CHAPTER 9

Monodromy

The transformation of monodromy generators for irreducible Fuchsian systems of Schlesinger canonical form under the middle convolution or the addition is studied by [**Kz**] and [**DR, DR2**] etc. A non-zero homomorphism of an irreducible single Fuchsian differential equation to an irreducible system of Schlesinger canonical form induces the isomorphism of their monodromies of the solutions (cf. Remark 1.14). In particular since any rigid local system is realized by a single Fuchsian differential equation, their monodromies naturally coincide with each other through the correspondence of their monodromy generators. The correspondence between the local monodromies and the global monodromies is described by [**DR2**], which we will review.

9.1. Middle convolution of monodromies

For given matrices $A_j \in M(n,\mathbb{C})$ for $j = 1,\dots,p$ the Fuchsian system

$$\frac{dv}{dx} = \sum_{j=1}^{p} \frac{A_j}{x-c_j} v \tag{9.1}$$

of Schlesinger canonical form (SCF) is defined. Put $A_0 = -A_1 - \cdots - A_p$ and $\mathbf{A} = (A_0, A_1, \dots, A_p)$ which is an element of

$$M(n,\mathbb{C})_0^{p+1} := \{(C_0,\dots,C_p) \in M(n,\mathbb{C})^{p+1}\,;\, C_0 + \cdots + C_p = 0\}, \tag{9.2}$$

The Riemann scheme of (9.1) is defined by

(9.3)

$$\begin{Bmatrix} x = c_0 = \infty & c_1 & \cdots & c_p \\ [\lambda_{0,1}]_{m_{0,1}} & [\lambda_{1,1}]_{m_{1,1}} & \cdots & [\lambda_{p,1}]_{m_{p,1}} \\ \vdots & \vdots & \vdots & \vdots \\ [\lambda_{0,n_0}]_{m_{0,n_0}} & [\lambda_{1,n_1}]_{m_{1,n_1}} & \cdots & [\lambda_{p,n_p}]_{m_{p,1}} \end{Bmatrix}, \quad [\lambda]_k := \begin{pmatrix} \lambda \\ \vdots \\ \lambda \end{pmatrix} \in M(1,k,\mathbb{C})$$

if

$$A_j \sim L(m_{j,1},\dots,m_{j,n_j};\lambda_{j,1},\dots,\lambda_{j,n_j}) \quad (j = 0,\dots,p)$$

under the notation (4.33). Here the Fuchs relation equals

$$\sum_{j=0}^{p}\sum_{\nu=1}^{n_j} m_{j,\nu}\lambda_{j,\nu} = 0. \tag{9.4}$$

We define that $\mathbf{A}$ is *irreducible* if a subspace V of $\mathbb{C}^n$ satisfies $A_jV \subset A_j$ for $j = 0,\dots,p$, then $V = \{0\}$ or $V = \mathbb{C}^n$. In general, $\mathbf{A} = (A_0,\dots,A_p)$, $\mathbf{A}' = (A'_0,\dots,A'_p) \in M(n,\mathbb{C})^{p+1}$, we denote by $\mathbf{A} \sim \mathbf{A}'$ if there exists $U \in GL(n,\mathbb{C})$ such that $A'_j = UA_jU^{-1}$ for $j = 0,\dots,p$.

For $(\mu_0,\dots,\mu_p) \in \mathbb{C}^{p+1}$ with $\mu_0+\cdots+\mu_p = 0$, the addition $\mathbf{A}' = (A'_0,\dots,A'_p) \in M(n,\mathbb{C})_0^{p+1}$ of $\mathbf{A}$ with respect to $(\mu_0,\dots,\mu_p)$ is defined by $A'_j = A_j + \mu_j$ for $j = 0,\dots,p$.

For a complex number μ the middle convolution $\bar{\mathbf{A}} := mc_\mu(\mathbf{A})$ of $\mathbf{A}$ is defined by $\bar{A}_j = \bar{A}_j(\mu)$ for $j = 1, \dots, p$ and $\bar{A}_0 = -\bar{A}_1 - \cdots - \bar{A}_p$ under the notation in §1.5. Then we have the following theorem.

THEOREM 9.1 ([**DR, DR2**]). *Suppose that* $\mathbf{A}$ *satisfies the conditions*

$$\bigcap_{\substack{1\le j\le p\\ j\ne i}} \ker A_j \cap \ker(A_0 - \tau) = \{0\} \qquad (i = 1, \dots, p,\ \forall\tau \in \mathbb{C}), \tag{9.5}$$

$$\bigcap_{\substack{1\le j\le p\\ j\ne i}} \ker {}^tA_j \cap \ker({}^tA_0 - \tau) = \{0\} \qquad (i = 1, \dots, p,\ \forall\tau \in \mathbb{C}). \tag{9.6}$$

i) *The tuple* $mc_\mu(\mathbf{A}) = (\bar{A}_0, \dots, \bar{A}_p)$ *also satisfies the same conditions as above with replacing* A_ν *by* $\bar{A}_\nu$ *for* $\nu = 0, \dots, p$*, respectively. Moreover we have*

$$mc_\mu(\mathbf{A}) \sim mc_\mu(\mathbf{A}') \quad \textit{if} \quad \mathbf{A} \sim \mathbf{A}', \tag{9.7}$$

$$mc_{\mu'} \circ mc_\mu(\mathbf{A}) \sim mc_{\mu+\mu'}(\mathbf{A}), \tag{9.8}$$

$$mc_0(\mathbf{A}) \sim \mathbf{A} \tag{9.9}$$

and $mc_\mu(\mathbf{A})$ *is irreducible if and only if* $\mathbf{A}$ *is irreducible.*

ii) (cf. [**O6**, Theorem 5.2]) *Assume*

$$\mu = \lambda_{0,1} \ne 0 \quad \textit{and} \quad \lambda_{j,1} = 0 \quad \textit{for} \quad j = 1, \dots, p \tag{9.10}$$

and

$$\lambda_{j,\nu} = \lambda_{j,1} \quad \textit{implies} \quad m_{j,\nu} \le m_{j,1} \tag{9.11}$$

for $j = 0, \dots, p$ *and* $\nu = 2, \dots, n_j$*. Then the Riemann scheme of* $mc_\mu(\mathbf{A})$ *equals*

$$\begin{Bmatrix} x = \infty & c_1 & \cdots & c_p \\ [-\mu]_{m_{0,1}-d} & [0]_{m_{1,1}-d} & \cdots & [0]_{m_{p,1}-d} \\ [\lambda_{0,2} - \mu]_{m_{0,2}} & [\lambda_{1,2} + \mu]_{m_{1,2}} & \cdots & [\lambda_{p,2} + \mu]_{m_{p,2}} \\ \vdots & \vdots & \vdots & \vdots \\ [\lambda_{0,n_0} - \mu]_{m_{0,n_0}} & [\lambda_{1,n_1} + \mu]_{m_{1,n_1}} & \cdots & [\lambda_{p,n_p} + \mu]_{m_{p,1}} \end{Bmatrix} \tag{9.12}$$

with

$$d := m_{0,1} + \cdots + m_{p,1} - (p-1)\operatorname{ord}\mathbf{m}. \tag{9.13}$$

EXAMPLE 9.2. The addition of

$$mc_{-\lambda_{0,1}-\lambda_{1,2}-\lambda_{2,2}}(\{\lambda_{0,2} - \lambda_{0,1}, \lambda_{0,1} + \lambda_{1,1} + \lambda_{2,2}, \lambda_{0,1} + \lambda_{1,2} + \lambda_{2,1}\})$$

with respect to $(-\lambda_{1,2} - \lambda_{2,2}, \lambda_{1,2}, \lambda_{2,2})$ give the Fuchsian system of Schlesinger canonical form

$$\frac{du}{dx} = \frac{A_1}{x}u + \frac{A_2}{x-1}u,$$

$$A_1 = \begin{pmatrix} \lambda_{1,1} & \lambda_{0,1} + \lambda_{1,2} + \lambda_{2,1} \\ & \lambda_{1,2} \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} \lambda_{2,2} & \\ \lambda_{0,1} + \lambda_{1,1} + \lambda_{2,2} & \lambda_{2,1} \end{pmatrix}.$$

with the Riemann scheme

$$\begin{Bmatrix} x = \infty & 0 & 1 \\ \lambda_{0,1} & \lambda_{1,1} & \lambda_{2,1} \\ \lambda_{0,2} & \lambda_{1,2} & \lambda_{2,2} \end{Bmatrix} \qquad (\lambda_{0,1} + \lambda_{0,2} + \lambda_{1,1} + \lambda_{1,2} + \lambda_{2,1} + \lambda_{2,2} = 0).$$

The system is invariant as $W(x; \lambda_{j,\nu})$-modules under the transformation $\lambda_{j,\nu} \mapsto \lambda_{j,3-\nu}$ for $j = 0, 1, 2$ and $\nu = 1, 2$.

Suppose $\lambda_{j,\nu}$ are generic complex numbers under the condition $\lambda_{0,1} + \lambda_{1,2} + \lambda_{2,1} = \lambda_{0,2} + \lambda_{1,1} + \lambda_{2,2} = 0$. Then A_1 and A_2 have a unique simultaneous eigenspace.

In fact, $A_1\binom{0}{1} = \lambda_{1,2}\binom{0}{1}$ and $A_2\binom{0}{1} = \lambda_{2,1}\binom{0}{1}$. Hence the system is not invariant as $W(x)$-modules under the transformation above and $\mathbf{A}$ is not irreducible in this case.

To describe the monodromies, we review the multiplicative version of these operations.

Let $\mathbf{M} = (M_0, \dots, M_p)$ be an element of

$$GL(n,\mathbb{C})_1^{p+1} := \{(G_0, \dots, G_p) \in GL(n,\mathbb{C})^{p+1}\,;\, G_p \cdots G_0 = I_n\}. \tag{9.14}$$

For $(\rho_0, \dots, \rho_p) \in \mathbb{C}^{p+1}$ satisfying $\rho_0 \cdots \rho_p = 1$, the *multiplication* of $\mathbf{M}$ with respect to ρ is defined by $(\rho_0 M_0, \dots, \rho_p M_p)$.

For a given $\rho \in \mathbb{C}^\times$, we define $\tilde{M}_j = \big(M_{j,\nu,\nu'}\big)_{\substack{1\le\nu\le n\\ 1\le\nu'\le p}} \in GL(pn,\mathbb{C})$ by

$$\tilde{M}_{j,\nu,\nu'} = \begin{cases} \delta_{\nu,\nu'} I_n & (\nu \ne j),\\ M_{\nu'} - 1 & (\nu = j,\ 1 \le \nu' \le j-1),\\ \rho M_j & (\nu = \nu' = j),\\ \rho(M_{\nu'} - 1) & (\nu = j,\ j+1 \le \nu' \le p). \end{cases}$$

Let $\bar{M}_j$ denote the quotient $\tilde{M}_j|_{\mathbb{C}^{pn}/V}$ of

$$\tilde{M}_j = \begin{pmatrix} I_n & & & & \\ & \ddots & & & \\ M_1 - 1 & \cdots & \rho M_j & \cdots & \rho(M_p - 1) \\ & & & \ddots & \\ & & & & I_n \end{pmatrix} \in GL(pn,\mathbb{C}) \tag{9.15}$$

for $j = 1, \dots, p$ and $M_0 = (M_p \dots M_1)^{-1}$. The tuple $\mathrm{MC}_\rho(\mathbf{M}) = (\bar{M}_0, \dots, \bar{M}_p)$ is called (the multiplicative version of) the middle convolution of $\mathbf{M}$ with respect to ρ. Here $V := \ker(\tilde{M} - 1) + \bigcap_{j=1}^p \ker(\tilde{M}_j - 1)$ with

$$\tilde{M} := \begin{pmatrix} M_1 & & \\ & \ddots & \\ & & M_p \end{pmatrix}.$$

Then we have the following theorem.

THEOREM 9.3 ([**DR, DR2**]). *Let* $\mathbf{M} = (M_0, \dots, M_p) \in GL(n,\mathbb{C})_1^{p+1}$. *Suppose*

$$\bigcap_{\substack{1\le\nu\le p\\ \nu\le i}} \ker(M_\nu - 1) \cap \ker(M_i - \tau) = \{0\} \qquad (1 \le i \le p,\ \forall\tau \in \mathbb{C}^\times), \tag{9.16}$$

$$\bigcap_{\substack{1\le\nu\le p\\ \nu\le i}} \ker({}^tM_\nu - 1) \cap \ker({}^tM_i - \tau) = \{0\} \qquad (1 \le i \le p,\ \forall\tau \in \mathbb{C}^\times). \tag{9.17}$$

i) *The tuple* $\mathrm{MC}_\rho(\mathbf{M}) = (\bar{M}_0, \dots, \bar{M}_p)$ *also satisfies the same conditions as above with replacing* M_ν *by* $\bar{M}_\nu$ *for* $\nu = 0, \dots, p$*, respectively. Moreover we have*

$$\mathrm{MC}_\rho(\mathbf{M}) \sim \mathrm{MC}_\rho(\mathbf{M}') \quad \textit{if} \;\; \mathbf{M} \sim \mathbf{M}', \tag{9.18}$$

$$\mathrm{MC}_{\rho'} \circ \mathrm{MC}_\rho(\mathbf{M}) \sim \mathrm{MC}_{\rho\rho'}(\mathbf{M}), \tag{9.19}$$

$$\mathrm{MC}_1(\mathbf{M}) \sim \mathbf{M} \tag{9.20}$$

and $\mathrm{MC}_\rho(\mathbf{M})$ *is irreducible if and only if* $\mathbf{M}$ *is irreducible.*

ii) *Assume*

$$M_j \sim L(m_{j,1},\dots,m_{j,n_j};\rho_{j,1},\dots,\rho_{j,n_j}) \quad \textit{for} \quad j=0,\dots,p, \tag{9.21}$$

$$\rho=\rho_{0,1}\neq 1 \quad \textit{and} \quad \rho_{j,1}=1 \quad \textit{for} \quad j=1,\dots,p \tag{9.22}$$

and

$$\rho_{j,\nu}=\rho_{j,1} \quad \textit{implies} \quad m_{j,\nu}\le m_{j,1} \tag{9.23}$$

for $j=0,\dots,p$ *and* $\nu=2,\dots,n_j$. *In this case, we say that* $\mathbf{M}$ *has a spectral type* $\mathbf{m}:=(\mathbf{m}_0,\dots,\mathbf{m}_p)$ *with* $\mathbf{m}_j=(m_{j,1},\dots,m_{j,n_j})$.

Putting $(\bar M_0,\dots,\bar M_p)=\mathrm{MC}_\rho(M_0,\dots,M_p)$, *we have*

$$\bar M_j \sim \begin{cases} L(m_{0,1}-d,m_{0,2},\dots,m_{0,n_0};\rho^{-1},\rho^{-1}\rho_{0,2},\dots\rho^{-1}\rho_{0,n_0}) & (j=0),\\ L(m_{j,1}-d,m_{j,2},\dots,m_{j,n_j};1,\rho\rho_{j,2},\dots\rho\rho_{j,n_j}) & (j=1,\dots,p). \end{cases} \tag{9.24}$$

Here d *is given by* (9.13).

Remark 9.4. i) We note that some $m_{j,1}$ may be zero in Theorem 9.1 and Theorem 9.3.

ii) It follows from Theorem 9.1 (resp. Theorem 9.3) and Scott's lemma that any irreducible tuple $\mathbf{A}\in M(n,\mathbb{C})_0^{p+1}$ (resp. $\mathbf{M}\in GL(n,\mathbb{C})_1^{p+1}$) can be connected by successive applications of middle convolutions and additions (resp. multiplications) to an irreducible tuple whose spectral type $\bar{\mathbf{m}}$ satisfies $\operatorname{ord}\bar{\mathbf{m}}=1$ or $d_{\max}(\bar{\mathbf{m}})\le 0$. Moreover the spectral type of an irreducible tuple $\mathbf{M}$ or $\mathbf{A}$ is irreducibly realizable in the sense in Definition 4.16 (cf. [**Ko**], [**CB**], [**O6**]),

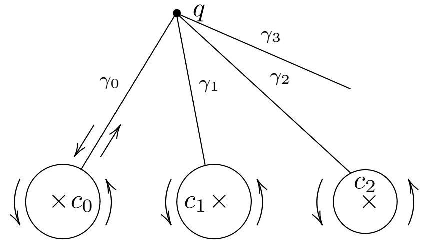
Definition 9.5. Let $\mathbf{M}=(M_0,\dots,M_p)\in GL(n,\mathbb{C})_1^{p+1}$. Suppose (9.21). Fix $\ell=(\ell_0,\dots,\ell_p)\in\mathbb{Z}_{\ge1}^{p+1}$ and define $\partial_\ell\mathbf{M}$ as follows.

$$\begin{aligned} \rho_j &:= \begin{cases} \rho_{j,\ell_j} & (0\le j\le p,\ 1\le \ell_j\le n_j),\\ \text{any complex number} & (0\le j\le p,\ n_j<\ell_j), \end{cases}\\ \rho &:= \rho_0\rho_1\dots\rho_p,\\ (M_0',\dots,M_p') &:= \mathrm{MC}_\rho(\rho_1\cdots\rho_p M_0,\rho_1^{-1}M_1,\rho_2^{-1}M_2,\dots,\rho_p^{-1}M_p),\\ \partial_\ell\mathbf{M} &:= (\rho_1^{-1}\cdots\rho_p^{-1}M_0',\rho_1M_1',\rho_2M_2,'\dots,\rho_pM_p'). \end{aligned}$$

Here we note that if $\ell=(1,\dots,1)$ and $\rho_{j,1}=1$ for $j=2,\dots,p$, $\partial_\ell\mathbf{M}=\mathrm{MC}_\rho(\mathbf{M})$.

Let $u(1),\dots,u(n)$ be independent solutions of (9.1) at a generic point q. Let γ_j be a closed path around c_j as in the following figure. Denoting the result of the analytic continuation of $\tilde u:=(u(1),\dots,u(n))$ along γ_j by $\gamma_j(\tilde u)$, we have a *monodromy generator* $M_j\in GL(n,\mathbb{C})$ such that $\gamma_j(\tilde u)=\tilde uM_j$. We call the tuple $\mathbf{M}=(M_0,\dots,M_p)$ the *monodromy* of (9.1) with respect to $\tilde u$ and $\gamma_0,\dots,\gamma_p$. The connecting path first going along γ_i and then going along γ_j is denoted by $\gamma_i\circ\gamma_j$.

(9.25)



$$\begin{aligned} \gamma_i\circ\gamma_j(\tilde u) &= \gamma_j(\tilde uM_i)\\ &= \gamma_j(\tilde u)M_i\\ &= \tilde uM_jM_i,\\ M_pM_{p-1}\cdots M_1M_0 &= I_n. \end{aligned}$$

The following theorem says that the monodromy of solutions of the system obtained by a middle convolution of the system (9.1) is a multiplicative middle convolution of that of the original system (9.1).

THEOREM 9.6 ([**DR2**]). *Let* $\mathrm{Mon}(\mathbf{A})$ *denote the monodromy of the equation* (9.1). *Put* $\mathbf{M} = \mathrm{Mon}(\mathbf{A})$. *Suppose* $\mathbf{M}$ *satisfies* (9.16) *and* (9.17) *and*

$$\operatorname{rank}(A_0 - \mu) = \operatorname{rank}(M_0 - e^{2\pi\sqrt{-1}\mu}), \tag{9.26}$$

$$\operatorname{rank}(A_j) = \operatorname{rank}(M_j - 1) \tag{9.27}$$

for $j = 1, \ldots, p$, *then*

$$\mathrm{Mon}\bigl(mc_\mu(\mathbf{A})\bigr) \sim \mathrm{MC}_{e^{2\pi\sqrt{-1}\mu}}\bigl(\mathrm{Mon}(\mathbf{A})\bigr). \tag{9.28}$$

Let $\mathcal{F}$ be a space of (multi-valued) holomorphic functions on $\mathbb{C} \setminus \{c_1, \ldots, c_p\}$ valued in $\mathbb{C}^n$ such that $\mathcal{F}$ satisfies (2.15), (2.16) and (2.17). For example the solutions of the equation (9.1) defines $\mathcal{F}$. Fixing a base $u = \bigl(u(1), \ldots, u(n)\bigr)$ of $\mathcal{F}(U)$ with $U \ni q$, we can define monodromy generators $(M_0, \ldots, M_p)$. Fix $\mu \in \mathbb{C}$ and put $\rho = e^{2\pi\sqrt{-1}\mu}$ and

$$v_j(x) = \begin{pmatrix} \int^{(x+,c_j+,x-,c_j-)} \frac{u(t)(x-t)^{\mu-1}}{t-c_1} dt \\ \vdots \\ \int^{(x+,c_j+,x-,c_j-)} \frac{u(t)(x-t)^{\mu-1}}{t-c_p} dt \end{pmatrix} \quad \text{and} \quad v(x) = \bigl(v_1(x), \ldots, v_p(x)\bigr).$$

Then $v(x)$ is a holomorphic function valued in $M(pn, \mathbb{C})$ and the pn column vectors of $v(x)$ define a *convolution* $\tilde{\mathcal{F}}$ of $\mathcal{F}$ and the following facts are shown by [**DR2**].

The monodromy generators of $\tilde{\mathcal{F}}$ with respect to the base $v(x)$ equals the *convolution* $\tilde{\mathbf{M}} = (\tilde{M}_0, \ldots, \tilde{M}_1)$ of $\mathbf{M}$ given by (9.15) and if $\mathcal{F}$ corresponds to the space of solutions of (1.79), $\tilde{\mathcal{F}}$ corresponds to that of the system of Schlesinger canonical form defined by $\bigl(\tilde{A}_0(\mu), \ldots, \tilde{A}_p(\mu)\bigr)$ in (1.81), which we denote by $\mathcal{M}_{\tilde{\mathbf{A}}}$.

The middle convolution $\mathrm{MC}_\rho(\mathbf{M})$ of $\mathbf{M}$ is the induced monodromy generators on the quotient space of $\mathbb{C}^{pn}/V$ where V is the maximal invariant subspace such the restriction of $\tilde{\mathbf{M}}$ on V is a direct sum of finite copies of 1-dimensional spaces with the actions $(\rho^{-1}, 1, \ldots, 1, \overset{j}{\breve{\rho}}, 1, \ldots, 1) \in GL(1, \mathbb{C})_1^{p+1}$ $(j = 1, \ldots, p)$ and $(1, 1, \ldots, 1)$. The system defined by the middle convolution $mc_\mu(\mathbf{A})$ is the quotient of the system $\mathcal{M}_{\tilde{\mathbf{A}}}$ by the maximal submodule such that the submodule is a direct sum of finite copies of the equations $(x - c_j)\frac{dw}{dx} = \mu w$ $(j = 1, \ldots, p)$ and $\frac{dw}{dx} = 0$.

Suppose $\mathbf{M}$ and $\mathrm{MC}_\rho(\mathbf{M})$ are irreducible and $\rho \neq 1$. Assume $\phi(x)$ is a function belonging to $\mathcal{F}$ such that it is defined around $x = c_j$ and corresponds to the eigenvector of the monodromy matrix M_j with the eigenvalue different from 1. Then the holomorphic continuation of $\Phi(x) = \int^{(x+,c_j+,x-,c_j-)} \frac{\phi(t)(t-x)^\mu}{t-c_j} dt$ defines the monodromy isomorphic to $\mathrm{MC}_\rho(\mathbf{M})$.

REMARK 9.7. We can define the monodromy $\mathbf{M} = (M_0, \ldots, M_p)$ of the universal model $P_{\mathbf{m}} u = 0$ (cf. Theorem 6.14) so that $\mathbf{M}$ is entire holomorphic with respect to the spectral parameters $\lambda_{j,\nu}$ and the accessory parameters g_i under the normalization $u(j)^{(\nu-1)}(q) = \delta_{j,\nu}$ for $j,\ \nu = 1, \ldots, n$ and $q \in \mathbb{C} \setminus \{c_1, \ldots, c_p\}$. Here $u(1), \ldots, u(n)$ are solutions of $P_{\mathbf{m}} u = 0$.

DEFINITION 9.8. Let P be a Fuchsian differential operator with the Riemann scheme (4.15) and the spectral type $\mathbf{m} = \bigl(m_{j,\nu}\bigr)_{\substack{0 \le j \le p \\ 1 \le \nu \le n_j}}$. We define that P is *locally non-degenerate* if the tuple of the monodromy generators $\mathbf{M} := (M_0, \ldots, M_p)$ satisfies

$$M_j \sim L(m_{j,1}, \ldots, m_{j,n_j}; e^{2\pi\sqrt{-1}\lambda_{j,1}}, \ldots, e^{2\pi\sqrt{-1}\lambda_{j,n_j}}) \quad (j = 0, \ldots, p), \tag{9.29}$$

which is equivalent to the condition that

$$\dim Z(M_j) = m_{j,1}^2 + \cdots + m_{j,n_j}^2 \quad (j = 0, \ldots, p). \tag{9.30}$$

Suppose $\mathbf{m}$ is irreducibly realizable. Let $P_{\mathbf{m}}$ be the universal operator with the Riemann scheme (4.15). We say that the parameters $\lambda_{j,\nu}$ and g_i are *locally non-degenerate* if the corresponding operator is locally non-degenerate.

Note that the parameters are locally non-degenerate if

$$\lambda_{j,\nu} - \lambda_{j,\nu'} \notin \mathbb{Z} \quad (j = 0, \ldots, p,\ 1 \le \nu < \nu' \le n_j).$$

Define P_t as in Remark 4.4 iv). Then we can define monodromy generator M_t of P_t at $x = c_j$ so that M_t holomorphically depend on t (cf. Remark 9.7). Then Remark 4.13 v) proves that (9.30) implies (9.29) for every j.

The following proposition gives a sufficient condition such that an operator is locally non-degenerate.

Proposition 9.9. *Let P be a Fuchsian differential operator with the Riemann scheme* (4.15) *and let M_j be the monodromy generator at $x = c_j$. Fix an integer j with $0 \le j \le p$. Then the condition*

$$\begin{aligned} &\lambda_{j,\nu} - \lambda_{j,\nu'} \notin \mathbb{Z} \ \ \textit{or}\ \ (\lambda_{j,\nu} - \lambda_{j,\nu'})(\lambda_{j,\nu} + m_{j,\nu} - \lambda_{j,\nu'} - m_{j,\nu'}) \le 0 \\ &\qquad \textit{for}\ \ 1 \le \nu \le n_j \ \ \textit{and}\ \ 1 \le \nu' \le n_j \end{aligned} \tag{9.31}$$

implies $\dim Z(M_j) = m_{j,1}^2 + \cdots + m_{j,n_j}^2$. *In particular, P is locally non-degenerate if* (9.31) *is valid for $j = 0, \ldots, p$.*

Here we remark that the following condition implies (9.31).

$$\lambda_{j,\nu} - \lambda_{j,\nu'} \notin \mathbb{Z} \setminus \{0\} \quad \textit{for}\ \ 1 \le \nu \le n_j \ \ \textit{and}\ \ 1 \le \nu' \le n_j. \tag{9.32}$$

Proof. For $\mu \in \mathbb{C}$ we put

$$N_\mu = \bigl\{\nu\,;\, 1 \le \nu \le n_j,\ \mu \in \{\lambda_{j,\nu}, \lambda_{j,\nu} + 1, \ldots, \lambda_{j,\nu} + m_{j,\nu} - 1\}\bigr\}.$$

If $N_\mu > 0$, we have a local solution $u_{\mu,\nu}(x)$ of the equation $Pu = 0$ such that

$$u_{\mu,\nu}(x) = (x - c_j)^\mu \log^\nu (x - c_j) + \mathcal{O}_{c_j}(\mu + 1, L_\nu) \ \text{ for } \ \nu = 0, \ldots, N_\mu - 1. \tag{9.33}$$

Here L_ν are positive integers and if $j = 0$, then x and $x - c_j$ should be replaced by $y = \frac{1}{x}$ and y, respectively.

Suppose (9.31). Put $\rho = e^{2\pi\mu i}$, $\mathbf{m}'_\rho = \{m_{j,\nu}\,;\, \lambda_{j,\nu} - \mu \in \mathbb{Z}\}$ and $\mathbf{m}'_\rho = \{m'_{\rho,1}, \ldots, m'_{\rho,n_\rho}\}$ with $m'_{\rho,1} \ge m'_{\rho,2} \ge \cdots \ge m'_{\rho,n_\rho} \ge 1$. Then (9.31) implies

$$n - \operatorname{rank}(M_j - \rho)^k \le \begin{cases} m'_{\rho,1} + \cdots + m'_{\rho,k} & (1 \le k \le n_\rho), \\ m'_{\rho,1} + \cdots + m'_{\rho,n_\rho} & (n_\rho < k). \end{cases} \tag{9.34}$$

The above argument proving (9.29) under the condition (9.30) shows that the left hand side of (9.34) is not smaller than the right hand side of (9.34). Hence we have the equality in (9.34). Thus we have (9.30) and we can assume that $L_\nu = \nu$ in (9.33). □

Theorem 9.3, Theorem 9.6 and Proposition 3.1 show the following corollary. One can also prove it by the same way as in the proof of [**DR2**, Theorem 4.7].

Corollary 9.10. *Let P be a Fuchsian differential operator with the Riemann scheme* (4.15). *Let* $\operatorname{Mon}(P)$ *denote the monodromy of the equation $Pu = 0$. Put* $\operatorname{Mon}(P) = (M_0, \ldots, M_p)$. *Suppose*

$$M_j \sim L(m_{j,1}, \ldots, m_{j,n_j}; e^{2\pi\sqrt{-1}\lambda_{j,1}}, \ldots, e^{2\pi\sqrt{-1}\lambda_{j,n_j}}) \ \ \textit{for}\ \ j = 0, \ldots, p. \tag{9.35}$$

In this case, P is said to be locally non-degenerate. Under the notation in Definition 5.7, we fix $\ell \in \mathbb{Z}_{\geq 1}^{p+1}$ and suppose (5.24)*. Assume moreover*

$$\mu_\ell \notin \mathbb{Z}, \tag{9.36}$$

$$m_{j,\nu} \leq m_{j,\ell_j} \ \ or \ \ \lambda_{j,\ell_j} - \lambda_{j,\nu} \notin \mathbb{Z} \quad (j = 0, \dots, p,\ \nu = 1, \dots, n_j). \tag{9.37}$$

Then we have

$$\mathrm{Mon}(\partial_\ell P) \sim \partial_\ell \,\mathrm{Mon}(P). \tag{9.38}$$

In particular, $\mathrm{Mon}(P)$ *is irreducible if and only if* $\mathrm{Mon}(\partial_\ell P)$ *is irreducible.*

9.2. Scott's lemma and Katz's rigidity

The results in this section are known but we will review them with their proof for the completeness of this paper.

Lemma 9.11 (Scott [**Sc**]). *Let $\mathbf{M} \in GL(n,\mathbb{C})_1^{p+1}$ and $\mathbf{A} \in M(n,\mathbb{C})_0^{p+1}$ under the notation* (9.2) *and* (9.14)*. Then*

$$\sum_{j=0}^{p} \mathrm{codim}\ker(M_j - 1) \geq \mathrm{codim} \bigcap_{j=0}^{p} \ker(M_j - 1) + \mathrm{codim} \bigcap_{j=0}^{p} \ker({}^tM_j - 1), \tag{9.39}$$

$$\sum_{j=0}^{p} \mathrm{codim}\ker A_j \geq \mathrm{codim} \bigcap_{j=0}^{p} \ker A_j + \mathrm{codim} \bigcap_{j=0}^{p} \ker {}^tA_j. \tag{9.40}$$

In particular, if $\mathbf{M}$ and $\mathbf{A}$ are irreducible, then

$$\sum_{j=0}^{p} \dim\ker(M_j - 1) \leq (p-1)n, \tag{9.41}$$

$$\sum_{j=0}^{p} \dim\ker A_j \leq (p-1)n. \tag{9.42}$$

Proof. Consider the following linear maps:

$$\begin{aligned}
&V = \mathrm{Im}(M_0 - 1) \times \cdots \times \mathrm{Im}(M_p - 1) \subset \mathbb{C}^{n(p+1)},\\
&\beta:\ \mathbb{C}^n \to V, \quad v \mapsto ((M_0 - 1)v, \dots, (M_p - 1)v),\\
&\delta:\ V \to \mathbb{C}^n, \quad (v_0, \dots, v_p) \mapsto M_p \cdots M_1 v_0 + M_p \cdots M_2 v_1 + \cdots + M_p v_{p-1} + v_p.
\end{aligned}$$

Since $M_p \cdots M_1(M_0 - 1) + \cdots + M_p(M_{p-1} - 1) + (M_p - 1) = M_p \cdots M_1 M_0 - 1 = 0$, we have $\delta \circ \beta = 0$. Moreover we have

$$\begin{aligned}
\sum_{j=0}^{p} M_p \cdots M_{j+1}(M_j - 1)v_j &= \sum_{j=0}^{p}\Big(1 + \sum_{\nu=j+1}^{p} (M_\nu - 1)M_{\nu-1}\cdots M_{j+1}\Big)(M_j - 1)v_j\\
&= \sum_{j=0}^{p}(M_j - 1)v_j + \sum_{\nu=1}^{p}\sum_{i=0}^{\nu-1}(M_\nu - 1)M_{\nu-1}\cdots M_{i+1}(M_i - 1)v_i\\
&= \sum_{j=0}^{p}(M_j - 1)\Big(v_j + \sum_{i=0}^{j-1} M_{j-1}\cdots M_{i+1}(M_i - 1)v_i\Big)
\end{aligned}$$

and therefore $\mathrm{Im}\,\delta = \sum_{j=0}^{p} \mathrm{Im}(M_j - 1)$. Hence

$$\dim \mathrm{Im}\,\delta = \mathrm{rank}(M_0 - 1, \dots, M_p - 1) = \mathrm{rank}\begin{pmatrix} {}^tM_0 - 1 \\ \vdots \\ {}^tM_p - 1 \end{pmatrix}$$

and

$$\begin{aligned}\sum_{j=0}^{p}\operatorname{codim}\ker(M_j-1)&=\dim V=\dim\ker\delta+\dim\operatorname{Im}\delta\\&\ge\dim\operatorname{Im}\beta+\dim\operatorname{Im}\delta\\&=\operatorname{codim}\bigcap_{j=0}^{p}\ker(M_j-1)+\operatorname{codim}\bigcap_{j=0}^{p}\ker({}^tM_j-1).\end{aligned}$$

Putting

$$\begin{aligned}&V=\operatorname{Im}A_0\times\cdots\times\operatorname{Im}A_p\subset\mathbb{C}^{n(p+1)},\\&\beta:\ \mathbb{C}^n\to V,\quad v\mapsto(A_0v,\dots,A_pv),\\&\delta:\ V\to\mathbb{C}^n,\quad(v_0,\dots,v_p)\mapsto v_0+v_1+\cdots+v_p,\end{aligned}$$

we have the claims for $\mathbf{A}\in M(n,\mathbb{C})^{p+1}$ in the same way as in the proof for $\mathbf{M}\in GL(n,\mathbb{C})_1^{p+1}$. □

COROLLARY 9.12 (Katz [**Kz**] and [**SV**]). *Let* $\mathbf{M}\in GL(n,\mathbb{C})_1^{p+1}$ *and put*

$$V_1:=\{\mathbf{H}\in GL(n,\mathbb{C})_1^{p+1}\,;\,\mathbf{H}\sim\mathbf{M}\},\tag{9.43}$$

$$V_2:=\{\mathbf{H}\in GL(n,\mathbb{C})_1^{p+1}\,;\,H_j\sim M_j\quad(j=0,\dots,p)\}.\tag{9.44}$$

Suppose $\mathbf{M}$ *is a generic point of the algebraic variety* V_2. *Then*

$$\dim V_1=\operatorname{codim}Z(\mathbf{M}),\tag{9.45}$$

$$\dim V_2=\sum_{j=0}^{p}\operatorname{codim}Z(M_j)-\operatorname{codim}Z(\mathbf{M}).\tag{9.46}$$

Here $Z(\mathbf{M}):=\bigcap_{j=0}^{p}Z(M_j)$ *and* $Z(M_i)=\{X\in M(n,\mathbb{C})\,;\,XM_j=M_jX\}$.

Suppose moreover that $\mathbf{M}$ *is irreducible. Then* $\operatorname{codim}Z(\mathbf{M})=n^2-1$ *and*

$$\sum_{j=0}^{p}\operatorname{codim}Z(M_j)\ge 2n^2-2.\tag{9.47}$$

Moreover $\mathbf{M}$ *is rigid, namely,* $V_1=V_2$ *if and only if* $\displaystyle\sum_{j=0}^{p}\operatorname{codim}Z(M_j)=2n^2-2$.

PROOF. The group $GL(n,\mathbb{C})$ transitively acts on V_1 as simultaneous conjugations and the Lie algebra of the isotropy group with respect to $\mathbf{M}$ is identified with $Z(\mathbf{M})$ and hence $\dim V_1=\operatorname{codim}Z(\mathbf{M})$.

The group $GL(n,\mathbb{C})^{p+1}$ naturally acts on $GL(n,\mathbb{C})^{p+1}$ by conjugations. Putting $L=\{(g_j)\in GL(n,\mathbb{C})^{p+1}\,;\,g_pM_pg_p^{-1}\cdots g_0M_0g_0^{-1}=M_p\cdots M_0\}$, V_2 is identified with $L/Z(M_0)\times\cdots\times Z(M_p)$, which is a subset of the homogeneous space

$$\{\mathbf{H}\in M(n,\mathbb{C})^{p+1}\,;\,H_j\sim M_j\ (j=0,\dots,p)\}\simeq GL(n,\mathbb{C})^{p+1}/Z(M_0)\times\cdots\times Z(M_p).$$

Denoting $g_j=\exp(tX_j)$ with $X_j\in M(n,\mathbb{C})$ and $t\in\mathbb{R}$ with $|t|\ll 1$ and defining $A_j\in\operatorname{End}\bigl(M(n,\mathbb{C})\bigr)$ by $A_jX=M_jXM_j^{-1}$, we can prove that the dimension of L equals the dimension of the kernel of the map

$$\gamma:M(n,\mathbb{C})^{p+1}\ni(X_0,\dots,X_p)\mapsto\sum_{j=0}^{p}A_p\cdots A_{j+1}(A_j-1)X_j$$

by looking at the tangent space of L at the identity element because

$$\exp(tX_p)M_p\exp(-tX_p)\cdots\exp(tX_0)M_0(-tX_0) - M_p\cdots M_0$$
$$= -t\Big(\sum_{j=0}^{p} A_p\cdots A_{j+1}(A_j-1)X_j\Big)M_p\cdots M_0 + o(t).$$

We have obtained in the proof of Lemma 9.11 that $\operatorname{codim}\ker\gamma = \dim\operatorname{Im}\gamma = \dim\sum_{j=0}^p \operatorname{Im}(A_j-1) = \operatorname{codim}\bigcap_{j=0}^p \ker({}^tA_j-1)$. We will see that $\bigcap_{j=0}^p \ker({}^tA_j-1)$ is identified with $Z(\mathbf{M})$ and hence $\operatorname{codim}\ker\gamma = \operatorname{codim} Z(\mathbf{M})$ and

$$\dim V_2 = \dim\ker\gamma - \sum_{j=0}^{p}\dim Z(M_j) = \sum_{j=0}^{p}\operatorname{codim} Z(M_j) - \operatorname{codim} Z(\mathbf{M}).$$

In general, fix $\mathbf{H}\in V_2$ and define $A_j\in\operatorname{End}\big(M(n,\mathbb{C})\big)$ by $X\mapsto M_jXH_j^{-1}$ for $j=0,\dots,p$. Note that $A_pA_{p-1}\cdots A_0$ is the identity map. If we identify $M(n,\mathbb{C})$ with its dual by the inner product trace XY for X, $Y\in M(n,\mathbb{C})$, ${}^tA_j$ are identified with the map $Y\mapsto H_j^{-1}YM_j$, respectively.

Fix $P_j\in GL(n,\mathbb{C})$ such that $H_j = P_jM_jP_j^{-1}$. Then

$$A_j(X)=X \Leftrightarrow M_jXH_j^{-1}=X \Leftrightarrow M_jX = XP_jM_jP_j^{-1} \Leftrightarrow M_jXP_j = XP_jM_j,$$
$${}^tA_j(X)=X \Leftrightarrow H_j^{-1}XM_j=X \Leftrightarrow XM_j = P_jM_jP_j^{-1}X \Leftrightarrow P_j^{-1}XM_j = M_jP_j^{-1}X$$

and $\operatorname{codim}\ker(A_j-1) = \operatorname{codim} Z(M_j)$.

In particular, we have $\bigcap_{j=0}^p \ker({}^tA_j-1)\simeq Z(\mathbf{M})$ if $H_j=M_j$ for $j=0,\dots,p$.

Suppose $\mathbf{M}$ is irreducible. Then $\operatorname{codim} Z(\mathbf{M}) = n^2-1$ and the inequality (9.47) follows from $V_1\subset V_2$. Moreover suppose $\sum_{j=0}^p\operatorname{codim} Z(M_i) = 2n^2-2$. Then Scott's lemma proves

$$\begin{aligned}2n^2-2 &= \sum_{j=0}^{p}\operatorname{codim}\ker(A_j-1)\\ &\ge n^2 - \dim\bigcap_{j=0}^{p}\{X\in M(n,\mathbb{C})\,;\, M_jX = XH_j\}\\ &\quad + n^2 - \dim\bigcap_{j=0}^{p}\{X\in M(n,\mathbb{C})\,;\, H_jX = XM_j\}.\end{aligned}$$

Hence there exists a non-zero matrix X such that $M_jX = XH_j$ $(j=0,\dots,p)$ or $H_jX=XM_j$ $(j=0,\dots,p)$. If $M_jX=XH_j$ (resp. $H_jX=XM_j$) for $j=0,\dots,p$, $\operatorname{Im}X$ (resp. $\ker X$) is M_j-stable for $j=0,\dots,p$ and hence $X\in GL(n,\mathbb{C})$ because $\mathbf{M}$ is irreducible. Thus we have $V_1=V_2$ and we get all the claims in the corollary. $\square$