CHAPTER 1

Fractional operations

In this chapter we define several operations on a Weyl algebra. The operations are elementary or well-known but their combinations will be important.

In §1.4 we review on the ordinary differential equations and the ring of ordinary differential operators. We give Lemma 1.10 which is elementary but assures the existence of a cyclic vector of a determined ordinary equation. In §1.5 we also review on certain system of differential equations of the first order.

1.1. Weyl algebra

Let $\mathbb{C}[x_1,\dots,x_n]$ denote the polynomial ring of n variables $x_1,\dots,x_n$ over $\mathbb{C}$ and let $\mathbb{C}(x_1,\dots,x_n)$ denote the quotient field of $\mathbb{C}[x_1,\dots,x_n]$. The *Weyl algebra* $W[x_1,\dots,x_n]$ of n variables $x_1,\dots,x_n$ is the algebra over $\mathbb{C}$ generated by $x_1,\dots,x_n$ and $\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}$ with the fundamental relation

$$[x_i,x_j]=[\tfrac{\partial}{\partial x_i},\tfrac{\partial}{\partial x_j}]=0,\quad [\tfrac{\partial}{\partial x_i},x_j]=\delta_{i,j}\qquad(1\le i,\ j\le n). \tag{1.1}$$

We introduce a Weyl algebra $W[x_1,\dots,x_n][\xi_1,\dots,\xi_n]$ with parameters $\xi_1,\dots,\xi_N$ by

$$W[x_1,\dots,x_n][\xi_1,\dots,\xi_N]:=\mathbb{C}[\xi_1,\dots,\xi_N]\underset{\mathbb{C}}{\otimes}W[x_1,\dots,x_n]$$

and put

$$\begin{aligned}W[x_1,\dots,x_n;\xi_1,\dots,\xi_N]&:=\mathbb{C}(\xi_1,\dots,\xi_N)\underset{\mathbb{C}}{\otimes}W[x_1,\dots,x_n],\\ W(x_1,\dots,x_n;\xi_1,\dots,\xi_N)&:=\mathbb{C}(x_1,\dots,x_n,\xi_1,\dots,\xi_N)\underset{\mathbb{C}[x_1,\dots,x_n]}{\otimes}W[x_1,\dots,x_n].\end{aligned}$$

Here we have

$$[x_i,\xi_\nu]=[\tfrac{\partial}{\partial x_i},\xi_\nu]=0\quad(1\le i\le n,\ 1\le\nu\le N), \tag{1.2}$$

$$\begin{aligned}\Big[\frac{\partial}{\partial x_i},\frac{g}{f}\Big]&=\frac{\partial}{\partial x_i}\Big(\frac{g}{f}\Big)\\ &=\frac{\frac{\partial g}{\partial x_i}\cdot f-g\cdot\frac{\partial f}{\partial x_i}}{f^2}\quad(f,\ g\in\mathbb{C}[x_1,\dots,x_n,\xi_1,\dots,\xi_N])\end{aligned} \tag{1.3}$$

and $[\frac{\partial}{\partial x_i},f]=\frac{\partial f}{\partial x_i}\in\mathbb{C}[x_1,\dots,x_n,\xi_1,\dots,\xi_N]$.

For simplicity we put $x=(x_1,\dots,x_n)$ and $\xi=(\xi_1,\dots,\xi_N)$ and the algebras $\mathbb{C}[x_1,\dots,x_n]$, $\mathbb{C}(x_1,\dots,x_n)$, $W[x_1,\dots,x_n][\xi_1,\dots,\xi_N]$, $W[x_1,\dots,x_n;\xi_1,\dots,\xi_N]$, $W(x_1,\dots,x_n;\xi_1,\dots,\xi_N)$ etc. are also denoted by $\mathbb{C}[x]$, $\mathbb{C}(x)$, $W[x][\xi]$, $W[x;\xi]$, $W(x;\xi)$ etc., respectively. Then

$$\mathbb{C}[x,\xi]\subset W[x][\xi]\subset W[x;\xi]\subset W(x;\xi). \tag{1.4}$$

The element P of $W(x;\xi)$ is uniquely written by

$$P=\sum_{\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{Z}^n_{\ge0}}p_\alpha(x,\xi)\frac{\partial^{\alpha_1+\cdots+\alpha_n}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}\qquad(p_\alpha(x,\xi)\in\mathbb{C}(x,\xi)). \tag{1.5}$$

Here $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots\}$. Similar we will denote the set of positive integers by $\mathbb{Z}_{>0}$. If $P \in W(x;\xi)$ is not zero, the maximal integer $\alpha_1 + \cdots + \alpha_n$ satisfying $p_\alpha(x,\xi) \neq 0$ is called the *order* of P and denoted by $\operatorname{ord} P$. If $P \in W[x;\xi]$, $p_\alpha(x,\xi)$ are polynomials of x with coefficients in $\mathbb{C}(\xi)$ and the maximal degree of $p_\alpha(x,\xi)$ as polynomials of x is called the *degree* of P and denoted by $\deg P$.

1.2. Laplace and gauge transformations and reduced representatives

First we will define some fundamental operations on $W[x;\xi]$.

DEFINITION 1.1. i) For a non-zero element $P \in W(x;\xi)$ we choose an element $(\mathbb{C}(x,\xi) \setminus \{0\})P \cap W[x;\xi]$ with the minimal degree and denote it by $\mathrm{R}\,P$ and call it a *reduced representative* of P. If $P = 0$, we put $\mathrm{R}\,P = 0$. Note that $\mathrm{R}\,P$ is determined up to multiples by non-zero elements of $\mathbb{C}(\xi)$.

ii) For a subset I of $\{1,\dots,n\}$ we define an automorphism L_I of $W[x;\xi]$:

$$\mathrm{L}_I(\tfrac{\partial}{\partial x_i}) = \begin{cases} x_i & (i \in I) \\ \frac{\partial}{\partial x_i} & (i \notin I) \end{cases}, \quad \mathrm{L}_I(x_i) = \begin{cases} -\frac{\partial}{\partial x_i} & (i \in I) \\ x_i & (i \notin I) \end{cases} \text{ and } \mathrm{L}_I(\xi_\nu) = \xi_\nu. \tag{1.6}$$

We put $\mathrm{L} = \mathrm{L}_{\{1,\dots,n\}}$ and call L the *Laplace transformation* of $W[x;\xi]$.

iii) Let $W_L(x;\xi)$ be the algebra isomorphic to $W(x;\xi)$ which is defined by the Laplace transformation

$$\mathrm{L} : W(x;\xi) \ \overset{\sim}{\to} \ W_L(x;\xi) \ \overset{\sim}{\to} \ W(x;\xi). \tag{1.7}$$

For an element $P \in W_L(x;\xi)$ we define

$$\mathrm{R}_L(P) := \mathrm{L}^{-1} \circ \mathrm{R} \circ \mathrm{L}(P). \tag{1.8}$$

Note that the element of $W_L(x;\xi)$ is a finite sum of products of elements of $\mathbb{C}[x]$ and rational functions of $(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n},\xi_1,\dots,\xi_N)$.

We will introduce an automorphism of $W(x;\xi)$.

DEFINITION 1.2 (gauge transformation). Fix an element $(h_1,\dots,h_n) \in \mathbb{C}(x,\xi)^n$ satisfying

$$\frac{\partial h_i}{\partial x_j} = \frac{\partial h_j}{\partial x_i} \qquad (1 \leq i,\ j \leq n). \tag{1.9}$$

We define an automorphism $\operatorname{Adei}(h_1,\dots,h_n)$ of $W(x;\xi)$ by

$$\begin{aligned} \operatorname{Adei}(h_1,\dots,h_n)(x_i) &= x_i && (i = 1,\dots,n), \\ \operatorname{Adei}(h_1,\dots,h_n)(\tfrac{\partial}{\partial x_i}) &= \tfrac{\partial}{\partial x_i} - h_i && (i = 1,\dots,n), \\ \operatorname{Adei}(h_1,\dots,h_n)(\xi_\nu) &= \xi_\nu && (\nu = 1,\dots,N). \end{aligned} \tag{1.10}$$

Choose functions f and g satisfying $\frac{\partial g}{\partial x_i} = h_i$ for $i = 1,\dots,n$ and put $f = e^g$ and

$$\operatorname{Ad}(f) = \operatorname{Ade}(g) = \operatorname{Adei}(h_1,\dots,h_n). \tag{1.11}$$

We will define a homomorphism of $W(x;\xi)$.

DEFINITION 1.3 (coordinate transformation). Let $\phi = (\phi_1,\dots,\phi_n)$ be an element of $\mathbb{C}(x_1,\dots,x_m,\xi)^n$ such that the rank of the matrix

$$\Phi := \Big(\frac{\partial \phi_j}{\partial x_i}\Big)_{\substack{1\leq i\leq m\\ 1\leq j\leq n}} \tag{1.12}$$

equals n for a generic point $(x,\xi) \in \mathbb{C}^{m+N}$. Let $\Psi = \big(\psi_{i,j}(x,\xi)\big)_{\substack{1\leq i\leq n\\ 1\leq j\leq m}}$ be an left inverse of Φ, namely, $\Psi\Phi$ is an identity matrix of size n and $m \geq n$. Then a homomorphism T_ϕ^* from $W(x_1,\dots,x_n;\xi)$ to $W(x_1,\dots,x_m;\xi)$ is defined by

$$
\begin{aligned}
&T_\phi^*(x_i)=\phi_i(x) && (1\le i\le n),\\
&T_\phi^*(\tfrac{\partial}{\partial x_i})=\sum_{j=1}^m \psi_{i,j}(x,\xi)\tfrac{\partial}{\partial x_j} && (1\le i\le n).
\end{aligned}
\tag{1.13}
$$

If $m>n$, we choose linearly independent elements $h_\nu=(h_{\nu,1},\dots,h_{\nu,m})$ of $\mathbb{C}(x,\xi)^m$ for $\nu=1,\dots,m-n$ such that $\psi_{i,1}h_{\nu,1}+\cdots+\psi_{i,m}h_{\nu,m}=0$ for $i=1,\dots,n$ and $\nu=1,\dots,m-n$ and put

$$
\mathcal{K}^*(\phi):=\sum_{\nu=1}^{m-n}\mathbb{C}(x,\xi)\sum_{j=1}^m h_{\nu,j}\tfrac{\partial}{\partial x_j}\in W(x;\xi). \tag{1.14}
$$

The meaning of these operations are clear as follows.

Remark 1.4. Let P be an element of $W(x;\xi)$ and let $u(x)$ be an analytic solution of the equation $Pu=0$ with a parameter ξ. Then under the notation in Definitions 1.1–1.2, we have $(\mathrm{R}\,P)u(x)=\big(\mathrm{Ad}(f)(P)\big)(f(x)u(x))=0$. Note that $\mathrm{R}\,P$ is defined up to the multiplications of non-zero elements of $\mathbb{C}(\xi)$.

If a Laplace transform

$$
(\mathcal{R}_k u)(x)=\int_C e^{-x_1t_1-\cdots-x_kt_k}u(t_1,\dots,t_k,x_{k+1},\dots,x_n)dt_1\cdots dt_k \tag{1.15}
$$

of $u(x)$ is suitably defined, then $\big(\mathrm{L}_{\{1,\dots,k\}}(\mathrm{R}\,P)\big)(\mathcal{R}_k u)=0$, which follows from the equalities $\frac{\partial \mathcal{R}_k u}{\partial x_i}=\mathcal{R}_k(-x_iu)$ and $0=\int_C\frac{\partial}{\partial t_i}\big(e^{-x_1t_1-\cdots-x_kt_k}u(t,x_{k+1},\dots)\big)dt=-x_i\mathcal{R}_k u+\mathcal{R}_k(\frac{\partial u}{\partial t_i})$ for $i=1,\dots,k$. Moreover we have

$$
f(x)\mathcal{R}_k\,\mathrm{R}\,Pu=f(x)\big(L_{\{1,\dots,k\}}(\mathrm{R}\,P)\big)(\mathcal{R}_k u)=\big(\mathrm{Ad}(f)L_{\{1,\dots,k\}}(\mathrm{R}\,P)\big)\big(f(x)\mathcal{R}_k u\big).
$$

Under the notation of Definition 1.3, we have $T_\phi^*(P)u(\phi_1(x),\dots,\phi_n(x))=0$ and $Qu\big(\phi_1(x),\dots,\phi_n(x)\big)=0$ for $Q\in\mathcal{K}^*(\phi)$.

Another transformation of $W[x;\xi]$ based on an integral transformation frequently used will be given in Proposition 13.2.

We introduce some notation for combinations of operators we have defined.

Definition 1.5. Retain the notation in Definitions 1.1–1.3 and recall that $f=e^g$ and $h_i=\frac{\partial g}{\partial x_i}$.

$$
\mathrm{RAd}(f)=\mathrm{RAde}(g)=\mathrm{RAdei}(h_1,\dots,h_n):=\mathrm{R}\circ\mathrm{Adei}(h_1,\dots,h_n), \tag{1.16}
$$

$$
\begin{aligned}
\mathrm{AdL}(f)&=\mathrm{AdeL}(h)=\mathrm{AdeiL}(h_1,\dots,h_n)\\
&:=\mathrm{L}^{-1}\circ\mathrm{Adei}(h_1,\dots,h_n)\circ\mathrm{L},
\end{aligned}
\tag{1.17}
$$

$$
\begin{aligned}
\mathrm{RAdL}(f)&=\mathrm{RAdeL}(h)=\mathrm{RAdeiL}(h_1,\dots,h_n)\\
&:=\mathrm{L}^{-1}\circ\mathrm{RAdei}(h_1,\dots,h_n)\circ\mathrm{L},
\end{aligned}
\tag{1.18}
$$

$$
\mathrm{Ad}(\partial_{x_i}^\mu):=\mathrm{L}^{-1}\circ\mathrm{Ad}(x_i^\mu)\circ\mathrm{L}, \tag{1.19}
$$

$$
\mathrm{RAd}(\partial_{x_i}^\mu):=\mathrm{L}^{-1}\circ\mathrm{RAd}(x_i^\mu)\circ\mathrm{L}\,. \tag{1.20}
$$

Here μ is a complex number or an element of $\mathbb{C}(\xi)$ and $\mathrm{Ad}(\partial_{x_i}^\mu)$ defines an endomorphism of $W_L(x;\xi)$.

We will sometimes denote $\frac{\partial}{\partial x_i}$ by ∂_{x_i} or ∂_i for simplicity. If $n=1$, we usually denote x_1 by x and $\frac{\partial}{\partial x_1}$ by $\frac{d}{dx}$ or ∂_x or ∂. We will give some examples.

Since the calculation $\mathrm{Ad}\bigl(x^{-\mu}\bigr)\partial = x^{-\mu}\circ\partial\circ x^{\mu} = x^{-\mu}(x^{\mu}\partial+\mu x^{\mu-1}) = \partial+\mu x^{-1}$ is allowed, the following calculation is justified by the isomorphism (1.7):

$$\begin{aligned}\mathrm{Ad}(\partial^{-\mu})x^m &= \partial^{-\mu}\circ x^m\circ\partial^{\mu}\\ &= (x^m\partial^{-\mu} + \tfrac{(-\mu)m}{1!}x^{m-1}\partial^{-\mu-1} + \tfrac{(-\mu)(-\mu-1)m(m-1)}{2!}x^{m-2}\partial^{-\mu-2}\\ &\qquad + \cdots + \tfrac{(-\mu)(-\mu-1)\cdots(-\mu-m+1)m!}{m!}\partial^{-\mu-m})\partial^{\mu}\\ &= \sum_{\nu=0}^{m}(-1)^{\nu}(\mu)_{\nu}\binom{m}{\nu}x^{m-\nu}\partial^{-\nu}.\end{aligned}$$

This calculation is in a ring of certain pseudo-differential operators according to Leibniz's rule. In general, we may put $\mathrm{Ad}(\partial^{-\mu})P = \partial^{-\mu}\circ P\circ\partial^{\mu}$ for $P\in W[x;\xi]$ under Leibniz's rule. Here m is a positive integer and we use the notation

$$(\mu)_{\nu} := \prod_{i=0}^{\nu-1}(\mu+i),\qquad \binom{m}{\nu} := \frac{\Gamma(m+1)}{\Gamma(m-\nu+1)\Gamma(\nu+1)} = \frac{m!}{(m-\nu)!\nu!}. \tag{1.21}$$

1.3. Examples of ordinary differential operators

In this paper we mainly study ordinary differential operators. We give examples of the operations we have defined, which are related to classical differential equations.

Example 1.6 ($n=1$). For a rational function $h(x,\xi)$ of x with a parameter ξ we denote by $\int h(x,\xi)dx$ the function $g(x,\xi)$ satisfying $\frac{d}{dx}g(x,\xi) = h(x,\xi)$. Put $f(x,\xi) = e^{g(x,\xi)}$ and define

$$\vartheta := x\frac{d}{dx}. \tag{1.22}$$

Then we have the following identities.

$$\mathrm{Adei}(h)\partial = \partial - h = \mathrm{Ad}(e^{\int h(x)dx})\partial = e^{\int h(x)dx}\circ\partial\circ e^{-\int h(x)dx}, \tag{1.23}$$

$$\mathrm{Ad}(f)x = x,\quad \mathrm{AdL}(f)\partial = \partial, \tag{1.24}$$

$$\mathrm{Ad}(\lambda f) = \mathrm{Ad}(f)\quad \mathrm{AdL}(\lambda f) = \mathrm{AdL}(f), \tag{1.25}$$

$$\mathrm{Ad}(f)\partial = \partial - h(x,\xi)\ \Rightarrow\ \mathrm{AdL}(f)x = x + h(\partial,\xi), \tag{1.26}$$

$$\mathrm{Ad}\bigl((x-c)^{\lambda}\bigr) = \mathrm{Ade}\bigl(\lambda\log(x-c)\bigr) = \mathrm{Adei}\bigl(\tfrac{\lambda}{x-c}\bigr), \tag{1.27}$$

$$\mathrm{Ad}\bigl((x-c)^{\lambda}\bigr)x = x,\quad \mathrm{Ad}\bigl((x-c)^{\lambda}\bigr)\partial = \partial - \tfrac{\lambda}{x-c}, \tag{1.28}$$

$$\mathrm{RAd}\bigl((x-c)^{\lambda}\bigr)\partial = \mathrm{Ad}\bigl((x-c)^{\lambda}\bigr)\bigl((x-c)\partial\bigr) = (x-c)\partial - \lambda, \tag{1.29}$$

$$\begin{aligned}\mathrm{RAdL}\bigl((x-c)^{\lambda}\bigr)x &= L^{-1}\circ\mathrm{RAd}\bigl((x-c)^{\lambda}\bigr)(-\partial)\\ &= L^{-1}\bigl((x-c)(-\partial)+\lambda\bigr)\\ &= (\partial-c)x+\lambda = x\partial - cx + 1 + \lambda,\end{aligned} \tag{1.30}$$

$$\mathrm{RAdL}\bigl((x-c)^{\lambda}\bigr)\partial = \partial,\quad \mathrm{RAdL}\bigl((x-c)^{\lambda}\bigr)\bigl((\partial-c)x\bigr) = (\partial-c)x+\lambda, \tag{1.31}$$

$$\mathrm{Ad}(\partial^{\lambda})\vartheta = \mathrm{AdL}(x^{\lambda})\vartheta = \vartheta+\lambda, \tag{1.32}$$

$$\mathrm{Ad}\bigl(e^{\frac{\lambda(x-c)^m}{m}}\bigr)x = x,\quad \mathrm{Ad}\bigl(e^{\frac{\lambda(x-c)^m}{m}}\bigr)\partial = \partial - \lambda(x-c)^{m-1}, \tag{1.33}$$

$$\mathrm{RAdL}\bigl(e^{\frac{\lambda(x-c)^m}{m}}\bigr)x = \begin{cases} x + \lambda(\partial-c)^{m-1} & (m\ge 1),\\ (\partial-c)^{1-m}x+\lambda & (m\le -1),\end{cases} \tag{1.34}$$

$$T^{*}_{(x-c)^m}(x) = (x-c)^m,\quad T^{*}_{(x-c)^m}(\partial) = \tfrac{1}{m}(x-c)^{1-m}\partial. \tag{1.35}$$

Here m is a non-zero integer and λ is a non-zero complex number.

Some operations are related to Katz's operations defined by [**Kz**]. The operation $\mathrm{RAd}\big((x-c)^\mu\big)$ corresponds to the *addition* given in [**DR**] and the operator

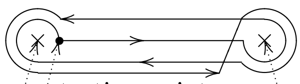
$$mc_\mu := \mathrm{RAd}(\partial^{-\mu}) = \mathrm{RAdL}(x^{-\mu}) \tag{1.36}$$

corresponds to Katz's *middle convolution* and the Euler transformation or the Riemann-Liouville integral (cf. [**Kh**, §5.1]) or the fractional derivation

$$(I_c^\mu(u))(x) = \frac{1}{\Gamma(\mu)}\int_c^x u(t)(x-t)^{\mu-1}dt. \tag{1.37}$$

Here c is suitably chosen. In most cases, c is a singular point of the multi-valued holomorphic function $u(x)$. The integration may be understood through an analytic continuation with respect to a parameter or in the sense of generalized functions. When $u(x)$ is a multi-valued holomorphic function on the punctured disk around c, we can define the complex integral

$$(\tilde I_c^\mu(u))(x) := \int^{(x+,c+,x-,c-)} u(z)(x-z)^{\mu-1}dz \tag{1.38}$$



through Pochhammer contour $(x+, c+, x-, c-)$ along a double loop circuit (cf. [**WW**, 12.43]). If $(z-c)^{-\lambda}u(z)$ is a meromorphic function in a neighborhood of the point c, we have

$$(\tilde I_c^\mu(u))(x) = \big(1 - e^{2\pi\lambda\sqrt{-1}}\big)\big(1 - e^{2\pi\mu\sqrt{-1}}\big)\int_c^x u(t)(x-t)^{\mu-1}dt. \tag{1.39}$$

For example, we have

$$\begin{aligned} I_c^\mu\big((x-c)^\lambda\big) &= \frac{1}{\Gamma(\mu)}\int_c^x (t-c)^\lambda (x-t)^{\mu-1}dt \\ &= \frac{(x-c)^{\lambda+\mu}}{\Gamma(\mu)}\int_0^1 s^\lambda(1-s)^{\mu-1}ds \quad (x-t=(1-s)(x-c)) \\ &= \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\mu+1)}(x-c)^{\lambda+\mu}, \end{aligned} \tag{1.40}$$

$$\tilde I_c^\mu\big((x-c)^\lambda\big) = \frac{4\pi^2 e^{\pi(\lambda+\mu)\sqrt{-1}}}{\Gamma(-\lambda)\Gamma(1-\mu)\Gamma(\lambda+\mu+1)}(x-c)^{\lambda+\mu+1}. \tag{1.41}$$

For $k \in \mathbb{Z}_{\ge 0}$ we have

$$\tilde I_c^\mu\big((x-c)^k\log(x-c)\big) = \frac{-4\pi^2 k! e^{\pi\lambda\sqrt{-1}}}{\Gamma(1-\mu)\Gamma(\mu+k+1)}(x-c)^{\mu+k+1}. \tag{1.42}$$

We note that since

$$\tfrac{d}{dt}\big(u(t)(x-t)^{\mu-1}\big) = u'(t)(x-t)^{\mu-1} - \tfrac{d}{dx}\big(u(t)(x-t)^{\mu-1}\big)$$

and

$$\begin{aligned} \tfrac{d}{dt}\big(u(t)(x-t)^{\mu}\big) &= u'(t)(x-t)^\mu - u(t)\tfrac{d}{dx}(x-t)^\mu \\ &= xu'(t)(x-t)^{\mu-1} - tu'(t)(x-t)^{\mu-1} - \mu u(t)(x-t)^{\mu-1}, \end{aligned}$$

we have

$$\begin{aligned} I_c^\mu(\partial u) &= \partial I_c^\mu(u), \\ I_c^\mu(\vartheta u) &= (\vartheta-\mu)I_c^\mu(u). \end{aligned} \tag{1.43}$$

REMARK 1.7. i) The integral (1.37) is naturally well-defined and the equalities (1.43) are valid if $\operatorname{Re}\lambda > 1$ and $\lim_{x\to c} x^{-1}u(x) = 0$. Depending on the definition of I_c^λ, they are also valid in many cases, which can be usually proved in this paper by analytic continuations with respect to certain parameters (for example, cf. (3.6)). Note that (1.43) is valid if I_c^μ is replaced by $\tilde I_c^\mu$ defined by (1.38).

ii) Let ϵ be a positive number and let $u(x)$ be a holomorphic function on

$$U_{\epsilon,\theta}^+ := \{x \in \mathbb{C}\,;\, |x - c| < \epsilon \text{ and } e^{-i\theta}(x - c) \notin (-\infty, 0]\}.$$

Suppose that there exists a positive number δ such that $|u(x)(x-c)^{-k}|$ is bounded on $\{x \in U_{\epsilon.\theta}^+\,;\, |\operatorname{Arg}(x - c) - \theta| < \delta\}$ for any $k > 0$. Note that the function $Pu(x)$ also satisfies this estimate for $P \in W[x]$. Then the integration (1.37) is defined along a suitable path $C : \gamma(t)$ $(0 \le t \le 1)$ such that $\gamma(0) = c$, $\gamma(1) = x$ and $|\operatorname{Arg}\bigl(\gamma(t) - c\bigr) - \theta| < \delta$ for $0 < t < \frac12$ and the equalities (1.43) are valid.

EXAMPLE 1.8. We apply additions, middle convolutions and Laplace transformations to the trivial ordinary differential equation

$$\frac{du}{dx} = 0, \tag{1.44}$$

which has the solution $u(x) \equiv 1$.

i) (Gauss hypergeometric equation). Put

$$\begin{aligned}
P_{\lambda_1,\lambda_2,\mu} &:= \operatorname{RAd}\bigl(\partial^{-\mu}\bigr) \circ \operatorname{RAd}\bigl(x^{\lambda_1}(1-x)^{\lambda_2}\bigr)\partial \\
&= \operatorname{RAd}(\partial^{-\mu}) \circ \mathrm{R}(\partial - \tfrac{\lambda_1}{x} + \tfrac{\lambda_2}{1-x}) \\
&= \operatorname{RAd}(\partial^{-\mu})\bigl(x(1-x)\partial - \lambda_1(1-x) + \lambda_2 x\bigr) \\
&= \operatorname{RAd}(\partial^{-\mu})\bigl((\vartheta - \lambda_1) - x(\vartheta - \lambda_1 - \lambda_2)\bigr) \\
&= \operatorname{Ad}(\partial^{-\mu})\bigl((\vartheta + 1 - \lambda_1)\partial - (\vartheta + 1)(\vartheta - \lambda_1 - \lambda_2)\bigr) \\
&= (\vartheta + 1 - \lambda_1 - \mu)\partial - (\vartheta + 1 - \mu)(\vartheta - \lambda_1 - \lambda_2 - \mu) \\
&= (\vartheta + \gamma)\partial - (\vartheta + \beta)(\vartheta + \alpha) \\
&= x(1-x)\partial^2 + \bigl(\gamma - (\alpha + \beta + 1)x\bigr)\partial - \alpha\beta
\end{aligned} \tag{1.45}$$

with

$$\begin{cases} \alpha = -\lambda_1 - \lambda_2 - \mu, \\ \beta = 1 - \mu, \\ \gamma = 1 - \lambda_1 - \mu. \end{cases} \tag{1.46}$$

We have a solution

$$\begin{aligned}
u(x) &= I_0^\mu(x^{\lambda_1}(1-x)^{\lambda_2}) \\
&= \frac{1}{\Gamma(\mu)}\int_0^x t^{\lambda_1}(1-t)^{\lambda_2}(x-t)^{\mu-1}dt \\
&= \frac{x^{\lambda_1+\mu}}{\Gamma(\mu)}\int_0^1 s^{\lambda_1}(1-s)^{\mu-1}(1-xs)^{\lambda_2}ds \quad (t = xs) \\
&= \frac{\Gamma(\lambda_1+1)x^{\lambda_1+\mu}}{\Gamma(\lambda_1+\mu+1)}F(-\lambda_2, \lambda_1+1, \lambda_1+\mu+1; x) \\
&= \frac{\Gamma(\lambda_1+1)x^{\lambda_1+\mu}(1-x)^{\lambda_2+\mu}}{\Gamma(\lambda_1+\mu+1)}F(\mu, \lambda_1+\lambda_2+\mu, \lambda_1+\mu+1; x) \\
&= \frac{\Gamma(\lambda_1+1)x^{\lambda_1+\mu}(1-x)^{-\lambda_2}}{\Gamma(\lambda_1+\mu+1)}F(\mu, -\lambda_2, \lambda_1+\mu+1; \frac{x}{x-1})
\end{aligned} \tag{1.47}$$

of the Gauss hypergeometric equation $P_{\lambda_1,\lambda_2,\mu}u = 0$ with the Riemann scheme

$$\begin{Bmatrix} x=0 & 1 & \infty & \\ 0 & 0 & 1-\mu & ;\ x \\ \lambda_1+\mu & \lambda_2+\mu & -\lambda_1-\lambda_2-\mu & \end{Bmatrix}, \tag{1.48}$$

which is transformed by the middle convolution mc_μ from the Riemann scheme

$$\begin{Bmatrix} x=0 & 1 & \infty & \\ \lambda_1 & \lambda_2 & -\lambda_1-\lambda_2 & ;x \end{Bmatrix}$$

of $x^{\lambda_1}(1-x)^{\lambda_2}$. Here using Riemann's P symbol, we note that

$$\begin{aligned}
&P\begin{Bmatrix} x=0 & 1 & \infty & \\ 0 & 0 & 1-\mu & ;\ x \\ \lambda_1+\mu & \lambda_2+\mu & -\lambda_1-\lambda_2-\mu & \end{Bmatrix}\\
&= x^{\lambda_1+\mu}P\begin{Bmatrix} x=0 & 1 & \infty & \\ -\lambda_1-\mu & 0 & \lambda_1+1 & ;\ x \\ 0 & \lambda_2+\mu & -\lambda_2 & \end{Bmatrix}\\
&= x^{\lambda_1+\mu}(1-x)^{\lambda_2+\mu}P\begin{Bmatrix} x=0 & 1 & \infty & \\ -\lambda_1-\mu & -\lambda_2-\mu & \lambda_1+\lambda_2+\mu+1 & ;\ x \\ 0 & 0 & \mu & \end{Bmatrix}\\
&= x^{\lambda_1+\mu}P\begin{Bmatrix} x=0 & 1 & \infty & \\ -\lambda_1-\mu & \lambda_1+1 & 0 & ;\ \dfrac{x}{x-1} \\ 0 & -\lambda_2 & \lambda_2+\mu & \end{Bmatrix}\\
&= x^{\lambda_1+\mu}(1-x)^{-\lambda_2}P\begin{Bmatrix} x=0 & 1 & \infty & \\ -\lambda_1-\mu & \lambda_1+\lambda_2+1 & -\lambda_2 & ;\ \dfrac{x}{x-1} \\ 0 & 0 & \mu & \end{Bmatrix}.
\end{aligned}$$

In general, the Riemann scheme and its relation to mc_μ will be studied in Chapter 4 and the symbol 'P' will be omitted for simplicity.

The function $u(x)$ defined by (1.47) corresponds to the characteristic exponent $\lambda_1+\mu$ at the origin and depends meromorphically on the parameters λ_1, λ_2 and μ. The local solutions corresponding to the characteristic exponents $\lambda_2+\mu$ at 1 and $-\lambda_1-\lambda_2-\mu$ at ∞ are obtained by replacing I_0^μ by I_1^μ and I_∞^μ, respectively.

When we apply $\mathrm{Ad}(x^{\lambda_1'}(x-1)^{\lambda_2'})$ to $P_{\lambda_1,\lambda_2,\mu}$, the resulting Riemann scheme is

$$\begin{Bmatrix} x=0 & 1 & \infty & \\ \lambda_1' & \lambda_2' & 1-\lambda_1'-\lambda_2'-\mu & ;\ x \\ \lambda_1+\lambda_1'+\mu & \lambda_2+\lambda_2'+\mu & -\lambda_1-\lambda_2-\lambda_1'-\lambda_2'-\mu, & \end{Bmatrix}. \tag{1.49}$$

Putting $\lambda_{1,1} = \lambda_1'$, $\lambda_{1,2} = \lambda_1+\lambda_1'+\mu$, $\lambda_{2,1} = \lambda_2'$, $\lambda_{2,2} = \lambda_2+\lambda_2'+\mu$, $\lambda_{0,1} = 1-\lambda_1'-\lambda_2'-\mu$ and $\lambda_{0,2} = -\lambda_1-\lambda_2-\lambda_1'-\lambda_2'-\mu$, we have the Fuchs relation

$$\lambda_{0,1}+\lambda_{0,2}+\lambda_{1,1}+\lambda_{1,2}+\lambda_{2,1}+\lambda_{2,2} = 1 \tag{1.50}$$

and the corresponding operator

$$\begin{aligned} P_\lambda = x^2(x-1)^2\partial^2 &+ x(x-1)\big((\lambda_{0,1}+\lambda_{0,2}+1)x+\lambda_{1,1}+\lambda_{1,2}-1\big)\partial \\ &+ \lambda_{0,1}\lambda_{0,2}x^2 + (\lambda_{2,1}\lambda_{2,2}-\lambda_{0,1}\lambda_{0,2}-\lambda_{1,1}\lambda_{1,2})x+\lambda_{1,1}\lambda_{1,2} \end{aligned} \tag{1.51}$$

has the Riemann scheme

$$\begin{Bmatrix} x=0 & 1 & \infty & \\ \lambda_{0,1} & \lambda_{1,1} & \lambda_{2,1} & ;\ x \\ \lambda_{0,2} & \lambda_{1,2} & \lambda_{2,2} & \end{Bmatrix}. \tag{1.52}$$

By the symmetry of the transposition $\lambda_{j,1}$ and $\lambda_{j,2}$ for each j, we have integral representations of other local solutions.

ii) (Airy equations). For a positive integer m we put

$$\begin{aligned} P_m &:= \mathrm{L} \circ \mathrm{Ad}(e^{\frac{x^{m+1}}{m+1}})\partial \\ &= \mathrm{L}(\partial - x^m) = x - (-\partial)^m. \end{aligned} \tag{1.53}$$

Thus the equation

$$\frac{d^m u}{dx^m} - (-1)^m x u = 0 \tag{1.54}$$

has a solution

$$u_j(x) = \int_{C_j} \exp\left(\frac{z^{m+1}}{m+1} - xz\right) dz \qquad (0 \le j \le m), \tag{1.55}$$

where the path C_j of the integration is

$$C_j : z(t) = e^{\frac{(2j-1)\pi\sqrt{-1}}{m+1} - t} + e^{\frac{(2j+1)\pi\sqrt{-1}}{m+1} + t} \quad (-\infty < t < \infty).$$

Here we note that $u_0(x) + \cdots + u_m(x) = 0$. The equation has the symmetry under the rotation $x \mapsto e^{\frac{2\pi\sqrt{-1}}{m+1}} x$.

iii) (Jordan-Pochhammer equation). For $\{c_1, \ldots, c_p\} \in \mathbb{C} \setminus \{0\}$ put

$$\begin{aligned} P_{\lambda_1,\ldots,\lambda_p,\mu} &:= \mathrm{RAd}(\partial^{-\mu}) \circ \mathrm{RAd}\Big(\prod_{j=1}^p (1 - c_j x)^{\lambda_j}\Big)\partial \\ &= \mathrm{RAd}(\partial^{-\mu}) \circ \mathrm{R}\Big(\partial + \sum_{j=1}^p \frac{c_j \lambda_j}{1 - c_j x}\Big) \\ &= \mathrm{RAd}(\partial^{-\mu})\Big(p_0(x)\partial + q(x)\Big) \\ &= \partial^{-\mu+p-1}\Big(p_0(x)\partial + q(x)\Big)\partial^{\mu} = \sum_{k=0}^p p_k(x)\partial^{p-k} \end{aligned}$$

with

$$\begin{aligned} p_0(x) &= \prod_{j=1}^p (1 - c_j x), \quad q(x) = p_0(x) \sum_{j=1}^p \frac{c_j \lambda_j}{1 - c_j x}, \\ p_k(x) &= \binom{-\mu + p - 1}{k} p_0^{(k)}(x) + \binom{-\mu + p - 1}{k - 1} q^{(k-1)}(x), \\ \binom{\alpha}{\beta} &:= \frac{\Gamma(\alpha + 1)}{\Gamma(\beta + 1)\Gamma(\alpha - \beta + 1)} \quad (\alpha, \beta \in \mathbb{C}). \end{aligned}$$

We have solutions

$$u_j(x) = \frac{1}{\Gamma(\mu)} \int_{\frac{1}{c_j}}^x \prod_{\nu=1}^p (1 - c_\nu t)^{\lambda_\nu} (x - t)^{\mu-1} dt \quad (j = 0, 1, \ldots, p,\ c_0 = 0)$$

of the Jordan-Pochhammer equation $P_{\lambda_1,\ldots,\lambda_p,\mu} u = 0$ with the Riemann scheme

$$\left\{\begin{matrix} x = \frac{1}{c_1} & \cdots & \frac{1}{c_p} & \infty & \\ [0]_{(p-1)} & \cdots & [0]_{(p-1)} & [1-\mu]_{(p-1)} & ; x \\ \lambda_1 + \mu & \cdots & \lambda_p + \mu & -\lambda_1 - \cdots - \lambda_p - \mu & \end{matrix}\right\}. \tag{1.56}$$

Here and hereafter we use the notation

$$[\lambda]_{(k)} := \begin{pmatrix} \lambda \\ \lambda + 1 \\ \vdots \\ \lambda + k - 1 \end{pmatrix} \tag{1.57}$$

for a complex number λ and a non-negative integer k. If the component $[\lambda]_{(k)}$ is appeared in a Riemann scheme, it means the corresponding local solutions with the exponents $\lambda+\nu$ for $\nu=0,\dots,k-1$ have a semisimple local monodromy when λ is generic.

1.4. Ordinary differential equations

We will study the ordinary differential equation

$$\mathcal{M}: Pu = 0 \tag{1.58}$$

with an element $P \in W(x;\xi)$ in this paper. The solution $u(x,\xi)$ of $\mathcal{M}$ is at least locally defined for x and ξ and holomorphically or meromorphically depends on x and ξ. Hence we may replace P by $\mathrm{R}\,P$ and we similarly choose P in $W[x;\xi]$.

We will identify $\mathcal{M}$ with the left $W(x;\xi)$-module $W(x;\xi)/W(x;\xi)P$. Then we may consider (1.58) as the fundamental relation of the generator u of the module $\mathcal{M}$.

The results in this section are standard and well-known but for our convenience we briefly review them.

1.4.1. Euclidian algorithm. First note that $W(x;\xi)$ is a (left) Euclidean ring. Let P, $Q \in W(x;\xi)$ with $P \neq 0$. Then there uniquely exists R, $S \in W(x;\xi)$ such that

$$Q = SP + R \qquad (\operatorname{ord} R < \operatorname{ord} P). \tag{1.59}$$

Hence we note that $\dim_{\mathbb{C}(x,\xi)}\bigl(W(x;\xi)/W(x;\xi)P\bigr) = \operatorname{ord} P$. We get R and S in (1.59) by a simple algorithm as follows. Put

$$P = a_n\partial^n + \cdots + a_1\partial + a_0 \quad\text{and}\quad Q = b_m\partial^m + \cdots + b_1\partial + b_0 \tag{1.60}$$

with $a_n \neq 0$, $b_m \neq 0$. Here a_n, $b_m \in \mathbb{C}(x,\xi)$. The division (1.59) is obtained by the induction on $\operatorname{ord} Q$. If $\operatorname{ord} P > \operatorname{ord} Q$, (1.59) is trivial with $S = 0$. If $\operatorname{ord} P \leq \operatorname{ord} Q$, (1.59) is reduced to the equality $Q' = S'P + R$ with $Q' = Q - a_n^{-1}b_m\partial^{m-n}P$ and $S' = S - a_n^{-1}b_m\partial^{m-n}$ and then we have S' and R satisfying $Q' = S'P + R$ by the induction because $\operatorname{ord} Q' < \operatorname{ord} Q$. The uniqueness of (1.59) is clear by comparing the highest order terms of (1.59) in the case when $Q = 0$.

By the standard Euclidean algorithm using the division (1.59) we have M, $N \in W(x;\xi)$ such that

$$MP + NQ = U,\ P \in W(x;\xi)U \quad\text{and}\quad Q \in W(x;\xi)U. \tag{1.61}$$

Hence in particular any left ideal of $W(x;\xi)$ is generated by a single element of $W[x;\xi]$, namely, $W(x;\xi)$ is a principal ideal domain.

Definition 1.9. The operators P and Q in $W(x;\xi)$ are defined to be *mutually prime* if one of the following equivalent conditions is valid.

(1.62) $W(x;\xi)P + W(x;\xi)Q = W(x;\xi)$,

(1.63) there exists $R \in W(x;\xi)$ satisfying $RQu = u$ for the equation $Pu = 0$,

(1.64) $\begin{cases}\text{the simultaneous equation } Pu = Qu = 0 \text{ has not a non-zero solution}\\ \text{for a generic value of } \xi.\end{cases}$

The operator S satisfying $W(x;\xi)P + W(x;\xi)Q = W(x;\xi)S$ is called the *greatest common left divisor* of P and Q and the operator T satisfying $W(x;\xi)P \cap W(x;\xi)Q = W(x;\xi)T$ is called the the *least common left multiple* of P and Q. These operators are defined uniquely up to the multiples of elements of $\mathbb{C}(x;\xi)\backslash\{0\}$.

Put $(P_1, P_2, P_3, S_1) = (Q, P, R, S)$ in (1.59). Then $\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} S_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} P_2 \\ P_3 \end{pmatrix}$ and in the same way we successively get $P_3, \ldots, P_N$ such that

$$\begin{aligned} &\begin{pmatrix} P_j \\ P_{j+1} \end{pmatrix} = \begin{pmatrix} S_j & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} P_{j+1} \\ P_{j+2} \end{pmatrix}, \\ &\operatorname{ord} P_j = \operatorname{ord} S_j + \operatorname{ord} P_{j+1}, \\ &\operatorname{ord} P_{j+2} < \operatorname{ord} P_{j+1} \quad \text{or} \quad P_{j+2} = 0 \end{aligned} \tag{1.65}$$

for $j = 1, 2, \ldots, N-1$ with $P_{N+1} = 0$. Putting

$$\begin{aligned} &\tilde{U}^{(j)} = \begin{pmatrix} U_{11}^{(j)} & U_{12}^{(j)} \\ U_{21}^{(j)} & U_{22}^{(j)} \end{pmatrix} := \tilde{S}_1 \cdots \tilde{S}_j, && \tilde{S}_j := \begin{pmatrix} S_j & 1 \\ 1 & 0 \end{pmatrix} \\ &\tilde{V}^{(j)} = \begin{pmatrix} V_{11}^{(j)} & V_{12}^{(j)} \\ V_{21}^{(j)} & V_{22}^{(j)} \end{pmatrix} := \tilde{S}_j^{-1} \cdots \tilde{S}_1^{-1}, && \tilde{S}_j^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -S_j \end{pmatrix}, \\ &U_{ii'} := U_{ii'}^{(N-1)}, && V_{ii'} := V_{ii'}^{(N-1)} \quad (1 \le i,\, i' \le 2), \end{aligned}$$

we have

$$\begin{aligned} P_1 &= U_{11} P_N, & P_N &= V_{11} P_1 + V_{12} P_2, \\ P_2 &= U_{21} P_N, & 0 &= V_{21} P_1 + V_{22} P_2. \end{aligned} \tag{1.66}$$

Note that

$$\begin{aligned} &U_{12}^{(j+2)} = U_{11}^{(j+1)} = U_{11}^{(j)} S_j + U_{12}^{(j)}, && U_{11}^{(1)} = S_1, && U_{12}^{(1)} = 1, \\ &U_{22}^{(j+2)} = U_{21}^{(j+1)} = U_{21}^{(j)} S_j + U_{22}^{(j)}, && U_{21}^{(1)} = 1, && U_{22}^{(1)} = 0, \\ &V_{11}^{(j+2)} = V_{21}^{(j+1)} = -S_j V_{21}^{(j)} + V_{11}^{(j)}, && V_{21}^{(1)} = 1, && V_{11}^{(1)} = 0, \\ &V_{12}^{(j+2)} = V_{22}^{(j+1)} = -S_j V_{22}^{(j)} + V_{12}^{(j)}, && V_{22}^{(1)} = -S_1, && V_{12}^{(1)} = 1. \end{aligned}$$

Hence by the relation $\operatorname{ord} S_j = \operatorname{ord} P_j - \operatorname{ord} P_{j+1}$, we inductively have

$$\begin{aligned} \operatorname{ord} U_{11}^{(j+1)} &= \operatorname{ord} V_{22}^{(j+1)} = \operatorname{ord} P_1 - \operatorname{ord} P_{j+1}, \\ \operatorname{ord} U_{21}^{(j+1)} &= \operatorname{ord} V_{21}^{(j+1)} = \operatorname{ord} P_2 - \operatorname{ord} P_{j+1} \end{aligned}$$

and therefore

$$\begin{aligned} &\operatorname{ord} U_{11} = \operatorname{ord} V_{22} = \operatorname{ord} P_1 - \operatorname{ord} P_N, && \operatorname{ord} U_{21} = \operatorname{ord} V_{21} = \operatorname{ord} P_2 - \operatorname{ord} P_N, \\ &\operatorname{ord} U_{12} = \operatorname{ord} V_{12} = \operatorname{ord} P_1 - \operatorname{ord} P_{N-1}, && \operatorname{ord} U_{22} = \operatorname{ord} V_{11} = \operatorname{ord} P_2 - \operatorname{ord} P_{N-1}. \end{aligned}$$

Moreover we have

$$T_1 P_1 + T_2 P_2 = 0 \quad \Leftrightarrow \quad (T_1, T_2) \in W(x;\xi)(V_{21}, V_{22}), \tag{1.67}$$

which is proved as follows. We have only to prove the implication $\Rightarrow$ in the above. Replacing (P_1, P_2) by (U_{11}, U_{21}), we may assume $\operatorname{ord} P_N = 0$. Suppose $T_1 P_1 + T_2 P_2 = 0$ and $T_1 \notin W(x;\xi) V_{21}$. Putting $T_1 = B V_{21} + A$ with $\operatorname{ord} A < \operatorname{ord} V_{21} = \operatorname{ord} P_2$, we have $(B V_{21} + A) P_1 + T_2 P_2 = 0$ and therefore $A P_1 + (P_2 - B V_{22}) P_2 = 0$. Hence for $j = 1$ we have non-zero operators A_j and B_j satisfying

$$A_j P_j + B_j P_{j+1} = 0, \ \operatorname{ord} A_j < \operatorname{ord} P_{j+1} \quad \text{and} \quad \operatorname{ord} B_j < \operatorname{ord} P_j.$$

Since $P_j = S_j P_{j+1} + P_{j+2}$, the above equality implies $(A_j S_j + B_j) P_{j+1} + A_j P_{j+2} = 0$ with $\operatorname{ord} A_j < \operatorname{ord} P_{j+1}$ and therefore the existence of the above non-zero (A_j, B_j) is inductively proved for $j = 1, 2, \ldots, N-1$. The relations $A_{N-1} P_{N-1} + B_{N-1} P_N = 0$ and $\operatorname{ord} B_{N-1} < \operatorname{ord} P_{N-1}$ contradict to the fact that $\operatorname{ord} P_N = 0$.

The operator $U := P_N$ is the greatest common left divisor of P and Q, which equals U in (1.61), and the operator $T := V_{21}P = -V_{22}Q \in W(x;\xi)$ is the least common left multiple of P and Q. Note that

$$\operatorname{ord} T + \operatorname{ord} U = \operatorname{ord} P + \operatorname{ord} Q. \tag{1.68}$$

1.4.2. cyclic vector. In general, for a positive integer m and any left $W(x;\xi)$-submodule $\mathcal{N}$ of $W(x;\xi)^m$, we can find elements $v_1, \dots, v_{m'} \in \mathcal{N}$ such that $\mathcal{N} = W(x;\xi)v_1 + \cdots + W(x;\xi)v_{m'}$ and $m' \le m$. In particular, any left $W(x;\xi)$-submodule of $W(x;\xi)^m$ is finitely generated.

This is proved by the induction on m. In fact, we can find $v_1 = (v_1^{(1)}, \dots, v_1^{(m)}) \in \mathcal{N}$ such that $\{v^{(1)} \mid (v^{(1)}, \dots, v^{(m)}) \in \mathcal{N}\} = w(x;\xi)v_1^{(1)}$ and then $\mathcal{N}$ is generated by v_1 and the elements generating $\mathcal{N}' = \{(0, v_2, \dots, v_m) \in \mathcal{N}\} \subset W(x;\xi)^{m-1}$.

Moreover we have the following.

$$\text{Any left } W(x;\xi)\text{-module } \mathcal{R} \text{ with } \dim_{\mathbb{C}(x,\xi)} \mathcal{R} < \infty \text{ is } \textit{cyclic}, \tag{1.69}$$

namely, it is generated by a suitable single element, which is called a *cyclic vector*. Hence any system of ordinary differential equations is isomorphic to a single differential equation under the algebra $W(x;\xi)$.

To prove (1.69) it is sufficient to show that the direct sum $\mathcal{M} \oplus \mathcal{N}$ of $\mathcal{M} : Pu = 0$ and $\mathcal{N} : Qv = 0$ is cyclic. In fact $\mathcal{M} \oplus \mathcal{N} = W(x;\xi)w$ with $w = u + (x-c)^n v \in \mathcal{M} \oplus \mathcal{N}$ and $n = \operatorname{ord} P$ if $c \in \mathbb{C}$ is generic. For the proof we have only to show $\dim_{\mathbb{C}(x,\xi)} W(x;\xi)w \ge m + n$ and we may assume that P and Q are in $W[x;\xi]$ and they are of the form (1.60). Fix ξ generically and we choose $c \in \mathbb{C}$ such that $a_n(c)b_m(c) \ne 0$. Since the function space $V = \{\phi(x) + (x-c)^n\varphi(x)\,;\, P\phi(x) = Q\varphi(x) = 0\}$ is of dimension $m+n$ in a neighborhood of $x = c$, $\dim_{\mathbb{C}(x;\xi)} W(x;\xi)w \ge m+n$ because the relation $Rw = 0$ for an operator $R \in W(x;\xi)$ implies $R\psi(x) = 0$ for $\psi \in V$.

Let $\mathcal{M}$ be a system of linear ordinary differential equations, namely, a finitely generated left $W(x;\xi)$-module. Then there exist finite elements $u_1, \dots, u_n$ of $\mathcal{M}$ such that $\mathcal{M} = W(x;\xi)u_1 + \cdots + W(x;\xi)u_n$. Then $\mathcal{N} := \{(P_1, \dots, P_n) \in W(x;\xi)^n \mid P_1u_1 + \cdots + P_nu_n = 0\}$ is generated by suitable elements $A_i = (A_{i,1}, \dots, A_{i,n}) \in \mathcal{N}$ $(1 \le i \le m)$ with $m \le n$. Then $\mathcal{M}$ is isomorphic to $W(x;\xi)^n/\mathcal{N}$ and $\mathcal{N} = W(x;\xi)A_1 + \cdots + W(x;\xi)A_m$.

We give a lemma, which implies (1.69) by putting $A = (A_{i,j})_{\substack{1\le i\le m\\1\le j\le n}}$ in the above.

Lemma 1.10. *Let $A \in M\bigl(m,n,W(x;\xi)\bigr)$. Here m and n are positive integers and $A \ne 0$. Then there exist $S \in GL\bigl(m, W(x;\xi)\bigr)$, $T \in GL\bigl(n, W(x;\xi)\bigr)$, $P \in W(x;\xi) \setminus \{0\}$ and $k \in \mathbb{Z}_{\ge 0}$ such that $(B_{i,j}) = B = SAT$ is the following form:*

$$B = SAT = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & P & \\ & & & & 0 \\ & & & & & \ddots \end{pmatrix}, \quad B_{i,j} = \begin{cases} 1 & (1 \le i = j \le k), \\ P & (i = j = k+1), \\ 0 & (i \ne j \ \text{ or } \ i > k+1). \end{cases} \tag{1.70}$$

Here k and $\operatorname{ord} P$ do not depend on the choice of S and T and in general, $M(m,n,R)$ denotes the linear space of matrices of size $m \times n$ whose elements are in R and when R is a ring with the unit, $GL(n,R)$ denotes the group whose elements are invertible matrices of $M(n,n,R)$.

Proof. Consider the following standard transformations of the matrix C in $M\bigl(m,n,W(x;\xi)\bigr)$ as in the linear algebra:

(1) Multiply a row of C from the left by a non-zero element of $\mathbb{C}(x;\xi)$.
(2) Choose two rows of C and permute them.

(3) Consider a row vector which equals a left multiplication of a row of C by an element of $W(x;\xi)$ and add it to another row of C.
(4) Multiply a column of C from the right by a non-zero element of $\mathbb{C}(x;\xi)$.
(5) Choose two columns of C and permute them.
(6) Consider a column vector which equals a right multiplication of a column of C by an element of $W(x;\xi)$ and add it to another column of C.

Let $\tilde{A}$ be a matrix obtained by a suitable successive applications of these transformation to A. First we will prove that we may assume $B = \tilde{A}$. Let d denote the minimal order of non-zero elements in the matrices obtained by successive applications of these transformations to A. We may assume $\tilde{A}_{1,1} \neq 0$ and $\operatorname{ord} \tilde{A}_{1,1} = d$. By suitable transformations (3) and (6), we may moreover assume $\tilde{A}_{i,1} = \tilde{A}_{1,j} = 0$ if $i \geq 2$ and $j \geq 2$ because of the minimality of d. Put $A' = \bigl(\tilde{A}_{i,j}\bigr)_{\substack{2\leq i\leq m\\ 2\leq j\leq n}}$. If $A' = 0$, then $B = \tilde{A}$.

We may assume $A' \neq 0$. If $d = 0$, we get B by the induction on m. Hence we may assume $d > 0$ and $\tilde{A}_{2,2} \neq 0$. Putting $d' = \operatorname{ord} \tilde{A}_{2,2} \geq d > 0$, we may moreover assume $\operatorname{ord}(\tilde{A}_{2,2} - \partial^{d'}) < d'$. Add the right multiplication of the second column of $\tilde{A}$ by x^s $(s = 0, 1, 2, \dots)$ to the first column. Then add the left multiplication of the first row by an element $-P \in W(x;\xi)$ to the second row. Then the $(2,1)$-element of the resulting matrix equals

$$\tilde{A}_{2,2}x^s - P\tilde{A}_{1,1}.$$

We can choose P so that $\operatorname{ord}(\tilde{A}_{2,2}x^s - P\tilde{A}_{1,1}) < d$. Then the minimality of d implies $\tilde{A}_{2,2}x^s \in W(x;\xi)\tilde{A}_{1,1}$. Put

$$x^{d'-s}\tilde{A}_{2,2}x^s = \tilde{A}_{2,2,0} + s\tilde{A}_{2,2,1} + \cdots + s^{d'}\tilde{A}_{2,2,d'}.$$

Here $\tilde{A}_{2,2,\nu} \in W(x;\xi)$ do not depend on s. Note that $\tilde{A}_{2,2,d'} = x^{d'}\tilde{A}_{2,2}$ and $\tilde{A}_{2,2,0} = 1$. The condition $\tilde{A}_{2,2}x^s \in W(x;\xi)\tilde{A}_{1,1}$ for $s = 0, 1, \dots$ implies $\tilde{A}_{2,2,\nu} \in W(x;\xi)\tilde{A}_{1,1}$, which contradicts to $\tilde{A}_{2,2,0} = 1$ because $d \geq 1$. Hence $A' = 0$.

Define a left $W(x;\xi)$-module by $\mathcal{M} = W(x;\xi)^n / \sum_{i=1}^m W(x;\xi)(A_{i,1}, \dots, A_{i,n})$ and put $\mathcal{M}' := \{u \in \mathcal{M} \mid \exists P \in W(x;\xi) \setminus \{0\} \text{ such that } Pu = 0\}$. Note that the above transformations give isomorphisms between finitely generated $W(x;\xi)$-modules. Note that $\dim_{W(x;\xi)} \mathcal{M}' = \operatorname{ord} P$ and $\mathcal{M}/\mathcal{M}' \simeq W(x;\xi)^{n-k-1}$ as left $W(x;\xi)$-modules. Thus we have the lemma by the following.

Suppose $W(x;\xi)^m$ is isomorphic to $W(x;\xi)^n$ as left $W(x;\xi)$ modules. Suppose moreover A gives the isomorphism. Then we have $\operatorname{ord} P = 1$ and $m = n$ by using the transformation of A into B. □

Corollary 1.11. i) *If m and n are positive integers satisfying $m \neq n$, then $W(x;\xi)^m$ is not isomorphic to $W(x;\xi)^n$ as left $W(x;\xi)$-modules.*

ii) *Any element of $GL\bigl(n, W(x;\xi)\bigr)$ is a product of fundamental matrices corresponding to the transformations* (1)–(6) *in the above proof.*

1.4.3. irreducibility. Lastly we give the following standard definition.

Definition 1.12. Fix $P \in W(x;\xi)$ with $\operatorname{ord} P > 0$. The equation (1.58) is *irreducible* if and only if one of the following equivalent conditions is valid.

(1.71) The left $W(x;\xi)$-module $\mathcal{M}$ is simple.

(1.72) The left $W(x;\xi)$-ideal $W(x;\xi)P$ is maximal.

(1.73) $P = QR$ with $Q,\ R \in W(x;\xi)$ implies $\operatorname{ord} Q \cdot \operatorname{ord} R = 0$.

(1.74) $\forall Q \notin W(x;\xi)P,\ \exists M,\ N \in W(x;\xi)$ satisfying $MP + NQ = 1$.

$$\begin{cases} ST \in W(x;\xi)P \text{ with } S,T \in W(x;\xi) \text{ and } \operatorname{ord} S < \operatorname{ord} P \\ \quad \Rightarrow S = 0 \text{ or } T \in W(x;\xi)P. \end{cases} \tag{1.75}$$

The equivalence of the above conditions is standard and easily proved. The last condition may be a little non-trivial.

Suppose (1.75) and $P = QR$ and $\operatorname{ord} Q \cdot \operatorname{ord} R \neq 0$. Then $R \notin W(x;\xi)P$ and therefore $Q = 0$, which contradicts to $P = QR$. Hence (1.75) implies (1.73).

Suppose (1.71), (1.74), $ST \in W(x;\xi)P$ and $T \notin W(x;\xi)P$. Then there exists P' such that $\{J \in W(x;\xi)\,;\, JT \in W(x;\xi)P\} = W(x;\xi)P'$, $\operatorname{ord} P' = \operatorname{ord} P$ and moreover $P'v = 0$ is also simple. Since $Sv = 0$ with $\operatorname{ord} S < \operatorname{ord} P'$, we have S=0.

In general, a system of ordinary differential equations is defined to be irreducible if it is simple as a left $W(x;\xi)$-module.

REMARK 1.13. Suppose the equation $\mathcal{M}$ given in (1.58) is irreducible.

i) Let $u(x,\xi)$ be a non-zero solution of $\mathcal{M}$, which is locally defined for the variables x and ξ and meromorphically depends on (x,ξ). If $S \in W[x;\xi]$ satisfies $Su(x,\xi) = 0$, then $S \in W(x;\xi)P$. Therefore $u(x,\xi)$ determines $\mathcal{M}$.

ii) Suppose $\operatorname{ord} P > 1$. Fix $R \in W(x;\xi)$ such that $\operatorname{ord} R < \operatorname{ord} P$ and $R \neq 0$. For $Q \in W(x;\xi)$ and a positive integer m, the condition $R^m Qu = 0$ is equivalent to $Qu = 0$. Hence for example, if $Q_1 u + \partial^m Q_2 u = 0$ with certain $Q_j \in W(x;\xi)$, we will allow the expression $\partial^{-m} Q_1 u + Q_2 u = 0$ and $\partial^{-m} Q_1 u(x,\xi) + Q_2 u(x,\xi) = 0$.

iii) For $T \notin W(x;\xi)P$ we construct a differential equation $Qv = 0$ satisfied by $v = Tu$ as follows. Put $n = \operatorname{ord} P$. We have $R_j \in W(x;\xi)$ such that $\partial^j Tu = R_j u$ with $\operatorname{ord} R_j < \operatorname{ord} P$. Then there exist $b_0, \dots, b_n \in \mathbb{C}(x,\xi)$ such that $b_n R_n + \cdots + b_1 R_1 + b_0 R_0 = 0$. Then $Q = b_n \partial^n + \cdots + b_1 \partial + b_0$.

1.5. Okubo normal form and Schlesinger canonical form

In this section we briefly explain the interpretation of Katz's middle convolution (cf. [**Kz**]) by [**DR**] and its relation to our fractional operations.

For constant square matrices T and A of size n', the ordinary differential equation

$$(xI_{n'} - T)\frac{du}{dx} = Au \tag{1.76}$$

is called *Okubo normal form* of Fuchsian system when T is a diagonal matrix. Then

$$mc_\mu\big((xI_{n'} - T)\partial - A\big) = (xI_{n'} - T)\partial - (A + \mu I_{n'}) \tag{1.77}$$

for generic $\mu \in \mathbb{C}$, namely, the system is transformed into

$$(xI_{n'} - T)\frac{du_\mu}{dx} = \big(A + \mu I_{n'}\big)u_\mu \tag{1.78}$$

by the operation mc_μ. Hence for a solution $u(x)$ of (1.76), the Euler transformation $u_\mu(x) = I_c^\mu(u)$ of $u(x)$ satisfies (1.78).

For constant square matrices A_j of size m and the *Schlesinger canonical form*

$$\frac{dv}{dx} = \sum_{j=1}^{p} \frac{A_j}{x - c_j} v \tag{1.79}$$

of a Fuchsian system of the Riemann sphere, we have

$$\frac{du}{dx} = \sum_{j=1}^{p} \frac{\tilde{A}_j(-1)}{x - c_j} u \quad \text{with} \quad u := \begin{pmatrix} \frac{v}{x-c_1} \\ \vdots \\ \frac{v}{x-c_p} \end{pmatrix}, \tag{1.80}$$

$$\tilde{A}_j(\mu) := \begin{array}{r@{}l} & \begin{array}{ccccccc} \hphantom{A_1} & \hphantom{\cdots} & \hphantom{A_{j-1}} & j & \hphantom{A_{j+1}} & \hphantom{\cdots} & \hphantom{A_p} \end{array} \\ j) & \begin{pmatrix} A_1 & \cdots & A_{j-1} & A_j+\mu & A_{j+1} & \cdots & A_p \end{pmatrix} \end{array} \tag{1.81}$$

since $\frac{v}{x-c_j} + (x-c_j)\frac{d}{dx}\frac{v}{x-c_j} = \frac{dv}{dx} = \sum_{\nu=1}^{p}\frac{A_\nu}{x-c_\nu}v$. Here $\tilde{A}_j$ are square matrices of size pm. The addition $\mathrm{Ad}\big((x-c_k)^{\mu_k}\big)$ transforms A_j into $A_j + \mu_k\delta_{j,k}I_m$ for $j = 1, \ldots, p$ in the system (1.79). Putting

$$A(\mu) = A(0) + \mu I_{pm} = \tilde{A}_1(\mu) + \cdots + \tilde{A}_p(\mu) \quad \text{and} \quad T = \begin{pmatrix} c_1 I_m & & \\ & \ddots & \\ & & c_p I_m \end{pmatrix},$$

the equation (1.80) is equivalent to (1.76) with $n' = pm$ and $A = A(-1)$. Define square matrices of size n' by

$$\tilde{A} := \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_p \end{pmatrix} \tag{1.82}$$

Then $\ker\tilde{A}$ and $\ker A(\mu)$ are invariant under $\tilde{A}_j(\mu)$ for $j = 1, \ldots, p$ and therefore $\tilde{A}_j(\mu)$ induce endomorphisms of $V := \mathbb{C}^{pm}/\big(\ker\tilde{A} + \ker A(\mu)\big)$, which correspond to square matrices of size $N := \dim V$, which we put $\bar{A}_j(\mu)$, respectively, under a fixed basis of V. Then the middle convolution mc_μ of (1.79) is the system

$$\frac{dw}{dx} = \sum_{j=1}^{p}\frac{\bar{A}_j(\mu)}{x-c_j}w \tag{1.83}$$

of rank N, which is defined and studied by [**DR, DR2**]. Here $\ker\tilde{A}\cap\ker A(\mu) = \{0\}$ if $\mu \neq 0$.

We define another realization of the middle convolution as in [**O5**, §2]. Suppose $\mu \neq 0$. The square matrices of size n'

$$A_j^\vee(\mu) := \begin{array}{r@{}l} & \begin{array}{c} j \end{array} \\ j) & \begin{pmatrix} A_1 \\ \vdots \\ A_j+\mu \\ \vdots \\ A_p \end{pmatrix} \end{array} \quad \text{and} \quad A^\vee(\mu) := A_1^\vee(\mu) + \cdots + A_p^\vee(\mu) \tag{1.84}$$

satisfy

$$\tilde{A}(A+\mu I_{n'}) = A^\vee(\mu)\tilde{A} = \Big(A_iA_j + \mu\delta_{i,j}A_i\Big)_{\substack{1\le i\le p\\ 1\le j\le p}} \in M(n',\mathbb{C}), \tag{1.85}$$

$$\tilde{A}(A+\mu I_{n'})\tilde{A}_j(\mu) = A_j^\vee(\mu)\tilde{A}(A+\mu I_{n'}). \tag{1.86}$$

Hence $w^\vee := \tilde{A}(A+\mu I_{n'})u$ satisfies

$$\frac{dw^\vee}{dx} = \sum_{j=1}^{p}\frac{A_j^\vee(\mu)}{x-c_j}w^\vee, \tag{1.87}$$

$$\sum_{j=1}^{p}\frac{A_j^\vee(\mu)}{x-c_j} = \left(\frac{A_i+\mu\delta_{i,j}I_m}{x-c_j}\right)_{\substack{1\le i\le p,\\ 1\le j\le p}}$$

and $\tilde{A}(A+\mu I_{n'})$ induces the isomorphism

$$\tilde{A}(A+\mu I_{n'}) : V = \mathbb{C}^{n'}/(\mathcal{K}+\mathcal{L}_\mu) \xrightarrow{\sim} V^\vee := \operatorname{Im}\tilde{A}(A+\mu I_{n'}) \subset \mathbb{C}^{n'}. \tag{1.88}$$

Hence putting $\bar{A}_j^\vee(\mu) := A_j^\vee(\mu)|_{V^\vee}$, the system (1.83) is isomorphic to the system

$$\frac{dw^\vee}{dx} = \sum_{j=1}^{p} \frac{\bar{A}_j^\vee(\mu)}{x-c_j} w^\vee \tag{1.89}$$

of rank N, which can be regarded as a middle convolution mc_μ of (1.79). Here

$$w^\vee = \begin{pmatrix} w_1^\vee \\ \vdots \\ w_p^\vee \end{pmatrix}, \quad w_j^\vee = \sum_{\nu=1}^{p} (A_j A_\nu + \mu\delta_{j,\nu})(u_\mu)_\nu \quad (j=1,\dots,p) \tag{1.90}$$

and if $v(x)$ is a solution of (1.79), then

$$w^\vee(x) = \Bigl(\sum_{\nu=1}^{p} (A_j A_\nu + \mu\delta_{j,\nu}) I_c^\mu\Bigl(\frac{v(x)}{x-c_\nu}\Bigr)\Bigr)_{j=1,\dots,p} \tag{1.91}$$

satisfies (1.89).

Since any non-zero homomorphism between irreducible $W(x)$-modules is an isomorphism, we have the following remark (cf. §1.4 and §3.2).

Remark 1.14. Suppose that the systems (1.79) and (1.89) are irreducible. Moreover suppose the system (1.79) is isomorphic to a single Fuchsian differential equation $P\tilde{u} = 0$ as left $W(x)$-modules and the equation $mc_\mu(P)\tilde{w} = 0$ is also irreducible. Then the system (1.89) is isomorphic to the single equation $mc_\mu(P)\tilde{w} = 0$ because the differential equation satisfied by $I_c^\mu(\tilde{u}(x))$ is isomorphic to that of $I_c^\mu(Q\tilde{u}(x))$ for a non-zero solution $v(x)$ of $P\tilde{u} = 0$ and an operator $Q \in W(x)$ with $Q\tilde{u}(x) \neq 0$ (cf. §3.2, Remark 5.4 iii) and Proposition 6.13).

In particular, if the systems are rigid and their spectral parameters are generic, all the assumptions here are satisfied (cf. Remark 4.17 ii) and Corollary 10.12).

Yokoyama [**Yo2**] defines extension and restriction operations among the systems of differential equations of Okubo normal form. The relation of Yokoyama's operations to Katz's operations is clarified by [**O7**], which shows that they are equivalent from the view point of the construction and the reduction of systems of Fuchsian differential equations.