Projective representations and spin characters of complex reflection groups $G(m,p,n)$ and $G(m,p,\infty)$, I

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Abstract. Let $G(m,p,n)$ be a complex reflection group and $R\big(G(m,p,n)\big)$ one of its representation group, and let $G(m,p,\infty)$ and $R\big(G(m,p,\infty)\big)$ be their inductive limits as $n\to\infty$. We study projective irreducible representations (= IRs) of $G(m,p,n)$ and their characters which we call *spin* characters of them. We study in particular projective IRs of generalized symmetric groups $G(m,1,n)$ and projective factor representations of $G(m,1,\infty)$ and their characters, and also limiting process as $n\to\infty$. Since $R\big(G(m,1,n)\big)$ is a special central extension of $G(m,1,n)$ by the Schur multiplier $Z=H^2\big(G(m,1,n),\boldsymbol{C}^\times\big)$, a projective IR π of $G(m,1,n)$ has its spin type, a character χ of Z, such that $\pi(z)=\chi(z)I$. In the latter part of the paper we study in detail the case of a certain spin type and also the relation to the non-spin case.[6]

0 Introduction

1. In this paper we study projective (or spin) representations of infinite family of complex reflection groups $G(m,p,n)$, $m>1$, $n\geq 4$, $p|m$, and their inductive limits $G(m,p,\infty)=\lim_{n\to\infty}G(m,p,n)$, and characters of such representations, called *spin characters*. Our principal aims or problems here are three-fold as

(A) Construct all the irreducible spin representations and calculate their characters for finite groups $G(m,p,n)$, $n\geq 4$.

(B) Analyse the limiting procedure of normalized irreducible characters of $G(m,p,n)$ as $n\to\infty$.

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[6] *2010 Mathematical Subject Classification*: Primary 20C25, 20F55; Secondary 20B99, 20E22, 20F05

Keywords: projective representation, spin representations, spin character, positive definite function, complex reflection group, generalized symmetric groups, limit of characters

(C) Determine all the spin characters of projective factor representations of finite type of infinite groups $G(m,p,\infty)$, and give the explicit character formula.

2. Thanks to Schur [Sch1], we know that a finite group G has a finite number of non-isomorphic representation groups. Take one of representation groups of G and denote it by $R(G)$. Then, any *projective* representation of G can be lifted up to a linear representation of $R(G)$, and any choice of representation group $R(G)$ of G gives an equivalent theory of projective representations of G. In this sense, we may call $R(G)$ a *universal* covering group of G, even though it is not unique.

A representation group $R(G)$ is a special kind of central extension of G by the abelian group $Z = H^2(G, \boldsymbol{C}^\times)$, called *Schur multiplier* of G. Schur multipliers are given by Ihara-Yokonuma [IhYo] for Weyl groups, by Davies-Morris [DaMo] for general symmetric groups $G(m,1,n)$, and by Read [Rea1] for general complex reflection groups $G(m,p,n)$. For each of such groups G, they have determined, on the way of calculating Schur multiplier $H^2(G, \boldsymbol{C}^\times)$, one of its representation groups, denoted by $R(G)$ here. These groups $R(G)$ are given by presenting a set of generators and a set of fundamental relations together, and to go further we are forced to manipulate these things well.

3. We have a constructive definition of an infinite family of complex reflection groups $G(m,p,n)$ as the wreath product groups as follows. For $p = 1$, $G(m,1,n) = \mathfrak{S}_n(\boldsymbol{Z}_m) = D_n(\boldsymbol{Z}_m) \rtimes \mathfrak{S}_n$, wreath product of $\boldsymbol{Z}_m$ with the symmetric group $\mathfrak{S}_n$, where $D_n(\boldsymbol{Z}_m)$ denotes the direct product (restricted direct product if $n = \infty$) of n copies of $\boldsymbol{Z}_m$. For $p > 1$, $G(m,p,n)$ are a special kind of normal subgroups $\mathfrak{S}_n(\boldsymbol{Z}_m)^{S(p)}$ of $\mathfrak{S}_n(\boldsymbol{Z}_m)$ (cf. §2).

We have studied, for this kind of wreath product groups, in case of $n = \infty$, characters of factor representations of finite type and construction of such kind of representations, and also in case $n < \infty$, construction of irreducible representations and explicit character formula for them, and furthermore the limiting procedure of characters when n tends to ∞. Our results in [Hir] and [HH1] ∼ [HH4] for these non-spin cases of $\mathfrak{S}_n(\boldsymbol{Z}_m)^{S(p)} = G(m,p,n)$ prepare a fundamental background of the spin case at present.

4. Among the category of $G(m,p,n)$, the groups $G(m,1,n)$ form an important subcategory called *generalized symmetric groups*, which were first introduced by Osima [Osi]. For the study on projective representations and spin characters, a generalized symmetric group $G(m,1,n)$ with $p = 1$ can be considered as a *mother group*, whereas complex reflection groups $G(m,p,n)$ with $p > 1$ as *her child groups*. The reason why we use this terminology is that many results for $G(m,p,n)$ follow from those for $G(m,1,n)$, as the results in §4 and §6 indicate it (cf. in particular, Theorem 6.2). For instance, in some cases, the restriction onto the subgroup $G(m,p,n)$ of a character on $G(m,1,n)$ is itself a character of $G(m,p,n)$.

5. The paper consists of two parts. Part I is the preparatory part for the whole of the future studies, which start from Part II and might continue so long, of the subject in the title of this paper. It is mainly devoted to the general theory of projective representations and spin characters of complex reflection groups. Concerning to them, when we first read Nazarov's paper [Naz, 1992] on projective representations of the infinite symmetric group $\mathfrak{S}_\infty = G(1,1,\infty)$, we have a strong impression about the explicit form of spin characters. Then we asked naturally several questions. In particular, *"Why spin characters of $\mathfrak{S}_\infty$ unilaterally have such unbalanced supports ?"* In the finite case of $\mathfrak{S}_n$, $n<\infty$, spin characters have unbalanced supports or rather balanced supports according as they are self-associate or non-self-associate (for the definition of self-associate, cf. Definition 8.1 in [II]), and so we ask *"What happened on the way of transition from the finite case of $\mathfrak{S}_n$, $n<\infty$, to the infinite case of $\mathfrak{S}_\infty$, $n=\infty$?"*

By definition (cf. §6.1), a normalized spin irreducible character of $G = G(m,p,n)$, $n<\infty$, or a normalized spin character of $G = G(m,p,\infty)$, is a function f on $R(G)$, normalized as $f(e)=1$, which satisfies the conditions (i), (ii) and (iii) below (then f satisfies automatically (iv) in addition):

(i) f is central $\big($or invariant under inner automorphisms of $R(G)\big)$,

(ii) positive definite,

(iii) extremal in the set $K_1\big(R(G)\big)$ of all functions satisfying (i)–(ii),

(iv) for some $\chi \in \widehat{Z}$ with $Z = H^2(G, \boldsymbol{C}^\times)$, f satisfies

$$f(zg') = \chi(z) f(zg') \quad \big(z \in Z, g' \in R(G)\big).$$

The central character $\chi \in \widehat{Z}$ above is called the *spin type* (or simply *type*) of f.

At the starting point of the series of our present studies for $G = G(m,p,n)$, we analyse questions similar as above in the following way. To be fundamental, we start from the most elementary assumption that a normalized central function f on $R(G)$ satisfies (iv), or has a certain spin type (and not assuming neither positive-definiteness nor extremality). Then we study how many important informations come out from this simplest assumption on f.

As basic foundations for the future, we study in §7, conjugacy relations (modulo Z) in $R(G)$. Then we prove that very important informations, principally on the evaluation of the support of f, come out from the above simple assumption: in §§8–9, for finite case $G = G(m,1,n)$, $n<\infty$, and in §10, for infinite case $G = G(m,1,\infty)$. As is proved partly in §6.6 and mainly in §11, these informations are crucial for factorisability of characters, and also in the infinite case of $G(m,1,\infty)$, for the validity of the criterion (EF) for a normalized central f with spin type to be a character, in each of different spin types. These informations help us to foresee and to carry out the explicit calculations of irreducible spin characters later. For the importance of evaluation of $\mathrm{supp}(f)$, also see §16.1 and [II, Tables 21.2 and 25.1] for example.

6. At the end of Part I, we add one section, §14, to extend general theory on the limiting process of (normalized) irreducible characters of an increasing sequence $H_1 \hookrightarrow H_2 \hookrightarrow \ldots \hookrightarrow H_n \hookrightarrow \ldots$ of finite groups, in particular the so-called

Vershik-Kerov ergodic method [VK]. In addition, to understand the situations better, we discuss also the case where H_n's are compact and H_n is imbedded into H_{n+1} continuously. This additional section serves as a preparatory section for the present papers [I] and [II] both. Put $H_\infty = \lim_{n\to\infty} H_n = \cup_{n\geq 1} H_n$. It is called *locally finite* in [Ker, p.5], when H_n's are finite, for which the following is known in [VK],

(b-1) *any normalized character f of H_∞ is a limit of normalized irreducible characters $\widetilde{\chi}_{\pi_n} := \chi_{\pi_n}/\dim\pi_n$ of IRs π_n of H_n.*

In the case of wreath product $H_n = \mathfrak{S}_n(T)$ of a compact group T with symmetric group $\mathfrak{S}_n$, this assertion is known by [HH5], [HH6], [HHH1] and [HoHH]. In the general case where H_n's are infinite compact groups, the assertion (b-1) and more are proved here in §14 (cf. Theorems 14.2 and 14.3 below).

We consider naturally a converse assertion to (b-1) as

(b-2) *if a series of normalized irreducible characters $\widetilde{\chi}_{\pi_n}$ has a pointwise limit f_∞, then it is necessarily a character of H_∞ . (??)*

For a locally finite group H_∞, Kerov wrote on the middle of p.11 of [Ker] as

> " Let us call a path $t \in T$ *regular* if the limits (5.2) exist. The corresponding limiting function φ_t is harmonic, though not necessarily extreme. ...."

As is explained in §14.5, this means that a pointwise limit $f_\infty = \lim_{n\to\infty} \widetilde{\chi}_{\pi_n}$ on H_∞ is not necessarily a character, and so (b-2) does not always hold.

We call a limit f_∞ a *bad* limit if it is not extremal or not continuous. In the case of symmetric groups $\mathfrak{S}_n \nearrow \mathfrak{S}_\infty$, the assertion (b-2) is true [VK]. It is also true in the case of wreath product groups $H_n = \mathfrak{S}_n(T)$ of a finite group T with symmetric groups $\mathfrak{S}_n$, as seen by [HH1] ∼ [HH4] and by [Boy]. On the contrary, when T is an infinite compact group, the assertion (b-2) never holds for $H_n = \mathfrak{S}_n(T)$ and $H_\infty = \mathfrak{S}_\infty(T)$, because explicit examples of bad limits are given in [HHH1] (cf. Theorems 6.1 and 7.1, loc. cit.) Actually we wish to establish (b-2) in the case of the universal covering groups $H_n = R\big(G(m,p,n)\big)$ of $G(m,p,n) = \mathfrak{S}_n(\boldsymbol{Z}_m)^{S(p)}$ and their limit $H_\infty = R\big(G(m,p,\infty)\big)$.

7. The paper is organized as follows. In Part I, after a preparation in §1, we give in §2, a constructive definition of $G(m,p,n)$ as the wreath products $\mathfrak{S}_n(\boldsymbol{Z}_m) = G(m,1,n)$ and their normal subgroups $\mathfrak{S}_n(\boldsymbol{Z}_m)^{S(p)} = G(m,p,n)$. In §3, representation groups $R(G)$ are explicitly given for generalized symmetric groups $G = G(m,1,n)$ and for their child groups $G = G(m,p,n)$, $p|m$, $p > 1$, by giving pairs of a set of generators and a set of fundamental relations, from the results of [DaMo] and [Rea1]. In §4, normal subgroups of $R\big(G(m,1,n)\big)$ corresponding to child groups are studied. By this it becomes clear that the study for mother groups $G(m,1,n)$ is fundamentally important so that they merit names of *mother groups*. In §5, the results in §§3–4 for the case of n finite are extended to the case of $n = \infty$. In §6 we collects general aspects of characters.

Since $R(G)$ is a special kind of central extension of G by $Z = H^2(G, \boldsymbol{C}^\times)$ as $1 \to Z \to R(G) \xrightarrow{\Phi} G \to 1$ (exact), any (irreducible) character f of $R(G)$ has its own spin type $\chi \in \widehat{Z}$ given as $f(zg') = \chi(z)f(g')$ $\big(z \in Z,\, g' \in R(G)\big)$. If χ is trivial, or $\chi = \mathbf{1}_Z$, then f is reduced to the base group G through the canonical homomorphism $\Phi : R(G) \to G \cong R(G)/Z$, and is essentially a non-spin character of G. On the other hand, we say $g', k' \in R(G)$ are *conjugate modulo* Z if there exists $h' \in R(G)$ such that $h'g'h'^{-1} = zk'$ for some $z \in Z$. In that case, we have $f(g') = \chi(z)f(k')$, and so f is fully determined if we know its values on a complete set of representatives of classes of *conjugacy modulo* Z. Going down on the base level G, we have $hgh^{-1} = k$ with $g = \Phi(g'), k = \Phi(k'), h = \Phi(h')$, and in this connection our former results for wreath product groups (e.g., [HH1] ∼ [HH4]) lead our calculations in §7 on the conjugation modulo Z. In §§8–9, starting from a simple assumption that a central function f has a spin type, the support of f defined as $\mathrm{supp}(f) := \{g' \in R(G); f(g') \neq 0\}$ is evaluated on the level of $G(m,1,n)$, according to the type χ and to the parity of m. In §10 these basic results on $\mathrm{supp}(f)$ for a central function f with spin type are extended to the case of $G = G(m,1,\infty)$, $n = \infty$. The properties of $\mathrm{supp}(f)$ depend heavily on the spin type χ of f.

A spin character f of $G = G(m,1,\infty)$ is said to be *factorizable* if

$$f(g'g'') = f(g')f(g'')$$

for $g', g'' \in R(G)$ with disjoint supports $\mathrm{supp}(g')$, $\mathrm{supp}(g'') \subset \boldsymbol{N}$, where $\mathrm{supp}(g') := \mathrm{supp}\big(\Phi(g')\big) \subset \boldsymbol{N}$. Factorisability holds in general for non-spin characters on the base level $G = \mathfrak{S}_\infty(\boldsymbol{Z}_m)$ and played important role in our previous works. But, for spin characters, this property does not necessarily hold for some spin types. We can examine this situation in §11 using the result in §10.

A projective representation π of G is called *of (spin) type* χ if $\pi(z) = \chi(z)I$ $\big(z \in Z = H^2(G, \boldsymbol{Z}^\times)\big)$, where I denotes the identity operator. In §12, for $G = G(m,1,\infty)$, we study if finite-dimensional spin representations exist or not for each spin type χ. By [DaMo], we have $H^2\big(G(m,1,\infty), \boldsymbol{Z}^\times\big) = \boldsymbol{Z}_2$ if m is odd, and $= \boldsymbol{Z}_2^3$ if m is even. So the case of m even is more complicated and interesting. Let the standard generators of $Z = \boldsymbol{Z}_2^3$ be z_1, z_2, z_3 (cf. Theorem 3.3), then spin type $\chi \in \widehat{Z}$ is expressed by $\beta = (\beta_1, \beta_2, \beta_3), \beta_i = \chi(z_i) = \pm 1$. Except non-spin type $\beta = (1,1,1)$, we have 7 different spin types. The results in Part I for $R\big(G(m,1,\infty)\big)$ with m even is summarized in §13 in Table 13.1, depending on spin types, separated into Cases I to VII.

8. In Part II, which is the start of our detailed studies, the case of spin type $\beta = (1,\, 1, -1)$, called Case VII, is studied along with the non-spin case with $\beta = (1,1,1)$, called Case VIII. These two cases have very intimate relation. In §15, we recall known results of the non-spin case. All the characters in Case VII can be obtained through a simple manner from those in Case VIII as is explained in §16, and thus we can arrive to an explicit character formula in Case VII, Type $(1,\, 1, -1)$, which is the final result for the problem (C) in this case. The relation of this result to the results of Dudko and Nessonov in [DuNe] on spin characters of $G(m,1,\infty)$, obtained by a different method, is simple in this case (cf. [II], §25).

Starting from §17, we study the problems (A) and (B) in Case VII. In §17, all the irreducible spin representations of $G(m,1,n)$ of type $\beta = (1,\ 1,-1)$ are constructed as induced representations. In §18, all the spin irreducible characters of this type are calculated. On the other hand, spin theory of generalized symmetric groups has been studied by Read [Rea1], Hoffman and Humphreys [HoHu1], Stembridge [Stem], and Morris and Jones [MoJo] etc. Our method here is quite different from theirs. In §19, the limiting process as $n \to \infty$ for series of normalized spin irreducible characters of $G(m,1,n)$ of this type, is studied.

A short summary of the essential part of Part I was reported in [HHH2].

Part I
General theory for complex reflection groups

1 Projective representations and representation groups

1.1. Projective representations and spin characters.

A *projective representation* ρ of a group G is by definition an assignment for each $g \in G$ a continuous operator $\rho(g)$ on a Hilbert space $V(\rho)$ satisfying

$$\rho(g)\rho(h) = r^{\rho}_{g,h}\,\rho(gh) \quad (g,h \in G), \tag{1.1}$$

where $r^{\rho}_{g,h} \in \boldsymbol{C}^{\times}$. The $\boldsymbol{C}^{\times}$-valued function $r^{\rho}_{g,h}$ on $G \times G$ is called the *factor set* of ρ. Here in this paper we treat only discrete groups and so there is no demand on the continuity of $g \to \rho(g)$, but we assume that each $\rho(g)$ is unitary. A character of a projective representation, if it exists in any sense, is called a *spin character* of G (and accordingly the representation itself is also called *spin*, as in [Mor]).

Two projective representations ρ and ρ' are mutually *equivalent* if there exists a bounded linear operator R from $V(\rho)$ onto $V(\rho')$ such that $R\,\rho(g)\,R^{-1} = \rho'(g)$ $(g \in G)$. They are called *associated* if their factor sets $r^{\rho}_{h,g}$ and $r^{\rho'}_{g,h}$ are mutually equivalent or if there exists a $\boldsymbol{C}^{\times}$-valued function c_g on G such that

$$r^{\rho}_{g,h} = \frac{c_g\, c_h}{c_{gh}} \cdot r^{\rho'}_{g,h} \quad (g,h \in G). \tag{1.2}$$

The equivalence class of $r^{\rho}_{g,h}$ is an element of the cohomology group $H^2(G,\boldsymbol{C}^{\times})$, which is called *Schur multiplier* of G.

Take a central extension $\widetilde{G}$ of G by an abelian subgroup Z:

$$(1.3) \qquad 1 \longrightarrow Z \longrightarrow \widetilde{G} \xrightarrow{\Phi} G \longrightarrow 1 \quad \text{(exact)},$$

where Φ denotes the natural homomorphism from $\widetilde{G}$ onto G. Let π be a unitary representation of $\widetilde{G}$ such that $\pi(z) = \lambda(z)\, I_{V(\pi)}$ $(z \in Z)$ with a non-zero scalar $\lambda(z)$ and the identity operator $I_{V(\pi)}$ on the representation space $V(\pi)$ of π. Take a section $\Psi : G \to \widetilde{G}$, that is, a map such that $\Phi \circ \Psi$ is the identity map on G, and put

$$(1.4) \qquad \rho(g) := \pi\big(\Psi(g)\big) \quad (g \in G),$$

then ρ is a projective representation of G with a factor set given by the character λ of Z as

$$(1.5) \qquad r^{\rho}_{g,h} := \lambda\big(\Psi(g)\,\Psi(h)\,\Psi(gh)^{-1}\big) \quad (g, h \in G).$$

A central extension $\widetilde{G}$ in (1.3) is called a *representation group* of G and denoted by $R(G)$ if it satisfies the following:

(PR1) any irreducible projective representation of G is equivalent to such a one which is associated to someone obtained as in (1.4)–(1.5);
(PR2) among central extensions with the property (PR1), it is minimal.

For any finite group G, Schur proved in [Sch1] that there exists a finite number of non-isomorphic representation groups of G, where (PR2) is replaced by "*the order* $|\widetilde{G}|$ *is the smallest.* He also proved that the central subgroup Z for any representation group is unique and isomorphic to Schur multiplier $H^2(G, \boldsymbol{C}^\times)$. Moreover he proved the following.

Lemma 1.1 ([Sch1, §5])**.** *Let G be a finite group. A central extension $\widetilde{G}$ in (1.3) is a representation group of G if and only if it satisfies* (1) *and* (2) *below:*

(1) *the central subgroup Z is contained in the commutator group $[\widetilde{G}, \widetilde{G}]$: $Z \subset [\widetilde{G}, \widetilde{G}]$;*

(2) *among such $\widetilde{G}$, the order $|\widetilde{G}|$ is the largest (which equals to $|G| \cdot |H^2(G, \boldsymbol{C}^\times)|$).*

Moreover the condition (2) *can be replaced by* (2′) *below:*

(2′) $|Z| = |H^2(G, \boldsymbol{C}^\times)|$.

We take one of representation groups of G and denote it by $R(G)$. Any *projective* representation of G can be lifted up to a linear representation of $R(G)$, and the study of projective representations of G is translated to the study of linear representations of $R(G)$. This situation is similar also for the study of spin characters. Any choice of representation group $R(G)$ of G gives an equivalent theory of projective representations of G. In this point of view, a representation group $R(G)$, even it is not necessarily unique, may be called as a *universal covering group* of G, comparing it to the case of connected Lie groups.

Definition 1.1. The *spin type* or simply *type* of a projective irreducible representation (=IR) π of G is a character χ of the central group $Z = H^2(G, \boldsymbol{C}^\times)$ such that $\pi(z) = \chi(z) I_{V(\pi)}$. A central (or an invariant) function f on $\widetilde{G}$, for instance a spin character, is called *of type* χ if

$$f(zg') = \chi(z)\, f(g') \quad (z \in Z,\ g' \in \widetilde{G}). \tag{1.6}$$

In the sequel we study projective representations and spin characters (cf. definition in §6.1) of complex reflection groups $G(m,p,n)$, and also of their inductive limits $G(m,p,\infty)$ as $n \to \infty$.

1.2. Cases of finite and infinite symmetric groups.

In [Sch3, Part I], it is proved that, for n-th symmetric group $\mathfrak{S}_n$, the Schur multiplier $H^2(\mathfrak{S}_n, \boldsymbol{C}^\times) = \boldsymbol{Z}_2$, and there exist two non-isomorphic representation groups $\mathfrak{T}_n, \mathfrak{T}'_n$ for $n \geq 4, n \neq 6$, and for $n = 6$ these two are isomorphic. Moreover, for n-th alternating group $\mathfrak{A}_n$, the cases of $n = 6, 7$ are exceptional, and for $n \geq 4, \neq 6, 7$, $H^2(\mathfrak{A}_n, \boldsymbol{C}^\times) = \boldsymbol{Z}_2$, and there exists unique representation group $\mathfrak{B}_n$ given as $\mathfrak{B}_n = [\mathfrak{T}_n, \mathfrak{T}_n] \cong [\mathfrak{T}'_n, \mathfrak{T}'_n]$.

Theorem 1.2 ([Sch3]). (i) *The n-th symmetric group $\mathfrak{S}_n$ is presented by a set of generators and fundamental relations as follows, where e denotes the identity element:*

- *set of generators:* $\{s_1, s_2, \ldots, s_{n-1}\}$ *with* $s_i = (i\ \ i{+}1)$ *simple reflections;*
- *set of fundamental relations:*

$$\text{(S-n)} \qquad \begin{cases} s_i^2 = e \ \ (1 \leq i \leq n-1), \quad (s_i s_{i+1})^3 = e \ \ (1 \leq i \leq n-2), \\ s_i s_j = s_j s_i \quad (|i-j| \geq 2). \end{cases}$$

(ii) *A representation group of $\mathfrak{S}_n$, which is denoted by $\mathfrak{T}'_n$ in [Sch3], is given by a set of generators and fundamental relations as follows:*

- *set of generators:* $\{z, r_1, r_2, \ldots, r_{n-1}\}$ *;*
- *set of fundamental relations:*

$$\text{(T-n)} \qquad \begin{cases} z^2 = e\,, \quad z r_i = r_i z \ \ (1 \leq i \leq n-1), \\ r_i^2 = e \ \ (1 \leq i \leq n-1), \quad (r_i r_{i+1})^3 = e \ \ (1 \leq i \leq n-2), \\ r_i r_j = z r_j r_i \quad (|i-j| \geq 2); \end{cases}$$

$$\{e\} \longrightarrow Z = \{z, e\} \longrightarrow \mathfrak{T}'_n \stackrel{\Phi}{\longrightarrow} \mathfrak{S}_n \longrightarrow \{e\},$$

with the natural homomorphism: $\Phi : \mathfrak{T}'_n \ni r_i \to s_i \in \mathfrak{S}_n$.

We denote $\mathfrak{T}'_n$ by $R(\mathfrak{S}_n)$ or $\widetilde{\mathfrak{S}}_n$ and prefer to use it in our study (rather than $\mathfrak{T}_n$). The infinite symmetric group $\mathfrak{S}_\infty$ is defined as the group of finite permutations of the set of natural numbers $\boldsymbol{N}$, then it is an inductive limit of $\mathfrak{S}_n$ as $n \to \infty$: $\mathfrak{S}_\infty := \lim\limits_{n\to\infty} \mathfrak{S}_n$. According to this, put $\mathfrak{T}'_\infty := \lim\limits_{n\to\infty} \mathfrak{T}'_n$, then we see that it is a representation group of $\mathfrak{S}_\infty$ and so denote it by $R(\mathfrak{S}_\infty)$ or $\widetilde{\mathfrak{S}}_\infty$:

$$(1.7) \qquad \mathfrak{S}_\infty = \lim_{n\to\infty} \mathfrak{S}_n, \quad R(\mathfrak{S}_\infty) = \lim_{n\to\infty} R(\mathfrak{S}_n).$$

Projective representations and spin characters of $\mathfrak{S}_\infty$ have been studied by Nazarov [Naz], where he has chosen groups $\mathfrak{T}_n$ and $\mathfrak{T}_\infty := \lim_{n\to\infty} \mathfrak{T}_n$ to be studied.

2 Wreath product groups and complex reflection groups

2.1. Wreath products of symmetric groups with finite abelian groups.

For a set I, denote by $\mathfrak{S}_I$ the group of finite permutations on I. For $I = \boldsymbol{I}_n := \{1, 2, \ldots, n\}$ or $I = \boldsymbol{I}_\infty := \boldsymbol{N}$, the suffices I are usually replaced by n or ∞ respectively: $\mathfrak{S}_{\boldsymbol{I}_n} = \mathfrak{S}_n$, $\mathfrak{S}_{\boldsymbol{N}} = \mathfrak{S}_\infty$. Take a finite abelian group T and define the wreath product groups $\mathfrak{S}_I(T)$ as follows:

$$(2.1) \qquad \mathfrak{S}_I(T) := D_I(T) \rtimes \mathfrak{S}_I, \quad D_I(T) := \prod_{j\in I}{}' T_j, \quad T_j := T \ \ (j \in I),$$

where $\prod'$ denotes the restricted direct product if I is infinite, and $\mathfrak{S}_I$ acts on $D_I(T)$ naturally by permuting the components. For a subgroup S of T, we have a canonical normal subgroup of $\mathfrak{S}_I(T)$ given as

$$(2.2) \qquad \mathfrak{S}_I(T)^S := D_I(T)^S \rtimes \mathfrak{S}_I, \quad D_I(T)^S := \{d \in D_I(T) \ ; \ P(d) \in S\},$$

where $P(d) := \prod_{j\in I} t_i$ for $d = (t_j)_{j\in I} \in D_I(T)$. Replacing $\mathfrak{S}_I$ by its subgroup $\mathfrak{A}_I$ consisting of even permutations, we define a subgroup $\mathfrak{A}_I(T)^S$ of $\mathfrak{S}_I(T)^S$ similarly. Later on, the index I is replaced by n or ∞ according to $I = \boldsymbol{I}_n$ or $I = \boldsymbol{N}$.

Now let $T = \boldsymbol{Z}_m$, understood as a multiplicative group. Then the groups $\mathfrak{S}_n(\boldsymbol{Z}_m)$ were introduced in [Osi] and called *generalized symmetric groups*. Any subgroup of $T = \boldsymbol{Z}_m$ is given as

$$(2.3) \qquad S(p) := \{t^p \ ; \ t \in T\} \cong \boldsymbol{Z}_{m/p} \qquad \text{for a divisor } p \text{ of } m.$$

We put $G(m, p, n) := \mathfrak{S}_n(\boldsymbol{Z}_m)^{S(p)}$ for n finite and also for $n = \infty$. Then we have

$$(2.4) \qquad G(m, p, \infty) = \lim_{n\to\infty} G(m, p, n) \qquad \text{for } \ p|m.$$

This is a constructive definition of finite and infinite complex reflection groups.

In [HH1], we have studied the characters of $\mathfrak{S}_\infty(T)$ and of $\mathfrak{S}_\infty(T)^S$ for any finite abelian group T, and in [HH2]–[HH3], similarly for the case of finite groups T. They serve as basic ingredient in our present study.

2.2. Classification of complex reflection groups.

A linear transformation on a complex finite-dimensional vector space V is called a *complex reflection* if it is of finite order and leaves a complex hyperplane invariant pointwise. A group G is called a *complex reflection group* if it is generated by complex reflections. In [ShTo], Shephard and Todd classified all finite complex reflection groups G acting irreducibly on V. In their classification, the groups $G(m,p,n)$ are divided into 3 infinite subfamilies numbered as 1 to 3 as follows, leaving other 34 exceptional groups aside:

1. symmetric groups $\mathfrak{S}_n = G(1,1,n)$;
2. $G(m,p,n) = \mathfrak{S}(\boldsymbol{Z}_m)^{S(p)},\ m > 1,\ n > 1,\ p|m,$
(this family contains generalized symmetric groups $G(m,1,n) = \mathfrak{S}_n(\boldsymbol{Z}_m)$) ;
3. cyclic groups $G(m,1,1) = \boldsymbol{Z}_m,\ G(m,p,1) = \boldsymbol{Z}_{m/p}$.

3 Representation groups of complex reflection groups

3.1 Representation groups of generalized symmetric groups

For a generalized symmetric group $G = G(m,1,n) = \mathfrak{S}_n(\boldsymbol{Z}_m)$, Davies and Morris [DaMo] gave its *Schur multiplier* $H^2(G,\boldsymbol{C}^\times)$ and also one of its representation groups. First we choose generators and fundamental relations as follows.

Proposition 3.1. *The generalized symmetric group $G(m,1,n) = \mathfrak{S}_n(\boldsymbol{Z}_m)$ is presented by*

- *set of generators :* $\{s_1, s_2, \ldots, s_{n-1},\ y_1, y_2, \ldots, y_n\}$,
  where y_j corresponds to a generator of $T_j = \boldsymbol{Z}_m$;
- *set of fundamental relations:*

$$\text{(ST-mn)}\qquad \begin{cases} \text{relations} \quad \text{(S-n)} \quad \text{for } \{s_1,\ldots,s_{n-1}\}, \\ y_j^{\,m} = e \qquad (1 \le j \le n), \\ y_j y_k = y_k y_j \qquad (j \ne k), \\ s_i y_i s_i^{-1} = y_{i+1},\ \ s_i y_{i+1} s_i^{-1} = y_i \quad (1 \le i \le n-1), \\ s_i y_j s_i^{-1} = y_j \qquad (j \ne i, i+1). \end{cases}$$

We can translate the result in [DaMo] as follows.

Theorem 3.2 (Case m odd)**.** *Suppose* $4 \le n$ *and* m *is odd.*

(i) *For* $G(m,1,n) = \mathfrak{S}_n(\boldsymbol{Z}_m)$, *a representation group* $R\big(G(m,1,n)\big)$ *is given as*

$$\{e\} \longrightarrow Z \longrightarrow R\big(G(m,1,n)\big) \xrightarrow{\Phi} G(m,1,n) \longrightarrow \{e\},$$

- *set of generators :* $\big\{z_1,\ r_i\,(1 \le i \le n-1),\ \eta_j\,(1 \le j \le n)\big\}$ *;*

$$\Phi(r_i) = s_i\ (1 \le i \le n-1),\ \Phi(\eta_j) = y_j\ (1 \le j \le n)\,;$$

- *set of fundamental relations :*

$$\begin{array}{ll}
\text{(i)} & z_1^{\,2} = e\,, \quad z_1 \ \text{central element}\,; \\
\text{(ii)} & \begin{cases} r_i^{\,2} = e \ \ (1 \le i \le n-1), \quad (r_i r_{i+1})^3 = e \ \ (1 \le i < n-1), \\ r_i r_j = z_1 r_j r_i \quad (|i-j| \ge 2), \end{cases} \\
\text{(iii)} & \eta_j^{\,m} = e \qquad (1 \le j \le n), \\
\text{(iv)} & \eta_j \eta_k = \eta_k \eta_j \ \ (j \ne k), \\
\text{(v)} & \begin{cases} r_i \eta_i r_i^{-1} = \eta_{i+1},\ r_i \eta_{i+1} r_i^{-1} = \eta_i \quad (1 \le i \le n-1), \\ r_i \eta_j r_i^{-1} = \eta_j \qquad (j \ne i, i+1)\,; \end{cases}
\end{array}$$

$$Z = H^2\big(G(m,1,n), \boldsymbol{C}^\times\big) = \langle z_1 \rangle \cong \boldsymbol{Z}_2.$$

(ii) *This representation group* $R\big(G(m,1,n)\big)$ *is isomorphic to the semidirect product of* $R(\mathfrak{S}_n)$ *with* $D_n(\boldsymbol{Z}_m)$ *as*

$$R\big(G(m,1,n)\big) \cong D_n(\boldsymbol{Z}_m) \rtimes R(\mathfrak{S}_n), \tag{3.1}$$

where $R(\mathfrak{S}_n) = \langle r_1, r_2, \ldots, r_{n-1} \rangle$ *acts on* $D_n(\boldsymbol{Z}_m) \cong \langle \eta_1, \eta_2, \ldots, \eta_n \rangle$ *through the quotient group* $R(\mathfrak{S}_n)/\langle z_1 \rangle \cong \mathfrak{S}_n$.

Theorem 3.3 (Case m even)**.** *Suppose* $4 \le n$ *and* m *is even. Then for* $G(m,1,n) = \mathfrak{S}_n(\boldsymbol{Z}_m)$, *a representation group* $R\big(G(m,1,n)\big)$ *is given as*

$$\{e\} \longrightarrow Z \longrightarrow R\big(G(m,1,n)\big) \xrightarrow{\Phi} G(m,1,n) \longrightarrow \{e\},$$

- *set of generators :* $\big\{z_1, z_2, z_3,\ r_i\,(1 \le i \le n-1),\ \eta_j\,(1 \le j \le n)\big\}$ *;*

$$\Phi(r_i) = s_i\ (1 \le i \le n-1),\ \Phi(\eta_j) = y_j\ (1 \le j \le n)\,;$$

- *set of fundamental relations :*

$$\begin{array}{ll}
\text{(i)} & z_i^{\,2} = e \ \ (1 \le i \le 3), \quad z_i \ \text{central element}\,; \\
\text{(ii)} & \begin{cases} r_i^{\,2} = e \ \ (1 \le i \le n-1), \quad (r_i r_{i+1})^3 = e \ \ (1 \le i < n-1), \\ r_i r_j = z_1 r_j r_i \quad (|i-j| \ge 2), \end{cases} \\
\text{(iii)} & \eta_j^{\,m} = e \qquad (1 \le j \le n), \\
\text{(iv)} & \eta_j \eta_k = z_2 \eta_k \eta_j \ \ (j \ne k), \\
\text{(v)} & \begin{cases} r_i \eta_i r_i^{-1} = \eta_{i+1},\ r_i \eta_{i+1} r_i^{-1} = \eta_i \quad (1 \le i \le n-1), \\ r_i \eta_j r_i^{-1} = z_3 \eta_j \qquad (j \ne i, i+1)\,; \end{cases}
\end{array}$$

$$Z = H^2\big(G(m,1,n), \boldsymbol{C}^\times\big) = \langle z_1, z_2, z_3 \rangle \cong \boldsymbol{Z}_2^3.$$

3.2 Representation groups of general complex reflection groups $G(m,p,n)$

For a general complex reflection group $G(m,p,n)$, Read [Rea1] gave its *Schur multiplier* and one of its representation groups. The main part of the former is given as below:

Table 3.1. Schur multiplier $H^2\big(G(m,p,n),\boldsymbol{C}^\times\big) \cong \boldsymbol{Z}_2^\ell$**,** $\ell = \ell(m,p,n)$, $5 \le n$**.**

CASE	p	$q=m/p$	$\ell(m,p,n)$
OO	odd	odd	1
OE	odd	even	3
EO	even	odd	2
EE	even	even	3

$\big($$n \ge 5$ is a stable range for the exponent $\ell = \ell(m,p,n)$$\big)$

To give a representation group in each case, we present the complex reflection group $G(m,p,n)$ by giving a set of generators and that of fundamental relations. Put

$$
(3.2) \qquad \begin{cases} x_1 := y_1^p & (\text{when } p=m,\ \ x_1 = e), \\ x_j := y_1^{-1}y_j & (2 \le j \le n). \end{cases}
$$

Proposition 3.4. *Let* $4 \le n < \infty$.

The complex reflection group $G(m,p,n) = \mathfrak{S}_n(\boldsymbol{Z}_m)^{S(p)}$ *is presented as follows:*

- *set of generators:* $\{s_1, s_2, \ldots, s_{n-1}\,;\ x_1, x_2, \ldots, x_n\}\,;$
- *set of fundamental relations:*

$$
\text{(ii)} \quad \begin{cases} s_i^2 = e \ \ (1 \le i \le n-1), \quad (s_is_{i+1})^3 = e \ \ (1 \le i \le n-2), \\ s_is_j = s_js_i \ \ (|i-j| \ge 2)\,; \end{cases}
$$

$$
\text{(iii)} \quad \begin{cases} x_1^q = e \quad \text{with } q = m/p\,, \\ x_j^m = e \quad (2 \le j \le n)\,; \end{cases}
$$

$$
\text{(iv)} \quad x_jx_k = x_kx_j \qquad (j \ne k)\,;
$$

$$
\text{(v-1)} \quad \begin{cases} s_ix_is_i^{-1} = x_{i+1} \\ s_ix_{i+1}s_i^{-1} = x_i \end{cases} \quad (2 \le i \le n-1),
$$
$$
s_ix_js_i^{-1} = x_j \qquad (2 \le i \le n-1,\ j \ne i, i+1,\ 1 \le j \le n),
$$

$$
\text{(v-2)} \quad \begin{cases} s_1x_1s_1^{-1} = x_1x_2^p, \\ s_1x_2s_1^{-1} = x_2^{-1}, \end{cases}
$$
$$
s_1x_js_1^{-1} = x_2^{-1}x_j \quad (3 \le j \le n).
$$

Note that the choice of generators in (3.2) is a little different from that of Read [Rea1] for the convenience of considering inductive limits as $n \to \infty$.

In [Rea1], the Schur multiplier $H^2\big(G(m,p,n)\big)$ is given for any possible (m,p,n), and on the way of calculating it, Read gave one of representation groups of $G(m,p,n)$, which we denote by $R\big(G(m,p,n)\big)$. It is written in a unified form, but we rewrite it separately in each of 4 cases in Table 3.1, since the structure of $R\big(G(m,p,n)\big)$ is one of fundamental ingredients of our study, and as our starting point it should be clearly written in our notation.

Theorem 3.5 (Case OO)**.** *Assume* $5 \le n < \infty$ *and* m *is odd.*
(i) *A representation group* $R\big(G(m,p,n)\big)$ *is given as*

- *set of generators:* $\{r_1, r_2, \dots, r_{n-1};\ w_1,\ w_2, \dots, w_{n-1}, w_n\}$,
$$\Phi(r_i) = s_i\ \ (1 \le n \le n-1), \quad \Phi(w_j) = x_j\ \ (1 \le j \le n)\,;$$
- *set of fundamental relations:*

$$\text{(i)} \qquad z_1^{\,2} = e, \quad z_1 \ \text{central element}\,;$$
$$\text{(ii)} \qquad \begin{cases} r_i^{\,2} = e \ \ (1 \le i \le n-1), \quad (r_i r_{i+1})^3 = e \ \ (1 \le i \le n-2), \\ r_i r_j = z_1 r_j r_i \ \ (|i-j| \ge 2)\,; \end{cases}$$
$$\text{(iii)} \qquad \begin{cases} w_1^{\,q} = e \quad \text{with } q = m/p, \\ w_j^{\,m} = e \quad (2 \le j \le n)\,; \end{cases}$$
$$\text{(iv)} \qquad w_j w_k = w_k w_j \qquad (j \ne k)\,;$$
$$\text{(v-1)} \qquad \begin{cases} r_i w_i r_i^{-1} = w_{i+1} \\ r_i w_{i+1} r_i^{-1} = w_i \end{cases} \quad (2 \le i \le n-1),$$
$$r_i w_j r_i^{-1} = w_j \qquad (2 \le i \le n-1,\ j \ne i, i+1,\ 1 \le j \le n)\,;$$
$$\text{(v-2)} \qquad \begin{cases} r_1 w_1 r_1^{-1} = w_1\, w_2^{\,p}, \\ r_1\, w_2\, r_1^{-1} = w_2^{-1}, \end{cases}$$
$$r_1 w_j r_1^{-1} = w_2^{-1} w_j \quad (3 \le j \le n).$$

(ii) *The representation group* $R\big(G(m,p,n)\big)$ *is isomorphic to the semidirect product of* $R(\mathfrak{S}_n)$ *with* $D_n(\boldsymbol{Z}_m)^{S(p)}$ *as*

$$\text{(3.3)} \qquad R\big(G(m,p,n)\big) \cong D_n(\boldsymbol{Z}_m)^{S(p)} \rtimes R(\mathfrak{S}_n),$$

where $R(\mathfrak{S}_n)$ *acts on* $D_n(\boldsymbol{Z}_m)^{S(p)} \cong \langle w_1, w_2, \dots, w_n\rangle$ *through the quotient group* $R(\mathfrak{S}_n)/\langle z_1\rangle \cong \mathfrak{S}_n$.

Theorem 3.6 (Case OE)**.** *Assume* $5 \le n < \infty$, p *be odd and* $q = m/p$ *be even* ($\therefore\ m = pq$ *even*). *A representation group* $R\big(G(m,p,n)\big)$ *in this case is given as follows:*

- *set of generators:* $\big\{z_1, z_2, z_3,\ r_1, r_2, \dots, r_{n-1},\ w_1,\ w_2, \dots, w_n\big\}\,;$
- *set of fundamental relations:*

$$\text{(i)} \qquad z_1^{\,2} = z_2^{\,2} = z_3^{\,2} = e, \quad z_i \ \text{central elements}\,;$$

(ii) $\begin{cases} r_i^{\,2} = e \ \ (1 \le i \le n-1), \quad (r_i r_{i+1})^3 = e \ \ (1 \le i \le n-2), \\ r_i r_j = z_1 r_j r_i \ \ (|i-j| \ge 2)\,; \end{cases}$

(iii) $\begin{cases} w_1^{\,q} = e & \text{with } q = m/p, \\ w_j^{\,m} = z_2^{\,m/2} & (2 \le j \le n)\,; \end{cases}$

(iv) $\quad w_j w_k = z_2 w_k w_j \qquad (j \ne k)\,;$

(v-1) $\begin{cases} r_i w_i r_i^{\,-1} = w_{i+1} \\ r_i w_{i+1} r_i^{\,-1} = w_i \end{cases} \quad (2 \le i \le n-1),$

$\quad r_i w_j r_i^{\,-1} = w_j \qquad (2 \le i \le n-1, \ j \ne i, i+1, \ 2 \le j \le n),$

$\quad r_i w_1 r_i^{\,-1} = z_3 w_1 \qquad (2 \le i \le n-1)\,;$

(v-2) $\begin{cases} r_1 w_1 r_1^{\,-1} = z_2^{\,(p-1)/2} z_3 \, w_1 \, w_2^{\,p}, \\ r_1 \, w_2 \, r_1^{\,-1} \ = \ w_2^{\,-1}, \end{cases}$

$\quad r_1 w_j r_1^{\,-1} = w_2^{\,-1} w_j \quad (3 \le j \le n)\,;$

$$Z = H^2\big(G(m,p,n), \boldsymbol{C}^\times\big) = \langle z_1, z_2, z_3 \rangle \cong \boldsymbol{Z}_2^{\,3}.$$

Theorem 3.7 (Case EO)**.** *Assume* $5 \le n < \infty$, *p be even and $q = m/p$ be odd ($\therefore$ m even). A representation group $R\big(G(m,p,n)\big)$ in this case is given as follows:*

- *set of generators:* $\big\{z_1, z_2, \ r_1, r_2, \ldots, r_{n-1}, \ w_1, \ w_2, \ldots, w_n\big\}$ *;*
- *set of fundamental relations:*

(i) $\quad z_1^{\,2} = e, \quad z_2^{\,2} = e, \quad z_i$ central elements$\,;$

(ii) $\begin{cases} r_i^{\,2} = e \ \ (1 \le i \le n-1), \quad (r_i r_{i+1})^3 = e \ \ (1 \le i \le n-2), \\ r_i r_j = z_1 \, r_j r_i \ \ (|i-j| \ge 2)\,; \end{cases}$

(iii) $\begin{cases} w_1^{\,q} = e & \text{with } q = m/p\,, \\ w_j^{\,m} = z_2^{\,m/2} & (2 \le j \le n)\,; \end{cases}$

(iv) $\begin{cases} w_j w_k = z_2 \, w_k w_j & (j \ne k, \ 2 \le j,k \le n)\,, \\ w_1 w_k = w_k w_1 & (2 \le k \le n)\,; \end{cases}$

(v-1) $\begin{cases} r_i w_i r_i^{\,-1} = w_{i+1} \\ r_i w_{i+1} r_i^{\,-1} = w_i \end{cases} \quad (2 \le i \le n-1),$

$\quad r_i w_j r_i^{\,-1} = w_j \qquad (2 \le i \le n-1, \ j \ne i, i+1, \ 1 \le j \le n)\,;$

(v-2) $\begin{cases} r_1 w_1 r_1^{\,-1} = z_2^{\,p/2} \, w_1 w_2^{\,p}, \\ r_1 w_2 r_1^{\,-1} = w_2^{\,-1}, \end{cases}$

$\quad r_1 w_j r_1^{\,-1} = w_2^{\,-1} w_j \quad (3 \le j \le n)\,;$

$$Z = H^2\big(G(m,p,n), \boldsymbol{C}^\times\big) = \langle z_1, z_2 \rangle \cong \boldsymbol{Z}_2^{\,2}.$$

Theorem 3.8 (Case EE)**.** *Assume* $5 \le n < \infty$ *and both p and q are even. A representation group $R\big(G(m,p,n)\big)$ is given as follows:*

- *set of generators:* $\big\{z_1, z_2, z_3, \ r_1, r_2, \ldots, r_{n-1}, \ w_1 \ w_2, \ldots w_n\big\}$ *;*

- *set of fundamental relations:*

(i) $z_1^{\,2} = z_2^{\,2} = z_3^{\,2} = e, \quad z_i$ central elements;

(ii) $\begin{cases} r_i^{\,2} = e \ \ (1 \le i \le n-1), \quad (r_i r_{i+1})^3 = e \ \ (1 \le i \le n-2), \\ r_i r_j = z_1\, r_j r_i \ \ (|i-j| \ge 2)\,; \end{cases}$

(iii) $\begin{cases} w_1^{\,q} = e \quad \text{with } q = m/p\,, \\ w_j^{\,m} = e \quad (2 \le j \le n)\,; \end{cases}$

(iv) $\begin{cases} w_j w_k = z_2\, w_k w_j \quad (j \ne k,\ 2 \le j,k \le n)\,, \\ w_1 w_k = w_k w_1 \qquad (2 \le k \le n)\,; \end{cases}$

(v-1) $\begin{cases} r_i w_i r_i^{-1} = w_{i+1} \\ r_i w_{i+1} r_i^{-1} = w_i \end{cases} \quad (2 \le i \le n-1),$

$r_i w_j r_i^{-1} = w_j \qquad (2 \le i \le n-1,\ j \ne i, i+1,\ 2 \le j \le n),$

$r_i w_1 r_i^{-1} = z_3\, w_1 \quad (2 \le i \le n-1)\,;$

(v-2) $\begin{cases} r_1 w_1 r_1^{-1} = z_2^{\,p/2} z_3\, w_1 w_2^{\,p}, \\ r_1 w_2 r_1^{-1} = w_2^{-1}, \end{cases}$

$r_1 w_j r_1^{-1} = w_2^{-1} w_j \quad (3 \le j \le n).$

$$Z = H^2\big(G(m,p,n), \boldsymbol{C}^\times\big) = \langle z_1, z_2, z_3 \rangle \cong \boldsymbol{Z}_2^{\,3}\,;$$

Note that, in Theorems 3.6 ∼ 3.8, $Z = \langle z_1 \rangle$, $\langle z_1, z_2, z_3 \rangle$, $\langle z_1, z_2 \rangle$, $\langle z_1, z_2, z_3 \rangle$, in respective cases is contained in $[R\big(G(m,p,n)\big), R\big(G(m,p,n)\big)]$ as is demanded in Lemma 1.1 (1).

4 Normal subgroups of $R\big(G(m,1,n)\big)$ corresponding to $G(m,p,n)$

In the exact sequence for the representation group of a generalized symmetric group:

$$\{e\} \longrightarrow Z \longrightarrow R\big(G(m,1,n)\big) \xrightarrow{\ \Phi\ } G(m,1,n) \longrightarrow \{e\}, \tag{4.1}$$

we take the full inverse image $\Phi^{-1}\big(G(m,p,n)\big)$, and compare it with the representation group $R\big(G(m,p,n)\big)$ of $G(m,p,n)$.

Theorem 4.1 (Cases OO)**.** *Let* $5 \le n < \infty$, *and both* p *and* $q = m/p$ *are odd. Put*

$$\begin{cases} w_1' = \eta_1^{\,p}, \\ w_j' = \eta_1^{-1}\eta_j \quad (2 \le j \le n), \end{cases} \tag{4.2}$$

in $R\big(G(m,1,n)\big)$. *Then the full inverse image* $\Phi^{-1}\big(G(m,p,n)\big)$ *is presented by*

$$\left\{\begin{array}{l}\textit{the set of generators } \{r_1, r_2, \ldots, r_{n-1}\,;\ w'_1, w'_2, \ldots, w'_n\} \textit{ and the set of} \\ \textit{fundamental relations obtained from that of Theorem 3.5 (Case OO) by} \\ \textit{replacing } w_j \textit{ with } w'_j.\end{array}\right.$$

The normal subgroup $\Phi^{-1}\big(G(m,p,n)\big) \subset R\big(G(m,1,n)\big)$ *is canonically isomorphic to the representation group* $R\big(G(m,p,n)\big)$ *under* $r_i \to r_i$ $(1 \le i \le n-1)$, $w'_j \to w_j$ $(1 \le j \le n)$.

Theorem 4.2 (Cases OE and EO)**.** *Let* $5 \le n < \infty$, *and assume that one of* p, q *is odd and the other is even. Put*

$$\left\{\begin{array}{l} w'_1 = \eta_1^{\,p}, \\ w'_j = z_3^{\,j-1}\eta_1^{-1}\eta_j \quad (2 \le j \le n). \end{array}\right. \tag{4.3}$$

in $R\big(G(m,1,n)\big)$. *Then the full inverse image* $\Phi^{-1}\big(G(m,p,n)\big) \subset R\big(G(m,1,n)\big)$ *is presented by*

$$\left\{\begin{array}{l}\textit{the set of generators } \{r_1, r_2, \ldots, r_{n-1}\,;\ w'_1, w'_2, \ldots, w'_n\} \textit{ and the set of} \\ \textit{fundamental relations obtained from that of Theorem 3.6 (Case OE) or} \\ \textit{that of Theorem 3.7 (Case EO) replacing by replacing } w_j \textit{ with } w'_j.\end{array}\right.$$

The normal subgroup $\Phi^{-1}\big(G(m,p,n)\big) \subset R\big(G(m,1,n)\big)$ *is canonically isomorphic to the representation group* $R\big(G(m,p,n)\big)$ *under* $r_i \to r_i$ $(1 \le i \le n-1)$, $w'_j \to w_j$ $(1 \le j \le n)$.

Theorem 4.3 (Case EE)**.** *Let* $5 \le n < \infty$, *and both* p, q *are even. Put* w'_j *be as in* (4.3) *above. Let* H' *be the subgroup of* $R\big(G(m,1,n)\big)$ *generated by the set*

$$\{r_1, r_2, \ldots, r_{n-1}\,;\ w'_1, w'_2, \ldots, w'_n\}. \tag{4.4}$$

Then the latter satisfies a set of relations obtained from that of Theorem 3.8 (Case EE) by replacing w_j *with* w'_j *and also by reducing* z_3 *to* e.

In this manner, the normal subgroup H' *of* $R\big(G(m,1,n)\big)$ *is canonically isomorphic to the quotient group* $R\big(G(m,p,n)\big)/\langle z_3\rangle$ *under the correspondence* $r_i \to r_i$ $(1 \le i \le n-1)$, $w'_j \to w_j \bmod \langle z_3\rangle$ $(1 \le j \le n)$.

Sketch of Proofs of Theorems 4.1 ∼ 4.3.

The assertion that the set $\{r_1, r_2, \ldots, r_{n-1}\,;\ w'_1, w'_2, \ldots, w'_n\}$ satisfies the corresponding fundamental relations of the set of generators $\{r_1, r_2, \ldots, r_{n-1};\, w_1, w_2, \ldots, w_n\}$ (resp. under modulo $\langle z_3\rangle$ for Theorem 4.3) can be proved by calculations.

Hence we know that the map $r_i \to r_i$ $(1 \le i \le n-1)$, $w_j \to w'_j$ $(1 \le j \le n)$ (resp. $w'_j \to w_j \bmod \langle z_3\rangle$ $(1 \le j \le n)$) from $R\big(G(m,p,n)\big)$ (resp. from $R\big(G(m,p,n)\big)/\langle z_3\rangle$ into $R\big(G(m,1,n)\big)$) is homomorphic.

To prove that this homomorphism is actually an isomorphism, it is enough to count the orders of both groups. □

Summarizing the main result, we obtain the following table.

Table 4.1. Inclusion relations between representation groups.

Case	**Inclusion Relation**
OO, OE, EO	$R\big(G(m,p,n)\big) \hookrightarrow R\big(G(m,1,n)\big)$
EE	$R\big(G(m,p,n)\big)/\langle z_3\rangle \cong H' \subset R\big(G(m,1,n)\big)$

Remark 4.1. Thus we see that, in Cases OO, OE and EO, if we obtain some results on (spin) characters of $R\big(G(m,1,n)\big)$, then by restriction we get the corresponding informations on (spin) characters of $R\big(G(m,p,n)\big)$, and similarly for the case of $n=\infty$ (cf. Theorem 7.1 in [HH4] or §§6.3 ∼ 6.4 and Theorem 6.2 below). Moreover the same assertion holds for

$$R\big(G(m,p,n)\big)/\langle z_3\rangle \cong H' \subset R\big(G(m,1,n)\big).$$

Additional discussions will help us to get rather complete information on (spin) characters f of $R\big(G(m,p,n)\big)$ for which $f(z_3g') = -f(g')$ $\big(g' \in R\big(G(m,p,n)\big)\big)$.

In this sense, we call generalized symmetric groups $G(m,1,n)$, $4 \le n < \infty$, as *mother groups* and $G(m,p,n)$ with $p>1$ as her *child groups* of $G(m,1,n)$. The study on projective representations and spin characters of mother groups is fundamental and plays a crucial role for studying the cases of child groups (cf. §16.1 below).

The situation is also similar in the case of infinite general complex reflection groups.

5 Infinite version $R\big(G(m,1,\infty)\big)$ and $R\big(G(m,p,\infty)\big)$

It can be proved that the inductive limits $\lim_{n\to\infty} R\big(G(m,1,n)\big)$ and $\lim_{n\to\infty} R\big(G(m,p,n)\big)$ are representation groups of $G(m,1,\infty)$ and $G(m,p,\infty)$ respectively, and so we can denote them by $R\big(G(m,1,\infty)\big)$ and $R\big(G(m,p,\infty)\big)$. Here we only list up the similar results as in §§3 ∼ 4 as follows.

5.1. Similarly as Proposition 3.1, the infinite generalized symmetric group $G(m,1,\infty) = \lim_{n\to\infty} G(m,1,n)$ has a presentation with

a set of generators $\{s_i\ (1 \le i < \infty)\,;\ y_j\ (1 \le j < \infty)\}$, and
a set of fundamental relations in (ST-mn) in Proposition 3.1, but replacing
"$1 \le i \le n-1$" by "$1 \le i < \infty$", and "$1 \le j \le n$" by "$1 \le j < \infty$".

5.2. Quite similar theorems as Theorem 3.2 (Case m odd) and Theorem 3.3 (Case m even) hold under the above replacements.

5.3. Similar result as in Table 3.1 holds for $G(m,p,\infty)$.

5.4. Similarly as Proposition 3.4, the infinite generalized symmetric group $G(m,p,\infty) = \lim_{n\to\infty} G(m,p,n)$, $p|m$, has a presentation with
a set of generators $\{s_i\ (1 \le i < \infty)\,;\ x_j\ (1 \le j < \infty)\}$, and
a set of fundamental relations in Proposition 3.4 but replacing
"$i \le n-1$" and "$i \le n-2$" by "$i < \infty$"; and "$j \le n$" by "$j < \infty$".

5.5. Quite similar theorems as Theorem 3.5 (Case OO), Theorem 3.6 (Case OE), Theorem 3.7 (Case EO) and Theorem 3.8 (Case EE) hold under the above replacements, and we have the following.

Theorem 5.1. *There exist canonical embeddings as normal subgroups:*

$$\begin{cases} R\big(G(m,p,\infty)\big) \hookrightarrow R\big(G(m,1,\infty)\big) & \text{(in Case OO, EO, OE)}, \\ R\big(G(m,p,\infty)\big)/\langle z_3\rangle \cong H' \subset R\big(G(m,1,\infty)\big) & \text{(in Case EE)}, \end{cases}$$

where H' denotes the subgroup of $R\big(G(m,p,\infty)\big)$ generated by $\{r_i\ (1 \le i < \infty),\ w'_j\ (1 \le j < \infty)\}$ in Case EE.

5.6. Similar remark as Remark 4.1 holds also for the relations between (spin) characters of a mother group $R\big(G(m,1,\infty)\big)$ and those of child groups $R\big(G(m,p,\infty)\big)$,
$p > 1, p|m$. This leads the direction of our study.

6 General aspects about characters of groups

6.1. Characters. We give here a definition of character in a certain narrow sense, and we will utilize it hereafter except otherwise clearly stated.

In general, for a topological group G, denote by $\mathcal{P}(G)$ the set of continuous positive definite functions on G, by $K(G)$ the set of $f \in \mathcal{P}(G)$ central or invariant under G. Put $K_1(G) := \{f \in K(G); f(e) = 1\}$ with the identity element e of G, and $E(G) := \mathrm{Extr}\big(K_1(G)\big)$ the set of all extremal points of the convex set $K_1(G)$. We call a function $f \in E(G)$ a *character* of G. It corresponds 1-1 way to the normalized character of a quasi-equivalence class of factor representation of finite type of G, that is, that of a finite-dimensional irreducible representation or of a II_1 factor representation (cf. e.g., [HH3]).

6.2. Induction of characters from a subgroup.

Let H be a subgroup of G. Corresponding to taking a diagonal matrix element of induced representation from H to G, we define *trivial extension* $\widetilde{f}$ of a positive definite function $f \in \mathcal{P}(H)$ as

$$\widetilde{f}(g) := \begin{cases} f(g) & \text{if } g \in H, \\ 0 & \text{if } g \notin H. \end{cases} \tag{6.1}$$

Then $\widetilde{f}$ is positive definite on G, and is continuous if H is open in G.

Moreover, in the case where G/H is compact, for a character f of H, we define an *inducing up* $F = \mathrm{Ind}_H^G f$ of f as

$$F(g) := \int_{G/H} \widetilde{f}(kgk^{-1})\, d\dot{k} \quad (g, k \in G), \tag{6.2}$$

where $\dot{k} = kH \in G/H$ and $d\dot{k}$ denotes the normalized invariant measure on G/H. The function F is positive definite but its continuity is to be discussed. This process $f \to F$ corresponds to the process of taking the normalized character of an induced representation.

6.3. Restriction of characters onto a normal subgroup.

Let N be a *normal* subgroup of G. For $g \in G$, the restriction of the inner automorphism $\iota(g)$ of G onto N is denoted by $\iota_{|N}(g)$, and the set of all $\iota_{|N}(g), g \in G$, is denoted by $\mathrm{Aut}_G(N)$.

Suppose a function F on G is invariant. Then its restriction $f := F|_N$ onto N is G-invariant or invariant under $\mathrm{Aut}_G(N) \supset \mathrm{Int}(N)$. Put

$$\begin{aligned} &K(N,G) := \{f \in K(N)\,;\, f \text{ is } G\text{-invariant}\}, \\ &K_1(N,G) := \{f \in K(N,G)\,;\, f(e) = 1\}, \\ &E(N,G) := \text{the set of all extremal elements in } K_1(N,G). \end{aligned} \tag{6.3}$$

Lemma 14 in [Tho1] asserts the following (more generally, see [HH6, Theorem 14.1]):

Lemma 6.1 ([Tho1, Lemma 14]). *Let G be a countable discrete group and N its normal subgroup. For any character $F \in E(G)$, its restriction $f = F|N$ belongs to $E(N,G)$.*

6.4. Restriction in the case of representation groups $R\big(G(m,1,\infty)\big)$.

Let $G = R\big(G(m,1,\infty)\big)$ and N be its normal subgroup given as follows:

(6-a) In case m is odd, or in case m is even and Case OO, OE or EO, put $N = R\big(G(m,p,\infty)\big)$ which is imbedded canonically into G as in Theorem 5.1;

(6-b) In case m is even and Case EE, put $N = R\big(G(m,p,\infty)\big)/\langle z_3\rangle$ imbedded canonically into G as in Theorem 5.1.

Theorem 6.2. (i) *The index* $[\mathrm{Aut}_G(N), \mathrm{Int}(N)] = p$.

(ii) *A function f on N is G-invariant if and only if it is invariant, and $E(N) = E(N,G)$.*

(iii) *The restriction map $E(G) \ni F \to f = F|_N \in E(N,G) = E(N)$ is surjective.*

Proof. (i) Here we give a proof only for Case OO, $n = \infty$. For the other cases we will give necessary comments afterwards. Let $\boldsymbol{Z}^{(\boldsymbol{N})} := \prod'_{i \in \boldsymbol{N}} \boldsymbol{Z}_i$ be the restricted direct product of additive groups $\boldsymbol{Z}_i = \boldsymbol{Z}$ $(i \in \boldsymbol{N})$, and for $a = (a_i)_{i\in \boldsymbol{N}} \in \boldsymbol{Z}^{(\boldsymbol{N})}$, put $d'(a) := \eta_1^{a_1}\eta_2^{a_2}\cdots \in G = R\big(G(m,1,\infty)\big)$ and $\mathrm{ord}\big(d'(a)\big) := a_1 + a_2 + \cdots$, where the product and the sum are actually finite since $a_i = 0$ except for a finite number of i's by definition. Define

$$\begin{cases} \boldsymbol{Z}^{(\boldsymbol{N})}(p) := \big\{a \in \boldsymbol{Z}^{(\boldsymbol{N})};\ a_1 + a_2 + \cdots \equiv 0 \ (\mathrm{mod}\ p)\big\},\ \widetilde{\mathfrak{S}}_\infty := \langle r_i\ (i \in \boldsymbol{N})\rangle, \\ \widetilde{D}_\infty := \big\langle \eta_j\ (j \in \boldsymbol{N})\big\rangle,\ \ \widetilde{D}_\infty^{S(p)} := \big\{d' \in \widetilde{D}_\infty\ ;\ \mathrm{ord}(d') \equiv 0\ (\mathrm{mod}\ p)\big\}, \end{cases}$$

Then there hold isomorphisms $\widetilde{D}_\infty \cong \boldsymbol{Z}^{(\boldsymbol{N})}$ and $\widetilde{D}_\infty^{S(p)} \cong \boldsymbol{Z}^{(\boldsymbol{N})}(p)$. Also we have semidirect product expressions as $G = \widetilde{D}_\infty \rtimes \widetilde{\mathfrak{S}}_\infty$ and $N = \widetilde{D}_\infty^{S(p)} \rtimes \widetilde{\mathfrak{S}}_\infty$, whence

$$\iota(G) = \iota\big(\widetilde{D}_\infty\big) \rtimes \iota\big(\widetilde{\mathfrak{S}}_\infty\big), \quad \iota(N) = \iota\big(\widetilde{D}_\infty^{S(p)}\big) \rtimes \iota\big(\widetilde{\mathfrak{S}}_\infty\big). \tag{6.4}$$

Hence $[\mathrm{Aut}_G(N) : \mathrm{Int}(N)] = \big[\iota(G)\big|_N : \iota(N)\big] = \big[\iota\big(\widetilde{D}_\infty\big)\big|_N : \iota\big(\widetilde{D}_\infty^{S(p)}\big)\big]$.

To calculate the latter index, we consider a map $\mathcal{C}$ on $\boldsymbol{Z}^{(\boldsymbol{N})}$ as

$$\mathcal{C}:\ \boldsymbol{Z}^{(\boldsymbol{N})} \ni a \to b = (b_i)_{i\in\boldsymbol{N}} \in \boldsymbol{Z}^{(\boldsymbol{N})}, \quad b_i := a_i - a_{i+1}\ \ (i \in \boldsymbol{N}). \tag{6.5}$$

Then we see that $\mathcal{C}$ is an automorphism of $\boldsymbol{Z}^{(\boldsymbol{N})}$, and the inverse $\mathcal{C}^{-1}(b) = a$ is given by the formula $a_i = \sum_{j\geq i} b_j$. Denote by $\boldsymbol{Z}^{(\boldsymbol{N})}((p))$ the image of $\boldsymbol{Z}^{(\boldsymbol{N})}(p)$ under $\mathcal{C}$.

On the other hand, we can prove, by calculation using Theorem 3.2, the following formula: with $w_1 = \eta_1^p, w_j = \eta_1^{-1}\eta_j$ $(j > 1)$,

$$\begin{cases} \iota\big(d'(a)\big)r_1 := \big(w_2^{-1}\big)^{b_1}\cdot r_1\,, \\ \iota\big(d'(a)\big)r_i := \big(w_i w_{i+1}^{-1}\big)^{b_i}\cdot r_i\ \ (i > 1), \end{cases} \tag{6.6}$$

and $\iota\big(d'(a)\big) =$ the identity on $\widetilde{D}_\infty$, where $b = \mathcal{C}(a)$. Moreover we see by calculation using Theorem 3.5 that this formula holds also for N with $a \in \boldsymbol{Z}^{(\boldsymbol{N})}(p)$.

Since $\mathcal{C}$ is bijective, we can define, for any $b = (b_i)_{i\in\boldsymbol{N}} \in \boldsymbol{Z}^{(\boldsymbol{N})}$ $\big($resp. $b \in \boldsymbol{Z}^{(\boldsymbol{N})}((p))\big)$, an automorphism $\Theta(b)$ of G (resp. of N) by putting

$$\begin{cases} \Theta(b)r_1 := \big(w_2^{-1}\big)^{b_1}\cdot r_1\,, \\ \Theta(b)r_i := \big(w_i w_{i+1}^{-1}\big)^{b_i}\cdot r_i\,, \end{cases} \tag{6.7}$$

and $\Theta(b) :=$ the identity on $\widetilde{D}_\infty$ (resp on $\widetilde{D}_\infty^{S(p)}$), and there holds on $\boldsymbol{Z}^{(\boldsymbol{N})}$ $\big($resp. between $\boldsymbol{Z}^{(\boldsymbol{N})}(p)$ and $\boldsymbol{Z}^{(\boldsymbol{N})}((p))\big)$.

$$\iota\big(d'(a)\big) = \Theta(b), \quad b = \mathcal{C}(a). \tag{6.8}$$

Thus we see that $\big[\iota\big(\widetilde{D}_\infty\big)\big|_N : \iota\big(\widetilde{D}_\infty^{S(p)}\big)\big] = \big[\boldsymbol{Z}^{(\boldsymbol{N})} : \boldsymbol{Z}^{(\boldsymbol{N})}(p)\big] = p$.

Now, for other cases, the proof is essentially similar with the help of Theorems 3.3 and 3.6 – 3.8. Only the difference is that there appear multiplicative factors coming from the central subgroups $\langle z_2, z_3\rangle$ for the case EO and EE, and $\langle z_2\rangle$ for the case OE.

(ii) For any $g' \in G, h' \in N$, if $g' \notin N$, there exists an element $g'' \in G$ commuting with h' modulo Z such that $g'g'' \in N$. Cf. Theorem 16.2 in [HH6].

(iii) We can discuss as for Theorem 15.1 in [HH6]. □

Theorem 6.2 above tells us that if we know all the spin characters for a generalized symmetric groups $G(m,1,\infty)$, *a mother group*, then the results for spin characters of complex reflection groups $G(m,p,\infty)$, *her child groups*, are obtained simply by restriction, except Case EE. The last case needs some more additional studies.

6.5. (Spin) types of projective representations and of spin characters.

For a complex reflection group $G(m,p,n), 5 \le n \le \infty$, take one of its representation groups as

$$\{e\} \longrightarrow Z \longrightarrow R\big(G(m,p,n)\big) \stackrel{\Phi}{\longrightarrow} G(m,p,n) \longrightarrow \{e\}. \tag{6.9}$$

A projective representation π of $G(m,p,n)$ is called *of (spin) type* $\chi \in \widehat{Z}$, if it satisfies $\pi(z) = \chi(z) I_{V(\pi)}$, where χ is a one-dimensional character of the central group Z. An irreducible representation or a factor representation of $R\big(G(m,p,n)\big)$ has its own type.

A character $f \in E\big(R\big(G(m,p,n)\big)\big)$ of $R\big(G(m,p,n)\big)$ has its own (spin) type $\chi \in \widehat{Z}$ because of its extremality, that is,

$$f(zg') = \chi(z)\, f(g') \quad \big(z \in Z,\, g' \in R\big(G(m,p,n)\big)\big). \tag{6.10}$$

For example, in the case of $G(m,p,n)$ with $q = m/p$ is even, a character χ of $Z = \langle z_1, z_2, z_3\rangle$ is given as

$$\chi(z_i) = \beta_i = \pm 1 \ \ (1 \le i \le 3), \qquad \beta := (\beta_1, \beta_2, \beta_3). \tag{6.11}$$

In the following, *type* χ is often called *type* β.

6.6. Factorisability of spin characters of complex reflection groups.

For a $g = (d,\sigma) \in G(m,1,n) = D_n(T) \rtimes \mathfrak{S}_n, T = \boldsymbol{Z}_m$, define supports $\mathrm{supp}(d)$, $\mathrm{supp}(\sigma)$, $\mathrm{supp}(g)$ as

$$\begin{cases} \mathrm{supp}(d) := \{i \in \boldsymbol{I}_n \,;\, t_i \ne e_T\} \ \text{ with } \ d = (t_i)_{i \in \boldsymbol{I}_n},\ t_i \in T_i = T, \\ \mathrm{supp}(\sigma) := \{j \in \boldsymbol{I}_n \,;\, \sigma(j) \ne j\}, \\ \mathrm{supp}(g) := \mathrm{supp}(d) \bigcup \mathrm{supp}(\sigma), \end{cases} \tag{6.12}$$

where e_T denotes the identity element of T. For a $g' \in R\big(G(m,p,n)\big)$ such that $g = \Phi(g') \in G(m,p,n) \subset G(m,1,n)$, we put $\mathrm{supp}(g') := \mathrm{supp}(g)$.

Definition 6.1. A normalized central (or invariant) positive definite function $f \in K_1(G')$, $G' = R\big(G(m,p,n)\big)$, $5 \le n \le \infty$, $p|m$, is called *factorizable* if

$$f(g'g'') = f(g')f(g''), \tag{6.13}$$

for any $g', g'' \in G'$ with $\mathrm{supp}(g') \bigcap \mathrm{supp}(g'') = \varnothing$.

We will check in §11 if the following criterion for $f \in K_1(G'), G' = R\big(G(m,1,\infty)\big)$, to be extremal, or to be a character, holds or not, for each of spin types of f separately:

(EF) f is extremal $\iff$ f is factorizable.

Denote by $F(G')$ the set of all factorizable $f \in K_1(G')$, then $F(G') \subset E(G')$ in general (see just below), and $F(G') = E(G')$ if the criterion (EF) holds.

In some of previous works, this criterion was proved and played important roles, for instance, for $G(1,1,\infty) = \mathfrak{S}_\infty$ in [Tho2], for spin characters of $\mathfrak{S}_\infty$ in [Naz], for $G(m,1,\infty) = \mathfrak{S}_\infty(\boldsymbol{Z}_m)$ in [HH1], for $G(m,p,\infty) = \mathfrak{S}_\infty(\boldsymbol{Z}_m)^{S(p)}$ in [HH2] and [HH4], and for $\mathfrak{S}_\infty(T)$ with T a compact group in [HH5]–[HH6]. Contrary to these cases, for Cases II, III and VII of the present groups $G' = R\big(G(m,1,\infty)\big)$, the criterion (EF) does not hold (cf. Theorem 11.1 below).

Before going into §§7–10 of rather long studies on conjugacies in $G'_n := R\big(G(m,1,n)\big)$, $4 \le n \le \infty$, and on supports of $f \in K_1^{\mathrm{Y}}(G'_n)$ for Cases Y=I $\sim$ VII, which prepare basic informations for our later study, we prove here impatiently the following theorem, borrowing some results in §7 in advance.

Theorem 6.3. *For the infinite group $R\big(G(m,1,\infty)\big)$, the implication*

factorizable $\Longrightarrow$ extremal

is always true. In other words, for $f \in K_1\big(R\big(G(m,1,\infty)\big)\big)$ of certain type $\chi \in \widehat{Z}$, "to be factorizable" is sufficient for "to be a character".

Proof. This assertion can be proved similarly as the "if"-part of the proof for Satz 1 in [Tho2, pp.42–44]. In fact, let $f \in K_1(G')$, $G' = R\big(G(m,1,\infty)\big)$, be in Case Y and of type χ^{Y}, then f is completely determined if its values for representative elements of conjugacy classes modulo Z is known. Let the notations be as in **7.1.3**, Case of $R\big(G(m,1,\infty)\big)$. The set $K_1^{\mathrm{Y}}(G')$ is compact and convex, and the parameter space $\mathcal{S}^{\mathrm{Y}}$ for factorizable f's are imbedded into it through $\mathcal{S}^{\mathrm{Y}} \ni s \to f_s \in K_1^{\mathrm{Y}}(G')$. We apply Gelfand's theorem on uniform convergence in the space of continuous functions $C(\mathcal{S}^{\mathrm{Y}})$ on the compact set $\mathcal{S}^{\mathrm{Y}}$, and the Choquet-Bishop-de Leeuw representation theorem using the set of extremal points $E^{\mathrm{Y}}(G')$ in $K_1^{\mathrm{Y}}(G')$. For more details, see e.g. [HH4, §15]. □

Note 6.1. The implication "factorizable $\Longrightarrow$ extremal" in Theorem 6.3 is not true for any finite group $R\big(G(m,1,n)\big)$, $n < \infty$. In fact, let $f = \delta_e$ the delta function supported by the identity element e. Then it is factorizable in the trivial sense, but it is the normalized character of the regular representation.

7 Conjugation in $R\big(G(m,p,n)\big)$ modulo Z

7.1 Conjugacy classes modulo Z of a representation group $R\big(G(m,p,n)\big)$

7.1.1. For mother groups $G(m,1,n)$ we take usually as $n \geq 4$, and for child groups $G(m,p,n), p > 1$, we take usually as $n \geq 5$ because of Theorems 3.2 ~ 3.3 and Theorems 3.4 ~ 3.8 respectively. Let f be an central positive definite function on $R\big(G(m,1,n)\big), 4 \leq n \leq \infty$, or on $R\big(G(m,p,n)\big), p > 1, 5 \leq n \leq \infty$, of type χ. Then

$$f(z\,h'g'h'^{-1}) = \chi(z)f(g') \quad \big(z \in Z,\ g', h' \in R\big(G(m,p,n)\big)\big). \tag{7.1}$$

Therefore f is totally determined if the values $f(g')$ are given on a complete set of representatives of conjugacy classes modulo Z. Here we say $g', g'' \in R\big(G(m,p,n)\big)$ are *mutually conjugate modulo Z* if $g' = z\,h'g'h'^{-1}$ for some $z \in Z$, $h' \in R\big(G(m,p,n)\big)$, where Z is $\langle z_1\rangle$ or $\langle z_1, z_2, z_3\rangle$ according as m is odd or even. As a general terminology, we call a function f on $R(G)$ a *spin function* of spin type $\chi \in \widehat{Z}$ if $f(zg') = \chi(z)f(g')$ $\big(z \in Z, g' \in R(G)\big)$.

To study spin characters of $G(m,p,n)$ (containing the case of $n = \infty$), we are asked

(7-1) to fix a complete set of representatives of conjugacy classes of $R\big(G(m, p,n)\big)$ modulo Z;

(7-2) to study structure of every conjugacy classes modulo Z of $R\big(G(m,p,n)\big)$;

(7-3) to fix a section Ψ from $G(m,p,n)$ to $R\big(G(m,p,n)\big)$, especially for elements in the set of representatives.

Then we will apply the results to a central function f on $R(G)$ with a certain non-trivial spin type χ, to evaluate its support and so on.

7.1.2. Case of $G(m,1,\infty)$. For the demand **(7-1)**, we recall the case of $G(m,1,n) = \mathfrak{S}_n(T), T = \boldsymbol{Z}_m$, $4 \leq n \leq \infty$, from [HH1]. Define for $g = (d,\sigma) \in G(m,1,n) = \mathfrak{S}_n(T)$, $d = (t_i)_{i\in \boldsymbol{I}_n} \in D_n(T)$, $t_i \in T_i = T$, $\sigma \in \mathfrak{S}_n$, their supports as in (6.12). An element $g = (d,\sigma) \in \mathfrak{S}_n(T)$ is called *basic* if

CASE 1: σ is cyclic and $\mathrm{supp}(d) \subset \mathrm{supp}(\sigma)$,

CASE 2: $\sigma = \mathbf{1}$ and for $d = (t_i)_{i\in\boldsymbol{I}_n}$, $t_q \neq e_T$ only for one $q \in \boldsymbol{I}_n$.

The element $(d, \mathbf{1})$ in Case 2 is denoted by $\xi_q = \xi_q(t_q) = (t_q, (q))$, where (q) denotes the symbolic permutation of length 1 consisting of one point q. An arbitrary element $g = (d,\sigma) \in \mathfrak{S}_n(T)$ is expressed as a product of basic elements as

$$g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r}g_1g_2\cdots g_s \tag{7.2}$$

with $g_j = (d_j, \sigma_j)$ in Case 1, and $q_1, q_2, \ldots, q_r, \mathrm{supp}(g_j)$ $(1 \leq j \leq s)$ are mutually disjoint. This is called the *standard decomposition* of g even though it is unique only up to the orders between ξ_{q_k}'s and between g_j's. For a basic component $g_j=(d_j,\sigma_j)$, put

$$(7.3)\qquad \begin{cases} \ell_j := \ell(\sigma_j) = |\mathrm{supp}(\sigma_j)|, \\ K_j := \mathrm{supp}(\sigma_j), \quad d_j := (t_i)_{i\in K_j}, \quad P(d_j) := \prod_{i\in K_j} t_i. \end{cases}$$

Theorem 7.1. *The conjugacy class of* $g \in \mathfrak{S}_n(T), T = \boldsymbol{Z}_m$, *is characterized by the set*

$$(7.4)\qquad \big\{\,(t_{q_k},1)\;(1\le k\le r),\;\; (P(d_j),\ell(\sigma_j)\,)\;(1\le j\le s)\big\}.$$

According to this, we prepare the set $\Omega := \{(t,\ell)\,;\, t\in T,\, \ell\ge 1\}$ as a fundamental ingredient of a set of representatives of conjugacy classes of the infinite group $G := G(m,1,\infty)$, and $[g]$ is parametrized by $\boldsymbol{n}(g) := \big(n_\omega(g)\big)_{\omega\in\Omega} \in \big(\boldsymbol{Z}_{\ge 0}\big)^{(\Omega)}$, where $n_\omega(g)$ denotes the multiplicity of $\omega=(t,\ell)$ in (7.4). For a central function f on G, if it is factorizable, similarly as in Definition 6.1, then it is expressed as

$$(7.5)\qquad f(g) = \prod_{\omega\in\Omega} s_\omega^{\,n_\omega(g)},$$

where $s_\omega = f(g_\omega)$ for a fixed representative $g_\omega\in G$ of $\omega\in\Omega$. Put $s=(s_\omega)_{\omega\in\Omega}$, then it belongs to the direct product $\prod_{\omega\in\Omega} D_\omega$ of unit discs $D_\omega=\{z\in\boldsymbol{C}; |z|\le 1\}$. Denote f in (7.5) by f_s. Then the positive definiteness of $f=f_s$ is expressed by the set of inequalities expressed by polynomials of finite number of $s_\omega, \overline{s_\omega}$, and so the subset $\mathcal{S}$ of $\prod_{\omega\in\Omega} D_\omega$ consisting s from $f=f_s\in K_1(G)$ is closed and so compact. In [HH4, §15], Theorem 12 asserts the validity of the criterion (EF) for $G=G(m,1,\infty)$. In that occasion, to prove the implication " factorizable ⇒ extremal", we utilize the compact set $\mathcal{S}$ in a clever manner following the idea of [Tho2].

7.1.3. Case of $R\big(G(m,1,\infty)\big)$. We can imitate this method for the covering group $G' := R\big(G(m,1,\infty)\big)$ too. However the situation is a little complicated. For each $\omega\in\Omega$, fix once for all a representative $g'_\omega\in G'$ such that $\Phi(g'_\omega)=g_\omega$ with Φ in (6.9). Denote the conjugacy class of $g'\in G'$ modulo Z by $[g']_Z$, then it corresponds bijectively to the conjugacy class $[g]$ of $g=\Phi(g')\in G$, hence it is parametrized by $\boldsymbol{n}(g') := \boldsymbol{n}(g) \in \big(\boldsymbol{Z}_{\geqslant 0}\big)^{(\Omega)}$ with $n_\omega(g') := n_\omega(g)$. Any element $g''\in[g'_\omega]$ is expressed as $g'' = z'h'g'_\omega h'^{-1}$ with $z'\in Z$, $h'\in G'$, and for $f\in K_1^{\mathrm{Y}}(G')$, we have, by (7.1), $f(g'')=\chi^{\mathrm{Y}}(z')s_\omega$ with $s_\omega := f(g'_\omega)$.

Now, for a $g'\in G'$, put $g=\Phi(g')$ and let (7.2) be its standard decomposition. Taking appropriate preimages ξ'_{q_k} $(k\in\boldsymbol{I}_r)$, g'_j $(j\in\boldsymbol{I}_s)$ of corresponding elements such as $\Phi(\xi'_{q_k})=\xi_{q_k}$ etc., we have a decomposition of g', called *standard* as

$$(7.6)\qquad g' = \xi'_{q_1}\xi'_{q_2}\cdots\xi'_{q_r} g'_1 g'_2\cdots g'_s\,.$$

Let the parameters in Ω corresponding to ξ'_{q_k} and g'_j be respectively be $\omega^{(k)}$ and $\omega^{[j]}$. Then there are $z^{(k)}$, $z^{[j]} \in Z$ such that $\xi'_{q_k} \sim z^{(k)} g'_{\omega^{(k)}}$ and $g'_j \sim z^{[j]} g'_{\omega^{[j]}}$, where $\sim$ denotes the conjugacy under G'. If an $f \in K_1^{\mathrm{Y}}(G')$ is factorizable, then

$$f(g') = \chi^{\mathrm{Y}}\big(z^{(1)} \cdots z^{(r)} z^{[1]} \cdots z^{[s]}\big) \cdot \prod_{\omega\in\Omega} s_\omega^{\,n_\omega(g')}. \tag{7.7}$$

Denote this f by f_s^{Y}, then the subset $\mathcal{S}^{\mathrm{Y}}$ of $\prod_{\omega\in\Omega} D_\omega$ consisting s from $f_s^{\mathrm{Y}} \in K_1^{\mathrm{Y}}(G')$ is closed and so compact. This fact is utilized to prove Theorem 6.3 above.

7.2 Conjugation around a $g' \in R\big(G(m,1,n)\big)$ modulo Z (preparation)

For the demand **(7-2)**, we proceed as follows. First recall the exact sequence for generalized symmetric group $G(m,1,n) = \mathfrak{S}_n(\boldsymbol{Z}_m)$, $n \geq 4$, as

$$\{e\} \longrightarrow Z \longrightarrow R\big(G(m,1,n)\big) \xrightarrow{\ \Phi\ } G(m,1,n) \longrightarrow \{e\}, \tag{7.8}$$

with $Z = \langle z_1\rangle$ if m is odd, and $Z = \langle z_1, z_2, z_3\rangle$ if m is even.

To prove the results in this section, we appeal to calculations using the following lemmas on the structure of representation groups.

Lemma 7.2. *Put, for* $j<k$, $r_{jk} := (r_j r_{j+1} \cdots r_{k-2}) r_{k-1} (r_{k-2} \cdots r_{j+1} r_j)$, $r_{kj} = r_{jk}$ *in* $R\big(G(m,1,n)\big)$. *Then,* $\Phi(r_{jk}) = (s_j s_{j+1} \cdots s_{k-2}) s_{k-1} (s_{k-2} \cdots s_{j+1} s_j) = (j\ \ k)$
$=: s_{jk}$ *the permutation of* j *and* k.

(i) $$\begin{cases} r_i r_{jk} r_i^{-1} = z_1 r_{jk} & (j,k \neq i, i+1), \\ r_i r_{ji} r_i^{-1} = r_{j,i+1}, \quad r_i r_{j,i+1} r_i^{-1} = r_{ji} & (j \neq i, i+1). \end{cases}$$

(ii) *In case* m *is even,* $$\begin{cases} r_{jk} \eta_j r_{jk}^{-1} = z_3^{\,k-j-1}\, \eta_k, \\ r_{jk} \eta_k r_{jk}^{-1} = z_3^{\,k-j-1}\, \eta_j, \\ r_{jk} \eta_i r_{jk}^{-1} = z_3\, \eta_i \qquad (i \neq j,k). \end{cases}$$

Proof. (i) $r_i r_p r_i^{-1} = z_1 r_p$ $(p \neq i, i+1)$,

$$r_i (r_{i-1} r_i r_{i+1}) r_i^{-1} = r_{i-1} r_i r_{i-1} \cdot r_{i+1} r_i = z_1 r_{i-1} r_i r_{i+1} \cdot r_{i-1} r_i,$$
$$r_i (r_{i+1} r_i r_{i-1}) r_i^{-1} = r_i r_{i+1} \cdot r_{i-1} r_i r_{i-1} = z_1 r_i r_{i-1} \cdot r_{i+1} r_i r_{i-1},$$

$$\begin{aligned} \therefore\ \ r_i r_{jk} r_i^{-1} &= \\ &= z_1^{2(i-1-j)+2} (r_j r_{j+1} \cdots r_{i-1} r_i r_{i+1})(r_{i-1} r_i) \times \\ &\quad r_i (r_{i+2} \cdots r_{k-2}) r_{k-1} (r_{k-2} \cdots r_{i+2}) r_i^{-1} \cdot (r_i r_{i-1})(r_{i+1} r_i r_{i-1} \cdots r_{j+1} r_j) \\ &= z_1 (r_j r_{j+1} \cdots r_{i-1} r_i r_{i+1})(r_{i+2} \cdots r_{k-2}) r_{k-1} (r_{k-2} \cdots r_{i+2}) \cdot \\ &\quad \cdot (r_{i+1} r_i r_{i-1} \cdots r_{j+1} r_j) = z_1 r_{jk}. \end{aligned}$$

Suppose $j < i$, then

$$r_i r_{ji} r_i \ \ = \ \ r_i (r_j r_{j+1} \cdots r_{i-2} r_{i-1} r_{i-2} \cdots r_{j+1} r_j) r_i$$

$$= \quad (r_j r_{j+1} \cdots r_{i-2}) r_i r_{i-1} r_i (r_{i-2} \cdots r_{j+1} r_j)$$
$$= \quad (r_j r_{j+1} \cdots r_{i-2}) r_{i-1} r_i r_{i-1} (r_{n-2} \cdots r_{j+1} r_j) = r_{j,i+1}.$$

(ii) $r_{jk} \eta_j r_{jk}^{-1} = (r_j r_{j+1} \cdots r_{k-2} r_{k-1} r_{k-2} \cdots r_{j+1} r_j) \eta_j \cdot$
$\cdot (r_j r_{j+1} \cdots r_{k-2} r_{k-1} r_{k-2} \cdots r_{j+1} r_j)$
$= (r_j r_{j+1} \cdots r_{k-2}) \eta_k (r_{k-2} \cdots r_{j+1} r_j) = z_3^{k-j-1} \eta_k.$ □

Take an element $g' \in R\big(G(m,1,n)\big), G(m,1,n) = \mathfrak{S}_n(\boldsymbol{Z}_m)$, and put

$$\mathcal{Z}(g') := \{h' \in R\big(G(m,1,n)\big) \ ; \ \Phi(h' g' h'^{-1}) = \Phi(g')\}, \tag{7.9}$$

which is the centralizer of g' modulo Z. We study the group $\mathcal{Z}(g')$ or more exactly the set of elements $z \in Z$ appearing as $h' g' h'^{-1} = z g'$ under the conjugations $g' \to h' g' h'^{-1}$ by $h' \in \mathcal{Z}(g')$. Put $g = \Phi(g')$, $h = \Phi(h')$, then $h \in Z(g) := \{h \in G(m,1,n); hgh^{-1} = g\}$. Then a set of generators of the centralizer $Z(g) \subset G(m,1,n)$ of g can be obtained rather easily, and using it we have the following.

Lemma 7.3. *Let $4 \le n \le \infty$. For $g' \in R\big(G(m,1,n)\big)$, let the standard decomposition of $g := \Phi(g') \in G(m,1,n)$ be as*

$$g = \Phi(g') = (d, \sigma) = \xi_{q_1} \xi_{q_2} \cdots \xi_{q_r} \, g_1 g_2 \cdots g_s \in \mathfrak{S}_n(\boldsymbol{Z}_m) = G(m,1,n).$$

Then, the stationary subgroup in $R\big(G(m,1,n)\big)$ of g', considered modulo the central subgroup Z, contains the following elements:

(i) r_k for $k, k+1 \notin \operatorname{supp}(g')$,
(ii) η_k for $k \notin \operatorname{supp}(g')$,
(iii) η_{q_i} $(1 \le i \le r)$,
(iv) g'_j such that $\Phi(g'_j) = g_j$ $(1 \le j \le s)$,
(v) $\widetilde{\eta}_j := \prod_{p \in \operatorname{supp}(\sigma_j)} \eta_p$ $(1 \le j \le s)$,

where the product for $\widetilde{\eta}_j$ depends on the order of taking product by a factor z_2^a.

Notation 7.1. For a cyclic permutation $\sigma \in \mathfrak{S}_n$, let $\ell(\sigma)$ be its length. For a general $\sigma \in \mathfrak{S}_n$, let $L(\sigma)$ be its length with respect to simple reflections, and take its cyclic decomposition $\sigma = \sigma_1 \cdots \sigma_r$, then $L(\sigma) \equiv \sum_{1 \le j \le r} \big(\ell(\sigma_j) - 1\big) \pmod 2$, and $\operatorname{sgn}(\sigma) = (-1)^{L(\sigma)}$. For $\sigma' \in R(\mathfrak{S}_n)$ with $\sigma = \Phi(\sigma') \in \mathfrak{S}_n$, we put

$$L(\sigma') := L(\sigma), \quad \operatorname{sgn}(\sigma') := \operatorname{sgn}(\sigma). \tag{7.10}$$

7.3 Conjugation around a $g' \in R\big(G(m,1,n)\big)$ modulo Z (Case of m odd)

Theorem 7.4 (Case m odd, $4 \le n < \infty$)**.** *For $g' \in R\big(G(m,1,n)\big), G(m,1,n) = \mathfrak{S}_n(\boldsymbol{Z}_m)$, $4 \le n < \infty$, m odd, let the standard decomposition of $g = \Phi(g') \in$*

$G(m,1,n)$ be as in (7.2)*, then the conjugations by elements in* (i) *and* (vi) *give the following conjugacy relations, and* (ii), (iii) *and* (v) *give trivial ones:*

$$\begin{array}{lll} \text{(i)} & r_k g' r_k^{-1} = z_1^{\,L(\sigma)} \cdot g' & \text{if } |\mathrm{supp}(g')| \le n-2 \text{ and } \mathrm{supp}(g') \not\ni k, k+1\,; \\ \text{(iv)} & g'_j g' g'_j{}^{-1} = z_1^{\,(L(\sigma)-L(\sigma_j))L(\sigma_j)} \cdot g' & (1 \le j \le s). \end{array}$$

Theorem 7.5 (Case m odd, $n=\infty$)**.** *For $g' \in R\big(G(m,1,\infty)\big), G(m,1,\infty) = \mathfrak{S}_\infty(\boldsymbol{Z}_m)$, let the standard decomposition of $g = \Phi(g') \in G(m,1,\infty)$ be as in* (7.2)*, then the conjugations by elements in in* (i) *and* (vi) *give the following conjugacy relations, and* (ii), (iii) *and* (v) *give trivial ones:*

$$\begin{array}{lll} \text{(i)} & r_k g' r_k^{-1} = z_1^{\,L(\sigma)} \cdot g', & \\ \text{(iv)} & g'_j g' g'_j{}^{-1} = z_1^{\,(L(\sigma)-L(\sigma_j))L(\sigma_j)} \cdot g' & (1 \le j \le s). \end{array}$$

7.4 Conjugation around a $g' \in R\big(G(m,1,n)\big)$ modulo Z (Case of m even)

Definition 7.1. For $d = y_1^{a_1} y_2^{a_2} \cdots y_n^{a_n} \in D_n(\boldsymbol{Z}_m)$, and for $d' \in \widetilde{D}_n(\boldsymbol{Z}_m) := \langle \eta_1, \ldots, \eta_n \rangle$ such that $d = \Phi(d')$, we put

$$\mathrm{ord}(d) := \sum_{1 \le j \le n} a_j \pmod m, \quad \mathrm{ord}(d') := \mathrm{ord}(d). \tag{7.11}$$

Theorem 7.6 (Case m even, $4 \le n < \infty$)**.** *For $g' \in R\big(G(m,1,n)\big), G(m,1,n) = \mathfrak{S}_n(\boldsymbol{Z}_m)$, $4 \le n < \infty$, m even, let the standard decomposition of $g = \Phi(g') \in G(m,1,n)$ be as in* (7.2)*, then the conjugations by elements in* (i) $\sim$ (v) *give the following conjugacy relations:*

$$\begin{array}{ll} \text{(i)} & r_k g' r_k^{-1} = z_1^{\,L(\sigma)} \cdot z_3^{\,\mathrm{ord}(d)} \cdot g', \ \text{ if } |\mathrm{supp}(g')| \le n-2 \text{ and } \mathrm{supp}(g') \not\ni k, k+1; \\ \text{(ii)} & \eta_k g' \eta_k^{-1} = z_2^{\,\mathrm{ord}(d)} \cdot z_3^{\,L(\sigma)} \cdot g', \quad \text{if } |\mathrm{supp}(g')| \le n-1 \text{ and } \mathrm{supp}(g') \not\ni k\,; \\ \text{(iii)} & \eta_{q_i} g' \eta_{q_i}^{-1} = z_2^{\,\mathrm{ord}(d)-\mathrm{ord}(\xi_{q_i})} \cdot z_3^{\,L(\sigma)} \cdot g' \qquad (1 \le i \le r), \\ \text{(iv)} & g'_j g' g'_j{}^{-1} = z_1^{\,(L(\sigma)-L(\sigma_j))L(\sigma_j)} \cdot z_2^{\,(\mathrm{ord}(d)-\mathrm{ord}(d_j))\mathrm{ord}(d_j)} \cdot \\ & \qquad \cdot z_3^{\,L(\sigma)\,\mathrm{ord}(d_j)+\mathrm{ord}(d)\,L(\sigma_j)} \cdot g' \qquad (1 \le j \le s), \\ \text{(v)} & \widetilde{\eta}_j g' \widetilde{\eta}_j^{-1} = z_2^{\,\mathrm{ord}(d)(L(\sigma_j)+1)+L(\sigma_j)-\mathrm{ord}(d_j)} \cdot z_3^{\,L(\sigma)(L(\sigma_j)+1)} \cdot g' \quad (1 \le j \le s). \end{array}$$

Proof. (i) Take $h' \in R\big(G(m,1,n)\big)$ such that $g'' = h' g' h'^{-1}$ satisfies $\mathrm{supp}(g'') \not\ni 1, 2$.

$$\begin{array}{ll} & r_1 g'' r_1^{-1} = z_1^{\,L(\sigma)} \cdot z_3^{\,\mathrm{ord}(d)} \cdot g'', \quad r_1 g'' r_1^{-1} = (r_1 h' r_1^{-1}) r_1 g' r_1^{-1} (r_1 h' r_1^{-1})^{-1}, \\ \therefore & z_1^{\,L(\sigma)} \cdot z_3^{\,\mathrm{ord}(d)} \cdot g'' = (h'^{-1} r_1 h') g' (h'^{-1} r_1 h')^{-1}. \end{array}$$

(v) Let $g' = g''g'_j g'''$, $\Phi(g'') = \xi_{q_1}\cdots\xi_{q_r}g_1\cdots g_{j-1}$, $\Phi(g'_j) = g_j = (d_j,\sigma_j)$, $\Phi(g''') = g_{j+1}\cdots g_s$, then with $K_j := \mathrm{supp}(\sigma_j)$

$$\widetilde{\eta}_j g' \widetilde{\eta}_j^{-1} = z_2^{(\mathrm{ord}(d)-\mathrm{ord}(d_j))\ell(\sigma_j)} z_3^{(L(\sigma)-L(\sigma_j))\ell(\sigma_j)} g'' \cdot \widetilde{\eta}_j g'_j \widetilde{\eta}_j^{-1} \cdot g''' =: \clubsuit$$

$$g'_j := \prod\nolimits_{p\in K_j} \eta_p^{a_p} \cdot \sigma'_j, \quad \Phi(\sigma'_j) = \sigma_j, \quad \widetilde{\eta}_j = \eta_1\eta_2\cdots\eta_\ell,$$

$$\eta_q \cdot \prod\nolimits_{p\in K_j} \eta_p^{a_p} \cdot \eta_q^{-1} = z_2^{\sum_{p\neq q} a_p} \prod\nolimits_{p\in K_j} \eta_p^{a_p}$$

$$\therefore \quad \widetilde{\eta}_j \cdot \prod\nolimits_{p\in K_j} \eta_p^{a_p} \cdot \widetilde{\eta}_j^{-1} = z_2^{\sum_q \sum_{p\neq q} a_p} \prod\nolimits_{p\in K_j} \eta_p^{a_p}$$

$$= z_2^{(|\mathrm{supp}(\sigma_j)|-1)\sum_p a_p} \prod\nolimits_{p\in K_j} \eta_p^{a_p} = z_2^{\mathrm{ord}(d_j)(\ell(\sigma_j)-1)} \prod\nolimits_{p\in K_j} \eta_p^{a_p},$$

put $\quad \widetilde{\eta}_j \cdot \sigma'_j \cdot \widetilde{\eta}_j^{-1} = z_2^{B} z_3^{C} \sigma'_j, \quad$ then,

$$\clubsuit = z_2^{(\mathrm{ord}(d)-\mathrm{ord}(d_j))\ell(\sigma_j)} z_3^{(L(\sigma)-L(\sigma_j))\ell(\sigma_j)} g'' \cdot z_2^{\mathrm{ord}(d_j)(\ell(\sigma_j)-1)} \big(z_2^{B} z_3^{C} g'_j\big) \cdot g'''$$

$$= z_2^{(\mathrm{ord}(d)-\mathrm{ord}(d_j))(L(\sigma_j)+1)+\mathrm{ord}(d_j)\,L(\sigma_j)+B} z_3^{(L(\sigma)-L(\sigma_j))(L(\sigma_j)+1)+C} g'$$

$$= z_2^{\mathrm{ord}(d)(L(\sigma_j)+1)-\mathrm{ord}(d_j)+B} z_3^{L(\sigma)(L(\sigma_j)+1)+C} g',$$

put $\quad \sigma'_j := r_1 r_2 \cdots r_{\ell-1} \quad$ with $\ell = \ell(\sigma_j)$,

$$\sigma_j = \Phi(\sigma'_j) = \Phi(r_1r_2\cdots r_{\ell-1}) = s_1s_2\ldots s_{\ell-1} = (1\ 2\ 3\ \ldots\ \ell),$$

$$\therefore \quad \widetilde{\eta}_j \cdot r_1r_2\cdots r_{\ell-1} = z_3^{(\ell-1)(\ell-2)} \cdot (r_1r_2\cdots r_{\ell-1}) \cdot \eta_\ell \cdot \eta_1\eta_2\cdots\eta_{\ell-1}$$

$$= z_2^{\ell-1}(r_1r_2\cdots r_{\ell-1}) \cdot \eta_1\eta_2\cdots\eta_{\ell-1}\eta_\ell = z_2^{L(\sigma_j)}(r_1r_2\cdots r_{\ell-1}) \cdot \widetilde{\eta}_j,$$

$$\therefore \quad B = L(\sigma_j), \ C = 0.$$

$$\therefore \quad \widetilde{\eta}_j g' \widetilde{\eta}_j^{-1} = z_2^{\mathrm{ord}(d)(L(\sigma_j)+1)+L(\sigma_j)-\mathrm{ord}(d_j)} z_3^{L(\sigma)(L(\sigma_j)+1)} \cdot g'.$$

□

In the proof, we have seen the following.

Lemma 7.7. *Let* $g'_j = \prod_{p\in K_j} \eta_p^{a_p} \cdot \sigma'_j$, $\Phi(\sigma'_j) = \sigma_j$, $\widetilde{\eta}_j = \prod_{p\in K_j} \eta_p$. *Then*

$$\widetilde{\eta}_j g'_j \widetilde{\eta}_j^{-1} = z_2^{(\mathrm{ord}(d_j)+1)L(\sigma_j)} g'_j. \tag{7.12}$$

Theorem 7.8 (Case m even, $n=\infty$)**.** *For* $g' \in R\big(G(m,1,\infty)\big)$, $G(m,1,\infty) = \mathfrak{S}_\infty(\boldsymbol{Z}_m)$, *$m$ even, let the standard decomposition of* $g = \Phi(g') \in G(m,1,\infty)$ *be as in* (7.2), *then the conjugations by elements in* (i) ∼ (v) *give the following conjugacy relations:*

(i) $r_k g' r_k^{-1} = z_1^{L(\sigma)} \cdot z_3^{\mathrm{ord}(d)} \cdot g'$,

(ii) $\eta_k g' \eta_k^{-1} = z_2^{\mathrm{ord}(d)} \cdot z_3^{L(\sigma)} \cdot g'$,

(iii) $\eta_{q_i} g' \eta_{q_i}^{-1} = z_2^{\mathrm{ord}(d)-\mathrm{ord}(\xi_{q_i})} \cdot z_3^{L(\sigma)} \cdot g' \qquad (1 \le i \le r)$,

(iv) $g'_j g' {g'_j}^{-1} = z_1^{(L(\sigma)-L(\sigma_j))L(\sigma_j)} \cdot z_2^{(\mathrm{ord}(d)-\mathrm{ord}(d_j))\mathrm{ord}(d_j)} \cdot$
$\cdot z_3^{L(\sigma)\,\mathrm{ord}(d_j)+\mathrm{ord}(d)\,L(\sigma_j)} \cdot g' \qquad (1 \le j \le s)$,

(v) $\widetilde{\eta}_j g' \widetilde{\eta}_j^{-1} = z_2^{\mathrm{ord}(d)(L(\sigma_j)+1)+L(\sigma_j)-\mathrm{ord}(d_j)} \cdot z_3^{L(\sigma)(L(\sigma_j)+1)} \cdot g' \quad (1 \le j \le s)$.

Note 7.1. When we look into details on conjugacy among g_j's and that among ξ_{q_i}'s, we may have some elements in $\mathcal{Z}(g')$ which are not generated by elements listed up in Lemma 7.3 as follows.

(i) Suppose $g_j = (d_j, \sigma_j)$ and $g_{j'} = (d_{j'}, \sigma_{j'})$ are similar, that is, there exists a $\tau \in \mathfrak{S}_n$ only permuting K_j and $K_{j'}$ such that $\tau\, g_j\, \tau^{-1} = g_{j'}$, i.e., $\tau\sigma_j\tau^{-1} = \sigma_{j'}$, $\tau d_j \tau^{-1} = d_{j'}$. Then $\tau \in Z(g)$. Take $\tau' \in R(G)$ such that $\Phi(\tau') = \tau$, then

$$\tau'\, g'\, \tau'^{-1} = z_1^{(L(\sigma)-1)L(\sigma_j)}\, z_2^{\mathrm{ord}(d_j)}\, z_3^{\mathrm{ord}(d)L(\sigma_j)}\, g'. \tag{7.13}$$

(ii) Suppose ξ_q and ξ_p are similar, that is, $\tau\xi_q\tau^{-1} = \xi_p$ with $\tau = (p\ q)$. Take an element τ' with $\tau = \Phi(\tau') \in Z(g)$, then

$$\tau'\, g'\, \tau'^{-1} = z_1^{L(\sigma)}\, z_2^{\mathrm{ord}(t_q)}\, z_3^{\mathrm{ord}(d)}\, g'. \tag{7.14}$$

8 Supports of spin characters of $G(m,1,n)$ (m odd)

8.1 Evaluation of supports of spin characters of $G(m,1,n)$, m odd

As basic foundations for later studies (in [I], [II] and so on), we are interested in checking that a central function f on $R(G)$ with a certain spin type (not necessarily a character) has what kind of restrictions on its support, in general. So, in consecutive sections, §§8 ∼10, the main objects are central spin functions f on $R(G)$ with certain spin types, containing spin characters in particular.

Here assume $4 \le n \le \infty$, m odd. An irreducible character f of $R(G)$, $G = G(m,1,n)$, has spin type χ: $\chi(z_1) =: \beta_1 = \pm 1$. According as $\beta_1 = -1$ or $\beta_1 = 1$, f is a *spin* or *non-spin* character.

We obtain the following from Theorems 7.4 and 7.5 respectively.

Lemma 8.1 (Case m odd, $4 \le n < \infty$)**.** *Let* $G = G(m,1,n) = \mathfrak{S}_n(\boldsymbol{Z}_m)$, $4 \le n < \infty$, *m odd. Let f be a central spin function on $R(G)$, in particular a spin character of G. For $g' \in R(G)$, let the standard decomposition of $g = \Phi(g') \in G$ be as in* (7.2).

(a) *Conjugacy relations in* (i) *and* (vi) *in Theorem 7.4 give*

$$\begin{array}{lll} \text{(i)} & f(g') = (-1)^{\,L(\sigma)}\, f(g') & \text{if } |\mathrm{supp}(g')| \le n-2\,; \\ \text{(iv)} & f(g') = (-1)^{\,(L(\sigma)-L(\sigma_j))L(\sigma_j)}\, f(g') & (1 \le j \le s). \end{array}$$

$$\text{(b)} \qquad f(g') \ne 0 \implies \begin{cases} L(\sigma) \equiv 0, \quad L(\sigma_j) \equiv 0 \ \ (1 \le j \le s)\,; & \text{or} \\ L(\sigma) \equiv 1, \quad |\mathrm{supp}(g')| \ge n-1. \end{cases}$$

Lemma 8.2 (Case m odd, $n = \infty$)**.** *Let* $G = G(m,1,\infty) = \mathfrak{S}_\infty(\boldsymbol{Z}_m)$, *$m$ odd. Let f be a central spin function on $R(G)$, in particular a spin character* *of* G. *For* $g' \in R\big(G(m,1,\infty)\big)$, *let the standard decomposition of* $g = \Phi(g') \in G(m,1,\infty)$ *be as in* (7.2).

(a) *Conjugacy relations in* (i) *and* (vi) *in Theorem 7.5 give*

$$\text{(i)}\quad f(g') = (-1)^{\,L(\sigma)}\, f(g'),$$
$$\text{(iv)}\quad f(g') = (-1)^{\,(L(\sigma)-L(\sigma_j))L(\sigma_j)}\, f(g') \qquad (1 \le j \le s).$$

(b) $\quad f(g') \neq 0 \implies L(\sigma) \equiv 0,\ \ L(\sigma_j) \equiv 0\ \ (1 \le j \le s)$.

8.2 Supports and factorizability of spin characters of $G(m,1,\infty)$, m odd

The results for $R(G)$, $G = G(m,1,\infty) = \mathfrak{S}_\infty(\boldsymbol{Z}_m)$, m odd, is summarized in Table 8.1 below. Note that $R(G) = D_\infty(\boldsymbol{Z}_m) \rtimes R(\mathfrak{S}_\infty)$ with $R(\mathfrak{S}_\infty) = \mathfrak{T}'_\infty$, and $G(1,1,\infty) = \mathfrak{S}_\infty$ for $m = 1$.

Table 8.1. On characters of $R\big(G(m,1,\infty)\big)$, m odd.

Case	$\chi(z_1) = \pm 1$ (spin) Type of factor represen-tation	Existence of spin finite-dimensional irred. represen. π ; Reason why	extremal $\Leftrightarrow$ factori-zable	Support (mod Z) of f: $f(g') \neq 0 \implies$ Condition for $g = \Phi(g')$ $= (d,\sigma), \sigma = \sigma_1\sigma_2\cdots\sigma_s$
I.odd	$\chi(z_1) = -1$ $R\big(\mathfrak{S}_\infty(\boldsymbol{Z}_m)\big)$ seed represen. in [DaMo]	$\neg\,\exists\,\pi$ $\because$ if $\exists\pi$, then $\mathrm{Ker}(\pi) \supset N \ni z_1$	YES	$\sigma \in \mathfrak{A}_\infty$, and $L(\sigma_i) \equiv 0\ (\forall i)$ the smallest normal subgr. $N = \Phi^{-1}\big(\mathfrak{A}_\infty(\boldsymbol{Z}_m)^e\big)$
II.odd	$\chi(z_1) = 1$ $\mathfrak{S}_\infty(\boldsymbol{Z}_m)$ char. formula in [HH1]	$\exists$ 1-dimensional characters : $\chi_{\varepsilon,\zeta}$ $(\varepsilon = 0,1;\ \zeta \in \widehat{\boldsymbol{Z}_m})$	YES	**No condition** the smallest non-trivial normal subgroup : $N = \mathfrak{A}_\infty(\boldsymbol{Z}_m)^e$
III.odd $R(\mathfrak{S}_\infty)$ $:= \mathfrak{T}'_\infty$	$\chi(z_1) = -1$ spin symmetric group $R(\mathfrak{S}_\infty)$, charac. formula in [Naz]	$\neg\,\exists\,\pi$ $\because$ if $\exists\pi$, then $\mathrm{Ker}(\pi) \supset N \ni z_1$	YES	$\sigma = \Phi(\sigma') \in \mathfrak{A}_\infty$, and $L(\sigma_i) \equiv 0\ (\forall i)$ the smallest normal subgr. $N = \Phi^{-1}(\mathfrak{A}_\infty)$
VI.odd $\mathfrak{S}_\infty$	$\chi(z_1) = 1$ the symmetric group $\mathfrak{S}_\infty$ cf. [Tho2]	$\exists$ characters $\chi_\varepsilon = \mathrm{sgn}^\varepsilon$ $(\varepsilon = 0,1)$	YES	**No condition** the smallest normal subgr. $N = \mathfrak{A}_\infty$

in Case II.odd, one-dimensional characters $\chi_{\varepsilon,\zeta}(g) = \mathrm{sgn}(\sigma)^\varepsilon \zeta\big(P(d)\big)$ for $g = (d,\sigma)$

• in the 2nd column, " the type (spin or non-spin) of a character f " ;

• in the 3rd column, " existence or non-existence of a finite-dimensional representations π, with reason " (if π exists, then $\mathrm{Ker}(\pi)$ necessarily contains $\{r_i r_{i'},\, \eta_j\eta_k^{-1}\}$ (cf. §12.1), and accordingly the smallest normal subgroup N containing $\Phi^{-1}\big(\mathfrak{A}_\infty(\boldsymbol{Z}_m)^e\big)$, given in the last column) ;

• in the 4th column, " does the criterion (EF) hold or not ? " (the answer YES in Case I.odd is obtained similarly as for $R(\mathfrak{S}_\infty)$, cf. §11);

• in the 5th column, " evaluation of the support $\mathrm{supp}(f)$ (mod Z) of a character f of $R\big(G(m,1,\infty)\big)$ ", and " the smallest proper normal subgroup N containing $\Phi^{-1}\big(\mathfrak{A}_\infty(\boldsymbol{Z}_m)^e\big)$ ".

9 Supports of spin characters of $G(m,1,n)$ (m even)

We evaluate the support $\mathrm{supp}(f)$ (mod Z) of a normalized irreducible character f of $R\big(G(m,1,n)\big)$, in each case of (spin) type χ. Here, to be more fundamental, we assume simply f is a central spin function with a non-trivial spin type on $R(G)$, $G = G(m,1,n)$, $4 \le n < \infty$, m even, and study what kind of restriction on its support we can get. Then the results can be applied immediately to characters. The results are important to calculate spin irreducible characters and also to study limiting behavior of a series of irreducible characters as $n \nearrow \infty$, and thus to get all the characters of the inductive limit group $R\big(G(m,1,\infty)\big)$.

From Theorem 7.6, we get the following.

Lemma 9.1 (Case I, Type $(-1,-1,-1)$). *Let f be a central function of this spin type.*

(a) *f satisfies a system of relations coming from* (i) $\sim$ (v) *of Theorem 7.6 as*

$$\text{(i)}\quad f(g') = (-1)^{L(\sigma)+\mathrm{ord}(d)}\, f(g') \qquad \text{if} \quad |\mathrm{supp}(g')| \le n-2\,;$$

$$\text{(ii)}\quad f(g') = (-1)^{\mathrm{ord}(d)+L(\sigma)}\, f(g') \qquad \text{if} \quad |\mathrm{supp}(g')| \le n-1\,;$$

$$\text{(iii)}\quad f(g') = (-1)^{\mathrm{ord}(d)-\mathrm{ord}(\xi_{q_i})+L(\sigma)}\, f(g') \qquad (1 \le i \le r)\,;$$

$$\text{(iv)}\quad f(g') = (-1)^{(L(\sigma)+1)L(\sigma_j)+(\mathrm{ord}(d)+1)\mathrm{ord}(d_j)+L(\sigma)\mathrm{ord}(d_j)+\mathrm{ord}(d)L(\sigma_j)}\, f(g') \qquad (1 \le j \le s)\,;$$

$$\text{(v)}\quad f(g') = (-1)^{\mathrm{ord}(d)(L(\sigma_j)+1)+L(\sigma_j)-\mathrm{ord}(d_j)+L(\sigma)(L(\sigma_j)+1)}\, f(g') \quad (1 \le j \le s).$$

(b) $f(g') \ne 0 \implies$

$$\begin{cases} \mathrm{ord}(d)+L(\sigma) \equiv 0, \quad \mathrm{ord}(\xi_{q_i}) \equiv 0\ (\forall i),\ \mathrm{ord}(d_j)+L(\sigma_j) \equiv 0\ (\forall j)\,; \\ \text{or,} \\ \mathrm{ord}(d)+L(\sigma) \equiv 1, \quad |\mathrm{supp}(g')| = n,\ \mathrm{ord}(\xi_{q_i}) \equiv 1\ (\forall i), \\ \qquad\qquad\qquad\qquad \mathrm{ord}(d_j) \equiv 1\ (\forall j). \end{cases}$$

We can discuss similarly in each case, and omitting part (a) of the assertion, only the part (b) is given in the form of a complete list, which is convenient to check and refer. The detailed data in the list will be also applied to the case of $n=\infty$ with necessary modifications: in the part (b) in each Case below, we omit the word ‘ or ’ after semicolon ‘ ; ’.

Case II, Type $(-1,-1,1)$**:** (b) $f(g')\neq 0 \implies$

$$\begin{cases} \mathrm{ord}(d)\equiv 0, L(\sigma)\equiv 0, & \mathrm{ord}(\xi_{q_i})\equiv 0\ (\forall i),\ \mathrm{ord}(d_j)+L(\sigma_j)\equiv 0\ (\forall j)\ ; \\ \mathrm{ord}(d)\equiv 0, L(\sigma)\equiv 1, & \qquad\qquad \varnothing\ ; \\ \mathrm{ord}(d)\equiv 1, L(\sigma)\equiv 0, & |\mathrm{supp}(g')|=n,\ \mathrm{ord}(\xi_{q_i})\equiv 1\ (\forall i),\ \mathrm{ord}(d_j)\equiv 1\ (\forall j), \\ & L(\sigma_j)\equiv 0\ (\forall j) \qquad (\Longrightarrow\ \ r+s\ \text{ odd})\ ; \\ \mathrm{ord}(d)\equiv 1, L(\sigma)\equiv 1, & |\mathrm{supp}(g')|=n,\ \mathrm{ord}(\xi_{q_i})\equiv 1\ (\forall i),\ \mathrm{ord}(d_j)\equiv 1\ (\forall j) \\ & (\Longrightarrow\ \ r+s\ \text{ odd}). \end{cases}$$

Case III, Type $(-1,1,-1)$**:** (b) $f(g')\neq 0 \implies$

$$\begin{cases} \mathrm{ord}(d)\equiv 0, L(\sigma)\equiv 0, & L(\sigma_j)\equiv 0\ (\forall j)\ ; \\ \mathrm{ord}(d)\equiv 0, L(\sigma)\equiv 1, & |\mathrm{supp}(g')|=n,\ r=0,\ \mathrm{ord}(d_j)\equiv 0\ (\forall j), \\ & L(\sigma_j)\equiv 1\ (\forall j) \qquad (\Longrightarrow\ \ s\ \text{ odd})\ ; \\ \mathrm{ord}(d)\equiv 1, L(\sigma)\equiv 0, & |\mathrm{supp}(g')|\geq n-1\ ; \\ \mathrm{ord}(d)\equiv 1, L(\sigma)\equiv 1, & |\mathrm{supp}(g')|=n,\ \mathrm{ord}(d_j)\equiv 1,\ L(\sigma_j)\equiv 1\ (\forall j) \\ & (\Longrightarrow\ \ s\ \text{ odd}). \end{cases}$$

Case IV, Type $(-1,1,1)$**:** (b) $f(g')\neq 0 \implies$

$$\begin{cases} L(\sigma)\equiv 0, & L(\sigma_j)\equiv 0\ (\forall j)\ ; \\ L(\sigma)\equiv 1, & |\mathrm{supp}(g')|\geq n-1,\ \mathrm{ord}(\xi_{q_i})\equiv 0\ (\forall i), \\ & \mathrm{ord}(d_j)\equiv 0\ (\forall j),\ L(\sigma_j)\equiv 0\ (\forall j). \end{cases}$$

Case V, Type $(1,-1,-1)$**:** (b) $f(g')\neq 0 \implies$

$$\begin{cases} \mathrm{ord}(d)\equiv 0, L(\sigma)\equiv 0, & \mathrm{ord}(\xi_{q_i})\equiv 0\ (\forall i),\ \mathrm{ord}(d_j)\equiv 0\ (\forall j),\ L(\sigma_j)\equiv 0\ (\forall j); \\ \mathrm{ord}(d)\equiv 0, L(\sigma)\equiv 1, & |\mathrm{supp}(g')|=n,\ \mathrm{ord}(\xi_{q_i})\equiv 1\ (\forall i),\ \mathrm{ord}(d_j)\equiv 1\ (\forall j) \\ & (\Longrightarrow\ \ r+s\ \text{even})\ ; \\ \mathrm{ord}(d)\equiv 1, L(\sigma)\equiv 0, & |\mathrm{supp}(g')|=n,\ \mathrm{ord}(\xi_{q_i})\equiv 1\ (\forall i),\ \mathrm{ord}(d_j)\equiv 1\ (\forall j), \\ & L(\sigma_j)\equiv 0\ (\forall j) \qquad (\Longrightarrow\ \ r+s\ \text{ odd})\ ; \\ \mathrm{ord}(d)\equiv 1, L(\sigma)\equiv 1, & |\mathrm{supp}(g')|\geq n-1,\ \mathrm{ord}(\xi_{q_i})\equiv 0\ (\forall i), \\ & \mathrm{ord}(d_j)+L(\sigma_j)\equiv 0\ (\forall j). \end{cases}$$

Case VI, Type $(1,-1,1)$**:** (b) $f(g')\neq 0 \implies$

$$\begin{cases} \mathrm{ord}(d)\equiv 0, & \mathrm{ord}(\xi_{q_i})\equiv 0\ (\forall i),\ \mathrm{ord}(d_j)\equiv 0\ (\forall j), \\ & L(\sigma_j)\equiv 0\ (\forall j) \quad (\Longrightarrow\ L(\sigma)\equiv 0)\ ; \\ \mathrm{ord}(d)\equiv 1, & |\mathrm{supp}(g')|=n,\ \mathrm{ord}(\xi_{q_i})\equiv 1\ (\forall i), \\ & \mathrm{ord}(d_j)\equiv 1\ (\forall j), \qquad (\Longrightarrow\ \ r+s\ \text{ odd}). \end{cases}$$

Case VII, Type $(1,\ 1,-1)$**:** (b) $f(g')\neq 0 \implies$

$$\begin{cases} \mathrm{ord}(d)\equiv 0, L(\sigma)\equiv 0, & \textbf{No additional condition}\,; \\ \mathrm{ord}(d)\equiv 0, L(\sigma)\equiv 1, & |\mathrm{supp}(g')| = n,\ r=0,\ s\ \text{odd},\ \mathrm{ord}(d_j)\equiv 0\ (\forall j), \\ & L(\sigma_j)\equiv 1\ (\forall j) \qquad (\Longrightarrow\ n\ \text{even})\ ; \\ \mathrm{ord}(d)\equiv 1, L(\sigma)\equiv 0, & |\mathrm{supp}(g')| \geq n-1,\ L(\sigma_j)\equiv 0\ (\forall j) \\ & (\Longrightarrow\ r+s\ \text{odd})\ ; \\ \mathrm{ord}(d)\equiv 1, L(\sigma)\equiv 1, & |\mathrm{supp}(g')| = n,\ r=0,\ s\ \text{odd},\ \mathrm{ord}(d_j)\equiv 1\ (\forall j), \\ & L(\sigma_j)\equiv 1\ (\forall j) \qquad (\Longrightarrow\ n\ \text{even}). \end{cases}$$

Table 9.1. For finite group $R(G_n)$, $G_n = \mathfrak{S}_n(\boldsymbol{Z}_m) = G(m,1,n)$,
$4 \le n < \infty,\ m = 2m'$**:** $1 \to Z = \langle z_1, z_2, z_3\rangle \to R(G_n) \xrightarrow{\Phi} G_n \to 1$:

<table>
<tr><td rowspan="3">Case Y</td><td rowspan="3">$(\beta_1,\beta_2,\beta_3)$ (spin) Type of factor represen-tation</td><td colspan="4">$f(g')\neq 0 \Longrightarrow$ Condition for $g=\Phi(g')=(d,\sigma)$ $=\xi_{q_1}\cdots\xi_{q_r}g_1\cdots g_s,\ \xi_{q_i}=(t_{q_i},(q_i)),\ g_j=(d_j,\sigma_j)$</td></tr>
<tr><td colspan="2">$\mathrm{ord}(d)+L(\sigma)\equiv 0$ (mod 2)</td><td colspan="2">$\mathrm{ord}(d)+L(\sigma)\equiv 1$ (mod 2)</td></tr>
<tr><td>$\mathrm{ord}(d)\equiv 0$
$L(\sigma)\equiv 0$</td><td>$\mathrm{ord}(d)\equiv 1$
$L(\sigma)\equiv 1$</td><td>$\mathrm{ord}(d)\equiv 0$
$L(\sigma)\equiv 1$</td><td>$\mathrm{ord}(d)\equiv 1$
$L(\sigma)\equiv 0$</td></tr>
<tr><td>I</td><td>(–1,–1,–1)
seed repre.
in [IhYo],
in [DaMo]</td><td colspan="2">$\mathrm{ord}(\xi_{q_i})\equiv 0\ \ (1\leqslant i\leqslant r)$
$\mathrm{ord}(d_j)+L(\sigma_j)\equiv 0\ (1\leqslant j\leqslant s)$</td><td colspan="2">$|\mathrm{supp}(g')|=n$
$\mathrm{ord}(\xi_{q_i})\equiv 1\ (1\leqslant i\leqslant r)$
$\mathrm{ord}(d_j)\equiv 1\ (1\leqslant j\leqslant s)$</td></tr>
<tr><td>II</td><td>$(-1,-1,1)$
seed repre.
in [DaMo]</td><td>$L(\sigma)\equiv 0$
$\mathrm{ord}(\xi_{q_i})\equiv 0\ (\forall i)$
$\mathrm{ord}(d_j)+L(\sigma_j)$
$\equiv 0\ (\forall j)$</td><td>$|\mathrm{supp}(g')|=n$
$r+s$ odd
$\mathrm{ord}(\xi_{q_i})\equiv 1\ (\forall i)$
$\mathrm{ord}(d_j)\equiv 1\ (\forall j)$
$L(\sigma)\equiv 1$</td><td>$\varnothing$</td><td>$|\mathrm{supp}(g')|=n$
$r+s$ odd
$\mathrm{ord}(\xi_{q_i})\equiv 1\ (\forall i)$
$\mathrm{ord}(d_j)\equiv 1\ (\forall j)$
$L(\sigma_j)\equiv 0\ (\forall j)$</td></tr>
<tr><td>III</td><td>$(-1,1,-1)$</td><td>$\subset\mathfrak{A}_n(T)^S$
$L(\sigma_j)\equiv 0\ (\forall j)$</td><td colspan="2">$n$ **even**, $|\mathrm{supp}(g')|=n$
$r=0,\ s$ odd,
$\mathrm{ord}(d_j)\equiv\mathrm{ord}(d)\ (\forall j)$
$L(\sigma_j)\equiv 1\ (\forall j),\quad g=g_1g_2\cdots g_s$</td><td>$|\mathrm{supp}(g')|\geqslant n-1$</td></tr>
<tr><td>IV</td><td>$(-1,\ 1,\ 1)$
seed repre.
in [IhYo],
in [DaMo]</td><td>$\subset\mathfrak{A}_n(T)^S$
$L(\sigma_j)\equiv 0\ (\forall j)$
$\sigma=\sigma_1\cdots\sigma_s$</td><td colspan="2">$|\mathrm{supp}(g')|\geqslant n-1$</td><td>$\subset\mathfrak{A}_n(T)$,
$L(\sigma_j)\equiv 0\ (\forall j)$
$\sigma=\sigma_1\cdots\sigma_s$</td></tr>
<tr><td>V</td><td>$(1,-1,-1)$</td><td>$\subset\mathfrak{A}_n(T)^S$
$\mathrm{ord}(\xi_{q_i})\equiv 0\ (\forall i)$
$\mathrm{ord}(d_j)\equiv 0\ (\forall j)$
$L(\sigma_j)\equiv 0\ (\forall j)$</td><td>$|\mathrm{supp}(g')|\geqslant n-1$
$\mathrm{ord}(\xi_{q_i})\equiv 0\ (\forall i)$
$\mathrm{ord}(d_j)+L(\sigma_j)$
$\equiv 0\ (\forall j)$
$L(\sigma)\equiv 1$</td><td>$|\mathrm{supp}(g')|=n$
$r+s$ even
$\mathrm{ord}(\xi_{q_i})\equiv 1\ (\forall i)$
$\mathrm{ord}(d_j)\equiv 1\ (\forall j)$
$L(\sigma)\equiv 1$</td><td>$|\mathrm{supp}(g')|=n$
$r+s$ odd
$\mathrm{ord}(\xi_{q_i})\equiv 1\ (\forall i)$
$\mathrm{ord}(d_j)\equiv 1\ (\forall j)$
$L(\sigma_j)\equiv 0\ (\forall j)$</td></tr>
<tr><td>VI</td><td>$(1,-1,\ 1)$</td><td>$\subset\mathfrak{A}_n(T)^S$
$\mathrm{ord}(\xi_{q_i})\equiv 0\ (\forall i)$
$\mathrm{ord}(d_j)\equiv 0\ (\forall j)$
$L(\sigma_j)\equiv 0\ (\forall j)$</td><td>$|\mathrm{supp}(g')|=n$
$r+s$ odd
$\mathrm{ord}(\xi_{q_i})\equiv 1\ (\forall i)$
$\mathrm{ord}(d_j)\equiv 1\ (\forall j)$</td><td>$\varnothing$</td><td>$|\mathrm{supp}(g')|=n$
$r+s$ odd
$\mathrm{ord}(\xi_{q_i})\equiv 1\ (\forall i)$
$\mathrm{ord}(d_j)\equiv 1\ (\forall j)$</td></tr>
<tr><td>VII</td><td>$(1,\ 1,-1)$
seed repre.
in [IhYo]</td><td>$\subset\mathfrak{A}_n(T)^S$
No other condition</td><td colspan="2">n **even**, $|\mathrm{supp}(g')|=n$
$r=0,\ s$ odd,
$\mathrm{ord}(d_j)\equiv\mathrm{ord}(d)\ (\forall j)$
$L(\sigma_j)\equiv 1\ (\forall j),\quad g=g_1g_2\cdots g_s$</td><td>$|\mathrm{supp}(g')|\geqslant n-1$
$L(\sigma_j)\equiv 0\ (\forall j)$</td></tr>
<tr><td>VIII</td><td>$(1,\ 1,\ 1)$</td><td colspan="4">(Case of non-spin characters of $\mathfrak{S}_n(\boldsymbol{Z}_m)$)
No condition</td></tr>
</table>

In the table above, notations are as follows: (spin) type is $(\beta_1, \beta_2, \beta_3)$ with $\chi(z_i) = \beta_i$ $(1 \le i \le 3)$ according to Case Y (Y=I$\sim$VIII); $L(\sigma)$ denote the length of $\sigma \in \mathfrak{S}_n$ with respect to simple reflections $\{s_1, s_2, \ldots, s_{n-1}\}$, and $\ell(\sigma_j)$ the length of a cycle σ_j; $S := S(2) = \{t^2; t \in T\} \cong \boldsymbol{Z}_{m/2}$, $T = \boldsymbol{Z}_m$, $\mathfrak{A}_n(T)^S := \{(d, \sigma) \in \mathfrak{S}_n(T); \sigma \in \mathfrak{A}_n, P(d) \in S\}$.

10 Supports of spin characters of infinite generalized symmetric groups $G(m, 1, \infty)$

Since the situation in the case of m odd is simple, we treat here mainly the case of m even. The case of m odd can be easily estimated from it. Let f be a central spin function on $R\big(G(m,1,\infty)\big)$ with a non-trivial spin type, in particular, a spin character of $G(m,1,\infty)$, m even.

An evaluation of support of f can be obtained from Theorem 7.8, similarly as for finite groups $G(m,1,n), n < \infty$. In each Case Y (Y = I, II, ..., VII) for $n = \infty$, the evaluation of $\mathrm{supp}(f)$ is given by the system of conditions listed up in §9.2 after "(b) $f(g') \Longrightarrow$" but without the restrictive conditions "$|\mathrm{supp}(g')| = n$" and "$|\mathrm{supp}(g')| \ge n-1$". Their proofs are by easy calculations and omitted here. We call this system of conditions as Condition Y.

The subset of the representation group $R\big(G(m,1,\infty)\big)$ consisting of g' satisfying (Condition Y) is denoted by $\mathcal{O}(\mathrm{Y})$. Then our results are stated as follows.

Theorem 10.1 (m even)**.** *Let f be a central function on $R\big(G(m,1,\infty)\big)$ with the spin type in Case* Y*. Then, in each Case* Y (Y = I, II, ..., VII)*,*

$$f(g') \neq 0 \Longrightarrow g' \in \mathcal{O}(\mathrm{Y}), \quad \text{or} \quad \mathcal{O}(\mathrm{Y}) \supset \mathrm{supp}(f). \tag{10.1}$$

Definition 10.1. A non-empty subset $\mathcal{O}$ of $R\big(G(m,1,\infty)\big)$ is called *weakly multiplicative* (resp. *multiplicative*) if it has the following property:

$$h', k' \in \mathcal{O}, \ \ \mathrm{supp}(h') \bigcap \mathrm{supp}(k') = \varnothing \implies h'k' \in \mathcal{O} \tag{10.2}$$

$$(\text{resp.} \quad h', k' \in \mathcal{O} \implies h'k' \in \mathcal{O}\ ). \tag{10.3}$$

Definition 10.2. A non-empty subset $\mathcal{O}$ of $R\big(G(m,1,\infty)\big)$, containing the central subgroup Z, is called *factorizable* if it has the following property: for any $g' \in \mathcal{O}$,

$$g' = h'k',\ h', k' \in R\big(G(m,1,\infty)\big),\ \mathrm{supp}(h') \bigcap \mathrm{supp}(k') = \varnothing \implies h', k' \in \mathcal{O}.$$

Definition 10.3. A non-empty subset $\mathcal{O}$ of $R\big(G(m,1,\infty)\big)$, containing the central subgroup Z, is called *commutatively factorizable for type* χ, $\chi \in \widehat{Z}$, if it has the following property: for any $g' \in \mathcal{O}$,

$$\left\{ \begin{array}{l} g' = h'k', \ \ h', k' \in R\big(G(m,1,\infty)\big), \ \ \mathrm{supp}(h') \bigcap \mathrm{supp}(k') = \varnothing \\ \qquad\qquad \implies h', k' \in \mathcal{O} \ \text{ and } \ h'k'\mathrm{Ker}(\chi) = k'h'\mathrm{Ker}(\chi). \end{array} \right. \tag{10.4}$$

Table 10.1. Subsets $\mathcal{O}(\mathrm{Y}) \subset R\big(G(m,1,\infty)\big)$, $G(m,1,\infty) = \mathfrak{S}_\infty(\boldsymbol{Z}_m)$, $\mathcal{O}(\mathrm{Y}) \supset \mathrm{supp}(f)$, $f \in E^{\mathrm{Y}}\big(R\big(G(m,1,\infty)\big)\big)$:

Type χ = Type β, $\beta = (\beta_1, \beta_2, \beta_3)$ with $\beta_i = \chi(z_i)$ for m even.

Subset $\mathcal{O}(\mathrm{Y})$ in Case Y	$(\beta_1,\beta_2,\beta_3)$ Type of factor representation	(Condition Y) for $\mathcal{O}(\mathrm{Y})$: $g = \Phi(g') = (d,\sigma) = \xi_{q_1}\cdots\xi_{q_r} g_1 \cdots g_s$, $g_j = (d_j, \sigma_j)$	$\mathcal{O}(\mathrm{Y})$ Factorizable	$\mathcal{O}(\mathrm{Y})$ commu. factorizable	$\mathcal{O}(\mathrm{Y})$ weakly multiplica.	$\mathcal{O}(\mathrm{Y})$ multiplicative
m	**ODD**					
$\mathcal{O}(\mathrm{I.odd})$	$\chi(z_1) = -1$	$L(\sigma_j) \equiv 0 \pmod 2$ $(1 \leqslant j \leqslant s)$	YES	YES	YES	NO
m	**EVEN**					
$\mathcal{O}(\mathrm{I})$	$(-1,-1,-1)$	$\mathrm{ord}(\xi_{q_i}) \equiv 0 \pmod 2$ $(\forall i)$ $\mathrm{ord}(d_j) + L(\sigma_j) \equiv 0$ $(\forall j)$	YES	YES	YES	NO
$\mathcal{O}(\mathrm{II})$	$(-1,-1,\ 1)$	$\mathrm{ord}(d) \equiv 0$, $L(\sigma) \equiv 0$, $\mathrm{ord}(\xi_{q_i}) \equiv 0$ $(\forall i)$, $\mathrm{ord}(d_j) + L(\sigma_j) \equiv 0$ $(\forall j)$	NO	NO	YES	NO
$\mathcal{O}(\mathrm{III})$	$(-1,\ 1,-1)$	$\mathrm{ord}(d) \equiv 0$, $L(\sigma_j) \equiv 0$ $(\forall j)$	NO	NO	YES	NO
$\mathcal{O}(\mathrm{IV})$	$(-1,\ 1,\ 1)$	$L(\sigma_j) \equiv 0$ $(\forall j)$, for $\sigma = \sigma_1\sigma_2\cdots\sigma_s$	YES	YES	YES	NO
$\mathcal{O}(\mathrm{V})$	$(1,-1,-1)$	$\mathrm{ord}(\xi_{q_i}) \equiv 0$ $(\forall i)$, $\mathrm{ord}(d_j) \equiv 0$ $(\forall j)$, $L(\sigma_j) \equiv 0$ $(\forall j)$	YES	YES	YES	NO
$\mathcal{O}(\mathrm{VI})$	$(1,-1,\ 1)$	$\mathrm{ord}(\xi_{q_i}) \equiv 0$ $(\forall i)$, $\mathrm{ord}(d_j) \equiv 0$ $(\forall j)$, $L(\sigma_j) \equiv 0$ $(\forall j)$	YES	YES	YES	NO
$\mathcal{O}(\mathrm{VII})$	$(1,\ 1,-1)$	$\mathrm{ord}(d) \equiv 0$, $L(\sigma) \equiv 0$	NO	NO	YES	YES
$\mathcal{O}(\mathrm{VIII})$	$(1,\ 1,\ 1)$ non-spin	**No condition**	YES	YES	YES	YES

Notation 10.1. In Case Y, denote by $E^{\mathrm{Y}}\big(R\big(G(m,1,\infty)\big)\big)$ the set of all normalized spin characters of Type χ^{Y} $\big($or of type $\beta = (\beta_1,\beta_2,\beta_3), \beta_i = \chi^{\mathrm{Y}}(z_i), 1 \le i \le 3\big)$ of $G(m,1,\infty)$, and by $K_1^{\mathrm{Y}}\big(R\big(G(m,1,\infty)\big)\big)$ the set of normalized central positive definite functions on $R\big(G(m,1,\infty)\big)$ of (spin) type χ^{Y}. Put

$$\widetilde{G}^{\mathrm{Y}}(m,1,\infty) := R\big(G(m,1,\infty)\big)/\mathrm{Ker}(\chi^{\mathrm{Y}}), \quad Z^{\mathrm{Y}} := Z/\mathrm{Ker}(\chi^{\mathrm{Y}}). \tag{10.5}$$

Then $Z^{\mathrm{Y}} \cong \boldsymbol{Z}_2$ is of order 2. Note that $E^{\mathrm{Y}}\big(R\big(G(m,1,\infty)\big)\big)$ is the set of extremal points of the convex set $K_1^{\mathrm{Y}}\big(R\big(G(m,1,\infty)\big)\big)$, and that a central positive definite function $f \in K_1^{\mathrm{Y}}\big(R\big(G(m,1,\infty)\big)\big)$ can be considered as a function on the quotient group $\widetilde{G}^{\mathrm{Y}}(m,1,\infty)$. We have a disjoint union

$$E\big(R\big(G(m,1,\infty)\big) = \bigsqcup_{\mathrm{Y}=\mathrm{I}\,\sim\,\mathrm{VIII}} E^{\mathrm{Y}}\big(R\big(G(m,1,\infty)\big),\tag{10.6}$$

and similarly for $K_1\big(R\big(G(m,1,\infty)\big)\big)$.

We have a central extension (or a covering group) of $G(m,1,\infty)$ as

$$\{e\} \longrightarrow Z^{\mathrm{Y}} \longrightarrow \widetilde{G}^{\mathrm{Y}}(m,1,\infty) \xrightarrow{\Phi^{\mathrm{Y}}} G(m,1,\infty) \longrightarrow \{e\} \quad \text{(exact)}. \tag{10.7}$$

Diagram 10.1. Covering groups of $G(m,1,\infty)$.

upper level	$R\big(G(m,1,\infty)\big)$	covering group
	$\downarrow$	degree of covering $= 2^2 = \lvert\mathrm{Ker}(\chi^{\mathrm{Y}})\rvert$
upper level	$\widetilde{G}^{\mathrm{Y}}(m,1,\infty)\big)$	**double covering group**
	$\downarrow$	degree of covering $= 2$
lower (non-spin) level	$G(m,1,\infty)$	basic group

When we consider a spin character f of Type $\chi = \chi^{\mathrm{Y}}$, we take $\mathcal{O} = \mathcal{O}(\mathrm{Y})$. If it is commutatively factorizable for type χ, the property in Definition 10.3 implies that the images of $h', k' \in \mathcal{O}(\mathrm{Y}) \subset R\big(G(m,1,\infty)\big)$ down in the quotient group $\widetilde{G}^{\mathrm{Y}}(m,1,\infty)$ are mutually commutative, and so $f(h'k') = f(k'h')$.

To check if the subset $\mathcal{O} = \mathcal{O}(\mathrm{Y})$ satisfies or not the conditions for the corresponding $\chi = \chi^{Y} \in \widehat{Z}$, we apply the fundamental relations listed up in §9.2.

Remark 10.1. Noting $(1\ 2\ 3)(2\ 3\ 4) = (1\ 2)(3\ 4)$, we see that the condition "$L(\sigma_j) \equiv 0\ (\forall j)$" is not "multiplicative", and accordingly that the subset $\mathcal{O}(\mathrm{Y})$ is not multiplicative except Y = VII, VIII.

For Cases Y, Y = II, III, VII, consider a strengthened condition (str-Condition Y) on $g' \in R\big(G(m,1,\infty)\big)$ demanding that each component ξ_{q_i}, g_j of $g = \Phi(g')$ satisfies itself (Condition Y), and denote by $\mathcal{O}^{\mathrm{str}}(\mathrm{Y})$ the subset of g' defined by (str-Condition Y).

Lemma 10.2. (i) *For Case* Y, Y = II, III*, or* VII*, the condition* (str-Condition Y) *for* $g' \in R\big(G(m,1,\infty)\big)$ *is given by*

$$\mathrm{ord}(\xi_{q_i}) \equiv 0 \ \ (1 \le i \le r), \ \ L(\sigma_j) \equiv 0, \ \ \mathrm{ord}(d_j) \equiv 0 \ \ (1 \le j \le s), \tag{str}$$

and the subsets $\mathcal{O}^{\mathrm{str}}(\mathrm{Y})$ *are all equal to the subset* $\mathcal{O}(\mathrm{str})$ *defined by the condition* (str) *above and equal to* $\mathcal{O}(\mathrm{V}) = \mathcal{O}(\mathrm{VI})$.

(ii) *In such Case* Y*, if* $f \in K_1^{\mathrm{Y}}\big(R\big(G(m,1,\infty)\big)\big)$ *is factorizable (cf. Definition 6.1), then we have* $\mathrm{supp}(f) \subset \mathcal{O}^{\mathrm{str}}(\mathrm{Y}) = \mathcal{O}(\mathrm{str})$.

Lemma 10.3. (i) *Put* $\mathcal{O}'(\mathrm{Y})$ *as*

$$\mathcal{O}'(\mathrm{Y}) := \begin{cases} \mathcal{O}(\mathrm{Y}) & \text{for } \mathrm{Y} = \mathrm{I.odd},\ \mathrm{I},\ \mathrm{IV},\ \mathrm{V},\ \mathrm{VI}\,, \\ \mathcal{O}^{\mathrm{srt}}(\mathrm{Y}) & \text{for } \mathrm{Y} = \mathrm{II},\ \mathrm{III},\ \mathrm{VII}\,. \end{cases} \tag{10.8}$$

For a $g' \in \mathcal{O}'(\mathrm{Y})$*, take any of its basic components* h' *such that* $h = \Phi(h')$ *equals one of* ξ_{q_i} $(1 \le i \le r)$ *and* g_j $(1 \le j \le s)$*. Let* $J := \mathrm{supp}(h')$ *and*

$$(10.9) \qquad G'_J := \{k' \in R\big(G(m,1,\infty)\big)\ ;\ \mathrm{supp}(k') \cap J = \varnothing\},$$

then h' *and* G'_J *commute with each other elementwise modulo* $\mathrm{Ker}(\chi^{\mathrm{Y}})$*, that is,* $h'k'\mathrm{Ker}(\chi^{\mathrm{Y}}) = k'h'\mathrm{Ker}(\chi^{\mathrm{Y}})$ $(k' \in G'_J)$*, where* $\chi^{\mathrm{Y}} \in \widehat{Z}$ *corresponds to Case* Y*. In other words, going down to the quotient* $\widetilde{G}^{\mathrm{Y}}(m,1,\infty) = R\big(G(m,1,\infty)\big)/\mathrm{Ker}(\chi^{\mathrm{Y}})$*, they commute with each other.*

(ii) *In Case* Y, Y = II, III, VII*, take* $g' \in \mathcal{O}(\mathrm{Y}) \setminus \mathcal{O}^{\mathrm{str}}(\mathrm{Y})$*, then* g' *has a basic component* h' *such that* h' *is not commutative with some elements of* G'_J *modulo* $\mathrm{Ker}(\chi^{\mathrm{Y}})$*.*

11 Factorisability for spin characters of $G(m,1,\infty)$

Contrary to the non-spin case (Case VIII) of infinite generalized symmetric groups $G(m,1,\infty) = \mathfrak{S}_\infty(\boldsymbol{Z}_m)$ in [HH1], the factorizability criterion (EF) for characters in §6.6 :

for $f \in K_1\big(R\big(G(m,1,\infty)\big)\big)$ *of certain type* $\chi \in \widehat{Z}$*,*

(EF) $\qquad f$ *extremal* $\Longleftrightarrow f$ *factorizable,*

does not necessarily hold for the spin case, that is, for representation groups $R\big(G(m,1,\infty)\big)$ if m is even. As a result of our study, the validity of factorizability criterion (EF) corresponds to that of the factorizability for the subset $\mathcal{O}(\mathrm{Y})$ in §10, as seen below. Recall that the implication "factorizable $\Longrightarrow$ extremal" is always true as seen in Theorem 6.3. In other words, for $f \in K_1\big(R\big(G(m,1,\infty)\big)\big)$ of certain type $\chi \in \widehat{Z}$, "to be factorizable" is sufficient for "to be a character".

Theorem 11.1. (i) *The converse implication "extremal* $\Longrightarrow$ *factorizable" holds in Case* Y, Y = I.odd, I, IV, V, *and* VI*, and the criterion (EF) holds in these cases.*

(ii) *The criterion (EF) does not holds in Case* VII.

Proof. (i) Since $f(zh') = \chi^{\mathrm{Y}}(z)f(h')$ $\big(z \in Z, h' \in R\big(G(m,1,\infty)\big)\big)$, f can be considered as a function on the quotient group $\widetilde{G}^{\mathrm{Y}}(m,1,\infty) = R\big(G(m,1,\infty)\big)/\mathrm{Ker}(\chi^{\mathrm{Y}})$, and we can and do discuss on $\widetilde{G}^{\mathrm{Y}}$ through modulo $\mathrm{Ker}(\chi^{\mathrm{Y}}) \subset Z$.

Consider Case Y, Y=I.odd, I, IV, V, or VI. Take $g' \in \widetilde{G}^{\mathrm{Y}}(m,1,\infty)$ such that $f(g') \ne 0$. Then g' belongs to the subset $\mathcal{O}(\mathrm{Y})$. Let $J = \mathrm{supp}(g')$ and put as in (10.9),

$$G'_J = \{h' \in \widetilde{G}^{\mathrm{Y}}(m,1,\infty);\ \mathrm{supp}(h') \cap J = \varnothing\}.$$

By Lemma 10.3, g' and G'_J commute with each other elementwise. Let N be the order of g', and $Z_N = \langle g' \rangle$ the cyclic group generated by g', then

$$Z_N \cap G'_J = \{e\} \quad \text{or} \quad Z_N \cap G'_J = Z^{\mathrm{Y}},$$

where $Z^{\mathrm{Y}} := Z/\mathrm{Ker}(\chi^{\mathrm{Y}})$, since $Z/\mathrm{Ker}(\chi^{\mathrm{Y}}) \cong \boldsymbol{Z}_2$. Then the direct product $Z_N \times G'_J$ is a subgroup of $\widetilde{G}^{\mathrm{Y}}$ in the first case, and $(Z_N \times G'_J)/Z^{\mathrm{Y}}$ is imbedded into $\widetilde{G}^{\mathrm{Y}}$ as such a one in the second case.

The group G'_J is naturally isomorphic to $\widetilde{G}^{\mathrm{Y}}(m,1,\infty)$ and any conjugacy class of the latter meets with the former. Hence the restriction $\widehat{f} = f|_{G'_J}$ onto G'_J determines f naturally, and $\widehat{f}$ is extremal in $K_1(G'_J)$ or $\widehat{f} \in E(G'_J)$.

Moreover we have $E(Z_N \times G'_J) = E(Z_N) \times E(G'_J)$, and thus we can follow the proof of "only if"-part of Satz 1 in [Tho2].

(ii) We know from Lemma 10.2 (ii) that if a spin character f in Case VII has support $\mathrm{supp}(f) \not\subset \mathcal{O}^{\mathrm{str}}(\mathrm{Y}) = \mathcal{O}(\mathrm{str})$, then it is not decomposable. Such a spin character is given by the normalized trace character $\widetilde{\chi}_{\pi_{2,\zeta_k}}$ of two-dimensional IRs (see Theorem 12.2 below).

Thus the proof of Theorem 11.1 is completed. □

Remark 11.1. In Cases II and III, the criterion (EF) does not hold as in Case VII, principally because of the property of $\mathcal{O}(\mathrm{Y})$ in Lemma 10.3 (ii), for Y = II, III, VII. But its proof is postponed until a succeeding paper, since it needs an example of characters with supports touching $\mathcal{O}(\mathrm{Y}) \setminus \mathcal{O}^{\mathrm{str}}(\mathrm{Y})$.

12 Finite-dimensional spin representations of $G(m,1,\infty)$

12.1 Spin type admitting finite-dimensional IRs for $R\big(G(m,1,\infty)\big)$

Let m be even, and π a finite-dimensional IR of $R\big(G(m,1,\infty)\big)$. Then π has its own type $\beta = (\beta_1, \beta_2, \beta_3)$ given by $\pi(z_i) = \beta_i I$ $(1 \le i \le 3)$ with the identity operator I.

Since $R\big(G(m,1,\infty)\big) = \lim_{n\to\infty} R\big(G(m,1,n)\big)$, there exists an n_0 such that π is irreducible already on a finite subgroup $R\big(G(m,1,n_0)\big)$. For $n > n_0$, elements of the form $r_i r_{i'}$ $(i, i' > n)$ or $\eta_j \eta_k^{-1}$ $(j, k > n)$ commute with $R\big(G(m,1,n_0)\big)$, and so

$$\pi(r_i r_{i'}) = \lambda_{i,i'} I, \qquad \pi(\eta_j \eta_k^{-1}) = \mu_{j,k} I\,, \tag{12.1}$$

where $\lambda_{i,i'}, \mu_{j,k} \in \boldsymbol{C}^\times$. From $(r_i r_{i+1})^3 = e$ and $(r_i r_{i'})^2 = z_1$ $(|i - i'| \ge 2)$, we get

$$(\lambda_{i,i+1})^3 = 1, \quad (\lambda_{i,i+1}\lambda_{i+1,i+2})^2 = \beta_1.$$

It follows from this that $\lambda_{i,i+1}\lambda_{i+1,i+2} = \beta_1$, and so $\beta_1^{\,3} = 1$ whence $\beta_1 = 1$. Thus we have $\lambda_{i,i+1} = 1$ $(i > n_0)$ and $\lambda_{i,i'} = 1$ $(i, i' > n_0, i \ne i')$. This forces that π is trivial on the subgroup $\Phi^{-1}\big(\mathfrak{A}_\infty\big) \subset R\big(G(m,1,\infty)\big)$. Thus,

$$\pi(r_i) = \pi(r_1) \quad (i \ge 2), \quad \pi(r_1) =: J \quad \text{with } J^2 = I. \tag{12.2}$$

From $r_j(\eta_j\eta_{j+1}^{-1})r_j = \eta_{j+1}\eta_j^{-1} = (\eta_j\eta_{j+1}^{-1})^{-1}$, $(\eta_j\eta_{j+1}^{-1})^m = z_2^{m(m-1)/2} = z_2^{m/2}$, $(\eta_j\eta_{j+2}^{-1})^m = z_2^{m/2}$, and $\eta_j\eta_k^{-1}\cdot\eta_k\eta_{k+1} = \eta_j\eta_{k+1}^{-1}$, we get

$$\mu_{j,j+1} = \mu_{j,j+1}^{-1}, \qquad (\mu_{j,j+1})^m = \beta_2^{m/2},$$
$$(\mu_{j,j+1}\mu_{j+1,j+2})^m = \beta_2^{m/2}, \quad \mu_{j,k}\mu_{k,k+1} = \mu_{j,k+1} \ \ (j<k),$$

whence $\beta_2^{m/2} = 1$, $(\mu_{j,j+1})^2 = 1$. Moreover from $\eta_j\eta_{j+1} = z_2\eta_{j+1}\eta_j$, we have $\eta_j\eta_{j+1}^{-1} = z_2\eta_{j+1}^{-1}\eta_j = z_2\eta_{j+1}^{-1}(\eta_j\eta_{j+1}^{-1})\eta_{j+1}$, and so $\mu_{j,j+1} = \beta_2\mu_{j,j+1}$, whence $\beta_2 = 1$.

Finally $r_k(\eta_j\eta_k^{-1})r_k^{-1} = z_3\eta_j\eta_{k+1}^{-1}$ $(j<k)$ gives $\mu_{j,k} = \beta_3\mu_{j,k+1} = \beta_3\mu_{j,k}\mu_{k,k+1}$, and

$$\mu_{k,k+1} = \beta_3 \ \ (\forall k), \quad \mu_{j,k} = \beta_3^{k-j} \ \ (j<k).$$

So $$\begin{cases} \pi(\eta_j) = \beta_3\pi(\eta_{j+1}), \quad \pi(\eta_j) = \beta_3^{j-1}\pi(\eta_1), \quad \pi(\eta_1) =: K, \\ \pi(r_1) = J, \quad J^2 = I, \quad K^m = I, \quad JK = \beta_3 KJ. \end{cases}$$

Thus we have a unique spin type $\beta = (\beta_1,\beta_2,\beta_3) = (1,\ 1,-1)$, Case VII, which may admit finite-dimensional representations.

12.2 Type $(1,\ 1,-1)$: Finite-dimensional irreducible spin representations

To obtain such representations, we are lead to look for pairs $\{K,J\}$ which satisfies

(12.3) $\quad J^2 = I, \quad K^m = I, \quad JK = -KJ, \quad \{J,K\}$ is irreducible.

Also we can treat it by the method of induced representations for the semidirect product group $R\big(\mathfrak{S}_n(\boldsymbol{Z}_m)\big)/\langle z_1,z_2\rangle$. However, in this reduced case where $\pi(r_i) = \pi(r_1) = J$ $(i\ge 2)$ and $\pi(\eta_j) = \beta_3^{j-1}\pi(\eta_1) = (-1)^{j-1}K$ $(j\ge 2)$, it is enough to consider a simpler group given as $H := D\rtimes R$ with $D := \langle z\rangle\times\langle\eta_1\rangle$, $R := \langle r_1\rangle$ and with fundamental relations

$$(12.4) \qquad \begin{cases} z^2 = e, \quad z \ \text{central}, \quad r_1^2 = e, \\ \eta_1^m = e, \quad r_1\eta_1 r_1 = z\eta_1. \end{cases}$$

Let $\widehat{D}^-$ be the set of characters χ of D for which $\chi(z) = -1$. Then

$$(12.5) \qquad \begin{cases} \widehat{D}^- \cong \{\zeta_0,\zeta_1,\ldots,\zeta_{m-1}\}, \quad \zeta_k(\eta_1) = \omega^k, \\ \omega = e^{2\pi i/m} = \text{a primitive } m\text{-th root of } 1. \end{cases}$$

The action of r_1 on $\widehat{D}^-$ is given by $\zeta_k \to \zeta_{k+m'}$ with $m' = m/2$, where the index $k+m'$ is understood modulo m. In fact,

$$\big(r_1(\zeta_k)\big)(\eta_1) = \zeta_k\big(r_1(\eta_1)\big) = -\zeta_k(\eta_1) = -\omega^k = \omega^{k+m'} = \zeta_{k+m'}(\eta_1).$$

In this way we get all finite-dimensional irreducible spin representations of H, and then those of $R\big(G(m,1,\infty)\big)$ as follows.

Theorem 12.1. *Let m be even.*

(i) *The type of spin factor representations of $G(m,1,\infty)$, which admits finite-dimensional irreducible ones, is uniquely $\beta=(1,\ 1,-1)$, Case VII.*

(ii) *For Case VII, Type $\beta=(1,\ 1,-1)$, all finite-dimensional irreducible spin representations are 2-dimensional. A complete set of representatives of their equivalence classes is given by $\{\pi_{2,\zeta_k}\ ;\ 0\le k<m'=m/2\}$, where*

$$\pi_{2,\zeta_k}(r_i)=\pi_{2,\zeta_k}(r_1)=\begin{pmatrix}0&1\\1&0\end{pmatrix}\quad(i\ge 2),\tag{12.6}$$

$$\pi_{2,\zeta_k}(\eta_j)=(-1)^{j-1}\pi_{2,\zeta_k}(\eta_1)=\begin{pmatrix}\omega^{k+(j-1)m'}&0\\0&\omega^{k+jm'}\end{pmatrix}\quad(j\ge 1).\tag{12.7}$$

Introduce new generators $\eta'_j:=z_3^{\,j-1}\eta_j$ $(j\in\boldsymbol{N})$ for $\widetilde{D}_\infty=\langle z_3,\eta_1,\eta_2,\dots\rangle$, then

$$\pi_{2,\zeta_k}(\eta'_j)=\begin{pmatrix}\zeta_k(\eta_1)&0\\0&-\zeta_k(\eta_1)\end{pmatrix}=\begin{pmatrix}\omega^k&0\\0&\omega^{k+m'}\end{pmatrix}\quad(j\in\boldsymbol{N}).$$

Put $\mathrm{ord}(d'):=\sum_{j\in\boldsymbol{N}}a_j$ for $d'=\prod_{j\in\boldsymbol{N}}{\eta'_j}^{a_j}\in\widetilde{D}_\infty$, then

$$\pi_{2,\zeta_k}(d')=\begin{pmatrix}(\omega^k)^{\mathrm{ord}(d')}&0\\0&(-\omega^k)^{\mathrm{ord}(d')}\end{pmatrix}.\tag{12.8}$$

We note that this explicit form of two-dimensional representations is a little different from those in [IhYo] and [DaMo] in appearance.

The trace character $f=\chi_{\pi_{2,\zeta_k}}$ is of type $\beta=(1,\ 1,-1)$ and it can be considered as a central function on $\widetilde{G}:=R\big(G(m,1,\infty)\big)/\langle z_1,z_2\rangle$ and

$$\widetilde{G}=\widetilde{D}_\infty\rtimes\mathfrak{S}_\infty,\quad \widetilde{D}_\infty=\langle z_3\rangle\times{\prod_{j\in\boldsymbol{N}}}'\langle\eta_j\rangle,\quad \langle\eta_j\rangle\cong\boldsymbol{Z}_m.\tag{12.9}$$

Theorem 12.2. *Let m be even, and π_{2,ζ_k} two-dimensional irreducible spin representation of $G(m,1,\infty)$ of Case VII given above. Then its trace character $\chi_{\pi_{2,\zeta_k}}$ is completely determined if its value is given for*

$$\widetilde{g}=(d',\sigma)\in\widetilde{G}\quad\text{with}\quad d'=z_3^{\,a}\prod_{j\in\boldsymbol{N}}{\eta'_j}^{a_j},\ \sigma\in\mathfrak{S}_\infty.\tag{12.10}$$

Then, with notations $L(\sigma)$ in §9 and $\mathrm{ord}(d'):=\sum_{j\in\boldsymbol{N}}a_j$,

$$\chi_{\pi_{2,\zeta_k}}(\widetilde{g})=\begin{cases}2\cdot(-1)^a\omega^{k\cdot\mathrm{ord}(d')}=2\cdot(-1)^a\zeta_k(\eta_1)^{\mathrm{ord}(d')}\\ \qquad\qquad\text{if }\ L(\sigma)\equiv 0,\ \mathrm{ord}(d')\equiv 0\,,\\ 0\qquad\qquad\text{otherwise.}\end{cases}\tag{12.11}$$

Note 12.1. If we keep to the original generators $\{\eta_1, \eta_2, \dots\}$ of $\widetilde{D}_\infty$, we should have a kind of sign function on $\widetilde{D}_\infty$ as follows: for $d' = \prod_{j\in\boldsymbol{N}} \eta_j^{a_j} \in \widetilde{D}_\infty$, put

$$\mathrm{sgn}(d') := \prod_{j\in\boldsymbol{N}} (-1)^{(j-1)a_j}, \quad \mathrm{ord}'(d') := \sum_{j\in\boldsymbol{N}} a_j\,,$$

then

$$\pi_{2,\zeta_k}(d') = \begin{pmatrix} \mathrm{sgn}(d')\,\omega^{k\cdot\mathrm{ord}'(d')} & 0 \\ 0 & \mathrm{sgn}(d')\,(-\omega^k)^{\mathrm{ord}'(d')} \end{pmatrix} \quad (j \in \boldsymbol{N}).$$

13 Summary of results for $R\big(G(m,1,\infty)\big)$, m even

For the convenience of later use, we summarize in the following table.

Table 13.1. For infinite group $R\big(G(m,1,\infty)\big)$.

Case Y	$(\beta_1,\beta_2,\beta_3)$ Type of factor representation	Existence of spin finite-dimensional irred. represen. π	extremal $\Leftrightarrow$ factorizable	$\mathrm{supp}(f):\ f(g')\neq 0 \implies$ Condition Y: $g=\Phi(g')=(d,\sigma)=\xi_{q_1}\cdots\xi_{q_r}g_1\cdots g_s$, $g_j=(d_j,\sigma_j)$
I	$(-1,-1,-1)$ seed represen. [IhYo], [DaMo]	$\neg\exists$ (not exist) π	YES	$\mathrm{ord}(\xi_{q_i})\equiv 0 \pmod 2\ (\forall i)$ i.e., $\xi_{q_i}=(t_{q_i},(q_i)), t_{q_i}\in S$, $\mathrm{ord}(d_j)+L(\sigma_j)\equiv 0\ (\forall j)$
II	$(-1,-1,\ 1)$ seed represen. in [DaMo]	$\neg\exists\,\pi$	NO$^{(*)}$	$\subset \mathfrak{A}_\infty(T)^S$, i.e., $\mathrm{ord}(d)\equiv L(\sigma)\equiv 0$, and $\mathrm{ord}(\xi_{q_i})\equiv 0\ (\forall i)$, $\mathrm{ord}(d_j)+L(\sigma_j)\equiv 0\ (\forall j)$
III	$(-1,\ 1,-1)$	$\neg\exists\,\pi$	NO$^{(*)}$	$\subset \mathfrak{A}_\infty(T)^S$, and $\mathrm{ord}(d)\equiv 0$, $L(\sigma_j)\equiv 0\ (\forall j)$
IV	$(-1,\ 1,\ 1)$ seed represen. [IhYo], [DaMo]	$\neg\exists\,\pi$	YES	$\subset \mathfrak{A}_\infty(T)$, and $L(\sigma_j)\equiv 0$, i.e., σ_j even $(\forall j)$ for $\sigma=\sigma_1\sigma_2\cdots\sigma_s$
V	$(1,-1,-1)$	$\neg\exists\,\pi$	YES	$\subset \mathfrak{A}_\infty(T)^S$, and $\mathrm{ord}(\xi_{q_i})\equiv 0\ (\forall i)$, $\mathrm{ord}(d_j)\equiv L(\sigma_j)\equiv 0\ (\forall j)$
VI	$(1,-1,\ 1)$	$\neg\exists\,\pi$	YES	$\subset \mathfrak{A}_\infty(T)^S$, and $\mathrm{ord}(\xi_{q_i})\equiv 0\ (\forall i)$, $\mathrm{ord}(d_j)\equiv L(\sigma_j)\equiv 0\ (\forall j)$
VII	$(1,\ 1,-1)$ seed represen. in [IhYo]	$\exists$ 2-dimensional irred. representa. π_{2,ζ_k} $(0\leqslant k<m/2)$	NO	$\subset \mathfrak{A}_\infty(T)^S$, i.e., $\mathrm{ord}(d)\equiv 0$, $L(\sigma)\equiv 0$
VIII	$(1,\ 1,\ 1)$ $\mathfrak{S}_\infty(\boldsymbol{Z}_m)$ char. formula in [HH1]	$\exists$ 1-dimensional character $\chi_{\varepsilon,\zeta}$ $(\varepsilon=0,1;\ \zeta\in\widehat{T})$	YES	**No condition**

$(*)$ A proof for this will be given in the succeeding paper.

In the table above, the following are given:

• in the second column, (spin) type of factor representation or of spin characters, and the information on the basic representations given in [IhYo] or [DaMo] which are called here as *seed representation*;

• in the 3rd column, information on finite-dimensional representations;

• in the 4th column, information on the validity of the criterion (EF);

• in the 5th column, (Condition Y) to define $\mathcal{O}(\mathrm{Y})$ for which $\mathrm{supp}(f) \subset \mathcal{O}(\mathrm{Y})$.

The results in Table 13.1 for $R\big(G(m,1,\infty)\big)$, m even, are complicated and accordingly very interesting.

14 Limits of irreducible characters of an increasing sequence of groups

Let $H_0 \hookrightarrow H_1 \hookrightarrow \ldots \hookrightarrow H_n \hookrightarrow H_{n+1} \hookrightarrow \ldots$ be an increasing sequence of compact groups such that the imbedding $H_n \hookrightarrow H_{n+1}$ is continuous, and put $H_\infty := \lim_{n\to\infty} H_n = \bigcup_{0\leqslant n<\infty} H_n$ be its inductive limit with the inductive limit topology. Then, H_∞ is a topological group which is discrete if all H_n are finite, and it is no more locally compact if H_n are not finite and not stable for n sufficiently large (cf. [TSH], §5). We summarize in this section results on limiting processes of (normalized) irreducible characters of H_n as $n \to \infty$, and extend them in some extent. When all H_n are finite groups, the infinite group H_∞ is called *locally finite* in [Ker, p.5]. For this kind of groups, the study was initiated by Vershik-Kerov [VK] for symmetric groups $\mathfrak{S}_n \nearrow \mathfrak{S}_\infty$, and by [HH1]–[HH3] and by [Boy] for the case of wreath product groups $\mathfrak{S}_n(T)$ with a finite group T, and also by [HH5], [HH6] and [HHH1] for more general case of $\mathfrak{S}_n(T)$ with a compact group T. More general branching graphs, not necessarily induced by inductive limit groups, are treated in [KOO] (Jack graph), [BO], [Ols], and [HoHH] (more general graphs) and so on.

Even though we need here only the case of locally finite groups, it is preferable to discuss together the case of infinite series of compact groups, for which we will prove below general results (Theorems 14.2 and 14.3) and give a comment on counter examples.

14.1. An increasing sequence of compact groups.

We assume that $H_0 = \{e\}$ the trivial group of the identity element e, and H_n are strictly increasing. For the duals $H_n^\wedge$ of H_n's, we introduce an adjacent relation $\nearrow$ as follows: for $\alpha \in H_n^\wedge$ and $\beta \in H_{n+1}^\wedge$, $\alpha \nearrow \beta$ means α is actually contained in $\beta|_{H_n}$ and denote by $\kappa(\alpha,\beta)$ its multiplicity. We put $\kappa(\alpha,\beta) = 0$ if $\alpha \nearrow \beta$ does not hold, for later conveniences. Put $\boldsymbol{G}_n := H_n^\wedge$ and $\boldsymbol{G} := \bigsqcup_{n\geqslant 0} \boldsymbol{G}_n$, where the unique element in $\boldsymbol{G}_0$ is denoted by $\varnothing$.

Let f_n be a central, continuous, positive-definite function on H_n, normalized as $f_n(e) = 1$. Then it has an absolutely convergent Fourier series expansion as

$$f_n(h) = \sum_{\alpha \in \boldsymbol{G}_n} \varphi_n(\alpha)\, \chi_\alpha(h) = \sum_{\alpha \in \boldsymbol{G}_n} \dim \alpha \cdot \varphi_n(\alpha) \cdot \widetilde{\chi}_\alpha(h) \quad (h \in H_n), \tag{14.1}$$

where χ_α is the usual trace character of α, and $\widetilde{\chi}_\alpha = \chi_\alpha / \dim \alpha$ is the normalized irreducible character with $\dim \alpha$ = dimension of α. Then

$$\varphi_n(\alpha) \geq 0, \qquad \sum_{\alpha \in \boldsymbol{G}_n} \dim \alpha \cdot \varphi_n(\alpha) = 1, \tag{14.2}$$

so that the support of φ_n, $\{\alpha \in \boldsymbol{G}_n; \varphi_n(\alpha) > 0\}$ is at most countable.

14.2. Branching graph and central measures.

A *branching graph* consists of the stratified vertex sets $\boldsymbol{G} = \bigsqcup_{n \geqslant 0} \boldsymbol{G}_n$ and the edges satisfying the following conditions, where $\boldsymbol{G}_n$ is called the vertexes of the n-th level.

(BG1) Two vertexes $\alpha, \beta \in \boldsymbol{G}$ can be adjacent only if they belong to consecutive levels. If $\alpha \in \boldsymbol{G}_n$ and $\beta \in \boldsymbol{G}_{n+1}$ are adjacent, we express it as $\alpha \nearrow \beta$ and call (α, β) the ingoing [resp. outgoing] edge of β [resp. α].

(BG2) $\boldsymbol{G}_0$ consists of the unique element $\varnothing$ that has no ingoing edges.

(BG3) For any vertex except $\varnothing$, its ingoing [resp. outgoing] edges form a nonempty finite [resp. non-empty (possibly infinite)] set.

(BG4) If $\alpha \nearrow \beta$ holds, the edge (α, β) carries multiplicity $\kappa(\alpha, \beta) > 0$.

For $\alpha \in \boldsymbol{G}_n, \beta \in \boldsymbol{G}_{n+1}$, we put $\kappa(\alpha, \beta) = 0$ if they are not adjacent.

A non-negative real-valued function φ on $\boldsymbol{G}$ is called *harmonic* if

$$\varphi(\alpha) = \sum_{\beta:\, \alpha \nearrow \beta} \kappa(\alpha, \beta)\, \varphi(\beta) \quad (\alpha \in \boldsymbol{G}), \tag{14.3}$$

and *normalized* as $\varphi(\varnothing) = 1$, and $\mathrm{supp}(\varphi)$ is *at most countable*.

Let $\mathfrak{T} = \mathfrak{T}(\boldsymbol{G})$ denote the set of all infinite paths on $\boldsymbol{G}$ starting at $\varnothing$, for which $t \in \mathfrak{T}$ is expressed as

$$t = \big(t(0) \nearrow t(1) \nearrow \cdots \nearrow t(n) \nearrow \cdots\big), \quad t(n) \in \boldsymbol{G}_n.$$

Its truncated path up to the n-th level is $t_n = \big(t(0) \nearrow t(1) \nearrow \cdots \nearrow t(n)\big)$. $\mathfrak{T}_n$ denotes the set of all finite paths up to n-th level. For a finite path u connecting $\alpha \in \boldsymbol{G}_m$ and $\beta \in \boldsymbol{G}_n$ $(m < n)$ as $\alpha = u(m) \nearrow \cdots \nearrow u(n) = \beta$, its *weight* w_u is defined by

$$w_u := \prod_{m \leq i < n} \kappa\big(u(i), u(i+1)\big), \tag{14.4}$$

and we define the *dimension function* d on $\boldsymbol{G}$ by

$$d(\alpha, \beta) := \sum_{\text{path } u:\, \alpha \nearrow \cdots \nearrow \beta} w_u. \tag{14.5}$$

From [HoHH, §2], we quote the following assertions (1°), (2°) and (3°).

(1°) *A harmonic function* φ *on* $\boldsymbol{G}$ *satisfies, for any* $m < n$ *and* $\alpha \in \boldsymbol{G}_m$,

$$\varphi(\alpha) = \sum_{\beta \in \boldsymbol{G}_n} d(\alpha, \beta)\, \varphi(\beta).$$

A subset $\boldsymbol{G}^0 \subset \boldsymbol{G}$, with edges inherited from $\boldsymbol{G}$, is called a *subgraph* if for any $\beta \in \boldsymbol{G}^0$, any path in $\boldsymbol{G}$ connecting $\varnothing$ to β is also a path in $\boldsymbol{G}^0$. For each $u = \big(u(0) \nearrow \cdots \nearrow u(n)\big) \in \mathfrak{T}_n$, put

$$C_u := \{t \in \mathfrak{T}\,;\ t(k) = u(k),\ \ 0 \le k \le n\},$$

and denote by $\mathfrak{B}(\mathfrak{T})$ the Borel field of $\mathfrak{T}$ generated by the set of all C_u's. A probability measure on measurable space $(\mathfrak{T}, \mathfrak{B}(\mathfrak{T}))$ is called *central* if it is supported by the path space $\mathfrak{T}(\boldsymbol{G}^0)$ of some countable subgraph $\boldsymbol{G}^0$ of $\boldsymbol{G}$, and

$$\frac{M(C_u)}{w_u} = \frac{M(C_v)}{w_v} \tag{14.6}$$

for all n and $u, v \in \mathfrak{T}_n$ with a common terminating vertex.

(2°) *There exists a bijective correspondence between the central probabilities* M *on* $\mathfrak{T}$ *and the harmonic functions* φ *on* $\boldsymbol{G}$ *through*

$$M(C_u) := w_u \cdot \varphi(\alpha) \tag{14.7}$$

for any $\alpha \in \boldsymbol{G}_n$ *and* $u \in \mathfrak{T}_n$ *such that* $u(n) = \alpha$ $(n \ge 0)$.

Define a random variable $X_n : \mathfrak{T} \to \boldsymbol{G}_n$ by $X_n(t) = t(n)$, then $\mathfrak{B}(\mathfrak{T})$ is generated by $X_1, X_2, \ldots$. Let $\mathfrak{B}_n$ be the sub-σ-field generated by $X_n, X_{n+1}, \ldots$ and put $\mathfrak{B}_\infty := \bigcap_{n \geqslant 0} \mathfrak{B}_n$.

(3°) *Let* M *be an extremal central probability on* $\mathfrak{T}$*. Then* M *is trivial on* $\mathfrak{B}_\infty$*, that is,* $M(B) = 0$ *or 1 for* $B \in \mathfrak{B}_\infty$.

14.3. Limit of Martin kernels on a branching graph.

On the branching graph $\boldsymbol{G}$, we regard the ratio

$$\frac{d(\alpha, \beta)}{d(\varnothing, \beta)}, \quad \alpha, \beta \in \boldsymbol{G}$$

as a *Martin kernel* on $\boldsymbol{G}$. Let M be a central probability on $\mathfrak{T} = \mathfrak{T}(\boldsymbol{G})$ which is supported by $\mathfrak{T}(\boldsymbol{G}^0)$ of a countable subgraph $\boldsymbol{G}^0$. Then M can be traced to probability M^0 on sub-σ-field $\mathfrak{B}^0 = \mathfrak{B}(\mathfrak{T}) \cap \mathfrak{T}(\boldsymbol{G}^0)$.

Theorem 14.1 [HoHH, Theorem 3.2]**.** *Let* M *be an extremal central probability measure on* $(\mathfrak{T}, \mathfrak{B}(\mathfrak{T}))$*, and* $\boldsymbol{G}^0$ *a countable subgraph associated with* M

as above. Let φ *be an extremal harmonic function on* $\boldsymbol{G}$ *associated with* M *as in* (2°). *Then, for* M*-almost sure* $t \in \mathfrak{T}$,

$$\lim_{n\to\infty} \frac{d\big(\alpha, t(n)\big)}{d\big(\varnothing, t(n)\big)} = \varphi(\alpha) \quad (\alpha \in \boldsymbol{G}^0). \tag{14.8}$$

14.4. Limits of irreducible characters.

As in **14.1**, let $H_0 \hookrightarrow \ldots \hookrightarrow H_n \hookrightarrow H_{n+1} \hookrightarrow \ldots$ be an increasing sequence of compact groups and define, $H_\infty = \lim_{n\to\infty} H_n$, $\boldsymbol{G} = \bigsqcup_{n\geqslant 0} \boldsymbol{G}_n$, $\boldsymbol{G}_n = H_n^\wedge$, $\kappa(\alpha,\beta) = [\beta|_{H_n} : \alpha]$ for $\alpha \in \boldsymbol{G}_n, \beta \in \boldsymbol{G}_{n+1}$, then

$$\chi_\beta|_{H_n} = \sum_{\alpha\in\boldsymbol{G}_n :\, \alpha\nearrow\beta} \kappa(\alpha,\beta)\,\chi_\alpha\,. \tag{14.9}$$

A *character* of $G = H_\infty$ is, by definition, an extremal element among the continuous, positive-definite, central and normalized functions on G. The following two are straight forward generalizations of Theorem 4.2 and Theorem 4.3 in [HoHH] respectively.

Theorem 14.2. *Let* $G = H_\infty$ *be the inductive limit of an increasing sequence* $H_n \hookrightarrow H_{n+1}$ *of compact groups such that each imbedding is continuous. Then there exist bijective correspondences between the following three objects:*

(1) *the set* $E(G)$ *of characters of* G,

(2) *the set of extremal harmonic functions* φ *on* $\mathfrak{T}(\boldsymbol{G})$,

(3) *the set of extremal central probabilities* M *on* $\mathfrak{T}(\boldsymbol{G})$.

To be precise, f *in* (1) *and* φ *in* (2) *are connected as*

$$f|_{H_n} = \sum_{\alpha\in\boldsymbol{G}_n} \varphi(\alpha)\,\chi_\alpha, \tag{14.10}$$

and the bijection between (2) *and* (3) *is given in* (2°).

Under the above bijective correspondences, we apply Theorem 14.1 to obtain the following result on limits of irreducible characters of H_n as $n \to \infty$.

Theorem 14.3. *Let* $G = H_\infty$ *be the inductive limit of an increasing sequence of compact groups* H_n *such that each imbedding is continuous. For any* $f \in E(G)$, *let* M *be the corresponding extremal central measure in Theorem 14.2. Then, for* M*-almost sure* $t \in \mathfrak{T}$, *the convergence*

$$\lim_{n\to\infty} \widetilde{\chi}_{t(n)} = f \tag{14.11}$$

is true and uniform on each H_k, $k \geq 1$.

Sketch of proof. We follow the proof of [HoHH, Theorem 4.3]. For $k < n$, there holds

$$\chi_\gamma|_{H_k} = \sum_{\beta \in \boldsymbol{G}_k} d\big(\beta, \gamma\big)\, \chi_\beta \quad (\gamma \in \boldsymbol{G}_n). \tag{14.12}$$

Under the correspondence $f \leftrightarrow \varphi \leftrightarrow M$ in Theorem 14.2, put $\boldsymbol{G}^0 = \operatorname{supp}\varphi$. Then $\boldsymbol{G}^0$ is a countable subgraph of $\boldsymbol{G}$, and we see from Theorem 14.1 that, for M-almost sure (= M-a.s.) $t \in \mathfrak{T}$,

$$\lim_{n\to\infty} \frac{d\big(\alpha, t(n)\big)}{d\big(\varnothing, t(n)\big)} = \varphi(\alpha) \quad (\alpha \in \boldsymbol{G}^0). \tag{14.13}$$

Take a path $t \in \mathfrak{T}(\boldsymbol{G}^0)$ satisfying (14.13), then

$$\alpha \in \boldsymbol{G}_k \text{ and } d\big(\alpha, t(n)\big) > 0 \implies \alpha \in \boldsymbol{G}_k^0 = \boldsymbol{G}_k \cap \boldsymbol{G}^0. \tag{14.14}$$

Put for $\alpha \in \boldsymbol{G}_k$, $k < n$,

$$Q_{t(n)}(\alpha) := \frac{d\big(\alpha, t(n)\big)}{d\big(\varnothing, t(n)\big)}\, d(\varnothing, \alpha), \quad Q(\alpha) := \varphi(\alpha)\, d(\varnothing, \alpha). \tag{14.15}$$

Then both are probabilities supported by $\boldsymbol{G}_k$. We estimate the difference of

$$\widetilde{\chi}_{t(n)}|_{H_k} = \sum_{\alpha \in \boldsymbol{G}_k^0} Q_{t(n)}(\alpha) \widetilde{\chi}_\alpha, \quad f|_{H_k} = \sum_{\alpha \in \boldsymbol{G}_k^0} Q(\alpha) \widetilde{\chi}_\alpha, \tag{14.16}$$

where the first equality follows from (14.12) and (14.15). For $\epsilon > 0$, there exists a finite subset $F \subset \boldsymbol{G}_k^0$ such that $1 - Q(F) < \epsilon$. Equality (14.13) shows that, for M-a.s. path $t \in \mathfrak{T}(\boldsymbol{G}^0)$, and n sufficiently large,

$$|Q_{t(n)}(F) - Q(F)| < \epsilon, \qquad \text{and,}$$
$$Q_{t(n)}(F^c) \le 1 - Q(F) + |Q_{t(n)}(F) - Q(F)| < 2\epsilon.$$

Putting them into (14.16), we have for $h \in H_k$,

$$|\widetilde{\chi}_{t(n)}(h) - f(h)| \le \sum_{\alpha \in F} |Q_{t(n)}(\alpha) - Q(\alpha)| + Q_{t(n)}(\boldsymbol{G}_k^0 \setminus F) + Q(\boldsymbol{G}_k^0 \setminus F) \le 4\epsilon.$$

Hence, for M-a.s. path t, $\displaystyle\lim_{n\to\infty} \sup_{h \in H_k} \big|\widetilde{\chi}_{t(n)}(h) - f(h)\big| = 0.$ □

The case of $H_n = U(n)$, the unitary group of order n, and $H_\infty = U(\infty) := \lim_{n\to\infty} U(n)$, is treated in [Ols].

14.5. Are limits of irreducible characters always extremal ?

Under the general setting of Theorem 14.3, assume that, along with a path $t \in \mathfrak{T}$, the pointwise limit

$$f_t(h) := \lim_{n\to\infty} \widetilde{\chi}_{t(n)}(h) \quad (h \in G = H_\infty) \tag{14.17}$$

exists on G, where $\widetilde{\chi}_{t(n)}$ denotes the normalized character of $t(n) \in \boldsymbol{G}_n = H_n^\wedge$. Here we are interested in paths t not picked up by any M-measure 1 subset indicated in that theorem and study the following assertion:

Assertion (??). *Suppose that the limit function f_t in (14.17) exists pointwise, then f_t is always a character of G, or f_t is continuous and extremal, i.e., $f_t \in E(G)$.*

This assertion is proved for infinite symmetric group $G = \mathfrak{S}_\infty$ by Vershik-Kerov already in [VK]. On the other hand, Kerov gives the following general comment in [Ker, p.11] for a locally finite group G:

> " Let us call a path $t \in T$ *regular* if the limits (5.2) exist. The corresponding limiting function φ_t is harmonic, though not necessarily extreme. ...... "

Here for a path $t = (\nu_1, \nu_2, \ldots, \nu_n, \ldots)$, $\nu_n \in \Gamma_n$, the limits (5.2) is

$$\varphi_t(\lambda) = \lim_{n\to\infty} \frac{d(\lambda, \nu_n)}{d(\emptyset, \nu_n)}, \quad \lambda \in \Gamma,$$

with $\Gamma = \bigsqcup_{n\geqslant 0} \Gamma_n$ a branching graph. For a locally finite group G, the existence of a limit φ_t is equivalent to the existence of a pointwise limit f_t of $\widetilde{\chi}_{t(n)}$ on G. We call a limit f_t a *bad* limit if it is not in $E(G)$. Then Kerov's above comment means that, *for certain locally finite groups G, there exist bad limits f_t on G.*

In [BO, p.5], Borodin-Olshanski give a comment as

> " In all examples of the graphs $\mathbb{G}$ considered in the present paper one can embed (the vertices of) $\mathbb{G}$ into $\Omega(\mathbb{G})$ in such a way that any point $\omega \in \Omega(\mathbb{G})$ can be approximated by a sequence of vertices $\{\lambda(n) \in \mathbb{G}_n\}_{n=1,2,\ldots}$, and for any such sequence
>
> $$K(\mu, \omega) = \lim_{n\to\infty} \frac{\dim_{\mathbb{G}}(\mu, \lambda(n))}{\dim_{\mathbb{G}} \lambda(n)}\,. \qquad "$$

not referring an assertion like Assertion (??). Here $\Omega(\mathbb{G})$ denotes the set of extreme points in the set of normalized, non-negative, harmonic functions on $\mathbb{G}$.

For wreath product groups $G = \mathfrak{S}_\infty(T) = H_\infty$ with $H_n = \mathfrak{S}_n(T)$ and T a finite group, Assertion (??) is affirmed by [HH1]–[HH3], and also by [Boy] by reducing it to the case of $\mathfrak{S}_\infty(\boldsymbol{Z}_p)$ (see [Boy] §2, especially Theorem 2). For the original case of $\mathfrak{S}_\infty$, see [VK], §1-4, especially Eqs (2) and (6).

For the general case of series of infinite compact group H_n, the following assertion, weaker than Assertion (??), is more natural:

Assertion (????). *If the limit function f_t in (14.17) exists compact uniformly, then f_t is always a character of G, or f_t is continuous and extremal, i.e., $f_t \in E(G)$.*

For the case of unitary groups $U(n) \nearrow U(\infty)$, this assertion is affirmed in [Ols] (Proposition 10.9). For the case of wreath product groups $\mathfrak{S}_n(T) \nearrow \mathfrak{S}_\infty(T)$ with any compact group T, we gave all the characters of G in [HH6], and studied limit process $\widetilde{\chi}_{t(n)} \to f_t$ in [HHH1] and [HoHH]. In [HHH1], it is proved that, if T is not finite, then Assertion (??) never holds since we can give explicitly pointwise limits f_t in (14.17), along paths $t \in \mathfrak{T}$, which are discontinuous, and so do not belong to $E(G)$ (*bad* limits). A necessary and sufficient condition on the path $t \in \mathfrak{T}$ is given for that f_t is actually a character (cf. Theorems 6.1 and 7.1, ibid.), and Assertion (????) is affirmed by Theorem 7.1, ibid.

As far as we know, a general affirmation to Assertion (??) has not yet been given for locally finite groups $G = H_\infty$, except the above mentioned case of wreath products $G = \mathfrak{S}_\infty(T)$ with T finite. In this connection, we are very much careful at this point for the covering groups $H_n = R\big(\mathfrak{S}_n(\boldsymbol{Z}_m)\big)$ of $G(m,1,n) = \mathfrak{S}_n(\boldsymbol{Z}_n)$ and $H_\infty = R\big(G(m,1,\infty)\big)$ (and also for $H_n = \mathfrak{S}_n(T_n)$ with finite groups T_n growing up to T_∞).

Part II

Detailed study in Case VII

15 Explicit formula for characters of $G(m,p,\infty)$ in Case VIII

Our results for characters of $G(m,p,n)=\mathfrak{S}_n(\boldsymbol{Z}_m)^{S(p)}$ and $G(m,p,\infty)=\mathfrak{S}_\infty(\boldsymbol{Z}_m)^{S(p)}$ in [HH1] prepare a background of the spin case at present. We review them briefly. Recall the definition of $\mathfrak{S}_I(T)$ in §2.1: for a finite abelian group T and its subgroup S,

$$\mathfrak{S}_I(T)=D_I(T)\rtimes\mathfrak{S}_I,\ \ D_I(T):=\prod\nolimits'_{i\in I}T_i\,,\ \ T_i=T\ (i\in I),$$

(15.1) $$\mathfrak{S}_I(T)^S=D_I(T)^S\rtimes\mathfrak{S}_I,\ D_I(T)^S:=\big\{d=(t_i)_{i\in I};P(d):=\prod\nolimits_{i\in I}t_i\in S\big\}.$$

Let $\widehat{T}$ be the dual of T, and χ_ε be the one-dimensional characters of $\mathfrak{S}_\infty$ given by $\chi_\varepsilon(\sigma)=\mathrm{sgn}_{\mathfrak{S}_\infty}(\sigma)^\varepsilon$ ($\sigma\in\mathfrak{S}_\infty$; $\varepsilon=0,1$). For the parameter of characters, we prepare a set $A:=\Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}};\mu\Big)$ satisfying the condition

(15.2) $$\begin{cases}\alpha_{\zeta,\varepsilon}\ \ (\zeta\in\widehat{T},\varepsilon\in\{0,1\}) & \begin{cases}\alpha_{\zeta,\varepsilon}=(\alpha_{\zeta,\varepsilon;p})_{p\in\boldsymbol{N}},\\ \alpha_{\zeta,\varepsilon;1}\ge\alpha_{\zeta,\varepsilon;2}\ge\alpha_{\zeta,\varepsilon;3}\ge\cdots\ge0,\end{cases}\\ \mu=(\mu_\zeta)_{\zeta\in\widehat{T}}\,,\quad \mu_\zeta\ge0\ (\zeta\in\widehat{T}),\quad \|\mu\|:=\sum_{\zeta\in\widehat{T}}\mu_\zeta\,,\\ \sum_{\zeta\in\widehat{T}}\sum_{\varepsilon\in\{0,1\}}\|\alpha_{\zeta,\varepsilon}\|+\|\mu\|=1,\quad \|\alpha_{\zeta,\varepsilon}\|:=\sum_{p\in\boldsymbol{N}}\alpha_{\zeta,\varepsilon;p}\,,\end{cases}$$

and the set of all such A's is denoted by $\mathcal{A}(\widehat{T})$.

Theorem 15.1 (cf. [HH2, Theorem 2])**.** *Let $G=\mathfrak{S}_\infty(T)$ be the wreath product of the infinite symmetric group $\mathfrak{S}_\infty$ with a finite abelian group T. Every normalized character of G is factorizable and is parametrized by a set $A\in\mathcal{A}(\widehat{T})$ as f_A. For a $g\in G$, let its standard decomposition be*

(15.3) $$g=\xi_{q_i}\xi_{q_2}\cdots\xi_{q_r}g_1g_2\cdots g_s,\quad \xi_{q_i}=(t_{q_i},(q_i)),\quad g_j=(d_j,\sigma_j).$$

Then the value $f_A(g)$ is given by the product of

(15.4) $$f_A(\xi_q)\ =\ \sum_{\zeta\in\widehat{T}}\left(\sum_{\varepsilon\in\{0,1\}}\sum_{p\in\boldsymbol{N}}\alpha_{\zeta,\varepsilon;p}\ +\ \mu_\zeta\right)\chi_\zeta(t_q),$$

(15.5) $$f_A(g_j)\ =\ \sum_{\zeta\in\widehat{T}}\left(\sum_{\varepsilon\in\{0,1\}}\sum_{p\in\boldsymbol{N}}(\alpha_{\zeta,\varepsilon;p})^{\ell(\sigma_j)}\,\chi_\varepsilon(\sigma_j)\right)\chi_\zeta\big(P(d_j)\big).$$

Theorem 15.2 ([HH2, Theorem 4], [HH6, Theorem 7.1]). *Let $S \subset T$ be a subgroup of a finite abelian group T, and $G^S = \mathfrak{S}_\infty(T)^S$ be the normal subgroup of $G = \mathfrak{S}_\infty(T)$ given in (15.1). Then the restriction of a character of G onto G^S is a character of G^S. Conversely any character of G^S is obtained by restriction from G.*

About the condition on the indices A that the restrictions $f_A|_{G^S}$ coincide with each other, see [HH6, Proposition 7.2].

In the case of $T = \boldsymbol{Z}_m$, as seen in §2.1, we have $S(p) = \{t^p\,;\, t \in T\} \cong \boldsymbol{Z}_{m/p}$, $p|m$, and

$$\mathfrak{S}_\infty(T) = G(m,1,\infty), \quad \mathfrak{S}_\infty(T)^{S(p)} = G(m,p,\infty), \quad \widehat{T} = \big\{\zeta_k\,;\, 0 \le k \le m-1\big\}.$$

16 Formula for spin characters of $G(m,1,\infty)$ in Case VII

16.1 Correspondence between (non-spin) characters of $\mathfrak{A}_\infty(\boldsymbol{Z}_m)^{S(2)}$ and spin characters of $G(m,1,\infty)$

Let m be even, and π_{2,ζ_k} 2-dimensional irreducible spin representation of $R\big(G(m,1,\infty)\big)$ of Case VII, Type $\beta = (1,\ 1, -1)$, given in Theorem 12.1. Put for $\mathrm{Y} = \mathrm{VII}$,

$$\begin{aligned} \widetilde{G} := \widetilde{G}^{\mathrm{Y}} &= R\big(G(m,1,\infty)\big)/\mathrm{Ker}(\chi^{\mathrm{Y}}) = R\big(G(m,1,\infty)\big)/\langle z_1, z_2\rangle, \\ &= \widetilde{D}_\infty \rtimes \mathfrak{S}_\infty, \quad \widetilde{D}_\infty = \langle z_3, \eta_j\ (j \in \boldsymbol{N})\rangle, \ \ \mathfrak{S}_\infty = \langle s_i\ (i \in \boldsymbol{N})\rangle, \end{aligned}$$

$$\{e\} \to \langle z_3\rangle \to \widetilde{G} \overset{\Phi^{\mathrm{Y}}}{\to} G(m,1,\infty) \to \{e\}, \ \ \Phi^{\mathrm{Y}}(\eta_j) = y_j\ (1 \le j < \infty), \tag{16.1}$$

with fundamental relations :

$$\begin{array}{ll} \text{(i)} & z_3^{\,2} = e\,, \quad z_3 \ \text{ central element}\,; \\ \text{(ii)} & \begin{cases} s_i^{\,2} = e \ \ (1 \le i < \infty), \quad (s_i s_{i+1})^3 = e \ \ (1 \le i < \infty), \\ s_i s_j = s_j s_i \quad (|i-j| \ge 2), \end{cases} \\ \text{(iii)} & \eta_j^{\,m} = e \qquad (1 \le j < \infty), \\ \text{(iv)} & \eta_j \eta_k = \eta_k \eta_j \ \ (j \ne k), \\ \text{(v)} & \begin{cases} s_i \eta_i s_i^{-1} = \eta_{i+1}, \ s_i \eta_{i+1} s_i^{-1} = \eta_i \quad (1 \le i < \infty), \\ s_i \eta_j s_i^{-1} = z_3 \eta_j \qquad (j \ne i, i+1)\,. \end{cases} \end{array}$$

The trace character $\chi_{\pi_{2,\zeta_k}}$ is a function on $\widetilde{G}$ given in (12.11) for $\widetilde{g} = (d', \sigma) \in \widetilde{G}$ with

$$d' = z_3^{\,a} \prod_{j \in \boldsymbol{N}} {\eta'_j}^{a_j} \in \widetilde{D}_\infty, \ \ \eta'_j = z_3^{\,j-1}\eta_j\ (j \in \boldsymbol{N}), \ \ \sigma \in \mathfrak{S}_\infty\,, \tag{16.2}$$

$$\chi_{\pi_{2,\zeta_k}}(\widetilde{g}) = \begin{cases} 2\cdot(-1)^a\,\omega^{k\,\mathrm{ord}(d')} & \text{if } L(\sigma)\equiv 0,\ \mathrm{ord}(d')\equiv 0\,, \\ 0 & \text{otherwise.} \end{cases} \tag{16.3}$$

We denote by $\widetilde{\chi}_{\pi_{2,\zeta_k}}$ the normalized character $\chi_{\pi_{2,\zeta_k}}/2$.

Consider the tensor product $\pi_{4,\zeta_k,\zeta_\ell} := \pi_{2,\zeta_k}\otimes\pi_{2,\zeta_\ell}$. Then it is a (non-spin) linear representation of $G = G(m,1,\infty)$.

Lemma 16.1. *The tensor product representation* $\pi_{4,\zeta_k,\zeta_\ell} = \pi_{2,\zeta_k}\otimes\pi_{2,\zeta_\ell}$ *splits into 4 one-dimensional characters of* $G = G(m,1,\infty) = \mathfrak{S}_\infty(\boldsymbol{Z}_m)$ *as*

$$\pi_{4,\zeta_k,\zeta_\ell} = \pi_{2,\zeta_k}\otimes\pi_{2,\zeta_\ell} = \sum_{\varepsilon=0,1}^{\oplus}\Big(\chi_{k+\ell}\cdot\mathrm{sgn}^\varepsilon \bigoplus \chi_{k+\ell+m'}\cdot\mathrm{sgn}^\varepsilon\Big)\,, \tag{16.4}$$

where the characters χ_k $(0\le k<m)$ *and* sgn^ε *on* G *is defined as*

$$\chi_k\big((d,\sigma)\big) := \zeta_k\big(P(d)\big),\ \ \mathrm{sgn}^\varepsilon\big((d,\sigma)\big) := \mathrm{sgn}^\varepsilon(\sigma)\quad \big((d,\sigma)\in G\big). \tag{16.5}$$

The character of $\pi_{4,\zeta_k,\zeta_\ell}$ *is given as: for* $g=(d,\sigma)\in G=\mathfrak{S}_\infty(\boldsymbol{Z}_m)$*,*

$$\chi_{\pi_{4,\zeta_k,\zeta_\ell}}(g) = \begin{cases} 4\cdot\omega^{(k+\ell)\mathrm{ord}(d)} & \text{if } L(\sigma)\equiv 0,\ \mathrm{ord}(d)\equiv 0\,, \\ 0 & \text{otherwise.} \end{cases} \tag{16.6}$$

In particular, in case $k=\ell=0$*, the normalized character* $\widetilde{\chi}_{\pi_{4,\zeta_0,\zeta_0}} = \big(\widetilde{\chi}_{\pi_{2,\zeta_0}}\big)^2$ *is the indicator function of the normal subgroup* $N = \mathfrak{A}_\infty(\boldsymbol{Z}_m)^{S(2)}$ *of* $G = \mathfrak{S}_\infty(\boldsymbol{Z}_m)$*.*

Let f be a spin character of $G(m,1,\infty)$ of Case VII. Then, as seen from Table 10.1, $\mathrm{supp}(f)\subset\mathcal{O}(\mathrm{VII}) = \Phi^{-1}(N)$, $N=\mathfrak{A}_\infty(\boldsymbol{Z}_m)^{S(2)}$, and f can be considered as a function on $\widetilde{G}$. Let $E^{\mathrm{VII}} := E^{\mathrm{VII}}\big(R\big(G(m,1,\infty)\big)\big)$ be the set of all spin characters of $G(m,1,\infty)$ of Case VII, Type $(1,\ 1,-1)$.

Definition 16.1 (Maps $\mathcal{M}$ and $\mathcal{N}$)**.** Between the set of normalized central positive definite functions $K_1(N)$ on $N=\mathfrak{A}_\infty(\boldsymbol{Z}_m)^{S(2)}$ and the set of such functions $K_1^{\mathrm{VII}}\big(R\big(G(m,1,\infty)\big)\big)$ of Case VII, we define maps $\mathcal{M}$ and $\mathcal{N}$ as follows: for $F\in K_1\big(\mathfrak{A}_\infty(\boldsymbol{Z}_m)^{S(2)}\big)$ and $f\in K_1^{\mathrm{VII}}\big(R\big(G(m,1,\infty)\big)\big)$, put

$$\begin{cases} \mathcal{M}(F)(g') := \widetilde{\chi}_{\pi_{2,\zeta_0}}(g')\cdot F(g), & \text{with } g=\Phi(g'), \\ \mathcal{N}(f)(g) := \widetilde{\chi}_{\pi_{2,\zeta_0}}(g')\cdot f(g'), & \text{with } g=\Phi(g'). \end{cases} \tag{16.7}$$

Recall that $K_1^{\mathrm{VII}}\big(R\big(G(m,1,\infty)\big)\big)$ is the set of normalized central positive definite functions f on $R\big(G(m,1,\infty)\big)$ with the homogeneity $f(z_ig') = \chi^{\mathrm{VII}}(z_i)f(g')$, $g'\in R\big(G(m,1,\infty)\big)$, where $\chi^{\mathrm{VII}}(z_i) = 1,\ 1,-1$ for $i=1,2,3$ respectively.

Note that similarly as for a spin character of Case VII,

for any $f\in K_1^{\mathrm{VII}}\big(R\big(G(m,1,\infty)\big)\big)$*, its supports* $\mathrm{supp}(f)$ *is contained in the* *subset* $\mathcal{O}(\mathrm{VII}) = \Phi^{-1}(N)$, $N = \mathfrak{A}_\infty(\boldsymbol{Z}_m)^{S(2)}$.

Moreover note that $\big(\widetilde{\chi}_{\pi_{2,\zeta_0}}\big)^2$ is the indicator function of $N \subset G(m,1,\infty)$ if considered as a function of $g = \Phi(g')$ through modulo Z, and is the indicator function of the subset $\mathcal{O}(\mathrm{VII}) = \Phi^{-1}(N)$ as a function in $g' \in R\big(G(m,1,\infty)\big)$.

These facts guarantee that $\mathcal{M}$ and $\mathcal{N}$ are mutually the inverse of the other and so both are bijective. Moreover since they are both linear, they map the sets of extremal points $E(N)$ and $E^{\mathrm{VII}}\big(R\big(G(m,1,\infty)\big)\big)$ mutually each other. Thus we obtain the following.

Theorem 16.2. *The map* $\mathcal{M}$ *from* $K_1\big(\mathfrak{A}_\infty(\boldsymbol{Z}_m)^{S(2)}\big)$ *to* $K_1^{\mathrm{VII}}\big(R\big(G(m,1,\infty)\big)\big)$ *maps characters of* $\mathfrak{A}_\infty(\boldsymbol{Z}_m)^{S(2)}$ *to spin characters of* $G(m,1,\infty)$ *of Case VII, Type* $(1,\ 1,-1)$, *bijectively:*

$$E\big(\mathfrak{A}_\infty(\boldsymbol{Z}_m)^{S(2)}\big) \underset{\mathcal{N}}{\overset{\mathcal{M}}{\rightleftarrows}} E^{\mathrm{VII}}\big(R\big(G(m,1,\infty)\big)\big). \tag{16.8}$$

16.2 Character formula for spin characters of $G(m,1,\infty)$ of Case VII

Theorem 16.3 ([HH2, Theorem 15], [HH6, Theorem 15.1])**.** *Let* T *be a finite abelian group and* S *a subgroup of* T. *For* $G = \mathfrak{S}_\infty(T)$, *define a normal subgroup* $N := \mathfrak{A}_\infty(T)^S$ *of* G *as*

$$\mathfrak{A}_\infty(T)^S = \big\{g = (d,\sigma) \in \mathfrak{S}_\infty(T) = D_\infty(T) \rtimes \mathfrak{S}_\infty\ ;\ \sigma \in \mathfrak{A}_\infty,\ P(d) \in S\big\}.$$

Then, for a character f *of* G, *its restriction* $f|_N$ *onto* N *is a character of* N, *and the map* $E(G) \ni f \to f|_N \in E(N)$ *is surjective.*

We apply this theorem to the case where $T = \boldsymbol{Z}_m$, $m = 2m'$ even, and $S = S(2) = \{t^2\,;\,t \in T\} \cong \boldsymbol{Z}_{m'}$. Let y be a generator of the cyclic group $\boldsymbol{Z}_m$ for which the product is multiplicatively written. Then

$$\widehat{T} = \big\{\zeta_k\ ;\ 0 \le k \le m-1\big\}, \quad \zeta_k(y) = \omega^k, \quad \omega = e^{2\pi i/m}. \tag{16.9}$$

Let $A = \Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}};\mu\Big) \in \mathcal{A}(\widehat{T})$ and f_A be as in Theorem 15.1. We have two involutive actions $\tau : A \to {}^tA$ and $\kappa : A \to R(\zeta_{m'})A$ on the parameter space $\mathcal{A}(\widehat{T})$ as

$${}^tA := \Big((\alpha'_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}};\mu'\Big) \ \text{ with } \ \alpha'_{\zeta,\varepsilon} = \alpha_{\zeta,\varepsilon+1},\ \mu'_\zeta = \mu_\zeta\ ; \tag{16.10}$$

$$R(\zeta_{m'})A := \Big((\alpha'_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}};\mu'\Big) \ \text{ with } \ \alpha'_{\zeta,\varepsilon} = \alpha_{\zeta_{m'}\zeta,\varepsilon},\ \mu'_\zeta = \mu_{\zeta_{m'}\zeta}\ , \tag{16.11}$$

where $\varepsilon + 1$ is calculated modulo 2. Then $\tau^2 = 1$, $\kappa^2 = 1$, $\tau\kappa = \kappa\tau$. The following is a part of [HH6, Theorem 16.2] but we add here a rather detailed proof.

Theorem 16.4. *Let* $m = 2m'$ *be even, and* $N = \mathfrak{A}_\infty(\boldsymbol{Z}_m)^{S(2)}$ *the normal subgroup of* $\mathfrak{S}_\infty(\boldsymbol{Z}_m) = G(m,1,\infty)$. *Then* $f_A\big|_N = f_{A'}\big|_N$ *for another parameter* $A' = \Big((\alpha'_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}; \mu'\Big)$ *if and only if*

$$A' \in \big\{\, A,\ \tau A,\ \kappa A, \tau\kappa A = \kappa\tau A \,\big\}. \tag{16.12}$$

Proof. For a $g \in N$, let its standard decomposition be $g = (d,\sigma) = \xi_{q_i}\xi_{q_2}\cdots\xi_{q_r}\cdot$
$g_1g_2\cdots g_s$, $\xi_{q_i} = (t_{q_i},(q_i))$, $g_j = (d_j,\sigma_j)$. Put $t_{q_i} = y^{b_i}$, $P(d_j) = y^{c_j}$. Note that $\chi_1(\sigma_j) = \mathrm{sgn}(\sigma_j) = (-1)^{\ell(\sigma_j)-1}$ and $\zeta_k(y^{b_i}) = \omega^{b_ik}, \zeta_k(y^{c_j}) = \omega^{c_jk}$. Then

$$\prod_{1\le j\le s}(-1)^{\ell(\sigma_j)-1} = 1, \qquad \sum_{1\le i\le r} b_i + \sum_{1\le j\le s} c_j \equiv 0 \ (\mathrm{mod}\ 2)\,; \tag{16.13}$$

$$f_A(g) \;=\; \prod_{1\le i\le r}\left\{\sum_{0\le k_i<m-1}\Big(\sum_{p\in\boldsymbol{N}}\big(\alpha_{\zeta_{k_i},0;p}+\alpha_{\zeta_{k_i},1;p}\big) \;+\; \mu_{\zeta_{k_i}}\Big)\zeta_{k_i}(y^{b_i})\right\}\times$$
$$\times\prod_{1\le j\le s}\left\{\sum_{0\le k'_j\le m-1}\Big(\sum_{p\in\boldsymbol{N}}\big((\alpha_{\zeta_{k'_j},0;p})^{\ell(\sigma_j)} - (-\alpha_{\zeta_{k'_j},1;p})^{\ell(\sigma_j)}\big)\Big)\zeta_{k'_j}(y^{c_j})\right\}.$$

Then the equality $f_A(g) = f_{{}^tA}(g)\ (g\in N)$ on N follows from

$$\prod_{1\le j\le s}\big((\alpha_{\zeta_{k'_j},0;p})^{\ell(\sigma_j)} - (-\alpha_{\zeta_{k'_j},1;p})^{\ell(\sigma_j)}\big) = \prod_{1\le j\le s}\big((\alpha_{\zeta_{k'_j},1;p})^{\ell(\sigma_j)} - (-\alpha_{\zeta_{k'_j},0;p})^{\ell(\sigma_j)}\big).$$

Now consider a transformation $R(\zeta_{m'})$ on $\widehat{T}$ given by $\zeta \to \zeta_{m'}\zeta$ or $\zeta_k \to \zeta_{k+m'}$ $(0\le k\le m-1)$ $(k+m'$ is counted modulo $m)$. Then

$$\zeta_k(y) = \omega^k \;\to\; \zeta_{k+m'}(y) = \omega^{k+m'} = -\omega^k = -\zeta_k(y),$$

$$\therefore \quad \prod_{1\le i\le r}\zeta_{k_i}(y^{b_i})\cdot\prod_{1\le j\le s}\zeta_{k'_j}(y^{c_j}) \;\to\; \prod_{1\le i\le r}\zeta_{k_i+m'}(y^{b_i})\cdot\prod_{1\le j\le s}\zeta_{k'_j+m'}(y^{c_j})$$
$$= \prod_{1\le i\le r}\zeta_{k_i}(y^{b_i})\cdot\prod_{1\le j\le s}\zeta_{k'_j}(y^{c_j}).$$

This proves that $f_A(g) = f_{R(\zeta_{m'})A}(g)$ for any $g\in N$.

To prove the converse, that is, $f_A|_N = f_{A'}|_N$ gives necessarily $A' = A$, ${}^tA$, $R(\zeta_{m'})A$, or $R(\zeta_{m'})({}^tA)$, we can discuss using the expansion of $f_A(g)$ into the sum of the product terms above, *so-called* monomial terms, for instance as in [HH4, §14.2]. We omit the details. □

Theorem 16.5. *Let* $m = 2m'$ *be even.*

(i) *Any spin character* f *of* $G(m,1,\infty)$ *of Case VII, Type* $\beta = (1,\ 1,-1)$, *is obtained as*

$$f = \mathcal{M}(F) = \widetilde{\chi}_{\pi_{2,\zeta_0}}\cdot F, \qquad F \in E\big(\mathfrak{A}_\infty(\boldsymbol{Z}_m)^{S(2)}\big). \tag{16.14}$$

(ii) *In turn, any character F of the group $N = \mathfrak{A}_\infty(\boldsymbol{Z}_m)^{S(2)}$ is obtained by restriction of a character*

$$(16.15)\qquad f_A \in E\big(\mathfrak{S}_\infty(\boldsymbol{Z}_m)\big),\quad A = \Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{\boldsymbol{Z}_m}\times\{0,1\}};\mu\Big),\ \mu = (\mu_\zeta)_{\zeta\in\widehat{\boldsymbol{Z}_m}},$$

where f_A is given in Theorem 15.1 by (15.4)–(15.5). *Moreover $f_A|_N = f_{A'}|_N$ if and only if $A' = A$, ${}^tA$, $R(\zeta_{m'})A$, or $R(\zeta_{m'})({}^tA)$.*

Note 16.1. The above theorem gives a parametrization of spin characters of type $(1,\,1,-1)$ of $G(m,1,\infty)$, and the relation to the work [DuNe] is direct as is reviewed in §25 of the paper [II].

Example 16.1. We have normalized characters of two-dimensional irreducible representations $f_k = \widetilde{\chi}_{\pi_{2,\zeta_k}}$ in $E^{\mathrm{VII}}\big(R(G(m,1,\infty))\big)$, $0 \le k \le m'-1$. The inverse image of f_k under $\mathcal{M}$ is $F_k = \mathcal{N}(f_k) \in E(N)$ a one-dimensional character given as follows: for $g = (d,\sigma) \in N = \mathfrak{A}_\infty(\boldsymbol{Z}_m)^{S(2)}$, we have $\mathrm{ord}(d) \equiv 0$, $L(\sigma) \equiv 0$, and

$$(16.16)\qquad F_k(g) = \omega^{k\,\mathrm{ord}(d)} = \zeta_k\big(P(d)\big)\,.$$

The characters $f_A \in E\big(G(m,1,\infty)\big)$, which gives F_k by restriction on N, are 4 one-dimensional characters $X_{\varepsilon,k'}$ given as : for $g = (d,\sigma) \in G(m,1,\infty) = \mathfrak{S}_\infty(\boldsymbol{Z}_m)$

$$(16.17)\qquad X_{\varepsilon,k'}(g) = \zeta_{k'}\big(P(d)\big)\ \mathrm{sgn}(\sigma)^\varepsilon \qquad (k' = k,\ k+m'\,;\ \varepsilon = 0,1),$$

which correspond to $A = \Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}};\mu\Big)$ with $\alpha_{\zeta_{k'},\varepsilon} = (1,0,0,,\ldots)$ and all other components in A are trivial, i.e., $\alpha_{\zeta',\varepsilon'} = \mathbf{0}$, $\mu = \mathbf{0}$.

17 Projective IRs of $G(m,1,n)$ in Case VII

From now on we study finite generalized symmetric groups $G(m,1,n)$. Projective representations of these groups have been studied by Read [Rea2], Hoffman and Humphrey [HoHu1] and Morris and Jones [MoJo]. Here we give a construction of projective IRs as induced representations using the semidirect product structure (17.1) below of their covering groups $\widetilde{G}(m,1,n)$ so that we can calculate completely their characters.

Let $n \ge 4$. Define a covering group $\widetilde{G}(m,1,n) = \widetilde{G}^{\mathrm{VII}}(m,1,n)$ of $G(m,1,n)$ by the set of generators and the set of fundamental relations as follows (cf. §16.1):

$$(17.1)\qquad \widetilde{G}(m,1,n) := R\big(G(m,1,n)\big)/\langle z_1, z_2\rangle = \widetilde{D}(m,1,n) \rtimes \mathfrak{S}_n,$$

$$(17.2)\qquad \widetilde{D}(m,1,n) := \langle z_3, \eta_j\ (1 \le j \le n)\rangle,\quad \mathfrak{S}_n = \langle s_i\ (1 \le i \le n-1)\rangle,$$

$$
\begin{array}{ll}
\text{(i)} & z_3^2 = e\,, \quad z_3 \text{ central element}\,; \\
\text{(ii)} & \begin{cases} s_i^2 = e \ (1 \le i \le n-1), \quad (s_i s_{i+1})^3 = e \ (1 \le i \le n-2), \\ s_i s_j = s_j s_i \quad (|i-j| \ge 2), \end{cases} \\
\text{(iii)} & \eta_j^m = e \qquad (1 \le j \le n), \\
\text{(iv)} & \eta_j \eta_k = \eta_k \eta_j \ \ (j \ne k), \\
\text{(v)} & \begin{cases} s_i \eta_i s_i^{-1} = \eta_{i+1},\ s_i \eta_{i+1} s_i^{-1} = \eta_i \quad (1 \le i \le n-1), \\ s_i \eta_j s_i^{-1} = z_3 \eta_j \qquad (j \ne i, i+1)\,. \end{cases}
\end{array}
$$

Then $\widetilde{G}(m, 1, \infty) = \lim_{n \to \infty} \widetilde{G}(m, 1, n)$ is the covering group of $G(m, 1, \infty)$ of Case VII.

In this section we construct IRs, and then in the next section (§17) we calculate their characters, and in the last section (§18) we study limiting process of characters as $n \to \infty$.

17.1 $\mathfrak{S}_n$-orbits in the dual of the abelian group $\widetilde{D}_n := \widetilde{D}(m, 1, n)$

To construct IRs of $\widetilde{G}(m, 1, n)$, we apply the standard method of induced representations for semidirect product groups (for a detailed account of the method, cf. e.g., [HHH1, §3.2]).

First we take the dual group $\widetilde{D}(m, 1, n)^\wedge$ of the abelian group $\widetilde{D}(m, 1, n) =: \widetilde{D}_n = \langle z_3 \rangle \times \langle \eta_1, \ldots, \eta_n \rangle$. Then, consider a complete system of representatives of its $\mathfrak{S}_n$-orbits for *spin* characters Y, or characters Y such that $Y(z_3) = -1$. Let

$$
(17.3) \qquad Y(z_3) = -1, \quad Y(\eta_j) = \omega^{b_j} \quad (0 \le b_j \le m-1,\ 1 \le j \le n).
$$

Denote this character by $Y = Y_{\boldsymbol{b}}$ with $\boldsymbol{b} = (b_1, b_2, \ldots, b_n)$.

Lemma 17.1. *The action of* $s_i \in \mathfrak{S}_n$ *on* $\widetilde{D}(m, 1, n)^\wedge$ *is given by*

$$
s_i \ :\ Y_{\boldsymbol{b}}(\cdot) \to Y_{\boldsymbol{b}'}(\cdot),\ \boldsymbol{b}' = (b_1', b_2', \ldots, b_n') :\ \begin{cases} b_i' = b_{i+1},\ b_{i+1}' = b_i, \\ b_j' = b_j + m' \ (j \ne i, i+1). \end{cases}
$$

Proof. From the above fundamental relations we have

$$
\begin{aligned}
s_i\big(Y_{\boldsymbol{b}}\big)(\eta_i) &= Y_{\boldsymbol{b}}(s_i^{-1}\eta_i s_i) = Y_{\boldsymbol{b}}(\eta_{i+1}) = \omega^{b_{i+1}}, \\
s_i\big(Y_{\boldsymbol{b}}\big)(\eta_{i+1}) &= Y_{\boldsymbol{b}}(s_i^{-1}\eta_{i+1} s_i) = Y_{\boldsymbol{b}}(\eta_i) = \omega^{b_i}, \\
s_i\big(Y_{\boldsymbol{b}}\big)(\eta_j) &= Y_{\boldsymbol{b}}(s_i^{-1}\eta_j s_i) = Y_{\boldsymbol{b}}(z_3\eta_j) = -\omega^{b_i} = \omega^{b_i + m'} \ \ (j \ne i, i+1). \quad \square
\end{aligned}
$$

For convenience of calculations, choose another set of generators of $\widetilde{D}_n$ as

$$
(17.4) \qquad \big\{\ z_3,\ \ \eta_j' = z_3^{j-1}\eta_j \ (1 \le j \le n)\ \big\},
$$

then the fundamental relations becomes

$$\begin{array}{ll}
(\mathrm{iii}') & {\eta'_j}^m = e \qquad (1 \le j \le n), \\
(\mathrm{iv}') & \eta'_j \eta'_k = \eta'_k \eta'_j \quad (j \ne k), \\
(\mathrm{v}') & \begin{cases} s_i \eta'_i s_i^{-1} = z_3\, \eta'_{i+1},\ s_i \eta'_{i+1} s_i^{-1} = z_3\, \eta'_i \quad (1 \le i \le n-1), \\ s_i \eta'_j s_i^{-1} = z_3\, \eta'_j \qquad (j \ne i, i+1)\,. \end{cases}
\end{array}$$

Accordingly $Y_{\boldsymbol{b}}(\eta'_j) = Y_{\boldsymbol{b}}(z_3^{j-1}\eta_j) = (-1)^{j-1}\omega^{b_j} = \omega^{b_j+(j-1)m'}$, and we change the parameter $\boldsymbol{b}$ to $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$, $\gamma_j := b_j + (j-1)m'$ (mod m), and put $Y^\gamma := Y_{\boldsymbol{b}}$. Then the transformation by s_i is described as

$$(17.5) \qquad s_i : Y^\gamma(\cdot) \to Y^{\gamma'}(\cdot),\ \gamma' = (\gamma'_1, \gamma'_2, \ldots, \gamma'_n) : \quad \text{with} \ \begin{cases} \gamma'_i = \gamma_{i+1} + m',\ \gamma'_{i+1} = \gamma_i + m', \\ \gamma'_j = \gamma_j + m' \ \ (j \ne i, i+1). \end{cases}$$

Note that every permutation in $\mathfrak{A}_n$ acts on γ naturally (without adding m''s), then we can obtain a set Γ of γ representing all $\mathfrak{S}_n$-orbits of $\widetilde{D}(m,1,n)^\wedge$ as follows. First we prepare a set of integers mutually different from each other modulo m' such that

$$(17.6) \qquad \begin{cases} 0 \le c_1 < c_2 < \ldots < c_K < m'\,, \\ 0 \le c_{K+1},\ \ldots,\ c_{K'} < m\,, \end{cases} \qquad \text{for } 1 \le K \le K' \le m,$$

and a partition of the interval $\boldsymbol{I}_n = \{1, 2, \ldots, n\} \subset \boldsymbol{N}$ into disjoint consecutive intervals as

$$(17.7) \qquad \begin{cases} \boldsymbol{I}_n = \Big(I_1 \sqcup I_2 \sqcup \ \cdots \ \sqcup I_K\Big) \bigsqcup \Big(I_{K+1} \sqcup \ \cdots \ \sqcup I_{K'}\Big), \\ |I_k| \ge 2 \ (1 \le k \le K), \quad |I_k| = 1 \ (K < k \le K'), \\ I_k = I_{k,+} \sqcup I_{k,-} \text{ (empty set admitted)} \ \ (1 \le k \le K), \\ \qquad\qquad \text{such that } j < j' \text{ for } j \in I_{k,+},\ j' \in I_{k,-}\ , \end{cases}$$

with only one possible exception $I_K = I_{K,-} \sqcup I_{K,+}$ when $|I_K| = 2$ (the order between $I_{K,+}$ and $I_{K,-}$ is reversed).

Then consider $\gamma = (\gamma_1, \gamma_2, \ldots \gamma_n)$ satisfying a condition

$$(\Gamma\text{-}1) \qquad \begin{cases} \gamma_j = c_k \ (j \in I_{k,+}),\ \ \gamma_j = c_k + m' \ (j \in I_{k,-}) \quad \text{for } 1 \le k \le K, \\ \gamma_j = c_k \ (I_k = \{j\}) \quad \text{for} \ \ K < k \le K'. \end{cases}$$

This condition on γ means that, on each interval I_k, γ_j $(j \in I_k)$ are ranged as

$$(17.8) \qquad \begin{cases} [c_k,\ \ldots\ , c_k,\ c_k + m',\ \ldots\ , c_k + m'], \\ \text{with one possible exception} \ \ [c_K + m', c_K] \quad \text{when } |I_K| = 2. \end{cases}$$

Furthermore we put the following condition:

$$(\Gamma\text{-}2) \qquad \begin{cases} \text{one possible exception can occur only when} \\ |I_{k,+}| = |I_{k,-}| = 1 \ \ (1 \le k \le K), \quad K' - K \le 1. \end{cases}$$

Denote by Γ the set of γ satisfying the conditions (Γ-1) $\sim$ (Γ-2).

Lemma 17.2. *Any $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$, $0 \le \gamma_j \le m-1$ $(1 \le j \le n)$, is conjugate under $\mathfrak{S}_n$ to one of $\gamma' \in \Gamma$. This means that any spin character Y of $\widetilde{D}(m,1,n)$ is conjugate under $\mathfrak{S}_n$ to some $Y^{\gamma'}, \gamma' \in \Gamma$.*

Example 17.1. A general $\gamma \in \Gamma$ looks like

$$(c_1, \ldots, c_1,\ c_1+m', \ldots, c_1+m'\,;\ \ldots\ ;\ c_K, \ldots, c_K,\ c_K+m', \ldots, c_K+m'\,;\ c_{K+1},\ \ldots\ , c_{K'}), \tag{17.9}$$

and exceptional γ's are

$$\begin{cases} (c_1, c_1+m'\,;\ \ldots\ ;\ c_{K-1}, c_{K-1}+m'\,;\ c_K+m', c_K\,;\ c_{K+1}) & \text{when } n = 2K+1; \\ (c_1, c_1+m'\,;\ \ldots\ ;\ c_{K-1}, c_{K-1}+m'\,;\ c_K+m', c_K) & \text{when } n = 2K. \end{cases}$$

Note 17.1. There remain still some more conjugacies among elements in Γ. To give exactly a complete set of representatives of conjugacy classes as a subset of Γ, the calculations are elementary but cumbersome and so omitted here.

17.2 Stationary subgroups of a character Y^γ of $\widetilde{D}(m,1,n)$

Let Y^γ be a character of $\widetilde{D}(m,1,n)$ with $\gamma \in \Gamma$ in (17.6)–(17.8) together with (Γ-1) $\sim$ (Γ-2) (cf. Example 17.1). Let us determine the stationary subgroup $\mathcal{S}(Y^\gamma) \subset \mathfrak{S}_n$ of Y^γ.

Let $\sigma \in \mathcal{S}(Y^\gamma)$. Since the integers $c_1, \ldots, c_{K'}$ are assumed to be different from each other modulo m', we see that

$$\sigma \in \prod_{1\le k\le K'} \mathfrak{S}_{I_k} = \prod_{1\le k\le K} \mathfrak{S}_{I_k}. \tag{17.10}$$

We see from (17.5) the following.

In case $\mathrm{sgn}(\sigma) = 1$, we have $\sigma \in \prod_{1\le k\le K} \big(\mathfrak{S}_{I_{k,+}} \times \mathfrak{S}_{I_{k,-}}\big)$.

In case $\mathrm{sgn}(\sigma) = -1$, we have $K' = K$ and $\boldsymbol{I}_n = \bigsqcup_{1\le k\le K} \big(I_{k,+} \sqcup I_{k,-}\big)$, and

$$\sigma(I_{k,\pm}) = I_{k,\mp} \quad (1 \le k \le K), \quad \text{and so} \tag{17.11}$$

$$|I_{k,+}| = |I_{k,-}| \ge 1 \ \ (1 \le k \le K). \tag{17.12}$$

Such a σ exists if $|I_{k,\pm}| \ge 2$ for some k. Suppose contrarily that $|I_{k,\pm}| = 1$ for any k, or

$$\gamma = \begin{cases} (c_1, c_1+m'\,;\ \ldots\ ;\ c_{K-1}, c_{K-1}+m'\,;\ c_K, c_K+m'), & \text{or} \\ (c_1, c_1+m'\,;\ \ldots\ ;\ c_{K-1}, c_{K-1}+m'\,;\ c_K+m', c_K). \end{cases} \tag{17.13}$$

Then, if K is odd, $\sigma = (1\ 2)(3\ 4)\cdots(2K-1\ \ 2K)$, and if K is even, no such σ exists.

With these results, we obtain the next proposition. Put, for disjoint subsets J_p $(1 \le p \le N)$ of $\boldsymbol{I}_n$,

$$\mathfrak{A}\Big(\prod\nolimits_{1\le p\le N}\mathfrak{S}_{J_p}\Big) := \mathfrak{A}_n \bigcap \prod\nolimits_{1\le p\le N}\mathfrak{S}_{J_p}\,. \tag{17.14}$$

Proposition 17.3.

(Case S-1) $\quad \mathcal{S}(Y^\gamma) = H^\gamma := \mathfrak{A}\Big(\prod\nolimits_{1\le k\le K}(\mathfrak{S}_{I_{k,+}} \times \mathfrak{S}_{I_{k,-}})\Big),$

if (a) $K' > K$, *or*

(b) $K' = K$ *and* $\begin{cases} |I_{k,+}| \ne |I_{k,-}| \text{ for some } k, \text{ or}, \\ |I_{k,\pm}| = 1 \ (1 \le k \le K),\ K \text{ even} \\ \qquad\qquad (n = 2K \le m). \end{cases}$

(Case S-2) $\quad \mathcal{S}(Y^\gamma) = H^\gamma \bigsqcup \sigma\, H^\gamma,\ \ H^\gamma = \mathfrak{A}\Big(\prod\nolimits_{1\le k\le K}(\mathfrak{S}_{I_{k,+}} \times \mathfrak{S}_{I_{k,-}})\Big),$

if $K' = K$, $|I_{k,+}| = |I_{k,-}|$ $(1 \le k \le K)$, *and* $|I_{k,\pm}| \ge 2$ *for some* k, *where*

$$\begin{cases} \sigma = \sigma_1\sigma_2\cdots\sigma_K, \quad \sigma_k \in \mathfrak{S}_{I_k} \ \ (\le k \le K), \\ \sigma_k(I_{k,\pm}) = I_{k,\mp} \ (1 \le k \le K), \quad \mathrm{sgn}(\sigma) = -1. \end{cases} \tag{17.15}$$

In this case $n = 2n'$. *When* n' *is odd,* s *can be taken as* $s^2 = e$, *and when* n' *is even,* s *can be taken as* $s = \tau s'$ *with* $s'^2 = e, \tau =$ *'a transposition in one of* $I_{k,\pm}$*'.*

(Case S-3) $\quad \mathcal{S}(Y^\gamma) = \{e, \sigma\} = H^\gamma \bigsqcup \sigma\, H^\gamma, \quad H^\gamma = \{e\},$

if $K' = K$, $|I_{k,+}| = |I_{k,-}| = 1$ $(1 \le k \le K)$, *with* K *odd, where*

$\sigma = \prod_{1\le k\le K}(i_{k,+}\ i_{k,-}),\ \ I_{k,\pm} = \{i_{k,\pm}\}\ (1 \le k \le K),$ *and* $n = 2K \le m.$

In this case $\sigma^2 = e$, $\mathrm{sgn}(\sigma) = -1$.

17.3 Relations between IRs of G and H, in case $G \supset H$ with $|G/H| = 2$

Let G be a finite group and H its subgroup with $|G/H| = 2$. Take an element $s \notin H$, then $G = H \sqcup sH$. Then, $G = sH \sqcup s^2H$, and so $s^2H = H$ and $s^2 \in H$, $sH = s^{-1}H$. Moreover $G = sHs^{-1} \sqcup sH$, and so $sHs^{-1} = H$ whence the subgroup H is normal. Then we can define a sign character ε on G as $\varepsilon : G \to G/H \cong \boldsymbol{Z}_2 \overset{\mathrm{sgn}}{\to} \{\pm 1\}$.

Take an IR $\rho \in \widehat{H}$ of H and consider the induced representation $\Pi = \mathrm{Ind}_H^G \rho$. The representation space $V(\Pi)$ of Π consists of $V(\rho)$-valued functions φ on G satisfying

$$\varphi(hg) = \rho(h)\varphi(g) \quad (h \in H, g \in G).$$

Take a system of representatives $\{e, s\}$, then the map $V(\Pi) \ni \varphi \to (\varphi(e), \varphi(s)) \in V(\rho) \times V(\rho)$ gives an isomorphism. On the latter space, Π is realized as follows:

$$\Pi(h) = \begin{pmatrix} \rho(h) & 0 \\ 0 & \rho^s(h) \end{pmatrix}, \quad \Pi(s) = \begin{pmatrix} 0 & I \\ \rho(s^2) & 0 \end{pmatrix}, \tag{17.16}$$

where $\rho^s(h) := \rho(shs^{-1})$ $(h \in H)$.

The following lemma gives relations between IRs of H and IRs of G, through inducing up from H to G and restricting from G onto H.

Lemma 17.4. *Let $|G/H| = 2$, and the notations be as above.*

(Case P-1). *Assume $\rho^s \cong \rho$. Then, for any $\tau \in G$, $\rho^\tau \cong \rho$. Let S be an intertwining operator such as $\rho^s(h) = S\,\rho(h)S^{-1}$ $(h \in H)$, then S can be chosen such that $\rho(s^2) = S^2$. The induced representation $\Pi = \mathrm{ind}_H^G \rho$ splits as*

$$\Pi \cong \Pi' \oplus \varepsilon \cdot \Pi', \qquad \begin{cases} \Pi'(h) = \big(\varepsilon \cdot \Pi'\big)(h) = \rho(h) \ \ (h \in H), \\ \Pi'(s) \ = \ S, \quad \big(\varepsilon \cdot \Pi'\big)(s) = -S. \end{cases} \tag{17.17}$$

$$\Pi' \not\cong \varepsilon \cdot \Pi', \quad \Pi'\big|_H \cong (\varepsilon \cdot \Pi')\big|_H \cong \rho \cong \rho^s. \tag{17.18}$$

$$\text{the character} \quad \chi_\Pi(g) = \begin{cases} 2\,\chi_\rho(g) & \text{if } g \in H, \\ 0 & \text{if } g \in sH. \end{cases} \tag{17.19}$$

(Case P-2). *Assume $\rho \not\cong \rho^s$. Then $\Pi = \mathrm{Ind}_H^G \rho$ is irreducible and*

$$\Pi \cong \varepsilon \cdot \Pi, \qquad \Pi\big|_H = (\varepsilon \cdot \Pi)\big|_H \cong \rho \oplus \rho^s. \tag{17.20}$$

$$\text{the character} \quad \chi_\Pi(g) = \begin{cases} \chi_\rho(g) + \chi_\rho(sgs^{-1}) & \text{if } g \in H, \\ 0 & \text{if } g \in sH. \end{cases} \tag{17.21}$$

17.4 IRs of $H = H^\gamma = \mathfrak{A}\big(\prod_{1 \le k \le K} \big(\mathfrak{S}_{I_{k,+}} \times \mathfrak{S}_{I_{k,-}}\big)\big)$

Take a $\gamma \in \Gamma$ in (Case S-2) or in (Case S-3), and put $\mathcal{K} = \mathcal{K}^\gamma := \{1, 2, \ldots, K\}$. We apply Lemma 17.4 to

$$G = \prod_{k \in \mathcal{K}} \big(\mathfrak{S}_{I_{k,+}} \times \mathfrak{S}_{I_{k,-}}\big) \quad \text{and} \quad H = \mathfrak{A}\Big(\prod_{k \in \mathcal{K}} \big(\mathfrak{S}_{I_{k,+}} \times \mathfrak{S}_{I_{k,-}}\big)\Big). \tag{17.22}$$

Take any $s \in G \setminus H$. Then, $\mathrm{sgn}(s) = -1$ and $\varepsilon(g) = \mathrm{sgn}(g)$ $(g \in G)$.

For a finite subset $I \subset \boldsymbol{N}$, let $\mathfrak{S}_I$ be the symmetric group acting on I. Then the set $\widehat{\mathfrak{S}_I}$ of equivalence classes of its IRs is parametrized by Young diagrams of size $|I|$. An IR π of $G = \prod_{k \in \mathcal{K}} \big(\mathfrak{S}_{I_{k,+}} \times \mathfrak{S}_{I_{k,-}}\big)$, identified with its equivalence class, is parametrized as π_Λ by a set Λ of Young diagrams as

$$\pi_\Lambda := \mathop{\boxtimes}_{k \in \mathcal{K}} \big(\pi_{\Lambda_{k,+}} \boxtimes \pi_{\Lambda_{k,-}}\big), \tag{17.23}$$

$$\Lambda = \big(\Lambda_{k,\epsilon}\big)_{k \in \mathcal{K}, \epsilon = \pm}, \quad \Lambda_{k,\epsilon} \text{ parametrizes } \widehat{\mathfrak{S}_{I_{k,\epsilon}}} \quad (\epsilon = \pm).$$

Moreover, denoting by ${}^t\Lambda_{k,\epsilon}$ the transposed of $\Lambda_{k,\epsilon}$, we have

$$\mathrm{sgn}\cdot\pi_\Lambda \cong \pi_{{}^t\Lambda}, \quad {}^t\Lambda := \left({}^t\Lambda_{k,\epsilon}\right)_{k\in\mathcal{K},\epsilon=\pm}. \tag{17.24}$$

Lemma 17.5 (cf. [Fro4, §2])**.**
Let $G = \prod_{k\in\mathcal{K}}\left(\mathfrak{S}_{I_{k,+}}\times\mathfrak{S}_{I_{k,-}}\right)$, $H = \mathfrak{A}\left(\prod_{k\in\mathcal{K}}\left(\mathfrak{S}_{I_{k,+}}\times\mathfrak{S}_{I_{k,-}}\right)\right)$, *and put* $\rho_\Lambda := \pi_\Lambda\big|_H$.

(Case TΛ-1). *Assume* ${}^t\Lambda\neq\Lambda$. *Then* $\begin{cases} \rho_{{}^t\Lambda}\cong\rho_\Lambda \quad \text{irreducible}, \\ \left(\rho_\Lambda\right)^s \cong \rho_\Lambda, \\ \mathrm{Ind}_H^G\rho_\Lambda \cong \pi_\Lambda\oplus\pi_{{}^t\Lambda}. \end{cases}$

(Case TΛ-2). *Assume* ${}^t\Lambda=\Lambda$. *Then*

$$\begin{cases} \rho_\Lambda \cong \rho_\Lambda^{(0)}\oplus\rho_\Lambda^{(1)}, \quad \rho_\Lambda^{(0)}\not\cong\rho_\Lambda^{(1)} \ \text{irreducible}, \\ \left(\rho_\Lambda^{(0)}\right)^s \cong \rho_\Lambda^{(1)}, \\ \mathrm{Ind}_H^G\rho_\Lambda^{(\alpha)} \cong \pi_\Lambda = \pi_{{}^t\Lambda} \ (\alpha=0,1). \end{cases}$$

17.5 Restriction of IRs of $\mathfrak{A}\Big(\prod_{b\in\mathcal{B}}\mathfrak{S}_{I_b}\Big)$ onto $\prod_{b\in\mathcal{B}}\mathfrak{A}_{I_b}$

Lemma 17.6 ([Fro4, §2])**.** *A complete representatives of equivalence classes of IRs for n-th alternating group* $\mathfrak{A}_n, n\geq 4$, *is given as follows: let* π_Λ *be an IR of* $\mathfrak{S}_n$ *parametrized by a Young diagram* Λ *of size* n, *then*

$$\begin{cases} \rho_\Lambda = \pi_\Lambda\big|_{\mathfrak{A}_n}, & \text{in case } {}^t\Lambda\neq\Lambda, \text{ in this case } \rho_{{}^t\Lambda}\cong\rho_\Lambda\ ; \\ \rho_\Lambda^{(0)},\ \ \rho_\Lambda^{(1)}, & \text{in case } {}^t\Lambda=\Lambda, \text{ where } \pi_\Lambda\big|_{\mathfrak{A}_n} = \rho_\Lambda^{(0)}\oplus\rho_\Lambda^{(1)}. \end{cases}$$

We assign an element $g\in\mathfrak{S}_n$ its *parity* $\nu=\nu(g)$ as $\nu=0$ or 1 depending on $g\in\mathfrak{A}_n$ or not, that is, $\mathrm{sgn}(g)=(-1)^\nu$. Also, putting $\mathcal{B}=\mathcal{B}^\gamma := \mathcal{K}\times\{\pm\} = \{1,2,\ldots,K\}\times\{\pm\}$, we give a parity $\nu(g)$ for an element $g\in G=\prod_{k\in\mathcal{K}}\left(\mathfrak{S}_{I_{k,+}}\times\mathfrak{S}_{I_{k,-}}\right)=\prod_{b\in\mathcal{B}}\mathfrak{S}_{I_b}$ as follows:

$$g = (g_{1,+},g_{1,-},\ldots,g_{K,+},g_{K,-}) = (g_b)_{b\in\mathcal{B}} \in \prod_{b\in\mathcal{B}}\mathfrak{S}_{I_b}, \tag{17.25}$$

$$\nu=\nu(g) := (\nu_b)_{b\in\mathcal{B}} \quad \text{with } \ \nu_b=\nu(g_b)\ (b\in\mathcal{B}). \tag{17.26}$$

We call $g\in G$ *even* or *odd* according as $|\nu| := \sum_{b\in\mathcal{B}}\nu_b$ is even or odd. Then $\mathrm{sgn}(g) := \prod_{b\in\mathcal{B}}\mathrm{sgn}(g_b) = (-1)^{|\nu|} = 1$ or $=-1$ according as g is even or odd, and H consists of even elements of G. According to the notation in Lemma 17.6, we put for each component $\mathfrak{S}_{I_b}$ $(b\in\mathcal{B})$ as $\pi_{\Lambda_b}\big|_{\mathfrak{A}_{I_b}} = \rho_{\Lambda_b}^{(0)}\oplus\rho_{\Lambda_b}^{(1)}$ in case ${}^t\Lambda_b=\Lambda_b$. Define a subgroup $H_0=H_0^\gamma$ of $H=H^\gamma$ as

$$H_0 = H_0^\gamma := \prod_{k\in\mathcal{K}}\left(\mathfrak{A}_{I_{k,+}}\times\mathfrak{A}_{I_{k,-}}\right) = \prod_{b\in\mathcal{B}}\mathfrak{A}_{I_b}. \tag{17.27}$$

Lemma 17.7. *In Case $T\Lambda$-2 in Lemma 17.5 or if ${}^t\Lambda = \Lambda$, the superfices $(0),(1)$ of irreducible components $\rho_\Lambda^{(0)}, \rho_\Lambda^{(1)}$ can be adjusted so that the following holds: for $\kappa = 0,1$,*

$$(17.28)\qquad \rho_\Lambda^{(\kappa)}\big|_{H_0} \cong \sum_{|\epsilon|\equiv\kappa\,(2)}^{\oplus} \bigotimes_{b\in\mathcal{B}} \rho_{\Lambda_b}^{(\epsilon_b)}, \qquad \epsilon := (\epsilon_b)_{b\in\mathcal{B}},\ |\epsilon| := \sum_{b\in\mathcal{B}} \epsilon_b,$$

where the orthogonal direct sum runs over all components satisfying $|\epsilon| \equiv \kappa \pmod 2$.

17.6 IRs of $\mathcal{S}(Y^\gamma) = H^\gamma \bigsqcup \sigma H^\gamma$ in Case S-2

In Case S-2, we apply again Lemma 17.4 to the triplet (G', H, σ) with

$$G' := \mathcal{S}(Y^\gamma) = H^\gamma \bigsqcup \sigma H^\gamma,\ H = H^\gamma = \mathfrak{A}\Big(\prod_{b\in\mathcal{B}} \mathfrak{S}_{I_b}\Big),\ \mathcal{B} = \mathcal{B}^\gamma = \mathcal{K}\times\{\pm\},$$

and refer to Diagrams 17.1 and 17.2 below to see inclusion relations of groups which interplay, and also refer rows of (Case S-2) in Table 17.1 below. Here $\gamma\in\Gamma$ satisfies with $\mathcal{K} = \{1,2,\ldots,K\}$

$$(17.29)\qquad K' = K,\quad |I_{k,+}| = |I_{k,-}|\ (k\in\mathcal{K}),\ \ |I_{k,\pm}| \geq 2 \text{ for some } k\in\mathcal{K},$$
$$\text{and}\quad \begin{cases} \sigma = \sigma_1\sigma_2\cdots\sigma_K, & \sigma_k \in \mathfrak{S}_{I_k}\ \ (k\in\mathcal{K}),\\ \sigma_k(I_{k,\pm}) = I_{k,\mp}\ (k\in\mathcal{K}),\ \operatorname{sgn}(\sigma) = -1. \end{cases}$$

We know IRs of H in Lemma 17.5. For an IR ρ of H, we study the symmetry $\rho \to \rho^\sigma$ with $\rho^\sigma(h) = \rho(\sigma h\sigma^{-1})$ $(h\in H)$. Let $\Lambda = \big((\Lambda_{k,+},\Lambda_{k,-})\big)_{k\in\mathcal{K}}$. Then, for $G = \prod_{k\in\mathcal{K}} \big(\mathfrak{S}_{I_{k,+}}\times\mathfrak{S}_{I_{k,-}}\big)$,

$$(17.30)\qquad \pi_\Lambda = \mathop{\boxtimes}_{k\in\mathcal{K}} \big(\pi_{\Lambda_{k,+}}\boxtimes\pi_{\Lambda_{k,-}}\big), \quad \big(\pi_\Lambda\big)^\sigma \cong \mathop{\boxtimes}_{k\in\mathcal{K}} \big(\pi_{\Lambda_{k,-}}\boxtimes\pi_{\Lambda_{k,+}}\big) =: \pi_{\Lambda^\sigma},$$
$$\text{with}\quad \Lambda^\sigma := \Big(\big(\Lambda_{1,-},\Lambda_{1,+}\big),\ \ldots,\ \big(\Lambda_{K,-},\Lambda_{K,+}\big)\Big) \text{ (exchange of } \Lambda_{k,+},\Lambda_{k,-}).$$

For $\Lambda = \big((\Lambda_{k,+},\Lambda_{k,-})\big)_{k\in\mathcal{K}}$, we call

$$\begin{cases} (\text{Case } \Lambda\Sigma\text{-1}) & \text{if } \Lambda^\sigma \neq \Lambda \text{ and } \Lambda^\sigma \neq {}^t\Lambda,\\ (\text{Case } \Lambda\Sigma\text{-2}) & \text{if } \Lambda^\sigma = \Lambda \text{ or } \Lambda^\sigma = {}^t\Lambda. \end{cases}$$

Case TΛ-1: ${}^t\Lambda \neq \Lambda$.

Since ${}^t(\Lambda^\sigma) = ({}^t\Lambda)^\sigma \neq \Lambda^\sigma$, both $\rho_\Lambda = \pi_\Lambda\big|_{H^\gamma}$ and $\rho_{\Lambda^\sigma} \cong \big(\rho_\Lambda\big)^\sigma \cong \big(\pi_\Lambda\big)^\sigma\big|_{H^\gamma}$ are irreducible.

• **(Case $\Lambda\Sigma$-1)** In this case, $\big(\rho_\Lambda\big)^\sigma \cong \rho_{\Lambda^\sigma} \ncong \rho_\Lambda$. By Lemma 17.4 (Case P-2) for $(G',H) = \big(\mathcal{S}(Y^\gamma), H^\gamma\big)$, we obtain an IR $\Pi(\Lambda)$ of the stationary subgroup $\mathcal{S}(Y^\gamma)$ through

$$(17.31)\qquad \Pi(\Lambda) := \operatorname{Ind}_{H^\gamma}^{\mathcal{S}(Y^\gamma)} \rho_\Lambda, \qquad \Lambda = \big(\Lambda_{k,\epsilon}\big)_{(k,\epsilon)\in\mathcal{B}},$$

$$\text{and} \qquad \Pi({}^t\Lambda) \cong \Pi(\Lambda) \not\cong \Pi(\Lambda^\sigma).$$

• **(Case ΛΣ-2)** In this case, $\big(\rho_\Lambda\big)^\sigma \cong \rho_\Lambda$. By Lemma 17.4 (Case P-1) for $(G', H) = \big(\mathcal{S}(Y^\gamma), H^\gamma\big)$, we obtain two non-equivalent IRs $\Pi(\Lambda, 0), \Pi(\Lambda, 1)$ of the stationary subgroup $\mathcal{S}(Y^\gamma)$ as the irreducible components of $\Pi(\Lambda) \cong \Pi({}^t\Lambda) \cong \Pi(\Lambda^\sigma)$ as

$$\Pi(\Lambda) := \mathrm{Ind}_{H^\gamma}^{\mathcal{S}(Y^\gamma)} \rho_\Lambda \ = \Pi(\Lambda)^{(0)} \oplus \Pi(\Lambda)^{(1)}, \ \ \Pi(\Lambda)^{(1)} \cong \mathrm{sgn} \cdot \Pi(\Lambda)^{(1)}. \tag{17.32}$$

Case TΛ-2: ${}^t\Lambda = \Lambda$.

In this case, ${}^t\Lambda_b = \Lambda_b$ for any $b = (k, \epsilon) \in \mathcal{B} = \mathcal{K} \times \{\pm\}$, and we have non-equivalent IRs $\rho_\Lambda^{(0)}, \rho_\Lambda^{(1)}$ of $H^\gamma \subset G$ by restriction from G as $\pi_\Lambda\big|_{H^\gamma} \cong \rho_\Lambda^{(0)} \oplus \rho_\Lambda^{(1)}$.

• **(Case ΛΣ-1)** In this case, $\Lambda^\sigma \neq \Lambda = {}^t\Lambda$ and so by Lemma 17.6

$$\big(\rho_\Lambda^{(\kappa)}\big)^\sigma \not\cong \rho_\Lambda^{(\kappa)} \quad (\kappa = 0, 1),$$

because $\pi_\Lambda|_{H^\gamma} \cong \rho_\Lambda^{(0)} \oplus \rho_\Lambda^{(1)}$, $(\pi_\Lambda)^\sigma \cong \pi_{\Lambda^\sigma}$, $\pi_{\Lambda^\sigma}|_{H^\gamma} \cong \rho_{\Lambda^\sigma}^{(0)} \oplus \rho_{\Lambda^\sigma}^{(1)}$. Therefore, by Lemma 17.4 (Case P-2), we get, for each $\kappa = 0, 1$, an IR $\Pi\big(\Lambda; (\kappa)\big)$ of the stationary subgroup $G' = \mathcal{S}(Y^\gamma)$ as

$$\Pi\big(\Lambda; (\kappa)\big) := \mathrm{Ind}_{H^\gamma}^{G'} \rho_\Lambda^{(\kappa)}, \ \ \Pi\big(\Lambda; (\kappa)\big) \cong \mathrm{sgn} \cdot \Pi\big(\Lambda; (\kappa)\big) \quad (\kappa = 0, 1), \tag{17.33}$$

each of which is equivalent to one of $\Pi\big(\Lambda^\sigma; (\kappa')\big)$ $(\kappa' = 0, 1)$ respectively.

Note 17.2. In (Case TΛ-2)+(Case ΛΣ-1), since $(\pi_\Lambda)^\sigma \not\cong \pi_\Lambda$ for $G = \prod_{b\in\mathcal{B}} \mathfrak{S}_{I_b}$, we see
from Lemma 17.4 (Case P-2), applied for inducing up π_Λ of G to $\widetilde{G} := G \sqcup \sigma G$, that $\widetilde{\Pi}(\Lambda) := \mathrm{Ind}_G^{\widetilde{G}} \pi_\Lambda$ is irreducible. Moreover, since $\mathrm{Ind}_{H^\gamma}^G \rho_\Lambda^{(\kappa)} \cong \pi_\Lambda$ $(\kappa = 0, 1)$, we have $\mathrm{Ind}_{G'}^{\widetilde{G}} \Pi(\Lambda, (\kappa)) \cong \mathrm{Ind}_{H^\gamma}^{\widetilde{G}} \rho_\Lambda^{(\kappa)} \cong \widetilde{\Pi}(\Lambda)$ irreducible for both $\kappa = 0, 1$.

Diagram 17.1. Relations among IRs
in (Case TΛ-2)+(Case ΛΣ-1): $\Lambda^\sigma \neq \Lambda = {}^t\Lambda$.

$$\begin{array}{ccccc}
 & & \widetilde{G} = G \sqcup \sigma G & & \widetilde{\Pi}(\Lambda) := \mathrm{Ind}_G^{\widetilde{G}} \pi_\Lambda \cong \mathrm{Ind}_G^{\widetilde{G}} \pi_{\Lambda^\sigma} \\
 & \nearrow & & \nwarrow & \\
\pi_\Lambda \quad G = \prod_{b\in\mathcal{B}} \mathfrak{S}_{I_b} & & & & G' = \mathcal{S}(Y^\gamma) = H^\gamma \sqcup \sigma H^\gamma \qquad \Pi\big(\Lambda; (0)\big), \ \Pi\big(\Lambda; (1)\big) \\
 & \nwarrow & & \nearrow & \\
 & & H^\gamma = \mathfrak{A}\big(\prod_{b\in\mathcal{B}} \mathfrak{S}_{I_b}\big) & & \rho_\Lambda^{(0)}, \ \rho_\Lambda^{(1)} \ : \ \pi_\Lambda\big|_{H^\gamma} = \rho_\Lambda^{(0)} \oplus \rho_\Lambda^{(1)}
\end{array}$$

• **(Case ΛΣ-2)**

In this case, $\Lambda = {}^t\Lambda = \Lambda^\sigma$ and so $\Lambda_{k,+} = {}^t\Lambda_{k,+} = \Lambda_{k,-}$ $(k \in \mathcal{K})$. For $H = H^\gamma = \mathfrak{A}\big(\prod_{b\in\mathcal{B}} \mathfrak{S}_{I_b}\big) \subset G$, we have two IRs $\rho_\Lambda^{(\kappa)}, \kappa = 0, 1$, through $\pi_\Lambda\big|_{H^\gamma} \cong$

$\rho_\Lambda^{(0)} \oplus \rho_\Lambda^{(1)}$. To get IRs of $G' = \mathcal{S}(Y^\gamma) = H \sqcup \sigma H$, we need to study if $\big(\rho_\Lambda^{(\kappa)}\big)^\sigma \cong \rho_\Lambda^{(\kappa)}$ or not, for each $\kappa = 0, 1$. For that, we apply Lemma 17.7. If $\big(\rho_\Lambda^{(\kappa)}\big)^\sigma \not\cong \rho_\Lambda^{(\kappa)}$, then $\mathrm{Ind}_H^{G'} \rho_\Lambda^{(\kappa)}$ is irreducible, and vice versa.

For any Λ_b, $b = (k, \epsilon) \in \mathcal{B}$, we have ${}^t\Lambda_b = \Lambda_b$, and $\pi_{\Lambda_b}\big|_{\mathfrak{A}_{I_b}} = \rho_{\Lambda_b}^{(0)} \oplus \rho_{\Lambda_b}^{(1)}$, and the IRs $\rho_\Lambda^{(\kappa)}$ $(\kappa = 0, 1)$ are defined by (17.28) using the normal subgroup $H_0 = \prod_{b \in \mathcal{B}} \mathfrak{A}_{I_b}$ of H. We adjust the superfices $(0), (1)$ of $\rho_{\Lambda_{k,-}}^{(0)}, \rho_{\Lambda_{k,-}}^{(1)}$ so that the representation $\big(\rho_{\Lambda_{k,-}}^{(\epsilon)}\big)^{\sigma_k}(h) := \rho_{\Lambda_{k,-}}^{(\epsilon)}(\sigma_k h \sigma_k^{-1})$ $(h \in \mathfrak{A}_{I_{k,+}})$ is equivalent to $\rho_{\Lambda_{k,+}}^{(\epsilon)}$. For an IR $T(\epsilon) := \otimes_{b \in \mathcal{B}} \rho_{\Lambda_b}^{(\epsilon_b)}$ with $\epsilon := (\epsilon_b)_{b \in \mathcal{B}}$, of H_0, put

$$T(\epsilon)^\sigma(h) := T(\epsilon)(\sigma h \sigma^{-1}) \quad (h \in H_0). \tag{17.34}$$

Then, since $\Lambda_{k,+} = \Lambda_{k,-}$ in the present case, we have exchanges of $\epsilon_{k,+}, \epsilon_{k,-}$ $(k \in \mathcal{K})$ as

$$T(\epsilon)^\sigma \cong T(\epsilon^\sigma) \quad \text{with} \ \ \epsilon^\sigma := \big((\epsilon_{k,-}, \epsilon_{k,+})\big)_{k \in \mathcal{K}} \ \ \text{for} \ \ \epsilon = \big((\epsilon_{k,+}, \epsilon_{k,-})\big)_{k \in \mathcal{K}}.$$

For a $g = (g_b)_{b \in \mathcal{B}} \in H = \mathfrak{A}\big(\prod_{b \in \mathcal{B}} \mathfrak{S}_{I_b}\big)$, let its parity be $\nu = (\nu_b)_{b \in \mathcal{B}}$, then $\sigma g \sigma^{-1}$ is an exchange of components of g for $(k,+), (k,-)$ for $k \in \mathcal{K}$ as

$$\sigma g \sigma^{-1} = \big((\sigma_k g_{k,-} \sigma_k^{-1}, \sigma_k g_{k,+} \sigma_k^{-1})\big)_{k \in \mathcal{K}} \quad \text{with parity} \ \ \nu^\sigma, \tag{17.35}$$

where ν^σ is defined similarly as ϵ^σ. Denote by $V\big(T(\epsilon)\big)$ the representation space of $T(\epsilon)$. Then the operator $\big(\rho_\Lambda^{(\kappa)}\big)^\sigma(g) = \rho_\Lambda^{(\kappa)}(\sigma g \sigma^{-1})$ sends $V\big(T(\epsilon)\big)$ onto $V\big(T\big(\epsilon + \nu^\sigma\big)\big)$, where $\epsilon + \nu^\sigma$ is calculated componentwise modulo 2. In fact, for $h = \prod_{b \in \mathcal{B}} h_b \in H_0$, $h_b \in \mathfrak{A}_{I_b}$, we have

$$\begin{aligned} &h' := g^{-1} h g = \prod\nolimits_{b \in \mathcal{B}} h'_b, \quad h'_b := g_b^{-1} h_b g_b \in \mathfrak{A}_{I_b}, \\ &\rho_{\Lambda_b}^{(\epsilon_b)}(h'_b) = \big(\rho_{\Lambda_b}^{(\epsilon_b)}\big)^{g_b^{-1}}(h), \quad \big(\rho_{\Lambda_b}^{(\epsilon_b)}\big)^{g_b^{-1}} \cong \rho_{\Lambda_b}^{(\epsilon_b + \nu_b)} \quad \text{with} \ \ \nu_b = \nu(g_b). \end{aligned}$$

Lemma 17.8. *Let $g \in H = \mathfrak{A}\big(\prod_{b \in \mathcal{B}} \mathfrak{S}_{I_b}\big)$ and ν be its parity.*

(i) *Under the action of $H_0 = \prod_{b \in \mathcal{B}} \mathfrak{A}_{I_b}$ through $\big(\rho_\Lambda^{(\kappa)}\big)^\sigma(h)$ $(h \in H_0)$, the operator $\big(\rho_\Lambda^{(\kappa)}\big)^\sigma(g) = \rho_\Lambda^{(\kappa)}(\sigma g \sigma^{-1})$ sends the subspace $V\big(T(\epsilon)\big)$ onto the subspace $V\big(T\big(\epsilon^\sigma + \nu\big)\big)$ in the space of $\rho_\Lambda^{(\kappa)}$ decomposed as*

$$V\big(\rho_\Lambda^{(\kappa)}\big) = \sum\nolimits_{|\epsilon| \equiv \kappa \, (2)}^{\oplus} V\big(T(\epsilon)\big). \tag{17.36}$$

(ii) *Under the action of H_0 through $\rho_\Lambda^{(\kappa)}(h)$ $(h \in H_0)$, the operator $\rho_\Lambda^{(\kappa)}(g)$ sends the subspace $V\big(T(\epsilon)\big)$ onto the subspace $V\big(T\big(\epsilon + \nu\big)\big)$ in $V\big(\rho_\Lambda^{(\kappa)}\big)$.*

Lemma 17.9. *In (Case TΛ-2)+(Case ΛΣ-2), the representation $\big(\rho_\Lambda^{(\kappa)}\big)^\sigma$ of $H = H^\gamma = \mathfrak{A}\big(\prod_{b \in \mathcal{B}} \mathfrak{S}_{I_b}\big)$ is not equivalent to $\rho_\Lambda^{(\kappa)}$. Accordingly, for each*

$\kappa = 0, 1$, *the representation of* $G' = \mathcal{S}(Y^\gamma) = H \sqcup \sigma H$ *induced from* $H = H^\gamma$ *is irreducible:*

$$\Pi\big(\Lambda; (\kappa)\big) := \mathrm{Ind}_{H^\gamma}^{\mathcal{S}(Y^\gamma)} \rho_\Lambda^{(\kappa)}, \quad \mathrm{sgn} \cdot \Pi\big(\Lambda; (\kappa)\big) \cong \Pi\big(\Lambda; (\kappa)\big) \quad (\kappa = 0, 1),$$

and $\Pi\big(\Lambda; (0)\big) \not\cong \Pi\big(\Lambda; (1)\big)$, $\Pi\big(\Lambda; (\kappa)\big)\big|_{H^\gamma} \cong \rho_\Lambda^{(\kappa)} \oplus \big(\rho_\Lambda^{(\kappa)}\big)^\sigma$.

Proof. The first assertion is obtained by comparing (i) and (ii) in Lemma 17.8.

The assertion for non-equivalency is obtained by comparing irreducible components of $\Pi(\Lambda)^{(\kappa)}\big|_{H_0} \cong \rho_\Lambda^{(\kappa)}\big|_{H_0} \oplus \big(\rho_\Lambda^{(\kappa)}\big)^\sigma\big|_{H_0}$ $(\kappa = 0, 1)$. □

Note 17.3. In (Case TΛ-2)+(Case ΛΣ-2), since $\pi(\Lambda)^\sigma \cong \pi(\Lambda)$ for $G = \prod_{b \in \mathcal{B}} \mathfrak{S}_{I_b}$, there exist, by Lemma 17.4 (Case P-1), two non-equivalent IRs

$$\widetilde{\Pi}(\Lambda)^{(0)}, \widetilde{\Pi}(\Lambda)^{(1)}$$

of $\widetilde{G} := G \sqcup \sigma G$ as $\widetilde{\Pi}(\Lambda) = \mathrm{Ind}_G^{\widetilde{G}} \pi_\Lambda = \widetilde{\Pi}(\Lambda)^{(0)} \oplus \widetilde{\Pi}(\Lambda)^{(1)}$. So we have the following diagram :

Diagram 17.2. Relations among IRs
in (Case TΛ-2)+(Case ΛΣ-2) : $\Lambda^\sigma = \Lambda = {}^t\Lambda$.

$$\begin{array}{ccccc}
 & & \widetilde{G} = G \sqcup \sigma G & & \widetilde{\Pi}(\Lambda)^{(0)}, \widetilde{\Pi}(\Lambda)^{(1)} : \widetilde{\Pi}(\Lambda) = \widetilde{\Pi}(\Lambda)^{(0)} \oplus \widetilde{\Pi}(\Lambda)^{(1)} \\
 & \nearrow & & \nwarrow & \\
\pi_\Lambda \quad G = \prod_{b \in \mathcal{B}} \mathfrak{S}_{I_b} & & & & G' = \mathcal{S}(Y^\gamma) = H^\gamma \sqcup \sigma H^\gamma \qquad \Pi\big(\Lambda; (0)\big), \ \Pi\big(\Lambda; (1)\big) \\
 & \nwarrow & & \nearrow & \\
 & & H^\gamma = \mathfrak{A}\big(\prod_{b \in \mathcal{B}} \mathfrak{S}_{I_b}\big) & & \rho_\Lambda^{(0)}, \ \rho_\Lambda^{(1)} : \ \pi_\Lambda\big|_{H^\gamma} = \rho_\Lambda^{(0)} \oplus \rho_\Lambda^{(1)}
\end{array}$$

17.7 Projective IRs of $G(m, 1, n)$ of Case VII, Type $(1,\ 1, -1)$

By the similar method as in [HHH1, §3], we are now on the last step of constructing all irreducible projective representations of $G(m, 1, n)$ of Case VII, Type $(1,\ 1, -1)$, by inducing up from $\widetilde{D}_n \rtimes \mathcal{S}(Y^\gamma)$ to $\widetilde{D}_n \rtimes \mathfrak{S}_n = \widetilde{G}(m, 1, n)$ with $\widetilde{D}_n := \widetilde{D}(m, 1, n)$.

Let us recall some notations. We have chosen a new set of generators $\big\{z_3, \widetilde{\eta}_j := z_3^{j-1}\eta_j \ (1 \le j \le n)\big\}$ for the abelian normal subgroup $\widetilde{D}_n$. The set Γ consists of γ satisfying the conditions (Γ-1) ∼ (Γ-2). For a $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) \in \Gamma$, a one-dimensional character Y^γ of $\widetilde{D}_n$ is defined as $Y^\gamma(z_3) = -1$, $Y^\gamma(\widetilde{\eta}_j) = \omega^{\gamma_j}$ $(1 \le j \le n)$, and $\mathcal{S}(Y^\gamma)$ denotes the set of $\tau \in \mathfrak{S}_n$ which preserves Y^γ.

Let Π be an IR of $\mathcal{S}(Y^\gamma)$, and consider an IR of $\widetilde{D}_n \rtimes \mathcal{S}(Y^\gamma)$ as

$$(17.37) \qquad Y^\gamma \boxdot \Pi \ : \ \widetilde{D}_n \rtimes \mathcal{S}(Y^\gamma) \ni (d', \tau) \longrightarrow Y^\gamma(d') \cdot \Pi(\tau).$$

Then, by inducing up, we get an IR of $\widetilde{G}(m,1,n)$ as

$$\text{(17.38)} \qquad \mathrm{Ind}\, Y^\gamma \boxdot \Pi := \mathrm{Ind}^{\widetilde{G}(m,1,n)}_{\widetilde{D}_n \rtimes \mathcal{S}(Y^\gamma)} Y^\gamma \boxdot \Pi\,.$$

For IRs $\mathrm{Ind}\, Y^\gamma \boxdot \Pi$ in the table below, recall Conditions (Γ-1), (Γ-2) and Example 17.1.

Table 17.1. IRs $\mathrm{Ind}\, Y^\gamma \boxdot \Pi$ **of** $\widetilde{G}(m,1,n)$ **in Case VII, Type** $(1,\ 1,-1)$**.**
$\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n),\ 0 \le K \le K',\ \mathcal{K} = \mathcal{K}^\gamma := \{1, 2, \ldots, K\}$,
$b = (k, \epsilon) \in \mathcal{B} = \mathcal{B}^\gamma := \mathcal{K} \times \{\pm\},\ \Lambda = \big(\Lambda_b\big)_{b \in \mathcal{B}}$.

Structure of $\gamma \in \Gamma$ for a character Y^γ of $\widetilde{D}(m,1,n)$	Stationary subgroup $\mathcal{S}(Y^\gamma) \subset \mathfrak{S}_n$	Relations of the set of Young diagrams Λ with ${}^t\Lambda$ and Λ^σ	IRs of $G' = \mathcal{S}(Y^\gamma)$	Name of IRs of $\widetilde{G}(m,1,n)$
(Case S-1) (a) $K' > K$, or (b) $K' = K$, and $(\exists k)\ \lvert I_{k,+}\rvert \ne \lvert I_{k,-}\rvert$, or $\begin{cases} \lvert I_{k,\pm}\rvert = 1\ (k \in \mathcal{K}) \\ K \text{ even } (n = 2K \leqslant m) \end{cases}$	$\mathcal{S}(Y^\gamma) =$ $H^\gamma =$ $\mathfrak{A}\big(\prod_{b \in \mathcal{B}} \mathfrak{S}_{I_b}\big)$	(Case TΛ-1) ${}^t\Lambda \ne \Lambda$	ρ_Λ $(\rho_{{}^t\Lambda} \cong \rho_\Lambda)$	$T(\gamma;\Lambda)$
		(Case TΛ-2) ${}^t\Lambda = \Lambda$	$\rho_\Lambda^{(\kappa)}$ $(\kappa = 0, 1)$	$T\big(\gamma;\Lambda;(\kappa)\big)$ $(\kappa = 0, 1)$
(Case S-2) $K' = K$, and $\begin{cases} \lvert I_{k,+}\rvert = \lvert I_{k,-}\rvert \\ \quad (k \in \mathcal{K}), \\ (\exists k)\ \lvert I_{k,\pm}\rvert \geqslant 2 \end{cases}$	$\mathcal{S}(Y^\gamma) =$ $H^\gamma \sqcup \sigma H^\gamma$ $(\mathrm{sgn}(\sigma) = -1)$	(Case TΛ-1; ΛΣ-1) ${}^t\Lambda \ne \Lambda$, $\Lambda^\sigma \ne \Lambda, {}^t\Lambda$	$\Pi(\Lambda) =$ $\mathrm{Ind}^{G'}_{H^\gamma} \rho_\Lambda$	$T(\gamma;\Lambda)$
		(Case TΛ-1; ΛΣ-2) ${}^t\Lambda \ne \Lambda$, $\Lambda^\sigma = \Lambda$ or ${}^t\Lambda$	$\Pi(\Lambda)^{(\kappa)}$ $(\kappa = 0, 1)$	$T(\gamma;\Lambda)^{(\kappa)}$ $(\kappa = 0, 1)$
		(Case TΛ-2) ${}^t\Lambda = \Lambda$	$\Pi(\Lambda;(\kappa))$ $= \mathrm{Ind}^{G'}_{H^\gamma} \rho_\Lambda^{(\kappa)}$ $(\kappa = 0, 1)$	$T\big(\gamma;\Lambda;(\kappa)\big)$ $(\kappa = 0, 1)$
(Case S-3) $K' = K$, $\begin{cases} \lvert I_{k,+}\rvert = \lvert I_{k,-}\rvert = 1 \\ \quad (k \in \mathcal{K}),\ K \text{ odd} \\ \quad (n = 2K \leqslant m) \end{cases}$	$\mathcal{S}(Y^\gamma) = \{e, \sigma\}$ $(H^\gamma = \{e\})$	$\Lambda = \varnothing$	sgn^μ $(\mu = 0, 1)$ $(\mathrm{sgn}(\sigma) = -1)$	$T(\gamma;\mathrm{sgn}^\mu)$ $(\mu = 0, 1)$

Theorem 17.10. *For every $\gamma \in \Gamma$, take IRs Π of the stationary subgroup $\mathcal{S}(Y^\gamma) \subset \mathfrak{S}_n$ which are listed in §17.4 and §17.6, and consider IRs of*

$$\widetilde{G}(m,1,n) = R\big(G(m,1,n)\big)/\langle z_1, z_2\rangle$$

obtained as $\mathrm{Ind}\, Y^\gamma \boxdot \Pi$. *Then any irreducible projective representations of $G(m,1,n)$ in Case VII, Type $(1,\ 1,-1)$, is equivalent to one of the above IRs.*

Example 17.2. Consider a special case of (Case S-1) (b), where $K' = K = 1$ and $(I_{1,+}, I_{1,-}) = (\boldsymbol{I}_n, \emptyset)$ or $(I_{1,+}, I_{1,-}) = (\emptyset, \boldsymbol{I}_n)$ for γ. Put $\ell = \gamma_1 = c_1$ or

$\ell = \gamma_1 = c_1 + m'$ accordingly, and $\zeta_\ell(y) := \omega^\ell$, $\zeta_\ell \in \widehat{\boldsymbol{Z}_m}$ a character of $\boldsymbol{Z}_m = \langle y \rangle$. Then $Y^\gamma(\eta'_j) = \omega^\ell$, $\eta'_j = z_3^{j-1}\eta_j$, or

$$Y^\gamma(\eta_j) = (-1)^{j-1}\omega^\ell = (-1)^{j-1}\zeta_\ell(\eta_j) \quad \text{under } T_j \ni \eta_j \to y \in \boldsymbol{Z}_m,$$

and $H^\gamma = \mathfrak{A}_n$. Take $\Lambda = \varnothing$ and $\Pi = \mathbf{1}_{\mathfrak{A}_n}$ the trivial representation of $\mathfrak{A}_n = H^\gamma$. Then the IR $\pi_{2,\zeta_\ell,n} := \mathrm{Ind}_{H^\gamma}^{\widetilde{G}(m,1,n)} Y^\gamma \boxdot \Pi$ is a two-dimensional representation given by

$$\pi_{2,\zeta_\ell,n}(\eta'_j) = \begin{pmatrix} \omega^\ell & 0 \\ 0 & -\omega^\ell \end{pmatrix}, \quad \pi_{2,\zeta_\ell,n}(s_i) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{17.39}$$

for $1 \le j \le n$ and $1 \le i \le n-1$. As the limit under $n \to \infty$, we get two-dimensional IR $\pi_{2,\zeta_\ell} = \lim_{n\to\infty} \pi_{2,\zeta_\ell,n}$ of $\widetilde{G}(m,1,\infty)$, and $\pi_{2,\zeta_\ell} \cong \pi_{2,\zeta_{\ell+m'}}$. These are nothing but the spin IRs π_{2,ζ_k} of $G(m,1,\infty)$ of Type $(1,\ 1,-1)$ given in Theorem 12.1.

18 Irreducible spin characters of $G(m,1,n), m$ even, Case VII, Type $(1,\ 1,-1)$

18.1 Irreducible spin characters of $G(m,1,n)$ of Type $(1,\ 1,-1)$

We can give explicitly all irreducible spin characters of $G(m,1,n)$ of Case VII, Type $(1,\ 1,-1)$, as induced characters from $\widetilde{D}_n \rtimes \mathcal{S}(Y^\gamma)$ to $\widetilde{D}_n \rtimes \mathfrak{S}_n = \widetilde{G}(m,1,n)$ for $\gamma \in \Gamma$.

We quote a general formula for the character of an induced representations with which our calculations are going on. For this, we set up newly some notations only for this subsection. Put $\widetilde{H}_n = \widetilde{D}_n \rtimes \mathcal{S}(Y^\gamma)$, $\widetilde{G}_n := \widetilde{D}_n \rtimes \mathfrak{S}_n = \widetilde{G}(m,1,n)$, $\pi_n := Y^\gamma \boxdot \Pi$ and $\Pi_n := \mathrm{Ind}_{\widetilde{H}_n}^{\widetilde{G}_n} \pi_n$. The normalized characters of π_n and Π_n are denoted respectively by $\widetilde{\chi}_{\pi_n}$ and $\widetilde{\chi}_{\Pi_n}$.

Then, for $g' \in \widetilde{G}_n$ not conjugate to an element in $\widetilde{H}_n$, $\widetilde{\chi}_{\Pi_n}(g') = 0$; and for $h' \in \widetilde{H}_n$,

$$\widetilde{\chi}_{\Pi_n}(h') = \frac{1}{|\widetilde{G}_n|} \sum_{g'' \in \widetilde{G}_n} \widetilde{\chi}_{\pi_n}\big(g'' h' g''^{-1}\big), \tag{18.1}$$

where $\widetilde{\chi}_{\pi_n}$ is extended from $\widetilde{H}_n$ to $\widetilde{G}_n$ by putting $= 0$ outside $\widetilde{H}_n$.

Let $h' = (d', \sigma) \in \widetilde{H}_n = \widetilde{D}_n \rtimes \mathcal{S}(Y^\gamma)$. For $g'' = (d'', \tau) \in \widetilde{G}_n = \widetilde{D}_n \rtimes \mathfrak{S}_n$, we have

$$g''h'g''^{-1} = d''\tau \cdot d'\sigma \cdot \tau^{-1} d''^{-1} = \Big(d'' \cdot \tau d' \tau^{-1} \cdot \kappa\, d''^{-1} \kappa^{-1},\ \tau\sigma\tau^{-1}\Big),$$

where $\kappa := \tau\sigma\tau^{-1}$. Suppose $g''h'g''^{-1} \in \widetilde{H}_n$. Then $\kappa = \tau\sigma\tau^{-1} \in \mathcal{S}(Y^\gamma)$, and so

$$Y^\gamma\big(\kappa d''\kappa^{-1}\big) = Y^\gamma\big(d''\big),$$

$$\therefore \quad \Pi_n(g''h'g''^{-1}) = Y^\gamma\big(\tau d'\tau^{-1}\big) \cdot \Pi(\tau\sigma\tau^{-1}). \tag{18.2}$$

Moreover, let $d' = {\eta'_1}^{a_1}{\eta'_2}^{a_2}\cdots{\eta'_n}^{a_n}$, then

$$\tau d'\tau^{-1} = z_3^{\,\mathrm{ord}(d')L(\tau)} \cdot {\eta'_{\tau(1)}}^{a_1}{\eta'_{\tau(2)}}^{a_2}\cdots{\eta'_{\tau(n)}}^{a_n}, \tag{18.3}$$

$$\therefore \quad Y^\gamma\big(\tau d'\tau^{-1}\big) = (-1)^{\,\mathrm{ord}(d')L(\tau)} \cdot \omega^{\gamma_{\tau(1)}a_1+\gamma_{\tau(2)}a_2+\cdots+\gamma_{\tau(n)}a_n}. \tag{18.4}$$

Theorem 18.1. *For IR* $\Pi_n = \mathrm{Ind}_{\widetilde{H}_n}^{\widetilde{G}_n}\pi_n$, $\pi_n = Y^\gamma \boxdot \Pi$, *of* $\widetilde{G}_n = \widetilde{G}(m,1,n)$, *its normalized character is given for* $h' \in \widetilde{H}_n = \widetilde{D}_n \rtimes \mathcal{S}(Y^\gamma)$ *by*

$$\widetilde{\chi}_{\Pi_n}(h') = \frac{1}{|\mathfrak{S}_n|}\sum_{\substack{\tau\in\mathfrak{S}_n\,;\\ \tau\sigma\tau^{-1}\in\mathcal{S}(Y^\gamma)}} Y^\gamma\big(\tau d'\tau^{-1}\big)\cdot\widetilde{\chi}_\Pi(\tau\sigma\tau^{-1}),$$

and $\widetilde{\chi}_{\Pi_n}(g') = 0$ *if* $g' \in \widetilde{G}_n$ *is not conjugate to any element in* $\widetilde{H}_n$*. Moreover the character value* $Y^\gamma\big(\tau d'\tau^{-1}\big)$ *is given by* (18.3)–(18.4)*.*

In the case where $\mathrm{ord}(d') \equiv 0 \pmod 2$, explicit calculations of the sum in the right hand side are similar as, and simpler than those in [HHH1, §4]. The character formula itself in this case is also similar to that in loc. cit. (cf. §19.1), and is omitted here since to write it down we should prepare still more some notations.

Except the above case we see from Table 9.1 that $\widetilde{\chi}_{\Pi_n}(g') \neq 0 \implies |\mathrm{supp}(g')| \geq n-1$, and so it has no influence to analyse the limit process of the normalized character $\widetilde{\chi}_{\Pi_n}$ as $n \to \infty$. The explicit calculation of the value $\widetilde{\chi}_{\Pi_n}(g')$ is elementary but rather cumbersome paying attention on the sign factor coming from $Y^\gamma(z_3) = -1$, and omitted here.

18.2 Tensor products of two-dimensional IRs of $\widetilde{G}(m,1,n)$ with an IR π of $G(m,1,n)$

We prove here a relation between irreducible spin characters of $G(m,1,n)$ of Type (1, 1, −1) and irreducible characters of $G(m,1,n)$. Using this relation we can translate the results for the latter to that for the former or for the covering group $\widetilde{G}(m,1,n)$ (see e.g., §19.2).

Let $\{\eta'_j = z_3^{\,j-1}\eta_j \ (1 \leq j \leq n)\}$ be new generators of $\widetilde{D}_n = \widetilde{D}(m,1,n) = \langle z_3, \eta_1, \eta_2, \ldots, \eta_n\rangle$, then we have, for two-dimensional IR $\pi_{2,\zeta_0,n}$ of $\widetilde{G}(m,1,n)$,

$$\pi_{2,\zeta_0,n}(\eta'_j) = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \ (j \in \boldsymbol{I}_n), \quad \pi_{2,\zeta_0,n}(\sigma) = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}^{L(\sigma)} \ (\sigma \in \mathfrak{S}_n). \tag{18.5}$$

Definition 18.1. A character $\mathrm{sgn}(\cdot)$ of the abelian group $\widetilde{D}_n = \langle z_3, \eta'_j \ (1 \le j \le n)\rangle$ is defined for $d' = z_3^b {\eta'_1}^{a_1} {\eta'_2}^{a_2} \cdots {\eta'_n}^{a_n} \in \widetilde{D}_n$,

$$\mathrm{sgn}(d') := (-1)^b \, (-1)^{\mathrm{ord}(d')}, \qquad \mathrm{ord}(d') = a_1 + a_2 + \cdots + a_n. \tag{18.6}$$

Let π be an IR of $G := G(m,1,n)$, and we study its tensor product $\Pi' := \pi_{2,\zeta_0,n} \otimes \pi$ which is a representation of the covering group $\widetilde{G} := \widetilde{G}(m,1,n)$.

From (18.5), we see that Π' is expressed as follows. Put $H := D_n(\boldsymbol{Z}_m) \rtimes \mathfrak{A}_n$ and take $s \in \mathfrak{S}_n$ such that $\mathrm{sgn}(s) = -1$, $s^2 = e$, then $G = H \sqcup sH$. For $h' = (d', \sigma) \in \widetilde{G}$, $d' = z_3^b {\eta'_1}^{a_1} {\eta'_2}^{a_2} \cdots {\eta'_n}^{a_n}$, put $h = \Phi(h') \in H$ and $\rho := \pi|_H$, then $\pi(h) = \rho(h)$, and

$$\Pi'(h') = \begin{pmatrix} (-1)^b \rho(h) & O \\ O & \mathrm{sgn}(d')\rho(h) \end{pmatrix}, \qquad \Pi'(s) = \begin{pmatrix} O & I \\ I & O \end{pmatrix}. \tag{18.7}$$

The character of Π' is given for $g' = (d', \sigma) \in \widetilde{D}_n \rtimes \mathfrak{S}_n$, $g = \Phi(g')$, as

$$\chi_{\Pi'}(g') = \chi_{\pi_{2,\zeta_0,n}}(g') \cdot \chi_\pi(g) = \begin{cases} (-1)^b \, 2 \, \chi_\pi(g) \\ \qquad \text{if } \ \mathrm{ord}(d') \equiv 0, \ L(\sigma) \equiv 0 \ (\mathrm{mod}\ 2)\,; \\ 0 \qquad \text{otherwise}. \end{cases} \tag{18.8}$$

Lemma 18.2. (i) *Let ρ be an IR of $H = D_n \rtimes \mathfrak{A}_n$. Then we have two IRs of $\widetilde{H} := \widetilde{D}_n \rtimes \mathfrak{A}_n$ as follows: for $h' = (d', \sigma) \in \widetilde{H}$, $d' = z_3^b \prod_{1 \le j \le n} {\eta'_j}^{a_j} \in \widetilde{D}_n$, $h = \Phi(h') = (d, \sigma) \in H$,*

$$h' \to (-1)^b \rho(h), \qquad h' \to \mathrm{sgn}(d')\rho(h) = (-1)^b \, (-1)^{\mathrm{ord}(d')} \rho(h).$$

These are not mutually equivalent.

(ii) *Let ρ be an IR of H such that $\rho^s \not\cong \rho$ for an $s \in \mathfrak{S}_n$, $\mathrm{sgn}(s) = -1$, $s^2 = e$. Then any two of the following four IRs of $\widetilde{H}$ are not mutually equivalent:*

$$h' \to (-1)^b \rho(h), \ \ h' \to (-1)^b \rho^s(h), \ \ h' \to \mathrm{sgn}(d')\rho(h), \ \ h' \to \mathrm{sgn}(d')\rho^s(h).$$

We give the proof in the Appendix and here we apply it.

Let π be an IR of $G := G(m,1,n) = D_n \rtimes \mathfrak{S}_n$.

(Case 18.2.1) If $\rho := \pi|_H$ is irreducible. Then $\rho^s \cong \rho$, $\rho^s(h) := \rho(shs^{-1})$. As is seen from (18.7), the tensor product $\Pi' = \pi_{2,\zeta_0,n} \otimes \pi$ is irreducible.

(Case 18.2.2) If $\rho = \pi|_H$ is reducible. Then $\rho = \rho_1 \oplus \rho_1^s$, $\rho_1^s \not\cong \rho_1$ with an IR ρ_1, and

$$\pi(h) = \begin{pmatrix} \rho_1(h) & 0 \\ 0 & \rho_1^s(h) \end{pmatrix}, \quad \pi(s) = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{18.9}$$

For $h' = (d',\sigma) \in \widetilde{H} := \widetilde{D}_n \rtimes \mathfrak{A}_n$, $d' = z_3{}^b \prod_{1\le j\le n} \eta_j'^{\,a_j} \in \widetilde{D}_n$, $h = \Phi(h') = (d,\sigma) \in H = D_n \rtimes \mathfrak{A}_n$,

$$\Pi'(h') = \begin{pmatrix} (-1)^b \pi(h) & 0 \\ 0 & \mathrm{sgn}(d')\pi(h) \end{pmatrix}$$
$$= \begin{pmatrix} (-1)^b\rho_1(h) & 0 & 0 & 0 \\ 0 & (-1)^b \rho_1{}^s(h) & 0 & 0 \\ 0 & 0 & \mathrm{sgn}(d')\rho_1(h) & 0 \\ 0 & 0 & 0 & \mathrm{sgn}(d')\rho_1{}^s(h) \end{pmatrix},$$

$$\Pi'(s) = \begin{pmatrix} 0 & \pi(s) \\ \pi(s) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \end{pmatrix}.$$

An intertwining operator U of $\Pi' = \pi_{2,\zeta_0,n} \otimes \pi$ is of the form $U = \mathrm{diag}(aI, bI, cI, dI)$, and $U\,\Pi'(s) = \Pi'(s)\,U$ gives us $(d,c,b,a) = (a,b,c,d)$, or $d = a, c = b$. Hence $U_1 = \mathrm{diag}(I,O,O,I)$, $U_2 = \mathrm{diag}(O,I,I,O)$ are intertwining projections. The IRs $\Pi_k' := U_k \Pi' U_k$ $(k = 1,2)$ and their characters are given as follows:

$$(18.10)\qquad \Pi_1'(h') = \begin{pmatrix} (-1)^b\rho_1(h) & 0 \\ 0 & \mathrm{sgn}(d')\rho_1{}^s(h) \end{pmatrix}, \quad \Pi_1'(s) = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix};$$

$$(18.11)\qquad \Pi_2'(h') = \begin{pmatrix} (-1)^b\rho_1{}^s(h) & 0 \\ 0 & \mathrm{sgn}(d')\rho_1(h) \end{pmatrix}, \quad \Pi_2'(s) = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

$$(18.12)\qquad \begin{cases} \chi_{\Pi_1'}(h') = (-1)^b \chi_{\rho_1}(h) + \mathrm{sgn}(d')\chi_{\rho_1}(shs^{-1}), \\ \chi_{\Pi_1'}(sh') = 0 \qquad \text{on } s\widetilde{H} = \widetilde{D}_n \rtimes s\mathfrak{A}_n\,; \end{cases}$$

$$(18.13)\qquad \begin{cases} \chi_{\Pi_2'}(h') = (-1)^b \chi_{\rho_1}(shs^{-1}) + \mathrm{sgn}(d')\chi_{\rho_1}(h), \\ \chi_{\Pi_2'}(sh') = 0 \qquad \text{on } s\widetilde{H} = \widetilde{D}_n \rtimes s\mathfrak{A}_n\,. \end{cases}$$

$$(18.14).\qquad \chi_{\Pi_2'}(h') = \chi_{\Pi_1'}(sh's^{-1}) = (-1)^{\mathrm{ord}(d')}\,\chi_{\Pi_1'}(h')$$
$$\text{for } h' = (d',\sigma) \in \widetilde{D}_n \rtimes \mathfrak{S}_n = \widetilde{G}(m,1,n).$$

Theorem 18.3. *Let π be an IR of $G(m,1,n) = D_n \rtimes \mathfrak{S}_n$ and $\Pi' = \pi_{2,\zeta_0,n} \otimes \pi$ the tensor product representation of $\widetilde{G}(m,1,n)$.*

(i) *Assume $\pi|_H$ be irreducible. Then Π' is irreducible and its character is given by* (18.8).

(ii) *Assume $\pi|_H$ be reducible. Then $\pi|_H \cong \rho_1 \oplus \rho_1{}^s$, and*

$$\Pi' \cong \Pi_1' \oplus \Pi_2', \qquad \Pi_2'(h') \cong (-1)^{\mathrm{ord}(d')}\,\Pi_1'(h'),$$

for $h' = (d',\sigma) \in \widetilde{D}_n \rtimes \mathfrak{S}_n = \widetilde{G}(m,1,n)$, $d' = z_3{}^b \prod_{1\le j\le n} \eta_j'^{\,a_j} \in \widetilde{D}_n$. Their characters are given as in (18.12)–(18.13) with $h = \Phi(h')$. Moreover,

if $\mathrm{ord}(d') \equiv 0 \pmod 2$, *then* $\chi_{\Pi'_1}(h') = \chi_{\Pi'_2}(h') = (-1)^b \chi_\pi(h)$;
if $L(\sigma) \equiv 1 \pmod 2$, *then* $\chi_{\Pi'_1}(h') = \chi_{\Pi'_2}(h') = 0$;
if $\mathrm{ord}(d') \equiv 1 \pmod 2$ *and* $|\mathrm{supp}(h')| \le n-2$, *then* $\chi_{\Pi'_1}(h') = \chi_{\Pi'_2}(h') = 0$.

Proof. For the last assertion in (ii), if $|\mathrm{supp}(h')| \le n-2$, take $p, q \in \boldsymbol{I}_n \setminus \mathrm{supp}(h')$. Then $s_0 = s \cdot (p\ q) \in \mathfrak{A}_n$ and

$$shs^{-1} = s_0 h s_0^{-1} \quad \text{(or more exactly} \quad sh's^{-1} = z_3^{\mathrm{ord}(d')} s_0 h' s_0^{-1}\,),$$

and therefore $\chi_{\rho_1}(shs^{-1}) = \chi_{\rho_1}(s_0 h s_0^{-1}) = \chi_{\rho_1}(h)$ $(h \in H = \Phi(\widetilde{H})\,)$. □

Note 18.1. In case of infinite group $G(m,1,\infty), n = \infty$, for a spin character f of Case VII, Type $(1,\ 1,-1)$, we see from Table 13.1 that, if $g' \in \mathrm{supp}(f)$, then $g = \Phi(g') = (d,\sigma) \in \mathfrak{A}_\infty(\boldsymbol{Z}_m)^{S(2)}$, or

(18.15) $$\mathrm{ord}(d') \equiv 0, \quad L(\sigma) \equiv 0 \pmod 2.$$

19 Limits of irreducible spin characters of $G(m,1,n)$ as $n \to \infty$, in Case VII

19.1 Limits of irreducible spin characters of $G(m,1,n)$

Let $\Pi_n, n \gg 0$, be a series of spin IRs of $G(m,1,n)$ of Type $(1,\ 1,-1)$, and denote by $\widetilde{\chi}_{\Pi_n}$ their normalized characters $\chi_{\Pi_n}/\dim \Pi_n$. Using the explicit form of $\widetilde{\chi}_{\Pi_n}$ on $\widetilde{G}(m,1,n)$ calculated by means of Theorem 18.1, we can determine their limits on $\widetilde{G}(m,1,\infty) = \lim_{n\to\infty} \widetilde{G}(m,1,n)$, and analyse the limit process. The method and the result are similar to those in [HH4, §§11 ∼ 14] and [HHH1, §§5 ∼ 8].

Here we note that, to analyse the limits, only the values $\widetilde{\chi}_{\Pi_n}(h')$, $h' = (d',\sigma) \in \widetilde{D}_n \rtimes \mathcal{S}(Y^\gamma)$, in the case where $\mathrm{ord}(d') \equiv 0$, $L(\sigma) \equiv 0 \pmod 2$, should be taken into account. Because in other cases, $\widetilde{\chi}_{\Pi_n}(h') \ne 0$ only when $|\mathrm{supp}(h')| \ge n-1$. Thus there is no special affection caused by $Y^\gamma(z_3) = -1$, to carry out the calculations.

Comparing with the result in Theorem 16.5, we get the following theorem.

Theorem 19.1. *Let m be even. Any normalized spin character f of $G(m,1,\infty)$ of Type $(1,\ 1,-1)$ is a pointwise limit of a series of normalized irreducible characters $\widetilde{\chi}_{\Pi_n}$ of $G(m,1,n)$ of the same type, i. e., for $g' \in \widetilde{G}(m,1,\infty) = R\big(G(m,1,n)\big)/\langle z_1, z_2\rangle$,*

(19.1) $$f(g') = \lim_{n\to\infty} \widetilde{\chi}_{\Pi_n}(g').$$

19.2 Second proof for limits of irreducible spin characters of Type $(1,\ 1,-1)$

By means of Theorem 18.3, we can give another proof of Theorem 19.1 and also clarify a criterion on the convergence. Let f be a spin character of $G(m,1,\infty) = \mathfrak{S}_\infty(\boldsymbol{Z}_m)$ in Case VII, Type $(1,\ 1,-1)$, or $f \in E^{\mathrm{VII}}\big(R(G(m,1,\infty))\big)$. Put

$$F(g) = \widetilde{\chi}_{\pi_{2,\zeta_0}}(g') \cdot f(g') \quad \big(g = \Phi(g'),\ g' \in \widetilde{G}(m,1,\infty)\big). \tag{19.2}$$

Then $F(g) = 0$ outside $\mathfrak{A}_\infty(\boldsymbol{Z}_m)^{S(2)}$, and by Theorem 16.5, its restriction $F' := F|_{\mathfrak{A}_\infty(\boldsymbol{Z}_m)^{S(2)}}$ is a character of $\mathfrak{A}_\infty(\boldsymbol{Z}_m)^{S(2)}$. Moreover, by Theorem 16.3, F' is in turn a restriction of a character F'' of $\mathfrak{S}_\infty(\boldsymbol{Z}_m)$.

For the wreath product group $\mathfrak{S}_\infty(\boldsymbol{Z}_m)$, which is an inductive limit of $\mathfrak{S}_n(\boldsymbol{Z}_m)$, the limits of normalized irreducible characters $\widetilde{\chi}_{\pi_n}$ of $\mathfrak{S}_n(\boldsymbol{Z}_m)$ are well studied, where π_n denotes an IR of $\mathfrak{S}_n(\boldsymbol{Z}_m)$ (cf. [HH1]–[HH4], [Boy] and more generally [HHH1] and [HoHH]). We know the following:

(19.2.1) any character of $\mathfrak{S}_\infty(\boldsymbol{Z}_m)$ is a pointwise limit of some series $\widetilde{\chi}_{\pi_n}$ as $n \to \infty$;

(19.2.2) a criterion is given for a series of irreducible characters $\widetilde{\chi}_{\pi_n}$ to be convergent.

By (19.2.1), we can take for F'' a series of irreducible characters χ_{π_n} such that $F'' = \lim_{n\to\infty} \widetilde{\chi}_{\pi_n}$ pointwise, where π_n is an IR of $\mathfrak{S}_n(\boldsymbol{Z}_m)$ for each $n \gg 0$.

(Case 19.2.3) If π_n is in Case 18.2.1, put $\Pi_n := \pi_{2,\zeta_0,n} \otimes \pi_n$, a spin IR of $\widetilde{G}(m,1,n)$.

(Case 19.2.4) If π_n is in Case 18.2.2, put Π_n be any of two irreducible components

$$\Pi'_1,\ \Pi'_2 \text{ of } \pi_{2,\zeta_0,n} \otimes \pi_n \text{ of } \widetilde{G}(m,1,n).$$

Then, by (18.8) and by Theorem 18.2 (ii) respectively, we have for $g' = (d',\sigma) \in \widetilde{D}_n \rtimes \mathfrak{S}_n = \widetilde{G}(m,1,n),\ d' = z_3{}^b \prod_{1\le j\le n} \eta_j'^{\,a_j} \in \widetilde{D}_n,\ g = \Phi(g')$,

In Case 19.2.3, $$\widetilde{\chi}_{\Pi_n}(g') = \begin{cases} (-1)^b\, \widetilde{\chi}_{\pi_n}(g) & \text{if } \mathrm{ord}(d') \equiv 0,\ L(\sigma) \equiv 0 \pmod 2, \\ 0 & \text{otherwise ;} \end{cases}$$

In Case 19.2.4, $$\widetilde{\chi}_{\Pi_n}(g') = \begin{cases} (-1)^b\, \widetilde{\chi}_{\pi_n}(g) & \text{if } \mathrm{ord}(d') \equiv 0,\ L(\sigma) \equiv 0 \pmod 2, \\ 0 & \text{if } |\mathrm{supp}(g')| \le n-2, \text{ in other cases.} \end{cases}$$

Hence $\lim_{n\to\infty} \widetilde{\chi}_{\Pi_n}(g') = (-1)^b F''(g) = (-1)^b F'(g) = (-1)^b F(g) = \widetilde{\chi}_{\pi_{2,\zeta_0}}(g') \cdot F(g) = f(g')$. Thus we obtain for $g' \in \widetilde{G}(m,1,\infty) = \lim_{n\to\infty} \widetilde{G}(m,1,n)$,

$$f(g') = \lim_{n\to\infty} \widetilde{\chi}_{\Pi_n}(g') \qquad \text{pointwise.}$$

20 Appendix. Proof of Lemma 18.2

Proof of (i). Two IRs are mutually equivalent if and only if

$$\chi_\rho(h) = (-1)^{\mathrm{ord}(d)}\,\chi_\rho(h), \quad h = (d,\sigma) \in D_n \rtimes \mathfrak{A}_n\,,$$

and, in turn, this condition is equivalent to $\chi_\rho\big((d,\sigma)\big) = 0$ if $\mathrm{ord}(d) \equiv 1 \pmod 2$. But this is not the case. In fact we can prove it by constructing IR ρ by Mackey type method as induced representations using semidirect product structure $H = D_n \rtimes \mathfrak{A}_n$ and then calculating χ_ρ explicitly as in [HH1]. Let us give the realization of IRs more in detail. A character of $D_n = \langle y_1, y_2, \ldots, y_n\rangle$ is given as

$$\chi_\gamma(y_j) = \omega^{\gamma_j} \;\; (j \in \boldsymbol{I}_n), \quad \omega = e^{2\pi/m}, \;\; \gamma := (\gamma_1, \gamma_2, \ldots, \gamma_n), \;\; 0 \le \gamma_j < m.$$

The action of $\sigma \in \mathfrak{A}_n$ on $\widehat{D_n}$ is given by $\sigma(\chi_\gamma) = \chi_{\sigma\gamma}$, where $(\sigma\gamma)_j = \gamma_{\sigma^{-1}(j)}$. Take the stationary subgroup $\mathcal{A}_\gamma \subset \mathfrak{A}_n$ of $\chi_\gamma \in \widehat{D_n}$ and then take its IR π_γ. Consider an IR $\chi_\gamma \boxdot \pi_\gamma$ of $D_n \rtimes \mathcal{A}_\gamma$ as $\big(\chi_\gamma \boxdot \pi_\gamma\big)(d,\sigma) := \chi_\gamma(d) \cdot \pi_\gamma(\sigma)$, and induce it up as

$$\Pi(\gamma, \pi_\gamma) := \mathrm{Ind}^H_{D_n \rtimes \mathcal{A}_\gamma}\big(\chi_\gamma \boxdot \pi_\gamma\big), \quad H = D_n \rtimes \mathfrak{A}_n\,. \tag{20.1}$$

Then we get an IR of H and any IR of H is equivalent to such a one. The mutual equivalences among $\Pi(\gamma,\pi_\gamma)$'s are all given by conjugations of the parameter (γ,π_γ) under the action of $\mathfrak{A}_n$. Thus the pair (γ,π_γ), modulo the conjugation under $\mathfrak{A}_n$, can be taken as a parameter of the dual $\widehat{H}$ of H.

Note that the character $d \to (-1)^{\mathrm{ord}(d)}$ of D_n is given by $\gamma^{(0)} := (m', m', \ldots, m')$, $m' = m/2$, and that $\chi_{\gamma^{(0)}}$ is $\mathfrak{A}_n$-invariant. Then we see that if $\rho \cong \Pi(\gamma,\pi_\gamma)$, then

$$(-1)^{\mathrm{ord}(\cdot)}\rho \cong \Pi(\gamma + \gamma^{(0)}, \pi_\gamma),$$

Since γ and $\gamma + \gamma^{(0)} = (\gamma_1 + m', \gamma_2 + m', \ldots, \gamma_n + m')$ are not conjugate under $\mathfrak{A}_n$, we have $\rho \not\cong (-1)^{\mathrm{ord}(\cdot)}\rho$. □

Proof of (ii). For $\rho = \Pi(\gamma,\pi_\gamma)$, let us determine the parameter of ρ^s. The representation space $V(\rho)$ for $\rho = \Pi(\gamma,\pi_\gamma)$ consists of $V(\pi_\gamma)$-valued function φ on H satisfying

$$\varphi(d\tau h) = \chi_\gamma(d)\pi_\gamma(\tau)\varphi(h) \quad (d \in D_n,\ \tau \in \mathcal{A}_\gamma,\ h \in H),$$

and the representation is given by $\rho(h_0)\varphi(h) = \varphi(hh_0)$ $(h_0, h \in H)$. Then ρ^s is given on the same space $V(\rho)$ as

$$\rho^s(h_0)\varphi(h) = \varphi(h \cdot s h_0 s^{-1}) \quad (h_0,\ h \in H). \tag{20.2}$$

Put $\psi(h) := \varphi(shs^{-1})$ $(h \in H)$. Then,

$$\psi(d_1\tau_1 h) \;=\; \chi_\gamma\big(s(d_1)\big)\pi_\gamma(s\tau_1 s^{-1})\varphi(shs^{-1}) \tag{20.3}$$

$$= s^{-1}(\chi_\gamma)(d_1)\pi_\gamma^s(\tau_1)\,\psi(h) = \chi_{s^{-1}\gamma}(d_1)\pi_\gamma^s(\tau_1)\,\psi(h).$$

Here $\tau_1 \in s^{-1}\mathcal{A}_\gamma s = \mathcal{A}_{s^{-1}\gamma}$, and π_γ^s is an IR of $\mathcal{A}_{s^{-1}\gamma}$ and so can be denoted as $\pi_{s^{-1}\gamma}$.

Moreover, through the map $\Psi : \varphi \to \psi$, define a representation of H on the space of ψ as $P(h) := \Psi \cdot \rho^s(h) \cdot \Psi^{-1}$. Then the formula (20.2) takes the form

$$P(h_0)\psi(h) \;\; = \;\; \psi(hh_0) \quad (h_0,\ h \in H).$$

Thus the representation P is identified as

$$P = \Pi(s^{-1}\gamma, \pi_{s^{-1}\gamma}), \quad P(h) = \Psi \cdot \rho^s(h) \cdot \Psi^{-1}.$$

Now returning to the assertion (ii), $\rho \not\cong \rho^s$ means that the parameters (γ, π_γ) and $(s^{-1}\gamma, \pi_{s^{-1}\gamma})$ are not mutually conjugate under $\mathfrak{A}_n$. Four IRs under question have respectively the following parameters:

$$(\gamma, \pi_\gamma), \quad (s^{-1}\gamma, \pi_{s^{-1}\gamma}), \quad (\gamma + \gamma^{(0)}, \pi_\gamma), \quad (s^{-1}\gamma + \gamma^{(0)}, \pi_{s^{-1}\gamma}).$$

We can choose s as $s = s_1 = (1\ \ 2)$. Then it can be seen easily that γ and $s^{-1}\gamma + \gamma^{(0)}$ cannot be conjugate under $\mathfrak{A}_n$.

This completes the proof of the assertion (ii). $\square$

List of definitions and symbols for [I] :

Definitions:

basic element g : **7.1**
criterion (EF) : **6.6**
CASE OO, OE, EO, EE : Table 3.1
Condition Y : just before Th. 10.1
Criterion (EF) : just after Def. 6.1
factorizable for f : Def. 6.1
factorizable for $\mathcal{O}$: Def. 10.2
length $\ell(\sigma)$ of a cycle : Notation 7.1
(weakly) multiplicative : Def. 10.1
spin type (*type*) of a projective IR: Def. 1.1
standard decomposition of g : (7.2)
standard decomposition of g' : (7.7)
type χ of central function : Def. 1.1
type χ, *type* $\beta = (\beta_1, \beta_2, \beta_3)$: **6.5**

Symbols:

$A = \Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}; \mu\Big)$: (15.2)
$\mathfrak{A}_I(T)^S$: **2.1, 9.1**
$\mathfrak{A}_\infty(\boldsymbol{Z}_m)^e$: **8.2, 9.1**
$\mathrm{Aut}_G(N)$: **6.3**
$D_I(T)$, $D_n(T)$: (2.1)
$D_I(T)^S$, $D_n(T)^S$: (2.2)
$\widetilde{D}_\infty$, $\widetilde{D}_\infty^{S(p)}$: **6.4**
$E(G)$: **6.1**
$E(N,G)$: (6.3)
$E^{\mathrm{Y}}\big(R\big(G(m,1,\infty)\big)\big)$: Notation 10.1
f_A : Th. 15.1
$g_j = (d_j, \sigma_j)$: (7.2)
$G(m,p,n) = \mathfrak{S}_n(\boldsymbol{Z}_m)^{S(p)}$: **2.1**
$G(m,p,\infty)$: **2.1**
$\widetilde{G}^{\mathrm{Y}}(m,1,\infty)$: Notation 10.1
$H' \subset R\big(G(m,1,n)\big)$: Th. 4.3
η_j $(j \in \boldsymbol{I}_n)$: Th's 3.2, 3.3
$\boldsymbol{I}_n = \{1,2,\ldots,n\}$
$K(G)$, $K_1(G)$: **6.1**
$K(N,G)$, $K_1(N,G)$: (6.3)
$K_1^{\mathrm{Y}}\big(R\big(G(m,1,\infty)\big)\big)$: Notation 10.1
$\ell_j = \ell(\sigma_j) = |\mathrm{supp}(\sigma_j)|$: (7.3)
$L(\sigma)$, $L(\sigma')$: Notation 7.1
$\mathcal{M}$, $\mathcal{N}$: Def. 15.1
$\xi_q = \xi_q(t_q) = (t_q, (q))$: (7.2)
$\mathcal{O}(\mathrm{Y})$: just before Th. 10.1
$\mathcal{O}'(\mathrm{Y})$: Lem. 10.3
$\mathrm{ord}(d)$, $\mathrm{ord}(d')$: Def. 7.1
π_{2,ζ_k} : (12.6)
$\mathcal{P}(G)$: **6.1**
r_i $(i \in \boldsymbol{I}_{n-1})$: Th's 3.2, 3.3
$R\big(G(m,p,n)\big)$: just before Th. 3.5
$R(\mathfrak{S}_n)$, $R(\mathfrak{S}_\infty)$: **1.2**
$\mathrm{sgn}(\sigma') = \mathrm{sgn}(\sigma)$: Notation 7.1
$\mathrm{sgn}(d')$ $(d' \in \widetilde{D}_n)$: Def. 18.1
$\mathfrak{S}_I(T)$, $\mathfrak{S}_n(T)$: (2.1)
$\mathfrak{S}_I(T)^S$, $\mathfrak{S}_n(T)^S$: (2.2)
$\widetilde{\mathfrak{S}}_n$, $\widetilde{\mathfrak{S}}_\infty$: **1.2**
$\mathrm{supp}(d)$, $\mathrm{supp}(\sigma)$, $\mathrm{supp}(g)$: (6.12)
Φ on $R\big(G(m,1,n)\big)$: Th's 3.2, 3.3
Φ^{Y} on $R\big(G(m,1,\infty)\big)$: (10.7)
w_j $(j \in \boldsymbol{I}_n)$: Th's 3.5–3.8
w'_j $(j \in \boldsymbol{I}_n)$: Th's 4.1–4.3
χ^{Y} : Notation 10.1
Ω : just after Th. 7.1
x_j $(j \in \boldsymbol{I}_n)$: Prop. 3.4
y_j $(j \in \boldsymbol{I}_n)$: Prop. 3.1
z_1, z_2, z_3 : Th's 3.3, 3.6, 3.8
$Z(g)$: just below (7.9)
$\mathcal{Z}(g')$: (7.9)