CHAPTER 12

Connection problem

12.1. Connection formula

For a realizable tuple $\mathbf{m} \in \mathcal{P}_{p+1}$, let $P_{\mathbf{m}}u = 0$ be a universal Fuchsian differential equation with the Riemann scheme

$$\left\{\begin{matrix} x=0 & c_1=1 & \cdots & c_j & \cdots & c_p=\infty \\ [\lambda_{0,1}]_{(m_{0,1})} & [\lambda_{1,1}]_{(m_{1,1})} & \cdots & [\lambda_{j,1}]_{(m_{j,1})} & \cdots & [\lambda_{p,1}]_{(m_{p,1})} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ [\lambda_{0,n_0}]_{(m_{0,n_0})} & [\lambda_{1,n_1}]_{(m_{1,n_1})} & \cdots & [\lambda_{j,n_j}]_{(m_{j,n_j})} & \cdots & [\lambda_{p,n_p}]_{(m_{p,n_p})} \end{matrix}\right\}. \tag{12.1}$$

The singular points of the equation are c_j for $j = 0, \ldots, p$. In this section we always assume $c_0 = 0$, $c_1 = 1$ and $c_p = \infty$ and $c_j \notin [0, 1]$ for $j = 2, \ldots, p-1$. We also assume that $\lambda_{j,\nu}$ are generic.

Definition 12.1 (connection coefficients). Suppose $\lambda_{j,\nu}$ are generic under the Fuchs relation. Let $u_0^{\lambda_{0,\nu_0}}$ and $u_1^{\lambda_{1,\nu_1}}$ be normalized local solutions of $P_{\mathbf{m}} = 0$ at $x = 0$ and $x = 1$ corresponding to the exponents λ_{0,ν_0} and λ_{1,ν_1}, respectively, so that $u_0^{\lambda_{0,\nu_0}} \equiv x^{\lambda_{0,\nu_0}} \mod x^{\lambda_{0,\nu_0}+1}\mathcal{O}_0$ and $u_1^{\lambda_{1,\nu_1}} \equiv (1-x)^{\lambda_{1,\nu_1}} \mod (1-x)^{\lambda_{1,\nu_1}+1}\mathcal{O}_1$. Here $1 \le \nu_0 \le n_0$ and $1 \le \nu_1 \le n_1$. If $m_{0,\nu_0} = 1$, $u_0^{\lambda_{0,\nu_0}}$ is uniquely determined and then the analytic continuation of $u_0^{\lambda_{0,\nu_0}}$ to $x = 1$ along $(0, 1) \subset \mathbb{R}$ defines a *connection coefficient* with respect to $u_1^{\lambda_{1,\nu_1}}$, which is denoted by $c(0 : \lambda_{0,\nu_0} \rightsquigarrow 1 : \lambda_{1,\nu_1})$ or simply by $c(\lambda_{0,\nu_0} \rightsquigarrow \lambda_{1,\nu_1})$. The connection coefficient $c(1 : \lambda_{1,\nu_1} \rightsquigarrow 0 : \lambda_{0.\nu_0})$ or $c(\lambda_{1,\nu_1} \rightsquigarrow \lambda_{0.\nu_0})$ of $u_1^{\lambda_{1,\nu_1}}$ with respect to $u_0^{\lambda_{0,\nu_0}}$ are similarly defined if $m_{1,\nu_1} = 1$.

Moreover we define $c(c_i : \lambda_{i,\nu_i} \rightsquigarrow c_j : \lambda_{j,\nu_j})$ by using a suitable linear fractional transformation T of $\mathbb{C} \cup \{\infty\}$ which transforms $\{c_i, c_j\}$ to $\{0, 1\}$ so that $T(c_\nu) \notin (0, 1)$ for $\nu = 0, \ldots, p$. If $p = 2$, we define the map T so that $T(c_k) = \infty$ for the other singular point c_k. For example if $c_j \notin [0, 1]$ for $j = 2, \ldots, p-1$, we put $T(x) = \frac{x}{x-1}$ to define $c(0 : \lambda_{0,\nu_0} \rightsquigarrow \infty : \lambda_{p,\nu_p})$ or $c(\infty : \lambda_{p,\nu_p} \rightsquigarrow 0 : \lambda_{0,\nu_0})$.

In the definition $u_0^{\lambda_{0,\nu_0}}(x) = x^{\lambda_{0,\nu_0}}\phi(x)$ with analytic function $\phi(x)$ at 0 which satisfies $\phi(0) = 1$ and if $\operatorname{Re}\lambda_{1,\nu_1} < \operatorname{Re}\lambda_{1,\nu}$ for $\nu \ne \nu_1$, we have

$$c(\lambda_{0,\nu_0} \rightsquigarrow \lambda_{1,\nu_1}) = \lim_{x\to 1-0}(1-x)^{-\lambda_{1,\nu_1}}u_0^{\lambda_{0,\nu_0}}(x) \qquad (x \in [0, 1)) \tag{12.2}$$

by the analytic continuation. The connection coefficient $c(\lambda_{0.\nu_0} \rightsquigarrow \lambda_{1,\nu_1})$ meromorphically depends on spectral parameters $\lambda_{j,\nu}$. It also holomorphically depends on accessory parameters g_i and singular points $\frac{1}{c_j}$ $(j = 2, \ldots, p-1)$ in a neighborhood of given values of parameters.

The main purpose in this section is to get the explicit expression of the connection coefficients in terms of gamma functions when $\mathbf{m}$ is rigid and $m_{0,\nu} = m_{1,\nu'} = 1$.

Fist we prove the following key lemma which describes the effect of a middle convolution on connection coefficients.

LEMMA 12.2. *Using the integral transformation* (1.37), *we put*

$$(T^{\mu}_{a,b}u)(x) := x^{-a-\mu}(1-x)^{-b-\mu}I^{\mu}_0 x^a(1-x)^b u(x), \tag{12.3}$$

$$(S^{\mu}_{a,b}u)(x) := x^{-a-\mu}I^{\mu}_0 x^a(1-x)^b u(x) \tag{12.4}$$

for a continuous function $u(x)$ *on* $[0,1]$. *Suppose* $\operatorname{Re} a \ge 0$ *and* $\operatorname{Re}\mu > 0$. *Under the condition* $\operatorname{Re} b + \operatorname{Re}\mu < 0$ *or* $\operatorname{Re} b + \operatorname{Re}\mu > 0$, $(T^{\mu}_{a,b}u)(x)$ *or* $S^{\mu}_{a,b}(u)(x)$ *defines a continuous function on* $[0,1]$, *respectively, and we have*

$$T^{\mu}_{a,b}(u)(0) = S^{\mu}_{a,b}(u)(0) = \frac{\Gamma(a+1)}{\Gamma(a+\mu+1)}u(0), \tag{12.5}$$

$$\frac{T^{\mu}_{a,b}(u)(1)}{T^{\mu}_{a,b}(u)(0)} = \frac{u(1)}{u(0)}C^{\mu}_{a,b}, \quad C^{\mu}_{a,b} := \frac{\Gamma(a+\mu+1)\Gamma(-\mu-b)}{\Gamma(a+1)\Gamma(-b)}, \tag{12.6}$$

$$\frac{S^{\mu}_{a,b}(u)(1)}{S^{\mu}_{a,b}(u)(0)} = \frac{1}{u(0)}\frac{\Gamma(a+\mu+1)}{\Gamma(\mu)\Gamma(a+1)}\int_0^1 t^a(1-t)^{b+\mu-1}u(t)dt. \tag{12.7}$$

PROOF. Suppose $\operatorname{Re} a \ge 0$ and $0 < \operatorname{Re}\mu < -\operatorname{Re} b$. Then

$$\begin{aligned}
&\Gamma(\mu)T^{\mu}_{a,b}(u)(x)\\
&\quad = x^{-a-\mu}(1-x)^{-b-\mu}\int_0^x t^a(1-t)^b(x-t)^{\mu-1}u(t)dt \quad (t = xs_1,\ 0<x<1)\\
&\quad = (1-x)^{-b-\mu}\int_0^1 s_1^a(1-s_1)^{\mu-1}(1-xs_1)^b u(xs_1)ds_1\\
&\quad = \int_0^1 s_1^a\Big(\frac{1-s_1}{1-x}\Big)^{\mu}\Big(\frac{1-xs_1}{1-x}\Big)^b u(xs_1)\frac{ds}{1-s_1}\\
&\quad = \int_0^1 (1-s_2)^a\Big(\frac{s_2}{1-x}\Big)^{\mu}\Big(1+\frac{xs_2}{1-x}\Big)^b u(x-xs_2)\frac{ds_2}{s_2} \qquad (s_1 = 1-s_2)\\
&\quad = \int_0^{\frac{1}{1-x}} \big(1-s(1-x)\big)^a s^{\mu}(1+xs)^b u\big(x-x(1-x)s\big)\frac{ds}{s} \quad (s_2 = (1-x)s).
\end{aligned}$$

Since

$$\big|s_1^a(1-s_1)^{\mu-1}(1-xs_1)^b u(xs_1)\big| \le \max\{(1-s_1)^{\operatorname{Re}\mu-1},1\}3^{-\operatorname{Re} b}\max_{0\le t\le 1}|u(t)|$$

for $0 \le s_1 < 1$ and $0 \le x \le \frac{2}{3}$, $T^{\mu}_{a,b}(u)(x)$ is continuous for $x \in [0,\frac{2}{3})$. We have

$$\big|\big(1-s(1-x)\big)^a s^{\mu-1}(1+xs)^b u\big(x-x(1-x)s\big)\big| \le s^{\operatorname{Re}\mu-1}(1+\tfrac{s}{2})^{\operatorname{Re} b}\max_{0\le t\le 1}|u(t)|$$

for $\frac{1}{2} \le x \le 1$ and $0 < s \le \frac{1}{1-x}$ and therefore $T^{\mu}_{a,b}(u)(x)$ is continuous for $x \in (\frac{1}{2},1]$. Hence $T^{\mu}_{a,b}(x)$ defines a continuous function on $[0,1]$ and

$$\begin{aligned}
T^{\mu}_{a,b}(u)(0) &= \frac{1}{\Gamma(\mu)}\int_0^1 (1-s_2)^a s_2^{\mu}u(0)\frac{ds_2}{s_2} = \frac{\Gamma(a+1)}{\Gamma(a+\mu+1)}u(0),\\
T^{\mu}_{a,b}(u)(1) &= \frac{1}{\Gamma(\mu)}\int_0^{\infty} s^{\mu}(1+s)^b u(1)\frac{ds}{s}
\end{aligned}$$

$(t = \frac{s}{1+s} = 1-\frac{1}{1+s},\ \frac{1}{1+s} = 1-t,\ 1+s = \frac{1}{1-t},\ s = \frac{1}{1-t}-1 = \frac{t}{1-t},\ \frac{ds}{dt} = -\frac{1}{(1-t)^2})$

$$= \frac{1}{\Gamma(\mu)}\int_0^1 \Big(\frac{t}{1-t}\Big)^{\mu-1}(1-t)^{-b-2}u(1)dt = \frac{\Gamma(-\mu-b)}{\Gamma(-b)}u(1).$$

The claims for $S^{\mu}_{a,b}$ are clear from

$$\Gamma(\mu)S^{\mu}_{a,b}(u)(x) = \int_0^1 s_1^a(1-s_1)^{\mu-1}(1-xs_1)^b u(xs_1)ds_1. \qquad \square$$

This lemma is useful for the middle convolution mc_μ not only when it gives a reduction but also when it doesn't change the spectral type.

EXAMPLE 12.3. Applying Lemma 12.2 to the solution

$$u_0^{\lambda_0+\mu}(x)=\int_0^x t^{\lambda_0}(1-t)^{\lambda_1}\Bigl(\prod_{j=2}^{p-1}\Bigl(1-\frac{t}{c_j}\Bigr)^{\lambda_j}\Bigr)(x-t)^{\mu-1}dt$$

of the Jordan-Pochhammer equation (cf. Example 1.8 iii)) with the Riemann scheme

$$\begin{Bmatrix} x=0 & c_1=1 & \cdots & c_j & \cdots & c_p=\infty\\ [0]_{(p-1)} & [0]_{(p-1)} & \cdots & [0]_{(p-1)} & \cdots & [1-\mu]_{(p-1)}\\ \lambda_0+\mu & \lambda_1+\mu & \cdots & \lambda_j+\mu & \cdots & -\sum_{\nu=0}^{p-1}\lambda_\nu-\mu \end{Bmatrix},$$

we have

$$c(0\!:\!\lambda_0+\mu\rightsquigarrow 1\!:\!\lambda_1+\mu)=\frac{\Gamma(\lambda_0+\mu+1)\Gamma(-\lambda_1-\mu)}{\Gamma(\lambda_0+1)\Gamma(-\lambda_1)}\prod_{j=2}^{p-1}\Bigl(1-\frac{1}{c_j}\Bigr)^{\lambda_j},$$

$$c(0\!:\!\lambda_0+\mu\rightsquigarrow 1\!:\!0)=\frac{\Gamma(\lambda_0+\mu+1)}{\Gamma(\mu)\Gamma(\lambda_0+1)}\int_0^1 t^{\lambda_0}(1-t)^{\lambda_1+\mu-1}\prod_{j=1}^{p-1}\Bigl(1-\frac{t}{c_j}\Bigr)^{\lambda_j}dt.$$

Moreover the equation $Pu=0$ with

$$P:=\mathrm{RAd}(\partial^{-\mu'})\,\mathrm{RAd}(x^{\lambda'})\,\mathrm{RAd}(\partial^{-\mu})\,\mathrm{RAd}(x^{\lambda_0}(1-x)^{\lambda_1})\partial$$

is satisfied by the generalized hypergeometric function ${}_3F_2$ with the Riemann scheme

$$\begin{Bmatrix} x=0 & 1 & \infty\\ 0 & [0]_{(2)} & 1-\mu'\\ \lambda'+\mu' & & 1-\lambda'-\mu-\mu'\\ \lambda_0+\lambda'+\mu+\mu' & \lambda_1+\mu+\mu' & -\lambda_0-\lambda_1-\lambda'-\mu-\mu' \end{Bmatrix}$$

corresponding to $111,21,111$ and therefore

$$\begin{aligned} c(\lambda_0+\lambda'+\mu+\mu'\rightsquigarrow\lambda_1+\mu+\mu')&=C^{\mu}_{\lambda_0,\lambda_1}\cdot C^{\mu'}_{\lambda_0+\lambda'+\mu,\lambda_1+\mu}\\ &=\frac{\Gamma(\lambda_0+\mu+1)\Gamma(-\lambda_1-\mu)}{\Gamma(\lambda_0+1)\Gamma(-\lambda_1)}\cdot\frac{\Gamma(\lambda_0+\lambda'+\mu+\mu'+1)\Gamma(-\lambda_1-\mu-\mu')}{\Gamma(\lambda_0+\lambda'+\mu+1)\Gamma(-\lambda_1-\mu)}\\ &=\frac{\Gamma(\lambda_0+\mu+1)\Gamma(\lambda_0+\lambda'+\mu+\mu'+1)\Gamma(-\lambda_1-\mu-\mu')}{\Gamma(\lambda_0+1)\Gamma(-\lambda_1)\Gamma(\lambda_0+\lambda'+\mu+1)}. \end{aligned}$$

We further examine the connection coefficient.

In general, putting $c_0=0$ and $c_1=1$ and $\lambda_1=\sum_{k=0}^p\lambda_{k,1}-1$, we have

$$\begin{Bmatrix} x=c_j\quad(j=0,\dots,p-1) & \infty\\ [\lambda_{j,\nu}-(\delta_{j,0}+\delta_{j,1})\lambda_{j,n_j}]_{(m_{j,\nu})} & [\lambda_{p,\nu}+\lambda_{0,n_0}+\lambda_{1,n_1}]_{(m_{0,\nu})} \end{Bmatrix}$$

$$\xrightarrow{x^{\lambda_{0,n_0}}(1-x)^{\lambda_{1,n_1}}}\begin{Bmatrix} x=c_j & \infty\\ [\lambda_{j,\nu}]_{(m_{j,\nu})} & [\lambda_{p,\nu}]_{(m_{p,\nu})} \end{Bmatrix}$$

$$\xrightarrow{x^{-\lambda_{0,1}}\prod_{j=1}^{p-1}(1-c_j^{-1}x)^{-\lambda_{j,1}}}\begin{Bmatrix} [0]_{(m_{j,1})} & [\lambda_{p,1}+\sum_{k=0}^{p-1}\lambda_{k,1}]_{(m_{p,1})}\\ [\lambda_{j,\nu}-\lambda_{j,1}]_{(m_{j,\nu})} & [\lambda_{p,\nu}+\sum_{k=0}^{p-1}\lambda_{k,1}]_{(m_{p,\nu})} \end{Bmatrix}$$

$$\xrightarrow{\partial^{1-\sum_{k=0}^p\lambda_{k,1}}}\begin{Bmatrix} [0]_{(m_{j,1}-d)} & [\lambda_{p,1}+\sum_{k=0}^{p-1}\lambda_{k,1}-2\lambda_1]_{(m_{p,1}-d)}\\ [\lambda_{j,\nu}-\lambda_{j,1}+\lambda_1]_{(m_{j,\nu})} & [\lambda_{p,\nu}+\sum_{k=0}^{p-1}\lambda_{k,1}-\lambda_1]_{(m_{p,\nu})} \end{Bmatrix}$$

$$\Bigl(d=\sum_{k=0}^p m_{k,1}-(p-1)n\Bigr)$$

$$\xrightarrow{x^{\lambda_{0,1}}\prod_{j=1}^{p-1}(1-c_j^{-1}x)^{\lambda_{j,1}}} \begin{Bmatrix} x=c_j & \infty \\ [\lambda_{j,1}]_{(m_{j,1}-d)} & [\lambda_{p,1}-2\lambda_1]_{(m_{p,1}-d)} \\ [\lambda_{j,\nu}+\lambda_1]_{(m_{j,\nu})} & [\lambda_{p,\nu}-\lambda_1]_{(m_{p,\nu})} \end{Bmatrix},$$

$$C^{\lambda_1}_{\lambda_{0,n_1}-\lambda_{0,1},\lambda_{1,n_1}-\lambda_{1,1}} = \frac{\Gamma(\lambda_{0,n_0}+\lambda_1-\lambda_{0,1}+1)\Gamma(\lambda_{1,1}-\lambda_{1,n_1}-\lambda_1)}{\Gamma(\lambda_{0,n_0}-\lambda_{0,1}+1)\Gamma(\lambda_{1,1}-\lambda_{1,n_1})}.$$

In general, the following theorem is a direct consequence of Definition 5.7 and Lemma 12.2.

THEOREM 12.4. *Put $c_0=0$, $c_1=1$ and $c_j\in\mathbb{C}\setminus\{0\}$ for $j=3,\dots,p-1$. By the transformation*

$$\mathrm{RAd}\Big(x^{\lambda_{0,1}}\prod_{j=1}^{p-1}\big(1-\frac{x}{c_j}\big)^{\lambda_{j,1}}\Big)\circ\mathrm{RAd}\Big(\partial^{1-\sum_{k=0}^{p}\lambda_{k,1}}\Big)\circ\mathrm{RAd}\Big(x^{-\lambda_{0,1}}\prod_{j=1}^{p-1}\big(1-\frac{x}{c_j}\big)^{-\lambda_{j,1}}\Big)$$

the Riemann scheme of a Fuchsian ordinary differential equation and its connection coefficient change as follows:

$$\begin{aligned}
\{\lambda_{\mathbf{m}}\} &= \big\{[\lambda_{j,\nu}]_{(m_{j,\nu})}\big\}_{\substack{0\le j\le p\\ 1\le\nu\le n_j}} = \begin{Bmatrix} x=c_j\ \ (j=0,\dots,p-1) & \infty \\ [\lambda_{j,1}]_{(m_{j,1})} & [\lambda_{p,1}]_{(m_{p,1})} \\ [\lambda_{j,\nu}]_{(m_{j,\nu})} & [\lambda_{p,\nu}]_{(m_{p,\nu})} \end{Bmatrix} \\
&\mapsto \{\lambda'_{\mathbf{m}'}\} = \Big\{[\lambda'_{j,\nu}]_{(m'_{j,\nu})}\Big\}_{\substack{0\le j\le p\\ 1\le\nu\le n_j}} \\
&= \begin{Bmatrix} x=c_j\ \ (j=0,\dots,p-1) & \infty \\ [\lambda_{j,1}]_{(m_{j,1}-d)} & [\lambda_{p,1}-2\sum_{k=0}^{p}\lambda_{k,1}+2]_{(m_{p,1}-d)} \\ [\lambda_{j,\nu}+\sum_{k=0}^{p}\lambda_{k,1}-1]_{(m_{j,\nu})} & [\lambda_{p,\nu}-\sum_{k=0}^{p}\lambda_{k,1}+1]_{(m_{p,\nu})} \end{Bmatrix}
\end{aligned}$$

with

$$\begin{aligned}
d &= m_{0,1}+\cdots+m_{p,1}-(p-1)\operatorname{ord}\mathbf{m},\\
m'_{j,\nu} &= m_{j,\nu}-d\delta_{\nu,1}\quad (j=0,\dots,p,\ \nu=1,\dots,n_j),\\
\lambda'_{j,1} &= \lambda_{j,1}\quad (j=0,\dots,p-1),\ \lambda'_{p,1}=-2\lambda_{0,1}-\cdots-2\lambda_{p-1,1}-\lambda_{p,1}+2,\\
\lambda'_{j,\nu} &= \lambda_{j,\nu}+\lambda_{0,1}+\lambda_{1,1}+\cdots+\lambda_{p,1}-1\quad (j=0,\dots,p-1,\ \nu=2,\dots,n_j),\\
\lambda'_{p,\nu} &= \lambda_{p,\nu}-\lambda_{0,1}-\cdots-\lambda_{p,1}+1
\end{aligned}$$

and if $m_{0,n_0}=1$ and $n_0>1$ and $n_1>1$, then

$$\frac{c'(\lambda'_{0,n_0}\rightsquigarrow\lambda'_{1,n_1})}{\Gamma(\lambda'_{0,n_0}-\lambda'_{0,1}+1)\Gamma(\lambda'_{1,1}-\lambda'_{1,n_1})} = \frac{c(\lambda_{0,n_0}\rightsquigarrow\lambda_{1,n_1})}{\Gamma(\lambda_{0,n_0}-\lambda_{0,1}+1)\Gamma(\lambda_{1,1}-\lambda_{1,n_1})}. \tag{12.8}$$

Applying the successive reduction by ∂_{max} to the above theorem, we obtain the following theorem.

THEOREM 12.5. *Suppose that a tuple $\mathbf{m}\in\mathcal{P}$ is irreducibly realizable and $m_{0,n_0}=m_{1,n_1}=1$ in the Riemann scheme* (12.1). *Then the connection coefficient satisfies*

$$\begin{aligned}
&\frac{c(\lambda_{0,n_0}\rightsquigarrow\lambda_{1,n_1})}{\bar c\big(\lambda(K)_{0,n_0}\rightsquigarrow\lambda(K)_{1,n_1}\big)}\\
&\quad=\prod_{k=0}^{K-1}\frac{\Gamma\big(\lambda(k)_{0,n_0}-\lambda(k)_{0,\ell(k)_0}+1\big)\cdot\Gamma\big(\lambda(k)_{1,\ell(k)_1}-\lambda(k)_{1,n_1}\big)}{\Gamma\big(\lambda(k+1)_{0,n_0}-\lambda(k+1)_{0,\ell(k)_0}+1\big)\cdot\Gamma\big(\lambda(k+1)_{1,\ell(k)_1}-\lambda(k+1)_{1,n_1}\big)}
\end{aligned}$$

under the notation in Definitions 5.12. *Here* $\bar c\big(\lambda(K)_{0,n_0} \rightsquigarrow \lambda(K)_{1,n_1}\big)$ *is a corresponding connection coefficient for the equation* $(\partial_{max}^K P_{\mathbf m})v = 0$ *with the fundamental spectral type* $f\mathbf m$*. We note that*

$$\begin{aligned}&\big(\lambda(k+1)_{0,n_0} - \lambda(k+1)_{0,\ell(k)_0} + 1\big) + \big(\lambda(k+1)_{1,\ell(k)_1} - \lambda(k+1)_{1,n_1}\big)\\ &\quad= \big(\lambda(k)_{0,n_0} - \lambda(k)_{0,\ell(k)_0} + 1\big) + \big(\lambda(k)_{1,\ell(k)_1} - \lambda(k)_{1,n_1}\big)\end{aligned} \tag{12.9}$$

for $k = 0,\dots,K-1$.

When $\mathbf m$ is rigid in the theorem above, we note that $\bar c(\lambda_{0,n_0}(K) \rightsquigarrow \lambda_{1,n_1}(K)) = 1$ and we have the following more explicit result.

Theorem 12.6. *Let* $\mathbf m \in \mathcal P$ *be a rigid tuple. Assume* $m_{0,n_0} = m_{1,n_1} = 1$, $n_0 > 1$ *and* $n_1 > 1$ *in the Riemann scheme* (12.1)*. Then*

$$c(\lambda_{0,n_0} \rightsquigarrow \lambda_{1,n_1}) = \frac{\displaystyle\prod_{\nu=1}^{n_0-1} \Gamma\big(\lambda_{0,n_0} - \lambda_{0,\nu} + 1\big) \cdot \prod_{\nu=1}^{n_1-1} \Gamma\big(\lambda_{1,\nu} - \lambda_{1,n_1}\big)}{\displaystyle\prod_{\substack{\mathbf m'\oplus\mathbf m''=\mathbf m\\ m'_{0,n_0}=m''_{1,n_1}=1}} \Gamma\big(|\{\lambda_{\mathbf m'}\}|\big) \cdot \prod_{j=2}^{p-1}\Big(1-\frac{1}{c_j}\Big)^{-\lambda(K)_{j,\ell(K)_j}}}, \tag{12.10}$$

$$\sum_{\substack{\mathbf m'\oplus\mathbf m''=\mathbf m\\ m'_{0,n_0}=m''_{1,n_1}=1}} m'_{j,\nu} = (n_1-1)m_{j,\nu} - \delta_{j,0}(1-n_0\delta_{\nu,n_0}) + \delta_{j,1}(1-n_1\delta_{\nu,n_1}) \qquad (1\le\nu\le n_j,\ 0\le j\le p) \tag{12.11}$$

under the notation in Definitions 4.12 and 5.12.

Proof. We may assume $\mathbf m$ is monotone and $\operatorname{ord}\mathbf m > 1$.

We will prove this theorem by the induction on $\operatorname{ord}\mathbf m$. Suppose

$$\mathbf m = \mathbf m' \oplus \mathbf m'' \ \text{ with }\ m'_{0,n_0} = m''_{1,n_1} = 1. \tag{12.12}$$

If $\partial_{\mathbf 1}\mathbf m'$ is not well-defined, then

$$\operatorname{ord}\mathbf m' = 1 \ \text{ and }\ m'_{j,1} = 1 \ \text{ for }\ j = 1,2,\dots,p \tag{12.13}$$

and $1 + m_{1,1} + \cdots + m_{p,1} - (p-1)\operatorname{ord}\mathbf m = 1$ because $\operatorname{idx}(\mathbf m,\mathbf m') = 1$ and therefore

$$d_{\mathbf 1}(\mathbf m) = m_{0,1}. \tag{12.14}$$

If $\partial_{\mathbf 1}\mathbf m''$ is not well-defined,

$$\begin{aligned}&\operatorname{ord}\mathbf m'' = 1 \ \text{ and }\ m''_{j,1} = 1 \ \text{ for }\ j = 0,2,\dots,p,\\ &d_{\mathbf 1}(\mathbf m) = m_{1,1}.\end{aligned} \tag{12.15}$$

Hence if $d_{\mathbf 1}(\mathbf m) < m_{0,1}$ and $d_{\mathbf 1}(\mathbf m) < m_{1,1}$, $\partial_{\mathbf 1}\mathbf m'$ and $\partial_{\mathbf 1}\mathbf m''$ are always well-defined and $\partial_{\mathbf 1}\mathbf m = \partial_{\mathbf 1}\mathbf m' \oplus \partial_{\mathbf 1}\mathbf m''$ and the direct decompositions (12.12) of $\mathbf m$ correspond to those of $\partial_{\mathbf 1}\mathbf m$ and therefore Theorem 12.4 shows (12.10) by the induction because we may assume $d_{\mathbf 1}(\mathbf m) > 0$. In fact, it follows from (5.15) that the gamma factors in the denominator of the fraction in the right hand side of (12.10) don't change by the reduction and the change of the numerator just corresponds to the formula in Theorem 12.4.

If $d_{\mathbf 1}(\mathbf m) = m_{0,1}$, there exists the direct decomposition (12.12) with (12.13) which doesn't correspond to a direct decomposition of $\partial_{\mathbf 1}\mathbf m$ but corresponds to the term $\Gamma(|\{\lambda_{\mathbf m'}\}|) = \Gamma(\lambda_{0,n_1}+\lambda_{1,1}+\cdots+\lambda_{p,1}) = \Gamma(\lambda'_{0,n_1}-\lambda'_{0,1}+1)$ in (12.8). Similarly if $d_{\mathbf 1}(\mathbf m) = m_{1,1}$, there exists the direct decomposition (12.12) with (12.15) and it corresponds to the term $\Gamma(|\{\lambda_{\mathbf m'}\}|) = \Gamma(1 - |\{\lambda_{\mathbf m''}\}|) = \Gamma(1 - \lambda_{0,1} - \lambda_{1,n_1} - \lambda_{2,1} - \cdots - \lambda_{p,1}) = \Gamma(\lambda'_{1,1} - \lambda'_{1,n_1})$ (cf. (12.21)). Thus Theorem 12.4 assures (12.10) by the induction on $\operatorname{ord}\mathbf m$.

Note that the above proof with (12.9) shows (12.18). Hence

$$\begin{aligned}\sum_{\substack{\mathbf{m}'\oplus\mathbf{m}''=\mathbf{m}\\ m'_{0,n_0}=m''_{1,n_1}=1}}|\{\lambda_{\mathbf{m}}\}| &= \sum_{\nu=1}^{n_0-1}(\lambda_{0,n_0}-\lambda_{0,\nu}+1)+\sum_{\nu=1}^{n_1-1}(\lambda_{1,\nu}-\lambda_{1,n_1})\\ &=(n_0-1)+(n_0-1)\lambda_{0,n_0}-\sum_{\nu=1}^{n_0-1}\lambda_{0,\nu}+\sum_{\nu=1}^{n_1-1}\lambda_{1,\nu}\\ &\quad+(n_1-1)\Bigl(\sum_{j=0}^{p}\sum_{\nu=1}^{n_j-\delta_{j,1}}m_{j,\nu}\lambda_{j,\nu}-n+1\Bigr)\\ &=(n_0+n_1-2)\lambda_{0,n_0}+\sum_{\nu=1}^{n_0-1}\bigl((n_1-1)m_{0,\nu}-1\bigr)\lambda_{0,\nu}\\ &\quad+\sum_{\nu=1}^{n_1-1}\bigl((n_1-1)m_{1,\nu}+1\bigr)\lambda_{1,\nu}+\sum_{j=2}^{p}\sum_{\nu=1}^{n_2}(n_1-1)m_{j,\nu}\lambda_{j,\nu}\\ &\quad+(n_0+n_1-2)-(n_1-1)\operatorname{ord}\mathbf{m}.\end{aligned}$$

The left hand side of the above first equation and the right hand side of the above last equation don't contain the term λ_{1,n_1} and therefore the coefficients of $\lambda_{j,\nu}$ in the both sides are equal, which implies (12.11). □

COROLLARY 12.7. *Retain the notation in* Theorem 12.6. *We have*

$$\#\{\mathbf{m}'\,;\,\mathbf{m}'\oplus\mathbf{m}''=\mathbf{m}\ \ \textit{with}\ \ m'_{0,n_0}=m''_{1,n_1}=1\}=n_0+n_1-2, \tag{12.16}$$

$$\sum_{\substack{\mathbf{m}'\oplus\mathbf{m}''=\mathbf{m}\\ m'_{0,n_0}=m''_{1,n_1}=1}}\operatorname{ord}\mathbf{m}'=(n_1-1)\operatorname{ord}\mathbf{m}, \tag{12.17}$$

$$\sum_{\substack{\mathbf{m}'\oplus\mathbf{m}''=\mathbf{m}\\ m'_{0,n_0}=m''_{1,n_1}=1}}|\{\lambda'_m\}|=\sum_{\nu=1}^{n_0-1}(\lambda_{0,n_0}-\lambda_{0,\nu}+1)+\sum_{\nu=1}^{n_1-1}(\lambda_{1,\nu}-\lambda_{1,n_1}). \tag{12.18}$$

Let $c(\lambda_{0,n_0}+t\rightsquigarrow\lambda_{1,n_1}-t)$ be the connection coefficient for the Riemann scheme $\bigl\{[\lambda_{j,\nu}+t(\delta_{j,0}\delta_{\nu,n_0}-\delta_{j,1}\delta_{\nu,n_1})]_{(m_{j,\nu})}\bigr\}$. Then

$$\lim_{t\to+\infty}c(0\!:\!\lambda_{0,n_0}+t\rightsquigarrow 1\!:\!\lambda_{1,n_1}-t)=\prod_{j=2}^{p-1}\bigl(1-c_j\bigr)^{\lambda(K)_{j,\ell(K)_j}}. \tag{12.19}$$

Under the notation in Theorem 10.13*, we have*

(12.20)

$$\prod_{\substack{\mathbf{m}'\oplus\mathbf{m}''=\mathbf{m}\\ m'_{0,n_0}=m''_{1,n_1}=1}}\Gamma\bigl(|\{\lambda_{\mathbf{m}'}\}|\bigr)=\prod_{\substack{\alpha_{\mathbf{m}'}\in\Delta(\mathbf{m})\\ m'_{0,n_0}+m'_{1,n_1}=1}}\Gamma\bigl(m'_{1,n_1}+(-1)^{m'_{1,n_1}}(\Lambda(\lambda)|\alpha_{\mathbf{m}'})\bigr).$$

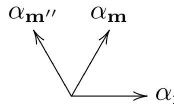
PROOF. We have (12.18) in the proof of Theorem 12.4 and then Stirling's formula and (12.18) prove (12.19). Putting $(j,\nu)=(0,n_0)$ in (12.11) and considering the sum $\sum_\nu$ for (12.11) with $j=1$, we have (12.16) and (12.17), respectively.

Comparing the proof of Theorem 12.6 with that of Theorem 10.13, we have (12.20). Proposition 7.9 also proves (12.20). □

REMARK 12.8. i) When we calculate a connection coefficient for a given rigid partition $\mathbf{m}$ by (12.10), it is necessary to get all the direct decompositions $\mathbf{m}=\mathbf{m}'\oplus\mathbf{m}''$ satisfying $m'_{0,n_0}=m''_{1,n_1}=1$. In this case the equality (12.16) is useful because we know that the number of such decompositions equals n_0+n_1-2, namely, the number of gamma functions appearing in the numerator equals that appearing in the denominator in (12.10).

ii) A direct decomposition $\mathbf{m} = \mathbf{m}' \oplus \mathbf{m}''$ for a rigid tuple $\mathbf{m}$ means that $\{\alpha_{\mathbf{m}'}, \alpha_{\mathbf{m}''}\}$ is a fundamental system of a root system of type A_2 in $\mathbb{R}\alpha_{\mathbf{m}'} + \mathbb{R}\alpha_{\mathbf{m}''}$ such that $\alpha_{\mathbf{m}} = \alpha_{\mathbf{m}'} + \alpha_{\mathbf{m}''}$ and

$$\begin{cases}(\alpha_{\mathbf{m}'}|\alpha_{\mathbf{m}'}) = (\alpha_{\mathbf{m}''}|\alpha_{\mathbf{m}''}) = 2,\\ (\alpha_{\mathbf{m}'}|\alpha_{\mathbf{m}''}) = -1.\end{cases}$$



iii) In view of Definition 4.12, the condition $\mathbf{m} = \mathbf{m}' \oplus \mathbf{m}''$ in (12.10) means

$$\big|\{\lambda_{\mathbf{m}'}\}\big| + \big|\{\lambda_{\mathbf{m}''}\}\big| = 1. \tag{12.21}$$

Hence we have

$$\begin{aligned} &c(\lambda_{0,n_0} \rightsquigarrow \lambda_{1,n_1}) \cdot c(\lambda_{1,n_1} \rightsquigarrow \lambda_{0,n_0}) \\ &= \frac{\displaystyle\prod_{\substack{\mathbf{m}'\oplus\mathbf{m}''=\mathbf{m}\\ m'_{0,n_0}=m''_{1,n_1}=1}} \sin\big(|\{\lambda_{\mathbf{m}'}\}|\pi\big)}{\displaystyle\prod_{\nu=1}^{n_0-1} \sin\big(\lambda_{0,\nu} - \lambda_{1,\nu}\big)\pi \cdot \prod_{\nu=1}^{n_1-1} \sin\big(\lambda_{1,\nu} - \lambda_{1,n_1}\big)\pi}. \end{aligned} \tag{12.22}$$

iv) By the aid of a computer, the author obtained the table of the concrete connection coefficients (12.10) for the rigid triplets $\mathbf{m}$ satisfying $\operatorname{ord}\mathbf{m} \le 40$ together with checking (12.11), which contains 4,111,704 independent cases (cf. §13.11).

v) Is there an interpretation of $\lambda(K)_{j,\ell(K)_j}$ in Theorem 12.6 as (12.20)?

12.2. An estimate for large exponents

The Gauss hypergeometric series

$$F(\alpha,\beta,\gamma;x) := \sum_{k=0}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)\cdot\beta(\beta+1)\cdots(\beta+k-1)}{\gamma(\gamma+1)\cdots(\gamma+k-1)\cdot k!}x^k$$

uniformly and absolutely converges for

$$x \in \overline{D} := \{x \in \mathbb{C}\,;\ |x| \le 1\} \tag{12.23}$$

if $\operatorname{Re}\gamma > \operatorname{Re}(\alpha+\beta)$ and defines a continuous function on $\overline{D}$. The continuous function $F(\alpha,\beta,\gamma+n;x)$ on $\overline{D}$ uniformly converges to the constant function 1 when $n \to +\infty$, which obviously implies

$$\lim_{n\to\infty} F(\alpha,\beta,\gamma+n;1) = 1 \tag{12.24}$$

and proves Gauss's summation formula (0.3) by using the recurrence relation

$$\frac{F(\alpha,\beta,\gamma;1)}{F(\alpha,\beta,\gamma+1;1)} = \frac{(\gamma-\alpha)(\gamma-\beta)}{\gamma(\gamma-\alpha-\beta)}. \tag{12.25}$$

We will generalize such convergence in a general system of ordinary differential equations of Schlesinger canonical form.

Under the condition

$$a > 0,\ b > 0 \text{ and } c > a + b,$$

the function $F(a,b,c;x) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!}x^k$ is strictly increasing continuous function of $x \in [0,1]$ satisfying

$$1 \le F(a,b,c;x) \le F(a,b,c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

and it increases if a or b or $-c$ increases. In particular, if

$$0 \le a \le N,\ 0 \le b \le N \text{ and } c > 2N$$

with a positive integer N, we have

$$\begin{aligned}
0 &\le F(a,b,c;x) - 1 \\
&\le \frac{\Gamma(c)\Gamma(c-2N)}{\Gamma(c-N)\Gamma(c-N)} - 1 = \frac{(c-N)_N}{(c-2N)_N} - 1 = \prod_{\nu=1}^{N} \frac{c-\nu}{c-N-\nu} - 1 \\
&\le \left(\frac{c-N}{c-2N}\right)^N - 1 = \left(1 + \frac{N}{c-2N}\right)^N - 1 \\
&\le N\left(1+\frac{N}{c-2N}\right)^{N-1} \frac{N}{c-2N}.
\end{aligned}$$

Thus we have the following lemma.

LEMMA 12.9. *For a positive integer N we have*

$$|F(\alpha,\beta,\gamma;x) - 1| \le \left(1 + \frac{N}{\operatorname{Re}\gamma - 2N}\right)^N - 1 \tag{12.26}$$

if

$$x \in \overline{D},\ |\alpha| \le N,\ |\beta| \le N \quad \text{and} \quad \operatorname{Re}\gamma > 2N. \tag{12.27}$$

PROOF. The lemma is clear because

$$\Big|\sum_{k=1}^{\infty} \frac{(\alpha)_k(\beta)_k}{(\gamma)_k k!} x^k\Big| \le \sum_{k=1}^{\infty} \frac{(|\alpha|)_k(|\beta|)_k}{(\operatorname{Re}\gamma)_k k!}|x|^k = F(|\alpha|,|\beta|,\operatorname{Re}\gamma - 2N; |x|) - 1 \qquad \square$$

For the Gauss hypergeometric equation

$$x(1-x)u'' + \big(\gamma - (\alpha+\beta+1)x\big)u' - \alpha\beta u = 0$$

we have

$$\begin{aligned}
(xu')' = u' + xu'' &= \frac{xu'}{x} + \frac{((\alpha+\beta+1)x - \gamma)u' + \alpha\beta u}{1-x} \\
&= \frac{\alpha\beta}{1-x}u + \left(\frac{1}{x} - \frac{\gamma}{x(1-x)} + \frac{\alpha+\beta+1}{1-x}\right)xu' \\
&= \frac{\alpha\beta}{1-x}u + \left(\frac{1-\gamma}{x} + \frac{\alpha+\beta-\gamma+1}{1-x}\right)xu'.
\end{aligned}$$

Putting

$$\tilde{u} = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} := \begin{pmatrix} u \\ \frac{xu'}{\alpha} \end{pmatrix} \tag{12.28}$$

we have

$$\tilde{u}' = \frac{\begin{pmatrix} 0 & \alpha \\ 0 & 1-\gamma \end{pmatrix}}{x}\tilde{u} + \frac{\begin{pmatrix} 0 & 0 \\ \beta & \alpha+\beta-\gamma+1 \end{pmatrix}}{1-x}\tilde{u}. \tag{12.29}$$

In general, for

$$v' = \frac{A}{x}v + \frac{B}{1-x}v$$

we have

$$\begin{aligned}
xv' &= Av + \frac{x}{1-x}Bv \\
&= Av + x\big(xv' + (B-A)v\big).
\end{aligned}$$

Thus

$$\begin{cases} xu_0' = \alpha u_1, \\ xu_1' = (1-\gamma)u_1 + x\bigl(xu_1' + \beta u_0 + (\alpha+\beta)u_1\bigr) \end{cases} \tag{12.30}$$

and the functions

$$\begin{cases} u_0 = F(\alpha,\beta,\gamma;x), \\ u_1 = \dfrac{\beta x}{\gamma} F(\alpha+1,\beta+1,\gamma+1;x) \end{cases} \tag{12.31}$$

satisfies (12.30).

THEOREM 12.10. *Let n, n_0 and n_1 be positive integers satisfying $n = n_0 + n_1$ and let $A = \begin{pmatrix} 0 & A_0 \\ 0 & A_1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ B_0 & B_1 \end{pmatrix} \in M(n,\mathbb{C})$ such that A_1, $B_1 \in M(n_1,\mathbb{C})$, $A_0 \in M(n_0,n_1,\mathbb{C})$ and $B_0 \in M(n_1,n_0,\mathbb{C})$. Let $D(\mathbf{0},\mathbf{m}) = D(\mathbf{0},m_1,\dots,m_{n_1})$ be the diagonal matrix of size n whose k-th diagonal element is m_{k-n_0} if $k > n_0$ and 0 otherwise. Let $u^{\mathbf{m}}$ be the local holomorphic solution of the system*

$$u = \frac{A - D(\mathbf{0},\mathbf{m})}{x}u + \frac{B - D(\mathbf{0},\mathbf{m})}{1-x}u \tag{12.32}$$

at the origin. Then if $\operatorname{Re} m_\nu$ are sufficiently large for $\nu = 1,\dots,n_1$, the Taylor series of $u^{\mathbf{m}}$ at the origin uniformly converge on $\overline{D} = \{x \in \mathbb{C};\, |x| \le 1\}$ and for a positive number C, the function $u^{\mathbf{m}}$ and their derivatives uniformly converge to constants on $\overline{D}$ when $\min\{\operatorname{Re} m_1,\dots,\operatorname{Re} m_{n_1}\} \to +\infty$ with $|A_{ij}| + |B_{ij}| \le C$. In particular, for $x \in \overline{D}$ and an integer N satisfying

$$\sum_{\nu=1}^{n_1} |(A_0)_{i\nu}| \le N,\ \sum_{\nu=1}^{n_1} |(A_1)_{i\nu}| \le N,\ \sum_{\nu=1}^{n_0} |(B_0)_{i\nu}| \le N,\ \sum_{\nu=1}^{n_1} |(B_1)_{i\nu}| \le N \tag{12.33}$$

we have

$$\max_{1\le\nu\le n} \bigl|u_\nu^{\mathbf{m}}(x) - u_\nu^{\mathbf{m}}(0)\bigr| \le \max_{1\le\nu\le n_0} |u_\nu^{\mathbf{m}}(0)| \cdot \frac{2^N(N+1)^2}{\min\limits_{1\le\nu\le n_1} \operatorname{Re} m_\nu - 4N - 1} \tag{12.34}$$

if $\operatorname{Re} m_\nu > 5N + 4$ for $\nu = 1,\dots,n_1$.

PROOF. Use the method of majorant series and compare to the case of Gauss hypergeometric series (cf. (12.30) and (12.31)), namely, $\lim_{c\to+\infty} F(a,b,c;x) = 1$ on $\overline{D}$ with a solution of the Fuchsian system

$$\begin{aligned} u' &= \frac{A}{x}u + \frac{B}{1-x}u, \\ A &= \begin{pmatrix} 0 & A_0 \\ 0 & A_1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ B_0 & B_1 \end{pmatrix}, \quad u = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}, \\ xv_0' &= A_0 v_1, \\ xv_1' &= x^2 v_1' + (1-x)A_1 v_1 + xB_0 v_0 + xB_1 v_1 \\ &= A_1 v_1 + x\bigl(xv_1' + B_0 v_0 + (B_1 - A_1)v_1\bigr) \end{aligned}$$

or the system obtained by the substitution $A_1 \mapsto A_1 - D(\mathbf{m})$ and $B_1 \mapsto B_1 - D(\mathbf{m})$. Fix positive real numbers α, β and γ satisfying

$$\begin{aligned}
&\alpha \ge \sum_{\nu=1}^{n_1} |(A_0)_{i\nu}| \quad (1 \le i \le n_0), \quad \beta \ge \sum_{\nu=1}^{n_0} |(B_0)_{i\nu}| \quad (1 \le i \le n_1),\\
&\alpha + \beta \ge \sum_{\nu=1}^{n_1} |(B_1 - A_1)_{i\nu}| \quad (1 \le i \le n_0),\\
&\gamma = \min\{\operatorname{Re} m_1, \dots, \operatorname{Re} m_{n_1}\} - 2 \max_{1 \le i \le n_1} \sum_{\nu=1}^{n_1} |(A_1)_{i\nu}| - 1 > \alpha + \beta.
\end{aligned}$$

Then the method of majorant series with Lemma 12.11, (12.30) and (12.31) imply

$$u_i^{\mathbf{m}} \ll \begin{cases} \max_{1 \le \nu \le n_0} |u_\nu^{\mathbf{m}}(0)| \cdot F(\alpha, \beta, \gamma; x) & (1 \le i \le n_0), \\ \frac{\beta}{\gamma} \cdot \max_{1 \le \nu \le n_0} |u_\nu^{\mathbf{m}}(0)| \cdot F(\alpha+1, \beta+1, \gamma+1; x) & (n_0 < i \le n), \end{cases}$$

which proves the theorem because of Lemma 12.9 with $\alpha = \beta = N$ as follows. Here $\sum_{\nu=0}^\infty a_\nu x^\nu \ll \sum_{\nu=0}^\infty b_\nu x^\nu$ for formal power series means $|a_\nu| \le b_\nu$ for $\nu \in \mathbb{Z}_{\ge 0}$.

Put $\bar{m} = \min\{\operatorname{Re} m_1, \dots, \operatorname{Re} m_{n_1}\} - 2N - 1$ and $L = \max_{1 \le \nu \le n_0} |u_\nu^{\mathbf{m}}(0)|$. Then $\gamma \ge \bar{m} - 2N - 1$ and if $0 \le i \le n_0$ and $x \le \overline{D}$,

$$\begin{aligned}
|u_i^{\mathbf{m}}(x) - u_i^{\mathbf{m}}(0)| &\le L \cdot \bigl(F(\alpha, \beta, \gamma; |x|) - 1\bigr)\\
&\le L\biggl(\Bigl(1 + \frac{N}{\bar{m} - 4N - 1}\Bigr)^N - 1\biggr)\\
&\le L\Bigl(1 + \frac{N}{\bar{m} - 4N - 1}\Bigr)^{N-1} \frac{N^2}{\bar{m} - 4N - 1} \le \frac{L2^{N-1}N^2}{\bar{m} - 4N - 1}.
\end{aligned}$$

If $n_0 < i \le n$ and $x \in \overline{D}$,

$$\begin{aligned}
|u_i^{\mathbf{m}}(x)| &\le \frac{\beta}{\gamma} \cdot LF(\alpha+1, \beta+1, \gamma+1; |x|)\\
&\le \frac{LN}{\bar{m} - 2N - 1}\biggl(\Bigl(1 + \frac{N+1}{\bar{m} - 4N - 3}\Bigr)^{N+1} + 1\biggr) \le \frac{LN(2^{N+1} + 1)}{\bar{m} - 2N - 1}.
\end{aligned}$$ □

LEMMA 12.11. *Let $A \in M(n, \mathbb{C})$ and put*

(12.35) $$|A| := \max_{1 \le i \le n} \sum_{\nu=1}^n |A_{i\nu}|.$$

If positive real numbers $m_1, \dots, m_n$ satisfy

(12.36) $$m_{min} := \min\{m_1, \dots, m_n\} > 2|A|,$$

we have

(12.37) $$|\bigl(kI_n + D(\mathbf{m}) - A\bigr)^{-1}| \le (k + m_{min} - 2|A|)^{-1} \qquad (\forall k \ge 0).$$

PROOF. Since

$$\begin{aligned}
\bigl|\bigl(D(\mathbf{m}) - A\bigr)^{-1}\bigr| &= \bigl|D(\mathbf{m})^{-1}(I_n - D(\mathbf{m})^{-1}A)^{-1}\bigr|\\
&= \Bigl|D(\mathbf{m})^{-1} \sum_{k=0}^\infty \bigl(D(\mathbf{m})^{-1}A\bigr)^k\Bigr|\\
&\le m_{\min}^{-1} \cdot \Bigl(1 + \frac{2|A|}{m_{\min}}\Bigr) \le (m_{\min} - 2|A|)^{-1},
\end{aligned}$$

we have the lemma by replacing m_ν by $m_\nu + k$ for $\nu = 1, \dots, n$. □

12.3. Zeros and poles of connection coefficients

In this section we examine the connection coefficients to calculate them in a different way from the one given in §12.1.

First review the connection coefficient $c(0:\lambda_{0,2}\rightsquigarrow 1:\lambda_{1,2})$ for the solution of Fuchsian differential equation with the Riemann scheme $\begin{Bmatrix} x=0 & 1 & \infty \\ \lambda_{0,1} & \lambda_{1,1} & \lambda_{2,1} \\ \lambda_{0,2} & \lambda_{1,2} & \lambda_{2,2}\end{Bmatrix}$. Denoting the connection coefficient $c(0\!:\!\lambda_{0,2}\rightsquigarrow 1\!:\!\lambda_{1,2})$ by $c(\begin{Bmatrix} \lambda_{0,1} & & \lambda_{1,1} & \lambda_{2,1} \\ \lambda_{0,2} & \rightsquigarrow & \lambda_{1,2} & \lambda_{2,2}\end{Bmatrix})$, we have

$$u_0^{\lambda_{0,2}} = c(\begin{Bmatrix} \lambda_{0,1} & & \lambda_{1,1} & \lambda_{2,1} \\ \lambda_{0,2} & \rightsquigarrow & \lambda_{1,2} & \lambda_{2,2}\end{Bmatrix})u_1^{\lambda_{1,2}} + c(\begin{Bmatrix} \lambda_{0,1} & & \lambda_{1,2} & \lambda_{2,1} \\ \lambda_{0,2} & \rightsquigarrow & \lambda_{1,1} & \lambda_{2,2}\end{Bmatrix})u_1^{\lambda_{1,1}}. \tag{12.38}$$

$$\begin{aligned} c(\begin{Bmatrix} \lambda_{0,1} & & \lambda_{1,1} & \lambda_{2,1} \\ \lambda_{0,2} & \rightsquigarrow & \lambda_{1,2} & \lambda_{2,2}\end{Bmatrix}) &= c(\begin{Bmatrix} \lambda_{0,1}-\lambda_{0,2} & & \lambda_{1,1}-\lambda_{1,2} & \lambda_{0,2}+\lambda_{1,2}+\lambda_{2,1} \\ 0 & \rightsquigarrow & 0 & \lambda_{0,2}+\lambda_{1,2}+\lambda_{2,2}\end{Bmatrix}) \\ &= F(\lambda_{0,2}+\lambda_{1,2}+\lambda_{2,1}, \lambda_{0,2}+\lambda_{1,2}+\lambda_{2,2}, \lambda_{0,2}-\lambda_{0,1}+1; 1) \end{aligned} \tag{12.39}$$

under the notation in Definition 12.1. As was explained in the first part of §12.2, the connection coefficient is calculated from

$$\lim_{n\to\infty} c(\begin{Bmatrix} \lambda_{0,1}-n & & \lambda_{1,1}+n & \lambda_{2,1} \\ \lambda_{0,2} & \rightsquigarrow & \lambda_{1,2} & \lambda_{2,2}\end{Bmatrix}) = 1 \tag{12.40}$$

and

$$\frac{c(\begin{Bmatrix} \lambda_{0,1} & & \lambda_{1,1} & \lambda_{2,1} \\ \lambda_{0,2} & \rightsquigarrow & \lambda_{1,2} & \lambda_{2,2}\end{Bmatrix})}{c(\begin{Bmatrix} \lambda_{0,1}-1 & & \lambda_{1,1}+1 & \lambda_{2,1} \\ \lambda_{0,2} & \rightsquigarrow & \lambda_{1,2} & \lambda_{2,2}\end{Bmatrix})} = \frac{(\lambda_{0,2}+\lambda_{1,1}+\lambda_{2,2})(\lambda_{0,2}+\lambda_{1,1}+\lambda_{2,1})}{(\lambda_{0,2}-\lambda_{0,1}+1)(\lambda_{1,1}-\lambda_{1,2})}. \tag{12.41}$$

The relation (12.40) is easily obtained from (12.39) and (12.24) or can be reduced to Theorem 12.10.

We will examine (12.41). For example, the relation (12.41) follows from the relation (12.25) which is obtained from

$$\begin{aligned} \gamma\bigl(\gamma-1-(2\gamma-\alpha-\beta-1)x\bigr)F(\alpha,\beta,\gamma;x) + (\gamma-\alpha)(\gamma-\beta)xF(\alpha,\beta,\gamma+1;x) \\ = \gamma(\gamma-1)(1-x)F(\alpha,\beta,\gamma-1;x) \end{aligned}$$

by putting $x=1$ (cf. [**WW**, §14.1]). We may use a shift operator as follows. Since

$$\begin{aligned} \frac{d}{dx}F(\alpha,\beta,\gamma;x) &= \frac{\alpha\beta}{\gamma}F(\alpha+1,\beta+1,\gamma+1;x) \\ &= c(\begin{Bmatrix} 1-\gamma & & \gamma-\alpha-\beta & \alpha \\ 0 & \rightsquigarrow & 0 & \beta\end{Bmatrix})\tfrac{d}{dx}u_1^0 + c(\begin{Bmatrix} 1-\gamma & & 0 & \alpha \\ 0 & \rightsquigarrow & \gamma-\alpha-\beta & \beta\end{Bmatrix})\tfrac{d}{dx}u_1^{\gamma-\alpha-\beta} \end{aligned}$$

and

$$\tfrac{d}{dx}u_1^{\gamma-\alpha-\beta} \equiv (\alpha+\beta-\gamma)(1-x)^{\gamma-\alpha-\beta-1} \mod (1-x)^{\gamma-\alpha-\beta}\mathcal{O}_1,$$

we have

$$\frac{\alpha\beta}{\gamma}c(\begin{Bmatrix} -\gamma & & 0 & \alpha+1 \\ 0 & \rightsquigarrow & \gamma-\alpha-\beta-1 & \beta+1\end{Bmatrix}) = (\alpha+\beta-\gamma)c(\begin{Bmatrix} 1-\gamma & & 0 & \alpha \\ 0 & \rightsquigarrow & \gamma-\alpha-\beta & \beta\end{Bmatrix}),$$

which also proves (12.41) because

$$\frac{c(\begin{Bmatrix} \lambda_{0,1} & & \lambda_{1,1} & \lambda_{2,1} \\ \lambda_{0,2} & \rightsquigarrow & \lambda_{1,2} & \lambda_{2,2}\end{Bmatrix})}{c(\begin{Bmatrix} \lambda_{0,1}-1 & & \lambda_{1,1}+1 & \lambda_{2,1} \\ \lambda_{0,2} & \rightsquigarrow & \lambda_{1,2} & \lambda_{2,2}\end{Bmatrix})} = \frac{c(\begin{Bmatrix} \lambda_{0,1}-\lambda_{0,2} & & 0 & \lambda_{0,2}+\lambda_{1,1}+\lambda_{2,1} \\ 0 & \rightsquigarrow & \lambda_{1,2}-\lambda_{1,1} & \lambda_{0,2}+\lambda_{1,1}+\lambda_{2,2}\end{Bmatrix})}{c(\begin{Bmatrix} \lambda_{0,1}-\lambda_{0,2}-1 & & 0 & \lambda_{0,2}+\lambda_{1,2}+\lambda_{2,1}+1 \\ 0 & \rightsquigarrow & \lambda_{1,2}-\lambda_{1,1}-1 & \lambda_{0,2}+\lambda_{1,2}+\lambda_{2,2}+1\end{Bmatrix})}.$$

Furthermore each linear term appeared in the right hand side of (12.41) has own meaning, which is as follows.

Examine the zeros and poles of the connection coefficient $c(\begin{Bmatrix} \lambda_{0,1} & & \lambda_{1,1} & \lambda_{2,1} \\ \lambda_{0,2} & \rightsquigarrow & \lambda_{1,2} & \lambda_{2,2}\end{Bmatrix})$. We may assume that the parameters $\lambda_{j,\nu}$ are generic in the zeros or the poles.

Consider the linear form $\lambda_{0,2}+\lambda_{1,1}+\lambda_{2,2}$. The local solution $u_0^{\lambda_{0,2}}$ corresponding to the characteristic exponent $\lambda_{0,2}$ at 0 satisfies a Fuchsian differential equation of order 1 which has the characteristic exponents $\lambda_{2,2}$ and $\lambda_{1,1}$ at ∞ and 1, respectively, if and only if the value of the linear form is 0 or a negative integer. In this case $c(\left\{\begin{smallmatrix}\lambda_{0,1} & & \lambda_{1,1} & \lambda_{2,1}\\ \lambda_{0,2} & \rightsquigarrow & \lambda_{1,2} & \lambda_{2,2}\end{smallmatrix}\right\})$ vanishes. This explains the term $\lambda_{0,2}+\lambda_{1,1}+\lambda_{2,2}$ in the numerator of the right hand side of (12.41). The term $\lambda_{0,2}+\lambda_{1,2}+\lambda_{2,2}$ is similarly explained.

The normalized local solution $u_0^{\lambda_{0,2}}$ has poles where $\lambda_{0,1}-\lambda_{0,2}$ is a positive integer. The residue at the pole is a local solution corresponding to the exponent $\lambda_{0,2}$. This means that $c(\left\{\begin{smallmatrix}\lambda_{0,1} & & \lambda_{1,1} & \lambda_{2,1}\\ \lambda_{0,2} & \rightsquigarrow & \lambda_{1,2} & \lambda_{2,2}\end{smallmatrix}\right\})$ has poles where $\lambda_{0,1}-\lambda_{0,2}$ is a positive integer, which explains the term $\lambda_{0,2}-\lambda_{0,1}+1$ in the denominator of the right hand side of (12.41).

There exists a local solution $a(\lambda)u_1^{\lambda_{1,1}}+b(\lambda)u_1^{\lambda_{1,2}}$ such that it is holomorphic for $\lambda_{j,\nu}$ and $b(\lambda)$ has a pole if the value of $\lambda_{1,1}-\lambda_{1,2}$ is a non-negative integer, which means $c(\left\{\begin{smallmatrix}\lambda_{0,1} & & \lambda_{1,1} & \lambda_{2,1}\\ \lambda_{0,2} & \rightsquigarrow & \lambda_{1,2} & \lambda_{2,2}\end{smallmatrix}\right\})$ has poles where $\lambda_{1,2}-\lambda_{1,1}$ is non-negative integer. This explains the term $\lambda_{1,1}-\lambda_{1,2}$ in the denominator of the right hand side of (12.41). These arguments can be generalized, which will be explained in this section.

Fist we examine the possible poles of connection coefficients.

PROPOSITION 12.12. *Let $Pu=0$ be a differential equation of order n with a regular singularity at $x=0$ such that P contains a holomorphic parameter $\lambda=(\lambda_1,\dots,\lambda_N)$ defined in a neighborhood of $\lambda^o=(\lambda_1^o,\dots,\lambda_N^o)$ in $\mathbb{C}^N$. Suppose that the set of characteristic exponents of P at $x=0$ equals $\{[\lambda_1]_{(m_1)},\dots,[\lambda_N]_{(m_N)}\}$ with $n=m_1+\cdots+m_N$ and*

$$\lambda_{2,1}^o:=\lambda_2^o-\lambda_1^o\in\mathbb{Z}_{\ge 0}\ \text{and}\ \lambda_i^o-\lambda_j^o\notin\mathbb{Z}\ \text{if}\ 1\le i<j\le N\ \text{and}\ j\ne 2. \tag{12.42}$$

Let $u_{j,\nu}$ be local solutions of $Pu=0$ uniquely defined by

$$u_{j,\nu}\equiv x^{\lambda_j+\nu}\mod x^{\lambda_j+m_j}\mathcal{O}_0\quad (j=1,\dots,m_j\ \text{and}\ \nu=0,\dots,m_j-1). \tag{12.43}$$

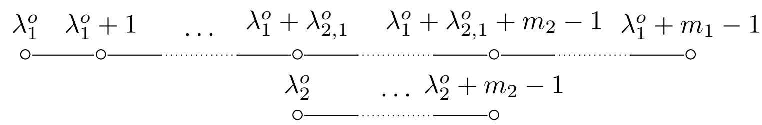
Note that $u_{j,\nu}=\sum_{k\ge 0}a_{k,j,\nu}(\lambda)x^{\lambda_j+\nu+k}$ with meromorphic functions $a_{k,j,\nu}(\lambda)$ of λ which are holomorphic in a neighborhood of λ^o if $\lambda_2-\lambda_1\ne\lambda_{2,1}^o$. Then there exist solutions $v_{j,\nu}$ with holomorphic parameter λ in a neighborhood of λ^o which satisfy the following relations. Namely

$$v_{j,\nu}=u_{j,\nu}\quad (3\le j\le N\ \text{and}\ \nu=0,\dots,m_j-1) \tag{12.44}$$

and when $\lambda_1^o+m_1\ge\lambda_2^o+m_2$,

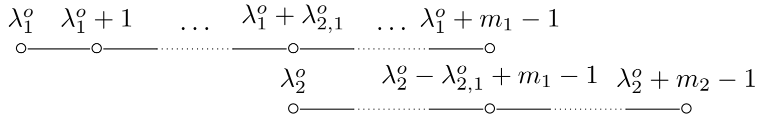
$$\begin{aligned} v_{1,\nu}&=u_{1,\nu} && (0\le\nu<m_1),\\ v_{2,\nu}&=\frac{u_{2,\nu}-u_{1,\nu+\lambda_{2,1}^o}}{\lambda_1-\lambda_2+\lambda_{2,1}^o}-\sum_{m_2+\lambda_{2,1}^o\le i<m_1}\frac{b_{\nu,i}u_{1,i}}{\lambda_1-\lambda_2+\lambda_{2,1}^o} && (0\le\nu<m_2) \end{aligned} \tag{12.45}$$

with the diagram



which illustrates some exponents and when $\lambda_1^o + m_1 < \lambda_2^o + m_2$,

(12.46)
$$
\begin{aligned}
v_{2,\nu} &= u_{2,\nu} \qquad (0 \le \nu < m_2),\\
v_{1,\nu} &= u_{1,\nu} - \sum_{\max\{0,m_1-\lambda_{2,1}^o\}\le i<m_2} \frac{b_{\nu,i}u_{2,i}}{\lambda_1-\lambda_2+\lambda_{2,1}^o} \qquad (0\le \nu<\min\{m_1,\lambda_{2,1}^o\}),\\
v_{1,\nu} &= \frac{u_{1,\nu}-u_{2,\nu-\lambda_{2,1}^o}}{\lambda_1-\lambda_2+\lambda_{2,1}^o} - \sum_{\max\{0,m_1-\lambda_{2,1}^o\}\le i<m_2} \frac{b_{\nu,i}u_{2,i}}{\lambda_1-\lambda_2+\lambda_{2,1}^o} \quad (\lambda_{2,1}^o\le\nu<m_1)
\end{aligned}
$$

with 

and here $b_{\nu,i} \in \mathbb{C}$. *Note that* $v_{j,\nu}$ $(1 \le j \le N,\ 0 \le \nu < m_j)$ *are linearly independent for any fixed* λ *in a neighborhood of* λ^o.

Proof. See §2.1 and the proof of Lemma 4.5 (and [**O3**, Theorem 6.5] in a more general setting) for the construction of local solutions of $Pu = 0$.

Note that $u_{j,\nu}$ for $j \ge 3$ are holomorphic with respect to λ in a neighborhood of $\lambda = \lambda^o$. Moreover note that the local monodromy generator M_0 of the solutions $Pu = 0$ at $x = 0$ satisfies $\prod_{j=1}^N (M_0 - e^{2\pi\sqrt{-1}\lambda_j}) = 0$ and therefore the functions $(\lambda_1 - \lambda_2 - \lambda_{2,1}^o)u_{j,\nu}$ of λ are holomorphically extended to the point $\lambda = \lambda^o$ for $j = 1$ and 2, and the values of the functions at $\lambda = \lambda^o$ are solutions of the equation $Pu = 0$ with $\lambda = \lambda^o$.

Suppose $\lambda_1^o + m_1 \ge \lambda_2^o + m_2$. Then $u_{j,\nu}$ $(j = 1, 2)$ are holomorphic with respect to λ at $\lambda = \lambda^o$ and there exist $b_{j,\nu} \in \mathbb{C}$ such that

$$
u_{2,\nu}|_{\lambda=\lambda^o} = u_{1,\nu+\lambda_{2,1}^o}|_{\lambda=\lambda^o} + \sum_{m_2+\lambda_{2,1}^o\le\nu<m_1} b_{\nu,i}\big(u_{1,i}|_{\lambda=\lambda^o}\big)
$$

and we have the proposition. Here

$$
u_{2,\nu}|_{\lambda=\lambda^o} \equiv x^{\lambda_2^o} + \sum_{m_2+\lambda_{2,1}^o\le\nu<m_1} b_{\nu,i}x^{\lambda_1^o+\nu} \mod x^{\lambda_1^o+m_1}\mathcal{O}_0.
$$

Next suppose $\lambda_1^o + m_1 < \lambda_2^o + m_2$. Then there exist $b_{j,\nu} \in \mathbb{C}$ such that

$$
\begin{aligned}
\big((\lambda_1-\lambda_2+\lambda_{2,1}^o)u_{1,\nu}\big)|_{\lambda=\lambda^o} &= \sum_{\max\{0,m_1-\lambda_{2,1}^o\}\le i<m_2} b_{\nu,i}\big(u_{2,i}|_{\lambda=\lambda^o}\big)\\
&\qquad\qquad (0\le\nu<\min\{m_1,\lambda_{2,1}^o\}),\\
u_{1,\nu}|_{\lambda=\lambda^o} &= \sum_{\max\{0,m_1-\lambda_{2,1}^o\}\le i<m_2} b_{\nu,i}\big(u_{2,i}|_{\lambda=\lambda^o}\big) \quad (\lambda_{2,1}^o\le\nu<m_1)
\end{aligned}
$$

and we have the proposition. □

The proposition implies the following corollaries.

Corollary 12.13. *Keep the notation and the assumption in* Proposition 12.12.

i) *Let* $W_j(\lambda, x)$ *be the Wronskian of* $u_{j,1}, \dots, u_{j,m_j}$ *for* $j = 1, \dots, N$. *Then* $(\lambda_1 - \lambda_2 + \lambda_{2,1}^o)^{\ell_1} W_1(\lambda)$ *and* $W_j(\lambda)$ *with* $2 \le j \le N$ *are holomorphic with respect to* λ *in a neighborhood of* λ^o *by putting*

$$
\ell_1 = \max\big\{0, \min\{m_1, m_2, \lambda_{2,1}^o, \lambda_{2,1}^o + m_2 - m_1\}\big\}. \tag{12.47}
$$

ii) *Let*

$$w_k=\sum_{j=1}^{N}\sum_{\nu=1}^{m_j}a_{j,\nu,k}(\lambda)u_{j,\nu,k}$$

be a local solution defined in a neighborhood of 0 *with a holomorphic* λ *in a neighborhood of* λ^o. *Then*

$$(\lambda_1-\lambda_2+\lambda_{2,1}^o)^{\ell_{2,j}}\det\Bigl(a_{j,\nu,k}(\lambda)\Bigr)_{\substack{1\le\nu\le m_j\\1\le k\le m_j}}$$

with

$$\begin{cases}\ell_{2,1}=\max\bigl\{0,\min\{m_1-\lambda_{2,1}^o,m_2\}\bigr\},\\ \ell_{2,2}=\min\{m_1,m_2\},\\ \ell_{2,j}=0\qquad(3\le j\le N)\end{cases}$$

are holomorphic with respect to λ *in a neighborhood of* λ^o.

Proof. i) Proposition 12.12 shows that $u_{j,\nu}$ $(2\le j\le N,\ 0\le\nu<m_j)$ are holomorphic with respect to λ at λ^o. The functions $u_{1,\nu}$ for $\min\{m_1,\lambda_{2,1}^o\}\le\nu\le m_1$ are same. The functions $u_{1,\nu}$ for $0\le\nu<\min\{m_1,\lambda_{2,1}^o\}$ may have poles of order 1 along $\lambda_2-\lambda_1=\lambda_{2,1}^o$ and their residues are linear combinations of $u_{2,i}|_{\lambda_2=\lambda_1+\lambda_{2,1}^o}$ with $\max\{0,m_1-\lambda_{2,1}^o\}\le i<m_2$. Since

$$\begin{aligned}&\min\bigl\{\#\{\nu\,;\,0\le\nu<\min\{m_1,\lambda_{2,1}^o\}\},\ \#\{i\,;\,\max\{0,m_1-\lambda_{2,1}^o\}\le i<m_2\}\bigr\}\\&\quad=\max\bigl\{0,\min\{m_1,\lambda_{2,1}^o,m_2,m_2-m_1+\lambda_{2,1}^o\}\bigr\},\end{aligned}$$

we have the claim.

ii) A linear combination of $v_{j,\nu}$ $(1\le j\le N,\ 0\le\nu\le m_j)$ may have a pole of order 1 along $\lambda_1-\lambda_2+\lambda_{2,1}^o$ and its residue is a linear combination of

$$\begin{aligned}&\Bigl(u_{1,\nu}+\sum_{m_2+\lambda_{2,1}^o\le i<m_1}b_{\nu+\lambda_{2,1}^o,i}u_{1,i}\Bigr)|_{\lambda_2=\lambda_1+\lambda_{2,1}^o}\quad(\lambda_{2,1}^o\le\nu<\min\{m_1,m_2+\lambda_{2,1}^o\}),\\&\Bigl(u_{2,\nu}+\sum_{\max\{0,m_1-\lambda_{2,1}^o\}\le i<m_2}b_{\nu+\lambda_{2,1}^o,i}u_{2,i}\Bigr)|_{\lambda_2=\lambda_1+\lambda_{2,1}^o}\quad(0\le\nu<m_1-\lambda_{2,1}^o),\\&\sum_{\max\{0,m_1-\lambda_{2,1}^o\}\le i<m_2}b_{\nu,i}u_{2,i}|_{\lambda_2=\lambda_1+\lambda_{2,1}^o}\quad(0\le\nu<\min\{m_1,\lambda_{2,1}^o\}).\end{aligned}$$

Since

$$\begin{aligned}&\#\bigl\{\nu\,;\,\lambda_{2,1}^o\le\nu<\min\{m_1,m_2+\lambda_{2,1}^o\}\bigr\}=\max\bigl\{0,\min\{m_1-\lambda_{2,1}^o,m_2\}\bigr\},\\&\#\{\nu\,;\,0\le\nu<m_1-\lambda_{2,1}^o\}\\&\quad+\min\bigl\{\#\{i\,;\,\max\{0,m_1-\lambda_{2,1}^o\}\le i<m_2\},\#\{\nu\,;\,0\le\nu<\min\{m_1,\lambda_{2,1}^o\}\}\bigr\}\\&=\min\{m_1,m_2\},\end{aligned}$$

we have the claim. □

Remark 12.14. If the local monodromy of the solutions of $Pu=0$ at $x=0$ is locally non-degenerate, the value of $(\lambda_1-\lambda_2+\lambda_{2,1}^o)^{\ell_1}W_1(\lambda)$ at $\lambda=\lambda^o$ does not vanish.

Corollary 12.15. *Let* $Pu=0$ *be a differential equation of order* n *with a regular singularity at* $x=0$ *such that* P *contains a holomorphic parameter* $\lambda=(\lambda_1,\dots,\lambda_N)$ *defined on* $\mathbb{C}^N$. *Suppose that the set of characteristic exponents of* P *at* $x=0$ *equals* $\bigl\{[\lambda_1]_{(m_1)},\dots,[\lambda_N]_{(m_N)}\bigr\}$ *with* $n=m_1+\cdots+m_N$. *Let* $u_{j,\nu}$ *be the solutions of* $Pu=0$ *defined by* (12.43).

i) *Let $W_1(x,\lambda)$ denote the Wronskian of $u_{1,1},\dots,u_{1,m_1}$. Then*

$$\frac{W_1(x,\lambda)}{\prod_{j=2}^N\prod_{0\le\nu<\min\{m_1,m_j\}}\Gamma(\lambda_1-\lambda_j+m_1-\nu)} \tag{12.48}$$

is holomorphic for $\lambda\in\mathbb{C}^N$.

ii) *Let*

$$v_k(\lambda)=\sum_{j=1}^N\sum_{\nu=1}^{m_j}a_{j,\nu,k}(\lambda)u_{j,\nu}\quad(1\le k\le m_1) \tag{12.49}$$

be local solutions of $Pu=0$ defined in a neighborhood of 0 which have a holomorphic parameter $\lambda\in\mathbb{C}^N$. Then

$$\frac{\det\Bigl(a_{1,\nu,k}(\lambda)\Bigr)_{\substack{1\le\nu\le m_1\\1\le k\le m_1}}}{\prod_{j=2}^N\prod_{1\le\nu\le\min\{m_1,m_j\}}\Gamma(\lambda_j-\lambda_1-m_1+\nu)} \tag{12.50}$$

is a holomorphic function of $\lambda\in\mathbb{C}^N$.

Proof. Let $\lambda_{j,1}^o\in\mathbb{Z}$. The order of poles of (12.48) and that of (12.50) along $\lambda_j-\lambda_1=\lambda_{j,1}^o$ are

$$\begin{aligned}&\#\{\nu\,;\,0\le\nu<\min\{m_1,m_j\}\ \text{ and }\ m_1-\lambda_{j,1}^o-\nu\le0\}\\&\quad=\#\{\nu\,;\,\max\{0,m_1-\lambda_{j,1}^o\}\le\nu<\min\{m_1,m_j\}\}\\&\quad=\max\bigl\{0,\min\{m_1,m_j,\lambda_{j,1}^o,\lambda_{j,1}^o+m_j-m_1\}\bigr\}\end{aligned}$$

and

$$\begin{aligned}&\#\{\nu\,;\,1\le\nu\le\min\{m_1,m_j\}\text{ and }\lambda_{j,1}^o-m_1+\nu\le0\}\\&\quad=\max\bigl\{0,\min\{m_1,m_j,m_1-\lambda_{j,1}^o\}\bigr\},\end{aligned}$$

respectively. Hence Corollary 12.13 assures this corollary. □

Remark 12.16. The product of denominator of (12.48) and that of (12.50) equals the periodic function

$$\prod_{j=2}^N(-1)^{[\frac{\min\{m_1,m_j\}}{2}]+1}\Bigl(\frac{\pi}{\sin(\lambda_1-\lambda_j)\pi}\Bigr)^{\min\{m_1,m_j\}}.$$

Definition 12.17 (generalized connection coefficient). Let $P_{\mathbf{m}}u=0$ be the Fuchsian differential equation with the Riemann scheme

$$\begin{Bmatrix}x=c_0=0 & c_1=1 & c_2 & \cdots & c_p=\infty\\ [\lambda_{0,1}]_{(m_{0,1})} & [\lambda_{1,1}]_{(m_{1,1})} & [\lambda_{2,1}]_{(m_{2,1})} & \cdots & [\lambda_{p,1}]_{(m_{p,1})}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ [\lambda_{0,n_0}]_{(m_{0,n_0})} & [\lambda_{1,n_1}]_{(m_{1,n_1})} & [\lambda_{2,n_2}]_{(m_{2,n_2})} & \cdots & [\lambda_{p,n_p}]_{(m_{p,n_p})}\end{Bmatrix}. \tag{12.51}$$

We assume $c_2,\dots,c_{p-1}\notin[0,1]$. Let $u_{0,\nu}^{\lambda_{0,\nu}+k}$ $(1\le\nu\le n_0,\ 0\le k<m_{0,\nu})$ and $u_{1,\nu}^{\lambda_{1,\nu}+k}$ $(1\le\nu\le n_1,\ 0\le k<m_{1,\nu})$ be local solutions of $P_{\mathbf{m}}u=0$ such that

$$\begin{cases}u_{0,\nu}^{\lambda_{0,\nu}+k}\equiv x^{\lambda_{0,\nu}+k} & \mod x^{\lambda_{0,\nu}+m_{0,\nu}}\mathcal{O}_0,\\ u_{1,\nu}^{\lambda_{1,\nu}+k}\equiv(1-x)^{\lambda_{1,\nu}+k} & \mod (1-x)^{\lambda_{1,\nu}+m_{1,\nu}}\mathcal{O}_1.\end{cases} \tag{12.52}$$

They are uniquely defined on $(0,1)\subset\mathbb{R}$ when $\lambda_{j,\nu}-\lambda_{j,\nu'}\notin\mathbb{Z}$ for $j=0,1$ and $1\le\nu<\nu'\le n_j$. Then the connection coefficients $c_{\nu,k}^{\nu',k'}(\lambda)$ are defined by

$$u_{0,\nu}^{\lambda_{0,\nu}+k}=\sum_{\nu',\,k'}c_{\nu,k}^{\nu',k'}(\lambda)u_{1,\nu'}^{\lambda_{1,\nu'}+k'}. \tag{12.53}$$

Note that $c_{\nu,k}^{\nu',k'}(\lambda)$ is a meromorphic function of λ when $\mathbf{m}$ is rigid.

Fix a positive integer n' and the integer sequences $1\le\nu_1^0<\nu_2^0<\cdots<\nu_L^0\le n_0$ and $1\le\nu_1^1<\nu_2^1<\cdots<\nu_{L'}^1\le n_1$ such that

$$n'=m_{0,\nu_1^0}+\cdots+m_{0,\nu_L^0}=m_{1,\nu_1^1}+\cdots+m_{1,\nu_{L'}^1}. \tag{12.54}$$

Then a *generalized connection coefficient* is defined by

(12.55)
$$\begin{aligned}&c\bigl(0:[\lambda_{0,\nu_1^0}]_{(m_{0,\nu_1^0})},\dots,[\lambda_{0,\nu_L^0}]_{(m_{0,\nu_L^0})}\rightsquigarrow 1:[\lambda_{1,\nu_1^1}]_{(m_{1,\nu_1^1})},\dots,[\lambda_{1,\nu_{L'}^1}]_{(m_{1,\nu_{L'}^1})}\bigr)\\&\quad:=\det\Bigl(c_{\nu,k}^{\nu',k'}(\lambda)\Bigr)_{\substack{\nu\in\{\nu_1^0,\dots,\nu_L^0\},\ 0\le k<m_{0,\nu}\\ \nu'\in\{\nu_1^1,\dots,\nu_{L'}^1\},\ 0\le k'<m_{1,\nu'}}}.\end{aligned}$$

The connection coefficient defined in §12.1 corresponds to the case when $n'=1$.

Remark 12.18. i) When $m_{0,1}=m_{1,1}$, Corollary 12.15 assures that

$$\frac{c\bigl(0:[\lambda_{0,1}]_{(m_{0,1})}\rightsquigarrow 1:[\lambda_{1,1}]_{(m_{1,1})}\bigr)}{\displaystyle\prod_{\substack{2\le j\le n_0\\0\le k<\min\{m_{0,1},\,m_{0,j}\}}}\Gamma(\lambda_{0,1}-\lambda_{0,j}+m_{0,1}-k)\cdot\prod_{\substack{2\le j\le n_1\\0<k\le\min\{m_{1,1},\,m_{1,j}\}}}\Gamma(\lambda_{1,j}-\lambda_{1,1}-m_{1,1}+k)}$$

is holomorphic for $\lambda_{j,\nu}\in\mathbb{C}$.

ii) Let $v_1,\dots,v_{n'}$ be generic solutions of $P_{\mathbf{m}}u=0$. Then the generalized connection coefficient in Definition 12.17 corresponds to a usual connection coefficient of the Fuchsian differential equation satisfied by the Wronskian of the n' functions $v_1,\dots,v_{n'}$. The differential equation is of order $\binom{n}{n'}$. In particular, when $n'=n-1$, the differential equation is isomorphic to the dual of the equation $P_{\mathbf{m}}=0$ (cf. Theorem 4.19) and therefore the result in §12.1 can be applied to the connection coefficient. The precise result will be explained in another paper.

Remark 12.19. The following procedure has not been completed in general. But we give a procedure to calculate the generalized connection coefficient (12.55), which we put $c(\lambda)$ here for simplicity when $\mathbf{m}$ is rigid.

(1) Let $\bar\epsilon=\bigl(\bar\epsilon_{j,\nu}\bigr)$ be the shift of the Riemann scheme $\{\lambda_{\mathbf{m}}\}$ such that

$$\begin{cases}\bar\epsilon_{0,\nu}=-1 & (\nu\in\{1,2,\dots,n_0\}\setminus\{\nu_1^0,\dots,\nu_L^0\}),\\ \bar\epsilon_{1,\nu}=1 & (\nu\in\{1,2,\dots,n_1\}\setminus\{\nu_1^1,\dots,\nu_{L'}^1\}),\\ \bar\epsilon_{j,\nu}=0 & (\text{otherwise}).\end{cases} \tag{12.56}$$

Then for generic λ we show that the connection coefficient (12.55) converges to a non-zero meromorphic function $\bar c(\lambda)$ of λ by the shift $\{\lambda_{\mathbf{m}}\}\mapsto\{(\lambda+k\bar\epsilon)_{\mathbf{m}}\}$ when $\mathbb{Z}_{>0}\ni k\to\infty$.

(2) Choose suitable linear functions $b_i(\lambda)$ of λ by applying Proposition 12.12 or Corollary 12.15 to $c(\lambda)$ so that $e(\lambda):=\prod_{i=1}^N\Gamma\bigl(b_i(\lambda)\bigr)^{-1}\cdot c(\lambda)\bar c(\lambda)^{-1}$ is holomorphic for any λ.

In particular, when $L = L' = 1$ and $\nu_1^0 = \nu_1^1 = 1$, we may put

$$\{b_i\} = \bigcup_{j=2}^{n_0} \bigl\{\lambda_{0,1} - \lambda_{0,j} + m_{0,1} - \nu \,;\, 0 \le \nu < \min\{m_{0,1}, m_{0,j}\}\bigr\}$$
$$\cup \bigcup_{j=2}^{n_1} \bigl\{\lambda_{1,j} - \lambda_{1,1} - m_{1,1} + \nu \,;\, 1 \le \nu \le \min\{m_{1,1}, m_{1,j}\}\bigr\}.$$

(3) Find the zeros of $e(\lambda)$ some of which are explained by the reducibility or the shift operator of the equation $P_{\mathbf{m}}u = 0$ and choose linear functions $c_i(\lambda)$ of λ so that $f(\lambda) := \prod_{i=1}^{N'} \Gamma\bigl(c_i(\lambda)\bigr) \cdot e(\lambda)$ is still holomorphic for any λ.

(4) If $N = N'$ and $\sum_i d_i(\lambda) = \sum_i c_i(\lambda)$, Lemma 12.20 assures $f(\lambda) = \bar{c}(\lambda)$ and

$$c(\lambda) = \frac{\prod_{i=1}^{N} \Gamma\bigl(b_i(\lambda)\bigr)}{\prod_{i=1}^{N} \Gamma\bigl(c_i(\lambda)\bigr)} \cdot \bar{c}(\lambda) \tag{12.57}$$

because $\frac{f(\lambda)}{f(\lambda+\epsilon)}$ is a rational function of λ, which follows from the existence of a shift operator assured by Theorem 11.2.

Lemma 12.20. *Let $f(t)$ be a meromorphic function of $t \in \mathbb{C}$ such that $r(t) = \frac{f(t)}{f(t+1)}$ is a rational function and*

$$\lim_{\mathbb{Z}_{>0} \ni k \to \infty} f(t+k) = 1. \tag{12.58}$$

Then there exists $N \in \mathbb{Z}_{\ge 0}$ and b_i, $c_i \in \mathbb{C}$ for $i = 1, \dots, n$ such that

$$b_1 + \cdots + b_N = c_1 + \cdots + c_N, \tag{12.59}$$

$$f(t) = \frac{\prod_{i=1}^{N} \Gamma(t + b_i)}{\prod_{i=1}^{N} \Gamma(t + c_i)}. \tag{12.60}$$

Moreover, if $f(t)$ is an entire function, then $f(t)$ is the constant function 1.

Proof. Since $\lim_{k\to\infty} r(t+k) = 1$, we may assume

$$r(t) = \frac{\prod_{i=1}^{N}(t + c_i)}{\prod_{i=1}^{N}(t + b_i)}$$

and then

$$f(t) = \frac{\prod_{i=1}^{N}\prod_{\nu=0}^{n-1}(t + c_i + \nu)}{\prod_{i=1}^{N}\prod_{\nu=0}^{n-1}(t + b_i + \nu)} f(t+n).$$

Since

$$\lim_{n\to\infty} \frac{n! n^{x-1}}{\prod_{\nu=0}^{n-1}(x+\nu)} = \Gamma(x),$$

the assumption implies (12.59) and (12.60).

We may assume $b_i \ne c_j$ for $1 \le i \le N$ and $1 \le j \le N$. Then the function (12.60) with (12.59) has a pole if $N > 0$. □

We have the following proposition for zeros of $c(\lambda)$.

Proposition 12.21. *Retain the notation in Remark 12.19 and fix λ so that*

$$\lambda_{j,\nu} - \lambda_{j,\nu'} \notin \mathbb{Z} \quad (j = 0,\ 1 \ \ \text{and} \ \ 0 \le \nu < \nu' \le n_j). \tag{12.61}$$

i) *The relation $c(\lambda) = 0$ is valid if and only if there exists a non-zero function*

$$v = \sum_{\substack{\nu \in \{\nu_1^0, \dots, \nu_L^0\} \\ 0 \le k < m_{0,\nu}}} C_{\nu,k} u_0^{\lambda_{0,\nu}+k} = \sum_{\substack{\nu \in \{1,\dots,n_1\} \setminus \{\nu_1^1,\dots,\nu_{L'}^1\} \\ 0 \le k < m_{1,\nu}}} C'_{\nu,k} u_1^{\lambda_{1,\nu}+k}$$

on $(0,1)$ *with* $C_{\nu,k}$, $C'_{\nu,k} \in \mathbb{C}$.

ii) *Fix a shift* $\epsilon = (\epsilon_{j,\nu})$ *compatible to* $\mathbf{m}$ *and let* $R_{\mathbf{m}}(\epsilon, \lambda)$ *be the shift operator in* Theorem 11.2. *Suppose* $R_{\mathbf{m}}(\epsilon, \lambda)$ *is bijective, namely,* $c_{\mathbf{m}}(\epsilon; \lambda) \neq 0$ (cf. Theorem 11.8). *Then* $c(\lambda + \epsilon) = 0$ *if and only if* $c(\lambda) = 0$

PROOF. Assumption (12.61) implies that $\{u_0^{\lambda_{0,\nu}+k}\}$ and $\{u_1^{\lambda_{1,\nu}+k}\}$ define sets of basis of local solutions of the equation $P_{\mathbf{m}} u = 0$. Hence the claim i) is clear from the definition of $c(\lambda)$.

Suppose $c(\lambda) = 0$ and $R_{\mathbf{m}}(\epsilon, \lambda)$ is bijective. Then applying the claim i) to $R_{\mathbf{m}}(\epsilon, \lambda)v$, we have $c(\lambda + \epsilon) = 0$. If $R_{\mathbf{m}}(\epsilon, \lambda)$ is bijective, so is $R_{\mathbf{m}}(-\epsilon, \lambda + \epsilon)$ and $c(\lambda + \epsilon) = 0$ implies $c(\lambda) = 0$. □

COROLLARY 12.22. *Let* $\mathbf{m} = \mathbf{m}' \oplus \mathbf{m}''$ *be a rigid decomposition of* $\mathbf{m}$ *such that*

$$\sum_{\nu \in \{\nu_1^0, \ldots, \nu_L^0\}} m'_{0,\nu} > \sum_{\nu \in \{\nu_1^1, \ldots, \nu_{L'}^1\}} m'_{1,\nu}. \tag{12.62}$$

Then $\Gamma(|\{\lambda_{\mathbf{m}'}\}|) \cdot c(\lambda)$ *is holomorphic under the condition* (12.61).

PROOF. When $|\{\lambda_{\mathbf{m}'}\}|$=0, we have the decomposition $P_{\mathbf{m}} = P_{\mathbf{m}''} P_{\mathbf{m}'}$ and hence $c(\lambda) = 0$. There exists a shift ϵ compatible to $\mathbf{m}$ such that

$$\sum_{j=0}^{p} \sum_{\nu=1}^{n_j} m'_{j,\nu} \epsilon_{j,\nu} = 1.$$

Let λ be generic under $|\{\lambda_{\mathbf{m}}\}| = 0$ and $|\{\lambda_{\mathbf{m}'}\}| \in \mathbb{Z} \setminus \{0\}$. Then Theorem 11.9) ii) assures $c_{\mathbf{m}}(\epsilon; \lambda) \neq 0$ and Proposition 12.21 proves the corollary. □

REMARK 12.23. Suppose that Remark 12.19 (1) is established. Then Proposition 12.12 and Proposition 12.21 with Theorem 11.8 assure that the denominator and the numerator of the rational function which equals $\frac{c(\lambda)}{c(\lambda+\bar{\epsilon})}$ are products of certain linear functions of λ and therefore (12.57) is valid with suitable linear functions $b_i(\lambda)$ and $c_i(\lambda)$ of λ satisfying $\sum_{i=1}^{N} b_i(\lambda) = \sum_{i=1}^{N} c_i(\lambda)$.

EXAMPLE 12.24 (generalized hypergeometric function). The generalized hypergeometric series (0.7) satisfies the equation $P_n(\alpha; \beta)u = 0$ given by (13.21) and [**Kh**, §4.1.2 Example 9] shows that the equation is isomorphic to the system of Okubo normal form

$$\left(x - \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}\right) \frac{d\tilde{u}}{dx} = \begin{pmatrix} -\beta_n & 1 & & & & \\ \alpha_{2,1} & 0 & 1 & & & \\ \alpha_{3,1} & & 1 & 1 & & \\ \vdots & & & \ddots & \ddots & \\ \alpha_{n-1,1} & & & & n-3 & 1 \\ \alpha_{n,1} & -c_{n-1} & -c_{n-2} & \cdots & -c_2 & -c_1+(n-2) \end{pmatrix} \tilde{u} \tag{12.63}$$

with

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \ u = u_1 \ \text{ and } \ \sum_{\nu=1}^{n} \alpha_\nu = \sum_{\nu=1}^{n} \beta_\nu.$$

Let us calculate the connection coefficient

$$c(0\!:\!0 \rightsquigarrow 1\!:\!-\beta_n) = \lim_{x \to 1-0} (1-x)^{\beta_n} {}_nF_{n-1}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_{n-1}; x) \quad (\operatorname{Re} \beta_n > 0).$$

Applying Theorem 12.10 to the system of Schlesinger canonical form satisfied by $\operatorname{Ad}\bigl((1-x)^{\beta_n}\bigr)$, the connection coefficient satisfies Remark 12.19 i) with $\bar{c}(\lambda) = 1$, namely,

$$\lim_{k \to +\infty} c(0\!:\!0 \rightsquigarrow 1\!:\!-\beta_n)|_{\alpha_j \mapsto \alpha_j + k,\ \beta_j \mapsto \beta_j + k \ \ (1 \le j \le n)} = 1. \tag{12.64}$$

Then Remark 12.19 ii) shows that $\prod_{j=1}^{n}\Gamma(\beta_j)^{-1}\cdot c(0\!:\!0 \rightsquigarrow 1\!:\!-\beta_n)$ is a holomorphic function of $(\alpha,\beta)\in\mathbb{C}^{n+(n-1)}$.

Corresponding to the Riemann scheme (0.8), the existence of rigid decompositions

$$\overbrace{1\cdots1}^{n};n-11;\overbrace{1\cdots1}^{n}=\overbrace{0\cdots0}^{n-1}1;10;0\cdots\overset{i}{\breve{1}}\cdots0\oplus\overbrace{1\cdots1}^{n-1}0;n-11;1\cdots\overset{i}{\breve{0}}\cdots1$$

for $i=1,\dots,n$ proves that $\prod_{i=1}^{n}\Gamma(\alpha_i)\cdot\prod_{j=1}^{n}\Gamma(\beta_j)^{-1}\cdot c(0:0\rightsquigarrow 1:-\beta_n)$ is also entire holomorphic. Then the procedure given in Remark 12.19 assures

$$c(0\!:\!0 \rightsquigarrow 1\!:\!-\beta_n)=\frac{\prod_{i=1}^{n}\Gamma(\beta_i)}{\prod_{i=1}^{n}\Gamma(\alpha_i)}. \tag{12.65}$$

We can also prove (12.65) as in the following way. Since

$$\frac{d}{dx}F(\alpha;\beta;x)=\frac{\alpha_1\cdots\alpha_n}{\beta_1\cdots\beta_{n-1}}F(\alpha_1+1,\dots,\alpha_n+1;\beta_1+1,\dots,\beta_{n-1}+1;x)$$

and

$$\frac{d}{dx}\bigl(1-x\bigr)^{-\beta_n}\bigl(1+(1-x)\mathcal{O}_1\bigr)=\beta_n\bigl(1-x\bigr)^{-\beta_n-1}\bigl(1+(1-x)\mathcal{O}_1\bigr),$$

we have

$$\frac{c(0\!:\!0 \rightsquigarrow 1\!:\!-\beta_n)}{c(0\!:\!0 \rightsquigarrow 1\!:\!-\beta_n)|_{\alpha_j\mapsto\alpha_j+1,\ \beta_j\mapsto\beta_j+1}}=\frac{\alpha_1\dots\alpha_n}{\beta_1\dots\beta_n},$$

which proves (12.65) because of (12.64).

A further study of generalized connection coefficients will be developed in another paper. In this paper we will only give some examples in §13.5 and §13.7.5.