CHAPTER 13

Examples

When we classify tuples of partitions in this chapter, we identify the tuples which are isomorphic to each other. For example, $21, 111, 111$ is isomorphic to any one of $12, 111, 111$ and $111, 21, 111$ and $21, 3, 111, 111$.

Most of our results in this paper are constructible and can be implemented in computer programs. Several reductions and constructions and decompositions of tuples of partitions and connections coefficients associated with Riemann schemes etc. can be computed by a program `okubo` written by the author (cf. §13.11).

In §13.1 and §13.2 we list fundamental and rigid tuples respectively, most of which are obtained by the program `okubo`.

In §13.3 and §13.4 we apply our fractional calculus to Jordan-Pochhammer equations and a hypergeometric family (generalized hypergeometric equations), respectively. Most of the results in these chapters are known but it will be useful to understand our unifying interpretation and apply it to general Fuchsian equations.

In §13.5 we study an even family and an odd family corresponding to Simpson's list [**Si**]. The differential equations of an even family appear in suitable restrictions of Heckman-Opdam hypergeometric systems and in particular the explicit calculation of a connection coefficient for an even family was the original motivation for the study of Fuchsian differential equations developed in this paper (cf. [**OS**]). We also calculate a generalized connection coefficient for an even family of order 4.

In §13.7, §13.8 and §13.9 we study the rigid Fuchsian differential equations of order not larger than 4 and those of order 5 or 6 and the equations belonging to 12 maximal series and some minimal series classified by [**Ro**] which include the equations in Yokoyama's list [**Yo**]. We list sufficient data from which we get some connection coefficients and the necessary and sufficient conditions for the irreducibility of the equations as is explained in §13.9.2.

In §13.6 we give some interesting identities of trigonometric functions as a consequence of the explicit value of connection coefficients.

We examine Appell hypergeometric equations in §13.10, which will be further discussed in another paper.

In §13.11 we explain computer programs `okubo` and a library of `Risa/Asir` which calculate the results described in this paper.

13.1. Basic tuples

The number of basic tuples and fundamental tuples (cf. Definition 6.15) with a given Pidx (cf. (4.27)) are as follows.

Pidx	0	1	2	3	4	5	6	7	8	9	10	11
# fund. tuples	1	4	13	36	67	103	162	243	305	456	578	720
# basic tuples	0	4	13	36	67	90	162	243	305	420	565	720
# basic triplets	0	3	9	24	44	56	97	144	163	223	291	342
# basic 4-tuples	0	1	3	9	17	24	45	68	95	128	169	239
maximal order	1	6	12	18	24	30	36	42	48	54	60	66

Note that if $\mathbf{m}$ is a basic tuple with $\operatorname{idx}\mathbf{m} < 0$, then

$$\operatorname{Pidx} k\mathbf{m} = 1 + k^2(\operatorname{Pidx}\mathbf{m} - 1) \qquad (k = 1, 2, \ldots). \tag{13.1}$$

Hence the non-trivial fundamental tuple $\mathbf{m}$ with $\operatorname{Pidx}\mathbf{m} \le 4$ or equivalently $\operatorname{idx}\mathbf{m} \ge -6$ is always basic.

The tuple $2\mathbf{m}$ with a basic tuple $\mathbf{m}$ satisfying $\operatorname{Pidx}\mathbf{m} = 2$ is a fundamental tuple and $\operatorname{Pidx} 2\mathbf{m} = 5$. The tuple $422, 44, 44, 44$ is this example.

13.1.1. $\operatorname{Pidx}\mathbf{m} = 1$, $\operatorname{idx}\mathbf{m} = 0$. There exist 4 basic tuples: (cf. [**Ko**], Corollary 6.3)

$\tilde{D}_4$: 11,11,11,11 $\tilde{E}_6$: 111,111,111 $\tilde{E}_7$: 22,1111,1111 $\tilde{E}_8$: 33,222,111111

They are not of Okubo type. The tuples of partitions of Okubo type with minimal order which are reduced to the above basic tuples are as follows.

$\tilde{D}_4$: 21,21,21,111 $\tilde{E}_6$: 211,211,1111 $\tilde{E}_7$: 32,2111,11111 $\tilde{E}_8$: 43,322,1111111

The list of simply reducible tuples of partitions whose indices of rigidity equal 0 is given in Example 6.18.

We list the number of realizable tuples of partitions whose indices of rigidity equal 0 according to their orders and the corresponding fundamental tuple.

ord	11,11,11,11	111,111,111	22,1111,1111	33,222,111111	total
2	1				1
3	1	1			2
4	4	1	1		6
5	6	3	1		10
6	21	8	5	1	35
7	28	15	6	1	50
8	74	31	21	4	130
9	107	65	26	5	203
10	223	113	69	12	417
11	315	204	90	14	623
12	616	361	205	37	1219
13	808	588	256	36	1688
14	1432	948	517	80	2977
15	1951	1508	659	100	4218
16	3148	2324	1214	179	6865
17	4064	3482	1531	194	9271
18	6425	5205	2641	389	14660
19	8067	7503	3246	395	19211
20	12233	10794	5400	715	29142

13.1.2. $\operatorname{Pidx}\mathbf{m} = 2$, $\operatorname{idx}\mathbf{m} = -2$. There are 13 basic tuples (cf. Proposition 6.10, [**O6**, Proposition 8.4]):

```
+2:11,11,11,11,11      3:111,111,21,21       *4:211,22,22,22
 4:1111,22,22,31       4:1111,1111,211        5:11111,11111,32
 5:11111,221,221       6:111111,2211,33      *6:2211,222,222
*8:22211,2222,44       8:11111111,332,44     10:22222,3331,55
*12:2222211,444,66
```

Here the number preceding to a tuple is the order of the tuple and the sign "*" means that the tuple is the one given in Example 7.51 ($D_4^{(m)}$, $E_6^{(m)}$, $E_7^{(m)}$ and $E_8^{(m)}$) and the sign "+" means $d(\mathbf{m}) < 0$.

The tuples $22211, 422, 422$ and $4211, 422, 2222$ are of Okubo type with the minimal order which are reduced to $2211, 222, 222$.

13.1.3. Pidx $\mathbf{m} = 3$, idx $\mathbf{m} = -4$. There are 36 basic tuples

+2:11,11,11,11,11,11	3:111,21,21,21,21	4:22,22,22,31,31
+3:111,111,111,21	+4:1111,22,22,22	4:1111,1111,31,31
4:211,211,22,22	4:1111,211,22,31	*6:321,33,33,33
6:222,222,33,51	+4:1111,1111,1111	5:11111,11111,311
5:11111,2111,221	6:111111,222,321	6:111111,21111,33
6:21111,222,222	6:111111,111111,42	6:222,33,33,42
6:111111,33,33,51	6:2211,2211,222	7:1111111,2221,43
7:1111111,331,331	7:2221,2221,331	8:11111111,3311,44
8:221111,2222,44	8:22211,22211,44	*9:3321,333,333
9:111111111,333,54	9:22221,333,441	10:1111111111,442,55
10:22222,3322,55	10:222211,3331,55	12:22221111,444,66
*12:33321,3333,66	14:2222222,554,77	*18:3333321,666,99

13.1.4. Pidx $\mathbf{m} = 4$, idx $\mathbf{m} = -6$. There are 67 basic tuples

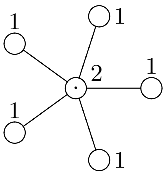
+2:11,11,11,11,11,11,11	3:21,21,21,21,21,21	+3:111,111,21,21,21
+4:22,22,22,22,31	4:211,22,22,31,31	4:1111,22,31,31,31
+3:111,111,111,111	+4:1111,1111,22,31	4:1111,211,22,22
4:211,211,211,22	4:1111,211,211,31	5:11111,11111,41,41
5:11111,221,32,41	5:221,221,221,41	5:11111,32,32,32
5:221,221,32,32	6:3111,33,33,33	6:2211,2211,2211
+6:222,33,33,33	6:222,33,33,411	6:2211,222,33,51
*8:431,44,44,44	8:11111111,44,44,71	5:11111,11111,221
5:11111,2111,2111	+6:111111,111111,33	+6:111111,222,222
6:111111,111111,411	6:111111,222,3111	6:21111,2211,222
6:111111,2211,321	6:2211,33,33,42	7:1111111,1111111,52
7:1111111,322,331	7:2221,2221,322	7:1111111,22111,43
7:22111,2221,331	8:11111111,3221,44	8:11111111,2222,53
8:2222,2222,431	8:2111111,2222,44	8:221111,22211,44
9:33111,333,333	9:3222,333,333	9:22221,22221,54
9:222111,333,441	9:111111111,441,441	10:22222,33211,55
10:1111111111,433,55	10:1111111111,4411,55	10:2221111,3331,55
10:222211,3322,55	12:222111111,444,66	12:333111,3333,66
12:33222,3333,66	12:222222,4431,66	*12:4431,444,444
12:111111111111,552,66	12:3333,444,552	14:33332,4442,77
14:22222211,554,77	15:33333,555,771	*16:44431,4444,88
16:333331,5551,88	18:33333111,666,99	18:3333222,666,99
*24:4444431,888,cc		

Here a, b, c, … represent 10,11,12,…, respectively.

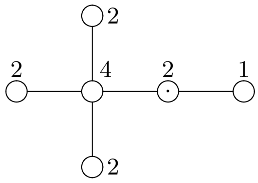
13.1.5. Dynkin diagrams of basic tuples whose indices of rigidity equals -2**.** We express the basic root $\alpha_{\mathbf{m}}$ for Pidx $\mathbf{m} = 2$ using the Dynkin diagram (See (7.11) for Pidx $\mathbf{m} = 1$). The circles in the diagram represent the simple roots in supp $\alpha_{\mathbf{m}}$ and two circles are connected by a line if the inner product of the corresponding simple roots is not zero. The number attached to a circle is the corresponding coefficient n or $n_{j,\nu}$ in the expression (7.12).

For example, if $\mathbf{m} = 22, 22, 22, 211$, then $\alpha_{\mathbf{m}} = 4\alpha_0 + 2\alpha_{0,1} + 2\alpha_{1,1} + 2\alpha_{2,1} + 2\alpha_{3,1} + \alpha_{3,2}$, which corresponds to the second diagram in the following.

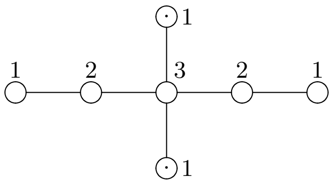
The circle with a dot at the center means a simple root whose inner product with $\alpha_{\mathbf{m}}$ does not vanish. Moreover the type of the root system $\Pi(\mathbf{m})$ (cf. (7.47)) corresponding to the simple roots without a dot is given. The symmetry of the equation describing the isomonodromic deformation of Fuchsian systems of Schlesinger canonical form with a given spectral type, which are induced from Katz's operation and Schlesinger transformations, is described by the Weyl group corresponding to the affinization of the Dynkin diagram with simple roots in Π_0 (cf. §13.1.6).



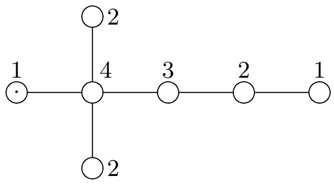
11, 11, 11, 11, 11 $\quad 5A_1$



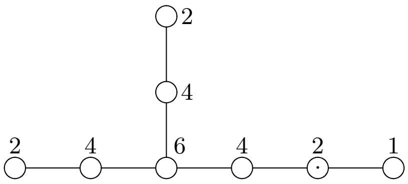
22, 22, 22, 211 $\quad D_4 + A_1$



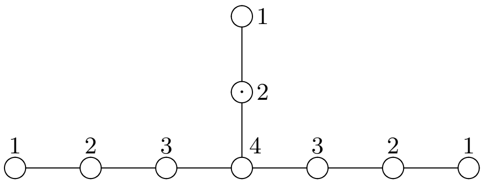
21, 21, 111, 111 $\quad A_5$



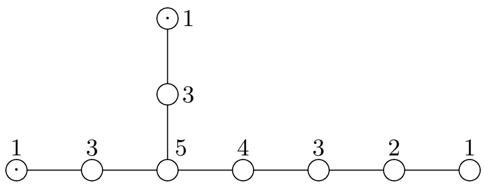
31, 22, 22, 1111 $\quad D_6$



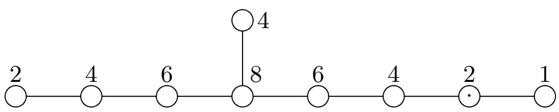
222, 222, 2211 $\quad E_6 + A_1$



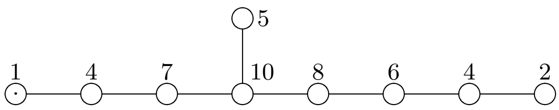
211, 1111, 1111 $\quad A_7 + A_1$



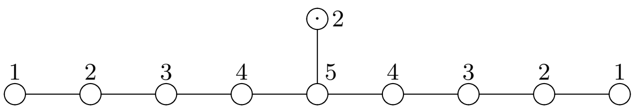
221, 221, 11111 $\quad D_7$



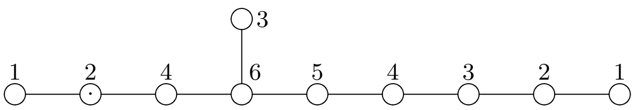
44, 2222, 22211 $\quad E_7 + A_1$



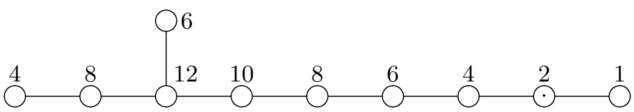
55, 3331, 22222 $\quad E_8$



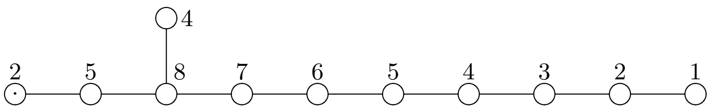
32, 11111, 111111 $\quad A_9$



33, 2211, 111111 $\quad D_8 + A_1$



66, 444, 2222211 $\quad E_8 + A_1$



44, 332, 11111111 D_{10}

13.1.6. Isomonodromic deformations. R. Fuchs [**Fu**] obtained the sixth Painlevé equation from the isomonodromic deformation of the second order Fuchsian equations with 4 essential regular singular points. The other classical Painlevé equations can be obtained from the degeneration of the sixth Painlevé equation and this procedure corresponds to the confluence of the Fuchsian equation. Form this view point the Garnier system corresponds to the equation describing the isomonodromic deformation of the Fuchsian system with the spectral type $11, 11, \cdots, 11$.

Haraoka-Filipuk [**HF**] proved that the equations describing isomonodromic deformations of Fuchsian systems of Schlesinger canonical form are invariant under the Katz's additions and middle convolutions. Hence it is important to study isomonodromic deformations of Fuchsian systems of Schlesinger canonical form with the fundamental spectral types. Moreover we ignore the Fuchsian systems with only three singular points because of the non-existence of their isomonodromic deformations. Among them the higher-dimensional Painlevé type equations corresponding to the following spectral types have been deeply studied (cf. [**FIS**]).

order	index	Painlevé type equation	partitions
2	$6-2p$	Garnier	$11, 11, \cdots, 11 \in \mathcal{P}^{(2)}_{p+1}$
$m+1$	$2-2m$	Fuji–Suzuki–Tsuda	$1^{m+1}, 1^{m+1}, m1, m1$
$2m$	$2-2m$	Sasano	$1^{2m}, m^2, m^2, 2m-11$
$2m$	$2-2m$	matrix Painlevé ($D_4^{(m)}$)	$m^2, m^2, m^2, mm-11$

When the index of rigidity equals -2, there are 4 fundamental spectral types that we should consider. They are in the above list and Sakai [**Sa**] calculates the Hamiltonian functions of the corresponding Painlevé type equations. Then the Painlevé type equations corresponding to the spectral types $111, 111, 21, 21$ and $1111, 22, 22, 211$ coincide with the Fuji-Suzuki system and the Sasano system, respectively, and the new system called matrix Painlevé system is obtained. These systems in the above list are now extensively studied together with their degenerations (cf. [**KNS**], [**FIS**], [**Ts**] etc.). Note that Katz's operations keeping their spectral types invariant induce so-called Bäcklund transformations of the Painlevé type equations.

13.2. Rigid tuples

13.2.1. Simpson's list. Simpson [**Si**] classified the rigid tuples containing the partition $11\cdots1$ into 4 types (Simpson's list), which follows from Proposition 6.17. They are H_n, EO_{2m}, EO_{2m+1} and X_6 in the following table.

See Remark 7.11 ii) for $[\Delta(\mathbf{m})]$ with these rigid tuples $\mathbf{m}$.

The simply reducible rigid tuple (cf. §6.5) which is not in Simpson's list is isomorphic to $21111, 222, 33$.

order	type	name	partitions
n	H_n	hypergeometric family	$1^n, 1^n, n-11$
$2m$	EO_{2m}	even family	$1^{2m}, mm-11, mm$
$2m+1$	EO_{2m+1}	odd family	$1^{2m+1}, mm1, m+1m$
6	$X_6 = \gamma_{6,2}$	extra case	$111111, 222, 42$
6	$\gamma_{6,6}$	(see §13.9.14)	$21111, 222, 33$
n	P_n	Jordan Pochhammer	$n-11, n-11, \ldots \in \mathcal{P}^{(n)}_{n+1}$

$H_1 = EO_1$, $H_2 = EO_2 = P_2$, $H_3 = EO_3$.

13.2.2. Isomorphic classes of rigid tuples. Let $\mathcal{R}_{p+1}^{(n)}$ be the set of rigid tuples in $\mathcal{P}_{p+1}^{(n)}$. Put $\mathcal{R}_{p+1} = \bigcup_{n=1}^{\infty} \mathcal{R}_{p+1}^{(n)}$, $\mathcal{R}^{(n)} = \bigcup_{p=2}^{\infty} \mathcal{R}_{p+1}^{(n)}$ and $\mathcal{R} = \bigcup_{n=1}^{\infty} \mathcal{R}^{(n)}$. The sets of isomorphic classes of the elements of $\mathcal{R}_{p+1}^{(n)}$ (resp. $\mathcal{R}_{p+1}$, $\mathcal{R}^{(n)}$ and $\mathcal{R}$) are denoted $\bar{\mathcal{R}}_{p+1}^{(n)}$ (resp. $\bar{\mathcal{R}}_{p+1}$, $\bar{\mathcal{R}}^{(n)}$ and $\bar{\mathcal{R}}$). Then the number of the elements of $\bar{\mathcal{R}}^{(n)}$ are as follows.

n	$\#\bar{\mathcal{R}}_3^{(n)}$	$\#\bar{\mathcal{R}}^{(n)}$	n	$\#\bar{\mathcal{R}}_3^{(n)}$	$\#\bar{\mathcal{R}}^{(n)}$	n	$\#\bar{\mathcal{R}}_3^{(n)}$	$\#\bar{\mathcal{R}}^{(n)}$
2	1	1	15	1481	2841	28	114600	190465
3	1	2	16	2388	4644	29	143075	230110
4	3	6	17	3276	6128	30	190766	310804
5	5	11	18	5186	9790	31	235543	371773
6	13	28	19	6954	12595	32	309156	493620
7	20	44	20	10517	19269	33	378063	588359
8	45	96	21	14040	24748	34	487081	763126
9	74	157	22	20210	36078	35	591733	903597
10	142	306	23	26432	45391	36	756752	1170966
11	212	441	24	37815	65814	37	907150	1365027
12	421	857	25	48103	80690	38	1143180	1734857
13	588	1177	26	66409	112636	39	1365511	2031018
14	1004	2032	27	84644	139350	40	1704287	2554015

13.2.3. Rigid tuples of order at most 8. We show all the rigid tuples whose orders are not larger than 8.

`2:11,11,11` (H_2: Gauss)

`3:111,111,21` ($H_3 : {}_3F_2$)
`3:21,21,21,21` (P_3)

`4:1111,1111,31` ($H_4 : {}_4F_3$)
`4:1111,211,22` (EO_4: even)
`4:211,211,211` (B_4, II_2, α_4)
`4:211,22,31,31` (I_4, II_2^*)
`4:22,22,22,31` ($P_{4,4}$)
`4:31,31,31,31,31` (P_4)

`5:11111,11111,41` ($H_5 : {}_5F_4$)
`5:11111,221,32` (EO_5: odd)
`5:2111,2111,32` (C_5)
`5:2111,221,311` (B_5, III_2)
`5:221,221,221` (α_5)
`5:221,221,41,41` (J_5)
`5:221,32,32,41`
`5:311,311,32,41` (I_5, III_2^*)
`5:32,32,32,32` ($P_{4,5}$)
`5:32,32,41,41,41` (M_5)
`5:41,41,41,41,41,41` (P_5)

`6:111111,111111,51` ($H_6 : {}_6F_5$)
`6:111111,222,42` ($D_6 = X_6$: extra)
`6:111111,321,33` (EO_6: even)
`6:21111,2211,42` (E_6)
`6:21111,222,33` ($\gamma_{6,6}$)
`6:21111,222,411` (F_6, IV)
`6:21111,3111,33` (C_6)
`6:2211,2211,33` (β_6)
`6:2211,2211,411` (G_6)
`6:2211,321,321`
`6:222,222,321` (α_6)
`6:222,3111,321`
`6:3111,3111,321` (B_6, II_3)
`6:2211,222,51,51` (J_6)
`6:2211,33,42,51`
`6:222,33,33,51`
`6:222,33,411,51`
`6:3111,33,411,51` (I_6, II_3^*)
`6:321,321,42,51`
`6:321,42,42,42`
`6:33,33,33,42` ($P_{4,6}$)
`6:33,33,411,42`
`6:33,411,411,42`
`6:411,411,411,42` (N_6, IV^*)

6:33,42,42,51,51 (M_6)
6:411,42,42,51,51
6:321,33,51,51,51 (K_6)
6:51,51,51,51,51,51,51 (P_6)

7:1111111,1111111,61 (H_7)
7:211111,2221,52 (D_7)
7:22111,22111,52 (E_7)
7:22111,3211,43
7:2221,2221,43 (β_7)
7:2221,322,421
7:2221,331,4111
7:31111,322,421
7:3211,3211,421
7:3211,322,4111
7:2221,2221,61,61 (J_7)
7:3211,331,52,61
7:322,331,511,61
7:322,43,52,52
7:331,43,511,52
7:4111,43,511,52
7:421,421,52,52
7:43,43,43,43 $(P_{4,7})$
7:331,331,61,61,61 (L_7)
7:43,43,43,61,61
7:511,511,52,52,61 (N_7)
7:52,52,52,61,61,61 (M_7)
7:1111111,331,43 (EO_7)
7:211111,322,43 (γ_7)
7:22111,2221,511 (F_7)
7:22111,331,421
7:2221,31111,43
7:2221,331,331
7:31111,31111,43 (C_7)
7:31111,331,4111 $(B_7,\ \mathrm{III}_3)$
7:3211,322,331
7:322,322,322 (α_7)
7:2221,43,43,61
7:322,322,52,61
7:322,421,43,61
7:331,331,43,61
7:4111,4111,43,61 $(I_7,\ \mathrm{III}_3^*)$
7:421,421,421,61
7:421,43,43,52
7:421,43,511,511
7:421,43,52,61,61
7:43,52,52,52,61
7:43,43,61,61,61,61 (K_7)
7:61,61,61,61,61,61,61,61 (P_7)

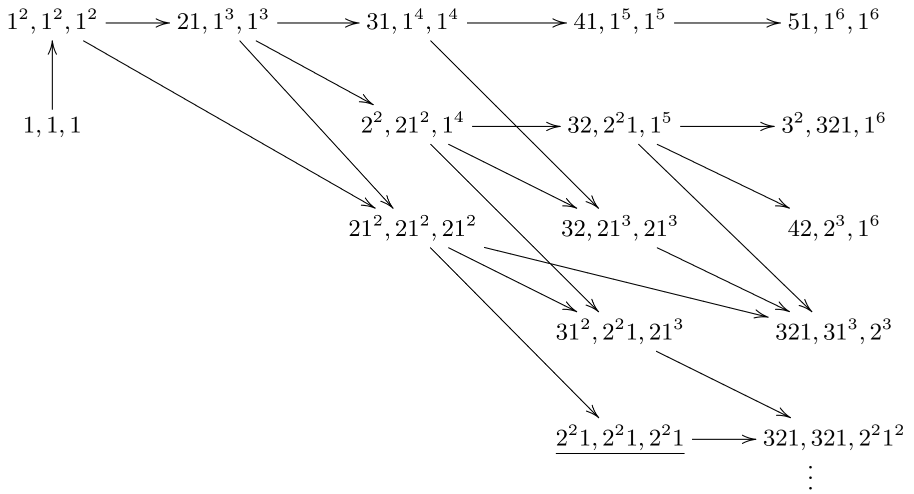
8:11111111,11111111,71 (H_8)
8:2111111,2222,62 (D_8)
8:2111111,422,44
8:221111,2222,611 (F_8)
8:221111,332,44 (γ_8)
8:22211,22211,611 (G_8)
8:22211,3311,44
8:22211,41111,44
8:22211,44,53,71
8:2222,32111,53
8:2222,3311,521
8:2222,422,431
8:311111,332,521
8:32111,32111,53
8:32111,3311,521
8:32111,422,431
8:3221,3311,5111
8:332,332,332 (α_8)
8:332,41111,422
8:3221,4211,431
8:3311,332,422
8:3311,4211,422
8:41111,4211,422
8:22211,2222,71,71 (J_8)
8:3221,332,62,71
8:3221,44,62,62
8:1111111,431,44 (EO_8)
8:2111111,332,53
8:221111,22211,62 (E_8)
8:221111,3311,53
8:221111,4211,44
8:22211,3221,53
8:22211,332,521
8:22211,431,431
8:2222,2222,53 $(\beta_{8,2})$
8:2222,3221,44 $(\beta_{8,4})$
8:2222,332,5111
8:311111,3221,53
8:311111,41111,44 (C_8)
8:32111,3221,44
8:32111,332,5111
8:3221,3221,521
8:3221,332,431
8:332,332,4211
8:332,4211,4211
8:3311,3311,431
8:3221,422,422
8:41111,41111,431 $(B_8,\ II_4)$
8:4211,4211,4211
8:2222,44,44,71
8:3221,44,521,71
8:3311,3311,62,71

`8:3311,332,611,71`
`8:3311,44,611,62`
`8:332,431,44,71`
`8:332,53,53,62`
`8:41111,44,611,62`
`8:4211,44,611,611`
`8:422,422,44,71`
`8:422,431,62,62`
`8:431,44,44,62`
`8:422,53,53,611`
`8:431,521,53,62`
`8:44,5111,521,62`
`8:44,521,53,53`
`8:5111,521,53,611`
`8:332,332,71,71,71`
`8:4211,44,62,71,71`
`8:431,53,53,71,71`
`8:44,53,611,62,71`
`8:521,53,62,62,71`
`8:53,62,62,62,62`
`8:53,53,62,71,71,71`
`8:611,62,62,62,71,71` (M_8)
`8:3311,431,53,71`
`8:332,422,53,71`
`8:332,44,611,611`
`8:41111,44,5111,71` $(I_8,\ II_4^*)$
`8:4211,422,53,71`
`8:4211,53,53,62`
`8:422,431,521,71`
`8:422,44,53,62`
`8:431,44,53,611`
`8:431,431,611,62`
`8:44,44,44,53` $(P_{4,8})$
`8:44,521,521,611`
`8:5111,5111,53,62`
`8:521,521,521,62`
`8:332,44,62,71,71`
`8:422,44,611,71,71`
`8:44,44,62,62,71`
`8:521,521,53,71,71`
`8:53,53,611,611,71`
`8:611,611,611,62,62` (N_8)
`8:431,44,71,71,71,71` (K_8)
`8:71,71,71,71,71,71,71,71,71` (P_8)

Here the underlined tuples are not of Okubo type (cf. (11.33)).

The tuples H_n, EO_n and X_6 are tuples in Simpson's list. The series $A_n = EO_n$, B_n, C_n, D_n, E_n, F_n, G_{2m}, I_n, J_n, K_n, L_{2m+1}, M_n and N_n are given in [**Ro**] and called submaximal series. The Jordan-Pochhammer tuples are denoted by P_n and the series H_n and P_n are called maximal series by [**Ro**]. The series $\alpha_n, \beta_n, \gamma_n$ and δ_n are given in [**Ro**] and called minimal series. See §13.9 for these series introduced by [**Ro**]. Then $\delta_n = P_{4,n}$ and they are generalized Jordan-Pochhammer tuples (cf. Example 10.5 and §13.9.13). Moreover II_n, II_n^*, III_n, III_n^*, IV and IV* are in Yokoyama's list in [**Yo**] (cf. §13.9.15).

Hierarchy of rigid triplets



Here the arrows represent certain operations ∂_ℓ of tuples given by Definition 5.7.

13.3. Jordan-Pochhammer family

We have studied the the Riemann scheme of Jordan-Pochhammer family P_n in Example 1.8 iii).

$\mathbf{m} = (p-11, p-11, \ldots, p-11) \in \mathcal{P}_{p+1}^{(p)}$

$$\begin{Bmatrix} x=0 & 1=\frac{1}{c_1} & \cdots & \frac{1}{c_{p-1}} & \infty \\ [0]_{(p-1)} & [0]_{(p-1)} & \cdots & [0]_{(p-1)} & [1-\mu]_{(p-1)} \\ \lambda_0+\mu & \lambda_1+\mu & \cdots & \lambda_{p-1}+\mu & -\lambda_0-\cdots-\lambda_{p-1}-\mu \end{Bmatrix}$$

$\Delta(\mathbf{m}) = \{\alpha_0,\, \alpha_0+\alpha_{j,1}\,;\, j=0,\ldots,p\}$

$[\Delta(\mathbf{m})] = 1^{p+1}\cdot(p-1)$

$P_p = H_1 \oplus P_{p-1} : p+1 = (p-1)H_1 \oplus H_1 : 1$

Here the number of the decompositions of a given type is shown after the decompositions. For example, $P_p = H_1 \oplus P_{p-1} : p+1 = (p-1)H_1 \oplus H_1 : 1$ represents the decompositions

$$\begin{aligned} \mathbf{m} &= 10,\ldots,\overset{\nu}{\breve{01}},\ldots,10 \oplus p-21,\ldots,\overset{\nu}{\breve{p-10}},\ldots,p-21 \qquad (\nu=0,\ldots,p) \\ &= (p-1)(10,\ldots,10)\oplus 01,\ldots,01. \end{aligned}$$

The differential equation $P_{P_p}(\lambda,\mu)u=0$ with this Riemann scheme is given by

$$P_{P_p}(\lambda,\mu) := \mathrm{RAd}\bigl(\partial^{-\mu}\bigr)\circ\mathrm{RAd}\Bigl(x^{\lambda_0}\prod_{j=1}^{p-1}(1-c_jx)^{\lambda_j}\Bigr)\partial$$

and then

$$\begin{aligned} &P_{P_p}(\lambda,\mu) = \sum_{k=0}^{p} p_k(x)\partial^{p-k}, \\ &p_k(x) := \binom{-\mu+p-1}{k}p_0^{(k)}(x) + \binom{-\mu+p-1}{k-1}q^{(k-1)}(x) \end{aligned} \tag{13.2}$$

with

$$p_0(x) = x\prod_{j=1}^{p-1}(1-c_jx), \quad q(x) = p_0(x)\Bigl(-\frac{\lambda_0}{x}+\sum_{j=1}^{p-1}\frac{c_j\lambda_j}{1-c_jx}\Bigr). \tag{13.3}$$

It follows from Theorem 10.10 that the equation is irreducible if and only if

$$\lambda_j\notin\mathbb{Z}\ \ (j=0,\ldots,p-1),\ \mu\notin\mathbb{Z}\ \text{ and }\ \lambda_0+\cdots+\lambda_{p-1}+\mu\notin\mathbb{Z}. \tag{13.4}$$

It follows from Proposition 11.13 that the shift operator defined by the map $u\mapsto\partial u$ is bijective if and only if

$$\mu\notin\{1,2,\ldots,p-1\}\ \text{ and }\ \lambda_0+\cdots+\lambda_{p-1}+\mu\neq 0. \tag{13.5}$$

The normalized solution at 0 corresponding to the exponent $\lambda_0+\mu$ is

$$\begin{aligned} u_0^{\lambda_0+\mu}(x) &= \frac{\Gamma(\lambda_0+\mu+1)}{\Gamma(\lambda_0+1)\Gamma(\mu)}\int_0^x\Bigl(t^{\lambda_0}\prod_{j=1}^{p-1}(1-c_jt)^{\lambda_j}\Bigr)(x-t)^{\mu-1}dt \\ &= \frac{\Gamma(\lambda_0+\mu+1)}{\Gamma(\lambda_0+1)\Gamma(\mu)}\int_0^x\sum_{m_1=0}^{\infty}\cdots\sum_{m_{p-1}=0}^{\infty}\frac{(-\lambda_1)_{m_1}\cdots(-\lambda_{p-1})_{m_{p-1}}}{m_1!\cdots m_{p-1}!} \\ &\qquad c_2^{m_2}\cdots c_{p-1}^{m_{p-1}}t^{\lambda_0+m_1+\cdots+m_{p-1}}(x-t)^{\mu-1}dt \end{aligned}$$

$$= \sum_{m_1=0}^{\infty} \cdots \sum_{m_{p-1}=0}^{\infty} \frac{(\lambda_0+1)_{m_1+\cdots+m_{p-1}}(-\lambda_1)_{m_1}\cdots(-\lambda_{p-1})_{m_{p-1}}}{(\lambda_0+\mu+1)_{m_1+\cdots+m_{p-1}} m_1!\cdots m_{p-1}!}$$
$$c_2^{m_2}\cdots c_{p-1}^{m_{p-1}} x^{\lambda_0+\mu+m_1+\cdots+m_{p-1}}$$
$$= x^{\lambda_0+\mu}\Big(1 - \frac{(\lambda_0+1)(\lambda_1 c_1+\cdots+\lambda_{p-1}c_{p-1})}{\lambda_0+\mu+1}x+\cdots\Big).$$

This series expansion of the solution is easily obtained from the formula in §3.1 (cf. Theorem 8.1) and Theorem 11.3 gives the contiguity relation

$$u_0^{\lambda_0+\mu}(x) = u_0^{\lambda_0+\mu}(x)\big|_{\lambda_1\mapsto\lambda_1-1} - \Big(\frac{\lambda_0}{\lambda_0+\mu}u_0^{\lambda_0+\mu}(x)\Big)\Big|_{\substack{\lambda_0\mapsto\lambda_0+1\\ \lambda_1\mapsto\lambda_1-1}}. \tag{13.6}$$

Lemma 12.2 with $a=\lambda_0$, $b=\lambda_1$ and $u(x)=\prod_{j=2}^{p-1}(1-c_jx)^{\lambda_j}$ gives the following connection coefficients

$$c(0:\lambda_0+\mu \rightsquigarrow 1:\lambda_1+\mu) = \frac{\Gamma(\lambda_0+\mu+1)\Gamma(-\lambda_1-\mu)}{\Gamma(\lambda_0+1)\Gamma(-\lambda_1)}\prod_{j=2}^{p-1}(1-c_j)^{\lambda_j},$$
$$c(0:\lambda_0+\mu \rightsquigarrow 1:0) = \frac{\Gamma(\lambda_0+\mu+1)}{\Gamma(\mu)\Gamma(\lambda_0+1)}\int_0^1 t^{\lambda_0}(1-t)^{\lambda_1+\mu-1}\prod_{j=2}^{p-1}(1-c_jt)^{\lambda_j}dt$$
$$= \frac{\Gamma(\lambda_0+\mu+1)\Gamma(\lambda_1+\mu)}{\Gamma(\mu)\Gamma(\lambda_0+\lambda_1+\mu+1)}F(\lambda_0+1,-\lambda_2,\lambda_0+\lambda_1+\mu+1;c_2) \qquad (p=3).$$

Here we have

$$u_0^{\lambda_0+\mu}(x) = \sum_{k=0}^{\infty} C_k(x-1)^k + \sum_{k=0}^{\infty} C'_k(x-1)^{\lambda_1+\mu+k} \tag{13.7}$$

for $0<x<1$ with $C_0 = c(0:\lambda_0+\mu \rightsquigarrow 1:0)$ and $C'_0 = c(0:\lambda_0+\mu \rightsquigarrow 1:\lambda_1+\mu)$. Since $\frac{d^k u_0^{\lambda_0+\mu}}{dx^k}$ is a solution of the equation $P_{P_p}(\lambda,\mu-k)u=0$, we have

$$C_k = \frac{\Gamma(\lambda_0+\mu+1)}{\Gamma(\mu-k)\Gamma(\lambda_0+1)k!}\int_0^1 t^{\lambda_0}(1-t)^{\lambda_1+\mu-k-1}\prod_{j=2}^{p-1}(1-c_jt)^{\lambda_j}dt. \tag{13.8}$$

When $p=3$,

$$C_k = \frac{\Gamma(\lambda_0+\mu+1)\Gamma(\lambda_1+\mu-k)}{\Gamma(\mu-k)\Gamma(\lambda_0+\lambda_1+\mu+1-k)k!}F(\lambda_0+1,-\lambda_2,\lambda_0+\lambda_1+\mu+1-k;c_2).$$

Put

$$u_{\lambda,\mu}(x) = \frac{1}{\Gamma(\mu)}\int_0^x \Big(t^{\lambda_0}\prod_{j=1}^{p-1}(1-c_jt)^{\lambda_j}\Big)(x-t)^{\mu-1}dt = \partial^{-\mu}v_\lambda,$$
$$v_\lambda(x) := x^{\lambda_0}\prod_{j=1}^{p-1}(1-c_jx)^{\lambda_j}.$$

We have

$$\begin{aligned} u_{\lambda,\mu+1} &= \partial^{-\mu-1}v_\lambda = \partial^{-1}\partial^{-\mu}v_\lambda = \partial^{-1}u_{\lambda,\mu},\\ u_{\lambda_0+1,\lambda_1,\ldots,\mu} &= \partial^{-\mu}v_{\lambda_0+1,\lambda_1,\ldots} = \partial^{-\mu}xv_\lambda = -\mu\partial^{-\mu-1}v_\lambda + x\partial^{-\mu}v_\lambda\\ &= -\mu\partial^{-1}u_{\lambda,\mu} + xu_{\lambda,\mu},\\ u_{\ldots,\lambda_j+1,\ldots} &= \partial^{-\mu}(1-c_jx)v_\lambda = \partial^{-\mu}v_\lambda + c_j\mu\partial^{-\mu-1}v_\lambda - c_jx\partial^{-\mu}v_\lambda\\ &= (1-c_jx)u_{\lambda,\mu} + c_j\mu\partial^{-1}u_{\lambda,\mu}. \end{aligned} \tag{13.9}$$

From these relations with $P_{P_p} u_{\lambda,\mu} = 0$ we have all the contiguity relations. For example

$$\begin{aligned} \partial u_{\lambda_0,\dots,\lambda_{p-1},\mu+1} &= u_{\lambda,\mu}, \\ \partial u_{\lambda_0+1,\dots,\lambda_{p-1},\mu} &= (x\partial + 1 - \mu) u_{\lambda,\mu}, \\ \partial u_{\dots,\lambda_j+1,\dots,\mu} &= \big((1-c_j x)\partial - c_j(1-\mu)\big) u_{\lambda,\mu} \end{aligned} \tag{13.10}$$

and

$$\begin{aligned} P_{P_p}(\lambda,\mu+1) &= \sum_{j=0}^{p-1} p_j(x)\partial^{p-j} + p_n \\ p_n &= (-1)^{p-1} c_1 \dots c_{p-1}\Big((-\mu-1)_p + (-\mu)_{p-1}\sum_{j=0}^{p-1}\lambda_j\Big) \\ &= c_1 \cdots c_{p-1}(\mu+2-p)_{p-1}(\lambda_0 + \cdots + \lambda_{p-1} - \mu - 1) \end{aligned}$$

and hence

$$\Big(\sum_{j=0}^{p-1} p_j(x)\partial^{p-j-1}\Big) u_{\lambda,\mu} = -p_n u_{\lambda,\mu+1} = -p_n \partial^{-1} u_{\lambda,\mu}.$$

Substituting this equation to (13.9), we have $Q_j \in W(x;\lambda,\mu)$ such that $Q_j u_{\lambda,\mu}$ equals $u_{(\lambda_\nu+\delta_{\nu,j})_{\nu=0,\dots,p-1},\mu}$ for $j = 0,\dots,p-1$, respectively. The operators $R_j \in W(x;\lambda,\mu)$ satisfying $R_j Q_j u_{\lambda,\mu} = u_{\lambda,\mu}$ are calculated by the Euclidean algorithm, namely, we find $S_j \in W(x;\lambda,\mu)$ so that $R_j Q_j + S_j P_{P_p} = 1$. Thus we also have $T_j \in W(x;\lambda,\mu)$ such that $T_j u_{\lambda,\mu}$ equals $u_{(\lambda_\nu-\delta_{\nu,j})_{\nu=0,\dots,p-1},\mu}$ for $j = 0,\dots,p-1$, respectively.

As is shown in §2.4 the *Versal Jordan-Pochhammer operator* $\tilde P_{P_p}$ is given by (13.2) with

$$p_0(x) = \prod_{j=1}^{p}(1-c_j x), \quad q(x) = \sum_{k=1}^{p}\lambda_k x^{k-1}\prod_{j=k+1}^{p}(1-c_j x). \tag{13.11}$$

If $c_1,\dots,c_p$ are different to each other, the Riemann scheme of $\tilde P_{P_p}$ is

$$\left\{\begin{matrix} x = \frac{1}{c_j}\ (j=1,\dots,p) & \infty \\ [0]_{(p-1)} & [1-\mu]_{(p-1)} \\ \displaystyle\sum_{k=j}^{p}\frac{\lambda_k}{c_j\prod_{\substack{1\le\nu\le k\\ \nu\ne j}}(c_j-c_\nu)} + \mu & \displaystyle\sum_{k=1}^{p}\frac{(-1)^k\lambda_k}{c_1\dots c_k} - \mu \end{matrix}\right\}.$$

The solution of $\tilde P_{P_p} u = 0$ is given by

$$u_C(x) = \int_C \Big(\exp\int_0^t \sum_{j=1}^{p}\frac{-\lambda_j s^{j-1}}{\prod_{1\le\nu\le j}(1-c_\nu s)}ds\Big)(x-t)^{\mu-1}dt.$$

Here the path C starting from a singular point and ending at a singular point is chosen so that the integration has a meaning. In particular when $c_1 = \cdots = c_p = 0$, we have

$$u_C(x) = \int_C \exp\Big(-\sum_{j=1}^{p}\frac{\lambda_j t^j}{j}\Big)(x-t)^{\mu-1}dt$$

and if $\lambda_p \ne 0$, the path C starts from ∞ to one of the p independent directions $\lambda_p^{-1}e^{\frac{2\pi\nu\sqrt{-1}}{p}+t}$ $(t \gg 1,\ \nu = 0,1,\dots,p-1)$ and ends at x.

Suppose $n = 2$. The corresponding Riemann scheme for the generic characteristic exponents and its construction from the Riemann scheme of the trivial equation $u' = 0$ is as follows:

$$\begin{Bmatrix} x=0 & 1 & \infty \\ b_0 & c_0 & a_0 \\ b_1 & c_1 & a_1 \end{Bmatrix} \qquad \text{(Fuchs relation: } a_0+a_1+b_0+b_1+c_0+c_1=1)$$

$$\xleftarrow{x^{b_0}(1-x)^{c_0}\partial^{-a_1-b_1-c_1}} \begin{Bmatrix} x=0 & 1 & \infty \\ -a_1-b_0-c_1 & -a_1-b_1-c_0 & -a_0+a_1+1 \end{Bmatrix}$$

$$\xleftarrow{x^{-a_1-b_0-c_1}(1-x)^{-a_1-b_1-c_0}} \begin{Bmatrix} x=0 & 1 & \infty \\ 0 & 0 & 0 \end{Bmatrix}.$$

Then our fractional calculus gives the corresponding equation

$$\begin{aligned} &x^2(1-x)^2u'' - x(1-x)\big((a_0+a_1+1)x+b_0+b_1-1\big)u' \\ &\quad + \big(a_0a_1x^2-(a_0a_1+b_0b_1-c_0c_1)x+b_0b_1\big)u = 0, \end{aligned} \tag{13.12}$$

the connection formula

$$c(0\!:\!b_1 \rightsquigarrow 1\!:\!c_1) = \frac{\Gamma(c_0-c_1)\Gamma(b_1-b_0+1)}{\Gamma(a_0+b_1+c_0)\Gamma(a_1+b_1+c_0)} \tag{13.13}$$

and expressions of its solution by the integral representation

$$\begin{aligned} &\int_0^x x^{b_0}(1-x)^{c_0}(x-s)^{a_1+b_1+c_1-1}s^{-a_1-c_1-b_0}(1-s)^{-a_1-b_1-c_0}ds \\ &= \frac{\Gamma(a_0+b_1+c_0)\Gamma(a_1+b_1+c_1)}{\Gamma(b_1-b_0+1)}x^{b_1}\phi_{b_1}(x) \end{aligned} \tag{13.14}$$

and the series expansion

$$\begin{aligned} &\sum_{n\ge 0}\frac{(a_0+b_1+c_0)_n(a_1+b_1+c_0)_n}{(b_1-b_0+1)_n n!}(1-x)^{c_0}x^{b_1+n} \\ &\quad = (1-x)^{c_0}x^{b_1}F(a_0+b_1+c_0,a_1+b_1+c_0,b_1-b_0-1;x). \end{aligned} \tag{13.15}$$

Here $\phi_{b_1}(x)$ is a holomorphic function in a neighborhood of 0 satisfying $\phi_{b_1}(0) = 1$ for generic spectral parameters. We note that the transposition of c_0 and c_1 in (13.15) gives a nontrivial equality, which corresponds to Kummer's relation of Gauss hypergeometric function and the similar statement is true for (13.14). In general, different procedures of reduction of a equation give different expressions of its solution.

13.4. Hypergeometric family

We examine the hypergeometric family H_n which corresponds to the equations satisfied by the generalized hypergeometric series (0.7). Its spectral type is in Simpson's list (cf. §13.2).

$\mathbf{m} = (1^n, n-11, 1^n) : {}_nF_{n-1}(\alpha,\beta;z)$

$$1^n, n-11, 1^n = 1,10,1 \oplus 1^{n-1}, n-21, 1^{n-1}$$

$$\begin{aligned} \Delta(\mathbf{m}) &= \{\alpha_0+\alpha_{0,1}+\cdots+\alpha_{0,\nu}+\alpha_{2,1}+\cdots+\alpha_{2,\nu'}\,; \\ &\qquad 0\le\nu<n,\ 0\le\nu'<n\} \end{aligned}$$

$$[\Delta(\mathbf{m})] = 1^{n^2}$$

$$H_n = H_1 \oplus H_{n-1} : n^2$$

$$H_n \xrightarrow[R2E0]{1} H_{n-1}$$

Since $\mathbf{m}$ is of Okubo type, we have a system of Okubo normal form with the spectral type $\mathbf{m}$. Then the above $R2E0$ represents the reduction of systems of equations of Okubo normal form due to Yokoyama [**Yo2**]. The number 1 on the arrow represents a reduction by a middle convolution and the number shows the difference of the orders.

$$\begin{Bmatrix} x=0 & 1 & \infty \\ \lambda_{0,1} & [\lambda_{1,1}]_{(n-1)} & \lambda_{2,1} \\ \vdots & & \vdots \\ \lambda_{0,n-1} & & \lambda_{2,n-1} \\ \lambda_{0,n} & \lambda_{1,2} & \lambda_{2,n} \end{Bmatrix}, \quad \begin{Bmatrix} x=0 & 1 & \infty \\ 1-\beta_1 & [0]_{(n-1)} & \alpha_1 \\ \vdots & & \vdots \\ 1-\beta_{n-1} & & \alpha_{n-1} \\ 0 & -\beta_n & \alpha_n \end{Bmatrix} \tag{13.16}$$

$$\sum_{\nu=1}^{n}(\lambda_{0,\nu}+\lambda_{2,\nu})+(n-1)\lambda_{1,1}+\lambda_{1,2}=n-1,$$

$$\alpha_1+\cdots+\alpha_n=\beta_1+\cdots+\beta_n.$$

It follows from Theorem 11.7 that the universal operators

$$P^0_{H_1}(\lambda),\ P^2_{H_1}(\lambda),\ P^0_{H_{n-1}}(\lambda),\ P^1_{H_{n-1}}(\lambda),\ P^2_{H_{n-1}}(\lambda).$$

are shift operators for the universal model $P_{H_n}(\lambda)u=0$.

The Riemann scheme of the operator

$$P=\mathrm{RAd}(\partial^{-\mu_{n-1}})\circ\mathrm{RAd}(x^{\gamma_{n-1}})\circ\cdots\circ\mathrm{RAd}(\partial^{-\mu_1})\circ\mathrm{RAd}(x^{\gamma_1}(1-x)^{\gamma'})\partial$$

equals

$$\begin{Bmatrix} x=0 & 1 & \infty \\ 0 & [0]_{(n-1)} & 1-\mu_{n-1} \\ (\gamma_{n-1}+\mu_{n-1}) & & 1-(\gamma_{n-1}+\mu_{n-1})-\mu_{n-2} \\ \sum_{j=n-2}^{n-1}(\gamma_j+\mu_j) & & 1-\sum_{j=n-2}^{n-1}(\gamma_j+\mu_j)-\mu_{n-3} \\ \vdots & & \vdots \\ \sum_{j=2}^{n-1}(\gamma_j+\mu_j) & & 1-\sum_{j=2}^{n-1}(\gamma_j+\mu_j)-\mu_1 \\ \sum_{j=1}^{n-1}(\gamma_j+\mu_j) & \gamma'+\sum_{j=1}^{n-1}\mu_j & -\gamma'-\sum_{j=1}^{n-1}(\gamma_j+\mu_j) \end{Bmatrix}, \tag{13.17}$$

which is obtained by the induction on n with Theorem 5.2 and corresponds to the second Riemann scheme in (13.16) by putting

$$\begin{aligned} \gamma_j &= \alpha_{j+1}-\beta_j && (j=1,\dots,n-2), && \gamma'=-\alpha_1+\beta_1-1, \\ \mu_j &= -\alpha_{j+1}+\beta_{j+1} && (j=1,\dots,n-1), && \mu_{n-1}=1-\alpha_n. \end{aligned} \tag{13.18}$$

The integral representation of the local solutions at $x=0$ (resp. 1 and ∞) corresponding to the exponents $\sum_{j=1}^{n-1}(\gamma_j+\mu_j)$ (resp. $\gamma'+\sum_{j=1}^{n-1}\mu_j$ and $-\gamma'-\sum_{j=1}^{n-1}(\gamma_j+\mu_j)$) are given by

$$I_c^{\mu_{n-1}}x^{\gamma_{n-1}}I_c^{\mu_{n-2}}\cdots I_c^{\mu_1}x^{\gamma_1}(1-x)^{\gamma'} \tag{13.19}$$

by putting $c=0$ (resp. 1 and ∞).

For simplicity we express this construction using additions and middle convolutions by

$$u=\partial^{-\mu_{n-1}}x^{\gamma_{n-1}}\cdots\partial^{-\mu_2}x^{\gamma_2}\partial^{-\mu_2}x^{\gamma_1}(1-x)^{\gamma'}. \tag{13.20}$$

For example, when $n = 3$, we have the solution

$$\int_c^x t^{\alpha_3-\beta_2}(x-t)^{1-\alpha_3}dt\int_c^t s^{\alpha_2-\beta_1}(1-s)^{-\alpha_1+\beta_1-1}(t-s)^{-\alpha_2-\beta_2}ds.$$

The operator corresponding to the second Riemann scheme is

$$P_n(\alpha;\beta) := \prod_{j=1}^{n-1}(\vartheta-\beta_j)\cdot\partial - \prod_{j=1}^{n}(\vartheta-\alpha_j). \tag{13.21}$$

This is clear when $n = 1$. In general, we have

$$\begin{aligned}
&\mathrm{RAd}(\partial^{-\mu})\circ\mathrm{RAd}(x^\gamma)P_n(\alpha,\beta)\\
&=\mathrm{RAd}(\partial^{-\mu})\circ\mathrm{Ad}(x^\gamma)\Big(\prod_{j=1}^{n-1}x(\vartheta+\beta_j)\cdot\partial-\prod_{j=1}^{n}x(\vartheta+\alpha_j)\Big)\\
&=\mathrm{RAd}(\partial^{-\mu})\Big(\prod_{j=1}^{n-1}(\vartheta+\beta_j-1-\gamma)(\vartheta-\gamma)-\prod_{j=1}^{n}x(\vartheta+\alpha_j-\gamma)\Big)\\
&=\mathrm{Ad}(\partial^{-\mu})\Big(\prod_{j=1}^{n-1}(\vartheta+\beta_j-\gamma)\cdot(\vartheta-\gamma+1)\partial-\prod_{j=1}^{n}(\vartheta+1)(\vartheta+\alpha_j-\gamma)\Big)\\
&=\prod_{j=1}^{n-1}(\vartheta+\beta_j-\gamma-\mu)\cdot(\vartheta-\gamma-\mu+1)\partial-\prod_{j=1}^{n}(\vartheta+1-\mu)\cdot(\vartheta+\alpha_j-\gamma-\mu)
\end{aligned}$$

and therefore we have (13.21) by the correspondence of the Riemann schemes with $\gamma = \gamma_n$ and $\mu = \mu_n$.

Suppose $\lambda_{1,1} = 0$. We will show that

$$\begin{aligned}
&\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{n}(\lambda_{2,j}-\lambda_{0,n})_k}{\prod_{j=1}^{n-1}(\lambda_{0,n}-\lambda_{0,j}+1)_k k!}x^{\lambda_{0,n}+k}\\
&\qquad= x^{\lambda_{0,n}}{}_nF_{n-1}\big((\lambda_{2,j}-\lambda_{0,n})_{j=1,\dots,n},(\lambda_{0,n}-\lambda_{0,j}+1)_{j=1,\dots,n-1};x\big)
\end{aligned}\tag{13.22}$$

is the local solution at the origin corresponding to the exponent $\lambda_{0,n}$. Here

$${}_nF_{n-1}(\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_{n-1};x)=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\cdots(\alpha_{n-1})_k(\alpha_n)_k}{(\beta_1)_k\cdots(\beta_{n-1})_k k!}x^k. \tag{13.23}$$

We may assume $\lambda_{0,1} = 0$ for the proof of (13.22). When $n = 1$, the corresponding solution equals $(1-x)^{-\lambda_{2,1}}$ and we have (13.22). Note that

$$\begin{aligned}
&I_0^\mu x^\gamma\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{n}(\lambda_{2,j}-\lambda_{0,n})_k}{\prod_{j=1}^{n-1}(\lambda_{0,n}-\lambda_{0,j}+1)_k k!}x^{\lambda_{0,n}+k}\\
&=\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{n}(\lambda_{2,j}-\lambda_{0,n})_k}{\prod_{j=1}^{n-1}(\lambda_{0,n}-\lambda_{0,j}+1)_k k!}\frac{\Gamma(\lambda_{0,n}+\gamma+k+1)}{\Gamma(\lambda_{0,n}+\gamma+\mu+k+1)}x^{\lambda_{0,n}+\gamma+\mu+k}\\
&=\frac{\Gamma(\lambda_{0,n}+\gamma+1)}{\Gamma(\lambda_{0,n}+\gamma+\mu+1)}\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{n}(\lambda_{2,j}-\lambda_{0,n})_k\cdot(\lambda_{0,n}+\gamma+1)_k\cdot x^{\lambda_{0,n}+\gamma+\mu+k}}{\prod_{j=1}^{n-1}(\lambda_{0,n}-\lambda_{0,j}+1)_k\cdot(\lambda_{0,n}+\gamma+\mu+1)_k k!}.
\end{aligned}$$

Comparing (13.17) with the first Riemann scheme under $\lambda_{0,1} = \lambda_{1,1} = 0$ and $\gamma = \gamma_n$ and $\mu = \mu_n$, we have the solution (13.22) by the induction on n. The contiguity relation in Theorem 11.3 corresponds to the identity

$$\begin{aligned}&{}_nF_{n-1}(\alpha_1,\dots,\alpha_{n-1},\alpha_n+1;\beta_1,\dots,\beta_{n-1};x)\\ &\quad={}_nF_{n-1}(\alpha_1,\dots,\alpha_n;\beta_1,\dots,\beta_{n-1};x)\\ &\qquad+\frac{\alpha_1\cdots\alpha_{n-1}}{\beta_1\cdots\beta_{n-1}}x\cdot{}_nF_{n-1}(\alpha_1+1,\dots,\alpha_n+1;\beta_1+1,\dots,\beta_{n-1}+1;x).\end{aligned}\tag{13.24}$$

The series expansion of the local solution at $x=1$ corresponding to the exponent $\gamma'+\mu_1+\cdots+\mu_{n-1}$ is a little more complicated.

For the Riemann scheme

$$\begin{Bmatrix} x=\infty & 0 & 1\\ -\mu_2+1 & [0]_{(2)} & 0\\ 1-\gamma_2-\mu_1-\mu_2 & & \gamma_2+\mu_2\\ -\gamma'-\gamma_1-\gamma_2-\mu_1-\mu_2 & \underline{\gamma'+\mu_1+\mu_2} & \gamma_1+\gamma_2+\mu_1+\mu_2\end{Bmatrix},$$

we have the local solution at $x=0$

$$\begin{aligned}&I_0^{\mu_2}(1-x)^{\gamma_2}I_0^{\mu_1}x^{\gamma'}(1-x)^{\gamma_1}=I_0^{\mu_2}(1-x)^{\gamma_2}\sum_{n=0}^\infty\frac{(-\gamma_1)_n}{n!}x^n\\ &=I_0^{\mu_2}\sum_{n=0}^\infty\frac{\Gamma(\gamma'+1+n)(-\gamma_1)_n}{\Gamma(\gamma'+\mu_1+1+n)n!}x^{\gamma'+\mu_1+n}(1-x)^{\gamma_2}\\ &=I_0^{\mu_2}\sum_{m,n=0}^\infty\frac{\Gamma(\gamma'+1+n)(-\gamma_1)_n(-\gamma_2)_m}{\Gamma(\gamma'+\mu_1+1+n)m!n!}x^{\gamma'+\mu_1+m+n}\\ &=\sum_{m,n=0}^\infty\frac{\Gamma(\gamma'+\mu_1+1+m+n)\Gamma(\gamma'+1+n)(-\gamma_1)_n(-\gamma_2)_m x^{\gamma'+\mu_1+\mu_2+m+n}}{\Gamma(\gamma'+\mu_1+\mu_2+1+m+n)\Gamma(\gamma'+\mu_1+1+n)m!n!}\\ &=\frac{\Gamma(\gamma'+1)x^{\gamma'+\mu_1+\mu_2}}{\Gamma(\gamma'+\mu_1+\mu_2+1)}\sum_{m,n=0}^\infty\frac{(\gamma'+\mu_1+1)_{m+n}(\gamma'+1)_n(-\gamma_1)_n(-\gamma_2)_m x^{m+n}}{(\gamma'+\mu_1+\mu_2+1)_{m+n}(\gamma'+\mu_1+1)_n m!n!}.\end{aligned}$$

Applying the last equality in (3.8) to the above second equality, we have

$$\begin{aligned}&I_0^{\mu_2}(1-x)^{\gamma_2}I_0^{\mu_1}x^{\gamma'}(1-x)^{\gamma_1}\\ &=\sum_{n=0}^\infty\frac{\Gamma(\gamma'+1+n)(-\gamma_1)_n}{\Gamma(\gamma'+\mu_1+1+n)n!}x^{\gamma'+\mu_1+\mu_2+n}(1-x)^{-\gamma_2}\\ &\quad\cdot\sum_{m=0}^\infty\frac{\Gamma(\gamma'+\mu_1+1+n)}{\Gamma(\gamma'+\mu_1+\mu_2+1+n)}\frac{(\mu_2)_m(-\gamma_2)_m}{(\gamma'+\mu_1+n+\mu_2+1)_m m!}\Big(\frac{x}{x-1}\Big)^m\\ &=\frac{\Gamma(\gamma'+1)x^{\gamma'+\mu_1+\mu_2}(1-x)^{-\gamma_2}}{\Gamma(\gamma'+\mu_1+\mu_2+1)}\sum_{m,n=0}^\infty\frac{(\gamma'+1)_n(-\gamma_1)_n(-\gamma_2)_m(\mu_2)_m}{(\gamma'+\mu_1+\mu_2+1)_{m+n}m!n!}x^n\Big(\frac{x}{x-1}\Big)^m\\ &=\frac{\Gamma(\gamma'+1)}{\Gamma(\gamma'+\mu_1+\mu_2+1)}\\ &\quad\cdot x^{\gamma'+\mu_1+\mu_2}(1-x)^{-\gamma_2}F_3\big(-\gamma_2,-\gamma_1,\mu_2,\gamma'+1;\gamma'+\mu_1+\mu_2+1;x,\frac{x}{x-1}\big),\end{aligned}$$

where F_3 is Appell's hypergeometric function (13.53).

Let $u_1^{-\beta_n}(\alpha_1,\dots,\alpha_n;\beta_1,\dots,\beta_{n-1};x)$ be the local solution of $P_n(\alpha,\beta)u=0$ at $x=1$ such that $u_1^{-\beta_n}(\alpha;\beta;x)\equiv(x-1)^{-\beta_n}\mod (x-1)^{1-\beta_n}\mathcal{O}_1$ for generic α and

β. Since the reduction

$$\begin{Bmatrix} \lambda_{0,1} & [0]_{(n-1)} & \lambda_{2,1} \\ \vdots & & \vdots \\ \lambda_{0,n} & \lambda_{1,2} & \lambda_{2,n} \end{Bmatrix} \xrightarrow{\partial_{max}} \begin{Bmatrix} \lambda'_{0,1} & [0]_{(n-2)} & \lambda'_{2,1} \\ \vdots & & \vdots \\ \lambda'_{0,n-1} & \lambda'_{1,2} & \lambda'_{2,n-1} \end{Bmatrix}$$

satisfies $\lambda'_{1,2} = \lambda_{1,2} + \lambda_{0,1} + \lambda_{0,2} - 1$ and $\lambda'_{0,j} + \lambda'_{2,j} = \lambda_{0,j+1} + \lambda_{2,j+1}$ for $j = 1, \dots, n-1$, Theorem 11.3 proves

$$\begin{aligned} u_1^{-\beta_n}(\alpha;\beta;x) &= u_1^{-\beta_n}(\alpha_1,\dots,\alpha_n+1;\beta_1,\dots,\beta_{n-1}+1;x) \\ &\quad + \frac{\beta_{n-1}-\alpha_n}{1-\beta_n} u_1^{1-\beta_n}(\alpha;\beta_1,\dots,\beta_{n-1}+1;x). \end{aligned} \tag{13.25}$$

The condition for the irreducibility of the equation equals

$$\lambda_{0,\nu} + \lambda_{1,1} + \lambda_{2,\nu'} \notin \mathbb{Z} \qquad (1 \le \nu \le n,\ 1 \le \nu' \le n), \tag{13.26}$$

which is easily proved by the induction on n (cf. Example 10.17 ii)). The shift operator under a compatible shift $(\epsilon_{j,\nu})$ is bijective if and only if

$$\lambda_{0,\nu} + \lambda_{1,1} + \lambda_{2,\nu'} \ \text{ and } \ \lambda_{0,\nu} + \epsilon_{0,\nu} + \lambda_{1,1} + \epsilon_{1,1} + \lambda_{2,\nu'} + \epsilon_{2,\nu'} \tag{13.27}$$

are simultaneously not integers or positive integers or non-positive integers for each $\nu \in \{1,\dots,n\}$ and $\nu' \in \{1,\dots,n\}$.

Connection coefficients in this example are calculated by [**Le**] and [**OTY**] etc. In this paper we get them by Theorem 12.6.

There are the following direct decompositions ($\nu = 1, \dots, n$).

$$\begin{aligned} 1\dots1\overline{1}; n-1\underline{1}; 1\dots1 &= 0\dots0\overline{1};\ \ 1\ \ \ \underline{0}; 0\dots0\overset{\nu}{\smile}\!\!\!10\dots0 \\ &\oplus 1\dots1\overline{0}; n-2\underline{1}; 1\dots101\dots1. \end{aligned}$$

These n decompositions $\mathbf{m} = \mathbf{m}' \oplus \mathbf{m}''$ satisfy the condition $m'_{0,n_0} = m''_{1,n_1} = 1$ in (12.10), where $n_0 = n$ and $n_1 = 2$. Since $n_0 + n_1 - 2 = n$, Remark 12.8 i) shows that these decompositions give all the decompositions appearing in (12.10). Thus we have

$$\begin{aligned} c(\lambda_{0,n} \rightsquigarrow \lambda_{1,2}) &= \frac{\prod_{\nu=1}^{n-1} \Gamma(\lambda_{0,n} - \lambda_{0,\nu} + 1) \cdot \Gamma(\lambda_{1,1} - \lambda_{1,2})}{\prod_{\nu=1}^{n} \Gamma(\lambda_{0,n} + \lambda_{1,1} + \lambda_{2,\nu})} = \prod_{\nu=1}^{n} \frac{\Gamma(\beta_\nu)}{\Gamma(\alpha_\nu)} \\ &= \lim_{x\to 1-0} (1-x)^{\beta_n} {}_nF_{n-1}(\alpha,\beta;x) \qquad (\operatorname{Re}\beta_n > 0). \end{aligned}$$

Other connection coefficients are obtained by the similar way.

$c(\lambda_{0,n} \rightsquigarrow \lambda_{2,n})$: When $n = 3$, we have

$$\begin{aligned} 11\overline{1},21,11\underline{1} = {}& 00\overline{1},10,100 \quad 00\overline{1},10,010 \quad 10\overline{1},11,110 \quad 01\overline{1},11,110 \\ \oplus{}& 110,11,01\underline{1} = 110,11,10\underline{1} = 010,10,00\underline{1} = 100,10,00\underline{1} \end{aligned}$$

In general, by the rigid decompositions

$$\begin{aligned} 1\cdots1\overline{1}\,,\, n-11\,,\, 1\cdots1\underline{1} &= 0\cdots0\underline{\overline{1}}\,,\quad 1\ \ 0\,,\, 0\dots\overset{i}{\smile}\!\!\!010\cdots0\underline{0} \\ &\oplus 1\cdots1\overline{0}\,,\, n-21\,,\, 1\cdots101\cdots1\underline{\overline{1}} \\ &= 1\cdots\overset{i}{\smile}\!\!\!101\cdots1\underline{\overline{1}}\,,\, n-21\,,\, 1\cdots1\underline{0} \\ &\oplus 0\dots010\cdots0\overline{0}\,,\quad 1\ \ 0\,,\, 0\cdots0\underline{\overline{1}} \end{aligned}$$

for $i = 1, \dots, n-1$ we have

$$c(\lambda_{0,n} \rightsquigarrow \lambda_{2,n}) = \prod_{k=1}^{n-1} \frac{\Gamma(\lambda_{2,k} - \lambda_{2,n})}{\Gamma\bigl(\bigl|\{\lambda_{0,n} \quad \lambda_{1,1} \quad \lambda_{2,k}\}\bigr|\bigr)}$$
$$\cdot \prod_{k=1}^{n-1} \frac{\Gamma(\lambda_{0,n} - \lambda_{0,k} + 1)}{\Gamma\Bigl(\Bigl|\begin{Bmatrix} (\lambda_{0,\nu})_{\substack{1\le\nu\le n\\ \nu\neq k}} & [\lambda_{1,1}]_{(n-2)} & (\lambda_{2,\nu})_{1\le\nu\le n-1} \\ & \lambda_{1,2} & \end{Bmatrix}\Bigr|\Bigr)}$$
$$= \prod_{k=1}^{n-1} \frac{\Gamma(\beta_k)\Gamma(\alpha_k - \alpha_n)}{\Gamma(\alpha_k)\Gamma(\beta_k - \alpha_n)}.$$

Moreover we have

$$c(\lambda_{1,2} \rightsquigarrow \lambda_{0,n}) = \frac{\Gamma\bigl(\lambda_{1,2} - \lambda_{1,1} + 1\bigr) \cdot \prod_{\nu=1}^{n-1} \Gamma\bigl(\lambda_{0,\nu} - \lambda_{0,n}\bigr)}{\prod_{j=1}^{n} \Gamma\Bigl(\Bigl|\begin{Bmatrix} (\lambda_{0,\nu})_{1\le\nu\le n-1} & [\lambda_{1,1}]_{(n-2)} & (\lambda_{2,\nu})_{1\le\nu\le n,\ \nu\neq j} \\ & \lambda_{1,2} & \end{Bmatrix}\Bigr|\Bigr)}$$
$$= \prod_{\nu=1}^{n} \frac{\Gamma(1-\beta_\nu)}{\Gamma(1-\alpha_\nu)}.$$

Here we use the notation in Definition 4.12 and denote

$$(\mu_\nu)_{1\le\nu\le n} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{pmatrix} \in \mathbb{C}^n \quad \text{and} \quad (\mu_\nu)_{\substack{1\le\nu\le n\\ \nu\neq i}} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_{i-1} \\ \mu_{i+1} \\ \vdots \\ \mu_n \end{pmatrix} \in \mathbb{C}^{n-1}$$

for complex numbers $\mu_1, \dots, \mu_n$.

We have

$$\begin{aligned} {}_nF_{n-1}(\alpha,\beta;x) &= \sum_{k=0}^{\infty} C_k(1-x)^k + \sum_{k=0}^{\infty} C'_k(1-x)^{k-\beta_n}, \\ C_0 &= {}_nF_{n-1}(\alpha,\beta;1) \quad (\operatorname{Re}\beta_n < 0), \\ C'_0 &= \prod_{\nu=1}^{n} \frac{\Gamma(\beta_\nu)}{\Gamma(\alpha_\nu)} \end{aligned} \tag{13.28}$$

for $0 < x < 1$ if α and β are generic. Since

$$\frac{d^k}{dx^k} {}_nF_{n-1}(\alpha,\beta;x)$$
$$= \frac{(\alpha_1)_k \cdots (\alpha_n)_k}{(\beta_1)_k \cdots (\beta_{n-1})_k} {}_nF_{n-1}(\alpha_1+k,\dots,\alpha_n+k,\beta_1+k,\dots,\beta_{n-1}+k;x),$$

we have

$$C_k = \frac{(\alpha_1)_k \cdots (\alpha_n)_k}{(\beta_1)_k \cdots (\beta_{n-1})_k k!} {}_nF_{n-1}(\alpha_1+k,\dots,\alpha_n+k,\beta_1+k,\dots,\beta_{n-1}+k;1). \tag{13.29}$$

We examine the monodromy generators for the solutions of the generalized hypergeometric equation. For simplicity we assume $\beta_i \notin \mathbb{Z}$ and $\beta_i - \beta_j \notin \mathbb{Z}$ for $i \neq j$. Then $u = (u_0^{\lambda_{0,1}}, \dots, u_0^{\lambda_{0,n}})$ is a base of local solution at 0 and the corresponding monodromy generator around 0 with respect to this base equals

$$M_0 = \begin{pmatrix} e^{2\pi\sqrt{-1}\lambda_{0,1}} & & \\ & \ddots & \\ & & e^{2\pi\sqrt{-1}\lambda_{0,n}} \end{pmatrix}$$

and that around ∞ equals

$$\begin{aligned} M_\infty &= \Bigl(\sum_{k=1}^n e^{2\pi\sqrt{-1}\lambda_{2,\nu}} c(\lambda_{0,i} \rightsquigarrow \lambda_{2,k}) c(\lambda_{2,k} \rightsquigarrow \lambda_{k,j})\Bigr)_{\substack{1\le i\le n\\ 1\le j\le n}} \\ &= \Bigl(\sum_{k=1}^n e^{2\pi\sqrt{-1}\lambda_{2,\nu}} \prod_{\nu\in\{1,\dots,n\}\setminus\{k\}} \frac{\sin 2\pi(\lambda_{0,i}+\lambda_{1,1}+\lambda_{2,\nu})}{\sin 2\pi(\lambda_{0,k}-\lambda_{0,\nu})} \\ &\qquad \cdot \prod_{\nu\in\{1,\dots,n\}\setminus\{j\}} \frac{\sin 2\pi(\lambda_{0,i}+\lambda_{1,1}+\lambda_{2,\nu})}{\sin 2\pi(\lambda_{2,j}-\lambda_{2,\nu})}\Bigr)_{\substack{1\le i\le n\\ 1\le j\le n}}. \end{aligned}$$

Lastly we remark that the versal generalized hypergeometric operator is

$$\begin{aligned} \tilde P &= \mathrm{RAd}(\partial^{-\mu_{n-1}}) \circ \mathrm{RAd}\bigl((1-c_1x)^{\frac{\gamma_{n-1}}{c_1}}\bigr) \circ \cdots \circ \mathrm{RAd}(\partial^{-\mu_1}) \\ &\quad \circ \mathrm{RAd}\Bigl((1-c_1x)^{\frac{\gamma_1}{c_1}+\frac{\gamma'}{c_1(c_1-c_2)}}(1-c_2x)^{\frac{\gamma'}{c_2(c_2-c_1)}}\Bigr)\partial \\ &= \mathrm{RAd}(\partial^{-\mu_{n-1}}) \circ \mathrm{RAdei}\bigl(\frac{\gamma_{n-1}}{1-c_1x}\bigr) \circ \cdots \circ \mathrm{RAd}(\partial^{-\mu_1}) \\ &\quad \circ \mathrm{RAdei}\Bigl(\frac{\gamma_1}{1-c_1x} + \frac{\gamma' x}{(1-c_1x)(1-c_2x)}\Bigr)\partial \end{aligned}$$

and when $n = 3$, we have the integral representation of the solutions

$$\int_c^x\int_c^t \exp\Bigl(-\int_c^s \frac{\gamma_1(1-c_2u)+\gamma' u}{(1-c_1u)(1-c_2u)}du\Bigr)(t-s)^{\mu_1-1}\bigl(1-c_1t\bigr)^{\frac{\gamma_2}{c_1}}(x-t)^{\mu_2-1}ds\,dt.$$

Here c equals $\frac{1}{c_1}$ or $\frac{1}{c_2}$ or ∞.

13.5. Even/Odd family

The system of differential equations of Schlesinger canonical form belonging to an even or odd family EO_n is concretely given by [**Gl**]. We will examine concrete connection coefficients of solutions of the single differential equation belonging to an even or odd family. The corresponding tuples of partitions and their reductions and decompositions are as follows.

$$\begin{aligned} m+1m, m^21, 1^{2m+1} &= 10, 10, 1 \oplus m^2, mm-11, 1^{2m} \\ &= 1^2, 1^20, 1^2 \oplus mm-1, (m-2)^21, 1^{2m-1} \\ m^2, mm-11, 1^{2m} &= 1, 100, 1 \oplus mm-1, (m-1)^21, 1^{2m-1} \\ &= 1^2, 110, 1^2 \oplus (m-1)^2, m-1m-21, 1^{2m-2} \end{aligned}$$

$$\begin{aligned} EO_n &= H_1 \oplus EO_{n-1} : 2n = H_2 \oplus EO_{n-2} : \binom{n}{2} \\ [\Delta(\mathbf{m})] &= 1^{\binom{n}{2}+2n} \\ EO_n &\xrightarrow[R1E0R0E0]{1} EO_{n-1} \\ EO_2 &= H_2, \quad EO_3 = H_3 \end{aligned}$$

The following operators are shift operators of the universal model $P_{EO_n}(\lambda)u = 0$:

$$P^2_{H_1}(\lambda),\ P^1_{EO_{n-1}}(\lambda),\ P^2_{EO_{n-1}}(\lambda),\ P^2_{H_2}(\lambda),\ P^1_{EO_{n-2}}(\lambda),\ P^2_{EO_{n-2}}(\lambda).$$

EO_{2m} ($\mathbf{m} = (1^{2m}, mm-11, mm)$: even family)

$$\begin{Bmatrix} x=\infty & 0 & 1 \\ \lambda_{0,1} & [\lambda_{1,1}]_{(m)} & [\lambda_{2,1}]_{(m)} \\ \vdots & [\lambda_{1,2}]_{(m-1)} & [\lambda_{2,2}]_{(m)} \\ \lambda_{0,2m} & \lambda_{1,3} & \end{Bmatrix},$$

$$\sum_{\nu=1}^{2m} \lambda_{0,\nu} + m(\lambda_{1,1}+\lambda_{2,1}+\lambda_{2,2}) + (m-1)\lambda_{1,2} + \lambda_{1,3} = 2m-1.$$

The rigid decompositions

$$\begin{aligned} &1\cdots 1\overline{1}\,,\ mm-1\underline{1}\,,\ mm \\ &= 0\cdots 0\overline{1}\,,\ 10\underline{0}\,,\ \overset{i}{\smile}{10} \oplus 1\cdots 1\overline{0}\,,\ m-1m-1\underline{1}\,,\ \overset{i}{\smile}{01} \\ &= 0\cdots \overset{j}{\smile}{1}\overline{1}\,,\ 11\underline{0}\,,\ 11 \oplus 1\cdots \overset{j}{\smile}{0}\overline{0}\,,\ m-1m-2\underline{1}\,,\ m-1m-1, \end{aligned}$$

which are expressed by $EO_{2m} = H_1 \oplus EO_{2m-1} = H_2 \oplus EO_{2m-2}$, give

$$c(\lambda_{0,2m} \rightsquigarrow \lambda_{1,3}) = \prod_{i=1}^{2} \frac{\Gamma(\lambda_{1,i}-\lambda_{1,3})}{\Gamma\bigl(\bigl|\{\lambda_{0,2m} \quad \lambda_{1,1} \quad \lambda_{2,i}\}\bigr|\bigr)} \cdot \prod_{j=1}^{2m-1} \frac{\Gamma(\lambda_{0,2m}-\lambda_{0,j}+1)}{\Gamma\Bigl(\Bigl|\begin{Bmatrix} \lambda_{0,j} & \lambda_{1,1} & \lambda_{2,1} \\ \lambda_{0,2m} & \lambda_{1,2} & \lambda_{2,2} \end{Bmatrix}\Bigr|\Bigr)},$$

$$\begin{aligned} c(\lambda_{1,3} \rightsquigarrow \lambda_{0,2m}) = &\prod_{i=1}^{2} \frac{\Gamma(\lambda_{1,3}-\lambda_{1,i}+1)}{\Gamma\Biggl(\Biggl|\begin{Bmatrix} & [\lambda_{1,1}]_{(m-1)} & [\lambda_{2,\nu}]_{(m)} \\ (\lambda_{0,\nu})_{1\le\nu\le 2m-1} & [\lambda_{1,2}]_{(m-1)} & [\lambda_{2,3-i}]_{(m-1)} \\ & \lambda_{1,3} & \end{Bmatrix}\Biggr|\Biggr)} \\ &\cdot \prod_{j=1}^{2m-1} \frac{\Gamma(\lambda_{0,j}-\lambda_{0,2m})}{\Gamma\Biggl(\Biggl|\begin{Bmatrix} & [\lambda_{1,1}]_{(m-1)} & [\lambda_{2,1}]_{(m-1)} \\ (\lambda_{0,\nu})_{\substack{1\le\nu\le 2m-1 \\ \nu\ne j}} & [\lambda_{1,2}]_{(m-2)} & [\lambda_{2,2}]_{(m-1)} \\ & \lambda_{1,3} & \end{Bmatrix}\Biggr|\Biggr)}. \end{aligned}$$

These formulas were obtained by the author in 2007 (cf. [**O6**]), which is a main motivation for the study in this paper. The condition for the irreducibility is

$$\begin{cases} \lambda_{0,\nu}+\lambda_{1,1}+\lambda_{2,k} \notin \mathbb{Z} & (1\le\nu\le 2m,\ k=1,2), \\ \lambda_{0,\nu}+\lambda_{0,\nu'}+\lambda_{1,1}+\lambda_{1,2}+\lambda_{2,1}+\lambda_{2,2}-1 \notin \mathbb{Z} & (1\le\nu<\nu'\le 2m,\ k=1,2). \end{cases}$$

The shift operator for a compatible shift $(\epsilon_{j,\mu})$ is bijective if and only if the values of each linear function in the above satisfy (11.30).

For the Fuchsian equation $\tilde{P}u = 0$ of type EO_4 with the Riemann scheme

$$\begin{Bmatrix} x=\infty & 0 & 1 & \\ [a_1]_{(2)} & b_1 & [0]_{(2)} & ;x \\ [a_2]_{(2)} & b_2 & c_1 & \\ & b_3 & c_2 & \\ & 0 & & \end{Bmatrix} \tag{13.30}$$

and the Fuchs relation

$$2a_1+2a_2+b_1+b_2+b_3+c_1+c_2 = 3 \tag{13.31}$$

we have the connection formula

$$c(0\!:\!0 \rightsquigarrow 1\!:\!c_2) = \frac{\Gamma(c_1-c_2)\Gamma(-c_2)\prod_{\nu=1}^{3}\Gamma(1-b_\nu)}{\Gamma(a_1)\Gamma(a_2)\prod_{\nu=1}^{3}\Gamma(a_1+a_2+b_\nu+c_1-1)}. \tag{13.32}$$

Let $\tilde{Q}$ be the Gauss hypergeometric operator with the Riemann scheme

$$\begin{Bmatrix} x=\infty & 0 & 1 \\ a_1 & 1-a_1-a_2-c_1 & 0 \\ a_2 & 0 & c_1 \end{Bmatrix}.$$

We may normalize the operators by

$$\tilde{P} = x^3(1-x)\partial^4 + \cdots \quad \text{and} \quad \tilde{Q} = x(1-x)\partial^2 + \cdots .$$

Then

$$\tilde{P} = \tilde{S}\tilde{Q} - \prod_{\nu=1}^{3}(a_1+a_2+b_\nu+c_1-1)\cdot\partial$$
$$\tilde{Q} = \bigl(x(1-x)\partial + (a_1+a_2+c_1-(a_1+a_2+1)x)\bigr)\partial - a_1a_2$$

with a suitable $\tilde{S}$, $\tilde{T} \in W[x]$ and $e \in \mathbb{C}$ and as is mentioned in Theorem 11.7, $\tilde{Q}$ is a shift operator satisfying

$$(13.33)\qquad \begin{Bmatrix} x=\infty & 0 & 1 & \\ [a_1]_{(2)} & b_1 & [0]_{(2)} & ;x \\ [a_2]_{(2)} & b_2 & c_1 & \\ & b_3 & c_2 & \\ & 0 & & \end{Bmatrix} \xrightarrow{\tilde{Q}} \begin{Bmatrix} x=\infty & 0 & 1 & \\ [a_1+1]_{(2)} & b_1-1 & [0]_{(2)} & ;x \\ [a_2+1]_{(2)} & b_2-1 & c_1 & \\ & b_3-1 & c_2-1 & \\ & 0 & & \end{Bmatrix}.$$

Let $u_0^0 = 1+\cdots$ and $u_1^{c_2} = (1-x)^{c_2}+\cdots$ be the normalized local solutions of $\tilde{P}u = 0$ corresponding to the characteristic exponents 0 at 0 and c_2 at 1, respectively. Then the direct calculation shows

$$\tilde{Q}u_0^0 = \frac{a_1a_2\prod_{\nu=1}^{3}(a_1+a_2+b_\nu+c_1-1)}{\prod_{\nu=1}^{3}(1-b_\nu)} + \cdots ,$$
$$\tilde{Q}u_1^{c_2} = c_2(c_2-c_1)(1-x)^{c_2-1} + \cdots .$$

Denoting by $c(a_1,a_2,b_1,b_2,b_3,c_1,c_2)$ the connection coefficient $c(0:0 \rightsquigarrow 1:c_2)$ for the equation with the Riemann scheme (13.30), we have

$$\frac{c(a_1,a_2,b_1,b_2,b_3,c_1,c_2)}{c(a_1+1,a_2+1,b_1-1,b_2-1,b_3-1,c_1,c_2-1)} = \frac{a_1a_2\prod_{\nu=1}^{3}(a_1+a_2+b_\nu+c_1-1)}{(c_1-c_2)(-c_2)\prod_{\nu=1}^{3}(1-b_\nu)},$$

which proves (13.32) since $\lim_{k\to\infty} c(a_1+k,a_2+k,b_1-k,b_2-k,b_3-k,c_1,c_2-k) = 1$. Note that the shift operator (13.33) is not bijective if and only if the equation

$$\tilde{Q}u = \prod_{\nu=1}^{3}(a_1+a_2+b_\nu+c_1-1)\cdot\partial u = 0$$

has a non-zero solution, which is equivalent to

$$a_1a_2\prod_{\nu=1}^{3}(a_1+a_2+b_\nu+c_1-1) = 0.$$

In fact, there is a shift operator

$$\tilde{R} = x^3(1-x)^2\partial^3 - x^2(1-x)\bigl(2a_1+2a_2+7)x+b_1+b_2+b_3-6\bigr)\partial^2 + \cdots \in W[x]$$

so that

$$\begin{aligned}\tilde{R}\tilde{Q} &= \big(x(1-x)\partial - (a_1+a_2+1)x + (a_1+a_2+c_1)\big)\tilde{P}\\ &\quad + a_1a_2\prod_{\nu=1}^{3}(a_1+a_2+b_\nu+c_1-1).\end{aligned}$$

By the transformation $x \mapsto \frac{x}{x-1}$ we have

$$\begin{Bmatrix} x=\infty & 0 & 1\\ [0]_{(2)} & 0 & [a_1]_{(2)}\\ c_1 & b_1 & [a_2]_{(2)}\\ c_2 & b_2 & \\ & b_3 & \end{Bmatrix}$$

$$\xrightarrow{(1-x)^{a_1}\partial^{1-a_1}(1-x)^{-a_1}} \begin{Bmatrix} x=\infty & 0 & 1\\ 2-2a_1 & & a_1\\ 1+c_1-a_1 & a_1+b_1-1 & [a_1+a_2-1]_{(2)}\\ 1+c_2-a_1 & a_1+b_2-1 & \\ & a_1+b_3-1 & \end{Bmatrix}$$

$$\xrightarrow{x^{1-a_1-b_1}(1-x)^{1-a_1-a_2}} \begin{Bmatrix} x=\infty & 0 & 1\\ a_2+b_1 & & 1-a_2\\ a_1+a_2+b_1+c_1-1 & 0 & [0]_{(2)}\\ a_1+a_2+b_1+c_2-1 & b_2-b_1 & \\ & b_3-b_1 & \end{Bmatrix}$$

and therefore Theorem 12.4 gives the following connection formula for (13.30):

$$\begin{aligned}c(0\!:\!b_1 \rightsquigarrow \infty\!:\!a_2) = \frac{\Gamma(b_1+1)\Gamma(a_1-a_2)}{\Gamma(a_1+b_1)\Gamma(1-a_2)} \cdot {}_3F_2(&a_2+b_1, a_1+a_2+b_1+c_1-1,\\ &a_1+a_2+b_1+c_2-1; b_1-b_2-1, b_1-b_3-1; 1).\end{aligned}$$

In the same way, we have

$$\begin{aligned}c(1\!:\!c_1 \rightsquigarrow \infty\!:\!a_2) = \frac{\Gamma(c_1+1)\Gamma(a_1-a_2)}{\Gamma(a_1+c_1)\Gamma(1-a_2)} \cdot {}_3F_2(&b_1-c_1, b_2-c_1, b_3-c_1;\\ &a_1+c_1, c_1-c_2+1; 1).\end{aligned}$$

REMARK 13.1. When the parameters are generic under the condition

$$1-a_1-a_2-b_1-c_1 \in \mathbb{Z}_{\geq 0}, \tag{13.34}$$

$\tilde{P}u=0$ has a solution such that its monodromy group is isomorphic to the solution of the hypergeometric equation $\tilde{Q}u=0$ and it has $1-a_1-a_2-b_1-c_1$ apparent singular points. This solution is constructed by a successive applications of the shift operators $\tilde{R}$ to Gauss hypergeometric function. This can be considered as a generalization of Proposition 11.15.

We will calculate generalized connection coefficients defined in Definition 12.17. In fact, we get

$$c(1\!:\![0]_{(2)} \rightsquigarrow \infty\!:\![a_2]_{(2)}) = \frac{\prod_{\nu=1}^{2}\Gamma(2-c_\nu)\cdot\prod_{i=1}^{2}\Gamma(a_1-a_2+i)}{\Gamma(a_1)\prod_{\nu=1}^{3}\Gamma(a_1+b_\nu)}, \tag{13.35}$$

$$c(\infty\!:\![a_2]_{(2)} \rightsquigarrow 1\!:\![0]_{(2)}) = \frac{\prod_{\nu=1}^{2}\Gamma(c_\nu-1)\cdot\prod_{i=0}^{1}\Gamma(a_2-a_1-i)}{\Gamma(1-a_1)\prod_{\nu=1}^{3}\Gamma(1-a_1-b_\nu)} \tag{13.36}$$

according to the procedure given in Remark 12.19, which we will explain.

The differential equation with the Riemann scheme $\left\{\begin{matrix} x=\infty & 0 & 1 \\ \alpha_1 & [0]_{(2)} & [0]_{(2)} \\ \alpha_2 & [\beta]_{(2)} & \gamma_1 \\ \alpha_3 & & \gamma_2 \\ \alpha_4 & & \end{matrix}\right\}$

is $Pu=0$ with

$$\begin{aligned} P = \prod_{j=1}^{4}\bigl(\vartheta+\alpha_j\bigr) + \partial\bigl(\vartheta-\beta\bigr)\bigl((\partial-2\vartheta+\gamma_1+\gamma_2-1)(\vartheta-\beta) \\ + \sum_{1\le i<j\le 3}\alpha_i\alpha_j-(\beta-2\gamma_1-2\gamma_2-4)(\beta-1)-\gamma_1\gamma_2+1\bigr). \end{aligned} \tag{13.37}$$

The equation $Pu=0$ is isomorphic to the system

$$\begin{gathered} \frac{d\tilde u}{dx} = \frac{A}{x}\tilde u + \frac{B}{x-1}\tilde u, \\ A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c \end{pmatrix},\ B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ s & 1 & a & 0 \\ r & t & 0 & b \end{pmatrix},\ \tilde u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \end{gathered} \tag{13.38}$$

by the correspondence

$$\begin{cases} u_1 = u, \\ u_2 = (x-1)xu'' + \bigl((1-a-c)x+a-1\bigr)u' - su, \\ u_3 = xu', \\ u_4 = x^2(x-1)u''' + \bigl((3-a-c)x^2+(a-2)x\bigr)u'' + (1-a-c-s)xu', \end{cases}$$

where we may assume $\operatorname{Re}\gamma_1 \ge \operatorname{Re}\gamma_2$ and

$$\begin{aligned} &\beta = c,\ \gamma_1 = a+1,\ \gamma_2 = b+2, \\ &\prod_{\nu=1}^{4}(\xi-\alpha_\nu) = \xi^4 + (a+b+2c)\xi^3 + \bigl((a+c)(b+c)-s-t\bigr)\xi^2 \\ &\qquad - \bigl((b+c)s+(a+c)t\bigr)\xi + st - r. \end{aligned}$$

Here s, t and r are uniquely determined from $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta,\gamma_1,\gamma_2$ because $b+c\ne a+c$. We remark that $\operatorname{Ad}(x^{-c})\tilde u$ satisfies a system of Okubo normal form.

Note that the shift of parameters $(\alpha_1,\dots,\alpha_4,\beta,\gamma_1,\gamma_2)\mapsto(\alpha_1,\dots,\alpha_4,\beta-1,\gamma_1+1,\gamma_2+1)$ corresponds to the shift $(a,b,c,s,t,r)\mapsto(a+1,b+1,c-1,s,t,r)$.

Let $u^j_{\alpha_1,\dots,\alpha_4,\beta,\gamma_1,\gamma_2}(x)$ be local holomorphic solutions of $Pu=0$ in a neighborhood of $x=0$ determined by

$$\begin{aligned} u^j_{\alpha_1,\dots,\alpha_4,\beta,\gamma_1,\gamma_2}(0) &= \delta_{j,0}, \\ \bigl(\tfrac{d}{dx}u^j_{\alpha_1,\dots,\alpha_4,\beta,\gamma_1,\gamma_2}\bigr)(0) &= \delta_{j,1} \end{aligned}$$

for $j=0$ and 1. Then Theorem 12.10 proves

$$\lim_{k\to\infty}\tfrac{d^\nu}{dx^\nu}u^0_{\alpha,\beta-k,\gamma_1+k,\gamma_1+k}(x) = \delta_{0,\nu}\quad(\nu=0,1,2,\dots)$$

uniformly on $\overline{D}=\{x\in\mathbb{C}\,;\,|x|\le 1\}$.

Put $u = v_{\alpha,\beta,\gamma_1,\gamma_2} = (\gamma_1-2)^{-1}u^1_{\alpha,\beta,\gamma}$. Then Theorem 12.10 proves

$$\begin{aligned} &\lim_{k\to\infty}\tfrac{d^\nu}{dx^\nu}v_{\alpha,\beta-k,\gamma_1+k,\gamma_2+k}(x) = 0\quad(\nu=0,1,2,\dots), \\ &\lim_{k\to\infty}\Bigl((x-1)x\tfrac{d^2}{dx^2} + \bigl((2-\beta-\gamma_1)x+\gamma_1+k-2\bigr)\tfrac{d}{dx} - s\Bigr)v_{\alpha,\beta-k,\gamma_1+k,\gamma_2+k}(x) = 1 \end{aligned}$$

uniformly on $\overline{D}$. Hence

$$\lim_{k\to\infty} \tfrac{d}{dx} u^1_{\alpha,\beta-k,\gamma+1,\gamma_1+k}(x) = 1$$

uniformly on $\overline{D}$. Thus we obtain

$$\lim_{k\to\infty} c(\infty\colon [a_2]_{(2)} \rightsquigarrow 1\colon [0]_{(2)})|_{a_1\mapsto a_1-k,\ c_1\mapsto c_1+k,\ c_2\mapsto c_2+k} = 1$$

for the connection coefficient in (13.36). Then the procedure given in Remark 12.19 and Corollary 12.22 with the rigid decompositions

$$\begin{aligned}2\underline{2}, 1111, \overline{2}11 &= 1\underline{2}, 0111, \overline{1}11 \oplus 1\underline{0}, 1000, 100 = 1\underline{2}, 1011, \overline{1}11 \oplus 1\underline{0}, 0100, \overline{1}00\\ &= 1\underline{2}, 1101, \overline{1}11 \oplus 1\underline{0}, 0010, \overline{1}00 = 1\underline{2}, 1101, \overline{1}11 \oplus 1\underline{0}, 0010, \overline{1}00\end{aligned}$$

prove (13.36). Corresponding to Remark 12.19 (4), we note

$$\sum_{\nu=1}^{2}(c_\nu - 1) + \sum_{i=0}^{1}(a_2 - a_1 - i) = (1 - a_1) + \sum_{\nu=1}^{3}(1 - a_1 - b_\nu)$$

because of the Fuchs relation (13.31). We can similarly obtain (13.35).

The holomorphic solution of $\tilde{P}u = 0$ at the origin is given by

$$u_0(x) = \sum_{m\ge 0,\ n\ge 0} \frac{(a_1 + a_2 + b_3 + c_2 - 1)_n \prod_{\nu=1}^{2}\big((a_\nu)_{m+n}(a_1 + a_2 + b_\nu + c_1 - 1)_m\big)}{(1 - b_1)_{m+n}(1 - b_2)_{m+n}(1 - b_3)_m m!n!} x^{m+n}$$

and it has the integral representation

$$\begin{aligned}u_0(x) = {}& \frac{\prod_{\nu=1}^{3}\Gamma(1 - b_\nu)}{\prod_{\nu=1}^{2}\big(\Gamma(a_\nu)\Gamma(1 - a_\nu - b_\nu)\Gamma(b_\nu + c_\nu + a_1 + a_2 - 1)\big)}\\ & \int_0^x \int_0^{s_0} \int_0^{s_1} x^{b_1}(x - s_0)^{-b_1-a_1} s_0^{b_2+a_1-1}(s_0 - s_1)^{-b_2-a_2}\\ & \cdot s_1^{b_3+a_2-1}(1 - s_1)^{-b_3-c_1-a_2-a_1+1}(s_1 - s_2)^{c_1+b_1+a_2+a_1-2}\\ & \cdot s_2^{b_2+c_2+a_2+a_1-2}(1 - s_2)^{-c_2-b_1-a_2-a_1+1} ds_2 ds_1 ds_0.\end{aligned}$$

The equation is irreducible if and only if any value of the following linear functions is not an integer.

$$\begin{array}{llllll} a_1 & a_2 \\ a_1 + b_1 & a_1 + b_2 & a_1 + b_3 & a_2 + b_1 & a_2 + b_2 & a_2 + b_3 \end{array}$$

$$\begin{array}{lll} a_1 + a_2 + b_1 + c_1 - 1 & a_1 + a_2 + b_1 + c_2 - 1 & a_1 + a_2 + b_2 + c_1 - 1\\ a_1 + a_2 + b_2 + c_2 - 1 & a_1 + a_2 + b_3 + c_1 - 1 & a_1 + a_2 + b_2 + c_2 - 1. \end{array}$$

In the same way we have the connection coefficients for odd family.

EO_{2m+1} ($\mathbf{m} = (1^{2m+1}, mm1, m + 1m)$: odd family)

$$\begin{Bmatrix} x = \infty & 0 & 1\\ \lambda_{0,1} & [\lambda_{1,1}]_{(m)} & [\lambda_{2,1}]_{(m+1)}\\ \vdots & [\lambda_{1,2}]_{(m)} & [\lambda_{2,2}]_{(m)}\\ \lambda_{0,2m+1} & \lambda_{1,3} & \end{Bmatrix}$$

$\sum_{\nu=1}^{2m+1}\lambda_{0,\nu}+m(\lambda_{1,1}+\lambda_{1,2}+\lambda_{2,2})+(m+1)\lambda_{2,1}+\lambda_{1,3}=2m.$

$$c(\lambda_{0,2m+1}\rightsquigarrow\lambda_{1,3})=\prod_{k=1}^{2}\frac{\Gamma\bigl(\lambda_{1,k}-\lambda_{1,3}\bigr)}{\Gamma\bigl(\bigl|\bigl\{\lambda_{0,2m+1}\quad\lambda_{1,k}\quad\lambda_{2,1}\bigr\}\bigr|\bigr)}\cdot\prod_{k=1}^{2m}\frac{\Gamma\bigl(\lambda_{0,2m+1}-\lambda_{0,k}+1\bigr)}{\Gamma\left(\left|\begin{Bmatrix}\lambda_{0,k} & \lambda_{1,1} & \lambda_{2,1}\\ \lambda_{0,2m+1} & \lambda_{1,2} & \lambda_{2,2}\end{Bmatrix}\right|\right)},$$

$$c(\lambda_{1,3}\rightsquigarrow\lambda_{0,2m+1})=\prod_{k=1}^{2}\frac{\Gamma\bigl(\lambda_{1,3}-\lambda_{1,k}+1\bigr)}{\Gamma\left(\left|\begin{Bmatrix} & [\lambda_{1,k}]_{(m)} & [\lambda_{2,1}]_{(m)}\\ (\lambda_{0,\nu})_{1\le\nu\le 2m} & [\lambda_{1,3-k}]_{(m-1)} & [\lambda_{2,2}]_{(m)}\\ & \lambda_{1,3} & \end{Bmatrix}\right|\right)}\cdot\prod_{k=1}^{2m}\frac{\Gamma\bigl(\lambda_{0,k}-\lambda_{0,2m+1}\bigr)}{\Gamma\left(\left|\begin{Bmatrix} & [\lambda_{1,1}]_{(m-1)} & [\lambda_{2,1}]_{(m)}\\ (\lambda_{0,\nu})_{\substack{1\le\nu\le 2m\\ \nu\ne k}} & [\lambda_{1,2}]_{(m-1)} & [\lambda_{2,2}]_{(m-1)}\\ & \lambda_{1,3} & \end{Bmatrix}\right|\right)}.$$

The condition for the irreducibility is

$$\begin{cases}\lambda_{0,\nu}+\lambda_{1,k}+\lambda_{2,1}\notin\mathbb{Z} & (1\le\nu\le 2m+1,\ k=1,2),\\ \lambda_{0,\nu}+\lambda_{0,\nu'}+\lambda_{1,1}+\lambda_{1,2}+\lambda_{2,1}+\lambda_{2,2}-1\notin\mathbb{Z} & (1\le\nu<\nu'\le 2m+1,\ k=1,2).\end{cases}$$

The same statement using the above linear functions as in the case of even family is valid for the bijectivity of the shift operator with respect to compatible shift $(\epsilon_{j,\nu})$.

We note that the operation $\mathrm{RAd}(\partial^{-\mu})\circ\mathrm{RAd}\bigl(x^{-\lambda_{1,2}}(1-x)^{-\lambda_{2,2}}\bigr)$ transforms the operator and solutions with the above Riemann scheme of type EO_n into those of type EO_{n+1}:

$$\begin{Bmatrix}\lambda_{0,1} & [\lambda_{1,1}]_{([\frac{n}{2}])} & [\lambda_{2,1}]_{([\frac{n+1}{2}])}\\ \vdots & [\lambda_{1,2}]_{([\frac{n-1}{2}])} & [\lambda_{2,2}]_{([\frac{n}{2}])}\\ \lambda_{0,n} & \lambda_{1,3} & \end{Bmatrix}$$

$$\xrightarrow{x^{-\lambda_{1,2}}(1-x)^{-\lambda_{2,2}}}\begin{Bmatrix}\lambda_{0,1}+\lambda_{1,2}+\lambda_{2,2} & [\lambda_{1,1}-\lambda_{1,2}]_{([\frac{n}{2}])} & [\lambda_{2,1}-\lambda_{2,2}]_{([\frac{n+1}{2}])}\\ \vdots & [0]_{([\frac{n-1}{2}])} & [0]_{([\frac{n}{2}])}\\ \lambda_{0,n}+\lambda_{1,2}+\lambda_{2,2} & \lambda_{1,3}-\lambda_{1,2} & \end{Bmatrix}$$

$$\xrightarrow{\partial^{-\mu}}\begin{Bmatrix}\lambda_{0,1}+\lambda_{1,2}+\lambda_{2,2}-\mu & [\lambda_{1,1}-\lambda_{1,2}+\mu]_{([\frac{n}{2}])} & [\lambda_{2,1}-\lambda_{2,2}+\mu]_{([\frac{n+1}{2}])}\\ \vdots & [\mu]_{([\frac{n+1}{2}])} & [\mu]_{([\frac{n+2}{2}])}\\ \lambda_{0,n}+\lambda_{1,2}+\lambda_{2,2}-\mu & \lambda_{1,3}-\lambda_{1,2}+\mu & \\ 1-\mu & & \end{Bmatrix}.$$

13.6. Trigonometric identities

The connection coefficients corresponding to the Riemann scheme of the hypergeometric family in §13.4 satisfy

$$\sum_{\nu=1}^{n}c(1:\lambda_{1,2}\rightsquigarrow 0:\lambda_{0,\nu})\cdot c(0:\lambda_{1,\nu}\rightsquigarrow 1:\lambda_{1,2})=1,$$

$$\sum_{\nu=1}^{n}c(\infty:\lambda_{2,i}\rightsquigarrow 0:\lambda_{0,\nu})\cdot c(0:\lambda_{0,\nu}\rightsquigarrow\infty:\lambda_{2,j})=\delta_{ij}.$$

These equations with Remark 12.8 iii) give the identities

$$\sum_{k=1}^{n}\frac{\prod_{\nu\in\{1,\dots,n\}}\sin(x_k-y_\nu)}{\prod_{\nu\in\{1,\dots,n\}\setminus\{k\}}\sin(x_k-x_\nu)}=\sin\Bigl(\sum_{\nu=1}^{n}x_\nu-\sum_{\nu=1}^{n}y_\nu\Bigr),$$

$$\sum_{k=1}^{n}\prod_{\nu\in\{1,\dots,n\}\setminus\{k\}}\frac{\sin(y_i-x_\nu)}{\sin(x_k-x_\nu)}\prod_{\nu\in\{1,\dots,n\}\setminus\{j\}}\frac{\sin(x_k-y_\nu)}{\sin(y_j-y_\nu)}=\delta_{ij}\quad(1\le i,\,j\le n).$$

We have the following identity from the connection coefficients of even/odd families.

$$\sum_{k=1}^{n}\sin(x_k+s)\cdot\sin(x_k+t)\cdot\prod_{\nu\in\{1,\dots,n\}\setminus\{k\}}\frac{\sin(x_k+x_\nu+2u)}{\sin(x_k-x_\nu)}$$
$$=\begin{cases}\sin\Bigl(nu+\sum_{\nu=1}^{n}x_\nu\Bigr)\cdot\sin\Bigl(s+t+(n-2)u+\sum_{\nu=1}^{n}x_\nu\Bigr) & \text{if } n=2m,\\ \sin\Bigl(s+(n-1)u+\sum_{\nu=1}^{n}x_\nu\Bigr)\cdot\sin\Bigl(t+(n-1)u+\sum_{\nu=1}^{n}x_\nu\Bigr) & \text{if } n=2m+1.\end{cases}$$

The direct proof of these identities using residue calculus is given by [**Oc**]. It is interesting that similar identities of rational functions are given in [**Gl**, Appendix] which studies the systems of Schlesinger canonical form corresponding to Simpson's list (cf. §13.2).

13.7. Rigid examples of order at most 4

13.7.1. order 1. $\underline{1,1,1}$

$$u(x)=x^{\lambda_1}(1-x)^{\lambda_2}\qquad\begin{Bmatrix}-\lambda_1-\lambda_2 & \lambda_1 & \lambda_2\end{Bmatrix}$$

13.7.2. order 2. $\underline{11,11,11}$: H_2 (Gauss) $[\Delta(\mathbf{m})]=1^4$

$$u_{H_2}=\partial^{-\mu_1}u(x)\qquad\begin{Bmatrix}-\mu_1+1 & 0 & 0\\ -\lambda_1-\lambda_2-\mu_1 & \lambda_1+\mu_1 & \lambda_2+\mu_1\end{Bmatrix}$$

13.7.3. order 3. There are two types.

$\underline{111,21,111}$: H_3 (${}_3F_2$) $[\Delta(\mathbf{m})]=1^9$

$$u_{H_3}=\partial^{-\mu_2}x^{\lambda_3}u_{H_2}$$
$$\begin{Bmatrix}1-\mu_2 & 0 & [0]_{(2)}\\ -\lambda_3-\mu_1-\mu_2+1 & \lambda_3+\mu_2 & \\ -\lambda_1-\lambda_2-\lambda_3-\mu_1-\mu_2 & \lambda_1+\lambda_3+\mu_1+\mu_2 & \lambda_2+\mu_1+\mu_2\end{Bmatrix}$$

$\underline{21,21,21,21}$: P_3 (Jordan-Pochhammer) $[\Delta(\mathbf{m})]=1^4\cdot 2$

$$u_{P_3}=\partial^{-\mu}x^{\lambda_0}(1-x)^{\lambda_1}(c_2-x)^{\lambda_2}$$
$$\begin{Bmatrix}[1-\mu]_{(2)} & [0]_{(2)} & [0]_{(2)} & [0]_{(2)}\\ -\lambda_0-\lambda_1-\lambda_2-\mu & \lambda_0+\mu & \lambda_1+\mu & \lambda_2+\mu\end{Bmatrix}$$

13.7.4. order 4. There are 6 types.

$\underline{211,211,211}$: α_2 $[\Delta(\mathbf{m})]=1^{10}\cdot 2$

$$\partial^{-\mu_2}x^{\lambda_3}(1-x)^{\lambda_4}u_{H_2}$$
$$\begin{Bmatrix}[-\mu_2+1]_{(2)} & [0]_{(2)} & [0]_{(2)}\\ -\mu_1-\lambda_3-\lambda_4-\mu_2+1 & \lambda_3+\mu_2 & \lambda_4+\mu_2\\ -\lambda_1-\lambda_2-\lambda_3-\lambda_4-\mu_1-\mu_2 & \lambda_1+\lambda_3+\mu_1+\mu_2 & \lambda_2+\lambda_4+\mu_1+\mu_2\end{Bmatrix}$$

$\underline{1111, 31, 1111}$: H_4 $({}_4F_3)$ $[\Delta(\mathbf{m})] = 1^{16}$

$$\partial^{-\mu_3} x^{\lambda_4} u_{H_3}$$

$$\begin{Bmatrix} -\mu_3 + 1 & 0 & [0]_{(3)} \\ -\lambda_4 - \mu_2 - \mu_3 + 1 & \lambda_4 & \\ -\lambda_3 - \lambda_4 - \mu_1 - \mu_2 - \mu_3 + 1 & \lambda_3 + \lambda_4 + \mu_2 + \mu_3 & \\ -\lambda_1 - \cdots - \lambda_4 - \mu_1 - \mu_2 - \mu_3 & \lambda_1 + \cdots + \lambda_4 + \mu_1 + \mu_2 + \mu_3 & \lambda_2 + \mu_1 + \mu_2 + \mu_3 \end{Bmatrix}$$

$\underline{211, 22, 1111}$: EO_4 $[\Delta(\mathbf{m})] = 1^{14}$

$$\partial^{-\mu_3}(1-x)^{-\lambda'} u_{H_3}, \quad \lambda' = \lambda_2 + \mu_1 + \mu_2$$

$$\begin{Bmatrix} \lambda_2 + \mu_1 - \mu_2 - \mu_3 + 1 & [0]_{(2)} & [-\lambda_2 - \mu_1 - \mu_2 + \mu_3]_{(2)} \\ \lambda_2 - \lambda_3 - \mu_3 + 1 & \lambda_3 + \mu_2 + \mu_3 & \\ -\lambda_1 - \lambda_3 - \mu_3 & \lambda_1 + \lambda_3 + \mu_1 + \mu_2 + \mu_3 & [0]_{(2)} \\ -\mu_3 + 1 & & \end{Bmatrix}$$

We have the integral representation of the local solution corresponding to the exponent at 0:

$$\int_0^x \int_0^t \int_0^s (1-t)^{-\lambda_2-\mu_1-\mu_2}(x-t)^{\mu_3-1} s^{\lambda_3} (t-s)^{\mu_2-1} u^{\lambda_1}(1-u)^{\lambda_2}(s-u)^{\mu-1} du\, ds\, dt.$$

$\underline{211, 22, 31, 31}$: I_4 $[\Delta(\mathbf{m})] = 1^6 \cdot 2^2$

$$\partial^{-\mu_2}(c_2 - x)^{\lambda_3} u_{H_2}$$

$$\begin{Bmatrix} [-\mu_2 + 1]_{(2)} & [0]_{(3)} & [0]_{(3)} & [0]_{(2)} \\ -\lambda_3 - \mu_1 - \mu_2 + 1 & & & [\lambda_3 + \mu_2]_{(2)} \\ -\lambda_1 - \lambda_2 - \lambda_3 - \mu_1 - \mu_2 & \lambda_1 + \mu_1 + \mu_2 & \lambda_2 + \mu_1 + \mu_2 & \end{Bmatrix}$$

$\underline{31, 31, 31, 31, 31}$: P_4 $[\Delta(\mathbf{m})] = 1^5 \cdot 3$

$$u_{P_4} = \partial^{-\mu} x^{\lambda_0}(1-x)^{\lambda_1}(c_2 - x)^{\lambda_2}(c_3 - x)^{\lambda_3}$$

$$\begin{Bmatrix} [-\mu + 1]_{(3)} & [0]_{(3)} & [0]_{(3)} & [0]_{(3)} & [0]_{(3)} \\ -\lambda_0 - \lambda_2 - \lambda_3 - \mu & \lambda_0 + \mu & \lambda_1 + \mu & \lambda_2 + \mu & \lambda_3 + \mu \end{Bmatrix}$$

$\underline{22, 22, 22, 31}$: $P_{4,4}$ $[\Delta(\mathbf{m})] = 1^8 \cdot 2$

$$\partial^{-\mu'} x^{-\lambda'_0}(1-x)^{-\lambda'_1}(c_2 - x)^{-\lambda'_2} u_{P_3}, \quad \lambda'_j = \lambda_j + \mu, \quad \mu' = \lambda_0 + \lambda_1 + \lambda_2 + 2\mu$$

$$\begin{Bmatrix} [1 - \mu']_{(3)} & [\lambda_1 + \lambda_2 + \mu]_{(2)} & [\lambda_0 + \lambda_2 + \mu]_{(2)} & [\lambda_0 + \lambda_1 + \mu]_{(2)} \\ -\lambda_0 - \lambda_1 - \lambda_2 & [0]_{(2)} & [0]_{(2)} & [0]_{(2)} \end{Bmatrix}$$

13.7.5. Tuple of partitions : 211, 211, 211. $[\Delta(\mathbf{m})] = 1^{10} \cdot 2$

$$211, 211, 211 = H_1 \oplus H_3 : 6 = H_2 \oplus H_2 : 4 = 2H_1 \oplus H_2 : 1$$

From the operations

$$\begin{Bmatrix} x = \infty & 0 & 1 \\ 1 - \mu_1 & 0 & 0 \\ -\alpha_1 - \beta_1 - \mu_1 & \underline{\alpha_1 + \mu_1} & \beta_1 + \mu_1 \end{Bmatrix}$$

$$\xrightarrow{x^{\alpha_2}(1-x)^{\beta_2}} \begin{Bmatrix} x = \infty & 0 & 1 \\ 1 - \alpha_2 - \beta_2 - \mu_1 & \alpha_2 & \beta_2 \\ -\alpha_1 - \alpha_2 - \beta_1 - \beta_2 - \mu_1 & \underline{\alpha_1 + \alpha_2 + \mu_1} & \beta_1 + \beta_2 + \mu_1 \end{Bmatrix}$$

$$\xrightarrow{\partial^{-\mu_2}} \begin{Bmatrix} x = \infty & 0 & 1 \\ [-\mu_2 + 1]_{(2)} & [0]_{(2)} & [0]_{(2)} \\ 1 - \beta_2 - \mu_1 - \mu_2 & \alpha_2 + \mu_2 & \beta_2 + \mu_2 \\ -\alpha_1 - \beta_1 - \beta_2 - \mu_1 - \mu_2 & \underline{\alpha_1 + \mu_1 + \mu_2} & \beta_1 + \beta_2 + \mu_1 + \mu_2 \end{Bmatrix}$$

$$\longrightarrow \begin{Bmatrix} x=\infty & 0 & 1 \\ [\lambda_{2,1}]_{(2)} & [\lambda_{0,1}]_{(2)} & [\lambda_{1,1}]_{(2)} \\ \lambda_{2,2} & \lambda_{0,2} & \lambda_{1,2} \\ \lambda_{2,3} & \lambda_{0,3} & \lambda_{1,3} \end{Bmatrix} \quad \text{with} \quad \sum_{j=0}^{2}(2\lambda_{j,1}+\lambda_{j,2}+\lambda_{j,3})=3,$$

we have the integral representation of the solutions as in the case of other examples we have explained and so here we will not discuss them. The universal operator of type $11, 11, 11$ is

$$Q = x^2(1-x)^2\partial^2 - (ax+b)x(1-x)\partial + (cx^2+dx+e).$$

Here we have

$$\begin{aligned} b &= \lambda'_{0,1}+\lambda'_{0,2}-1, & e &= \lambda'_{0,1}\lambda'_{0,2},\\ -a-b &= \lambda'_{1,1}+\lambda'_{1,2}-1, & c+d+e &= \lambda'_{1,1}\lambda'_{1,2},\\ & & c &= \lambda'_{2,1}\lambda'_{2,2},\\ \lambda'_{0,1} &= \alpha_2, & \lambda'_{0,2} &= \alpha_1+\alpha_2+\mu_1,\\ \lambda'_{1,1} &= \beta_2, & \lambda'_{1,2} &= \beta_1+\beta_2+\mu_2,\\ \lambda'_{2,1} &= 1-\beta_2-\mu_1-\mu_2, & \lambda'_{2,2} &= -\alpha_1-\beta_1-\beta_2-\mu_1-\mu_2 \end{aligned}$$

corresponding to the second Riemann scheme in the above. The operator corresponding to the tuple $211, 211, 211$ is

$$\begin{aligned} P &= \mathrm{RAd}(\partial^{-\mu_2})Q\\ &= \mathrm{RAd}(\partial^{-\mu_2})\Big((\vartheta-\lambda'_{0,1})(\vartheta-\lambda'_{0,2})\\ &\quad + x\big(-2\vartheta^2+(2\lambda'_{0,1}+2\lambda'_{0,2}+\lambda'_{1,1}+\lambda'_{1,2}-1)\vartheta+\lambda'_{1,1}\lambda'_{1,2}-\lambda'_{0,1}\lambda'_{0,2}-\lambda'_{2,1}\lambda'_{2,2}\big)\\ &\quad + x^2(\vartheta+\lambda'_{2,1})(\vartheta+\lambda'_{2,2})\Big)\\ &= \partial^2(\vartheta-\lambda'_{0,1}-\mu_2)(\vartheta-\lambda'_{0,2}-\mu_2)\\ &\quad + \partial(\vartheta-\mu_2+1)\big(-2(\vartheta-\mu_2)^2+(2\lambda'_{0,1}+2\lambda'_{0,2}+\lambda'_{1,1}+\lambda'_{1,2}-1)(\vartheta-\mu_2)\\ &\quad + \lambda'_{1,1}\lambda'_{1,2}-\lambda'_{0,1}\lambda'_{0,2}-\lambda'_{2,1}\lambda'_{2,2}\big)\\ &\quad + (\vartheta-\mu_2+1)(\vartheta-\mu_2+2)(\vartheta+\lambda'_{2,1}-\mu_2)(\vartheta+\lambda'_{2,2}-\mu_2). \end{aligned}$$

The condition for the irreducibility:

$$\begin{cases} \lambda_{0,1}+\lambda_{1,1}+\lambda_{2,1}\notin\mathbb{Z},\\ \lambda_{0,\nu}+\lambda_{1,1}+\lambda_{2,1}\notin\mathbb{Z},\ \lambda_{0,1}+\lambda_{1,\nu}+\lambda_{2,1}\notin\mathbb{Z},\ \lambda_{0,1}+\lambda_{1,1}+\lambda_{2,\nu}\notin\mathbb{Z}\ \ (\nu=2,3),\\ \lambda_{0,1}+\lambda_{0,2}+\lambda_{1,1}+\lambda_{1,\nu}+\lambda_{2,1}+\lambda_{2,\nu'}\notin\mathbb{Z}\ \ (\nu,\,\nu'\in\{2,3\}). \end{cases}$$

There exist three types of direct decompositions of the tuple and there are 4 direct decompositions which give the connection coefficient $c(\lambda_{0,3}\rightsquigarrow\lambda_{1,3})$ by the formula (12.10) in Theorem 12.6:

$$\begin{aligned} 21\overline{1}, 21\underline{1}, 211 &= 00\overline{1}, 100, 100 \oplus 210, 11\underline{1}, 111\\ &= 11\overline{1}, 210, 111 \oplus 100, 00\underline{1}, 100\\ &= 10\overline{1}, 110, 110 \oplus 110, 10\underline{1}, 101\\ &= 10\overline{1}, 110, 101 \oplus 110, 10\underline{1}, 110 \end{aligned}$$

Thus we have

$$c(\lambda_{0,3} \rightsquigarrow \lambda_{1,3}) = \frac{\prod_{\nu=1}^{2} \Gamma(\lambda_{0,3} - \lambda_{0,\nu} + 1)}{\Gamma(\lambda_{0,3} + \lambda_{1,1} + \lambda_{2,1}) \cdot \Gamma(1 - \lambda_{0,1} - \lambda_{1,3} - \lambda_{2,1})} \cdot \frac{\prod_{\nu=1}^{2} \Gamma(\lambda_{1,\nu} - \Gamma_{1,3})}{\prod_{\nu=2}^{3} \Gamma(\lambda_{0,1} + \lambda_{0,3} + \lambda_{1,1} + \lambda_{1,2} + \lambda_{2,1} + \lambda_{2,\nu} - 1)}.$$

We can calculate generalized connection coefficient defined in Definition 12.17:

$$c([\lambda_{0,1}]_{(2)} \rightsquigarrow [\lambda_{1,1}]_{(2)}) = \frac{\prod_{\nu=2}^{3} \big(\Gamma(\lambda_{0,1} - \lambda_{0,\nu} + 2) \cdot \Gamma(\lambda_{1,\nu} - \lambda_{1,1} - 1)\big)}{\prod_{\nu=2}^{3} \big(\Gamma(\lambda_{0,1} + \lambda_{1,\nu} + \lambda_{2,1}) \cdot \Gamma(1 - \lambda_{0,\nu} - \lambda_{1,1} - \lambda_{2,1})\big)}.$$

This can be proved by the procedure given in Remark 12.19 as in the case of the formula (13.36). Note that the gamma functions in the numerator of this formula correspond to Remark 12.19 (2) and those in the denominator correspond to the rigid decompositions

$$\begin{aligned} \underline{2}11, \overline{2}11, 211 &= \underline{1}00, \overline{0}10, 100 \oplus \underline{1}11, \overline{2}01, 111 = \underline{1}00, \overline{0}01, 100 \oplus \underline{1}11, \overline{2}10, 111 \\ &= \underline{2}10, \overline{1}11, 111 \oplus \underline{0}01, \overline{1}00, 100 = \underline{2}01, \overline{1}11, 111 \oplus \underline{0}10, \overline{1}00, 100. \end{aligned}$$

The equation $Pu = 0$ with the Riemann scheme $\begin{Bmatrix} x=\infty & 0 & 1 \\ [\lambda_{0,1}]_{(2)} & [0]_{(2)} & [0]_{(2)} \\ \lambda_{0,2} & \lambda_{1,2} & \lambda_{2,2} \\ \lambda_{0,3} & \lambda_{1,3} & \lambda_{2,3} \end{Bmatrix}$ is isomorphic to the system

$$\tilde{u}' = \frac{A}{x}\tilde{u} + \frac{B}{x-1}\tilde{u}, \quad \tilde{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}, \quad u_1 = u,$$

$$A = \begin{pmatrix} 0 & 0 & c_1 & 0 \\ 0 & 0 & 0 & c_1 \\ 0 & 0 & a_1 & b_1 - b_2 - c_2 \\ 0 & 0 & 0 & a_2 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -a_1 - b_2 + c_1 & -b_1 + b_2 + c_2 & b_2 & 0 \\ -a_1 + a_2 + c_2 & -a_2 - b_1 + c_1 & a_1 - a_2 - c_2 & b_1 \end{pmatrix},$$

$$\begin{cases} a_1 = \lambda_{1,2}, \\ a_2 = \lambda_{1,3}, \\ b_1 = \lambda_{2,2} - 2, \\ b_2 = \lambda_{2,3} - 1, \\ c_1 = -\lambda_{0,1}, \\ c_2 = \lambda_{0,1} + \lambda_{0,2} + \lambda_{1,2} + \lambda_{2,2} - 1 \end{cases}$$

when $\lambda_{0,1}(\lambda_{0,1} + \lambda_{2,2})(\lambda_{0,1} + \lambda_{0,2} + \lambda_{1,2} + \lambda_{2,3} - 2) \neq 0$. Let $u(x)$ be a holomorphic solution of $Pu = 0$ in a neighborhood of $x = 0$. By a direct calculation we have

$$u_1(0) = \frac{(a_1 - 1)(a_2 - 1)}{(b_1 - c_1 + 1)(b_1 - b_2 - c_2)c_1} u'(0) + \frac{(a_2 + b_2 + c_2 - 1)a_1 - (c_1 + c_2)a_2 + (a_2 - a_1 + c_2)b_1 - (c_2 + 1)b_2 - c_2^2 + c_1}{(b_1 - c_1 + 1)(b_1 - b_2 - c_2)} u(0).$$

Since the shift described in Remark 12.19 (1) corresponds to the shift

$$(a_1, a_2, b_1, b_2, c_1, c_2) \mapsto (a_1 - k, a_2 - k, b_1 + k, b_2 + k, c_1, c_2),$$

it follows from Theorem 12.10 that

$$\lim_{k\to\infty} c([\lambda_{0,1}]_{(2)} \rightsquigarrow [\lambda_{1,1}]_{(2)})\Big|_{\substack{\lambda_{0,2}\mapsto\lambda_{0,2}-k,\ \lambda_{0,3}\mapsto\lambda_{0,3}-k\\ \lambda_{1,2}\mapsto\lambda_{1,2}+k,\ \lambda_{1,3}\mapsto\lambda_{1,3}+k}} = 1$$

as in the proof of (13.36) because $u_1(0) \sim \frac{k}{(b_1-b_2-c_2)c_1}u'(0) + Cu(0)$ with $C \in \mathbb{C}$ when $k \to \infty$. Thus we can calculate this generalized connection coefficient by the procedure described in Remark 12.19.

Using (3.8), we have the series expansion of the local solution at $x = 0$ corresponding to the exponent $\alpha_1 + \mu_1 + \mu_2$ for the Riemann scheme parametrized by α_i, β_i and μ_i with $i = 1, 2$.

$$\begin{aligned}
&I_0^{\mu_2} x^{\alpha_2}(1-x)^{\beta_2} I_0^{\mu_1} x^{\alpha_1}(1-x)^{\beta_1}\\
&= I_0^{\mu_2} \frac{\Gamma(\alpha_1+1)}{\Gamma(\alpha_1+\mu+1)} \sum_{n=0}^{\infty} \frac{(\alpha_1+1)_n(-\beta_1)_n}{(\alpha_1+\mu+1)_n n!} x^{\alpha_2}(1-x)^{\beta_2} x^{\alpha_1+\mu+n}\\
&= \frac{\Gamma(\alpha_1+1)\Gamma(\alpha_1+\alpha_2+\mu_1+1)x^{\alpha_1+\alpha_2+\mu_1+\mu_2}}{\Gamma(\alpha_1+\mu_1+1)\Gamma(\alpha_1+\alpha_2+\mu_1+\mu_2+1)}\\
&\quad \cdot \sum_{m,n=0}^{\infty} \frac{(\alpha_1+1)_n(\alpha_1+\alpha_2+\mu_1+1)_{m+n}(-\beta_1)_n(-\beta_2)_m}{(\alpha_1+\mu_1+1)_n(\alpha_1+\alpha_2+\mu_1+\mu_2+1)_{m+n} n! m!} x^{m+n}\\
&= \frac{\Gamma(\alpha_1+1)\Gamma(\alpha_1+\alpha_2+\mu_1+1)x^{\alpha_1+\alpha_2+\mu_1+\mu_2}(1-x)^{-\beta_2}}{\Gamma(\alpha_1+\mu_1+1)\Gamma(\alpha_1+\alpha_2+\mu_1+\mu_2+1)}\\
&\quad \cdot \sum_{m,\,n=0}^{\infty} \frac{(\alpha_1+1)_n(\alpha_1+\alpha_2+\mu_1+1)_n(\mu_2)_m(-\beta_1)_n(-\beta_2)_m}{(\alpha_1+\mu_1+1)_n(\alpha_1+\alpha_2+\mu_1+\mu_2+1)_{m+n} m! n!} x^n \Big(\frac{x}{x-1}\Big)^m.
\end{aligned}$$

Note that when $\beta_2 = 0$, the local solution is reduced to a local solution of the equation at $x = 0$ satisfied by the hypergeometric series ${}_3F_2(\alpha_1', \alpha_2', \alpha_3'; \beta_1', \beta_2'; x)$ and when $\alpha_2 = 0$, it is reduced to a local solution of the equation corresponding to the exponent at $x = 1$ with free multiplicity.

Let $u_0(\alpha_1, \alpha_2, \beta_1, \beta_2, \mu_1, \mu_2; x)$ be the local solution normalized by

$$u_0(\alpha, \beta, \mu; x) - x^{\alpha_1+\alpha_2+\mu_1+\mu_2} \in x^{\alpha_1+\alpha_2+\mu_1+\mu_2+1}\mathcal{O}_0$$

for generic α, β, μ. Then we have the contiguity relation

$$\begin{aligned}
u_0(\alpha, \beta_1 - 1, \beta_2, \mu; x) = u_0(\alpha, \beta, \mu; x) + \frac{(\alpha_1+1)(\alpha_1+\alpha_2+\mu_1+1)}{(\alpha_1+\mu_1+1)(\alpha_1+\alpha_2+\mu_1+\mu_2+1)}\\
\cdot u_0(\alpha_1+1, \alpha_2, \beta_1-1, \beta_2, \mu; x).
\end{aligned}$$

13.7.6. Tuple of partitions : 211, 22, 31, 31. $[\Delta(\mathbf{m})] = 1^6 \cdot 2$

$$\begin{aligned}
211, 22, 31, 31 &= H_1 \oplus P_3 : 4 = H_2 \oplus H_2 : 2 = 2H_1 \oplus H_2 : 2\\
&= 010, 10, 10, 10 \oplus 201, 12, 21, 21 = 010, 01, 10, 10 \oplus 201, 21, 21, 21\\
&= 001, 10, 10, 10 \oplus 210, 12, 21, 21 = 001, 01, 10, 10 \oplus 210, 21, 21, 21\\
&= 110, 11, 11, 20 \oplus 101, 11, 20, 11 = 110, 11, 20, 11 \oplus 101, 11, 11, 20\\
&= 200, 20, 20, 20 \oplus 011, 02, 11, 11\\
&\xrightarrow{\partial_{max}} 011, 02, 11, 11
\end{aligned}$$

$$\begin{Bmatrix} x=0 & \frac{1}{c_1} & \frac{1}{c_2} & \infty \\ [\lambda_{0,1}]_{(3)} & [\lambda_{1,1}]_{(3)} & [\lambda_{2,1}]_{(2)} & [\lambda_{3,1}]_{(2)} \\ \lambda_{0,2} & \lambda_{1,2} & \lambda_{2,2} & [\lambda_{3,2}]_{(2)} \\ & & \lambda_{2,3} & \end{Bmatrix}$$

$$\xrightarrow{x^{-\lambda_{0,1}}(1-c_1x)^{-\lambda_{1,1}}(1-c_2x)^{-\lambda_{2,1}}}$$

$$\begin{Bmatrix} x=0 & \frac{1}{c_1} & \frac{1}{c_2} & \infty \\ [0]_{(3)} & [0]_{(3)} & [0]_{(2)} & [\lambda_{3,1}+\lambda_{0,1}+\lambda_{1,1}+\lambda_{2,1}]_{(2)} \\ \lambda_{0,2}-\lambda_{0,1} & \lambda_{1,2}-\lambda_{1,1} & \lambda_{2,2}-\lambda_{2,1} & [\lambda_{3,2}+\lambda_{0,1}+\lambda_{1,1}+\lambda_{2,1}]_{(2)} \\ & & \lambda_{2,3}-\lambda_{2,1} & \end{Bmatrix}$$

$$\xrightarrow{\partial^{-\lambda'_1}}$$

$$\begin{Bmatrix} x=0 & \frac{1}{c_1} & \frac{1}{c_2} & \infty \\ 0 & 0 & & \\ \lambda_{0,2}+\lambda'_1-\lambda_{0,1} & \lambda_{1,2}+\lambda'_1-\lambda_{1,1} & \lambda_{2,2}+\lambda'_1-\lambda_{2,1} & [\lambda_{3,2}-\lambda_{3,1}+1]_{(2)} \\ & & \lambda_{2,3}+\lambda'_1-\lambda_{2,1} & \end{Bmatrix}$$

The condition for the irreducibility:

$$\begin{cases} \lambda_{0,1}+\lambda_{1,1}+\lambda_{2,\nu}+\lambda_{3,\nu'}\notin\mathbb{Z} \quad (\nu\in\{1,2,3\},\ \nu'\in\{1,2\}), \\ \lambda_{0,1}+\lambda_{0,2}+2\lambda_{1,1}+\lambda_{2,1}+\lambda_{2,\nu}+\lambda_{3,1}+\lambda_{3,2}\notin\mathbb{Z} \quad (\nu\in\{2,3\}), \end{cases}$$

$$c(\lambda_{0,2}\rightsquigarrow\lambda_{1,2}) = \frac{\Gamma(\lambda_{0,2}-\lambda_{0,1}+1)\Gamma(\lambda_{1,2}-\lambda_{1,1})(1-\frac{c_2}{c_1})^{\lambda_{2,1}}}{\prod_{\nu=2}^{3}\Gamma(\lambda_{0,1}+\lambda_{0,2}+2\lambda_{1,1}+\lambda_{2,1}+\lambda_{2,\nu}+\lambda_{3,1}+\lambda_{3,2}-1)},$$

$$c(\lambda_{0,2}\rightsquigarrow\lambda_{2,3}) = \prod_{\nu=1}^{2}\frac{\Gamma(\lambda_{2,3}-\lambda_{2,\nu})}{\Gamma(1-\lambda_{0,1}-\lambda_{1,1}-\lambda_{2,3}-\lambda_{3,\nu})} \cdot\frac{\Gamma(\lambda_{0,2}-\lambda_{0,1}+1)(1-\frac{c_1}{c_2})^{\lambda_{1,1}}}{\Gamma(\lambda_{0,1}+\lambda_{0,2}+2\lambda_{1,1}+\lambda_{2,1}+\lambda_{2,2}+\lambda_{3,1}+\lambda_{3,2}-1)}.$$

13.7.7. Tuple of partitions : $22,22,22,31$. $\quad [\Delta(\mathbf{m})] = 1^8\cdot 2$

$$\begin{aligned} 22,22,22,31 &= H_1\oplus P_3 : 8 = 2(11,11,11,20)\oplus 00,00,00,(-1)1 \\ &= 10,10,10,10\oplus 12,12,12,21 = 10,10,01,10\oplus 12,12,21,21 \\ &= 10,01,10,10\oplus 12,21,12,21 = 10,01,01,10\oplus 12,21,21,21 \\ &= 01,10,10,10\oplus 21,12,12,21 = 01,10,01,10\oplus 21,12,21,21 \\ &= 01,01,10,10\oplus 21,21,12,21 = 01,01,01,10\oplus 21,21,21,21 \\ &\xrightarrow{2} 12,12,12,21 \end{aligned}$$

The condition for the irreducibility:

$$\begin{cases} \lambda_{0,i}+\lambda_{1,j}+\lambda_{2,k}+\lambda_{3,1}\notin\mathbb{Z} \quad (i,\ j,\ k\in\{1,2\}), \\ \lambda_{0,1}+\lambda_{0,2}+\lambda_{1,1}+\lambda_{1,2}+\lambda_{2,1}+\lambda_{2,2}+\lambda_{3,1}+\lambda_{3,2}\notin\mathbb{Z}. \end{cases}$$

13.8. Other rigid examples with a small order

First we give an example which is not of Okubo type.

13.8.1. $221,221,221$. The Riemann Scheme and the direct decompositions are

$$\begin{Bmatrix} x=0 & 1 & \infty \\ [\lambda_{0,1}]_{(2)} & [\lambda_{1,1}]_{(2)} & [\lambda_{2,1}]_{(2)} \\ [\lambda_{0,2}]_{(2)} & [\lambda_{1,2}]_{(2)} & [\lambda_{2,2}]_{(2)} \\ \lambda_{0,3} & \lambda_{1,3} & \lambda_{2,3} \end{Bmatrix}, \qquad \sum_{j=0}^{2}(2\lambda_{j,1}+2\lambda_{j,2}+\lambda_{j,3}) = 4,$$

$$
\begin{aligned}
[\Delta(\mathbf{m})] &= 1^{14}\cdot 2\\
22\bar{1},22\underline{1},221 &= H_1\oplus 211,211,211:8 \qquad 6=|2,2,2|\\
&= H_2\oplus H_3 : 6 \qquad 11=|21,22,22|\\
&= 2H_2\oplus H_1 : 1\\
&= 10\bar{1},110,110\oplus 120,11\underline{1},111 = 01\bar{1},110,110\oplus 210,11\underline{1},111\\
&= 11\bar{1},120,111\oplus 110,10\underline{1},110 = 11\bar{1},210,111\oplus 110,01\underline{1},110\\
&\to 121,121,121
\end{aligned}
$$

and a connection coefficient is give by

$$
\begin{aligned}
c(\lambda_{0,3}\rightsquigarrow\lambda_{1,3}) = \prod_{\nu=1}^{2}\Bigg(&\frac{\Gamma(\lambda_{0,3}-\lambda_{0,\nu}+1)}{\Gamma(\lambda_{0,\nu}+\lambda_{0,3}+\lambda_{1,1}+\lambda_{1,2}+\lambda_{2,1}+\lambda_{2,2}-1)}\\
&\cdot\frac{\Gamma(\lambda_{1,\nu}-\lambda_{1,3})}{\Gamma(2-\lambda_{0,1}-\lambda_{0,2}-\lambda_{1,\nu}-\lambda_{1,3}-\lambda_{2,1}-\lambda_{2,2})}\Bigg).
\end{aligned}
$$

Using this example we explain an idea to get all the rigid decompositions $\mathbf{m}=\mathbf{m}'\oplus\mathbf{m}''$. Here we note that $\operatorname{idx}(\mathbf{m},\mathbf{m}')=1$. Put $\mathbf{m}=221,221,221$. We may assume $\operatorname{ord}\mathbf{m}'\le\operatorname{ord}\mathbf{m}''$.

Suppose $\operatorname{ord}\mathbf{m}'=1$. Then $\mathbf{m}'$ is isomorphic to $1,1,1$ and there exists tuples of indices (ℓ_0,ℓ_1,ℓ_2) such that $m'_{j,\nu}=\delta_{j,\ell_j}$. Then $\operatorname{idx}(\mathbf{m},\mathbf{m}')=m_{0,\ell_0}+m_{1,\ell_1}+m_{1,\ell_2}-(3-2)\operatorname{ord}\mathbf{m}\cdot\operatorname{ord}\mathbf{m}'$ and we have $m_{0,\ell_0}+m_{1,\ell_1}+m_{1,\ell_2}=6$. Hence $(m_{0,\ell_0},m_{1,\ell_1},m_{1,\ell_2})=(2,2,2)$, which is expressed by $6=|2,2,2|$ in the above. Since $\ell_j=1$ or 2 for $0\le j\le 2$, it is clear that there exist 8 rigid decompositions with $\operatorname{ord}\mathbf{m}'=1$.

Suppose $\operatorname{ord}\mathbf{m}'=2$. Then $\mathbf{m}'$ is isomorphic to $11,11,11$ and there exists tuples of indices $(\ell_{0,1},\ell_{0,2},\ell_{1,1},\ell_{1,2},\ell_{2,1},\ell_{2,2})$ which satisfies $\sum_{j=0}^{2}\sum_{\nu=1}^{2}m_{j,\ell_\nu}=(3-2)\operatorname{ord}\mathbf{m}\cdot\operatorname{ord}\mathbf{m}'+1=11$. Hence we may assume $(\ell_{0,1},\ell_{0,2},\ell_{1,1},\ell_{1,2},\ell_{2,1},\ell_{2,2})=(2,1,2,2,2,2)$ modulo obvious symmetries, which is expressed by $11=|21,22,22|$. There exist 6 rigid decompositions with $\operatorname{ord}\mathbf{m}'=2$.

In general, this method to get all the rigid decompositions of $\mathbf{m}$ is useful when $\operatorname{ord}\mathbf{m}$ is not big. For example if $\operatorname{ord}\mathbf{m}\le 7$, $\mathbf{m}'$ is isomorphic to $1,1,1$ or $11,11,11$ or $21,111,111$.

The condition for the irreducibility is given by Theorem 10.10 and it is

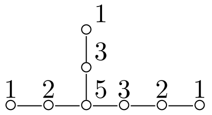
$$
\begin{cases}
\lambda_{0,i}+\lambda_{1,j}+\lambda_{2,k}\notin\mathbb{Z} & (i,\ j,\ k\in\{1,2\}),\\
\sum_{j=0}^{2}\sum_{\nu=1}^{2}\lambda_{j,\nu}+(\lambda_{i,3}-\lambda_{i,k})\notin\mathbb{Z} & (i\in\{0,1,2\},\ k\in\{1,2\}).
\end{cases}
$$

13.8.2. Other examples. Theorem 12.6 shows that the connection coefficients between local solutions of rigid differential equations which correspond to the eigenvalues of local monodromies with free multiplicities are given by direct decompositions of the tuples of partitions $\mathbf{m}$ describing their spectral types.

We list the rigid decompositions $\mathbf{m}=\mathbf{m}'\oplus\mathbf{m}''$ of rigid indivisible $\mathbf{m}$ in $\mathcal{P}^{(5)}\cup\mathcal{P}_3^{(6)}$ satisfying $m_{0,n_0}=m_{1,n_1}=m'_{0,n_0}=m''_{1,n_1}=1$. The positions of m_{0,n_0} and m_{1,n_1} in $\mathbf{m}$ to which Theorem 12.6 applies are indicated by an overline and an underline, respectively. The number of decompositions in each case equals n_0+n_1-2 and therefore the validity of the following list is easily verified.

We show the tuple $\partial_{max}\mathbf{m}$ after $\to$. The type $[\Delta(\mathbf{m})]$ of $\Delta(\mathbf{m})$ is calculated by (7.42), which is also indicated in the following with this calculation. For example, when $\mathbf{m}=311,221,2111$, we have $d(\mathbf{m})=2$, $\mathbf{m}'=\partial\mathbf{m}=111,021,0111$, $[\Delta(s(111,021,0111))]=1^9$, $\{m'_{j,\nu}-m'_{j,1}\in\mathbb{Z}_{>0}\}\cup\{2\}=\{1,1,1,1,2,2\}$ and hence $[\Delta(\mathbf{m})]=1^9\times 1^4\cdot 2^2=1^{13}\cdot 2^2$, which is a partition of $h(\mathbf{m})-1=17$.

Here we note that $h(\mathbf{m})$ is the sum of the numbers attached the Dynkin diagram



corresponding to $\alpha_{\mathbf{m}} \in \Delta_+$.

All the decompositions of the tuple $\mathbf{m}$ corresponding to the elements in $\Delta(\mathbf{m})$ are given, by which we easily get the necessary and sufficient condition for the irreducibility (cf. Theorem 10.13 and §13.9.2).

$$\operatorname{ord}\mathbf{m} = 5$$

$$\begin{aligned}
311,221,2111 &= 100,010,0001 \oplus 211,211,2110 && 6 = |3,2,1| \\
&= 100,001,1000 \oplus 211,220,1111 && 6 = |3,1,2| \\
&= 101,110,1001 \oplus 210,111,1110 && 11 = |31,22,21| \\
&= 2(100,100,1000) \oplus 111,021,0111 \\
&\xrightarrow{2} 111,021,0111 \\
[\Delta(\mathbf{m})] &= 1^9 \times 1^4 \cdot 2^2 = 1^{13} \cdot 2^2 \\
\mathbf{m} &= H_1 \oplus 211,211,211 : 6 = H_1 \oplus EO_4 : 1 = H_2 \oplus H_3 : 6 = 2H_1 \oplus H_3 : 2 \\
311,22\bar{1},211\underline{1} &= 211,211,2110 \oplus 100,010,0001 = 211,121,2110 \oplus 100,100,0001 \\
&= 100,001,1000 \oplus 211,220,1111 \\
&= 210,111,1110 \oplus 101,110,1001 = 201,111,1110 \oplus 110,110,1001 \\
31\bar{1},221,211\underline{1} &= 211,211,2110 \oplus 100,010,0001 = 211,121,2110 \oplus 100,100,0001 \\
&= 201,111,1110 \oplus 110,110,1001 \\
&= 101,110,1010 \oplus 210,111,1101 = 101,110,1100 \oplus 210,111,1011
\end{aligned}$$

$$\begin{aligned}
32,211\bar{1},211\underline{1} &= 22,1111,2110 \oplus 10,1000,0001 = 10,0001,1000 \oplus 22,2110,1111 \\
&= 11,1001,1010 \oplus 21,1110,1101 = 11,1001,1100 \oplus 21,1110,1011 \\
&= 21,1101,1110 \oplus 11,1010,1001 = 21,1011,1110 \oplus 11,1100,1001 \\
&\xrightarrow{2} 12,0111,0111 \\
[\Delta(\mathbf{m})] &= 1^9 \times 1^7 \cdot 2 = 1^{16} \cdot 2 \\
\mathbf{m} &= H_1 \oplus H_4 : 1 = H_1 \oplus EO_4 : 6 = H_2 \oplus H_3 : 9 = 2H_1 \oplus H_3 : 1
\end{aligned}$$

$$\begin{aligned}
22\bar{1},22\underline{1},41,41 &= 001,100,10,10 \oplus 220,121,31,31 = 001,010,10,10 \oplus 220,211,31,31 \\
&= 211,220,31,31 \oplus 010,001,10,10 = 121,220,31,31 \oplus 100,001,10,10 \\
&\xrightarrow{2} 021,021,21,21 \\
[\Delta(\mathbf{m})] &= 1^4 \cdot 2 \times 1^4 \cdot 2^3 = 1^6 \cdot 2^4 \\
\mathbf{m} &= H_1 \oplus 22,211,31,31 : 4 = H_2 \oplus H_3 : 2 = 2H_1 \oplus P_3 : 4 \\
22\bar{1},221,4\underline{1},41 &= 001,100,10,10 \oplus 220,121,31,31 = 001,010,10,10 \oplus 220,211,31,31 \\
&= 111,111,30,21 \oplus 110,110,11,20
\end{aligned}$$

$$\begin{aligned}
22\bar{1},32,32,4\underline{1} &= 101,11,11,20 \oplus 120,21,21,21 = 011,11,11,20 \oplus 210,21,21,21 \\
&= 001,10,10,10 \oplus 220,22,22,31 \\
&\xrightarrow{2} 021,12,12,21 \\
[\Delta(\mathbf{m})] &= 1^4 \cdot 2 \times 1^3 \cdot 2^2 = 1^7 \cdot 2^3
\end{aligned}$$

$$\begin{aligned}
\mathbf{m} &= H_1 \oplus 22,22,22,31:1 = H_1 \oplus 211,22,31,31:4 = H_2 \oplus P_3:2\\
&= 2H_1 \oplus P_3:2
\end{aligned}$$

$$\begin{aligned}
31\overline{1},31\underline{1},32,41 &= 001,100,10,10 \oplus 310,211,22,31 = 211,301,22,31 \oplus 100,001,10,10\\
&= 101,110,11,20 \oplus 210,201,21,21 = 201,210,21,21 \oplus 110,101,11,20\\
&\xrightarrow{3} 011,011,02,11\\
[\Delta(\mathbf{m})] &= 1^4 \times 1^4 \cdot 2 \cdot 3 = 1^8 \cdot 2 \cdot 3\\
\mathbf{m} &= H_1 \oplus 211,31,22,31:4 = H_2 \oplus P_3:4\\
&= 2H_1 \oplus H_3:1 = 3H_1 \oplus H_2:1\\
31\overline{1},311,32,4\underline{1} &= 001,100,10,10 \oplus 301,211,22,31\\
&= 101,110,11,20 \oplus 210,201,21,21 = 101,101,11,20 \oplus 210,210,21,21
\end{aligned}$$

$$\begin{aligned}
32,32,4\overline{1},4\underline{1},41 &= 11,11,11,20,20 \oplus 21,21,30,21,21\\
&= 21,21,21,30,21 \oplus 11,11,20,11,20\\
&\xrightarrow{3} 02,02,11,11,11\\
[\Delta(\mathbf{m})] &= 1^4 \times 2^2 \cdot 3 = 1^4 \cdot 2^2 \cdot 3\\
\mathbf{m} &= H_1 \oplus P_4:1 = H_2 \oplus P_3:3 = 2H_1 \oplus P_3:2 = 3H_1 \oplus H_2:1
\end{aligned}$$

$\operatorname{ord}\mathbf{m} = 6$ and $\mathbf{m} \in \mathcal{P}_3$

$$\begin{aligned}
321,3111,222 &= 311,2111,221 \oplus 010,1000,001 \qquad 7 = |2,3,2|\\
&= 211,2110,211 \oplus 110,1001,011 \qquad 13 = |32,31,22|\\
&= 210,1110,111 \oplus 111,2001,111\\
&\xrightarrow{2} 121,1111,022 \to 111,0111,012\\
[\Delta(\mathbf{m})] &= 1^{14} \times 1 \cdot 2^3 = 1^{15} \cdot 2^3\\
\mathbf{m} &= H_1 \oplus 311,2111,221:3 = H_2 \oplus 211,211,211:6 = H_3 \oplus H_3:6\\
&= 2H_1 \oplus EO_4:3\\
32\overline{1},311\underline{1},222 &= 211,2110,211 \oplus 110,1001,011 = 211,2110,121 \oplus 110,1001,101\\
&= 211,2110,112 \oplus 110,1001,110\\
&= 111,2100,111 \oplus 210,1011,111 = 111,2010,111 \oplus 210,1101,111\\
321,311\overline{1},311\underline{1} &= 221,2111,3110 \oplus 100,1000,0001 = 100,0001,1000 \oplus 221,3110,2111\\
&= 211,2101,2110 \oplus 110,1010,1001 = 211,2011,2110 \oplus 110,1101,1001\\
&= 110,1001,1100 \oplus 211,2110,2011 = 110,1001,1010 \oplus 211,2110,2101\\
&\xrightarrow{3} 021,0111,0111\\
[\Delta(\mathbf{m})] &= 1^9 \times 1^7 \cdot 2 \cdot 3 = 1^{16} \cdot 2 \cdot 3\\
\mathbf{m} &= H_1 \oplus 221,2111,311:6 = H_1 \oplus 32,2111,2111:1\\
&= H_2 \oplus 211,211,211:9 = 2H_1 \oplus H_4:1 = 3H_1 \oplus H_3:1
\end{aligned}$$

$$\begin{aligned}
32\overline{1},311\underline{1},3111 &= 221,3110,2111 \oplus 100,0001,1000 = 001,1000,1000 \oplus 320,2111,2111\\
&= 211,2110,2110 \oplus 110,1001,1001 = 211,2110,2011 \oplus 110,1001,1100\\
&= 211,2110,2011 \oplus 110,1001,1100
\end{aligned}$$

$$
\begin{aligned}
32\bar{1}, 32\underline{1}, 2211 &= 211, 220, 1111 \oplus 110, 101, 1100 = 101, 110, 1100 \oplus 220, 211, 1111\\
&= 111, 210, 1110 \oplus 210, 111, 1101 = 111, 210, 1101 \oplus 210, 111, 1110\\
&\xrightarrow{2} 121, 121, 0211 \to 101, 101, 0011\\
[\Delta(\mathbf{m})] &= 1^{10} \cdot 2 \times 1^{4} \cdot 2^{2} = 1^{14} \cdot 2^{3}\\
\mathbf{m} &= H_1 \oplus 311, 221, 2111 : 4 = H_1 \oplus 221, 221, 221 : 2\\
&= H_2 \oplus EO_4 : 2 = H_2 \oplus 211, 211, 211 : 4 = H_3 \oplus H_3 : 2\\
&= 2H_1 \oplus 211, 211, 211 : 2 = 2(110, 110, 1100) \oplus 101, 101, 0011 : 1\\
321, 32\bar{1}, 221\underline{1} &= 221, 221, 2210 \oplus 100, 100, 0001 = 110, 101, 1100 \oplus 211, 220, 1111\\
&= 211, 211, 2110 \oplus 110, 110, 0101 = 211, 211, 1210 \oplus 110, 110, 1001\\
&= 210, 111, 1110 \oplus 111, 210, 1101
\end{aligned}
$$

$$
\begin{aligned}
41\bar{1}, 221\underline{1}, 2211 &= 311, 2210, 2111 \oplus 100, 0001, 0100 = 311, 2210, 1211 \oplus 100, 0001, 1000\\
&= 101, 1100, 1100 \oplus 310, 1111, 1111 = 201, 1110, 1110 \oplus 210, 1101, 1101\\
&= 201, 1110, 1101 \oplus 210, 1101, 1110\\
&\xrightarrow{2} 211, 0211, 0211 \to 011, 001, 0011\\
[\Delta(\mathbf{m})] &= 1^{10} \cdot 2 \times 1^{4} \cdot 2^{3} = 1^{14} \cdot 2^{4}\\
\mathbf{m} &= H_1 \oplus 311, 221, 2211 : 8 = H_2 \oplus H_4 : 2 = H_3 \oplus H_3 : 4\\
&= 2H_1 \oplus 211, 211, 211 : 4\\
411, 221\bar{1}, 221\underline{1} &= 311, 2111, 2210 \oplus 100, 0100, 0001 = 311, 1211, 2210 \oplus 100, 1000, 0001\\
&= 100, 0001, 0100 \oplus 311, 2210, 2111 = 100, 0001, 1000 \oplus 311, 2210, 1211\\
&= 201, 1101, 1110 \oplus 210, 1110, 1101 = 210, 1101, 1110 \oplus 201, 1110, 1101
\end{aligned}
$$

$$
\begin{aligned}
41\bar{1}, 222, 2111\underline{1} &= 311, 221, 21110 \oplus 100, 001, 00001 = 311, 212, 21110 \oplus 100, 010, 00001\\
&= 311, 122, 21110 \oplus 100, 100, 00001 = 201, 111, 11100 \oplus 210, 111, 10011\\
&= 201, 111, 11010 \oplus 210, 111, 10101 = 201, 111, 10110 \oplus 210, 111, 11001\\
&\xrightarrow{2} 211, 022, 01111 \to 111, 012, 00111\\
[\Delta(\mathbf{m})] &= 1^{14} \times 1^{4} \cdot 2^{3} = 1^{18} \cdot 2^{3}\\
\mathbf{m} &= H_1 \oplus 311, 221, 2111 : 12 = H_3 \oplus H_3 : 6 = 2H_1 \oplus EO_4 : 3
\end{aligned}
$$

$$
\begin{aligned}
42, 221\bar{1}, 2111\underline{1} &= 32, 2111, 21110 \oplus 10, 0100, 00001 = 32, 1211, 21110 \oplus 10, 1000, 00001\\
&= 10, 0001, 10000 \oplus 32, 2210, 11111 = 31, 1111, 11110 \oplus 11, 1100, 10001\\
&= 21, 1101, 11100 \oplus 21, 1110, 10011 = 21, 1101, 11010 \oplus 21, 1110, 10101\\
&= 21, 1101, 10110 \oplus 21, 1110, 11001\\
&\xrightarrow{2} 22, 0211, 01111 \to 12, 0111, 00111\\
[\Delta(\mathbf{m})] &= 1^{14} \times 1^{6} \cdot 2^{2} = 1^{20} \cdot 2^{2}\\
\mathbf{m} &= H_1 \oplus 32, 2111, 2111 : 8 = H_1 \oplus EO_4 : 2 = H_2 \oplus H_4 : 4\\
&= H_3 \oplus H_3 : 6 = 2H_1 \oplus EO_4 : 2
\end{aligned}
$$

$$
\begin{aligned}
33, 311\bar{1}, 2111\underline{1} &= 32, 2111, 21110 \oplus 01, 1000, 00001 = 23, 2111, 21110 \oplus 10, 1000, 00001\\
&= 22, 2101, 11110 \oplus 11, 1010, 10001 = 22, 2011, 11110 \oplus 11, 1100, 10001\\
&= 11, 1001, 11000 \oplus 22, 2110, 10111 = 11, 1001, 10100 \oplus 22, 2110, 11011
\end{aligned}
$$

$$
\begin{aligned}
&= 11,1001,10010 \oplus 22,2110,11101 \\
&\xrightarrow{2} 13,1111,01111 \\
[\Delta(\mathbf{m})] &= 1^{16} \times 1^4 \cdot 2^2 = 1^{20} \cdot 2^2 \\
\mathbf{m} &= H_1 \oplus 32,2111,2111 : 8 = H_2 \oplus EO_4 : 12 = 2H_1 \oplus H_4 : 2
\end{aligned}
$$

$$
\begin{aligned}
32\overline{1},311\underline{1},3111 &= 221,3110,2111 \oplus 100,0001,1000 = 001,1000,1000 \oplus 320,2111,2111 \\
&= 211,2110,2110 \oplus 110,1001,1001 = 211,2110,2101 \oplus 110,1001,1010 \\
&= 211,2110,2011 \oplus 110,1001,1100 \\
&\xrightarrow{3} 021,0111,0111 \\
[\Delta(\mathbf{m})] &= 1^9 \times 1^7 \cdot 2 \cdot 3 = 1^{16} \cdot 2 \cdot 3 \\
\mathbf{m} &= H_1 \oplus 221,2111,311 : 6 = H_1 \oplus 32,2111,2111 : 1 \\
&= H_2 \oplus 211,211,211 : 9 = 2H_1 \oplus H_4 : 1 = 3H_1 \oplus H_3 : 1 \\
321,311\overline{1},311\underline{1} &= 100,0001,1000 \oplus 221,3110,2111 = 221,2111,3110 \oplus 100,1000,0001 \\
&= 211,2101,2110 \oplus 110,1010,1001 = 211,2011,2110 \oplus 110,1100,1001 \\
&= 110,1001,1100 \oplus 211,2110,2011 = 110,1001,1010 \oplus 211,2110,2101
\end{aligned}
$$

$$
\begin{aligned}
33,221\overline{1},221\underline{1} &= 22,1111,2110 \oplus 11,1100,1001 = 22,1111,1210 \oplus 11,1100,0101 \\
&= 21,1101,1110 \oplus 12,1110,1011 = 12,1101,1110 \oplus 21,1110,1011 \\
&= 11,1001,1100 \oplus 22,1210,1111 = 11,0101,1100 \oplus 22,2110,1111 \\
&\xrightarrow{1} 23,1211,1211 \to 21,1011,1011 \\
[\Delta(\mathbf{m})] &= 1^{16} \cdot 2 \times 1^4 = 1^{20} \cdot 2 \\
\mathbf{m} &= H_1 \oplus 32,2111,2111 : 8 = H_2 \oplus EO_4 : 8 = H_3 \oplus H_3 : 4 \\
&= 2(11,1100,1100) \oplus 11,0011,0011 : 1
\end{aligned}
$$

We show all the rigid decompositions of the following simply reducible partitions of order 6, which also correspond to the reducibility of the universal models.

$$
\begin{aligned}
42,222,111111 &= 32,122,011111 \oplus 10,100,100000 \\
&= 21,111,111000 \oplus 21,111,000111 \\
&\xrightarrow{1} 32,122,011111 \to 22,112,001111 \to 12,111,000111 \\
[\Delta(\mathbf{m})] &= 1^{28} \\
\mathbf{m} &= H_1 \oplus EO_5 : 18 = H_3 \oplus H_3 : 10
\end{aligned}
$$

$$
\begin{aligned}
33,222,21111 &= 23,122,11111 \oplus 10,100,10000 \\
&= 22,112,10111 \oplus 11,110,11000 \\
&= 21,111,11100 \oplus 12,111,10011 \\
&\xrightarrow{1} 23,122,11111 \to 22,112,01111 \to 12,111,00111 \\
[\Delta(\mathbf{m})] &= 1^{24} \\
\mathbf{m} &= H_1 \oplus EO_5 : 6 = H_2 \oplus EO_4 : 12 = H_3 \oplus H_3 : 6
\end{aligned}
$$

13.9. Submaximal series and minimal series

The rigid tuples $\mathbf{m} = \{m_{j,\nu}\}$ satisfying

$$\#\{m_{j,\nu}\,;\, 0 < m_{j,\nu} < \operatorname{ord}\mathbf{m}\} \ge \operatorname{ord}\mathbf{m} + 5 \tag{13.39}$$

are classified by Roberts [**Ro**]. They are the tuples of type H_n and P_n which satisfy

$$\#\{m_{j,\nu}\,;\, 0 < m_{j,\nu} < \operatorname{ord}\mathbf{m}\} = 2\operatorname{ord}\mathbf{m} + 2 \tag{13.40}$$

and those of 13 series $A_n = EO_n$, B_n, C_n, D_n, E_n, F_n, G_{2m}, I_n, J_n, K_n, L_{2m+1}, M_n, N_n called *submaximal series* which satisfy

$$\#\{m_{j,\nu}\,;\, 0 < m_{j,\nu} < \operatorname{ord}\mathbf{m}\} = \operatorname{ord}\mathbf{m} + 5. \tag{13.41}$$

The series H_n and P_n are called *maximal series*.

We examine these rigid series and give enough information to analyze the series, which will be sufficient to construct differential equations including their confluences, integral representation and series expansion of solutions and get connection coefficients and the condition of their reducibility.

In fact from the following list we easily get all the direct decompositions and Katz's operations decreasing the order. The number over an arrow indicates the difference of the orders. We also indicate Yokoyama's reduction for systems of Okubo normal form using extension and restriction, which are denoted E_i and R_i ($i = 0, 1, 2$), respectively (cf. [**Yo2**]). Note that the inverse operations of E_i are R_i, respectively. In the following we put

$$\begin{aligned} u_{P_m} &= \partial^{-\mu} x^{\lambda_0}(1-x)^{\lambda_1}(c_2 - x)^{\lambda_2}\cdots(c_{m-1}-x)^{\lambda_{m-1}},\\ u_{H_2} &= u_{P_2},\\ u_{H_{m+1}} &= \partial^{-\mu^{(m)}} x^{\lambda_0^{(m)}} u_{H_m}. \end{aligned} \tag{13.42}$$

We give all the decompositions

$$\mathbf{m} = \big(\operatorname{idx}(\mathbf{m}',\mathbf{m})\cdot\mathbf{m}'\big) \oplus \mathbf{m}'' \tag{13.43}$$

for $\alpha_{\mathbf{m}'} \in \Delta(\mathbf{m})$. Here some $m''_{j,\nu}$ may be negative if $\operatorname{idx}(\mathbf{m}',\mathbf{m}) > 1$ (cf. Remark 7.11 i)) and we will not distinguish between $\mathbf{m}' \oplus \mathbf{m}''$ and $\mathbf{m}'' \oplus \mathbf{m}'$ when $\operatorname{idx}(\mathbf{m}',\mathbf{m}) = 1$. Moreover note that the inequality assumed for the formula $[\Delta(\mathbf{m})]$ below assures that the given tuple of partition is monotone.

13.9.1. B_n. ($B_{2m+1} = \mathrm{III}_m$, $B_{2m} = \mathrm{II}_m$, $B_3 = H_3$, $B_2 = H_2$)

$$\begin{aligned} u_{B_{2m+1}} &= \partial^{-\mu'}(1-x)^{\lambda'}u_{H_{m+1}}\\ m^21, m+11^m, m1^{m+1} &= 10,10,01 \oplus mm-11, m1^m, m1^m\\ &= 01,10,10 \oplus m^2, m1^m, m-11^{m+1}\\ &= 1^20,11,11 \oplus (m-1)^21, m1^{m-1}, m-11^m\\ [\Delta(B_{2m+1})] &= 1^{(m+1)^2} \times 1^{m+2}\cdot m^2 = 1^{m^2+3m+3}\cdot m^2\\ B_{2m+1} &= H_1 \oplus B_{2m} \qquad : 2(m+1)\\ &= H_1 \oplus C_{2m} \qquad : 1\\ &= H_2 \oplus B_{2m-1} \qquad : m(m+1)\\ &= mH_1 \oplus H_{m+1} \qquad : 2 \end{aligned}$$

$$\begin{aligned} u_{B_{2m}} &= \partial^{-\mu'}x^{\lambda'}(1-x)^{\lambda''}u_{H_m}\\ mm-11, m1^m, m1^m &= 100,01,10 \oplus (m-1)^21, m1^{m-1}, m-11^m\\ &= 001,10,10 \oplus mm-10, m-11^m, m-11^m\\ &= 110,11,11 \oplus m-1m-21, m-11^{m-1}, m-11^{m-1} \end{aligned}$$

$$
\begin{aligned}
[\Delta(B_{2m})] &= 1^{m^2} \times 1^{2m+1} \cdot (m-1) = 1^{(m+1)^2} \cdot (m-1) \cdot m \\
B_{2m} &= H_1 \oplus B_{2m-1} && : 2m \\
&= H_1 \oplus C_{2m-2} && : 1 \\
&= H_2 \oplus B_{2m-2} && : m^2 \\
&= (m-1)H_1 \oplus H_{m+1} && : 1 \\
&= mH_1 \oplus H_m && : 1
\end{aligned}
$$

$$
B_{2m+1} \xrightarrow[R2E0]{m} H_{m+1}, \quad B_n \xrightarrow{1} B_{n-1}, \quad B_n \xrightarrow{1} C_{n-1}
$$

$$
B_{2m} \xrightarrow[R1E0]{m} H_m, \quad B_{2m} \xrightarrow{m-1} H_{m+1}
$$

13.9.2. An example. Using the example of type B_{2m+1}, we explain how we get explicit results from the data written in §13.9.1.

The Riemann scheme of type B_{2m+1} is

$$
\begin{Bmatrix}
\infty & 0 & 1 \\
[\lambda_{0,1}]_{(m)} & [\lambda_{1,1}]_{(m+1)} & [\lambda_{2,1}]_{(m)} \\
[\lambda_{0,2}]_{(m)} & \lambda_{1,2} & \lambda_{2,2} \\
\lambda_{0,3} & \vdots & \vdots \\
 & \lambda_{1,m+1} & \lambda_{2,m+2}
\end{Bmatrix},
$$

$$
\sum_{j=0}^{p} \sum_{\nu=1}^{n_j} m_{j,\nu}\lambda_{j,\nu} = 2m \qquad \text{(Fuchs relation)}.
$$

Theorem 10.13 says that the corresponding equation is irreducible if and only if any value of the following linear functions is not an integer.

$$
\begin{aligned}
L_{i,\nu}^{(1)} &:= \lambda_{0,i} + \lambda_{1,1} + \lambda_{2,\nu} \qquad (i = 1, 2, \quad \nu = 2, \dots, m+2), \\
L^{(2)} &:= \lambda_{0,3} + \lambda_{1,1} + \lambda_{2,1}, \\
L_{\mu,\nu}^{(3)} &:= \lambda_{0,1} + \lambda_{0,2} + \lambda_{1,1} + \lambda_{1,\mu} + \lambda_{2,1} + \lambda_{2,\nu} - 1 \\
&\qquad\qquad (\mu = 2, \dots, m+1, \quad \nu = 2, \dots, m+2), \\
L_i^{(4)} &:= \lambda_{0,i} + \lambda_{1,1} + \lambda_{2,1} \qquad (i = 1, 2).
\end{aligned}
$$

Here $L_{i,\nu}^{(1)}$ (resp. $L^{(2)}$ etc.) correspond to the terms $10, 01, 01$ and $H_1 \oplus B_{2m} : 2(m+1)$ (resp. $01, 10, 10$ and $H_1 \oplus C_{2m} : 1$ etc.) in §13.9.1.

It follows from Theorem 6.14 and Theorem 10.13 that the Fuchsian differential equation with the above Riemann scheme belongs to the universal equation $P_{B_{2m+1}}(\lambda)u = 0$ if

$$
L_i^{(4)} \notin \{-1, -2, \dots, 1-m\} \quad (i = 1,\ 2).
$$

Theorem 12.6 says that the connection coefficient $c(\lambda_{1,m+1} \rightsquigarrow \lambda_{2,m+2})$ equals

$$
\frac{\prod_{\mu=1}^{m} \Gamma(\lambda_{1,m+1} - \lambda_{1,\mu} + 1) \cdot \prod_{\mu=1}^{m+1} \Gamma(\lambda_{2,\nu} - \lambda_{1,m+2})}{\prod_{i=1}^{2} \Gamma(1 - L_{i,m+2}^{(1)}) \cdot \prod_{\nu=2}^{m+1} \Gamma(L_{m+1,\nu}^{(3)}) \cdot \prod_{\mu=2}^{m} \Gamma(1 - L_{\mu,m+2}^{(3)})}
$$

and

$$
c(\lambda_{1,m+1} \rightsquigarrow \lambda_{0,3}) = \frac{\prod_{\mu=1}^{m} \Gamma(\lambda_{1,m+1} - \lambda_{1,\mu} + 1) \cdot \prod_{i=1}^{2} \Gamma(\lambda_{0,i} - \lambda_{0,3})}{\Gamma(1 - L^{(2)}) \cdot \prod_{\nu=2}^{m+1} \Gamma(L_{m+1,\nu}^{(3)})},
$$

$$
c(\lambda_{2,m+2} \rightsquigarrow \lambda_{0,3}) = \frac{\prod_{\nu=1}^{m+1} \Gamma(\lambda_{2,m+2} - \lambda_{1,\nu} + 1) \cdot \prod_{i=1}^{2} \Gamma(\lambda_{0,i} - \lambda_{0,3})}{\prod_{i=1}^{2} \Gamma(L_{m+2}^{(1)}) \cdot \prod_{\nu=2}^{m+1} \Gamma(L_{m+1,\nu}^{(3)})}.
$$

It follows from Theorem 11.7 that the universal operators

$$P^0_{H_1}(\lambda)\quad P^2_{H_1}(\lambda)\quad P^0_{B_{2m}}(\lambda)\quad P^1_{B_{2m}}(\lambda)\quad P^2_{B_{2m}}(\lambda)\quad P^1_{C_{2m}}(\lambda)\quad P^2_{C_{2m}}(\lambda)$$
$$P^1_{H_2}(\lambda)\quad P^2_{H_2}(\lambda)\quad P^0_{B_{2m-1}}(\lambda)\quad P^1_{B_{2m-1}}(\lambda)\quad P^2_{B_{2m-1}}(\lambda)$$

define shift operators $R_{B_{2m+1}}(\epsilon,\lambda)$ under the notation in the theorem.

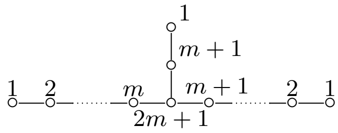
We also explain how we get the data in §13.9.1. Since $\partial_{max} : B_{2m+1} = \mathbf{m} := mm1, m+11^m, m1^{m+1} \to H_{m+1} = \mathbf{m}' := 0m1, 11^m, 01^{m+1}$, the equality (7.42) shows

$$\begin{aligned}[\Delta(B_{2m+1})] &= [\Delta(H_{m+1})] \cup \{d_{1,1,1}(\mathbf{m})\} \cup \{m'_{j,\nu} - m'_{j,1} > 0\}\\ &= 1^{(m+1)^2} \times m^1 \times 1^{m+2}\cdot m^1 = 1^{(m+1)^2} \times 1^{m+2}\cdot m^2 = 1^{m^2+3m+3}\cdot m^2.\end{aligned}$$

Here we note that $\{m'_{j,\nu} - m'_{j,1} > 0\} = \{m, 1, 1^{m+1}\} = 1^{m+2}\cdot m^1$ and $[\Delta(H_{m+1})]$ is given in §13.4.

We check (7.44) for $\mathbf{m}$ as follows:

$$\begin{aligned}h(\mathbf{m}) &= 2(1+\cdots+m) + (2m+1) + 2(m+1) + 1\\ &= m^2+5m+4,\end{aligned}$$
$$\textstyle\sum_{i\in[\Delta(\mathbf{m})]} i = (m^2+3m+3) + 2m = m^2+5m+3.$$



The decompositions $mH_1 \oplus H_{m+1}$ and $H_1 \oplus B_{2m}$ etc. in §13.9.1 are easily obtained and we should show that they are all the decompositions (13.43), whose number is given by $[\Delta(B_{2m+1})]$. There are 2 decompositions of type $mH_1 \oplus H_{m+1}$, namely, $B_{2m+1} = mm1, m+11^m, m1^{m+1} = m(100,10,10) \oplus \cdots = m(010,10,10) \oplus \cdots$, which correspond to $L^{(4)}_i$ for $i = 1$ and 2. Then the other decompositions are of type $\mathbf{m}' \oplus \mathbf{m}''$ with rigid tuples $\mathbf{m}'$ and $\mathbf{m}''$ whose number equals m^2+3m+3. The numbers of decompositions $H_1 \oplus B_{2m}$ etc. given in §13.9.1 are easily calculated which correspond to $L^{(1)}_{i,\nu}$ etc. and we can check that they give the required number of the decompositions.

13.9.3. C_n**.** ($C_4 = EO_4$, $C_3 = H_3$, $C_2 = H_2$)

$$\begin{aligned}u_{C_{2m+1}} &= \partial^{-\mu'} x^{\lambda'} u_{H_{m+1}}\\ m+1m, m1^{m+1}, m1^{m+1} &= 10,01,10 \oplus m^2, m1^m, m-11^{m+1}\\ &= 11,11,11 \oplus m(m-1), m-11^m m-11^m\\ [\Delta(C_{2m+1})] &= 1^{(m+1)^2} \times 1^{2m+2}\cdot m\cdot(m-1)\\ &= 1^{(m+1)(m+3)}\cdot m\cdot(m-1)\\ C_{2m+1} &= H_1 \oplus C_{2m} &&: 2m+2\\ &= H_2 \oplus C_{2m-2} &&: (m+1)^2\\ &= mH_1 \oplus H_{m+1} &&: 1\\ &= (m-1)H_1 \oplus H_{m+2} &&: 1\end{aligned}$$

$$\begin{aligned}u_{C_{2m}} &= \partial^{-\mu'} x^{\lambda'} (1-x)^{-\lambda_1-\mu-\mu^{(2)}-\cdots-\mu^{(m)}} u_{H_{m+1}}\\ m^2, m1^m, m-11^{m+1} &= 1,10,01 \oplus mm-1, m-11^{m-1}, m-11^{m-1}\\ &= 1^2,11,11 \oplus (m-1)^2, m-11^{m-1}, m-21^m\\ [\Delta(C_{2m})] &= 1^{(m+1)^2} \times 1^{m+1}\cdot(m-1)^2 = 1^{m^2+3m+2}\cdot(m-1)^2\\ C_{2m} &= H_1 \oplus C_{2m-1} &&: 2m+2\\ &= H_2 \oplus C_{2m-2} &&: m(m+1)\\ &= (m-1)H_1 \oplus H_{m+1} &&: 2\end{aligned}$$

$$C_{2m+1} \underset{R2E0R0E0}{\xrightarrow{m}} H_{m+1}, \quad C_{2m+1} \xrightarrow{m-1} H_{m+2}$$
$$C_{2m} \underset{R1E0R0E0}{\xrightarrow{m-1}} H_{m+1}, \quad C_n \xrightarrow{1} C_{n-1}$$

13.9.4. D_n**.** ($D_6 = X_6$: Extra case, $D_5 = EO_5$)

$$\begin{aligned}
u_{D_5} &= \partial^{-\mu_5}(1-x)^{-\lambda_3-\mu_3-\mu_4} u_{E_4} \\
u_{D_6} &= \partial^{-\mu_6}(1-x)^{-\lambda_1-\mu-\mu_5} u_{D_5} \\
u_{D_n} &= \partial^{-\mu_n}(1-x)^{-\lambda'_n} u_{D_{n-2}} \quad (n \geq 7) \\
(2m-1)2, 2^m 1, 2^{m-2}1^5 &= 10, 01, 10 \oplus (2m-2)2, 2^m, 2^{m-3}1^6 \\
&= 10, 10, 01 \oplus (2m-2)2, 2^{m-1}1^2, 2^{m-3}1^4 \\
&= (m-1)1, 1^m 0, 1^{m-2}1^2 \oplus m1, 1^m 1, 1^{m-2}1^3 \\
m \geq 2 \;\Rightarrow\; [\Delta(D_{2m+1})] &= 1^{6m+2} \cdot 2^{(m-1)(m-3)} \times 1^6 \cdot 2^{2m-3} = 1^{6m+8} \cdot 2^{m(m-2)} \\
D_{2m+1} &= H_1 \oplus D_{2m} && : m-2 \\
&= H_1 \oplus E_{2m} && : 5m \\
&= H_m \oplus H_{m+1} && : 10 \\
&= 2H_1 \oplus D_{2m-1} && : m(m-2)
\end{aligned}$$

$$\begin{aligned}
(2m-2)2, 2^m, 2^{m-3}1^6 &= 10, 1, 01 \oplus (2m-3)2, 2^{m-1}1, 2^{m-3}1^5 \\
&= (m-1)1, 1^m, 1^{m-3}1^3 \oplus (m-1)1, 1^m, 1^{m-3}1^3 \\
m \geq 3 \;\Rightarrow\; [\Delta(D_{2m})] &= 1^{6m+6} \cdot 2^{(m-1)(m-4)} \times 1^6 \cdot 2^{2m-4} = 1^{6m+10} \cdot 2^{m(m-3)} \\
D_{2m} &= H_1 \oplus D_{2m-1} && : 6m \\
&= H_m \oplus H_m && : 10 \\
&= 2H_1 \oplus D_{2m-2} && : m(m-3)
\end{aligned}$$

$$D_n \underset{R2E0}{\xrightarrow{2}} D_{n-2}, \quad D_n \xrightarrow{1} D_{n-1}, \quad D_{2m+1} \xrightarrow{1} E_{2m}$$

13.9.5. E_n**.** ($E_5 = C_5$, $E_4 = EO_4$, $E_3 = H_3$)

$$\begin{aligned}
u_{E_3} &= x^{-\lambda_0-\mu-\mu_3}\partial^{-\mu_3}(1-x)^{\lambda'_3} u_{H_2} \\
u_{E_4} &= \partial^{-\mu_4} u_{E_3} \\
u_{E_n} &= \partial^{-\mu_n}(1-x)^{\lambda'_n} u_{E_{n-2}} \quad (n \geq 5) \\
(2m-1)2, 2^{m-1}1^3, 2^{m-1}1^3 &= 10, 01, 10 \oplus (2m-2)2, 2^{m-1}1^2, 2^{m-2}1^4 \\
&= (m-1)1, 1^{m-1}1, 1^{m-1}1 \oplus m1, 1^{m-1}1^3, 1^{m-1}1^2 \\
&= (m-2)1, 1^{m-1}0, 1^{m-1}0 \oplus (m+1)1, 1^{m-1}1^2, 1^{m-1}1^3 \\
m \geq 2 \;\Rightarrow\; [\Delta(E_{2m+1})] &= 1^{6m-2} \cdot 2^{(m-2)^2} \times 1^6 \cdot 2^{2m-3} = 1^{6m+4} \cdot 2^{(m-1)^2} \\
E_{2m+1} &= H_1 \oplus E_{2m} && : 6(m-1) \\
&= H_{m-1} \oplus H_{m+2} && : 1 \\
&= H_m \oplus H_{m+1} && : 9 \\
&= 2H_1 \oplus E_{2m-1} && : (m-1)^2
\end{aligned}$$

$$\begin{aligned}
(2m-2)2, 2^{m-1}1^2, 2^{m-2}1^4 &= 10, 10, 01 \oplus (2m-3)2, 2^{m-2}1^3, 2^{m-2}1^3 \\
&= 10, 01, 10 \oplus (2m-3)2, 2^{m-1}1, 2^{m-3}1^5
\end{aligned}$$

$$
\begin{aligned}
&= (m-2)1, 1^{m-1}0, 1^{m-2}1 \oplus m1, 1^{m-1}1^2, 1^{m-2}1^3\\
&= (m-1)1, 1^{m-1}1, 1^{m-2}1^2 \oplus (m-1)1, 1^{m-1}1, 1^{m-2}1^2\\
m \geq 2 \ \Rightarrow\ [\Delta(E_{2m})] &= 1^{6m-4} \cdot 2^{(m-2)(m-3)} \times 1^6 \cdot 2^{2m-4}\\
&= 1^{6m+2} \cdot 2^{(m-1)(m-2)}\\
E_{2m} &= H_1 \oplus E_{2m-1} && : 4(m-1)\\
&= H_1 \oplus D_{2m-1} && : 2(m-2)\\
&= H_{m-1} \oplus H_{m+1} && : 4\\
&= H_m \oplus H_m && : 6\\
&= 2H_1 \oplus E_{2m-2} && : (m-1)(m-2)
\end{aligned}
$$

$$E_n \xrightarrow[R2E0]{2} E_{n-2}, \quad E_n \xrightarrow{1} E_{n-1}, \quad E_{2m} \xrightarrow{1} D_{2m-1}$$

13.9.6. F_n**.** ($F_5 = B_5$, $F_4 = EO_4$, $F_3 = H_3$)

$$
\begin{aligned}
u_{F_3} &= u_{H_3}\\
u_{F_4} &= \partial^{-\mu_4}(1-x)^{-\lambda_1-\lambda_0^{(3)}-\mu^{(3)}} u_{F_3}\\
u_{F_n} &= \partial^{-\mu_n}(1-x)^{\lambda'_n} u_{F_{n-2}} \quad (n \geq 5)\\
(2m-1)1^2, 2^m1, 2^{m-1}1^3 &= 10, 10, 01 \oplus (2m-2)1^2, 2^{m-1}1^2, 2^{m-1}1^2\\
&= 10, 01, 10 \oplus (2m-2)1^2, 2^m, 2^{m-2}1^4\\
&= (m-1)1, 1^m0, 1^{m-1}1 \oplus m1, 1^m1, 1^{m-1}1^2\\
m \geq 1 \ \Rightarrow\ [\Delta(F_{2m+1})] &= 1^{4m+1} \cdot 2^{(m-1)(m-2)} \times 1^4 \cdot 2^{2m-2} = 1^{4m+5} \cdot 2^{m(m-1)}\\
F_{2m+1} &= H_1 \oplus G_{2m} && : 3m\\
&= H_1 \oplus F_{2m} && : m-1\\
&= H_m \oplus H_{m+1} && : 6\\
&= 2H_1 \oplus F_{2m-1} && : m(m-1)
\end{aligned}
$$

$$
\begin{aligned}
(2m-2)1^2, 2^m, 2^{m-2}1^4 &= 10, 1, 01 \oplus (2m-3)1^2, 2^{m-1}1, 2^{m-2}1^3\\
&= (m-1)1, 1^m, 1^{m-2}1^2 \oplus (m-1)1, 1^m, 1^{m-2}1^2\\
m \geq 2 \ \Rightarrow\ [\Delta(F_{2m})] &= 1^{4m+2} \cdot 2^{(m-1)(m-3)} \times 1^4 \cdot 2^{2m-3} = 1^{4m+6} \cdot 2^{m(m-2)}\\
F_{2m} &= H_1 \oplus F_{2m-1} && : 4m\\
&= H_m \oplus H_m && : 6\\
&= 2H_1 \oplus F_{2m-2} && : m(m-2)
\end{aligned}
$$

$$F_n \xrightarrow[R2E0]{2} F_{n-2}, \quad F_n \xrightarrow{1} F_{n-1}, \quad F_{2m+1} \xrightarrow{1} G_{2m}$$

13.9.7. G_{2m}**.** ($G_4 = B_4$)

$$
\begin{aligned}
u_{G_2} &= u_{H_2}\\
u_{G_{2m}} &= \partial^{-\mu_{2m}}(1-x)^{\lambda'_{2m}} u_{G_{2m-2}}\\
(2m-2)1^2, 2^{m-1}1^2, 2^{m-1}1^2 &= 10, 01, 01 \oplus (2m-3)1^2, 2^{m-1}1, 2^{m-2}1^3\\
&= (m-2)1, 1^{m-1}0, 1^{m-1}0 \oplus m1, 1^{m-1}1^2, 1^{m-1}1^2
\end{aligned}
$$

$$m \geq 2 \ \Rightarrow\ [\Delta(G_{2m})] = 1^{4m-2}\cdot 2^{(m-2)^2} \times 1^4 \cdot 2^{2m-3} = 1^{4m+2}\cdot 2^{(m-1)^2}$$

$$\begin{aligned} G_{2m} &= H_1 \oplus F_{2m-1} && : 4m \\ &= H_{m-1}\oplus H_{m+1} && : 2 \\ &= 2H_1 \oplus G_{2m-2} && : (m-1)^2 \end{aligned}$$

$$G_{2m} = H_1 \oplus F_{2m-1} = H_{m-1}\oplus H_{m+1}$$

$$G_{2m} \underset{R2E0}{\xrightarrow{2}} G_{2(m-1)}, \quad G_{2m} \xrightarrow{1} F_{2m-1}$$

13.9.8. I_n**.** $(I_{2m+1} = \mathrm{III}^*_m,\ I_{2m} = \mathrm{II}^*_m,\ I_3 = P_3)$

$$u_{I_{2m+1}} = \partial^{-\mu'}x^{\lambda'}(c-x)^{\lambda''}u_{H_m}$$

$$\begin{aligned} &(2m)1, m+1m, m+11^m, m+11^m \\ &\quad = 10,10,10,01 \oplus (2m-1)1, mm, m1^m, m+11^{m-1} \\ &\quad = 20,11,11,11 \oplus (2m-2)1, mm-1, m1^{m-1}, m1^{m-1} \end{aligned}$$

$$[\Delta(I_{2m+1})] = 1^{m^2}\times 1^{2m}\cdot m\cdot(m+1) = 1^{m^2+2m}\cdot m\cdot (m+1)$$

$$\begin{aligned} I_{2m+1} &= H_1 \oplus I_{2m} && : 2m \\ &= H_2 \oplus I_{2m-1} && : m^2 \\ &= mH_1 \oplus H_{m+1} && : 1 \\ &= (m+1)H_1 \oplus H_m && : 1 \end{aligned}$$

$$u_{I_{2m}} = \partial^{-\mu'}(1-cx)^{\lambda''}u_{H_m}$$

$$\begin{aligned} &(2m-1)1, mm, m1^m, m+11^{m-1} \\ &\quad = 10,01,01,10 \oplus (2m-2)1, mm-1, m1^{m-1}, m1^{m-1} \\ &\quad = 20,11,11,11 \oplus (2m-3)1, m-1m-1, m-11^{m-1}, m1^{m-2} \end{aligned}$$

$$[\Delta(I_{2m})] = 1^{m^2}\times 1^m\cdot m^2 = 1^{m(m+1)}\cdot m^2$$

$$\begin{aligned} I_{2m} &= H_1 \oplus I_{2m-1} && : 2m \\ &= H_2 \oplus I_{2m-2} && : m(m-1) \\ &= mH_1 \oplus H_m && : 2 \end{aligned}$$

$$I_{2m+1} \xrightarrow{m+1} H_m, \quad I_{2m+1} \xrightarrow{m} H_{m+1}, \quad I_{2m} \xrightarrow{m} H_m, \quad I_n \xrightarrow{1} I_{n-1}$$

$$I_{2m+1} \underset{R1E0}{\longrightarrow} I_{2m} \underset{R2E0}{\longrightarrow} I_{2m-2}$$

13.9.9. J_n**.** $(J_4 = I_4,\ J_3 = P_3)$

$$\begin{aligned} u_{J_2} &= (c-x)^{\lambda'}u_{H_2} \\ u_{J_3} &= u_{P_3} \\ u_{J_n} &= \partial^{-\mu'_n}x^{\lambda'_n}u_{J_{n-2}} \quad (n\geq 4) \end{aligned}$$

$$\begin{aligned} &(2m)1, (2m)1, 2^m1, 2^m1 \\ &\quad = 10,10,01,10 \oplus (2m-1)1, (2m-1)1, 2^m, 2^{m-1}11 \\ &\quad = (m-1)1, m0, 1^m0, 1^m0 \oplus (m+1), m1, 1^m1, 1^m1 \end{aligned}$$

$$[\Delta(J_{2m+1})] = 1^{2m}\cdot 2^{(m-1)^2}\times 1^2\cdot 2^{2m-1} = 1^{2m+2}\cdot 2^{m^2}$$

$$\begin{aligned} J_{2m+1} &= H_1 \oplus J_{2m} && : 2m \\ &= H_m \oplus H_{m+1} && : 2 \\ &= 2H_1 \oplus J_{2m-2} && : m^2 \end{aligned}$$

$$\begin{aligned}
&(2m-1)1,(2m-1)1,2^m,2^{m-1}1^2\\
&\qquad = 10,10,1,01 \oplus (2m-2)1,(2m-2)1,2^{m-1}1,2^{m-1}1\\
&\qquad = (m-1)1,m0,1^m,1^{m-1}1 \oplus m0,(m-1)1,1^m,1^{m-1}1\\
[\Delta(J_{2m})] &= 1^{2m}\cdot 2^{(m-1)(m-2)} \times 1^2\cdot 2^{2m-2} = 1^{2m+2}\cdot 2^{m(m-1)}\\
J_{2m} &= H_1 \oplus J_{2m-1} \qquad : 2m\\
&= H_m \oplus H_m \qquad : 2\\
&= 2H_1 \oplus J_{2m-2} \qquad : m(m-1)
\end{aligned}$$

$$J_n \xrightarrow[R2E0]{2} J_{n-2}\ (n\geq 6),\quad J_n \xrightarrow{1} J_{n-1}$$

13.9.10. K_n**.** ($K_5 = M_5$, $K_4 = I_4$, $K_3 = P_3$)

$$u_{K_{2m+1}} = \partial^{\mu+\lambda'+\lambda''}(c'-x)^{\lambda'}(c''-x)^{\lambda''}u_{P_m}$$

$$m+1m,m+1m,(2m)1,(2m)1,(2m)1,\ldots \in \mathcal{P}^{(2m+1)}_{m+3}$$

$$= 11,11,11,20,20,\ldots \oplus mm-1,mm-1,(2m-1)0,(2m-2)1,(2m-2)1,\ldots$$

$$[\Delta(K_{2m+1})] = 1^{m+1}\cdot(m-1)\times m^2\cdot(m+1) = 1^{m+1}\cdot(m-1)\cdot m^2\cdot(m+1)$$

$$\begin{aligned}
K_{2m+1} &= H_2 \oplus K_{2m-1} \qquad : m+1\\
&= (m-1)H_1 \oplus P_{m+2} \qquad : 1\\
&= mH_1 \oplus P_{m+1} \qquad : 2\\
&= (m+1)H_1 \oplus P_m \qquad : 1
\end{aligned}$$

$$u_{K_{2m}} = \partial^{-\mu'}(c'-x)^{\lambda'}u_{P_m}$$

$$mm,mm-11,(2m-1)1,(2m-1)1,\ldots \in \mathcal{P}^{(2m)}_{m+2}$$

$$= 01,001,10,10,10,\ldots \oplus mm-1,mm-10,(2m-2)1,(2m-2)1,\ldots$$

$$= 11,110,11,20,20,\ldots \oplus m-1m-1,m-1m-21,(2m-2)0,(2m-3)1,\ldots$$

$$[\Delta(K_{2m})] = 1^{m+1}\cdot(m-1)\times 1\cdot(m-1)\cdot m^2 = 1^{m+2}\cdot(m-1)^2\cdot m^2$$

$$\begin{aligned}
K_{2m} &= H_1 \oplus K_{2m-1} \qquad : 2\\
&= H_2 \oplus K_{2m-2} \qquad : m\\
&= (m-1)H_1 \oplus P_{m+1} \qquad : 2\\
&= mH_1 \oplus P_m \qquad : 2
\end{aligned}$$

$$K_{2m+1} \xrightarrow{m+1} P_m,\quad K_{2m+1} \xrightarrow[R1]{m} P_{m+1},\quad K_{2m+1} \xrightarrow{m-1} P_{m+2}$$

$$K_{2m} \xrightarrow[R1]{m} P_m,\quad K_{2m} \xrightarrow{m-1} P_{m+1},\quad K_{2m} \xrightarrow{1} K_{2m-1}$$

13.9.11. L_{2m+1}**.** ($L_5 = J_5$, $L_3 = H_3$)

$$u_{L_{2m+1}} = \partial^{-\mu'}x^{\lambda'}u_{P_{m+1}}$$

$$mm1,mm1,(2m)1,(2m)1,\ldots \in \mathcal{P}^{(2m+1)}_{m+2}$$

$$= 001,010,10,10,\ldots \oplus mm0,mm-11,(2m-1)1,(2m-1)1,\ldots$$

$$= 110,110,11,20,\ldots \oplus m-1m-10,m-1m-11,(2m-1)0,(2m-2)1,\ldots$$

$$[\Delta(L_{2m+1})] = 1^{m+2}\cdot m\times 1^2\cdot m^3 = 1^{m+4}\cdot m^4$$

$$\begin{aligned}
L_{2m+1} &= H_1 \oplus K_{2m} \qquad : 4\\
&= H_2 \oplus L_{2m-1} \qquad : m\\
&= mH_1 \oplus P_{m+1} \qquad : 4
\end{aligned}$$

$$L_{2m+1} = H_1 \oplus K_{2m}, \quad L_{2m+1} = H_2 \oplus L_{2m-1}$$

$$L_{2m+1} \xrightarrow[R2E0]{m} P_{m+1}, \quad L_{2m+1} \xrightarrow{1} K_{2m}$$

13.9.12. M_n**.** ($M_5 = K_5$, $M_4 = I_4$, $M_3 = P_3$)

$$u_{M_{2m+1}} = \partial^{\mu+\lambda'_3+\cdots+\lambda'_{m+2}}(c_3-x)^{\lambda'_3}\cdots(c_{m+2}-x)^{\lambda'_{m+2}}u_{H_2}$$

$$\begin{aligned}
&(2m)1,(2m)1,(2m)1,(2m-1)2,(2m-1)2,\ldots \in \mathcal{P}^{(2m+1)}_{m+3}\\
&\quad = m-11, m0, m0, m-11, m-11, \ldots \oplus m+10, m1, m1, m1, m1, \ldots\\
&\quad = m-10, m-10, m-10, m-21, m-21, \ldots\\
&\qquad \oplus m+11, m+11, m+11, m+11, m+11, \ldots\\
&[\Delta(M_{2m+1})] = 1^4 \times 2^m \cdot (2m-1) = 1^4 \cdot 2^m \cdot (2m-1)
\end{aligned}$$

$$\begin{aligned}
M_{2m+1} &= P_{m-1} \oplus P_{m+2} && : 1\\
&= P_m \oplus P_{m+1} && : 3\\
&= 2H_1 \oplus M_{2m-1} && : m\\
&= (2m-1)H_1 \oplus H_2 && : 1
\end{aligned}$$

$$u_{M_{2m}} = \partial^{-\mu'}(c_3-x)^{\lambda'_3}\cdots(c_{m+1}-x)^{\lambda'_{m+1}}u_{H_2}$$

$$\begin{aligned}
&(2m-2)1^2,(2m-1)1,(2m-1)1,(2m-2)2,\ldots \in \mathcal{P}^{(2m)}_{m+2}\\
&\quad = 01, 10, 10, 10, \ldots \oplus (2m-2)1,(2m-2)1,(2m-2)1,(2m-3)2,\ldots\\
&\quad = m-21, m-10, m-10, m-21, \ldots \oplus m1, m1, m1, m1, \ldots\\
&\quad = m-11, m-11, m0, m-11, \ldots \oplus m-11, m0, m-11, m-11, \ldots\\
&[\Delta(M_{2m})] = 1^4 \times 1^2 \cdot 2^{m-1} \cdot (2m-2) = 1^6 \cdot 2^{m-1} \cdot (2m-2)
\end{aligned}$$

$$\begin{aligned}
M_{2m} &= H_1 \oplus M_{2m-1} && : 2\\
&= P_{m-1} \oplus P_{m+1} && : 2\\
&= P_m \oplus P_m && : 2\\
&= 2H_1 \oplus M_{2m-2} && : m-1\\
&= (2m-2)H_1 \oplus H_2 && : 1
\end{aligned}$$

$$M_n \xrightarrow{n-2} H_2, \quad M_n \xrightarrow{2} M_{n-2}, \quad M_{2m} \xrightarrow[R1E0]{1} M_{2m-1} \xrightarrow[R1]{} M_{2m-3}$$

13.9.13. N_n**.** ($N_6 = \mathrm{IV}^*$, $N_5 = I_5$, $N_4 = G_4$, $N_3 = H_3$)

$$u_{N_{2m+1}} = \partial^{-\mu'}x^{\lambda'}(c_3-x)^{\lambda'_3}\cdots(c_{m+1}-x)^{\lambda'_{m+1}}u_{H_2}$$

$$\begin{aligned}
&(2m-1)1^2,(2m-1)1^2,(2m)1,(2m-1)2,(2m-1)2,\ldots \in \mathcal{P}^{(2m+1)}_{m+2}\\
&\quad = 10, 01, 10, 10, 10\ldots\\
&\qquad \oplus (2m-2)1^2,(2m-1)1,(2m-1)1,(2m-2)2,(2m-2)2,\ldots\\
&\quad = m-11, m-11, m0, m-11, m-11, \ldots \oplus m1, m1, m1, m1, m1, \ldots\\
&[\Delta(N_{2m+1})] = 1^4 \times 1^4 \cdot 2^{m-1} \cdot (2m-1) = 1^8 \cdot 2^{m-1} \cdot (2m-1)
\end{aligned}$$

$$\begin{aligned}
N_{2m+1} &= H_1 \oplus M_{2m} && : 4\\
&= P_m \oplus P_{m+1} && : 4\\
&= 2H_1 \oplus N_{2m-1} && : m-1\\
&= (2m-1)H_1 \oplus H_2 && : 1
\end{aligned}$$

$$u_{N_{2m}} = \partial^{-\mu'} x^{\lambda'_0}(1-x)^{\lambda'_1}(c_3-x)^{\lambda'_3}\cdots(c_m-x)^{\lambda'_m} u_{H_2} \quad (m \ge 2)$$

$$\begin{aligned}
&(2m-2)1^2, (2m-2)1^2, (2m-2)1^2, (2m-2)2, (2m-2)2, \ldots \in \mathcal{P}_{m+1}^{(2m)}\\
&\quad = 01, 10, 10, 10, 10\ldots\\
&\qquad \oplus (2m-2)1, (2m-3)1^2, (2m-3)1^2, (2m-3)2, (2m-3)2, \ldots\\
&\quad = m-11, m-11, m-11, m-11, m-11, \ldots\\
&\qquad \oplus m-11, m-11, m-11, m-11, m-11, \ldots
\end{aligned}$$

$$[\Delta(N_{2m})] = 1^4 \times 1^6 \cdot 2^{m-2} \cdot (2m-2) = 1^{10} \cdot 2^{m-2} \cdot (2m-2)$$

$$\begin{aligned}
N_{2m} &= H_1 \oplus N_{2m-1} && : 6\\
&= P_m \oplus P_m && : 4\\
&= 2H_1 \oplus N_{2m-2} && : m-2\\
&= (2m-2)H_1 \oplus H_2 && : 1
\end{aligned}$$

$$N_n \xrightarrow{n-2} H_2, \quad N_n \xrightarrow{2} N_{n-2}, \quad N_{2m+1} \xrightarrow[R1E0]{1} M_{2m}, \quad N_{2m} \xrightarrow[R1E0]{1} N_{2m-1}$$

13.9.14. minimal series. The tuple $11, 11, 11$ corresponds to Gauss hypergeometric series, which has three parameters. Since the action of additions is easily analyzed, we consider the number of parameters of the equation corresponding to a rigid tuple $\mathbf{m} = \big(m_{j,\nu}\big)_{\substack{0\le j\le p\\ 1\le\nu\le n_j}} \in \mathcal{P}_{p+1}^{(n)}$ modulo additions and the Fuchs condition equals

$$n_0 + n_1 + \cdots + n_p - (p+1). \tag{13.44}$$

Here we assume that $0 < m_{j,\nu} < n$ for $1 \le \nu \le n_j$ and $j = 0, \ldots, p$.

We call the number given by (13.44) the *effective length* of $\mathbf{m}$. The tuple $11, 11, 11$ is the unique rigid tuple of partitions whose effective length equals 3. Since the reduction ∂_{max} never increase the effective length and the tuple $\mathbf{m} \in \mathcal{P}_3$ satisfying $\partial_{max} = 11, 11, 11$ is $21, 111, 111$ or $211, 211, 211$, it is easy to see that the non-trivial rigid tuple $\mathbf{m} \in \mathcal{P}_3$ whose effective length is smaller than 6 is H_2 or H_3.

The rigid tuple of partitions with the effective length 4 is also uniquely determined by its order, which is

$$\begin{aligned}
P_{4,2m+1} &: m+1m, m+1m, m+1m, m+1m\\
P_{4,2m} &: m+1m-1, mm, mm, mm
\end{aligned} \tag{13.45}$$

with $m \in \mathbb{Z}_{>0}$. Here $P_{4,2m+1}$ is a generalized Jordan-Pochhammer tuple in Example 10.5 i).

In fact, if $\mathbf{m} \in \mathcal{P}$ is rigid with the effective length 4, the argument above shows $\mathbf{m} \in \mathcal{P}_4$ and $n_j = 2$ for $j = 0, \ldots, 3$. Then $2 = \sum_{j=0}^3 m_{j,1}^2 + \sum_{j=0}^3 (n - m_{j,1})^2 - 2n^2$ and $\sum_{j=0}^3 (n - 2m_{j,1})^2 = 4$ and therefore $\mathbf{m} = P_{4,2m+1}$ or $P_{4,2m}$.

We give decompositions of $P_{4,n}$:

$$\begin{aligned}
&m+1, m; m+1, m; m+1, m; m+1, m\\
&\quad = k, k+1; k+1, k; k+1, k; k+1, k\\
&\qquad \oplus m-k+1, m-k-1; m-k, m-k; m-k, m-k; m-k, m-k\\
&\quad = 2(k+1, k; k+1, k; k+1, k; \ldots)\\
&\qquad \oplus m-2k-1, m-2k; m-2k-1, m-2k; m-2k-1, m-2k; \ldots
\end{aligned}$$

$$[\Delta(P_{4,2m+1})] = 1^{4m-4} \cdot 2^{m-1} \times 1^4 \cdot 2 = 1^{4m} \cdot 2^m$$

$$\begin{aligned}
P_{4,2m+1} &= P_{4,2k+1} \oplus P_{4,2(m-k)} && : 4 \quad (k = 0, \ldots, m-1)\\
&= 2P_{4,2k+1} \oplus P_{4,2m-4k-1} && : 1 \quad (k = 0, \ldots, m-1)
\end{aligned}$$

Here $P_{k,-n} = -P_{k,n}$ and in the above decompositions there appear "tuples of partitions" with negative entries corresponding formally to elements in Δ^{re} with (7.12) (cf. Remark 7.11 i)).

It follows from the above decompositions that the Fuchsian equation with the Riemann scheme

$$\begin{Bmatrix} \infty & 0 & 1 & c_3 \\ [\lambda_{0,1}]_{(m+1)} & [\lambda_{1,1}]_{(m+1)} & [\lambda_{2,1}]_{(m+1)} & [\lambda_{3,1}]_{(m+1)} \\ [\lambda_{0,2}]_{(m)} & [\lambda_{1,2}]_{(m)} & [\lambda_{2,1}]_{(m)} & [\lambda_{3,2}]_{(m)} \end{Bmatrix}$$

$$\sum_{j=0}^{4}\big((m+1)\lambda_{j,1}+m\lambda_{j,2}\big)=2m \qquad \text{(Fuchs relation).}$$

is irreducible if and only if

$$\sum_{j=0}^{4}\sum_{\nu=1}^{2}\big(k+\delta_{\nu,1}+(1-2\delta_{\nu,1})\delta_{j,i}\big)\lambda_{j,\nu}\notin\mathbb{Z} \qquad (i=0,1,\ldots,5,\ k=0,1,\ldots,m).$$

When $\mathbf{m}=P_{4,2m}$, we have the following.

$$\begin{aligned}
&m+1,m-1;m,m;m,m;m,m\\
&\quad=k+1,k;k+1,k;k+1,k;k+1,k\\
&\qquad\oplus m-k,m-k-1;m-k-1,m-k;m-k-1,m-k;m-k-1,m-k\\
&\quad=2(k+1,k-1;k,k;k,k;k,k)\\
&\qquad\oplus m-2k-1,m-2k+1;m-2k,m-2k;m-2k,m-2k;m-2k;m-2k\\
&[\Delta(P_{4,2m})]=1^{4m-4}\cdot 2^{m-1}\times 1^4=1^{4m}\cdot 2^{m-1}\\
&\quad P_{4,2m}=P_{4,2k+1}(=k+1,k;k+1,k;\ldots)\oplus P_{4,2m-2k+1} && :4\ \ (k=0,\ldots,m-1)\\
&\qquad\quad=2P_{4,2k}\oplus P_{4,2m-4k} && :1\ \ (k=1,\ldots,m-1)
\end{aligned}$$

$$P_{4,n}\xrightarrow{1}P_{4,n-1},\ P_{4,2m+1}\xrightarrow{2}P_{4,2m-1}$$

Roberts [**Ro**] classifies the rigid tuples $\mathbf{m}\in\mathcal{P}_{p+1}$ so that

$$\frac{1}{n_0}+\cdots+\frac{1}{n_p}\geq p-1. \tag{13.46}$$

They are tuples $\mathbf{m}$ in 4 series α, β, γ, δ, which are close to the tuples $r\tilde{E}_6$, $r\tilde{E}_7$, $r\tilde{E}_8$ and $r\tilde{D}_4$ with $r=1,2,\ldots$, namely, $(n_0,\ldots,n_p)=(3,3,3)$, $(2,2,4)$, $(2,3,6)$ and $(2,2,2,2)$, respectively (cf. (7.46)), and the series are called *minimal series*. Then $\delta_n=P_{4,n}$ and the tuples in the other three series belong to $\mathcal{P}_3$. For example, the tuples $\mathbf{m}$ of type α are

$$\begin{aligned}
\alpha_{3m}&=m+1mm-1,m^3,m^3, && \alpha_3=H_3,\\
\alpha_{3m\pm1}&=m^2m\pm1,m^2m\pm1,m^2m\pm1, && \alpha_4=B_4,
\end{aligned} \tag{13.47}$$

which are characterized by the fact that their effective lengths equal 6 when $n \geq 4$. As in other series, we have the following:

$$\alpha_n \xrightarrow{1} \alpha_{n-1}, \quad \alpha_{3m+1} \xrightarrow{2} \alpha_{3m-1}$$
$$[\Delta(\alpha_{3m})] = [\Delta(\alpha_{3m-1})] \times 1^5, \ [\Delta(\alpha_{3m-1})] = [\Delta(\alpha_{3m-2})] \times 1^4,$$
$$[\Delta(\alpha_{3m-2})] = [\Delta(\alpha_{3m-4})] \times 1^6 \cdot 2$$
$$[\Delta(\alpha_{3m-1})] = [\Delta(\alpha_2)] \times 1^{10(m-1)} \cdot 2^{m-1} = 1^{10m-6} \cdot 2^{m-1}$$
$$[\Delta(\alpha_{3m})] = 1^{10m-1} \cdot 2^{m-1}$$
$$[\Delta(\alpha_{3m-2})] = 1^{10m-10} \cdot 2^{m-1}$$

$$\begin{aligned}
\alpha_{3m} &= m+1mm-1, m^3, m^3 \\
&= kkk-1, k^2k-1, k^2k-1 \\
&\oplus (m-k+1)(m-k)(m-k), (m-k)^2(m-k+1), (m-k)^2(m-k+1) \\
&= k+1k-1k, k^3, k^3 \\
&\oplus (m-k+1)(m-k)(m-k-1), (m-k)^3, (m-k)^3 \\
&= 2(k+1kk-1, k^3, k^3) \\
&\oplus (m-2k-1)(m-2k)(m-2k+1), (m-2k)^3, (m-2k)^3
\end{aligned}$$

$$\begin{aligned}
\alpha_{3m} &= \alpha_{3k-1} \oplus \alpha_{3(m-k)+1} && : 9 \quad (k = 1, \dots, m) \\
&= \alpha_{3k} \oplus \alpha_{3(m-k)} && : 1 \quad (k = 1, \dots, m-1) \\
&= 2\alpha_{3k} \oplus \alpha_{3(m-2k)} && : 1 \quad (k = 1, \dots, m-1)
\end{aligned}$$

$$\begin{aligned}
\alpha_{3m-1} &= mmm-1, mmm-1, mmm-1 \\
&= kk-1k-1, kk-1k-1, kk-1k-1 \\
&\oplus (m-k)(m-k+1)(m-k), (m-k)(m-k+1)(m-k), \cdots \\
&= k+1kk-1, k^3, k^3 \\
&\oplus (m-k-1)(m-k)(m-k), (m-k)(m-k)(m-k-1), \cdots \\
&= 2(kkk-1, kkk-1, kkk-1) \\
&\oplus (m-2k)(m-2k)(m-2k+1), (m-2k)(m-2k)(m-2k+1), \cdots
\end{aligned}$$

$$\begin{aligned}
\alpha_{3m-1} &= \alpha_{3k-2}(= k, k-1, k-1; \cdots) \oplus \alpha_{3(m-k)+1} && : 4 \quad (k = 1, \dots, m) \\
&= \alpha_{3k} \oplus \alpha_{3(m-k)-1} && : 6 \quad (k = 1, \dots, m-1) \\
&= 2\alpha_{3k-1} \oplus \alpha_{3(m-2k)+1} && : 1 \quad (k = 1, \dots, m-1)
\end{aligned}$$

$$\begin{aligned}
\alpha_{3m-2} &= mm-1m-1, mm-1m-1, mm-1m-1 \\
&= kkk-1, kkk-1, kkk-1 \\
&\oplus (m-k)(m-k-1)(m-k), (m-k)(m-k-1)(m-k), \cdots \\
&= k+1kk-1, k^3, k^3 \\
&\oplus (m-k-1)(m-k-1)(m-k), (m-k)(m-k-1)(m-k-1), \cdots \\
&= 2(kk-1k-1, kk-1k-1, kk-1k-1) \\
&\oplus (m-2k)(m-2k+1)(m-2k+1), (m-2k)(m-2k+1)(m-2k+1), \cdots
\end{aligned}$$

$$\begin{aligned}
\alpha_{3m-2} &= \alpha_{3k-1}(= k, k-1, k-1; \cdots) \oplus \alpha_{3(m-k)-1} && : 4 \quad (k = 1, \dots, m-1) \\
&= \alpha_{3k} \oplus \alpha_{3(m-k)-2} && : 6 \quad (k = 1, \dots, m-1) \\
&= 2\alpha_{3k-2} \oplus \alpha_{3(m-2k)+2} && : 1 \quad (k = 1, \dots, m-1)
\end{aligned}$$

The analysis of the other minimal series

$$
\begin{aligned}
\beta_{4m,2} &= (2m+1)(2m-1), m^4, m^4 && \beta_{4,2} = H_4\\
\beta_{4m,4} &= (2m)^2, m^4, (m+1)m^2(m-1) && \beta_{4,4} = EO_4\\
\beta_{4m\pm1} &= (2m)(2m\pm1), (m\pm1)m^3, (m\pm1)m^3 && \beta_5 = C_5,\ \beta_3 = H_3\\
\beta_{4m+2} &= (2m+1)^2, (m+1)^2m^2, (m+1)^2m^2
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{6m,2} &= (3m+1)(3m-1), (2m)^3, m^6 && \gamma_{6,2} = D_6 = X_6\\
\gamma_{6m,3} &= (3m)^2, (2m+1)(2m)(2m-1), m^6 && \gamma_{6,3} = EO_6\\
\gamma_{6m,6} &= (3m)^2, (2m)^3, (m+1)m^4(m-1)\\
\gamma_{6m\pm1} &= (3m)(3m\pm1), (2m)^2(2m\pm1), m^5(m\pm1) && \gamma_5 = EO_5\\
\gamma_{6m\pm2} &= (3m\pm1)(3m\pm1), (2m)(2m\pm1)^2, m^4(m\pm1)^2 && \gamma_4 = EO_4\\
\gamma_{6m+3} &= (3m+2)(3m+1), (2m+1)^3, (m+1)^3m^3 && \gamma_3 = H_3
\end{aligned}
$$

and general $P_{p+1,n}$ will be left to the reader as an exercise.

13.9.15. Relation between series. We have studied the following sets of families of spectral types of Fuchsian differential equations which are closed under the irreducible subquotients in the Grothendieck group.

$\{H_n\}$	(hypergeometric family)	
$\{P_n\}$	(Jordan-Pochhammer family)	
$\{A_n = EO_n\}$	(even/odd family)	
$\{B_n,\ C_n,\ H_n\}$	(3 singular points)	
$\{C_n,\ H_n\}$	(3 singular points)	
$\{D_n,\ E_n,\ H_n\}$	(3 singular points)	
$\{F_n,\ G_{2m},\ H_n\}$	(3 singular points)	
$\{I_n,\ H_n\}$	(4 singular points)	
$\{J_n,\ H_n\}$	(4 singular points)	
$\{K_n,\ P_n\}$	($[\frac{n+5}{2}]$ singular points)	
$\{L_{2m+1},\ K_n,\ P_n\}$	($m+2$ singular points)	
$\{M_n,\ P_n\}$	($[\frac{n+5}{2}]$ singular points)	$\supset \{M_{2m+1},\ P_n\}$
$\{N_n,\ M_n,\ P_n\}$	($[\frac{n+3}{2}]$ singular points)	$\supset \{N_{2m+1},\ M_n.\ P_n\}$
$\{P_{4,n} = \delta_n\}$	(4 effective parameters)	
$\{\alpha_n\}$	(6 effective parameters and 3 singular points)	

Yokoyama classified $\mathbf{m} = \big(m_{j,\nu}\big)_{\substack{0\le j\le p\\ 1\le\nu\le n_j}} \in \mathcal{P}_{p+1}$ such that

(13.48) $\mathbf{m}$ is irreducibly realizable,

(13.49) $m_{0,1} + \cdots + m_{p-1,1} = (p-1)\operatorname{ord}\mathbf{m}$ (i.e. $\mathbf{m}$ is of Okubo type),

(13.50) $m_{j,\nu} = 1 \quad (0 \le j \le p-1,\ 2 \le \nu \le n_j)$.

The tuple $\mathbf{m}$ satisfying the above conditions is in the following list given by [**Yo**, Theorem 2] (cf. [**Ro**]).

Yokoyama	type	order	p+1	tuple of partitions
I_n	H_n	n	3	$1^n, n-11, 1^n$
I_n^*	P_n	n	$n+1$	$n-11, n-11, \ldots, n-11$
II_n	B_{2n}	$2n$	3	$n1^n, n1^n, nn-11$
II_n^*	I_{2n}	$2n$	4	$n1^n, n+11^{n-1}, 2n-11, nn$
III_n	B_{2n+1}	$2n+1$	3	$n1^{n+1}, n+11^n, nn1$
III_n^*	I_{2n+1}	$2n+1$	4	$n+11^n, n+11^n, (2n)1, n+1n$
IV	F_6	6	3	$21111, 411, 222$
IV^*	N_6	6	4	$411, 411, 411, 42$

13.10. Appell's hypergeometric functions

First we recall the Appell hypergeometric functions.

$$F_1(\alpha;\beta,\beta';\gamma;x,y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_{m+n}m!n!}x^m y^n, \tag{13.51}$$

$$F_2(\alpha;\beta,\beta';\gamma,\gamma';x,y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_m(\gamma')_n m!n!}x^m y^n, \tag{13.52}$$

$$F_3(\alpha,\alpha';\beta,\beta';\gamma;x,y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n}m!n!}x^m y^n, \tag{13.53}$$

$$F_4(\alpha;\beta;\gamma,\gamma';x,y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{m+n}}{(\gamma)_m(\gamma')_n m!n!}x^m y^n. \tag{13.54}$$

They satisfy the following equations

$$\Big((\vartheta_x+\vartheta_y+\alpha)(\vartheta_x+\beta) - \partial_x(\vartheta_x+\vartheta_y+\gamma-1)\Big)F_1 = 0, \tag{13.55}$$

$$\Big((\vartheta_x+\vartheta_y+\alpha)(\vartheta_x+\beta) - \partial_x(\vartheta_x+\gamma-1)\Big)F_2 = 0, \tag{13.56}$$

$$\Big((\vartheta_x+\alpha)(\vartheta_x+\beta) - \partial_x(\vartheta_x+\vartheta_y+\gamma-1)\Big)F_3 = 0, \tag{13.57}$$

$$\Big((\vartheta_x+\vartheta_y+\alpha)(\vartheta_x+\vartheta_y+\beta) - \partial_x(\vartheta_x+\gamma-1)\Big)F_4 = 0. \tag{13.58}$$

Similar equations hold under the symmetry $x \leftrightarrow y$ with $(\alpha,\beta,\gamma) \leftrightarrow (\alpha',\beta',\gamma')$.

13.10.1. Appell's F_1.

First we examine F_1. Put

$$\begin{aligned} u(x,y) &:= \int_0^x t^\alpha(1-t)^\beta(y-t)^{\gamma-1}(x-t)^{\lambda-1}dt \qquad (t=xs) \\ &= \int_0^1 x^{\alpha+\lambda+1}s^\alpha(1-xs)^\beta(y-xs)^{\gamma-1}(1-s)^{\lambda-1}ds \\ &= x^{\alpha+\lambda}y^{\gamma-1}\int_0^1 s^\alpha(1-s)^{\lambda-1}(1-xs)^\beta\Big(1-\frac{y}{x}s\Big)^{\gamma-1}ds, \\ h_x &:= x^\alpha(x-1)^\beta(x-y)^{\gamma-1}. \end{aligned}$$

Since the left ideal of $\overline{W}[x,y]$ is not necessarily generated by a single element, we want to have good generators of $\mathrm{RAd}(\partial_x^{-\lambda})\circ\mathrm{RAd}(h_x)\big(W[x,y]\partial_x + W[x,y]\partial_y\big)$ and we have

$$
\begin{aligned}
&P := \mathrm{Ad}(h_x)\partial_x = \partial_x - \frac{\alpha}{x} - \frac{\beta}{x-1} - \frac{\gamma-1}{x-y},\\
&Q := \mathrm{Ad}(h_x)\partial_y = \partial_y + \frac{\gamma-1}{x-y},\\
&R := xP + yQ = x\partial_x + y\partial_y - (\alpha+\gamma-1) - \frac{\beta x}{x-1},\\
&S := \partial_x(x-1)R = (\vartheta_x+1)(\vartheta_x+\vartheta_y-\alpha-\beta-\gamma+1) - \partial_x(\vartheta_x+\vartheta_y-\alpha-\gamma+1)\\
&T := \partial_x^{-\lambda}\circ S\circ\partial_x^{\lambda}\\
&\quad = (\vartheta_x-\lambda+1)(\vartheta_x+\vartheta_y-\alpha-\beta-\gamma-\lambda+1) - \partial_x(\vartheta_x+\vartheta_y-\alpha-\gamma-\lambda+1)
\end{aligned}
$$

with

$$a = -\alpha-\beta-\gamma-\lambda+1,\ b = 1-\lambda,\ c = 2-\alpha-\gamma-\lambda.$$

This calculation shows the equation $Tu(x,y)=0$ and we have a similar equation by changing $(x,y,\gamma,\lambda)\mapsto(y,x,\lambda,\gamma)$. Note that $TF_1(a;b,b';c;x,y)=0$ with $b'=1-\gamma$.

Putting

$$
\begin{aligned}
v(x,z) &= I_{0,x}^{\mu}(x^{\alpha}(1-x)^{\beta}(1-zx)^{\gamma-1})\\
&= \int_0^x t^{\alpha}(1-t)^{\beta}(1-zt)^{\gamma-1}(x-t)^{\mu-1}dt\\
&= x^{\alpha+\mu}\int_0^1 s^{\alpha}(1-xs)^{\beta}(1-xzs)^{\gamma-1}(1-s)^{\mu-1}ds,
\end{aligned}
$$

we have

$$
\begin{aligned}
u(x,y) &= y^{\gamma-1}v(x,\tfrac{1}{y}),\\
t^{\alpha}(1-t)^{\beta}(1-zt)^{\gamma-1} &= \sum_{m,n=0}^{\infty}\frac{(-\beta)_m(1-\gamma)_n}{m!n!}t^{\alpha+m+n}z^n,\\
v(x,z) &= \sum_{m,n=0}^{\infty}\frac{\Gamma(\alpha+m+n+1)(-\beta)_m(1-\gamma)_n}{\Gamma(\alpha+\mu+m+n+1)m!n!}x^{\alpha+\gamma+m+n}z^n\\
&= x^{\alpha+\mu}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\mu+1)}\sum_{m,n=0}^{\infty}\frac{(\alpha+1)_{m+n}(-\beta)_m(1-\gamma)_n}{(\alpha+\mu+1)_{m+n}m!n!}x^{m+n}z^n\\
&= x^{\alpha+\mu}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\mu+1)}F_1(\alpha+1;-\beta,1-\gamma;\alpha+\mu+1;x,xz).
\end{aligned}
$$

Using a versal addition to get the Kummer equation, we introduce the functions

$$
\begin{aligned}
v_c(x,y) &:= \int_0^x t^{\alpha}(1-ct)^{\frac{\beta}{c}}(y-t)^{\gamma-1}(x-t)^{\lambda-1},\\
h_{c,x} &:= x^{\alpha}(1-cx)^{\frac{\beta}{c}}(x-y)^{\gamma-1}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&R := \mathrm{Ad}(h_{c,x})(\vartheta_x+\vartheta_y) = \vartheta_x+\vartheta_y-(\alpha+\gamma-1)+\frac{\beta x}{1-cx},\\
&S := \partial_x(1-cx)R\\
&\quad = (\vartheta_x+1)\bigl(\beta-c(\vartheta_x+\vartheta_y-\alpha-\gamma+1)\bigr)+\partial_x(\vartheta_x+\vartheta_y-\alpha-\gamma+1),\\
&T := \mathrm{Ad}(\partial^{-\lambda})R\\
&\quad = (\vartheta_x-\lambda+1)\bigl(\beta-c(\vartheta_x+\vartheta_y-\lambda-\alpha-\gamma+1)\bigr)+\partial_x(\vartheta_x+\vartheta_y-\lambda-\alpha-\gamma+1)
\end{aligned}
$$

and hence $u_c(x,y)$ satisfies the differential equation

$$\Big(x(1-cx)\partial_x^2 + y(1-cx)\partial_x\partial_y \\ + \big(2-\alpha-\gamma-\lambda+(\beta+\lambda-2+c(\alpha+\gamma+\lambda-1))x\big)\partial_x + (\lambda-1)\partial_y \\ - (\lambda-1)(\beta+c(\alpha+\gamma+\lambda-1))\Big)u = 0.$$

13.10.2. Appell's F_4. To examine F_4 we consider the function

$$v(x,y) := \int_\Delta s^{\lambda_1}t^{\lambda_2}(st-s-t)^{\lambda_3}(1-sx-ty)^\mu ds\,dt$$

and the transformation

$$J_x^\mu(u)(x) := \int_\Delta u(t_1,\dots,t_n)(1-t_1x_1-\cdots-t_nx_n)^\mu dt_1\cdots dt_n \tag{13.59}$$

for function $u(x_1,\dots,u_n)$. For example the region Δ is given by

$$v(x,y) = \int_{s\le 0,\ t\le 0} s^{\lambda_1}t^{\lambda_2}(st-s-t)^{\lambda_3}(1-sx-ty)^\mu ds\,dt.$$

Putting $s\mapsto s^{-1}$, $t\mapsto t^{-1}$ and $|x|+|y|<c<\frac{1}{2}$, Aomoto [**Ao**] shows

$$\begin{aligned}&\int_{c-\infty i}^{c+\infty i}\int_{c-\infty i}^{c+\infty i} s^{-\gamma}t^{-\gamma'}(1-s-t)^{\gamma+\gamma'-\alpha-2}\Big(1-\frac{x}{s}-\frac{y}{t}\Big)^{-\beta}ds\,dt\\ &\quad= -\frac{4\pi^2\Gamma(\alpha)}{\Gamma(\gamma)\Gamma(\gamma')\Gamma(\alpha-\gamma-\gamma'+2)}F_4(\alpha;\beta;\gamma,\gamma';x,y),\end{aligned} \tag{13.60}$$

which follows from the integral formula

$$\begin{aligned}&\frac{1}{(2\pi i)^n}\int_{\frac{1}{n+1}-\infty i}^{\frac{1}{n+1}+\infty i}\cdots\int_{\frac{1}{n+1}-\infty i}^{\frac{1}{n+1}+\infty i}\prod_{j=1}^n t_j^{-\alpha_j}\Big(1-\sum_{j=1}^n t_j\Big)^{-\alpha_{n+1}}dt_1\cdots dt_n\\ &\quad= \frac{\Gamma\big(\sum_{j=1}^{n+1}\alpha_j-n\big)}{\prod_{j=1}^{n+1}\Gamma(\alpha_j)}.\end{aligned} \tag{13.61}$$

Since

$$J_x^\mu(u) = J_x^{\mu-1}(u) - \sum x_\nu J_x^{\mu-1}(x_\nu u)$$

and

$$\begin{aligned}&\frac{d}{dt_i}\big(u(t)(1-\sum t_\nu x_\nu)^\mu\big)\\ &\qquad= \frac{du}{dt_i}(t)(1-\sum t_\nu x_\nu)^\mu - \mu u(t)x_i(1-\sum t_\nu x_\nu)^{\mu-1},\end{aligned}$$

we have

$$\begin{aligned}
J_x^\mu(\partial_i u)(x) &= \mu x_i J_x^{\mu-1}(u)(x)\\
&= -x_i \int t_i^{-1} u(t) \frac{d}{dx_i}\bigl(1-\sum x_\nu t_\nu\bigr)^\mu dt\\
&= -x_i \frac{d}{dx_i} J_x^\mu\bigl(\frac{u}{x_i}\bigr)(x),\\
J_x^\mu\bigl(\partial_i(x_i u)\bigr) &= -x_i\partial_i J_x^\mu(u),\\
J_x^\mu(\partial_i u) &= \mu x_i J_x^{\mu-1}(u)\\
&= \mu x_i J_x^\mu(u) + \mu x_i \sum x_\nu J_x^{\mu-1}(x_\nu u)\\
&= \mu x_i J_x^\mu(u) + x_i \sum J_x^\mu\bigl(\partial_\nu(x_\nu u)\bigr)\\
&= \mu x_i J_x^\mu(u) - x_i \sum x_\nu \partial_\nu J_x^\mu(u)
\end{aligned}$$

and therefore

$$J_x^\mu(x_i\partial_i u) = (-1 - x_i\partial_i) J_x^\mu(u), \tag{13.62}$$

$$J_x^\mu(\partial_i u) = x_i\bigl(\mu - \sum x_\nu\partial_\nu\bigr) J_x^\mu(u). \tag{13.63}$$

Thus we have

Proposition 13.2. *For a differential operator*

$$P = \sum_{\substack{\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{Z}_{\geq 0}^n\\ \beta=(\beta_1,\dots,\beta_n)\in\mathbb{Z}_{\geq 0}^n}} c_{\alpha,\beta}\partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}\vartheta_1^{\beta_1}\cdots\vartheta_n^{\beta_n}, \tag{13.64}$$

we have

$$\begin{aligned}
&J_x^\mu\bigl(Pu(x)\bigr) = J_x^\mu(P) J_x^\mu\bigl(u(x)\bigr),\\
&J_x^\mu(P) := \sum_{\alpha,\,\beta} c_{\alpha,\beta} \prod_{k=1}^n \bigl(x_k(\mu - \sum_{\nu=1}^n \vartheta_\nu)\bigr)^{\alpha_k} \prod_{k=1}^n \bigl(-\vartheta_k - 1\bigr)^{\beta_k}.
\end{aligned} \tag{13.65}$$

Using this proposition, we obtain the system of differential equations satisfied by $J_x^\mu(u)$ from that satisfied by $u(x)$. Denoting the Laplace transform of the variable $x = (x_1, \dots, x_n)$ by L_x (cf. Definition 1.1), we have

$$J_x^\mu L_x^{-1}(\vartheta_i) = \vartheta_i, \quad J_x^\mu L_x^{-1}(x_i) = x_i\bigl(\mu - \sum_{\nu=1}^n \vartheta_\nu\bigr). \tag{13.66}$$

We have

$$\begin{aligned}
&\mathrm{Ad}\bigl(x^{\lambda_1}y^{\lambda_2}(xy-x-y)^{\lambda_3}\bigr)\partial_x = \partial_x - \frac{\lambda_1}{x} - \frac{\lambda_3(y-1)}{xy-x-y},\\
&\mathrm{Ad}\bigl(x^{\lambda_1}y^{\lambda_2}(xy-x-y)^{\lambda_3}\bigr)\partial_y = \partial_y - \frac{\lambda_2}{y} - \frac{\lambda_3(x-1)}{xy-x-y},\\
&\mathrm{Ad}\bigl(x^{\lambda_1}y^{\lambda_2}(xy-x-y)^{\lambda_3}\bigr)\bigl(x(x-1)\partial_x\bigr)\\
&\qquad = x(x-1)\partial_x - \lambda_1(x-1) - \frac{\lambda_3(x-1)(xy-x)}{xy-x-y},\\
&\mathrm{Ad}\bigl(x^{\lambda_1}y^{\lambda_2}(xy-x-y)^{\lambda_3}\bigr)\bigl(x(x-1)\partial_x - y\partial_y\bigr)\\
&\qquad = x(x-1)\partial_x - y\partial_y - \lambda_1(x-1) - \lambda_2 - \lambda_3(x-1)\\
&\qquad = x\vartheta_x - \vartheta_x - \vartheta_y - (\lambda_1+\lambda_3)x + \lambda_1 - \lambda_2 + \lambda_3,
\end{aligned}$$

$$\begin{aligned}
&\partial_x\,\mathrm{Ad}\bigl(x^{\lambda_1}y^{\lambda_2}(xy-x-y)^{\lambda_3}\bigr)\bigl(x(x-1)\partial_x - y\partial_y\bigr)\\
&\qquad = \partial_x x(\vartheta_x - \lambda_1 - \lambda_3) - \partial_x(\vartheta_x + \vartheta_y - \lambda_1 + \lambda_2 - \lambda_3)
\end{aligned} \tag{13.67}$$

and

$$\begin{aligned}&J^{\mu}_{x,y}\big(\partial_x x(\vartheta_x-\lambda_1-\lambda_3)-\partial_x(\vartheta_x+\vartheta_y-\lambda_1+\lambda_2-\lambda_3)\big)\\&\quad=\vartheta_x(1+\vartheta_x+\lambda_1+\lambda_3)-x(-\mu+\vartheta_x+\vartheta_y)(2+\vartheta_x+\vartheta_y+\lambda_1-\lambda_2+\lambda_3).\end{aligned}$$

Putting

$$T:=(\vartheta_x+\vartheta_y-\mu)(\vartheta_x+\vartheta_y+\lambda_1-\lambda_2+\lambda_3+2)-\partial_x(\vartheta_x+\lambda_1+\lambda_3+1)$$

with

$$\alpha=-\mu,\quad \beta=\lambda_1-\lambda_2+\lambda_3+2,\quad \gamma=\lambda_1+\lambda_3+2,$$

we have $Tv(x,y)=0$ and moreover it satisfies a similar equation by replacing $(x,y,\lambda_1,\lambda_3,\gamma)$ by $(y,x,\lambda_3,\lambda_1,\gamma')$. Hence $v(x,y)$ is a solution of the system of differential equations satisfied by $F_4(\alpha;\beta;\gamma,\gamma';x,y)$.

In the same way we have

$$\mathrm{Ad}\big(x^{\beta-1}y^{\beta'-1}(1-x-y)^{\gamma-\beta-\beta'-1}\big)\vartheta_x=\vartheta_x-\beta+1+\frac{(\gamma-\beta-\beta'-1)x}{1-x-y},$$

$$\begin{aligned}&\mathrm{Ad}\big(x^{\beta-1}y^{\beta'-1}(1-x-y)^{\gamma-\beta-\beta'-1}\big)(\vartheta_x-x(\vartheta_x+\vartheta_y))\\&\quad=\vartheta_x-x(\vartheta_x+\vartheta_y)-\beta+1+(\gamma-3)x\\&\quad=(\vartheta_x-\beta+1)-x(\vartheta_x+\vartheta_y-\gamma+3),\\&J^{\mu}_{x,y}\big(\partial_x(\vartheta_x-\beta+1)-\partial_x x(\vartheta_x+\vartheta_y-\gamma+3)\big)\\&\quad=x(-\vartheta_x-\vartheta_y+\mu)(-\vartheta_x-\beta)+\vartheta_x(-2-\vartheta_x-\vartheta_y-\gamma+3)\\&\quad=x\Big((\vartheta_x+\vartheta_y-\mu)(\vartheta_x+\beta)-\partial_x(\vartheta_x+\vartheta_y+\gamma-1)\Big).\end{aligned}\tag{13.68}$$

which is a differential operator killing $F_1(\alpha;\beta,\beta';\gamma;x,y)$ by putting $\mu=-\alpha$ and in fact we have

$$\begin{aligned}&\iint_{\substack{s\ge0,\ t\ge0\\1-s-t\ge0}}s^{\beta-1}t^{\beta'-1}(1-s-t)^{\gamma-\beta-\beta'-1}(1-sx-ty)^{-\alpha}ds\,dt\\&\quad=\iint_{\substack{s\ge0,\ t\ge0\\1-s-t\ge0}}\sum_{m,\,n=0}^{\infty}s^{\beta+m-1}t^{\beta'+n-1}(1-s-t)^{\gamma-\beta-\beta'-1}\frac{(\alpha)_{m+n}x^my^n}{m!n!}ds\,dt\\&\quad=\sum_{m,\,n=0}^{\infty}\frac{\Gamma(\beta+m)\Gamma(\beta'+n)\Gamma(\gamma-\beta-\beta')}{\Gamma(\gamma+m+n)}\cdot\frac{(\alpha)_{m+n}}{m!n!}x^my^n\\&\quad=\frac{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')}{\Gamma(\gamma)}F_1(\alpha;\beta,\beta';\gamma;x,y).\end{aligned}$$

Here we use the formula

$$\iint_{\substack{s\ge0,\ t\ge0\\1-s-t\ge0}}s^{\lambda_1-1}t^{\lambda_2-1}(1-s-t)^{\lambda_3-1}ds\,dt=\frac{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)}{\Gamma(\lambda_1+\lambda_2+\lambda_3)}.\tag{13.69}$$

13.10.3. Appell's F_3. Since

$$\begin{aligned}T_3&:=J_y^{-\alpha'}x^{-1}J_x^{-\alpha}\big(\partial_x(\vartheta_x-\beta+1)-\partial_x x(\vartheta_x+\vartheta_y-\gamma+3)\big)\\&=J_y^{-\alpha'}\big((-\vartheta_x-\alpha)(-\vartheta_x-\beta)+\partial_x(-\vartheta_x+\vartheta_y-\gamma+2)\big)\\&=(\vartheta_x+\alpha)(\vartheta_x+\beta)-\partial_x(\vartheta_x+\vartheta_y+\gamma-1)\end{aligned}$$

with (13.68), the operator T_3 kills the function

$$\begin{aligned}
&\iint_{\substack{s\ge0,\ t\ge0\\ 1-s-t\ge0}} s^{\beta-1}t^{\beta'-1}(1-s-t)^{\gamma-\beta-\beta'-1}(1-xs)^{-\alpha}(1-yt)^{-\alpha'}ds\,dt\\
&= \iint_{\substack{s\ge0,\ t\ge0\\ 1-s-t\ge0}} \sum_{m,\,n=0}^{\infty} s^{\beta+m-1}t^{\beta'+n-1}(1-s-t)^{\gamma-\beta-\beta'-1}\frac{(\alpha)_m(\alpha')_n x^m y^n}{m!n!}ds\,dt\\
&= \sum_{m,\,n=0}^{\infty}\frac{\Gamma(\beta+m)\Gamma(\beta'+n)\Gamma(\gamma-\beta-\beta')(\alpha)_m(\alpha')_n}{\Gamma(\gamma+m+n)m!n!}x^my^n\\
&= \frac{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')}{\Gamma(\gamma)}F_3(\alpha,\alpha';\beta,\beta';\gamma;x,y).
\end{aligned}$$

Moreover since

$$\begin{aligned}
T_3' &:= \mathrm{Ad}(\partial_x^{-\mu})\,\mathrm{Ad}(\partial_y^{-\mu'})\big((\vartheta_x+1)(\vartheta_x-\lambda_1-\lambda_3)-\partial_x(\vartheta_x+\vartheta_y-\lambda_1+\lambda_2-\lambda_3)\big)\\
&= (\vartheta_x+1-\mu)(\vartheta_x-\lambda_1-\lambda_3-\mu)-\partial_x(\vartheta_x+\vartheta_y-\lambda_1+\lambda_2-\lambda_3-\mu-\mu')
\end{aligned}$$

with (13.67) and

$$\alpha=-\lambda_1-\lambda_3-\mu,\quad \beta=1-\mu,\quad \gamma=-\lambda_1+\lambda_2-\lambda_3-\mu-\mu'+1,$$

the function

$$u_3(x,y) := \int_\infty^y\int_\infty^x s^{\lambda_1}t^{\lambda_2}(st-s-t)^{\lambda_3}(x-s)^{\mu-1}(y-t)^{\mu'-1}ds\,dt \tag{13.70}$$

satisfies $T_3'u_3(x,y)=0$. Hence $u_3(x,y)$ is a solution of the system of the equations that $F_3(\alpha,\alpha';\beta,\beta';\gamma;x,y)$ satisfies.

13.10.4. Appell's F_2.

Since

$$\begin{aligned}
&\partial_x\,\mathrm{Ad}\big(x^{\lambda_1-1}(1-x^{\lambda_2-1})\big)x(1-x)\partial_x\\
&\quad= \partial_x x(1-x)\partial_x-(\lambda_1-1)\partial_x+\partial_x(\lambda_1+\lambda_2-2)x\\
&\quad= \partial_x x(-\vartheta_x+\lambda_1+\lambda_2-2)+\partial_x(\vartheta-\lambda_1+1)
\end{aligned}$$

and

$$\begin{aligned}
T_2 &:= J_{x,y}^{\mu}\big(\partial_x x(-\vartheta_x+\lambda_1+\lambda_2-2)+\partial_x(\vartheta_x-\lambda_1+1)\big)\\
&= -\vartheta_x(\vartheta_x+1+\lambda_1+\lambda_2-2)+x(\mu-\vartheta_x-\vartheta_y)(-1-\vartheta_x-\lambda_1+1)\\
&= x\Big((\vartheta_x+\lambda_1)(\vartheta_x+\vartheta_y-\mu)-\partial_x(\vartheta_x+\lambda_1+\lambda_2-1)\Big)
\end{aligned}$$

with

$$\alpha=-\mu,\quad \beta=\lambda_1,\quad \gamma=\lambda_1+\lambda_2,$$

the function

$$\begin{aligned}
u_2(x,y) &:= \int_0^1\int_0^1 s^{\lambda_1-1}(1-s)^{\lambda_2-1}t^{\lambda_1'-1}(1-t)^{\lambda_2'-1}(1-xs-yt)^{\mu}ds\,dt\\
&= \int_0^1\int_0^1 \sum_{m,\,n=0}^{\infty} s^{\lambda_1+m-1}(1-s)^{\lambda_2-1}t^{\lambda_1'+n-1}(1-t)^{\lambda_2'-1}\frac{(-\mu)_{m+n}}{m!n!}x^my^n ds\,dt\\
&= \sum_{m,\,n=0}^{\infty}\frac{\Gamma(\lambda_1+m)\Gamma(\lambda_2)}{\Gamma(\lambda_1+\lambda_2+m)}\frac{\Gamma(\lambda_1'+n)\Gamma(\lambda_2')}{\Gamma(\lambda_1'+\lambda_2'+m)}\frac{(-\mu)_{m+n}}{m!n!}x^my^n\\
&= \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_1')\Gamma(\lambda_2')}{\Gamma(\lambda_1+\lambda_2)\Gamma(\lambda_1'+\lambda_2')}\sum_{m,\,n=0}^{\infty}\frac{(\lambda_1)_m(\lambda_1')_n(-\mu)_{m+n}}{(\lambda_1+\lambda_2)_m(\lambda_1'+\lambda_2')_n m!n!}x^my^n
\end{aligned}$$

is a solution of the equation $T_2u=0$ that $F_2(\alpha;\beta,\beta';\gamma,\gamma';x,y)$ satisfies.

Note that the operator $\tilde{T}_3$ transformed from T_3' by the coordinate transformation $(x,y)\mapsto(\frac{1}{x},\frac{1}{y})$ equals

$$\begin{aligned}\tilde{T}_3&=(-\vartheta_x+\alpha)(-\vartheta_x+\beta)-x(-\vartheta_x)(-\vartheta_x-\vartheta_y+\gamma-1)\\&=(\vartheta_x-\alpha)(\vartheta_x-\beta)-x\vartheta_x(\vartheta_x+\vartheta_y-\gamma+1)\end{aligned}$$

and the operator

$$\mathrm{Ad}(x^{-\alpha}y^{-\alpha'})\tilde{T}_3=\vartheta_x(\vartheta_x+\alpha-\beta)-x(\vartheta_x+\alpha)(\vartheta_x+\vartheta_y+\alpha+\alpha'-\gamma+1)$$

together with the operator obtained by the transpositions $x\leftrightarrow y$, $\alpha\leftrightarrow\alpha'$ and $\beta\leftrightarrow\beta'$ defines the system of the equations satisfied by the functions

$$\begin{cases}F_2(\alpha+\alpha'-\gamma+1;\alpha,\alpha';\alpha-\beta+1,\alpha'-\beta'+1;x,y),\\x^{-\alpha'}y^{-\alpha'}F_3(\alpha,\alpha';\beta,\beta';\gamma;\frac{1}{x},\frac{1}{y}),\end{cases}\tag{13.71}$$

which also follows from the integral representation (13.70) with the transformation $(x,y,s,t)\mapsto(\frac{1}{x},\frac{1}{y},\frac{1}{s},\frac{1}{t})$.

13.11. `Okubo` and `Risa/Asir`

Most of our results in this paper are constructible and they can be explicitly calculated and implemented in computer programs.

The computer program `okubo` [**O8**] written by the author handles combinatorial calculations in this paper related to tuples of partitions. It generates basic tuples (cf. §13.1) and rigid tuples (cf. §13.2), calculates the reductions originated by Katz and Yokoyama, the position of accessory parameters in the universal operator (cf. Theorem 6.14 iv)) and direct decompositions etc.

The author presented Theorem 12.6 in the case when $p=3$ as a conjecture in the fall of 2007, which was proved in May in 2008 by a completely different way from the proof given in §12.1, which is a generalization of the original proof of Gauss's summation formula of the hypergeometric series explained in §12.3. The original proof of Theorem 12.6 in the case when $p=3$ was reduced to the combinatorial equality (12.16). The author verified (12.16) by `okubo` and got the concrete connection coefficients for the rigid tuples $\mathbf{m}$ satisfying $\operatorname{ord}\mathbf{m}\le 40$. Under these conditions ($\operatorname{ord}\mathbf{m}\le 40$, $p=3$, $m_{0,n_0}=m_{1,n_1}=1$) there are 4,111,704 independent connection coefficients modulo obvious symmetries and it took about one day to got all of them by a personal computer with `okubo`.

Several operations on differential operators such as additions and middle convolutions defined in Chapter 1 can be calculated by a computer algebra and the author wrote a program for their results under `Risa/Asir`, which gives a reduction procedure of the operators (cf. Definition 5.12), integral representations and series expansions of the solutions (cf. Theorem 8.1), connection formulas (cf. Theorem 12.5), differential operators (cf. Theorem 6.14 iv)), the condition of their reducibility (cf. Corollary 10.12 i)), contiguity relations (cf. Theorem 11.3 ii)) etc. for any given spectral type or Riemann scheme (0.11) and displays the results using TeX. This program for Risa/Asir written by the author contains many useful functions calculating rational functions, Weyl algebra and matrices. These programs can be obtained from

`http://www.math.kobe-u.ac.jp/Asir/asir.html`
`ftp://akagi.ms.u-tokyo.ac.jp/pub/math/muldif`
`ftp://akagi.ms.u-tokyo.ac.jp/pub/math/okubo`.