Chapter 8

Deformation spaces of real projective structures on 2-orbifolds of negative Euler characteristic: An introduction

The main purpose here is to introduce real projective structures on 2-orbifolds to the readers. The theoretical aspects are not completely written here but the readers can find them in articles mentioned. Additionally, we discuss the computational aspect of this theory in a more detailed way.

First, we will give some introduction to real projective structures on orbifolds with relationships to hyperbolic structures. Next, we give some examples of real projective structures on annuli, a torus with one-hole and the orbifolds based on a triangle.

We also give a survey of real projective structures on manifolds (and orbifolds) from a historical point of view: the Hilbert metrics, the topological work of Choi (1994a,b) and Goldman (1990), the gauge theory point of view using Higgs bundles, the Hitchin's conjecture and the group theoretical work of Benoist (2001).

Next, we study real projective structures on 2-orbifolds of negative Euler characteristic. We present Theorem 8.3.1 characterizing the topology of the deformation spaces of convex real projective structures on 2-orbifolds of negative Euler characteristic. Next, we study the relationship between the deformation spaces and the Hitchin-Teichmüller components of the spaces of $\mathbb{PGL}(3, \mathbb{R})$-characters in Section 8.3.1. We try to now understand the deformation spaces of real projective structures on orbifolds. We discuss the geometric constructions available for such structures and the elementary 2-orbifolds and their real projective structures using the work of Goldman (1990). From these, we should be able to prove Theorem 8.3.1 characterizing the topology of the deformation space of real projective structures on 2-orbifolds. However, we do not present the full detail.

8.1 Introduction to real projective orbifolds

Let X be the real projective plane $\mathbb{RP}^n$ and G the group $\mathbb{PGL}(n+1, \mathbb{R})$ of *collineations*, i.e., projective automorphisms of $\mathbb{RP}^n$. An $\mathbb{RP}^n$*-structure* or *real projective structure* on an n-dimensional orbifold Σ is an $(\mathbb{RP}^n, \mathbb{PGL}(n+1, \mathbb{R}))$-structure on Σ. Two $\mathbb{RP}^n$-structures on Σ are equivalent if an isotopy from the identity map

I_Σ of Σ induces one from the other. A *real projective orbifold* or a $\mathbb{RP}^n$*-orbifold* is an orbifold with this structure. The deformation space $\mathbb{RP}^n(\Sigma)$ of $\mathbb{RP}^n$-structures on Σ is the space of equivalence classes of $\mathbb{RP}^n$-structures with appropriate topology.

A hyperbolic space can be represented by the Klein model. We have a standard ellipsoid in $\mathbb{RP}^n$ bounding a convex open domain Ω: This set corresponds to the space of rays in a convex cone in $\mathbb{R}^{n+1}$ given by the equation

$$x_0 > \sqrt{x_1^2 + \cdots + x_n^2}.$$

Then the hyperbolic isometry group is precisely the subgroup of $\mathbb{PGL}(n+1, \mathbb{R})$ acting on Ω, and a discrete group Γ of isometries becomes a discrete group of projective automorphisms. The quotient Ω/Γ has a real projective structure. These are called *hyperbolic real projective structures.* (See Section 3.1.6 for details.)

Given a hyperspace in $\mathbb{RP}^n$, we recall that the complement has the natural affine structure whose geodesic structure extends to projective ones. We call this the *affine subspace.* (See Section 3.1.4 for details.) A domain Ω in $\mathbb{RP}^n$ is *convex* if it forms a convex domain in the affine subspace or equals $\mathbb{RP}^n$ itself. (We can prove this by taking an inverse image in $\mathbb{R}^{n+1}$ with components that are convex cones and we use supporting hyperplanes. See the book [Berger (2009)] for details.) An open domain Ω is *properly convex* if it is contained in some bounded convex closed domain in an affine subspace of $\mathbb{RP}^n$. For a convex domain Ω, this is equivalent to the fact that Ω does not contain a complete 1-dimensional affine space, i.e., a complete affine line. If the boundary of a convex domain Ω does not contain a straight segment, then Ω is said to be *strictly convex.*

In fact, for any convex open domain Ω and Γ acting on Ω cocompactly and properly discontinuously, we obtain a real projective 2-orbifold.

Define $\mathbf{S}^n = (\mathbb{R}^{n+1} - \{O\})/\sim$ where $v \sim w$ iff $v = kw$ for $k > 0$. $\mathbf{S}^n$ has a real projective structure as a double cover of $\mathbb{RP}^n$. A *real projective sphere* $\mathbf{S}^n$ is $\mathbf{S}^n$ with the real projective structure and has a group of projective automorphisms $\mathrm{Aut}(\mathbf{S}^n)$ isomorphic to the group $\mathbb{SL}_\pm(n+1, \mathbb{R})$ of linear maps of determinant ± 1.

A closed real projective orbifold is said to *convex* if any arc in a relative homotopy class can be homotoped to a line relative to the end points. It is *properly convex* if it does not contain a complete affine line, i.e., a subspace projectively isomorphic to a complete real line. A closed real projective orbifold is convex if and only if it is diffeomorphic to Ω/Γ or $\mathbf{S}^n/\Gamma$ for a convex domain Ω in an affine subspace and a real projective automorphism group Γ acting on it or on the real projective sphere $\mathbf{S}^n$ properly discontinuously. It is *properly convex* if and only if it is diffeomorphic to Ω/Γ where Ω is a properly convex domain (Choi, 1994a,b).

There are closed convex real projective orbifolds that are not hyperbolic, which we will state later in detail.

A closed 2-orbifold Σ with $\chi(\Sigma) < 0$ with an $\mathbb{RP}^2$-structure is convex if and only if it is projectively diffeomorphic to the quotient of a properly convex domain in an affine patch by a properly discontinuous action of a group of projective automorphisms.

An arc in Σ that is locally a line is called *geodesic* or *projective geodesic*. If each component $\partial\Sigma$ is locally a line, Σ is said to have geodesic boundary. A closed curve in Σ whose lift develops into a line connecting the unique attracting and repelling fixed points of its holonomy is said to be a *principal* closed geodesic.

When $\partial\Sigma \neq \emptyset$, boundary components are required to be principal geodesic.

Let us discuss for $\mathbb{RP}^2$. A projective automorphism is said to be *positive hyperbolic* if it is diagonalizable and the maximum and minimum modulus eigenvalues are positive and have multiplicity one. Let A be a positive hyperbolic projective automorphism. The conjugation invariants of a positive hyperbolic element A are eigenvalues $\lambda_1, \lambda_2, \lambda_3$ with

$$0 < \lambda_1 < \lambda_2 < \lambda_3, \lambda_1\lambda_2\lambda_3 = 1.$$

Thus, A has three fixed points in $\mathbb{RP}^2$, that are noncollinear, one of which is an attracting fixed point, another one is a repelling one, and the last one is a saddle-type one. There are three A-invariant lines bounding four triangles in $\mathbb{RP}^2$.

The space of invariants for positive hyperbolic matrices is given by $0 < \lambda_1, 0 < \lambda_1 < \lambda_2, \lambda_1\lambda_2^2 < 1$. Here λ_1 and λ_2 completely characterize the conjugacy classes. We denote the region by D, homeomorphic to an open disk. Another way to describe this space is by the *Goldman invariants* of A given by $\lambda = \lambda_1, \tau = \lambda_2 + \lambda_3$. These satisfy

$$0 < \lambda < 1, \frac{2}{\sqrt{\lambda}} < \tau < \lambda + \frac{1}{\lambda^2}.$$

In general, a projective automorphism of $\mathbb{RP}^n$ is represented by a matrix with determinant ± 1 where the largest norm eigenvalue is positive. A projective automorphsim is *positive proximal* if the largest and smallest norm eigenvalues of the corresponding matrix are positive and of multiplicity one.

The following is a summary of the most general results about the geometry of convex real projective manifolds and orbifolds, following [Benoist (2008)]. (Historically, these results were obtained by Kuiper (1954), Benzecri (1960, 1962), Koszul (1965, 1968) and so on.) Recall that group is *hyperbolic* if its Caley graph is Gromov hyperbolic, and a closed curve is *essential* if the fundamental group of the closed curve injects.)

Theorem 8.1.1. *Let Ω/Γ be a closed n-dimensional real projective orbifold M where Ω is a properly convex domain in an affine subspace of $\mathbb{RP}^n$ and Γ is a discrete group of real projective automorphisms acting on Ω and is a hyperbolic group.*

- *Ω is strictly and properly convex.*
- *The holonomy of each essential closed curve is positive proximal with exactly two fixed points in* $\mathrm{bd}\Omega$ *which are an attracting fixed point and a repelling one and acts on the open line in Ω connecting the two fixed points.*
- *Each essential closed curve in M is realized by a closed geodesic.*

- *Suppose that the essential closed curve is homotopic to a simple closed curve. If M is an orientable* 2*-orbifold, then there exists a unique closed geodesic isotopic to it which is a principal closed geodesic or it double covers a segment with two endoints in singularites of order two. If Ω/Γ is not orientable, the closed geodesic is either simple or it double-covers a segment as above or a simple closed geodesic.*
- $\mathrm{bd}\Omega$ *is $C^{1,\alpha}$ and is an ellipsoid if C^2.* (Benzecri)

Note that these hold for each hyperbolic surfaces as well where the corresponding group to Γ is considered a subgroup of $\mathbb{PSO}(1, 2)$ and Ω is the interior of a conic.

The following theorem states in the surface case, convex ones are the most important ones. (Choi, 1994a,b).

Theorem 8.1.2. *Let Σ be a compact orientable real projective surface with principal geodesic or empty boundary and $\chi(\Sigma) < 0$. Then Σ has a collection of mutually disjoint simple closed geodesics the components of whose complement have closures that are properly convex real projective surfaces with principal geodesic boundary of negative Euler characteristic or elementary annuli.*

(See Section 8.1.1 for the definition of elementary annuli.)

From this, we obtained later in the paper [Choi and Goldman (1997)].

Theorem 8.1.3. *The deformation space of real projective structures on a closed orientable surface of genus g, $g > 1$, is an infinite countable union of open cells of dimension* $16g - 16$.

8.1.1 ***Examples of real projective 2-orbifolds.***

We recall the terminology and facts in Section 3.1.4:

8.1.1.1 *Elementary annuli*

Let ϑ be a collineation represented by a diagonal matrix with distinct positive eigenvalues. Then it has three fixed points in $\mathbb{RP}^2$: an attracting fixed point of the action of $\langle\vartheta\rangle$, a repelling fixed point, and a saddle-type fixed point. Three lines passing through two of them are ϑ-invariant, as are four open triangles bounded by them. Choosing two open sides of an open triangle ending at an attracting fixed point or a repelling fixed point simultaneously, their union is acted properly and freely upon by $\langle\vartheta\rangle$. The quotient space is diffeomorphic to an annulus. The $\mathbb{RP}^2$-surface projectively diffeomorphic to the quotient space is said to be an *elementary annulus*. (See the left part of Figure 8.1.)

A principal geodesic boundary is one connecting an attracting fixed point of ϑ with a repelling one. This definition is independent of orientation. There is a unique principal geodesic component among the two components. The other component is

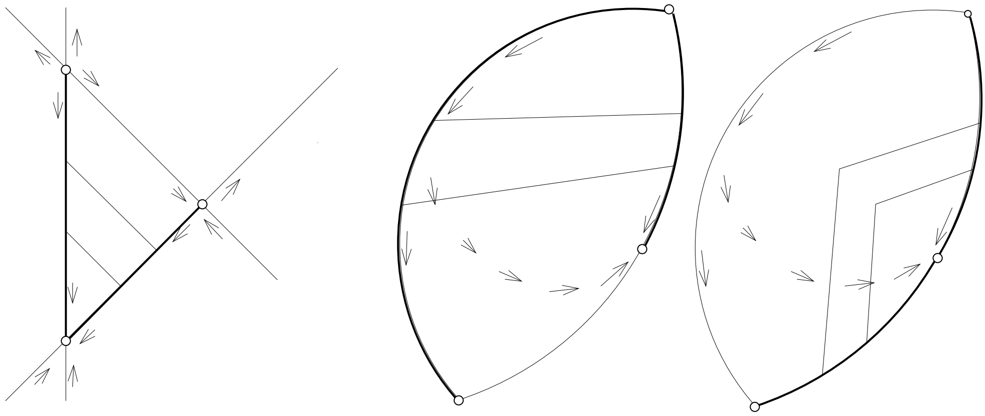

Fig. 8.1 Elementary annuli of hyperbolic and two quasi-hyperbolic types as quotients of domains and actions on them. The thicker lines indicate the included boundary components.

said to be *weak*.

A *pasting* of two boundary components of real projective surfaces with geodesic boundary can be described as attaching and projectively identifying thin regular neighborhoods of the geodesics in some ambient open surface. The necessary condition for pasting to take place is that the holonomy of the generator of the fundamental group of the boundary component is conjugate to the holonomy of the corresponding generator for the other boundary component. This is also the sufficient condition when the boundary components are principal geodesic. (Also, if both boundary components have complete affine lifts, it is also sufficient.)

A real projective annulus with geodesic boundary can be obtained by pasting the above elementary annuli along geodesic boundary of same types.

Goldman showed that each annulus with principal geodesic boundary is obtained by pasting elementary annuli. (See also the article [Sullivan and Thurston (1983)].) In fact, we can draw an arc in $\mathbb{RP}^2$ in a certain manner as in Figure 8.2 and obtain an annulus. This corresponds to the pasting construction.

One can also have an annulus with geodesic boundary where ϑ is *quasi-hyperbolic*, i.e., represented by a non-diagonalizable matrix with two positive eigenvalues. It has two fixed points in $\mathbb{RP}^2$. One is a repelling or attracting fixed point, say x and the other y. A 1-dimensional subspace m passing through x and y is invariant by ϑ. ϑ has an attracting and repelling fixed points x and y on m. There is another ϑ-invariant subspace l of dimension one with unique fixed point y on it. ϑ acts as a translation on $l - \{y\}$ identified with a complete affine line. (See Figure 8.1.)

Let L be a component of $\mathbb{RP}^2 - l - m$. An elementary annulus is the quotient of $L \cup m - \{x, y\}$ or the quotient of $L \cup l' \cup m_1$ for a unique component m_1 of $m - \{x, y\}$ and the component l' of $l - \{y\}$ adjacent to L so that a segment s connecting a

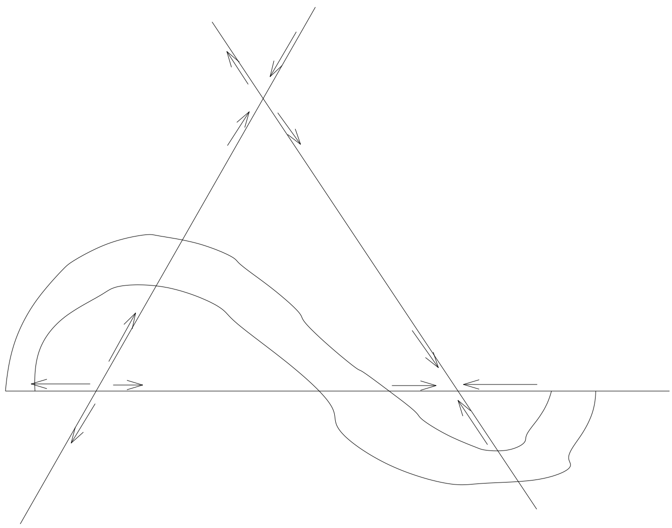

Fig. 8.2 Any immersed arc so that the directions of the action arrows do not change as it crosses the invariant lines corresponds to an annulus with geodesic boundary. To see this, simply act by g and the two arcs will bound a strip glued to an annulus. This was discovered by Goldman (1977). For a fixed holonomy, one can classify them by a free semigroup of rank two.

point of m_1 to l' is disjoint from $\vartheta(l)$. (Note that a wrong choice would give us a non-Hausdorff space.)

We note that the elementary annuli of quasi-hyperbolic type do not occur in convex real projective closed surfaces or 2-orbifolds of negative Euler characteristic. (See [Choi (1994b)].)

8.1.1.2 π-*Annuli*

Let ϑ be a hyperbolic projective automorphism. Take two adjacent ϑ-invariant triangles with three open sides of them all ending in an attracting fixed point or a repelling fixed point. Then the quotient of the union by $\langle\vartheta\rangle$ is diffeomorphic to an annulus. The projectively diffeomorphic surfaces are said to be π-*annuli* (Choi, 1994a,b).

A *reflection* in $\mathbb{RP}^2$ is an involution fixing a line and an isolated point. A reflection in a projective space is determined uniquely by a line of fixed points and a fixed point outside the line with a matrix conjugate to a diagonal matrix with entries $1, 1, -1$.

There is a reflection sending one triangle to the other inducing an order-two group. The quotient map is an orbifold map, and the quotient space carries an orbifold structure so that one boundary component is made of mirror points. Thus, the π-annulus is a double of an elementary annulus with a silvered boundary component.

Now, let ϑ be a quasi-hyperbolic projective automorphism. Then we define one of the two types of annuli to be a π-annulus: that is, an elementary annulus of quasi-hyperbolic type with the lifts of two boundary components ending at a common point.

Also the pasting of two elementary annuli of quasi-hyperbolic type along the boundary components corresponding to the complete affine lines is another type of a π-annulus. (See Figure 8.3.)

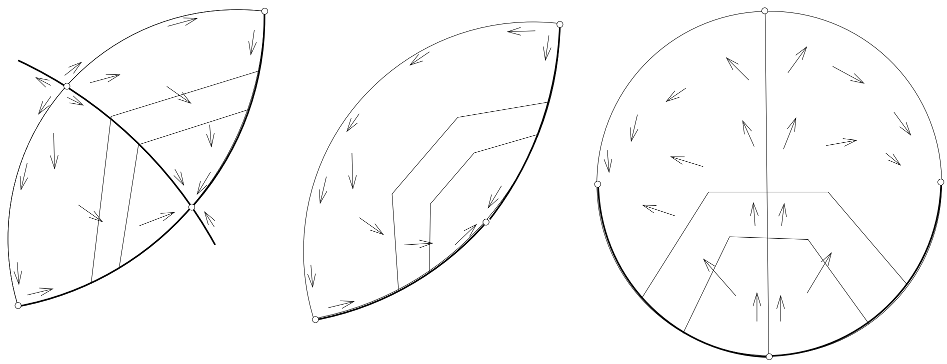

Fig. 8.3 The π-annuli of hyperbolic type and two of quasi-hyperbolic type.

We mention that Nagano and Yagi (1974) and Goldman (1977) essentially classified the real projective structures on annuli, Möbius bands, tori and Klein bottles. To this date, the work was not yet generalized to 2-orbifolds of Euler characteristic zero. The topology of the deformation spaces are still unknown. See [Baues and Goldman (2005)] also.

8.1.1.3 *An example*: *a bending torus with a disk removed*

Consider $\mathbb{H}^2$ as the inside of a standard ellipse in $\mathbb{RP}^2$ given by the set of null vectors in $\mathbb{R}^3$ with the standard Lorentzian metric from the quadratic form $x_0^2 - x_1^2 - x_2^2$.

Take an orientable hyperbolic 2-orbifold S. Then $S = \mathbb{H}^2/\Gamma$ for a discrete subgroup $\Gamma \subset \mathbf{Isom}(\mathbb{H}^2) = \mathbb{PSO}(1,2) \subset \mathbb{PGL}(3,\mathbb{R})$. Thus, S is identified with a quotient space of a convex open domain in $\mathbb{RP}^2$. Here, $\mathbb{H}^2$ is represented by the Klein model; i.e., it is identified with the standard unit disk in $\mathbb{RP}^2$.

Let S be an orientable hyperbolic closed 2-orbifold or a hyperbolic compact 2-orbifold with geodesic boundary. We can deform this to a parameter of nonhyperbolic real projective surfaces by so-called "bending" first discovered by Thurston (1977). Again denote by $\pi_1(S)$ the group of deck transformations of the universal cover $\tilde{S}$ of S.

An essential simple closed curve is homotopic to a simple closed geodesic by Theorem 8.1.1. Let S contain a simple closed geodesic c.

We have that $\tilde{S}$ is identified with $\mathbb{H}^2$. The inverse image L of c is a disjoint union of straight lines ending in $\mathrm{bd}\mathbb{H}^2$. Take a component l and the other components are of form $g(l)$ for $g \in \pi_1(S)$. Let the cyclic group generated by $\gamma \in \pi_1(S)$ acts on l so that it corresponds to the covering $l \to c$, where l and c are oriented along γ.

We can find an element of $\mathbb{PGL}(3, \mathbb{R})$ namely an element of $\mathbf{Isom}(\mathbb{H}^2)$ that preserves $\mathbb{H}^2$ and sends l to any segment. Therefore, we choose a projective coordinate system so that l has endpoints $[0, 1, 1]$ and $[0, -1, 1]$. Then γ is now represented as a matrix with eigenvalues $\lambda, 1/\lambda$, and 1 at respective points $[0, 1, 1]$, $[0, -1, 1]$, and $[1, 0, 0]$ for $\lambda > 1$.

Then any projective transformation η with a diagonalizable matrix with eigenvalues $a, 1/(ab)$, and b respectively at the above points commutes with γ. For each component $g(l)$ of L for $g \in \pi_1(S)$, we glue the relative closure of the left adjacent component C of $\mathbb{H}^2 - L$ with a right adjacent component C' by η.

This construction amounts to the following "cut and paste" construction: Cut S by a simple closed curve c and obtain $S - c$. Complete it by the induced path metric to $\bar{S}$ with two boundary components c'_1 and c'_2. Find an open ambient real projective 2-orbifold S' containing c'_1 and c'_2. Now, η induces a real projective diffeomorphism η' from an open neighborhood N_1 of c'_1 in S' to one N_2 of c'_2 in S'. Let S_1 be the copy of $S - c$ in S', we take $S_1 \cup N_1 \cup N_2$ in S', and we identify N_1 an N_2 by η'. The resulting 2-orbifold S' is still diffeomorphic to S.

This construction is said to be a *projective bending* of S. For each nonidentity η, we obtain a projective bending. For a parameter of η, we obtain a parameter of bendings. The resulting projective 2-orbifold S' is still properly convex (Goldman, 1990).

In fact, we could have started with any orientable compact properly convex 2-orbifold with geodesic boundary. Each simple closed curve is realized as a simple closed geodesic.

As a specific example, we consider a torus with one hole, i.e., a genus-one orientable hyperbolic surface with one boundary component where S decomposes into one pair-of-pants. We obtain various pictures of deformations and the convex domains that cover the deformed real projective surface.

Let us explain some explicit construction that can be obtained by some computer algebra systems. We did the computation with Mathematica™.

A hyperbolic pair-of-pants with geodesic boundary is first constructed: In $\mathbb{H}^2$ find a geodesic l_1 passing $[0, 0, 1]$ with endpoints $[1, 0, 1]$ and $[-1, 0, 1]$ and another geodesic l_2 passing $[0, 0, 1]$ with endpoints $[0, 1, 1]$ and $[0, -1, 1]$. We find a matrix A acting on l_1 with eigenvalues $\lambda, 1/\lambda$, and 1 for $\lambda > 1$ with respective fixed points $[-1, 0, 1]$, $[1, 0, 1]$, and $[0, 1, 0]$ and K acting on l_2 with eigenvalues $\mu, 1/\mu, 1$ for $\mu > 1$ with respective eigenvectors $[0, 1, 1]$, $[0, -1, 1]$, and $[1, 0, 0]$. Let $B = KA^{-1}K^{-1}$.

For λ, μ sufficiently large, one can make $A^{-1}(l_2)$ and $B(l_2)$ are disjoint geodesics. $C := BA$ has an invariant geodesic l_3 meeting $A^{-1}(l_2)$ and $B(l_2)$ at distinct points and containing the shortest segment between them. Then

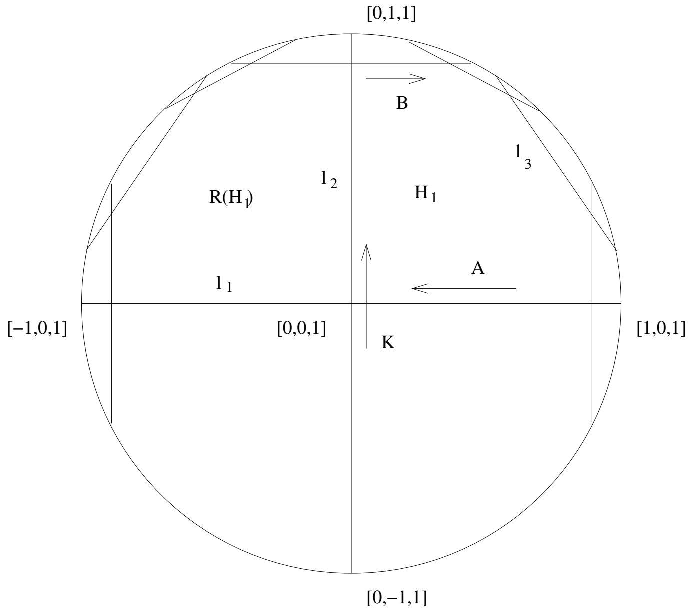

Fig. 8.4 The diagram for a torus with one-hole.

$l_1, l_2, K(l_1), B(l_2), l_3, A^{-1}(l_2)$ bound a hexagon H_1. Reflect H_1 using a reflection R fixing l_2 and $[1, 0, 0]$. The free group $F_2 := \langle A^2, B^2\rangle$ acts freely and properly discontinuously on $\mathbb{H}^2$ and $\bigcup_{g\in F_2} g(H_1 \cup R(H_1))$ forms a universal cover of a pair-of-pants P with geodesic boundary corresponding to A^2, B^2, A^2B^2. This constructs one pair-of-pants. (Actually, this is a double of a hexagonal 2-orbifold with three silvered edges and three boundary components.)

Consider the group generated by A^2, B^2, K. Then this generates a group Γ and $\mathbb{H}^2/\Gamma$ is diffeomorphic to a torus with one hole. (We are attaching the boundary component corresponding to A with that of B by K here.)

Let η be a matrix commuting with A with eigenvalues $\delta, \eta, 1/(\delta\eta)$ and eigenvectors at $[-1, 0, 1]$ $[0, 1, 0]$, and $[1, 0, 1]$. The bending by η corresponds to changing K to $K\eta$. This gives us a two-parameter space of bendings. (See Bending1.nb and Bending2.nb)

Another computations of bending constructions are given by Pat Hooper. See `http://merganser.math.gvsu.edu/~david/~reed03/~projects/hooper/` containing an applet of bendings with parameters. (This was a student project in "Mathematical Graphics: Introduction to Java" in the MSRI Summer School - Reed College, July 13 - July 26, 2003. `http://www.math.ubc.ca/~cass/msri-summer-school/`)

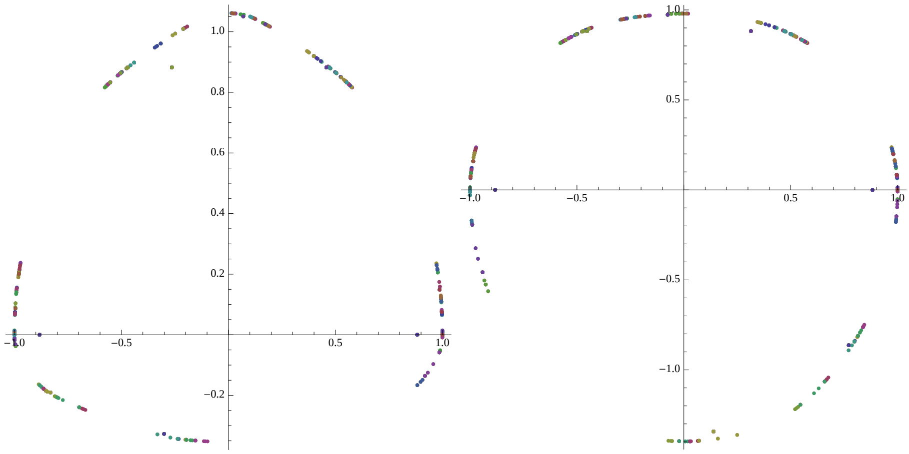

Fig. 8.5 The orbits of bent real projective structures.

8.1.1.4 *Projective triangle reflection groups due to Kac and Vinberg*

Next, we discuss the examples due to Kac and Vinberg (1967). These examples provided the first class of real projective 2-orbifolds and surfaces that are properly convex but not hyperbolic.

Consider a hyperbolic triangle reflection group. $\mathbb{H}^2$ contains a hyperbolic triangle with vertices v_1, v_2, v_3 with respective angles $\pi/p, \pi/q, \pi/r$ satisfying $1/p + 1/q + 1/r < 1$. Let R_1, R_2, and R_3 denote the projective reflections at the edges opposite v_1, v_2, and v_3 respectively. Then we obtain

$$(R_1R_2)^r = \mathrm{I}, (R_2R_3)^p = \mathrm{I}, \text{ and } (R_3R_1)^q = \mathrm{I}. \tag{8.1}$$

A triangle determines the sides of the reflections. We choose the reflection points p_1, p_2, p_3 for the sides e_1, e_2, e_3 respectively. Call the resulting reflections R_1, R_2, and R_3 respectively. They need to satisfy the relations 8.1. Putting the vertices v_1, v_2, and v_3 to $[1, 0, 0], [0, 1, 0]$, and $[0, 0, 1]$ respectively, we obtain the matrices of R_1, R_2, and R_3 as below:

$$R_1 = \begin{pmatrix} -1 & 0 & 0 \\ 2b_1 & 1 & 0 \\ 2c_1 & 0 & 1 \end{pmatrix}, R_2 = \begin{pmatrix} 1 & 2a_2 & 0 \\ 0 & -1 & 0 \\ 0 & 2c_2 & 1 \end{pmatrix}, \text{ and } R_3 = \begin{pmatrix} 1 & 0 & 2a_3 \\ 0 & 1 & 2b_3 \\ 0 & 0 & -1 \end{pmatrix} \tag{8.2}$$

where we have $p_1 = (-1, b_1, c_1), p_2 = (a_2, -1, c_2)$, and $p_3 = (a_3, b_3, -1)$.

The necessary and sufficient condition for R_1R_2 to be of order r for $r \geq 2$ is that

$$4a_2b_1 - 1 = \mathrm{tr}(R_1R_2) = \mathrm{tr}(R_1R_2)^{-1} = 1 + 2\cos 2\pi/r \text{ if } r > 2$$

and $a_2 = 0, b_1 = 0$ for $r = 2$. Thus, we obtain

$$4a_2b_1 = 2 + 2\cos 2\pi/r \text{ if } r > 2 \ \text{ or } a_2 = 0, b_1 = 0 \text{ if } r = 2, \tag{8.3}$$

$$4b_3c_2 = 2 + 2\cos 2\pi/p \text{ if } p > 2 \ \text{ or } b_3 = 0, c_2 = 0 \text{ if } q = 2, \text{ and} \tag{8.4}$$

$$4a_3c_1 = 2 + 2\cos 2\pi/q \text{ if } q > 2 \ \text{ or } a_3 = 0, c_1 = 0 \text{ if } r = 2. \tag{8.5}$$

From this, we obtain that if $p, q, r > 2$, then there is a one-parameter space of solutions of the above equations. This gives us a one-parameter space of real projective structures on the disk-orbifold with corner-reflectors of orders p, q, r. We mention that a single parameter value corresponds to the hyperbolic structure (Vinberg, 1971; Kac and Vinberg, 1967).

If any of p, q, r is 2, then there is just one solution. This corresponds to the hyperbolic structure. We computed some examples in TrianglegroupProj.nb and TrianglegroupProj2.nb. See Figure 8.6 for developing images.

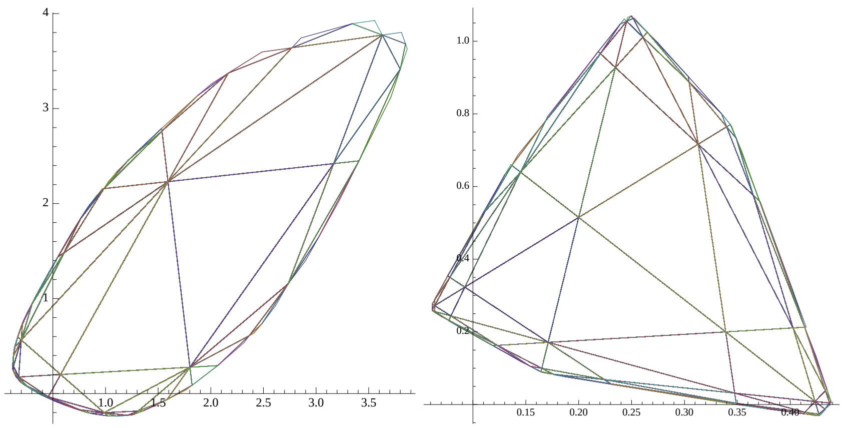

Fig. 8.6 The developing images of two triangle reflection 2-orbifolds of order $(3, 5, 5)$ and $(3, 3, 4)$.

8.2 A survey of real projective structures on surfaces of negative Euler characteristic.

In this section, we sketch some histories of real projective structures.

Historically, Cartan (1924) defined projectively flat structures or real projective structures on manifolds as structures that are "geodesically Euclidean but with no metrics". More precisely, a projectively flat structure on a manifold is given as a torsion-free projectively flat affine connection. "Projectively flat" here means that the connection has same geodesics structures as Euclidean metrics up to reparametrizations.

Later Ehresmann [Pradines (2007)] and Thurston (1977) identified this structure as being a maximal atlas of charts to $\mathbb{RP}^n$ with transition maps in $\mathbb{PGL}(n+1, \mathbb{R})$; that is, it is a geometric structure modeled on $(\mathbb{RP}^n, \mathbb{PGL}(n+1, \mathbb{R}))$. (For an introduction, see the article [Sullivan and Thurston (1983)].)

Kuiper (1954) first studied the convex real projective structures on closed surfaces and showed that they are either a real projective sphere, a real projective plane, a torus or a Klein bottle that is a quotient space of an open triangular domain in $\mathbb{RP}^2$ or is a quotient surface of genus $g, g > 1$ of a properly and strictly convex open domain in $\mathbb{RP}^2$ by a discrete group of projective automorphisms. Benzecri (1960) later generalized this to n-dimensional convex real projective manifolds.

Koszul (1965) showed that convexity is preserved for a closed convex real projective manifold if one deformed the projective structures by a sufficently small amount.

As shown above, Kac and Vinberg (1967) were first to find examples of convex projective surfaces that are not hyperbolic by deforming. The examples are based on Coxeter groups. (See Section 8.1.1.)

Kobayashi (1984) studied metrics on projective manifolds: Given a connected real projective manifold$M$, he considers projective maps

$$l \subset \mathbb{RP}^1 \to M$$

from a bounded interval l and take maximal ones. Using the Hilbert metric of l, he defines the Kobayashi metric. Kobayashi metric is a metric if and only if M has no complete real lines if and only if M is projectively isomorphic to Ω/Γ where Ω is a properly convex domain in $\mathbb{RP}^n$.

In this case, the Kobayashi metric is Finsler and a Hilbert metric given by

$$d(p, q) = |\log[o, s, q, p]|$$

for $p, q \in \Omega$ and o and s are end points of the maximal line containing p, q and o, q separates p, s (See Section 3.1.4.) If $\Omega = \mathbb{H}^n$, the metric is the standard hyperbolic metric. (See Figure 8.7.)

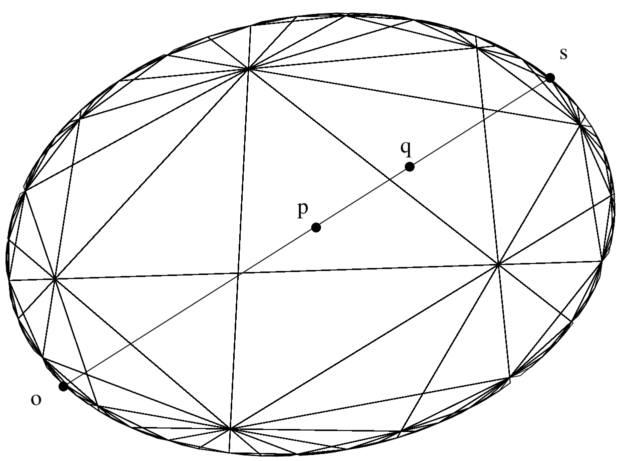

Fig. 8.7 The figure illustrating the cross ratios and the Hilbert metric. The boundary is conic here so that the metric is really a hyperbolic one.

8.2.1 *Topological work*

Nagano and Yagi (1974) classified affine structures on tori, and Goldman (1977) classified projective structures on annuli with geodesic boundary in his senior thesis (Sullivan and Thurston, 1983).

There is a construction called *grafting*: On a closed orientable convex real projective surface of negative Euler characteristic, an essential simple closed curve is homotopic to a simple closed projective geodesic. (See the article [Choi (1994a)] for details.) We cut along the geodesic and complete it to obtain a surface with two new geodesic boundary components. We paste by projective maps the above annuli with principal geodesic boundary to the boundary components with conjugate holonomies. That is, one can insert this type of annuli into a closed convex projective surface to obtain non-convex projective surfaces.

The convex decomposition theorem [Choi (1994a,b)] shows that a closed orientable real projective surface of negative Euler characteristic can be constructed from a closed convex orientable convex real projective surface of negative Euler characteristic by grafting.

Goldman (1990) classified convex projective structures on closed orientable surfaces. Let Σ be a closed orientable surface of genus $g > 1$ and let $\mathcal{CD}(\Sigma)$ denote the deformation space of convex real projective structures on Σ. Then $\mathcal{CD}(\Sigma)$ is homeomorphic to an open cell of dimension $16g - 16 = -8\chi(\Sigma)$. He gave an explicit parameterization to construct back any real projective surface diffeomorphic to Σ. This and Theorem 8.1.2 imply Theorem 8.1.3. Here, the classification is a constructive one.

8.2.2 *The gauge theory and projective structures.*

Atiyah and Bott (1983) studied self-dual connections on surfaces. Corlette (1988) showed that the space of flat connections for manifolds can be realized as the space of harmonic maps to certain symmetric space bundles.

8.2.3 *Hitchin's conjecture and the generalizations.*

Let G be the adjoint group of the split real form of a complex simple group. Hitchin (1992) used the Higgs fields on principal G-bundles over surfaces to obtain parametrizations of flat G-connections over surfaces.

Let Σ be a closed 2-orbifold of negative Euler characteristic. Recall from Chapter 6, the space of homomorphisms

$$\mathbf{Hom}(\pi_1(\Sigma), G)/G.$$

We denote by $\mathbf{Hom}^+(\pi_1(\Sigma), G)$ the subspace of representations which act completely reducibly on Lie algebra of G. It includes the subspace of irreducible representations. (A representation acts *completely reducibly* if every invariant subspace has a complementary invariant subspace. See [Hitchin (1992)] and [Corlette (1988)].)

A Higgs bundle is a pair (V, Φ) where V is a holomorphic vector bundle over a fixed Riemann surface Σ and Φ is a holomorphic section of $\mathrm{End}V \otimes K$ where K is the canonical line bundle. A Teichmüller space $\mathcal{T}(\Sigma)$ is mapped locally homeomorphically by hol to a component of the space

$$\mathbf{Hom}^+(\pi_1(\Sigma), \mathbb{PSL}(2, \mathbb{R}))/\mathbb{PSL}(2, \mathbb{R})$$

of conjugacy classes of Fuchsian discrete faithful irreducible representations by Theorem 6.2.2. A hyperbolic surface naturally corresponds to a conjugacy class of a discrete faithful and irreducible representation $\Gamma \to \mathbb{PSL}(2, \mathbb{R})$ for its fundamental group Γ. Thus, hol is a homeomorphism to the component.

The *Hitchin-Teichmüller component* is a component of

$$\mathbf{Hom}^+(\pi_1(\Sigma), G)/G$$

containing the compositions of form

$$\pi_1(\Sigma) \to \Gamma \to \mathbb{PSL}(2, \mathbb{R}) \to G. \tag{8.6}$$

where the first map is a Fuchsian representation and the second map is the natural irreducible representation $\mathbb{PSL}(2, \mathbb{R}) \to G$ of Kostant. (See Section 4 of [Hitchin (1992)].)

To find a flat connection on a given Higgs bundle, we solve for a unitary connection A

$$F_A + [\Phi, \Phi^*] = 0$$

given a holomorphic section $\Phi \in \mathrm{End}V \otimes K$. The theory of holomorphic sections of holomorphic bundles shows that the Hitchin-Teichmüller component is homeomorphic to an open cell of dimension $(2g-2)\dim G^r$.

Now we restrict our attention to $\mathbb{PGL}(n, \mathbb{R})$. For $n > 2$, Hitchin proved that

$$\mathbf{Hom}^+(\pi_1(\Sigma), \mathbb{PGL}(n, \mathbb{R}))/\mathbb{PGL}(n, \mathbb{R})$$

has three connected components if n is odd and six components if n is even.

A Fuchsian representation is a representation $\pi_1(\Sigma) \to \mathbb{PSL}(2, \mathbb{R})$ with image Γ such that $\mathbb{H}^2/\Gamma$ is homeomorphic to Σ. $\mathbb{PSL}(2, \mathbb{R})$ can be identified as an irreducible subgroup of $\mathbb{PSL}(n, \mathbb{R})$.

A Hitchin representation in $\mathbb{PSL}(n, \mathbb{R})$ is a representation which deforms to a Fuchsian representation, i.e., the ones of form

$$\Gamma \to \mathbb{PSL}(2, \mathbb{R}) \to \mathbb{PSL}(n, \mathbb{R}), \tag{8.7}$$

i.e., those in the Hitchin-Teichmüller component.

A convex projective surface is of form Ω/Γ. Hence, there is a representation $\pi_1(\Sigma) \to \Gamma$ determined only up to conjugation by $\mathbb{PGL}(3, \mathbb{R})$. This gives us a map

$$\mathrm{hol} : \mathcal{CD}(\Sigma) \to \mathbf{Hom}(\pi_1(\Sigma), \mathbb{PGL}(3, \mathbb{R}))/\mathbb{PGL}(3, \mathbb{R}).$$

This map was known to be a local-homeomorphism by Ehresmann and Thurston as in Section 6.2.3.2 and is injective to an open subset as shown by Goldman (1990).

Recall that $\mathbb{PGL}(3,\mathbb{R})$, $\mathbb{PSL}(3,\mathbb{R})$, and $\mathbb{SL}(3,\mathbb{R})$ are isomorphic to one another. When Σ is orientable, we obtain a local homeomorphism

$$\mathrm{hol} : \mathcal{CD}(\Sigma) \to \mathbf{Hom}(\pi_1(\Sigma), \mathbb{SL}(3,\mathbb{R}))/\mathbb{SL}(3,\mathbb{R}).$$

The map is in fact a homeomorphism onto the Hitchin-Teichmüller component as shown by Choi and Goldman (1997). (See Section 6.2.3.)

This result was naturally but unexpectedly extended in the early 2000s to the higher-Teichmüller theory developed by Labourie (2006) and Burger, Iozzi, Labourie, Wienhard (2005); however, we will not elaborate on this rather large and rapidly growing topic.

8.2.4 *Group theory and representations*

As stated earlier, Benzecri (1960), Kac and Vinberg (1967), and Koszul (1965) started to study the deformations of representations $\Gamma \to \mathbb{PGL}(n+1,\mathbb{R})$ from the discrete faithful representation $\Gamma \to \mathbb{PSO}(n,1)$ corresponding to hyperbolic manifolds. There is a well-known deformation due to Thurston called *bending* for projective and conformally flat structures: Given a totally-geodesic submanifold S of codimension one in a convex real projective manifold M so that the holonomy homomorphism h restricts in $\pi_1(S)$ to one fixing a point in $\mathbb{RP}^n$, we have a centralizing element η in $\mathbb{PGL}(n+1,\mathbb{R})$ in a one-parameter family of such elements. We can remove S from M and complete it to obtain a manifold with two copies of S as boundary component. Using the centralizing elements, we can re-glue in one-parameter ways. (See Section 8.1.1.3.)

Johnson and Millson (1987) found that certain hyperbolic manifolds have deformation spaces of projective structures that are singular by studying one with many totally geodesic submanifolds codimension one meeting transversally. (They also worked out this for conformally flat structures.)

An element γ of $\mathbb{GL}(m,\mathbb{R})$ is *proximal* if there is an eigenvalue of multiplicity one which is of largest modulus among eigenvalues. Recall that γ is positive proximal if γ is proximal and the largest modulus eigenvalue is positive. A subgroup Γ of $\mathbb{GL}(m,\mathbb{R})$ is *positive proximal* if every proximal element is positive proximal. (This means that it has a pair of an attracting and a repelling fixed point in $\mathbb{RP}^{m-1}$.) We say that Γ *divides* Ω if its image in $\mathbb{PGL}(m,\mathbb{R})$ acts on a properly convex domain $\Omega \subset \mathbb{RP}^{m-1}$ properly discontinuously but not necessarily freely so that the quotient space is compact.

Theorem 8.2.1. *Let Γ be an irreducible torsion-free subgroup of $\mathbb{GL}(m,\mathbb{R})$. Then Γ divides a strictly convex domain Ω if and only if Γ is positive proximal and discrete. If Ω is not a domain bounded by a conic, then Γ maps to a Zariski dense subgroup in $\mathbb{PGL}(m,\mathbb{R})$ under the projection $\mathbb{GL}(m,\mathbb{R}) \to \mathbb{PGL}(m,\mathbb{R})$.*

This is proved by Benoist (2000).

The recent work of Benoist (papers "Convex divisibles I-IV") proves the following theorem. (See also the survey article [Benoist (2001)].)

Theorem 8.2.2. *Let* Γ *be a discrete torsion-free subgroup of* $\mathbb{GL}(m, \mathbb{R})$ *dividing an open convex domain* Ω *in* $\mathbb{RP}^{m-1}$*. Let* C *be the corresponding cone on* $\mathbb{R}^m$*. The projectivization* Γ_0 *of* Γ *is the isomorphic image group in* $\mathbb{PGL}(m, \mathbb{R})$*. Then the following holds*

- *One of the following is true exclusively:*
  - C *is a product, i.e., a product of irreducible cones in subspaces,*
  - C *is homogeneous; i.e.,* Γ_0 *is Zariski dense in a copy of* $\mathbb{PSO}(1, 1-m)$ *in* $\mathbb{PGL}(m, \mathbb{R})$ *acting on* Ω *transitively,*
  - Γ_0 *is Zariski dense in* $\mathbb{PGL}(m, \mathbb{R})$*.*
- *If the virtual center of* Γ_0 *is trivial, i.e., every finite-index subgroup of* Γ_0 *has a trivial center, then*

  $$E_{\Gamma_0} = \{\rho \in H_{\Gamma_0} | \text{ The image of } \rho \text{ divides a convex open domain in } \mathbb{RP}^{m-1}\}$$

  is closed in

  $$H_{\Gamma_0} := \mathbf{Hom}(\Gamma_0, \mathbb{PGL}(m, \mathbb{R})).$$

  The openness was obtained by Koszul (1965)*.*
- *Let* Γ_0 *be as above. Then the following conditions are equivalent:*
  - Ω *is strictly convex.*
  - $\mathrm{bd}\Omega$ *is* C^1*.*
  - Γ *is a hyperbolic group.*
  - *The geodesic flow on* Ω/Γ *is Anosov.*

Benzecri (1960) showed that the boundary of Ω is C^1 or is an ellipsoid for closed convex projective manifolds. (See also [Goldman (1988)].)

This completes our survey. However, there were further developments of significance by Cooper, Long, and Thistlethwaite (2007, 2006) which we cannot cover here.

8.3 Real projective structures on 2-orbifolds of negative Euler characteristic.

We begin the study of the deformation spaces of real projective structures on 2-orbifolds.

Recall the orbifold Euler characteristic of orbifolds, a signed sum of the number of open cells with weights given by 1 divided by the orders of groups associated to the open cells. Let Σ be a connected compact 2-orbifold with $\chi(\Sigma) < 0$. The subspace of the deformation space $\mathbb{RP}^2(\Sigma)$ of $\mathbb{RP}^2$-structures on Σ corresponding to convex ones is denoted by $\mathcal{CD}(\Sigma)$ and the closed subspace corresponding to hyperbolic projective structures is denoted by $\mathcal{T}(\Sigma)$, identified as the Teichmüller space of Σ as defined by Thurston (1977). Then we see that $\mathcal{T}(\Sigma)$ is a subspace of $\mathcal{CD}(\Sigma)$, and $\mathcal{CD}(\Sigma)$ is an open subset of $\mathbb{RP}^2(\Sigma)$.

Theorem 8.3.1 (Choi, Goldman). *Let Σ be a compact 2-orbifold with $\chi(\Sigma) < 0$ and $\partial\Sigma = \emptyset$. Then the deformation space $\mathcal{CD}(\Sigma)$ of convex $\mathbb{RP}^2$-structures on Σ is homeomorphic to an open cell of dimension*

$$-8\chi(X_\Sigma) + (6k_c - 2b_c) + (3k_r - b_r)$$

where X_Σ is the underlying space of Σ, k_c is the number of cone-points, k_r the number of corner-reflectors, b_c the number of cone-points of order two, and b_r the number of corner-reflectors of order two.

Let us denote by $C_{\mathcal{T}}(\Sigma)$ the unique component of

$$\mathbf{Hom}(\pi_1(\Sigma), \mathbb{PGL}(3,\mathbb{R}))$$

containing the holonomy homomorphisms of hyperbolic $\mathbb{RP}^2$-structures on Σ. Then $C_{\mathcal{T}}(\Sigma)$ is also a component of

$$\mathbf{Hom}(\pi_1(\Sigma), \mathbb{PGL}(3,\mathbb{R}))$$

in the part

$$\mathbf{Hom}(\pi_1(\Sigma), \mathbb{PGL}(3,\mathbb{R}))^+$$

where $\mathbb{PGL}(3,\mathbb{R})$ acts properly, and $C_{\mathcal{T}}/\mathbb{PGL}(3,\mathbb{R})$ is the *Hitchin-Teichmüller component* as described by Hitchin (1992). We prove:

Theorem 8.3.2. *Let Σ be a closed 2-orbifold with negative Euler characteristic. Then*

$$\mathrm{hol} : \mathcal{CD}(\Sigma) \to C_{\mathcal{T}}(\Sigma)/\mathbb{PGL}(3,\mathbb{R})$$

is a homeomorphism, and $C_{\mathcal{T}}(\Sigma)$ consists of discrete faithful representations of $\pi_1(\Sigma)$.

Corollary 8.3.3. *The Hitchin-Teichmüller component $C_{\mathcal{T}}(\Sigma)/\mathbb{PGL}(3,\mathbb{R})$ is homeomorphic to an open cell of the dimension as above in Theorem 8.3.1.*

We study small 2-orbifolds with rigid hyperbolic structures; i.e., ones with the Teichmüller spaces consisting of singletons.

Corollary 8.3.4.

- *The sphere Σ with cone-points of order p, q, r satisfying $p \leq q \leq r, 1/p + 1/q + 1/r < 1$ has as its Teichmüller space a single point.*
  - *If $p = 2$, then so is $\mathcal{CD}(\Sigma)$.*
  - *If $p > 2$, then $\mathcal{CD}(\Sigma)$ is homeomorphic to $\mathbb{R}^2$.*

- *Let Σ be a 2-orbifold whose underlying space is a disk and with one cone point of order p and a corner-reflector of order q so that $1/p + 1/2q < 1/2$ has as its Teichmüller space a single point.*
  - *If $q = 2$, then so is $\mathcal{CD}(\Sigma)$.*
  - *If $q > 2$, then $\mathcal{CD}(\Sigma)$ is homeomorphic to $\mathbb{R}$.*
- *Let Σ be a 2-orbifold whose underlying space is a disk and with three corner-reflectors of order $p \leq q \leq r$, $1/p + 1/q + 1/r < 1$. Then $\mathcal{T}(\Sigma)$ is a single point.*
  - *If $p = 2$, then so is $\mathcal{CD}(\Sigma)$.*
  - *If $p > 2$, then $\mathcal{CD}(\Sigma)$ is homeomorphic to $\mathbb{R}$.*

8.3.1 ***Real projective 2-orbifolds and the Hitchin-Teichmüller components***

From now on, we are concerned with explaining the proof of Theorem 8.3.1 but we will not prove it actually.

By an $\mathbb{RP}^2$-*structure* or *projectively flat structure* on a 2-orbifold Σ we mean an $(\mathbb{RP}^2, \mathbb{PGL}(3, \mathbb{R}))$-structure on Σ. From now on, we look at $\mathbb{RP}^2$-orbifolds, that is, 2-orbifolds with $\mathbb{RP}^2$-structures. Here, we require that the boundary components of a surface with a real projective structure are always principal geodesic.

We define the deformation spaces of $\mathbb{RP}^2$-structures on 2-orbifolds, describe local properties, and define convex $\mathbb{RP}^2$-structures (when the 2-orbifolds are boundaryless).

We discuss the relationship between the $\mathbb{RP}^2$-structures and holonomy representations. First, we deduce that the deformation space is Hausdorff from the corresponding property of the holonomy representation variety. Next, we discuss convex $\mathbb{RP}^2$-structures. We show that the deformation space of convex $\mathbb{RP}^2$-structures on a 2-orbifold is an open subset of the full deformation space. We identify the deformation space of convex $\mathbb{RP}^2$-structures on a 2-orbifold with a subset of the space of conjugacy classes of representations of its fundamental group using the above relationship.

8.3.1.1 *Types of Singularities*

Recall that an automorphism of $\mathbb{RP}^2$ is a *reflection* if its matrix is conjugate to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

A reflection has a line of fixed points and an isolated fixed point, which is said to be the *reflection point*. An automorphism of $\mathbb{RP}^2$ is said to be a *rotation of order*

n, $n = 2, 3, \dots$, if its matrix is conjugate to

$$\begin{bmatrix} \cos\frac{2\pi}{n} & -\sin\frac{2\pi}{n} & 0 \\ \sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

A rotation has a unique isolated fixed point, called a *rotation point*, and an invariant line. A one-parameter family of invariant ellipses fills the complement in $\mathbb{RP}^2$ of the rotation point and the invariant line. A rotation of order two is a reflection also and conversely.

For $\mathbb{RP}^2$-orbifolds, the singular points have the neighborhoods with model open sets and finite group actions corresponding to one of the following:

(i) A mirror point: An open disk in $\mathbb{RP}^2$ meeting a line of fixed points of a reflection.
(ii) A cone-point of order n: An open disk in $\mathbb{RP}^2$ containing a rotation point of the rotation of order n.
(iii) A corner-reflector of order n: An open disk in $\mathbb{RP}^2$ containing the intersection point of the lines of fixed points of two reflections g_1 and g_2 generating a dihedral group of order $2n$.

8.3.1.2 *The deformation spaces and holonomy*

We recall some facts from the general (G, X)-structures. (See Chapter 6 for details) We define the deformation space $\mathbb{RP}^2(\Sigma)$ of $\mathbb{RP}^2$-structures on a connected 2-orbifold Σ with principal geodesic boundary as follows (assuming Σ is connected and has empty boundary): Give the C^1-topology to the set $\hat{\mathcal{S}}(\Sigma)$ of all developing pairs $(\mathbf{dev}, h)$ on $\tilde{\Sigma}$. Two pairs $(\mathbf{dev}, h)$ and $(\mathbf{dev}', h')$ are *equivalent under isotopy* if there exists a self-diffeomorphism f of the universal cover $\tilde{\Sigma}$ of Σ commuting with the deck transformations so that $\mathbf{dev}' = \mathbf{dev} \circ f$ and $h' = h$. (We can easily show that $\hat{\mathcal{S}}(\Sigma)$ is homeomorphic to $\mathcal{S}(\Sigma)$ in Section 6.2.1.) We denote by $\mathbb{RP}^{2*}(\Sigma)$ the space of equivalence classes with the quotient topology.

The pairs $(\mathbf{dev}, h)$ and $(\mathbf{dev}', h')$ are equivalent under the $\mathbb{PGL}(3, \mathbb{R})$-action if there exists an element g of $\mathbb{PGL}(3, \mathbb{R})$ so that $\mathbf{dev}' = g \circ \mathbf{dev}$ and $h'(\cdot) = gh(\cdot)g^{-1}$. The quotient space of $\mathbb{RP}^{2*}(\Sigma)$ under the $\mathbb{PGL}(3, \mathbb{R})$-equivalence relation is denote by $\mathbb{RP}^2(\Sigma)$.

Another interpretation of the deformation space is to consider all $\mathbb{RP}^2$-structures on Σ and quotient by the isotopies. One can easily obtain a one-to-one correspondence between the above two spaces.

If two $\mathbb{RP}^2$-structures are distinct up to isotopy, they are *isotopically distinct*. Isotopically distinct $\mathbb{RP}^2$-structures represent different points in the deformation spaces. An example is a pair of $\mathbb{RP}^2$-orbifolds with non-conjugate holonomy homomorphisms (see [Choi (2004)] for details).

By forgetting $\mathbf{dev}$ from the pair $(\mathbf{dev}, h)$, we obtain an induced map

$$\mathrm{hol}' : \mathbb{RP}^{2*}(\Sigma) \to \mathbf{Hom}(\pi_1(\Sigma), \mathbb{PGL}(3, \mathbb{R}))$$

to the space of homomorphisms of $\pi_1(\Sigma)$ since the isotopy does not change the holonomy homomorphism.

Since Σ is a compact 2-orbifold, we see that $\pi_1(\Sigma)$ is a finitely presented group by Corollary 4.7.2. From now on, we denote

$$H(\Sigma) = \mathbf{Hom}(\pi_1(\Sigma), \mathbb{PGL}(3, \mathbb{R}))$$

for the $\mathbb{R}$-algebraic subset of $\mathbb{PGL}(3, \mathbb{R})^n$ satisfying the relations corresponding to the relations of the presentation of $\pi_1(\Sigma)$ where n is the number of the generators of $\pi_1(\Sigma)$.

Choi (2004) shows that the map $\mathcal{H}'$ is a local homeomorphism since $\pi_1(\Sigma)$ is finitely presented. (See Section 6.2.3 for detail.)

Let U^n denote the open subset of $\mathbb{PGL}(3, \mathbb{R})^n$ consisting of $(X_1, \ldots, X_n)$ such that no line in $\mathbb{R}^3$ is simultaneously invariant under $X_1, \ldots, X_n$. The $\mathbb{PGL}(3, \mathbb{R})$-action is proper and free on the set

$$U(\Sigma) := H(\Sigma) \cap U^n$$

(Goldman, 1990).

Theorem 8.3.5. *Let Σ be a connected closed 2-orbifold with $\chi(\Sigma) < 0$. Then $\mathbb{RP}^2(\Sigma)$ has the structure of Hausdorff real analytic variety modeled on $U(\Sigma)/\mathbb{PGL}(3, \mathbb{R})$, and the induced map*

$$\mathrm{hol} : \mathbb{RP}^2(\Sigma) \to U(\Sigma)/\mathbb{PGL}(3, \mathbb{R})$$

is a homeomorphism onto an open subset.

8.3.2 *Understanding the deformation space of real projective structures*

8.3.2.1 *The deformation space of 2-orbifolds*

Here, we discuss how to use the above facts to study the deformation space of a given 2-orbifold, in a manner parallel to the Teichmüller space cases. We do not provide the complete proofs here. (See [Choi and Goldman (2005)] for more details.)

Recall that a principal geodesic is a geodesic that lifts to an arc developing to a straight line connecting an attracting fixed point and a repelling fixed point of its holonomy automorphism. A full 1-orbifold is *principal* if an inverse image of it in the universal cover develop into a straight line joining an attracting fixed point and a repelling fixed point of the composition of holonomies of the two reflections.

Recall that the projective invariant of a principal closed geodesic c of a real projective 2-orbifold is given by a point in the domain $D(c)$. The projective invariant of a principal full 1-orbifold c is given as the cross-ratio of the four points in its lift given by the two reflection points and the end points. Hence, we let $D(c)$ be identified with $\mathbb{R}^+$ by taking the absolute values of the logarithms of the cross-ratios.

As in Chapter 7, we can decompose an orientable compact convex real projective 2-orbifold Σ with $\chi(\Sigma) < 0$ and principal geodesic boundary by a mutually disjoint family of essential simple closed principal geodesics or geodesic principal full 1-orbifolds $c_1, \ldots, c_n$ so that the orbifold Euler characteristic of the completion of each component of $\Sigma - c_1 - \cdots - c_n$ is negative. The completed 2-orbifolds have all principal geodesic boundary. Moreover, these 2-orbifolds are *elementary* in the sense that we cannot apply the above steps any more.

8.3.2.2 *Geometric constructions.*

To understand this, let S be a 2-orbifold with principal boundary components. The pasting map f is defined on open neighborhood U of the union of the associated boundary components in an ambient open 2-orbifold S' where f satisfies the equation $\tilde{f} \circ \vartheta = \vartheta' \circ \tilde{f}$ where $\tilde{f}$ is a lift of f defined on $\tilde{U}$ the inverse image and ϑ and ϑ' are respective deck transformations acting on components of the inverse images in $\tilde{S}'$ of boundary components of S to be pasted by $\tilde{f}$.

In the real projective structures, it is sufficient that f is a locally projective map in some ambient real projective surface, the boundary components are principal, and ϑ and ϑ' have the same projective invariants described above.

Actually, we can think of the above condition as $f \circ h(c) = h(c') \circ f$ where $h(c)$ and $h(c')$ are holonomy of the closed curves c and c' and the boundary components are principal: The equation is necessary since if the pasting succeeded, then the equation holds. The additional principal geodesic condition is then the sufficient condition.

The geodesics and the full 1-orbifolds are principal always when we are splitting and pasting. (Actually, we need this condition so that the result of pasting is properly convex when the initial real projective 2-orbifolds are properly convex. See [Choi and Goldman (2005)] or [Goldman (1990)])

We describe how to construct convex real projective structures on a larger 2-orbifold from smaller ones. Recall the type of topological constructions with 1-orbifolds from Chapter 7. Suppose that they are boundary components of 2-orbifolds whose components have negative Euler characteristics.

(A)(I) Pasting or crosscapping along a simple closed curve.
(A)(II) Silvering or folding along a simple closed curve.
(B)(I) Pasting along two full 1-orbifolds.
(B)(II) Silvering or folding along a full 1-orbifold.

Now we suppose that the simple closed curves and 1-orbifolds are geodesic and try to obtain geometric versions of the above.

Suppose that the involved 1-orbifolds are geodesic boundary components of a properly convex real projective 2-orbifold with principal geodesic boundary.

(A)(I) For pasting two closed geodesics, we have an $\mathbb{R}^2$-amount of real projective automorphisms to do this. They would create convex real projective structures inequivalent in the deformation spaces. (Here the invariants of two closed geodesics have to be the same.) The possible projective automorphisms B satisfy $AB = BA'$ where A and A' are holonomies of the two closed geodesics. The equation becomes $AB' = B'A$ since we can define $A' = P^{-1}AP$ for $B' = BP^{-1}$ and an invertible P. The solution space of B' is the space of commuting matrices of A and hence is parametrized by $\mathbb{R}^2$.

(A)(I) For cross-capping, we have a unique pasting map. The map must be a real projective automorphism that preserves the orientation of the boundary component but reverses the normal direction and whose second power is the holonomy of the boundary component. The equation is $AB = BA$ and $B^2 = A$ where A is the holonomy of the principal boundary component and B is the pasting map. There is no condition on A other than its positive hyperbolicity. B has eigenvalues that are square roots of those of A and one of middle absolute value has a negative eigenvalue. B is determined since A is positive hyperbolic.

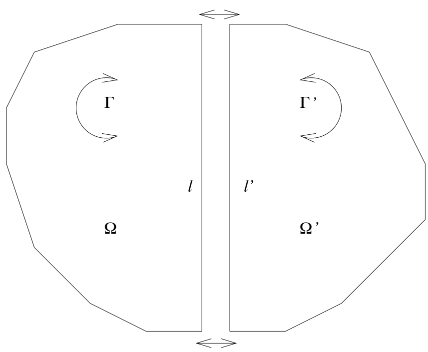

Fig. 8.8 (A)(I) Pasting of two closed principal geodesics

(A)(II)(i) For folding a closed geodesics, we have an $\mathbb{R}$-amount of real projective automorphism f to do this. They would create convex real projective structures inequivalent in the deformation space. The choice depends on the choice of two fixed points of the pasting map. The equation is $AB = BA^{-1}$ and $B^2 = \mathrm{I}$ and B fixes a point p of the principal geodesic l invariant under A. B is uniquely determined by the fixed point p and vice-versa since B switches the two eigenvectors of A and acts on the eigenspace of A of dimension-one as a reflection. Here, $BAB^{-1} = A^{-1}$ and $A \sim A^{-1}$. Therefore, A has eigenvalues $\lambda, 1, \lambda^{-1}$. This is a restriction on the holonomy type of boundary components that we can do folding on.

(A)(II)(ii) For silvering, we have a unique real projective automorphism of order 2 that reverses the normal direction but fixes the points of the boundary component and commutes with the holonomy of the boundary component. The equation is $AB = BA$ and $B^2 = \mathrm{I}$ and B fixes each point of the principal geodesic of A and acts on an eigenspace of dimension one as a reflection. Then B is a unique reflection.

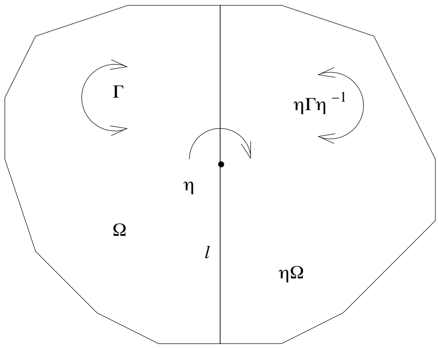

Fig. 8.9 (A)(II)(i) Folding a principal closed geodesic.

(B)(I) For pasting along two geodesic full 1-orbifolds, we have an $\mathbb{R}$-parameter ways to do this, and the invariants of the 2-orbifolds have to be the same: The boundary full 1-orbifolds have holonomy A_1 and A_2 associated with each boundary points, i.e., silvered points where $A_1^2 = A_2^2 = \mathrm{I}$. A_1A_2 represents a closed curve around the full 1-orbifold lifting to a simple closed curve in a double cover of our properly convex real projective 2-orbifold. Hence, it is positive hyperbolic. The equation is $A_iB = BA_i'$ for $i = 1, 2$ and A_i and A_i' are the holonomy elements of the generators of the local groups of the two boundary points of the full 1-orbifolds, acting on the principle geodesics. Moreover, $A_1A_2B = BA_1'A_2'$ since A_1A_2 and $A_1'A_2'$ are corresponding closed paths to become homotopic after pasting. (This corresponds to the cross-ratio invariants of the two full 1-orbifolds being the same.) Since $A_i' = P^{-1}A_iP$ for $i = 1, 2$, the above equation becomes $A_iB' = B'A_i$ and $A_1A_2B' = B'A_1A_2$ for $B' = BP^{-1}$ and $i = 1, 2$. Since A_1A_2 is positive hyperbolic, the solution space is homeomorphic to $\mathbb{R}$ as B' fixes each point of the principal geodesic of A_1A_2 and acts on the eigenspace of A_1A_2 whose corresponding point in $\mathbb{RP}^2$ is not on the geodesic.

(B)(II)(i) For silvering, we have a unique real projective automorphism since there is a unique projective automorphism commuting with the reflections at the end and fixing each point of the boundary component. The equation is $A_iB = BA_i$ and $A_1A_2B = BA_1A_2$ and $B^2 = \mathrm{I}$ and B fixes each point of the principal geodesic fixed by A_1A_2 and acts on an eigenspace of A_1A_2. Here, B is uniquely determined.

(B)(II)(ii) For folding, the full 1-orbifold ends at boundary points. The projective automorphism must send the the full 1-orbifold to itself and make the boundary segments to extend each other where they are sent. There is a unique such automorphism. The equation is $A_1B = BA_2$ and $B^2 = \mathrm{I}$ and B fixes a point of the principal geodesic. Here, B is uniquely determined. (This is similar to (A)(II).)

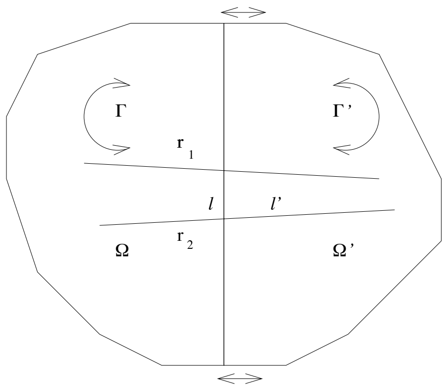

Fig. 8.10 (B)(II)(i) Pasting full 1-orbifolds

8.3.2.3 *Elementary 2-orbifolds and their real projective structures.*

To prove Theorem 8.3.1, we need to study the deformation space of elementary orbifolds and use results in Section 8.3.2.2. The details are in [Choi and Goldman (2005)]. We partially discuss the deformation space of convex real projective structures on elementary orbifolds with principal geodesic boundary. We discuss more about the computational aspects.

8.3.2.4 *A pair-of-pants*

We first discuss a pair-of-pants P. The deformation space was first studied by Goldman (1990). The geodesic boundary components of a convex real projective surface P with principal geodesic boundary are first oriented by a boundary orientation.

Recall that $D(c)$ for a boundary component c of a real projective surface denote the space of invariants (λ, τ) satisfying

$$0 < \lambda < 1 \text{ and } \frac{2}{\sqrt{\lambda}} < \tau < \lambda + \frac{1}{\lambda^2}.$$

Given a hyperbolic automorphism ϑ of $\mathbb{RP}^2$, we have that the invariant for ϑ is given by taking the smallest eigenvalue and the sum of the two other eigenvalues. We define $\mathcal{CD}(\partial P)$ as the product space $\prod_{i=1}^{3} D(c_i)$ where c_i are boundary components of P. Goldman (1990) proved that

$$\mathcal{F} : \mathcal{CD}(P) \to \mathcal{CD}(\partial P)$$

for a pair-of-pants P is a principal $\mathbb{R}^2$-fibration for a pair-of-pants P where $\mathcal{F}$ is given by sending the structure to the invariants of $h(c_i)$ for the boundary components c_1, c_2, c_3 of P.

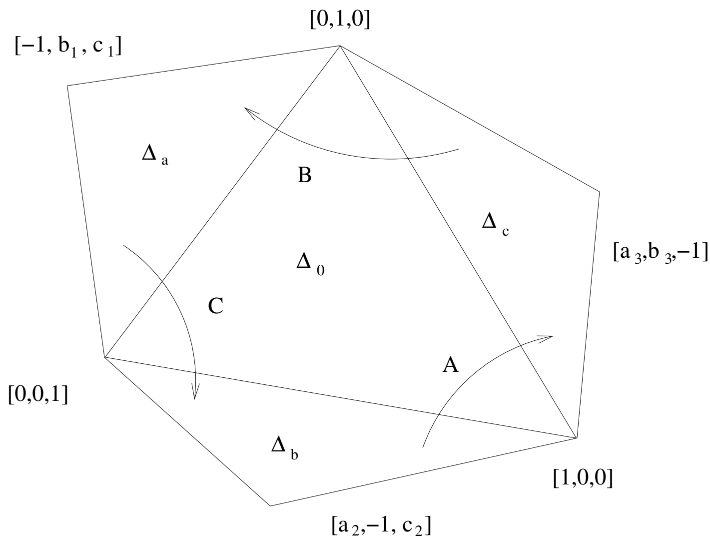

Fig. 8.11 The four adjacent triangles used to understand the convex real projective pair-of-pants

We explain this: Give P an orientation and the induced orientation on ∂P as well. There is a lamination with three leaves that tend to the boundary components in its end and it wraps around each boundary component in the reverse direction to the orientation. We can straighten each leaf so that it is a geodesic. This is accomplished by the fact that P is convex. $P - \partial P$ is a union of two triangles $\mathcal{T}_1$ and $\mathcal{T}_2$ bounded by three lines and vertices removed. In the universal cover $\tilde{P}$, we have a tessellation by these triangles. Under the developing map, each triangle is mapped to a triangle with vertices removed in $\mathbb{RP}^2$. Take one triangle T_0 and adjacent ones T_1, T_2, T_3. Notice that T_0 is in one class of triangles corresponding to $\mathcal{T}_1$ or $\mathcal{T}_2$ and T_1, T_2, and T_3 correspond to the other one.

There exists a deck transformation A sending T_1 to T_2 and B sending T_2 to T_3 and C sending T_3 to T_1. We have $CBA = \mathrm{I}$. In fact, A, B, C correspond to closed curves homotopic to the boundary components in the oriented direction. Since the developing map is a homeomorphism, A, B, and C correspond to elements of $\mathbb{PGL}(3, \mathbb{R})$.

Note the isomorphism $\mathbb{SL}(3, \mathbb{R})$ with $\mathbb{PGL}(3, \mathbb{R})$. We think of A, B, C as matrices of determinant 1 abusing notations.

We can put T_0 to a standard triangle with vertices: $[1, 0, 0]$, $[0, 1, 0]$, and $[0, 0, 1]$ by a projective automorphism and then we obtain:

- T_1 has vertices $[-1, b_1, c_1]$, $[0,1,0]$, and $[0,0,1]$,
- T_2 has vertices $[1,0,0]$, $[a_2, -1, c_2]$, and $[0,0,1]$, and
- T_3 has vertices $[1,0,0]$, $[0,1,0]$, and $[a_3, b_3, -1]$.

Here $b_1, c_1, a_2, c_2, a_3, b_3 > 0$. This position is not canonical. We can still act by transformations with diagonal matrices. Thus, we may assume that $b_1 = 2, c_1 = 2$ without loss of generality. (See Figure 8.11.)

The matrices must be of form

$$A := \begin{bmatrix} \alpha_1 & \alpha_1 a_2 + \gamma_1 c_2 a_3 & \gamma_1 a_3 \\ 0 & -\beta_1 + \gamma_1 b_3 c_2 & \gamma_1 b_3 \\ 0 & -\gamma_1 c_2 & -\gamma_1 \end{bmatrix},$$

$$B := \begin{bmatrix} -\alpha_2 & 0 & -\alpha_2 a_3 \\ -\alpha_2 b1 & \beta_2 & \beta_2 b_3 + \alpha_2 a_3 b_1 \\ \alpha_2 c_1 & 0 & -\gamma_2 + \alpha_2 a_3 c_1 \end{bmatrix}, \text{ and} \tag{8.8}$$

$$C := \begin{bmatrix} -\alpha_3 + \beta_3 a_2 b_1 & \beta_3 a_2 & 0 \\ -\beta_3 b_1 & -\beta_3 & 0 \\ \gamma_3 c_1 + \beta_3 b_1 c_2 & \beta_3 c_2 & \gamma_3 \end{bmatrix}$$

where $\alpha_i, \beta_i, \gamma_i > 0$ are positive real numbers satisfying equation:

$$\alpha_1\alpha_1\alpha_3 = 1, \beta_1\beta_2\beta_3 = 1, \gamma_1\gamma_2\gamma_3 = 1, \alpha_1\beta_1\gamma_1 = 1, \alpha_2\beta_2\gamma_2 = 1, \alpha_3\beta_3\gamma_3 = 1.$$

This follows since the determinants must be 1 and $CBA = \mathrm{I}$.

Solving for $CBA = \mathrm{I}$, we obtain without difficulty: Given $l_1, l_{1,2}, l_2, l_{2,2}, l_3, l_{3,2}$ square roots of smallest positive eigenvalues of A, B, C respectively so that $l_i < l_{i,2}$ for $i = 1,2,3$, we have two parameter solutions for $s > 0, t > 0$:

$$\begin{aligned}
&\alpha_1 = l_1^2, \alpha_2 = \frac{l_3}{l_1 l_2 s}, \alpha_3 = s\frac{l_2}{l_3 l_1}, \\
&\beta_1 = s\frac{l_3}{l_1 l_2}, \beta_2 = l_2^2, \beta_3 = \frac{l_1}{l_2 l_3 s}, \\
&\gamma_1 = \frac{l_2}{l_3 l_1 s}, \gamma_2 = s\frac{l_1}{l_2 l_3}, \gamma_3 = l_3^2, \\
&a_2 = t, a_3 = 2, b_1 = \frac{1}{t}\left(1 + \frac{l_2 l_3}{l_1}\tau_3 s + \frac{l_2^2}{l_1^2}s^2\right), b_3 = 2, \\
&c_1 = \frac{1}{2}\left(1 + \frac{l_1 l_2}{l_3}\tau_2 s + \frac{l_1^2}{l_3^2}s^2\right), \text{ and} \\
&c_2 = \frac{1}{2}\left(1 + \frac{l_3 l_1}{l_2}\tau_1 s + \frac{l_3^2}{l_2^2}s^2\right)
\end{aligned} \tag{8.9}$$

where $\tau_1 = l_{1,2}^2 + \frac{1}{(l_1^2 l_{1,2})^2}, \tau_2 = l_{2,2}^2 + \frac{1}{(l_2^2 l_{2,2})^2}$, and $\tau_3 = l_{3,2}^2 + \frac{1}{(l_3^2 l_{3,2})^2}$ hold. The importance of the solution is that we can choose arbitrary boundary invariants $l_i, l_{i,2}$ for $i = 1,2,3$, there exists two parameter family of solutions parameterized by $s, t > 0$ proving that $\mathcal{F}$ is a principal $\mathbb{R}^2$-bundle projection. (See Triangle10.nb for computations here.)

8.3.2.5 *Small orbifolds*

Let P be an orbifold that is either an annulus with a singularity $p, p \geq 2$, or a disk with singularity $p, q, p > q \geq 2$ and a sphere with singularity p, q, r, $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$. As above, we can divide P into two triangles by segments ending at singular points or winding around components of the boundary in the opposite direction to the boundary orientation. We introduce transformations A, B, and C as above.

In case of the sphere $S_{p,q,r}$ with singularities p, q, r, $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$, we obtain from the paper [Choi and Goldman (2005)] as the solution space parameterized by $s > 0, t > 0$:

$$\begin{aligned}
\alpha_1 =&1, \alpha_2 = \frac{1}{s}, \alpha_3 = s, \\
\beta_1 =&s, \beta_2 = 1, \beta_3 = \frac{1}{s}, \\
\gamma_1 =&\frac{1}{s}, \gamma_2 = s, \gamma_3 = 1 \\
a_2 =&t, a_3 = 2, b_1 = \frac{1}{t}(1 + \tau_3 s + s^2), b_3 = 2, \\
c_1 =&\frac{1}{2}(1 + \tau_2 s + s^2), \text{ and } c_2 = \frac{1}{2}(1 + \tau_1 s + s^2) \\
\text{where } \tau_1 =&2\cos\left(\frac{2\pi}{p}\right), \tau_2 = 2\cos\left(\frac{2\pi}{q}\right), \text{and } \tau_3 = 2\cos\left(\frac{2\pi}{r}\right)
\end{aligned} \tag{8.10}$$

hold.

We do not examin the other cases because of length. The interested readers can download some mathematica files from the authors webpages. See Triangle5graphic.nb,Triangle10graphic.nb, and Triangle10graphicII.nb. We give some examples of the developing images.

8.4 Notes

For computations, one can experiment with various packages that the author and Gye-Seon Lee wrote. Gye-Seon developed from the maple package I wrote. These contain computations where one or more of the cone-point orders are two. One has to be careful about adjusting the coordinates since some points would develop across the line at infinity. This creates problems. But theoretically, a well-chosen affine space would contain the convex domain. These packages will be maintained at `http://mathsci.kaist.ac.kr/~schoi/MSJbook2012.html`.

As a historical note, the closedness of the deformation spaces of real projective structures on closed surfaces of genus ≥ 2 was questioned by Thurston and was given to the author as a doctoral thesis problem in 1985.

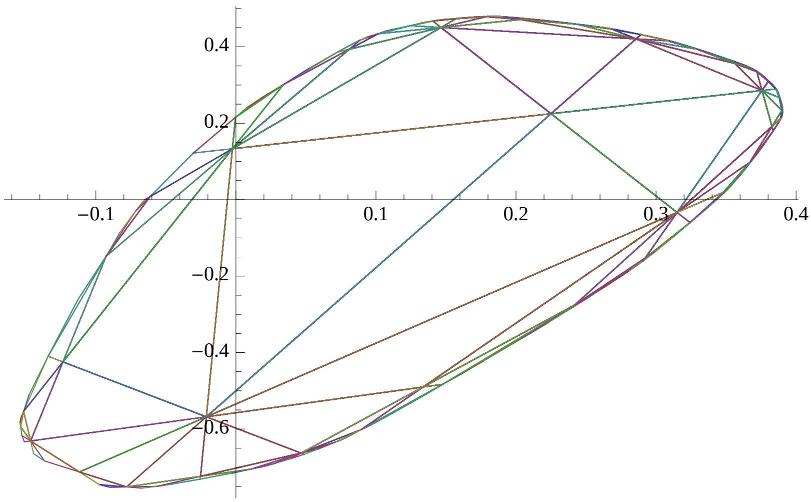

Fig. 8.12 The developing figure of a sphere with cone-points of order 3, 5, 5. See Triangle5graphic.nb

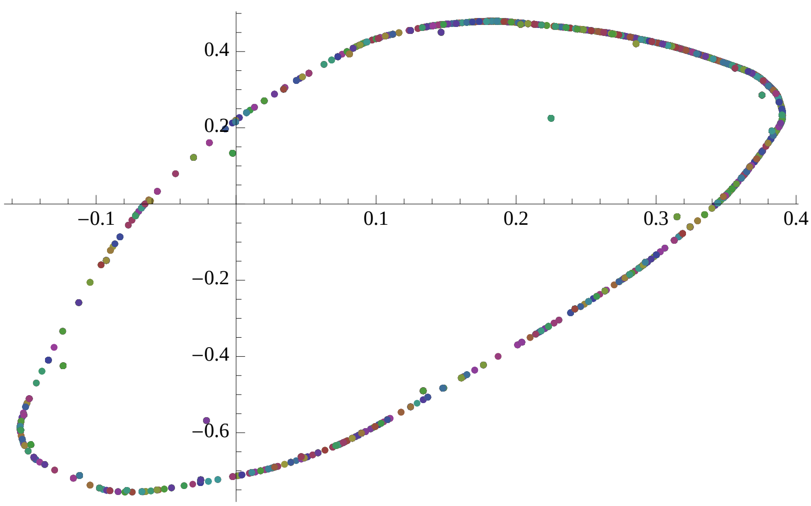

Fig. 8.13 The orbit points of a sphere with cone-points of order 3, 5, 5. See Triangle5graphic.nb

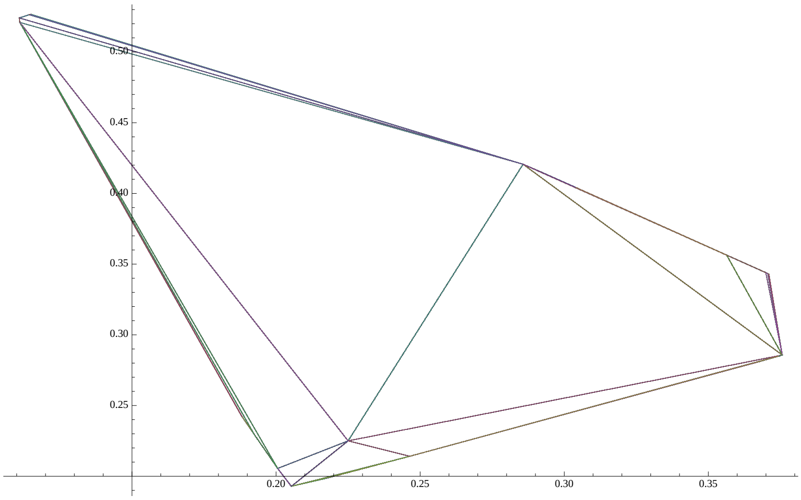

Fig. 8.14 The developing figure of an annulus with cone-points of order 3. See Triangle10graphicII.nb

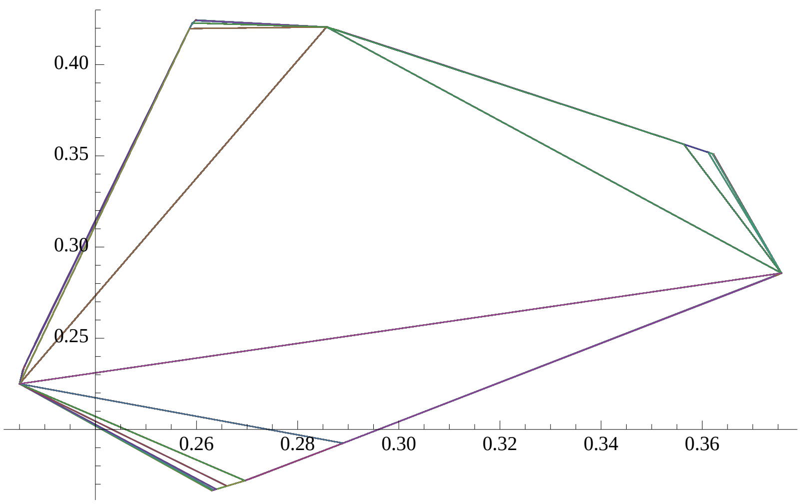

Fig. 8.15 The developing figure of a pair-of-pants. See Triangle10graphic.nb