1.1, Notation 4.1
$\mathcal{L}\cdot\mathcal{R}$: **5.2**
$\xi_q = (t_q, q)$: (4.1)
$\mho_n^+,\ \mho_n^-$: (8.22)
$\mathrm{ord}(d'),\ \mathrm{ord}(d)$: Notation 4.1
$\pi^0 \boxdot \pi^1,\ \Pi(\pi^0, \pi^1)$: (3.4)
$P_\gamma(d')$: Def. 6.3
$[P_\gamma]$ equivalence class of P_γ : **7.1**
$P_\gamma^+,\ P_\gamma^-$: (6.17), (6.18)
$P_0,\ P_+,\ P_-$: **6.4**
$\rho_n,\ \rho_{n,+},\ \rho_{n,-}$: Th. 5.6
$\nabla_n,\ \nabla_n^-,\ \nabla',\ \nabla''_n,\ \nabla_n^{\mathrm{II}},\ \nabla_n^{\mathrm{II}\pm}$: **8.1**
$r_i\ (i \in \boldsymbol{I}_{n-1})$: Th. 1.2
$r_i^{\mathrm{I}}(d'),\ r_i^{\mathrm{II}}(d')\ (d' \in \widetilde{D}_n)$: **7.1**, **7.2**
r_{ij} : Def. 1.1
$R(\mathfrak{A}_n) = \mathfrak{B}_n$: Th. 2.7
$R(G)$: **1.1**
$\mathrm{Rep}(\widetilde{D}_n), \mathrm{Rep}^+(\widetilde{D}_n), \mathrm{Rep}^-(\widetilde{D}_n)$: Th. 6.5
$\sqsubset$: Lem. 17.3
$S(P_\gamma),\ \mathcal{S}(P_\gamma)$: just before Th. 7.2
$\mathcal{S}(P_\gamma^+),\ \mathcal{S}(P_\gamma^-)$: Lem. 12.2
$\mathfrak{S}_I(T),\ \mathfrak{S}_I(T)^S$: **2.1.1**
$S(p) \subset \boldsymbol{Z}_m$: **2.1.1**
$\widetilde{\mathfrak{S}}_n = \mathfrak{T}'_n$: Th. 1.2
$\mathrm{sgn}(\sigma') = \mathrm{sgn}(\sigma)$: Notation 4.1
$\mathrm{sgn}_{\widetilde{D}}(d')$: Def. 6.1, (16.35)
$\mathrm{supp}(g),\ \mathrm{supp}(g')$: Notation 2.1
$\mathrm{supp}(d),\ \mathrm{supp}(\sigma)$: Notation 2.1
$\tau_k\gamma\ (\gamma \in \Gamma_n)$: Def. 6.4
$T'_j\ (j \in \boldsymbol{I}_{2N+1})$: **15.1**
$\widehat{T}^0$: **11.5**
$\overset{\mathrm{I}}{\rtimes},\ \overset{\mathrm{II}}{\rtimes}$: **3**, STEP1
$\Phi_{\mathfrak{S}} : \widetilde{\mathfrak{S}}_n \to \mathfrak{S}_n$: Th. 1.2
Φ : Th's 2.2–2.3
$\Phi_D : \widetilde{D}(m,n) \to D(m,n)$: **4**
$\Phi^{\mathrm{Y}},\ \Phi_{\mathrm{Y}}$: Diagram 2.1
$\Psi : \widetilde{D}(m,n) \to \mathcal{F}_n$: Lem. 5.3
$\chi_\gamma(d')\ \big(\gamma \in \Gamma_n,\ d' \in \widetilde{D}(m,n)\big)$: Def. 6.1
$X'_j\ (j \in \boldsymbol{I}_{2N+1})$: **15.1**
$y_i\ (i \in \boldsymbol{I}_n)$: Prop. 2.1
$Y_j\ (j \in \boldsymbol{I}_{2n'+1})$: (5.13)
$Y'_j = (-1)^{j-1}Y_j$: **7.1**
$\boldsymbol{Y}_n(T),\ \boldsymbol{Y}_n(T)^0$: **11.5**
$\boldsymbol{Y}_n^{\mathfrak{A}}(T)^0,\ \boldsymbol{Y}_n^{\mathfrak{A}}(T)^{0,1},\ \boldsymbol{Y}_n^{\mathfrak{A}}(T)^{0,2}$: **12.3**
$z_1 \in \widetilde{\mathfrak{S}}_n$: Th. 1.2
$Z = \langle z_1, z_2, z_3 \rangle$: Th. 2.3