CHAPTER 2

Confluences

In this chapter we first review on regular singularities of ordinary differential equations and then we give a procedure for constructing irregular singularities by confluences of regular singular points.

2.1. Regular singularities

In this section we review fundamental facts related to the regular singularities of the ordinary differential equations.

2.1.1. Characteristic exponents. The ordinary differential equation

$$a_n(x)\tfrac{d^n u}{dx^n} + a_{n-1}(x)\tfrac{d^{n-1}u}{dx^{n-1}} + \cdots + a_1(x)\tfrac{du}{dx} + a_0(x)u = 0 \tag{2.1}$$

of order n with meromorphic functions $a_j(x)$ defined in a neighborhood of $c \in \mathbb{C}$ has a singularity at $x = c$ if the function $\frac{a_j(x)}{a_n(x)}$ has a pole at $x = c$ for a certain j. The singular point $x = c$ of the equation is a *regular singularity* if it is a removable singularity of the functions $b_j(x) := (x-c)^{n-j}a_j(x)a_n(x)^{-1}$ for $j = 0, \ldots, n$. In this case $b_j(c)$ are complex numbers and the n roots of the *indicial equation*

$$\sum_{j=0}^{n} b_j(c)s(s-1)\cdots(s-j+1) = 0 \tag{2.2}$$

are called the *charactersitic exponents* of (2.1) at c.

Let $\{\lambda_1, \ldots, \lambda_n\}$ be the set of these characteristic exponents at c.

If $\lambda_j - \lambda_1 \notin \mathbb{Z}_{>0}$ for $1 < j \le n$, then (2.1) has a unique solution $(x-c)^{\lambda_1}\phi_1(x)$ with a holomorphic function $\phi_1(x)$ in a neighborhood of c satisfying $\phi_1(c) = 1$.

The singular point of the equation which is not regular singularity is called *irregular singularity*.

Definition 2.1. The regular singularity and the characteristic exponents for the differential operator

$$P = a_n(x)\tfrac{d^n}{dx^n} + a_{n-1}(x)\tfrac{d^{n-1}}{dx^{n-1}} + \cdots + a_1(x)\tfrac{d}{dx} + a_0(x) \tag{2.3}$$

are defined by those of the equation (2.1), respectively. Suppose P has a regular singularity at c. We say P is *normalized at c* if $a_n(x)$ is holomorphic at c and

$$a_n(c) = a_n^{(1)}(c) = \cdots = a_n^{(n-1)}(c) = 0 \quad\text{and}\quad a_n^{(n)}(c) \neq 0. \tag{2.4}$$

In this case $a_j(x)$ are analytic and have zeros of order at least j at $x = c$ for $j = 0, \ldots, n-1$.

2.1.2. Local solutions. The ring of convergent power series at $x = c$ is denoted by $\mathcal{O}_c$ and for a complex number μ and a non-negative integer m we put

$$\mathcal{O}_c(\mu, m) := \bigoplus_{\nu=0}^{m} (x-c)^{\mu}\log^{\nu}(x-c)\mathcal{O}_c. \tag{2.5}$$

Let P be a differential operator of order n which has a regular singularity at $x=c$ and let $\{\lambda_1,\cdots,\lambda_n\}$ be the corresponding characteristic exponents. Suppose P is normalized at c. If a complex number μ satisfies $\lambda_j-\mu\notin\{0,1,2,\dots\}$ for $j=1,\dots,n$, then P defines a linear bijective map

$$P:\mathcal{O}_c(\mu,m)\ \xrightarrow{\sim}\ \mathcal{O}_c(\mu,m) \tag{2.6}$$

for any non-negative integer m.

Let $\hat{\mathcal{O}}_c$ be the ring of formal power series $\sum_{j=0}^{\infty}a_j(x-c)^j$ $(a_j\in\mathbb{C})$ of x at c. For a domain U of $\mathbb{C}$ we denote by $\mathcal{O}(U)$ the ring of holomorphic functions on U. Put

$$B_r(c):=\{x\in\mathbb{C}\,;\,|x-c|<r\} \tag{2.7}$$

for $r>0$ and

$$\hat{\mathcal{O}}_c(\mu,m):=\bigoplus_{\nu=0}^{m}(x-c)^{\mu}\log^{\nu}(x-c)\hat{\mathcal{O}}_c, \tag{2.8}$$

$$\mathcal{O}_{B_r(c)}(\mu,m):=\bigoplus_{\nu=0}^{m}(x-c)^{\mu}\log^{\nu}(x-c)\mathcal{O}_{B_r(c)}. \tag{2.9}$$

Then $\mathcal{O}_{B_r(c)}(\mu,m)\subset\mathcal{O}_c(\mu,m)\subset\hat{\mathcal{O}}_c(\mu,m)$.

Suppose $a_j(x)\in\mathcal{O}\big(B_r(c)\big)$ and $a_n(x)\neq 0$ for $x\in B_r(c)\setminus\{c\}$ and moreover $\lambda_j-\mu\notin\{0,1,2,\dots\}$, we have

$$P:\mathcal{O}_{B_r(c)}(\mu,m)\ \xrightarrow{\sim}\ \mathcal{O}_{B_r(c)}(\mu,m), \tag{2.10}$$

$$P:\hat{\mathcal{O}}_c(\mu,m)\ \xrightarrow{\sim}\ \hat{\mathcal{O}}_c(\mu,m). \tag{2.11}$$

The proof of these results are reduced to the case when $\mu=m=c=0$ by the translation $x\mapsto x-c$, the operation $\mathrm{Ad}\big(x^{-\mu}\big)$, and the fact $P(\sum_{j=0}^{m}f_j(x)\log^j x)=(Pf_m(x))\log^j x+\sum_{j=0}^{m-1}\phi_j(x)\log^j x$ with suitable $\phi_j(x)$ and moreover we may assume

$$P=\prod_{j=0}^{n}(\vartheta-\lambda_j)-xR(x,\vartheta),$$

$$xR(x,\vartheta)=x\sum_{j=0}^{n-1}r_j(x)\vartheta^j\quad\big(r_j(x)\in\mathcal{O}\big(B_r(c)\big)\big).$$

When $\mu=m=0$, (2.11) is easy and (2.10) and hence (2.6) are also easily proved by the method of majorant series (for example, cf. [**O1**]).

For the differential operator

$$Q=\tfrac{d^n}{dx^n}+b_{n-1}(x)\tfrac{d^{n-1}}{dx^{n-1}}+\cdots+b_1(x)\tfrac{d}{dx}+b_0(x)$$

with $b_j(x)\in\mathcal{O}\big(B_r(c)\big)$, we have a bijection

$$\begin{array}{cccc} Q: & \mathcal{O}\big(B_r(c)\big) & \xrightarrow{\sim} & \mathcal{O}\big(B_r(c)\big)\oplus\mathbb{C}^n \\ & \Psi & & \Psi \\ & u(x) & \mapsto & Pu(x)\oplus\big(u^{(j)}(c)\big)_{0\le j\le n-1} \end{array} \tag{2.12}$$

because $Q(x-c)^n$ has a regular singularity at $x=c$ and the characteristic exponents are $-1,-2,\dots,-n$ and hence (2.10) assures that for any $g(x)\in\mathbb{C}[x]$ and $f(x)\in\mathcal{O}\big(B_r(c)\big)$ there uniquely exists $v(x)\in\mathcal{O}\big(B_r(c)\big)$ such that $Q(x-c)^n v(x)=f(x)-Qg(x)$.

If $\lambda_\nu-\lambda_1\notin\mathbb{Z}_{>0}$, the characteristic exponents of $R:=\mathrm{Ad}\big((x-c)^{-\lambda_1-1}\big)P$ at $x=c$ are $\lambda_\nu-\lambda_1-1$ for $\nu=1,\dots,n$ and therefore $R=S(x-c)$ with a differential operator R whose coefficients are in $\mathcal{O}\big(B_r(c)\big)$. Then there exists $v_1(x)\in\mathcal{O}\big(B_r(c)\big)$

such that $-S1 = S(x-c)v_1(x)$, which means $P\big((x-c)^{\lambda_1}(1+(x-c)v_1(x))\big) = 0$. Hence if $\lambda_i - \lambda_j \notin \mathbb{Z}$ for $1 \le i < j \le n$, we have solutions $u_\nu(x)$ of $Pu = 0$ such that

$$u_\nu(x) = (x-c)^{\lambda_\nu}\phi_\nu(x) \tag{2.13}$$

with suitable $\phi_\nu \in \mathcal{O}\big(B_r(c)\big)$ satisfying $\phi_\nu(c) = 1$ for $\nu = 1, \dots, n$.

Put $k = \#\{\nu\,;\,\lambda_\nu = \lambda_1\}$ and $m = \#\{\nu\,;\,\lambda_\nu - \lambda_1 \in \mathbb{Z}_{\ge 0}\}$. Then we have solutions $u_\nu(x)$ of $Pu = 0$ for $\nu = 1, \dots, k$ such that

$$u_\nu(x) - (x-c)^{\lambda_1}\log^{\nu-1}(x-c) \in \mathcal{O}_{B_r(c)}(\lambda_1+1, m-1). \tag{2.14}$$

If $\mathcal{O}_{B_r(c)}$ is replaced by $\hat{\mathcal{O}}_c$, the solution

$$u_\nu(x) = (x-c)^{\lambda_1}\log^{\nu-1}(x-c) + \sum_{i=1}^{\infty}\sum_{j=0}^{m-1} c_{\nu,i,j}(x-c)^{\lambda_1+i}\log^j(x-c) \in \hat{\mathcal{O}}_c(\lambda_1, m-1)$$

is constructed by inductively defining $c_{\nu,i,j} \in \mathbb{C}$. Since

$$\begin{aligned} P\Big(\sum_{i=N+1}^{\infty}\sum_{j=0}^{m-1} c_{\nu,i,j}(x-c)^{\lambda_1+i}\log^j(x-c)\Big) &= -P\Big((x-c)^{\lambda_1}\log^{\nu-1}(x-c) \\ &+ \sum_{i=1}^{N} c_{\nu,i,j}(x-c)^{\lambda_1+i}\log^j(x-c)\Big) \in \mathcal{O}_{B_r(c)}(\lambda_1+N, m-1) \end{aligned}$$

for an integer N satisfying $\mathrm{Re}(\lambda_\ell - \lambda_1) < N$ for $\ell = 1, \dots, n$, we have

$$\sum_{i=N+1}^{\infty}\sum_{j=0}^{m-1} c_{\nu,i,j}(x-c)^{\lambda_1+i}\log^j(x-c) \in \mathcal{O}_{B_r(c)}(\lambda_1+N, m-1)$$

because of (2.10) and (2.11), which means $u_\nu(x) \in \mathcal{O}_{B_r(c)}(\lambda_1, m)$.

2.1.3. Fuchsian differential equations. The regular singularity at ∞ is similarly defined by that at the origin under the coordinate transformation $x \mapsto \frac{1}{x}$. When $P \in W(x)$ and the singular points of P in $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ are all regular singularities, the operator P and the equation $Pu = 0$ are called *Fuchsian*. Let $\overline{\mathbb{C}}'$ be the subset of $\overline{\mathbb{C}}$ deleting singular points $c_0, \dots, c_p$ from $\overline{\mathbb{C}}$. Then the solutions of the equation $Pu = 0$ defines a map

$$\mathcal{F} :\ \overline{\mathbb{C}}' \supset U : (\text{simply connected domain}) \mapsto \mathcal{F}(U) \subset \mathcal{O}(U) \tag{2.15}$$

by putting $\mathcal{F}(U) := \{u(x) \in \mathcal{O}(U)\,;\, Pu(x) = 0\}$. Put

$$U_{j,\epsilon,R} = \begin{cases} \{x = c_j + re^{\sqrt{-1}\theta}\,;\, 0 < r < \epsilon,\ R < \theta < R + 2\pi\} & (c_j \ne \infty) \\ \{x = re^{\sqrt{-1}\theta}\,;\, r > \epsilon^{-1},\ R < \theta < R + 2\pi\} & (c_j = \infty). \end{cases}$$

For simply connected domains U, $V \subset \overline{\mathbb{C}}'$, the map $\mathcal{F}$ satisfies

$$\mathcal{F}(U) \subset \mathcal{O}(U) \quad\text{and}\quad \dim \mathcal{F}(U) = n, \tag{2.16}$$

$$V \subset U \ \Rightarrow\ \mathcal{F}(V) = \mathcal{F}(U)|_V, \tag{2.17}$$

$$\begin{cases} \exists\epsilon > 0,\ \forall\phi \in \mathcal{F}(U_{j,\epsilon,R}),\ \exists C > 0, \exists m > 0 \text{ such that} \\ |\phi(x)| < \begin{cases} C|x-c_j|^{-m} & (c_j \ne \infty,\ x \in U_{j,\epsilon,R}), \\ C|x|^m & (c_j = \infty,\ x \in U_{j,\epsilon,R}) \\ & \quad\text{for } j = 0, \dots, p,\ \forall R \in \mathbb{R}. \end{cases} \end{cases} \tag{2.18}$$

Then we have the bijection

(2.19)

$$\begin{array}{ccc}\Big\{\partial^n+\sum_{j=0}^{n-1}a_j(x)\partial^j\in W(x)\,:\,\text{Fuchsian}\Big\} & \xrightarrow{\sim} & \big\{\mathcal{F}\text{ satisfying (2.16)–(2.18)}\big\}\\ \Cup & & \Cup\\ P & \mapsto & \big\{U\mapsto\{u\in\mathcal{O}(U)\,;\,Pu=0\}\big\}.\end{array}$$

Here if $\mathcal{F}(U)=\sum_{j=1}^n\mathbb{C}\phi_j(x)$,

$$a_j(x)=(-1)^{n-j}\frac{\det\Phi_j}{\det\Phi_n}\quad\text{with}\quad\Phi_j=\begin{pmatrix}\phi_1^{(0)}(x)&\cdots&\phi_n^{(0)}(x)\\ \vdots&\vdots&\vdots\\ \phi_1^{(j-1)}(x)&\cdots&\phi_n^{(j-1)}(x)\\ \phi_1^{(j+1)}(x)&\cdots&\phi_n^{(j+1)}(x)\\ \vdots&\vdots&\vdots\\ \phi_1^{(n)}(x)&\cdots&\phi_n^{(n)}(x)\end{pmatrix}. \tag{2.20}$$

The elements $\mathcal{F}_1$ and $\mathcal{F}_2$ of the right hand side of (2.19) are naturally identified if there exists a simply connected domain U such that $\mathcal{F}_1(U)=\mathcal{F}_2(U)$.

Let

$$P=\partial^n+a_{n-1}(x)\partial^{n-1}+\cdots+a_0(x)$$

be a Fuchsian differential operator with $p+1$ regular singular points $c_0=\infty$, $c_1,\dots,c_p$ and let $\lambda_{j,1},\dots,\lambda_{j,n}$ be the characteristic exponents of P at c_j, respectively. Since $a_{n-1}(x)$ is holomorphic at $x=\infty$ and $a_{n-1}(\infty)=0$, there exists $a_{n-1,j}\in\mathbb{C}$ such that $a_{n-1}(x)=-\sum_{j=1}^p\frac{a_{n-1,j}}{x-c_j}$. For $b\in\mathbb{C}$ we have $x^n(\partial^n-bx^{-1}\partial^{n-1})=\vartheta^n-\big(b+\frac{n(n-1)}{2}\big)\vartheta^{n-1}+b_{n-2}\vartheta^{n-2}+\cdots+b_0$ with $b_j\in\mathbb{C}$. Hence we have

$$\lambda_{j,1}+\cdots+\lambda_{j,n}=\begin{cases}-\sum_{j=1}^p a_{n-1,j}-\frac{n(n-1)}{2} & (j=0),\\ a_{n-1,j}+\frac{n(n-1)}{2} & (j=1,\dots,p),\end{cases}$$

and the *Fuchs relation*

$$\sum_{j=0}^p\sum_{\nu=1}^n\lambda_{j,\nu}=\frac{(p-1)n(n-1)}{2}. \tag{2.21}$$

Suppose $Pu=0$ is reducible. Then $P=SR$ with $S,R\in W(x)$ so that $n'=\operatorname{ord}R<n$. Since the solution $v(x)$ of $Rv=0$ satisfies $Pv(x)=0$, R is also Fuchsian. Note that the set of m characteristic exponents $\{\lambda'_{j,\nu}\,;\,\nu=1,\dots,n'\}$ of $Rv=0$ at c_j is a subset of $\{\lambda_{j,\nu}\,;\,\nu=1,\dots,n\}$. The operator R may have other singular points $c'_1,\dots,c'_q$ called *apparent singular points* where any local solutions at the points is analytic. Hence the set characteristic exponents at $x=c'_j$ are $\{\lambda'_{j,\nu}\ \ \nu=1,\dots,n'\}$ such that $0\le\mu_{j,1}<\mu_{j,2}<\cdots<\mu_{j,n'}$ and $\mu_{j,\nu}\in\mathbb{Z}$ for $\nu=1,\dots,n'$ and $j=1,\dots,q$. Since $\mu_{j,1}+\cdots+\mu_{j,n'}\ge\frac{n'(n'-1)}{2}$, the Fuchs relation for R implies

$$\mathbb{Z}\ni\sum_{j=0}^p\sum_{\nu=1}^{n'}\lambda'_{j,\nu}\le\frac{(p-1)n'(n'-1)}{2}. \tag{2.22}$$

Fixing a generic point q and paths γ_j around c_j as in (9.25) and moreover a base $\{u_1,\dots,u_n\}$ of local solutions of the equation $Pu=0$ at q, we can define monodromy generators $M_j\in GL(n,\mathbb{C})$. We call the tuple $\mathbf{M}=(M_0,\dots,M_p)$ the *monodromy* of the equation $Pu=0$. The monodromy $\mathbf{M}$ is defined to be *irreducible* if there exists no subspace V of $\mathbb{C}^n$ such that $M_jV\subset V$ for $j=0,\dots,p$ and $0<\dim V<n$, which is equivalent to the condition that P is irreducible.

Suppose $Qv = 0$ is another Fuchsian differential equation of order n with the same singular points. The monodromy $\mathbf{N} = (N_0, \ldots, N_p)$ is similarly defined by fixing a base $\{v_1, \ldots, v_n\}$ of local solutions of $Qv = 0$ at q. Then

$$\begin{aligned}\mathbf{M} \sim \mathbf{N} &\overset{\text{def}}{\Leftrightarrow} \exists g \in GL(n, \mathbb{C}) \text{ such that } N_j = gM_jg^{-1} \ (j = 0, \ldots, p)\\ &\Leftrightarrow Qv = 0 \text{ is } W(x)\text{-isomorphic to } Pu = 0.\end{aligned} \tag{2.23}$$

If $Qv = 0$ is $W(x)$-isomorphic to $Pu = 0$, the isomorphism defines an isomorphism between their solutions and then $N_j = M_j$ under the bases corresponding to the isomorphism.

Suppose there exists $g \in GL(n, \mathbb{C})$ such that $N_j = gM_jg^{-1}$ for $j = 0, \ldots, p$. The equations $Pu = 0$ and $Qu = 0$ are $W(x)$-isomorphic to certain first order systems $U' = A(x)U$ and $V' = B(x)V$ of rank n, respectively. We can choose bases $\{U_1, \ldots, U_n\}$ and $\{V_1, \ldots, V_n\}$ of local solutions of $PU = 0$ and $QV = 0$ at q, respectively, such that their monodromy generators corresponding γ_j are same for each j. Put $\tilde{U} = (U_1, \ldots, U_n)$ and $\tilde{V} = (V_1, \ldots, V_n)$. Then the element of the matrix $\tilde{V}\tilde{U}^{-1}$ is holomorphic at q and can be extended to a rational function of x and then $\tilde{V}\tilde{U}^{-1}$ defines a $W(x)$-isomorphism between the equations $U' = A(x)U$ and $V' = B(x)V$.

Example 2.2 (apparent singularity). The differential equation

$$x(x-1)(x-c)\tfrac{d^2u}{dx^2} + (x^2 - 2cx + c)\tfrac{du}{dx} = 0 \tag{2.24}$$

is a special case of Heun's equation (6.19) with $\alpha = \beta = \lambda = 0$ and $\gamma = \delta = 1$. It has regular singularities at 0, 1, c and ∞ and its Riemann scheme equals

$$\begin{Bmatrix} x = \infty & 0 & 1 & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{Bmatrix}. \tag{2.25}$$

The local solution at $x = c$ corresponding to the characteristic exponent 0 is holomorphic at the point and therefore $x = c$ is an apparent singularity, which corresponds to the zero of the Wronskian $\det \Phi_n$ in (2.20). Note that the equation (2.24) has the solutions 1 and $c \log x + (1 - c) \log(x - 1)$.

The equation (2.24) is not $W(x)$-isomorphic to Gauss hypergeometric equation if $c \neq 0$ and $c \neq 1$, which follows from the fact that c is a modulus of the isomorphic classes of the monodromy. It is easy to show that any tuple of matrices $\mathbf{M} = (M_0, M_1, M_2) \in GL(2, \mathbb{C})$ satisfying $M_2M_1M_0 = I_2$ is realized as the monodromy of the equation obtained by applying a suitable addition $\mathrm{RAd}\bigl(x^{\lambda_0}(1 - x)^{\lambda_1}\bigr)$ to a certain Gauss hypergeometric equation or the above equation.

2.2. A confluence

The non-trivial equation $(x - a)\frac{du}{dx} = \mu u$ obtained by the addition $\mathrm{RAd}\bigl((x - a)^{\mu}\bigr)\partial$ has a solution $(x - a)^{\mu}$ and regular singularities at $x = c$ and ∞. To consider the confluence of the point $x = a$ to ∞ we put $a = \frac{1}{c}$. Then the equation is

$$\bigl((1 - cx)\partial + c\mu\bigr)u = 0$$

and it has a solution $u(x) = (1 - cx)^{\mu}$.

The substitution $c = 0$ for the operator $(1 - cx)\partial + c\mu \in W[x; c, \mu]$ gives the trivial equation $\frac{du}{dx} = 0$ with the trivial solution $u(x) \equiv 1$. To obtain a nontrivial equation we introduce the parameter $\lambda = c\mu$ and we have the equation

$$\bigl((1 - cx)\partial + \lambda\bigr)u = 0$$

with the solution $(1-cx)^{\frac{\lambda}{c}}$. The function $(1-cx)^{\frac{\lambda}{c}}$ has the holomorphic parameters c and λ and the substitution $c=0$ gives the equation $(\partial+\lambda)u=0$ with the solution $e^{-\lambda x}$. Here $(1-cx)\partial+\lambda = \mathrm{RAdei}\big(\frac{\lambda}{1-cx}\big)\partial = \mathrm{RAd}\big((1-cx)^{\frac{\lambda}{c}}\big)\partial$.

This is the simplest example of the confluence and we define a confluence of simultaneous additions in this section.

2.3. Versal additions

For a function $h(c,x)$ with a holomorphic parameter $c\in\mathbb{C}$ we put

$$\begin{aligned} h_n(c_1,\dots,c_n,x) &:= \frac{1}{2\pi\sqrt{-1}}\int_{|z|=R}\frac{h(z,x)dz}{\prod_{j=1}^n(z-c_j)} \\ &= \sum_{k=1}^n \frac{h(c_k,x)}{\prod_{1\le i\le n,\, i\ne k}(c_k-c_i)} \end{aligned} \tag{2.26}$$

with a sufficiently large $R>0$. Put

$$h(c,x) := c^{-1}\log(1-cx) = -x-\frac{c}{2}x^2-\frac{c^2}{3}x^3-\frac{c^3}{4}x^4-\cdots. \tag{2.27}$$

Then

$$(1-cx)h'(c,x) = -1 \tag{2.28}$$

and

$$\begin{aligned} h_n'(c_1,\dots,c_n,x)\prod_{1\le i\le n}(1-c_ix) &= -\sum_{k=1}^n\frac{\prod_{1\le i\le n,\, i\ne k}(1-c_ix)}{\prod_{1\le i\le n,\, i\ne k}(c_k-c_i)} \\ &= -x^{n-1}. \end{aligned} \tag{2.29}$$

The last equality in the above is obtained as follows. Since the left hand side of (2.29) is a holomorphic function of $(c_1,\dots,c_n)\in\mathbb{C}^n$ and the coefficient of x^m is homogeneous of degree $m-n+1$, it is zero if $m<n-1$. The coefficient of x^{n-1} proved to be -1 by putting $c_1=0$. Thus we have

$$h_n(c_1,\dots,c_n,x) = -\int_0^x\frac{t^{n-1}dt}{\prod_{1\le i\le n}(1-c_it)}, \tag{2.30}$$

$$\begin{aligned} &e^{\lambda_n h_n(c_1,\dots,c_n,x)}\circ\Big(\prod_{1\le i\le n}(1-c_ix)\Big)\partial\circ e^{-\lambda_n h_n(c_1,\dots,c_n,x)} \\ &\qquad = \Big(\prod_{1\le i\le n}(1-c_ix)\Big)\partial+\lambda_nx^{n-1}, \end{aligned} \tag{2.31}$$

$$e^{\lambda_n h_n(c_1,\dots,c_n,x)} = \prod_{k=1}^n\Big(1-c_kx\Big)^{\frac{\lambda_n}{c_k\prod_{\substack{1\le i\le n\\ i\ne k}}(c_k-c_i)}}. \tag{2.32}$$

Definition 2.3 (versal addition). We put

$$\begin{aligned} \mathrm{AdV}_{(\frac{1}{c_1},\dots,\frac{1}{c_p})}(\lambda_1,\dots,\lambda_p) &:= \mathrm{Ad}\left(\prod_{k=1}^p\Big(1-c_kx\Big)^{\sum_{n=k}^p\frac{\lambda_n}{c_k\prod_{\substack{1\le i\le n\\ i\ne k}}(c_k-c_i)}}\right) \\ &= \mathrm{Adei}\left(-\sum_{n=1}^p\frac{\lambda_nx^{n-1}}{\prod_{i=1}^n(1-c_ix)}\right), \end{aligned} \tag{2.33}$$

$$\mathrm{RAdV}_{(\frac{1}{c_1},\dots,\frac{1}{c_p})}(\lambda_1,\dots,\lambda_p) = \mathrm{R}\circ\mathrm{AdV}_{(\frac{1}{c_1},\dots,\frac{1}{c_p})}(\lambda_1,\dots,\lambda_p). \tag{2.34}$$

We call $\mathrm{RAdV}_{(\frac{1}{c_1},\dots,\frac{1}{c_p})}(\lambda_1,\dots,\lambda_p)$ a *versal addition* at the p points $\frac{1}{c_1},\dots,\frac{1}{c_p}$.

Putting

$$h(c,x) := \log(x-c),$$

we have

$$h'_n(c_1,\dots,c_n,x)\prod_{1\le i\le n}(x-c_i) = \sum_{k=1}^{n}\frac{\prod_{1\le i\le n,\ i\ne k}(x-c_i)}{\prod_{1\le i\le n,\ i\ne k}(c_k-c_i)} = 1$$

and the *confluence of additions around the origin* is defined by

$$\begin{aligned}(2.35)\qquad \mathrm{AdV}^0_{(a_1,\dots,a_p)}(\lambda_1,\dots,\lambda_p) &:= \mathrm{Ad}\left(\prod_{k=1}^{p}(x-a_k)^{\sum_{n=k}^{p}\frac{\lambda_n}{\prod_{\substack{1\le i\le n\\ i\ne k}}(a_k-a_i)}}\right)\\ &= \mathrm{Adei}\left(\sum_{n=1}^{p}\frac{\lambda_n}{\prod_{1\le i\le n}(x-a_i)}\right),\end{aligned}$$

$$(2.36)\qquad \mathrm{RAdV}^0_{(a_1,\dots,a_p)}(\lambda_1,\dots,\lambda_p) = \mathrm{R}\circ\mathrm{AdV}^0_{(a_1,\dots,a_p)}(\lambda_1,\dots,\lambda_p).$$

REMARK 2.4. Let $g_k(c,x)$ be meromorphic functions of x with the holomorphic parameter $c=(c_1,\dots,c_p)\in\mathbb{C}^p$ for $k=1,\dots,p$ such that

$$g_k(c,x)\in\sum_{i=1}^{p}\mathbb{C}\frac{1}{1-c_ix}\quad\text{if}\quad 0\ne c_i\ne c_j\ne 0\quad(1\le i<j\le p,\ 1\le k\le p).$$

Suppose $g_1(c,x),\dots,g_p(c,x)$ are linearly independent for any fixed $c\in\mathbb{C}^p$. Then there exist entire functions $a_{i,j}(c)$ of $c\in\mathbb{C}^p$ such that

$$g_k(x,c) = \sum_{n=1}^{p}\frac{a_{k,n}(c)x^{n-1}}{\prod_{i=1}^{n}(1-c_ix)}$$

and $\bigl(a_{i,j}(c)\bigr)\in GL(p,\mathbb{C})$ for any $c\in\mathbb{C}^p$ (cf. [**O3**, Lemma 6.3]). Hence the versal addition is essentially unique.

2.4. Versal operators

If we apply a middle convolution to a versal addition of the trivial operator ∂, we have a *versal Jordan-Pochhammer* operator.

$$\begin{aligned}(2.37)\qquad P &:= \mathrm{RAd}(\partial^{-\mu})\circ\mathrm{RAdV}_{(\frac{1}{c_1},\dots,\frac{1}{c_p})}(\lambda_1,\dots,\lambda_p)\partial\\ &= \mathrm{RAd}(\partial^{-\mu})\circ\mathrm{R}\Bigl(\partial+\sum_{k=1}^{p}\frac{\lambda_kx^{k-1}}{\prod_{\nu=1}^{k}(1-c_\nu x)}\Bigr)\\ &= \partial^{-\mu+p-1}\Bigl(p_0(x)\partial+q(x)\Bigr)\partial^{\mu} = \sum_{k=0}^{p}p_k(x)\partial^{p-k}\end{aligned}$$

with

$$p_0(x) = \prod_{j=1}^{p}(1-c_jx),\quad q(x) = \sum_{k=1}^{p}\lambda_kx^{k-1}\prod_{j=k+1}^{p}(1-c_jx),$$

$$p_k(x) = \binom{-\mu+p-1}{k}p_0^{(k)}(x) + \binom{-\mu+p-1}{k-1}q^{(k-1)}(x).$$

We naturally obtain the integral representation of solutions of the versal Jordan-Pochhammer equation $Pu=0$, which we show in the case $p=2$ as follows.

EXAMPLE 2.5. We have the *versal Gauss hypergeometric* operator

$$\begin{aligned}
P_{c_1,c_2;\lambda_1,\lambda_2,\mu} &:= \mathrm{RAd}(\partial^{-\mu})\circ \mathrm{RAdV}_{(\frac{1}{c_1},\frac{1}{c_2})}(\lambda_1,\lambda_2)\partial\\
&= \mathrm{RAd}(\partial^{-\mu})\circ \mathrm{RAd}\left((1-c_1x)^{\frac{\lambda_1}{c_1}+\frac{\lambda_2}{c_1(c_1-c_2)}}(1-c_2x)^{\frac{\lambda_2}{c_2(c_2-c_1)}}\right)\\
&= \mathrm{RAd}(\partial^{-\mu})\circ \mathrm{RAdei}\left(-\tfrac{\lambda_1}{1-c_1x}-\tfrac{\lambda_2x}{(1-c_1x)(1-c_2x)}\right)\partial\\
&= \mathrm{RAd}(\partial^{-\mu})\circ \mathrm{R}\left(\partial+\tfrac{\lambda_1}{1-c_1x}+\tfrac{\lambda_2x}{(1-c_1x)(1-c_2x)}\right)\\
&= \mathrm{Ad}(\partial^{-\mu})\left(\partial(1-c_1x)(1-c_2x)\partial+\partial(\lambda_1(1-c_2x)+\lambda_2x)\right)\\
&= \big((1-c_1x)\partial+c_1(\mu-1)\big)\big((1-c_2x)\partial+c_2\mu\big)\\
&\quad+\lambda_1\partial+(\lambda_2-\lambda_1c_2)(x\partial+1-\mu)\\
&= (1-c_1x)(1-c_2x)\partial^2\\
&\quad+\big((c_1+c_2)(\mu-1)+\lambda_1+(2c_1c_2(1-\mu)+\lambda_2-\lambda_1c_2)x\big)\partial\\
&\quad+(\mu-1)(c_1c_2\mu+\lambda_1c_2-\lambda_2),
\end{aligned}$$

whose solution is obtained by applying I_c^μ to

$$K_{c_1,c_2;\lambda_1,\lambda_2}(x)=(1-c_1x)^{\frac{\lambda_1}{c_1}+\frac{\lambda_2}{c_1(c_1-c_2)}}(1-c_2x)^{\frac{\lambda_2}{c_2(c_2-c_1)}}$$

The equation $Pu=0$ has the Riemann scheme

$$\begin{Bmatrix} x=\frac{1}{c_1} & \frac{1}{c_2} & \infty & \\ 0 & 0 & 1-\mu & ;\,x\\ \frac{\lambda_1}{c_1}+\frac{\lambda_2}{c_1(c_1-c_2)}+\mu & \frac{\lambda_2}{c_2(c_2-c_1)}+\mu & -\frac{\lambda_1}{c_1}+\frac{\lambda_2}{c_1c_2}-\mu & \end{Bmatrix}. \tag{2.38}$$

Thus we have the following well-known confluent equations

$$\begin{aligned}
P_{c_1,0;\lambda_1,\lambda_2,\mu} &= (1-c_1x)\partial^2+\big(c_1(\mu-1)+\lambda_1+\lambda_2x\big)\partial-\lambda_2(\mu-1), && \text{(Kummer)}\\
K_{c_1,0;\lambda_1,\lambda_2} &= (1-c_1x)^{\frac{\lambda_1}{c_1}+\frac{\lambda_2}{c_1^2}}\exp(\tfrac{\lambda_2x}{c_1}),\\
P_{0,0;0,-1,\mu} &= \partial^2-x\partial+(\mu-1), && \text{(Hermite)}\\
\mathrm{Ad}(e^{\frac14x^2})P_{0,0;0,1,\mu} &= (\partial-\tfrac12x)^2+x(\partial-\tfrac12x)-(\mu-1)\\
&= \partial^2+(\tfrac12-\mu-\tfrac{x^2}{4}), && \text{(Weber)}\\
K_{0,0;0,\mp1} &= \exp\Big(\int_0^x\pm t\,dt\Big)=\exp(\pm\tfrac{x^2}{2}).
\end{aligned}$$

The solution

$$\begin{aligned}
D_{-\mu}(x) &:= (-1)^{-\mu}e^{\frac{x^2}{4}}I_\infty^\mu(e^{-\frac{x^2}{2}})=\frac{e^{\frac{x^2}{4}}}{\Gamma(\mu)}\int_x^\infty e^{-\frac{t^2}{2}}(t-x)^{\mu-1}dt\\
&= \frac{e^{\frac{x^2}{4}}}{\Gamma(\mu)}\int_0^\infty e^{-\frac{(s+x)^2}{2}}s^{\mu-1}ds=\frac{e^{-\frac{x^2}{4}}}{\Gamma(\mu)}\int_0^\infty e^{-xs-\frac{t^2}{2}}s^{\mu-1}ds\\
&\sim x^{-\mu}e^{-\frac{x^2}{4}}\,{}_2F_0(\tfrac{\mu}{2},\tfrac{\mu}{2}+\tfrac12;-\tfrac{2}{x^2})=\sum_{k=0}^\infty x^{-\mu}e^{-\frac{x^2}{4}}\frac{(\frac{\mu}{2})_k(\frac{\mu}{2}+\frac12)_k}{k!}\big(-\tfrac{2}{x^2}\big)^k
\end{aligned}$$

of Weber's equation $\frac{d^2u}{dx^2}=(\frac{x^2}{4}+\mu-\frac12)u$ is called a parabolic cylinder function (cf. [**WW**, §16.5]). Here the above last line is an asymptotic expansion when $x\to+\infty$.

The normal form of Kummer equation is obtained by the coordinate transformation $y=x-\frac{1}{c_1}$ but we also obtain it as follows:

$$P_{c_1;\lambda_1,\lambda_2,\mu}:=\mathrm{RAd}(\partial^{-\mu})\circ\mathrm{R}\circ\mathrm{Ad}(x^{\lambda_2})\circ\mathrm{AdV}_{\frac{1}{c_1}}(\lambda_1)\partial$$

$$
\begin{aligned}
&= \mathrm{RAd}(\partial^{-\mu}) \circ \mathrm{R}\bigl(\partial - \tfrac{\lambda_2}{x} + \tfrac{\lambda_1}{1-c_1x}\bigr) \\
&= \mathrm{Ad}(\partial^{-\mu})\bigl(\partial x(1-c_1x)\partial - \partial(\lambda_2 - (\lambda_1 + c_1\lambda_2)x)\bigr) \\
&= (x\partial + 1 - \mu)\bigl((1-c_1x)\partial + c_1\mu) - \lambda_2\partial + (\lambda_1 + c_1\lambda_2)(x\partial + 1 - \mu) \\
&= x(1-c_1x)\partial^2 + \bigl(1 - \lambda_2 - \mu + (\lambda_1 + c_1(\lambda_2 + 2\mu - 2))x\bigr)\partial \\
&\quad + (\mu - 1)\bigl(\lambda_1 + c_1(\lambda_2 + \mu)\bigr),
\end{aligned}
$$

$$
\begin{aligned}
P_{0;\lambda_1,\lambda_2,\mu} &= x\partial^2 + (1 - \lambda_2 - \mu + \lambda_1 x)\partial + \lambda_1(\mu - 1), \\
P_{0;-1,\lambda_2,\mu} &= x\partial^2 + (1 - \lambda_2 - \mu - x)\partial + 1 - \mu \qquad \text{(Kummer)}, \\
K_{c_1;\lambda_1,\lambda_2}(x) &:= x^{\lambda_2}(1-c_1x)^{\frac{\lambda_1}{c_1}}, \quad K_{0;\lambda_1,\lambda_2}(x) = x^{\lambda_2}\exp(-\lambda_1 x).
\end{aligned}
$$

The Riemann scheme of the equation $P_{c_1;\lambda_1,\lambda_2,\mu}u = 0$ is

$$
\left\{\begin{matrix} x = 0 & \frac{1}{c_1} & \infty & \\ 0 & 0 & 1-\mu & ;\, x \\ \lambda_2 + \mu & \frac{\lambda_1}{c_1} + \mu & -\frac{\lambda_1}{c_1} - \lambda_2 - \mu & \end{matrix}\right\} \tag{2.39}
$$

and the local solution at the origin corresponding to the characteristic exponent $\lambda_2 + \mu$ is given by

$$
I_0^\mu(K_{c_1;\lambda_1,\lambda_2})(x) = \frac{1}{\Gamma(\mu)}\int_0^x t^{\lambda_2}(1-c_1t)^{\frac{\lambda_1}{c_1}}(x-t)^{\mu-1}dt.
$$

In particular, we have a solution

$$
\begin{aligned}
u(x) = I_0^\mu(K_{0;-1,\lambda_2})(x) &= \frac{1}{\Gamma(\mu)}\int_0^x t^{\lambda_2}e^t(x-t)^{\mu-1}dt \\
&= \frac{x^{\lambda_2+\mu}}{\Gamma(\mu)}\int_0^1 s^{\lambda_2}(1-s)^{\mu-1}e^{xs}ds \qquad (t = xs) \\
&= \frac{\Gamma(\lambda_2+1)x^{\lambda_2+\mu}}{\Gamma(\lambda_2+\mu+1)}\,{}_1F_1(\lambda_2+1, \mu+\lambda_2+1; x)
\end{aligned}
$$

of the Kummer equation $P_{0;-1,\lambda_2,\mu}u = 0$ corresponding to the exponent $\lambda_2 + \mu$ at the origin. If $c_1 \notin (-\infty, 0]$ and $x \notin [0, \infty]$ and $\lambda_2 \notin \mathbb{Z}_{\ge 0}$, the local solution at $-\infty$ corresponding to the exponent $-\lambda_2 - \frac{\lambda_1}{c_1} - \mu$ is given by

$$
\begin{aligned}
&\frac{1}{\Gamma(\mu)}\int_{-\infty}^x (-t)^{\lambda_2}(1-c_1t)^{\frac{\lambda_1}{c_1}}(x-t)^{\mu-1}dt \\
&= \frac{(-x)^{\lambda_2}}{\Gamma(\mu)}\int_0^\infty \Bigl(1 - \frac{s}{x}\Bigr)^{\lambda_2}\bigl(1 + c_1(s-x)\bigr)^{\frac{\lambda_1}{c_1}}s^{\mu-1}ds \qquad (s = x - t) \\
&\xrightarrow[c_1\to+0]{\lambda_1=-1} \\
&\frac{(-x)^{\lambda_2}}{\Gamma(\mu)}\int_0^\infty \Bigl(1 - \frac{s}{x}\Bigr)^{\lambda_2}e^{x-s}s^{\mu-1}ds \\
&= \frac{(-x)^{\lambda_2}e^x}{\Gamma(\mu)}\int_0^\infty s^{\mu-1}e^{-s}\Bigl(1 - \frac{s}{x}\Bigr)^{\lambda_2}ds \\
&\sim \sum_{n=0}^\infty \frac{\Gamma(\mu+n)\Gamma(-\lambda_2+n)}{\Gamma(\mu)\Gamma(-\lambda_2)n!x^n}(-x)^{\lambda_2}e^x = (-x)^{\lambda_2}e^x\,{}_2F_0(-\lambda_2, \mu; \tfrac{1}{x}).
\end{aligned}
$$

Here the above last line is an asymptotic expansion of a rapidly decreasing solution of the Kummer equation when $\mathbb{R} \ni -x \to +\infty$. The Riemann scheme of the equation $P_{0;-1,\lambda_2,\mu}u = 0$ can be expressed by

$$\begin{Bmatrix} x=0 & \infty & (1) \\ 0 & 1-\mu & 0 \\ \lambda_2+\mu & -\lambda_2 & -1 \end{Bmatrix}. \tag{2.40}$$

In general, the expression $\begin{Bmatrix} \infty & (r_1) & \cdots & (r_k) \\ \lambda & \alpha_1 & \cdots & \alpha_k \end{Bmatrix}$ with $0 < r_1 < \cdots < r_k$ means the existence of a solution $u(x)$ satisfying

$$u(x) \sim x^{-\lambda} \exp\Bigl(-\sum_{\nu=1}^{k} \alpha_\nu \frac{x^{r_\nu}}{r_\nu}\Bigr) \quad \text{for} \quad |x| \to \infty \tag{2.41}$$

under a suitable restriction of $\operatorname{Arg} x$. Here $k \in \mathbb{Z}_{\geq 0}$ and λ, $\alpha_\nu \in \mathbb{C}$.