Appendix

In Appendix we give a theorem which is proved by K. Nuida. The author greatly thanks to K. Nuida for allowing the author to put the theorem with its proof in this chapter.

Let (W, S) be a *Coxeter system*. Namely, W is a group with the set S of generators and under the notation $S = \{s_i\,;\, i \in I\}$, the fundamental relations among the generators are

$$s_k^2 = (s_i s_j)^{m_{i,j}} = e \quad \text{and} \quad m_{i,j} = m_{j,i} \quad \text{for} \quad \forall i,\ j,\ k \in I \quad \text{satisfying} \quad i \neq j. \tag{15.1}$$

Here $m_{i,j} \in \{2, 3, 4, \ldots\} \cup \{\infty\}$ and the condition $m_{i,j} = \infty$ means $(s_i s_j)^m \neq e$ for any $m \in \mathbb{Z}_{>0}$. Let E be a real vector space with the basis set $\Pi = \{\alpha_i\,;\, i \in I\}$ and define a symmetric bilinear form $(\ |\ )$ on E by

$$(\alpha_i|\alpha_i) = 2 \quad \text{and} \quad (\alpha_i|\alpha_j) = -2\cos\frac{\pi}{m_{i,j}}. \tag{15.2}$$

Then the *Coxeter group* W is naturally identified with the reflection group generated by the reflections s_{α_i} with respect to α_i $(i \in I)$. The set Δ_Π of the roots of (W, S) equals $W\Pi$, which is a disjoint union of the set of positive roots $\Delta_\Pi^+ := \Delta_\Pi \cap \sum_{\alpha\in\Pi} \mathbb{Z}_{\geq 0}\alpha$ and the set of negative roots $\Delta_\Pi^- := -\Delta_\Pi^+$. For $w \in W$ the length $L(w)$ is the minimal number k with the expression $w = s_{i_1} s_{i_2} \cdots s_{i_k}$ $(i_1, \ldots, i_k \in I)$. Defining $\Delta_\Pi(w) := \Delta_\Pi^+ \cap w^{-1}\Delta_\Pi^-$, we have $L(w) = \#\Delta_\Pi(w)$.

Fix β and $\beta' \in \Delta_\Pi$ and put

$$W_{\beta'}^{\beta} := \{w \in W\,;\, \beta' = w\beta\} \quad \text{and} \quad W^\beta := W_\beta^\beta. \tag{15.3}$$

THEOREM 15.1 (K. Nuida). *Retain the notation above. Suppose $W_{\beta'}^{\beta} \neq \emptyset$ and there exist no sequence $s_{i_1}, s_{i_2}, \ldots s_{i_k}$ of elements of S such that*

$$\begin{cases} k \geq 3, \\ s_{i_\nu} \neq s_{i'_\nu} \quad (1 \leq \nu < \nu' \leq k), \\ m_{i_\nu, i_{\nu+1}} \text{ and } m_{i_1, i_k} \text{ are odd integers} \quad (1 \leq \nu < k). \end{cases} \tag{15.4}$$

Then an element $w \in W_{\beta'}^{\beta}$ is uniquely determined by the condition

$$L(w) \leq L(v) \quad (\forall v \in W_{\beta'}^{\beta}). \tag{15.5}$$

PROOF. Put $\Delta_\Pi^\beta := \{\gamma \in \Delta_\Pi^+\,;\, (\beta|\gamma) = 0\}$. First note that the following lemma.

LEMMA 15.2. *If $w \in W_{\beta'}^{\beta}$ satisfies* (15.5), *then $w\Delta_\Pi^\beta \subset \Delta_\Pi^+$.*

In fact, if $w \in W_{\beta'}^{\beta}$ satisfies (15.5) and there exists $\gamma \in \Delta_\Pi^\beta$ satisfying $w\gamma \in \Delta_\Pi^-$, then there exists j for a minimal expression $w = s_{i_1} \cdots s_{i_{L_\Pi(w)}}$ such that $s_{i_{j+1}} \cdots s_{j_{L_\Pi(w)}}\gamma = \alpha_{i_j}$, which implies $W_{\beta'}^{\beta} \ni v := ws_\gamma = s_{i_1} \cdots s_{i_{j-1}} s_{i_{j+1}} \cdots s_{i_{L_\Pi(w)}}$ and contradicts to (15.5).

It follows from [**Br**] that the assumption (15.4) implies that W^β is generated by $\{s_\gamma\,;\,\gamma\in\Delta_\Pi^\beta\}$. Putting

$$\Pi^\beta=\Delta_\Pi^\beta\setminus\{r_1\gamma_1+r_2\gamma_2\in\Delta_\Pi^\beta\,;\,\gamma_2\notin\mathbb{R}\gamma_1,\ \gamma_j\in\Delta_\Pi^\beta \text{ and } r_j>0 \text{ for } j=1,2\}$$

and $S^\beta=\{s_\gamma\,;\,\gamma\in\Pi^\beta\}$, the pair (W^β,S^β) is a Coxeter system and moreover the minimal length of the expression of $w\in W^\beta$ by the product of the elements of S^β equals $\#\big(\Delta_\Pi^\beta\cap w^{-1}\Delta_\Pi^-\big)$ (cf. [**Nu**, Theorem 2.3]).

Suppose there exist two elements w_1 and $w_2\in W_{\beta'}^\beta$ satisfying $L(w_j)\le L(v)$ for any $v\in W_{\beta'}^\beta$ and $j=1,\,2$. Since $e\neq w_1^{-1}w_2\in W^\beta$, there exists $\gamma\in\Delta_\Pi^\beta$ such that $w_1^{-1}w_2\gamma\in\Delta_\Pi^-$. Since $-w_1^{-1}w_2\gamma\in\Delta_\Pi^\beta$, Lemma 15.2 assures $-w_2\gamma=w_1(-w_1^{-1}w_2\gamma)\in\Delta_\Pi^+$, which contradicts to Lemma 15.2. □

The above proof shows the following corollary.

COROLLARY 15.3. *Retain the assumption in* Theorem 15.1. *For an element* $w\in W_{\beta'}^\beta$, *the condition* (15.5) *is equivalent to* $w\Delta_\Pi^\beta\subset\Delta_\Pi^+$.

Let $w\in W_{\beta'}^\beta$ *satisfying* (15.5). *Then*

$$W_{\beta'}^\beta=w\big\langle s_\gamma\,;\,(\gamma|\beta)=0,\ \gamma\in\Delta_\Pi^+\big\rangle. \tag{15.6}$$