Chapter 4

Topology of orbifolds

This section begins by reviewing the theory of the compact group actions on manifolds. Then we move on to define orbifold and their maps. We also cover the groupoid definition. We discuss the differentiable structures on orbifolds and the triangulation of orbifolds following the book [Verona (1984)]. We expose the covering theory using the fiber-product approach following Thurston and the path-approach following Haefliger. We make some computations of the fundamental groups. Finally, we relate the fundamental groups with the covering spaces.

We tried to make the abstract definitions into more concrete forms here; however, in many respect, the abstract definitions give us a more accurate sense of what an orbifold is. (For examples, see the article [Lerman (2010)].) This section is somewhat technical but essential to the developments later.

4.1 Compact group actions

Although we need only the result for finite group actions, we will study the situations when G is a compact Lie group. Let X be a space. We are given a group action $G \times X \to X$ with $e(x) = x$ for all x and $gh(x) = g(h(x))$. That is, we have a homomorphism $G \to \mathbf{Diff}(X)$ so that the product operation corresponds to the composition. In this case, X with the action is said to be a G-space.

An *equivariant* map $\phi : X \to Y$ between G-spaces is a map so that $\phi(g(x)) = g(\phi(x))$ for all $x \in X$. An *isotropy subgroup* G_x is defined as $\{g \in G|g(x) = x\}$. We note that $G_{g(x)} = gG_xg^{-1}$ and $G_x \subset G_{\phi(x)}$ for an equivariant map ϕ.

Theorem 4.1.1 (Tietze-Gleason Theorem). *Let G be a compact group acting on a normal space X with a closed invariant set A. Let G also act linearly on $\mathbb{R}^n$. Then any equivariant map $\phi : A \to \mathbb{R}^n$ extends to an equivariant map $\phi : X \to \mathbb{R}^n$.*

An *orbit* of a point x of X is $G(x) = \{g(x)|g \in G\}$. Then we see that $G/G_x \to G(x)$ is one-to-one and onto continuous function. Therefore, the orbit type is given by the conjugacy class of G_x in G. The set of orbit types form a set partially ordered by the reversing the inclusion ordering of the conjugacy classes of subgroups of G.

Denote by X/G the space of orbits with the quotient topology.

For $A \subset X$, define $G(A) = \bigcup_{g\in G} g(A)$ is the *saturation* of A.

- $\pi : X \to X/G$ is an open, closed, and proper map.
- X/G is Hausdorff since G is compact.
- X is compact iff X/G is compact.
- X is locally compact iff X/G is locally compact.

We list some examples:

- Let $X = G\times Y$ and G acts as a product. Then every orbit is homeomorphic to G and the stabilizers are all trivial groups.
- For k, q relatively prime, the action of $\mathbb{Z}_k$ on the unit sphere $\mathbf{S}^3$ in the complex space $\mathbb{C}^2$ is generated by a matrix
$$\begin{bmatrix} e^{2\pi i/k} & 0 \\ 0 & e^{2\pi q i/k} \end{bmatrix}.$$
The quotient space is a Lens space.
- We also consider $\mathbf{S}^1$-actions on $\mathbf{S}^3$ given by
$$\begin{bmatrix} e^{2\pi k i\theta} & 0 \\ 0 & e^{2\pi q i\theta} \end{bmatrix}$$
Then it has three orbit types.
- Consider in general the torus T^n-action on $\mathbb{C}^n$ given by
$$(c_1, \dots, c_n)(y_1, \dots, y_n) = (c_1y_1, \dots, c_ny_n), |c_i| = 1, y_i \in \mathbb{C}.$$
Then there is a homeomorphism $h : \mathbb{C}^n/T^n \to (\mathbb{R}^+)^n$ given by sending
$$(y_1, \dots, y_n) \mapsto (|y_1|^2, \dots, |y_n|^2)$$
where $\mathbb{R}^+ := \{x \in \mathbb{R}|x \geq 0\}$. (In other words, $(\mathbb{R}^+)^n$ is the closure of the positive 2^n-tant of $\mathbb{R}^n$.) The interiors of sides represent different orbit types.
- Let H be a closed subgroup of Lie group G. Let H act on G by the left action. The left-coset space G/H is the orbit space where G acts on the right also.
- Given a G-action on a space X and $x \in X$, let G_x be the stabilizer of x. A map $G/G_x \to G(x)$ given by $gG_x \mapsto g(x)$ is a homeomorphism if G is compact.
- The twisted product: let X be a right G-space and Y a left G-space. A left action is given by $g(x, y) = (xg^{-1}, gy)$. The twisted product $X \times_G Y$ is the quotient space.
- Let $p : X \to B$ is a principal bundle with G acting on the right. Let F be a left G-space. Now G acts on the right on $X \times F$ by $g(x, f) = (xg, g^{-1}(f))$. Then $X \times_G F$ is the associated bundle. (See Section 2.4.2.1.)

Example 4.1 (Bredon). Let G be the rotation group $\mathbb{SO}(3, \mathbb{R})$, and let X be the vector space of symmetric matrices of trace 0 (hence orthogonally diagonalizable). Suppose that we act by conjugation $G \times X \to X$ given by $g(m) = gmg^{-1}, m \in X$ and each $g \in G$. By linear algebra, we prove that two symmetric matrices are in the same orbit if they have the same eigenvalues with multiplicities. Hence the orbit space is in a one-to-one correspondence with the set of triples (a, b, c) so that $a \geq b \geq c$ and $a+b+c = 0$. The second space is a 2-dimensional cone in $\mathbb{R}^3$. This is homeomorphic to X/G. The isotropy group of a diagonal matrix with three distinct eigenvalues is the group of diagonal matrices with entries ± 1 which is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. The isotropy group of a diagonal matrix with exactly two distinct eigenvalues is the group of matrices decomposing into an orthogonal 2×2-matrix and ± 1.

A point x of a space X with a group G acting on it is *stationary* if the stabilizer of x is G.

Example 4.2 (Conner-Floyd). There is an action of $\mathbb{Z}_r$ for $r = pq$, p, q relatively prime, on an Euclidean space of large dimensions without stationary points. This is accomplished in following steps. We sketch the construction here.

- Find a simplicial action $\mathbb{Z}_{pq}$ on $\mathbf{S}^3$ seen as a join $\mathbf{S}^1 \star \mathbf{S}^1$ without fixed points obtained by joining action of $\mathbb{Z}_p$ on the first factor circle and $\mathbb{Z}_q$ on the second factor circle.
- Find an equivariant simplicial map $h : \mathbf{S}^3 \to \mathbf{S}^3$ which is homotopically trivial.
- Build the infinite mapping cylinder using h infinitely many times which is contactible and embed it in an Euclidean space of high-dimensions where $\mathbb{Z}_{pq}$ acts orthogonally.
- Find the contractible neighborhood. Taking the product with the real line makes it into a Euclidean space. Now on this space $\mathbb{Z}_{pq}$ acts orthogonally as well.

4.1.1 *Tubes and slices*

For a compact group action, we need to establish the notion of tubes and slices. These are modeled on twisted product actions: Let G be a compact Lie group, X a right G-space, and S a left G-space. Then $X \times_G S$ is defined as the quotient space of $X \times S$ where $[xg, y] \sim [x, gy]$ for $g \in G$, $x \in X$, and $y \in S$.

Let H be a closed subgroup of G and let A be a left H-space. shThen $G \times_H A$ is a left G-space by the action $g[g', a] = [gg', a]$ where $g, g' \in G, a \in A$ as this sends equivalence classes to themselves. The inclusion $A \to G \times_H A$ induces a homeomorphism $A/H \to (G \times_H A)/G$.

The isotropy subgroup at $[e, a]$ for $a \in A$ and e the identity element of G is computed as follows: $[e, a] = g[e, a] = [g, a] = [h^{-1}, h(a)]$ for $h \in H$. Thus,

$G_{[e,a]} = H_a$ where H_a is the stabilizer of a in H.

As an example, let $G = \mathbf{S}^1$ and A be the unit-disk and $H = \mathbb{Z}_3$ generated by $e^{2\pi i/3}$. G and H act in standard manners in A. Then consider $G \times_H A$. The result is homeomorphic to a solid torus fibered with circles. Each non-central circle is mapped around the quotient solid torus three times and the central circle goes around once.

Let X be a G-space and P an orbit of type G/H. A *tube about* the orbit P is a G-equivariant embedding $G \times_H A \to X$ onto an open neighborhood of P where A is a space where H acts on. We note the following:

- Every orbit passes the image of $e \times A$ where e is the identity of G.
- P equals $G(x)$ for $x = [e, a]$ where a is the stationary point of H in A.
- For general points $x = [e, b]$, not necessarily stationary, we have $G_x = H_b \subset H$.

Let $x \in X$. Suppose that S is a set containing x such that $G_x(S) = S$; i.e., the stabilizer of x acts on S. Then S is said to be a *slice* if $G \times_{G_x} S \to X$ so that $[g, s] \to g(s)$ is a tube about $G(x)$. It is easy to see that S is a slice if and only if S is the image of $e \times A$ for some tube.

Let $x \in S$ and $H = G_x$. Then the following statements are equivalent:

- There is a tube $\phi : G \times_H A \to X$ about $G(x)$ such that $\phi([e, A]) = S$.
- S is a slice at x.
- $G(S)$ is an open neighborhood of $G(x)$ and there is an equivariant retraction $f : G(S) \to G(x)$ with $f^{-1}(x) = S$.

Let X be a completely regular G-space. Let $x_0 \in X$ have an isotropy group H in G. Find an orthogonal representation of G in $\mathbb{R}^n$ with a point v_0 whose isotropy group is H, which always exists by a compact group representation theory. There is an equivalence of orbits $G(x_0)$ and $G(v_0)$. We extend this to a neighborhood by Tietze-Gleason theorem. For $\mathbb{R}^n$, we find the equivariant retraction given by Lemma 5.1 of Chapter 1 of the book [Bredon (1972)]. Transferring this on X, we obtain:

Theorem 4.1.2 (Gleason, Montgomery-Yang). *Let X be a completely regular G-space. There is a tube about any orbit of a completely regular G-space with G compact.*

If G is a finite group acting on a manifold, then a tube is a union of disjoint open sets and a slice is an open subset where G_x acts on.

Theorem 4.1.3 (Path-lifting and the covering homotopy theorem). *Let X be a G-space and G a compact Lie group.*

- *Let $f : I \to X/G$ be any path. Then there exists a lifting $f' : I \to X$ so that $\pi \circ f' = f$.*

- *Assume that every open subspace of X/G is paracompact. Let $f : X \to Y$ be an equivariant map. Let $f' : X/G \to Y/G$ be an induced map. Let $F' : X/G \times I \to Y/G$ be a homotopy preserving orbit types that starts at f'. Then there is an equivariant $F : X \times I \to Y$ lifting F' starting at f. Moreover, any two such liftings of F' differ by composition with a self-equivalence of $X \times I$ covering the identity of $X/G \times I$ and equal to identity on $X \times \{0\}$.*
- *If G is finite and X a smooth manifold with a smooth G-action and if the functions have locally smooth lifts, then the lifts can be chosen to be also smooth. If the derivative of a smooth path with locally smooth lifts is never zero, then the lift is unique up to the action of G.*

4.1.2 *Locally smooth actions*

Let M be a G-space with G a compact Lie group, and let P be an orbit of type G/H. and V a vector space where H acts orthogonally. Then a *linear* tube in M is a tube of the form $\phi : G \times_H V \to M$.

Let S be a slice. S is a *linear slice* if $G \times_{G_x} S \to M$ given by $[g, s] \to g(s)$ is equivalent to a linear tube. In other words, this is the case if the G_x-space S is equivalent to the orthogonal G_x-space.

If there is a linear tube about each orbit, then M is said to be *locally smooth.*

Lemma 4.1.4. *Under the above assumptions, there exists a maximal orbit type G/H for G. (That is, H is conjugate to a subgroup of each isotropy group.)*

Proof. In each tube, there is a maximal orbit type in it and we find the union of maximal orbits in it has to be dense and open. For intersection of two tubes, the union of maximal orbits has to be dense and open in both tubes. Thus, the maximal orbit of a tube is of the maximal orbit type in M. □

The maximal orbits so obtained in a tube are called *principal orbits.*

4.1.3 *Manifolds as quotient spaces.*

Finally, we wish to understand about the quotient spaces. Let M be a smooth manifold (not necessarily connected), and G a compact Lie group acting smoothly on M. We denote by M^* the quotient space M/G. (This is a notation used for this book.) If G is finite, then this is equivalent to the fact that each $i_g : M \to M$ given by $x \mapsto g(x)$ is a diffeomorphism, and the following theorem holds if the dimension of M is ≤ 2.

Theorem 4.1.5. *Let n be the dimension of M and d the dimension of the maximal orbit. Then $M^* = M/G$ is a manifold with boundary if $n - d \leq 2$.*

Proof. Let $k = n - d$ be the codimension of the principal orbits. Consider a linear tube $G \times_K V$ where K is a subgroup of G acting on V. The orbit space $(G \times_K V)/G = (G \times_K V)^*$ is congruent to V^* where $V^* = V/K$. Let S be the unit sphere in V. Then V^* is a cone over S^*. We have that $\dim M^* = \dim V^* = \dim S^* + 1$.

If $k = 0$, then M^* is discrete. If M is a sphere, then M^* is one or two points. (Here, we regard a disconnected 0-sphere as a sphere also.)

If $k = 1$, then M^* is locally a cone over one or two points by the previous steps. Hence M^* is a 1-manifold. If $k = 2$, then M^* is locally a cone over an arc or a circle as S^* is a 1-manifold by the previous step. □

Example 4.1.6. Consider the $\mathbb{Z}_2$-action on $\mathbb{R}^3$ generated by the antipodal map $\vec{x} \mapsto -\vec{x}$. The result is not a manifold.

4.1.4 *Smooth actions are locally smooth*

Recall smooth actions. Let G be a compact Lie group acting smoothly on a manifold M. Then there exists a G-invariant Riemannian metric on M. Then $G(x)$ is a smooth manifold where $G/G_x \to G(x)$ is a diffeomorphism. Recall the exponential map for Riemannian manifolds: For any vector $X \in T_pM$, there is a unique geodesic γ_X with tangent vector at p equal to X. The exponential map $\exp : T_pM \to M$ is defined by $X \mapsto \gamma_X(1)$.

Lemma 4.1.7. *A G-invariant metric on M can always be constructed so that ∂M is totally geodesic.*

Proof. We start with any smooth Riemannian metric μ on M. Next, we integrate to obtain the Riemannian metric $\mu_1 = \int_{g\in G} g^*\mu dg$ on M using the Haar measure on G. Now μ_1 will extend to a Riemannian metric on an open manifold M' containing M. Find a tube T of ∂M in M', i.e., an open neighborhood of ∂M and a submanifold diffeomorphic to $\partial M \times (-\epsilon, \epsilon)$, $\epsilon > 0$. By taking a sufficiently small tube, we assume that μ_1 extends to a metric on T. Here, we assume that the exponential map from the normal bundle of ∂M to T is a diffeomorphism. (See Chapter 4 of the book Hirsch (1976) for details.) Then there exists an antipodal map $\sigma : T \to T$ fixing ∂M by sending a point x of T with a shortest geodesic γ perpendicular to ∂M with $\gamma(\delta) = x$ to $\gamma(-\delta)$ in T again. We may assume that $\sigma(T) = T$. Considering geodesics perpendicular to ∂M, we find that the commutativity $\sigma \circ g = g \circ \sigma$ holds. By comparing distances between two points and their images under σ, we see that $\sigma^*\mu_1$ is also G-invariant in T. We form the G-invariant Riemannian metric $\sigma^*\mu_1 + \mu_1$. Since σ is an isometric involution of this metric, it follows that ∂M is totally geodesic. (For the proof, we followed a note of Francis (2010) here.) Now we use a G-invariant partition of unity to form a metric in M' and hence on M. □

If A is a G-invariant smooth submanifold, then A has an open G-invariant tubular neighborhood. This follows by using the normal bundle to A and the exponential map restricted to the normal bundle N_A. Then this map is a local diffeomorphism in a neighborhood N of A in N_A. By taking the same radius open balls in the normal bundle, we obtain the invariant tubular neighborhood as its image.

Proposition 4.1.8. *Let M be a manifold with boundary ∂M. The smooth action of a compact Lie group is locally smooth.*

Proof. We use the fact that orbits are smooth submanifolds and the above statements and that normal bundles are linear tubes. $\square$

Theorem 4.1.9 (Newman's theorem). *Let M be a connected topological n-manifold. Then there is a finite open covering $\mathcal{U}$ of the one-point compactification of M such that there is no effective action of a compact Lie group with each orbit contained in some member of $\mathcal{U}$.*

The proof follows from algebraic topology.

Corollary 4.1.10. *If G is a compact Lie group acting effectively on M, then the set of fixed points M^G is nowhere dense.*

4.1.5 *Equivariant triangulation*

Illman (1978) proved:

Theorem 4.1.11. *Let G be a finite group. Let M be a smooth G-manifold with or without boundary. Then we have:*

- *There exists an equivariant simplicial complex K and a smooth equivariant triangulation $h : K \to M$.*
- *If $h : K \to M$ and $h_1 : L \to M$ are smooth triangulations of M, there exist equivariant subdivisions K' and L' of K and L, respectively, such that K' and L' are G-isomorphic.*

4.2 The definition of orbifolds

Let X be a Hausdorff second countable topological space. Let n be fixed. Consider a connected open subset $\tilde{U}$ in $\mathbb{R}^n$ with a finite group G acting smoothly on it and a G-invariant map $\phi : \tilde{U} \to U$ for an open subset U of X inducing a homeomorphism $\tilde{U}/G \to U$. ϕ or $(\tilde{U}, \phi)$ is an *orbifold chart*, $\tilde{U}$ or $U = \phi(\tilde{U})$ is a *model neighborhood or model open set*, $(\tilde{U}, G)$ is a *model pair*, and $(\tilde{U}, G, \phi)$ is a *chart* or a *model triple*. An *embedding* $i : (\tilde{U}, G, \phi) \to (\tilde{V}, H, \psi)$ is a smooth embedding $i : \tilde{U} \to \tilde{V}$ with $\phi = \psi \circ i$ which induces the inclusion map $U \to V$ for $U = \phi(\tilde{U})$ and $V = \phi(\tilde{V})$.

- Equivalently, i is an embedding inducing the inclusion map $U \to V$ and inducing an injective homomorphism $i^* : G \to H$ so that $i \circ g = i^*(g) \circ i$ for every $g \in G$. $i^*(G)$ will act on the open set that is the image of i.
- Note here i can be changed to $h \circ i$ for any $h \in H$. The images of $h \circ i$ will be disjoint for representatives h for $H/i^*(G)$. Conversely, any embedding $i' : \tilde{U} \to \tilde{V}$ lifting an inclusion $U \to V$ equals $h \circ i$ for $h \in H$. (See Proposition A.1 of the article [Moerdijk and Pronk (1999)].)

Definition 4.2.1. Let $\mathbb{R}_+ := \{x | x \geq 0\}$. Define $\mathbb{R}^n_+$ as the n-fold product of $\mathbb{R}_+$. A *cell* is a nonempty intersection of a convex open set in $\mathbb{R}^n$ with $\mathbb{R}^n_+$.

Two model triples $(\tilde{U}, G, \phi)$ and $(\tilde{V}, H, \psi)$ are *compatible* if for every $x \in U \cap V$ and open sets $U = \phi(\tilde{U})$ and $V = \psi(\tilde{V})$, there is an open neighborhood W of x in $U \cap V$ and the model triple $(\tilde{W}, K, \mu)$ with $\mu(\tilde{W}) = W$ such that there are embeddings to $(\tilde{U}, G, \phi)$ and $(\tilde{V}, H, \psi)$. (One can assume that W is a component of $U \cap V$.)

- Since G acts smoothly, G acts freely on an open dense subset of $\tilde{U}$.
- An *orbifold atlas* on X is a family of compatible model triples $\{(\tilde{U}, G, \phi)\}$ so that the family of open sets of form $\phi(\tilde{U})$ covers X.
- Two orbifold atlases are *compatible* if model triples in one atlas are compatible with model triples in the other atlas.
- Atlases form a partially ordered set by the inclusion relation. It has a maximal element.
- Given an atlas, we obtain a unique maximal atlas containing it by Zorn's lemma.
- An *orbifold* $\mathcal{O}$ is a topological space X with a maximal orbifold atlas. We say that X is the *underlying space* of $\mathcal{O}$ and write $X = |\mathcal{O}|$ and we say that $\mathcal{O}$ is based on $|\mathcal{O}|$.
- One can obtain an atlas of linear charts only: that is, charts of form $(\tilde{U}, G, \phi)$ where $\tilde{U}$ is an open subset of $\mathbb{R}^n$ and $G \subset \mathbb{O}(n, \mathbb{R})$. For each point $x \in \tilde{U}$, one can find a finite subgroup G_x stabilizing the point and a suitable G_x-invariant neighborhood in $\tilde{U}$. Then G_x acts linearly up to a choice O_x of coordinate charts since a smooth action is locally smooth, i.e., linear and orthogonal, by Proposition 4.1.8. (Note, if x is in the boundary, then O_x can be identified with an open set intersected with an upper-half space and G_x is acting orthogonally on the half-space.) We call such a chart (O_x, G_x, ϕ) a *linear chart*. Therefore, given an orbifold atlas, there is a compatible orbifold atlas consisting of only linear charts.
- G_x is called a *local group*. If the local group G_x is not trivial, then x is said to be *singular*.
- If we have $\tilde{U}$ with G acting freely, we can drop this from the atlas and replace with many charts with trivial group.

- A map $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ where $\mathcal{U}$ and $\mathcal{V}$ are maximal atlases is *smooth* if for each point $x \in X$, there is a model triple $(\tilde{U}, G, \phi) \in \mathcal{U}$ with $x \in U = \phi(\tilde{U})$ and a model triple $(\tilde{V}, H, \psi) \in \mathcal{V}$ with $f(x) \in V$ so that $f(U) \subset V = \psi(\tilde{V})$ and f lifts to a smooth map $\tilde{f} : \tilde{U} \to \tilde{V}$. In this case, f is said to be an *orbifold-map.*
- If above f has local lifts $\tilde{f} : \tilde{U} \to \tilde{V}$ that is an immersion for the every pairs of model triples as above, then f is said to be an *orbifold-immersion.*
- Two orbifolds are *diffeomorphic* if there is a smooth orbifold-map with a smooth inverse orbifold-map.

Sometimes, the orbifolds are called effective (or reduced) orbifolds as we defined here (Adem, Leida, and Ruan, 2007). There are ineffective orbifolds, where for a model neighborhood $(\tilde{U}, G, \phi)$ the group G is allowed to be not effective on $\tilde{U}$. This is not a well-studied area.

We also note our convention that an orbifold has certain topological property if the underlying space has that property.

Definition 4.1. A covering of an orbifold is *good* if each model neighborhood is connected, the open set in the triple is homeomorphic to a cell, and the group acts linear orthogonally and the intersection of any finite collection again has such properties.

We will show later that each orbifold has a good cover. (See Proposition 4.4.2.)

Given an orbifold $\mathcal{O}$, if we allow some open sets $\tilde{U}$ in model triples to be open subsets of the closed upper half space $\mathbb{R}^{n-1} \times \mathbb{R}_+$, then the orbifold has *boundary.* A *boundary subset* of an orbifold is the subset of the underlying space orbifolds where each element is so that each of its inverse image points in the model open sets goes to the boundary of $\mathbb{R}^{n-1} \times \mathbb{R}_+$ under charts. The complement of the boundary is the *interior* of the orbifold. If a finite group G acts on a subspace V, we denote by $G|V$ the homomorphism image of G as restrictions $\{g|V | g \in G\}$. The boundary has an orbifold structure also by restricting each model triple $(\tilde{U}, G, \phi)$ to $(\tilde{U} \cap V, G|V, \phi|V)$ for $V = \mathbb{R}^{n-1} \times \{0\}$ whenever $\tilde{U} \cap V \neq \emptyset$ as the model triples are all compatible. The *boundary* of an orbifold is the boundary subset with this orbifold structure. (We will show that the boundary is a suborbifold. See Definition 4.2.2.)

A compact orbifold with empty boundary is said to be a *closed* orbifold.

4.2.1 *Local groups and the singular set*

Let $x \in X$. A *local group* G_x of x is obtained by taking a model triple $(\tilde{U}, G, \phi)$ for x and finding the stabilizer G_y of y for an inverse image point y of x.

- This is independently defined up to conjugacy for any choice of y.
- We reason as follows: Smaller charts will give you the smaller or identical conjugacy class. The stabilizer group eventually does not change under

taking smaller and smaller charts up to conjugations. Thus, one can take a linear chart. Once a linear chart is achieved, the local group is well-defined up to conjugacy (Thus, as an abstract group with an action.)

The *singular set* is a set of points where G_x is not trivial. In each chart, the set of fixed points of each subgroup of G_x is a closed submanifold.

Let (O_x, G_x) and (O_y, G_y) be two charts. Subgroups H of G_x and H' of G_y are *strictly topologically conjugate* if there is a chart (U_z, G_z) with morphisms into (O_x, G_x) and (O_y, G_y) in the orbifold atlas so that H and H' correspond to conjugate subgroups in G_z. H and H' are *topologically conjugate* if there exists a sequence $H_1 = H, H_1, \cdots, H_n = H'$ where H_i and H_{i+1} are strictly topologically conjugate.

The connected maximal subset of the singular set where the topological conjugacy class of the stabilizer G_x of each of its element x is constant is a relatively closed submanifold. Thus X becomes a stratified smooth topological space where each stratum is given by the connected component of the set where the smooth topological conjugacy classes of subgroups of local groups G_x for $x \in X$ is constant. (Here, a *stratified space* is a space that is a union of disjoint relatively closed connected submanifolds. A *stratum* is one of these submanifolds. See Section 4.5.1.)

Because G_y is trivial for y in a dense open subset of O_x, a generic point of an orbifold has a trivial local group. Hence, there exists a dense open subset in the underlying space of an orbifold that is nonsingular. The set of singular points is nowhere dense also.

The singularity of a 1-orbifold is unique: a silvered point. Its neighborhood is modeled on an open interval where $\mathbb{Z}_2$ acts as a reflection group fixing a point. Thus, a connected 1-orbifold has a base space homeomorphic to the circle $\mathbf{S}^1$ or an interval (half-open, open, or closed) and is diffeomorphic to $\mathbf{S}^1$, a closed interval with one or two silvered points, a half-open interval with one or no silvered point, or an open interval.

To classify the singular points of 2-orbifolds, we classify finite groups in $\mathbb{O}(2, \mathbb{R})$ acting on open subsets of $\mathbb{R}^2$ since we are looking at finite subgroups of $\mathbb{GL}(2, \mathbb{R})$: These are as follows: $\mathbb{Z}_2$ acting as a reflection group or a rotation group generated by a rotation of angle π, cyclic groups C_n of order $n \geq 3$ and dihedral groups D_n of order $2n \geq 4$. The singular points of a two-dimensional orbifold fall into three types:

(i) The mirror point: $\mathbb{R}^2/\mathbb{Z}_2$ where $\mathbb{Z}_2$ is generated by the reflection on the y-axis.

(ii) The cone-points of order n: $\mathbb{R}^2/\mathbb{Z}_n$ where $\mathbb{Z}_n$ acting by rotations by angles $2\pi m/n$ for integers m.

(iii) The corner-reflector of order n: $\mathbb{R}^2/D_n$ where D_n is the dihedral group generated by reflections about two lines meeting at an angle π/n. (Note that D_n is of order $2n$. However, the order of the corner-reflector itself is n.)

From this, we see that the underlying space of a 2-orbifold is a surface with corner since each model neighborhood is diffeomorphic to a surface with corner by above. (This also follows from the proof of Theorem 4.1.5. See the beginning of Section 4.5 for the definition.)

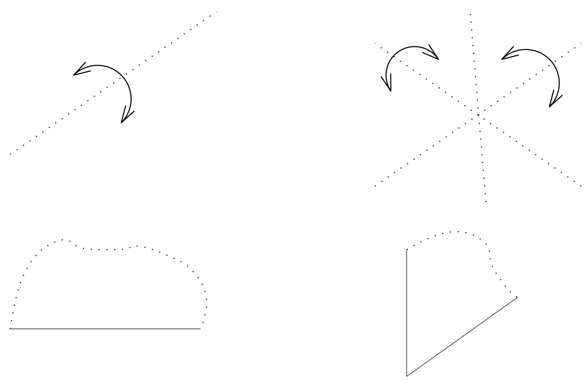

Fig. 4.1 The actions here are isometries on $\mathbb{R}^2$.

Definition 4.2. Given two orbifolds X and Y, we find a natural product orbifold-structure on $|X| \times |Y|$ where $|X|$ and $|Y|$ are the respective underlying spaces. We assume that the boundary of one of X or Y is empty. For a point $(x, y) \in |X| \times |Y|$, an orbifold neighborhood is $U \times V$ for respective model neighborhoods U and V of x and y where $(\tilde{U}, G, \phi)$ is the model triple for x and $(\tilde{V}, H, \psi)$ is one for y with $\phi(\tilde{U}) = U$ and $\psi(\tilde{V}) = V$. The group $G \times H$ acts on $\tilde{U} \times \tilde{V}$, and $(\tilde{U} \times \tilde{V}, G \times H, \phi \times \psi)$ is the model triple for (x, y). Then these charts $\phi \times \psi$ form an atlas of $|X| \times |Y|$ giving us an orbifold structure. We denote the orbifold by $X \times Y$ and call it the product orbifold of X and Y.

If both $\partial X \neq \emptyset$ and $\partial Y \neq \emptyset$, then we can put on $|X| \times |Y|$ an orbifold-structure with corner. (See Section 4.5.2 for detail.)

Definition 4.2.2. A *suborbifold* Y of an orbifold X is an embedded subset such that for each point y in Y and a chart $(\tilde{V}, G, \phi)$ of X for a neighborhood V of y there is a chart for y given by $(P, G|P, \phi)$ where P is a closed submanifold of $\tilde{V}$ where G also acts on and $G|P$ is the image of the restriction homomorphism of G to P. (We caution the readers that $G \to G|P$ is sometimes not injective.)

Clearly, an open subset inherits an orbifold structure to make them into a suborbifold, and the boundary of an orbifold is a suborbifold. (See Remark 4.2.5.)

A suborbifold in our sense is a "suborbifold" in the sense of Definition 2.3 of the book [Adem, Leida, and Ruan (2007)], which is easy to show from the definitions. However, our definition is strictly stronger. Also our definition is strictly weaker than the one in Section 6.1 of the book [Kapovich (2009)]. (Actually, we should say our suborbifolds are "strong" suborbifolds. However, we do not need their definition.) The basic reason for our definition is so that we wish do surgeries along the suborbifolds in later sections.

Let I be the orbifold based on $[0, \epsilon)$ with 0 given the silvered point structure. Then $I \times I$ is a 2-orbifold covered by $(-\epsilon, \epsilon) \times (-\epsilon, \epsilon)$ with $\mathbb{Z}_2^2$ acting on it by two reflections about the axes. That is, $I \times I$ has a corner-reflector of order 2 at which two silvered edges meet. The diagonal $\delta \subset [0, \epsilon) \times [0, \epsilon)$ can be given a suborbifold structure in the sense of Definition 2.3 of the book [Adem, Leida, and Ruan (2007)] by Example 2.6 of the same book. However, the inverse image of δ in $(-\epsilon, \epsilon) \times (-\epsilon, \epsilon)$ is not an embedded arc, i.e, a union of two transversal arcs, and it cannot be a suborbifold in our sense.

Now consider $J = \{0\} \times I$. Then J is given an orbifold-structure with one-silvered point. Then J is a suborbifold in our sense. However, J is not a suborbifold in the sense of Section 6.1 of the book [Kapovich (2009)]. The reason is that the local groups are required to be mutually isomorphic in the later case.

Clearly, manifolds are orbifolds. But as an orbifold, it might carry more charts. By an abuse of notations, a manifold in this paper will mean a manifold with the extended collection of charts as orbifolds: To explain, in general, let G be a finite group acting on a manifold M smoothly and freely. Then M/G is a manifold with an orbifold structure with an atlas of charts based on some H-invariant open set in M diffeomorphic to an open subset of $\mathbb{R}^n$ and a subgroup H of G as a model. For example, $\mathbb{RP}^n$, $n \geq 2$, will have a chart with the $\mathbb{Z}_2$-action on it.

Conversely, we say that an orbifold is a *manifold* if there is an atlas in the orbifold atlas with model triples with trivial groups only. A submanifold of a manifold has a suborbifold structure when the manifold is considered as an orbifold.

4.2.2 *Examples*: *good orbifolds*

Let $M = T^n$ and $\mathbb{Z}_2$ act on it with generator acting by $-I$. For $n = 2$, $M/\mathbb{Z}_2$ is topologically a sphere and has four singular points. For $n = 4$, we obtain a Kummer surface with sixteen singular points. In general, a regular branched covering of a surface by another surface gives us an orbifold structure.

Theorem 4.2.3. *Let M be an n-orbifold with boundary possibly empty and Γ be a discrete group of orbifold-diffeomorphisms of M acting properly discontinuously but not necessarily freely. Then the quotient space M/Γ has a natural structure of an orbifold.*

Proof. For each point $x \in M$, the stabilizer group Γ_x is a finite group since x has a neighborhood U whose closure is compact. Since G_x is finite, we form an open neighborhood $\bigcup_{g \in G_x} g(U)$ of x. By taking U sufficiently small, we may assume that U has a model triple (V, G, ϕ) for an open subset V in $\mathbb{R}^n$ or in a half-space

$\mathbb{R}^{n-1} \times \mathbb{R}_+$. Now, U has a finite group G_U acting on it. Each element $g : U \to U$ is an embedding and hence lifts to a diffeomorphism $\tilde{g} : V \to V$.

Let G_V be the finite group generated by these lifts and G. Then it follows that $g \in G_V$ for a homeomorphism of V iff $\phi \circ g = h \circ \phi$ for $h \in G_x$. Let $p : M \to M/\Gamma$ be the quotient map. Hence $(V, G_V, p \circ \phi)$ is then a model triple of $p(x) \in M/\Gamma$.

The sets of these types form an atlas of M/Γ and hence give us an orbifold structure. □

We say that M/Γ is a *quotient orbifold* of an orbifold M. In fact, in many cases orbifolds are of this form. If M is a manifold, they are called "good orbifolds". We will talk about these later.

- Consider the Euclidean plane $\mathbb{R}^2$ and the discrete group generated by order-two rotations at $(k+n, l+m)$ for $n, m \in \mathbb{Z}^2$ and fixed real numbers $k, l > 0$.
- Cut a rectangle of height 1 and length 2 containing two fixed points rotations on the top side and two the bottom side respectively. We glue by an isometry given by the composition of the two rotations on the top side, which is identical with that of the two rotations at the bottom side. We obtain an annulus. (See Figure 4.2.)
- Then we crease the top circle and the bottom circle at the cone-points and glue by the order 2 rotations. (This is called "folding". See Section 5.2.1.)
- Thus, the Poincaré polyhedron theorem exactly fits into this situation.
- We can modify this construction easily by taking a nonstandard $\mathbb{Z}_2$-lattice. This might be a good exercise for readers.

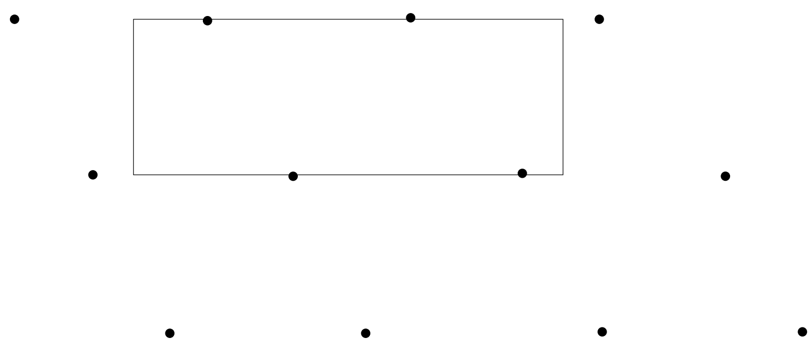

Fig. 4.2 The rectangle and the fixed points

This type of orbifold is an example of an Euclidean orbifold which is a quotient orbifold of the euclidean space by a wall paper group. (We call "pillows" tetrapaks to emphasize the Euclidean structure.) See Figures 4.4 and 4.5.

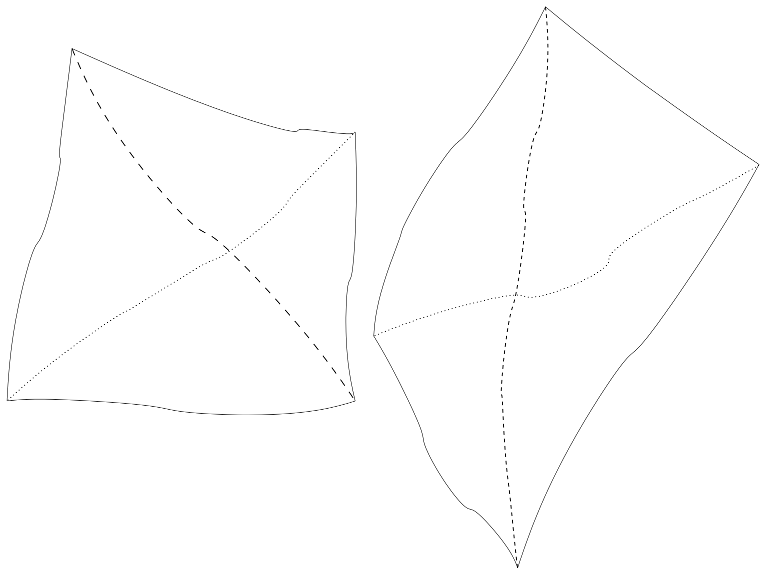

Fig. 4.3 Tetrapaks often called "pillows".

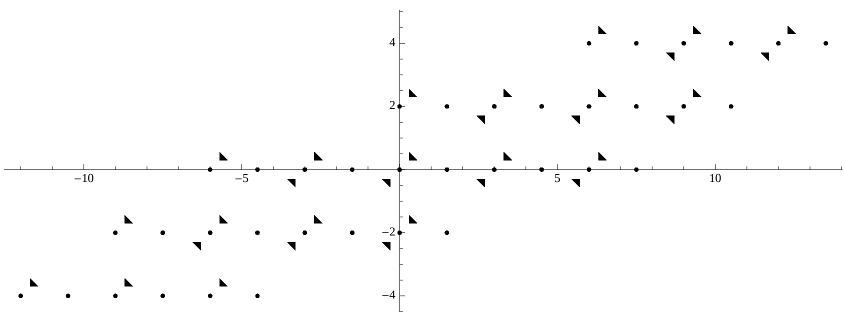

Fig. 4.4 A wall paper group p2: The points are fixed points of some elements of order 2 including generators and a triangle is mapped by various elements of the group. See wall2a.nb

4.2.3 *Examples*: *silvering*

Given a manifold M with boundary, we obtain a *doubled* $\hat{M}$ by taking $M \times \mathbb{Z}_2/\sim$ where $(x, 0) \sim (y, 1)$ if and only if $x = y \in \partial M$. A $\mathbb{Z}_2$-action $\hat{M}$ is induced by $(x, 0) \mapsto (x, 1)$ and $(x, 1) \mapsto (x, 0)$ for $x \in M$. We build a collar neighborhood of ∂M in M diffeomorphic to $\partial M \times [0, \epsilon)$. Then the $\mathbb{Z}_2$-action here can be extended to $\partial M \times (-\epsilon, \epsilon)$ by $(x, t) \to (x, -t)$. This is a smooth action. Hence, we can double M as a smooth manifold $\hat{M}$ and obtain a smooth $\mathbb{Z}_2$-action. Thus, M can be given a smooth orbifold structure modeled on $\mathbb{Z}_2$-invariant open subsets of $\hat{M}$ with $\mathbb{Z}_2$-action or open subsets of M^o with trivial group actions.

Now the boundary of M became now a set of singular points, called *silvered*

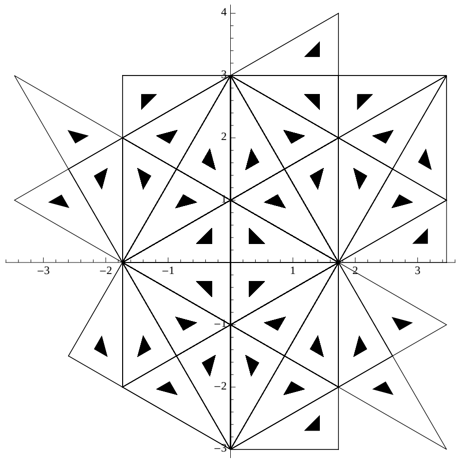

Fig. 4.5 A $(2, 3, 6)$-triangle reflection group. The fundamental domain is one of the bigger triangles and an inside triangle is mapped to many other by various elements of the group. See wall17a.nb

points. Actually, we can do this for the interior U of a properly and smoothly embedded submanfold of ∂M. Define $\hat{M}_U$ as $M \times \mathbb{Z}_2/\sim$ where $(x, 0) \sim (y, 1)$ if and only if $x = y \in U$. Then we can find an orbifold structure on M with U silvered in the above way. (See also Proposition 4.4.3.)

Example 4.2.4. Consider a surface with corner, its boundary that is a union of smooth arcs ending at corner points, and the set of its corner points.

- We choose some collection of these arcs $\alpha_1, \dots, \alpha_n$ and finite set of points in the interior $q_1, \dots, q_m$.
- We let the set of points where the the endpoints of half-arcs of the arcs in the collection coincide be called *distinguished corner points.* Denote them by $p_1, \dots, p_l$. Each p_i is given an order n_i, $n_i \geq 2$. Let each point q_i be given orders m_i, $m_i \geq 2$. If α_i is a loop, then its unique endpoint is a distinguished corner point.
- We give a Riemannian metric on a neighborhood N of the boundary by ϕ-equivariantly immersing the universal cover of the neighborhood into the Euclidean space $\mathbb{E}^2$ so that the boundary arcs are geodesic, the angle at each distinguished corner point p_i is π/n_i and at the non-distinguished corner points the angles are $\pi/2$, where the homomorphism $\phi : \pi_1(N) \to \mathbf{Isom}(\mathbb{R}^2)$ can be chosen.
- Then each point of the arc α_i is silvered by taking as a model open set a small open ball in $\mathbb{E}^2$ containing its image and invariant under the reflection about the image of α_i.
- At each point p_i, we take a model open set as a small open ball in $\mathbb{E}^2$ containing its image and invariant under the two reflections about the images

of α_k and α_l ending there forming an angle π/n_i for some k, l.
- At q_i, we model its open neighborhood by an open ball with a cyclic action by $\mathbb{Z}_{m_i}$. The neighborhood here is chosen to be disjoint from ones of the boundary points.
- For other points, we model an open neighborhood of the point disjoint from boundary or $\{q_1, \ldots, q_m\}$ by an open set in $\mathbb{E}^2$ without any group actions.
- Finally, we see that then these charts are compatible and hence gives rise to an orbifold structure.

Remark 4.2.5. When we say the boundary or interior of an orbifold, we do not mean the boundary or interior of the underlying space. They are different concepts. Of course the boundary of an orbifold is in the boundary of the underlying space but the converse is not necessarily true. For example, supposing that the underlying space is a topological manifold, a silvered $(n-1)$-dimensional open manifold in the boundary of the underlying space is in the interior of the orbifold. The interior of the underlying space is in the interior of the orbifold but the converse is not necessarily true.

4.3 The definition as a groupoid

We will try to avoid the definitions using the category theory as related to the theory of stacks in algebraic geometry as much as possible and use the more concrete set theoretic approach. See for example the articles [Moerdijk (2002); Moerdijk and Pronk (1997); Pohl (2010); Lerman (2010)] and Chapter III$\mathcal{G}$ of the book [Bridson and Haefliger (1999)] and the book [Adem, Leida, and Ruan (2007)]. (See the articles [Haefliger (1990, 1984a); Haefliger and Quach (1984b)] also for the beginning of this.) However, there are many reasons to learn orbifolds as groupoids since this framework provides us with more tools and insights from the category theory and even from the smooth manifold theory in the categorical setting. These definitions are mainly introduced to study sheaf theoretic considerations and bundles and so on. (The main reason we are introducing these definitions is to explain the path approach to covering spaces following Haefliger.)

Here, we will try to minimize the theoretical aspect. In spite of the technical nature, readers somewhat acquainted with the category theory will recognize that these definitions are very concrete. Only the abstract nature of the category theory comes when discussing the equivalences of these structures.

We follow mostly the expositions in the book [Adem, Leida, and Ruan (2007)] and the paper [Moerdijk (2002)].

4.3.1 *Groupoids*

A *topological groupoid* G consists of a space G_0 of objects and a space G_1 of arrows with five continuous maps:

- a source map $s : G_1 \to G_0$,
- a target map $t : G_1 \to G_0$,
- an associative composition map $m : G_{1s} \times_t G_1 \to G_1$ where

$$G_{1s} \times_t G_1 := \{(h, g) \in G_1 \times G_1 | s(h) = t(g)\}.$$

- a unit map $u : G_0 \to G_1$ so that $su(x) = x = tu(x)$ and $gu(x) = g$ if $s(g) = x$ and $u(x)g = g$ if $t(g) = x$, and
- an inverse map $i : G_1 \to G_1$ so that if $g : x \to y$, then $i(g) : y \to x$ and $i(g)g = u(x)$ and $gi(g) = u(y)$.

It will be convenient to think of these arrows at points as restrictions of maps to the singletons. Given a topological groupoid G, we will denote by G_0 the space of objects and G_1 the space of arrows. The arrow $u(x)$ in G_1 from a point x of G_0 to itself is denoted by I_x.

A *Lie groupoid* is one G where G_0 and G_1 are smooth manifolds and the five maps are smooth and s and t are submersions. (This implies that $G_{1s} \times_t G_1$ is a smooth manifold.)

Let M be a smooth manifold. If $G_0 = G_1 = M$ and every arrow is of form I_x for $x \in G_0$, then this is the *unit groupoid* on M.

As a simple example, let a Lie group K act smoothly on a smooth manifold M. The *action Lie groupoid* L is given by $L_0 = M$ and $L_1 = K \times M$ with s as the projection to the M factor and t as the action $K \times M \to M$. The unit map is the inclusion map $x \mapsto (e, x)$ for the unit element e of K. The inverse map $K \times M \to K \times M$ is given by $(g, x) \mapsto (g^{-1}, g(x))$.

If K is the trivial group, we obtain the unit Lie groupoid.

- Given a groupoid G, we define the *isotropy group* at x to be the set of all arrows from x to itself; i.e.,

$$\begin{aligned} G_x &= \{g \in G_1 | (s, t)(g) = (x, x)\} \\ &= (s, t)^{-1}(x, x) \\ &= s^{-1}(x) \cap t^{-1}(x) \subset G_1. \end{aligned}$$

- A *homomorphism* of Lie groupoids $\phi : H \to G$ is a pair of smooth maps $\phi_0 : H_0 \to G_0$ and $\phi_1 : H_1 \to G_1$ commuting with all structure maps.
- The fiber-product: Given two homomorphisms $\phi : H \to G, \psi : K \to G$ of Lie groupoids, we define the *fiber product* $H \times_G K$ to be the Lie groupoid whose objects are (y, g, z) for $y \in H_0, z \in K_0$, and arrow $g : \phi(y) \to \psi(z)$ and whose arrows $(y, g, z) \to (y', g', z')$ are pairs (h, k) of arrows $h : y \to y', k : z \to z'$ so that $g'\phi(h) = \psi(k)g$.

An *étale* map of a Lie groupoid is a homomorphism $\phi : G \to H$ so that $\phi_0 : G_0 \to H_0$ is a local homeomorphism. A homomorphism of Lie groupoids $\phi : H \to G$ is an *equivalence* if

- $t \circ \pi_1 : G_{1s} \times_\phi H_0 \to G_0$ is a surjective submersion.
- the square

$$\begin{array}{ccc} H_1 & \overset{\phi}{\to} & G_1 \\ (s,t)\downarrow & & \downarrow(s,t) \\ H_0 \times H_0 & \overset{\phi\times\phi}{\to} & G_0 \times G_0 \end{array}$$

  is a fiber product of manifolds.

We can show that two groupoids are equivalent if and only if they are Morita equivalent; i.e., there exists another groupoid and equivalences from it to the two groupoids. This essentially means that there is a larger groupoid containing both.

4.3.1.1 *A nerve of a groupoid and the homotopy groups*

Let G be a Lie groupoid. Define

$$G_n = \{(g_1, \dots, g_n) | g_i \in G_1, s(g_i) = t(g_{i+1})\}$$

as a fiber product. The face operator $d_i : G_n \to G_{n-1}$ is defined by sending $(g_1, \dots, g_n)$ to $(g_1, \dots, g_i g_{i+1}, \dots, g_n)$. This forms an abstract simplicial manifold, said to be the *nerve* of the groupoid G.

The classifying space BG is defined to be the geometric realization as a simplicial complex (Adem, Leida, and Ruan, 2007). We will not give much details here.

4.3.2 *An abstract definition*

- A groupoid G is *proper* if $s \times t : G_1 \to G_0 \times G_0$ is proper.
- A groupoid G is *étale* if s and t are local diffeomorphisms.
- A groupoid G is *foliation* if each isotropy group G_x is discrete.
- An *orbifold groupoid* is a proper étale Lie groupoid.

If G is an étale groupoid, then any arrow $g : x \to y$ in G induces a well-defined germ of a diffeomorphism $\tilde{g} : U_x \to V_y$ for neighborhoods U_x of x and V_y of y in G_0, defined as $\tilde{g} = t \circ \hat{g}$, where $\hat{g} : U_x \to G_1$ is a section of the source map $s : G_1 \to G_0$ with $\hat{g}(x) = g$. (By étale property, such sections exist.) We call an étale groupoid G *effective* (or *reduced*) if the assignment $g \mapsto \tilde{g}$ is faithful; or equivalently, if for each point $x \in G_0$ this map $g \mapsto \tilde{g}$ defines an injective group homomorphism $G_x \to \mathbf{Diff}(U_x)$.

Some authors define proper foliation Lie groupoids to be orbifold groupoids. However, they are equivalent under a Morita equivalence. Orbifold groupoids are usually effective groupoids. Also, G_x is finite for each point $x \in G_0$ if G is a proper foliation groupoid.

The set $ts^{-1}(x) = \{y | \exists z \in G_1, z : x \to y\}$ is called the *orbit* of x. The *orbit space* $|\mathcal{G}|$ of a groupoid $\mathcal{G}$ is the quotient space of its space of objects G_0 under the equivalence relation $x \sim y$ if and only if x and y are in the same orbit.

Theorem 4.3.1.

- *Let $\mathcal{G}$ be a proper effective étale groupoid. Then its orbit space $|\mathcal{G}|$ can be given the structure of an orbifold.*
- *Two effective orbifold groupoids $\mathcal{G}$ and $\mathcal{G}'$ represent the same orbifold up to isomorphism if and only if they are Morita equivalent.*

We do not prove this theorem (Adem, Leida, and Ruan, 2007); however, we show below that an orbifold gives rise to a proper effective étale groupoid.

Example 4.3. Let M be a smooth orbifold with the locally finite covering $\mathcal{U}$ by model neighborhoods in the orbifold atlas and the underlying space X. Each nonempty finite intesection of the members of $\mathcal{U}$ has a model (U, G, ϕ) in the orbifold atlas for some domain $U \subset \mathbb{R}^n$, a finite group G acting on it effectively, and ϕ inducing a homeomorphism U/G to its image. Let M_0 be the disjoint union of the model open sets in $\mathbb{R}^n$ of all finite intersections of members of $\mathcal{U}$, and let M_1 be the set of arrows obtained by restrictions to points in M_0 of all embeddings $U \to V$ for model triples (U, G, ψ) and (V, H, ϕ) lifting the inclusion maps and their compositions and the inverse arrows. (Here, it is possible that $U = V$ and $G = H$.) Also, we include I_x for all $x \in M_0$. Then the space of orbits is homeomorphic to X and M_0 and M_1 contain all the information of the atlas. The fact that this is a proper effective étale groupoid follows by checking the above definitions.

We note the alternative definition:

Definition 4.3. An *orbifold structure* on a paracompact Hausdorff space X consists of orbifold groupoid $\mathcal{G}$ and a homeomorphism $f : |\mathcal{G}| \to X$. Two orbifold structures $(\mathcal{G}, f)$ and $(\mathcal{H}, g : |\mathcal{H}| \to X)$ are *equivalent* if there is a groupoid equivalence $\phi : \mathcal{H} \to \mathcal{G}$ inducing the homeomorphism $|\phi| : |\mathcal{H}| \to X = |\mathcal{G}|$ so that $f \circ |\phi| = g$.

4.3.2.1 *Examples*

Let a discrete group Γ act on a connected manifold X properly discontinuously. Then (Γ, X) has an orbifold structure. We think of it as a groupoid where X_0 is given as X itself and X_1 as the space of arrows sending $x \to \gamma(x)$ for $x \in X_0$ for $\gamma \in \Gamma$. Hence, there are cardinality of Γ of components of X_1 homeomorphic to X_0. (This is the good orbifold discussed above. See Theorem 4.2.3.)

We obtain a 2-orbifold from a compact orientable Seifert fibered 3-manifold M: We choose X_0 to be the union of finitely open disks that are disjoint and bounded away from one anoother and each flow line meets at least one of them. We choose X_1 to be the space of flow lines with both end points in these disks.

The fiber order of a closed flow curve is the order of the germ of the return map to a transversal disk along the curve.

The orbifold X will be a 2-dimensional orbifold with cone-points whose orders are the fiber-orders of the corresponding closed flow lines.

4.3.2.2 *Actions of a Lie groupoid*

Let G be an orbifold groupoid. A *left G-space* is a manifold E equipped with an action by G: Such an action is given by two maps: an anchor $\pi : E \to G_0$ and an action $\mu : G_1 \times_{G_0} E \to E$.

- This map is defined on (g, f) with $\pi(f) = s(g)$ and written $\mu(g, f) = g.f$ for $f \in E$.
- It satisfies the action identity: $\pi(g.f) = t(g)$, $\mathrm{I}_x.f = f$, and $g.(h.f) = (g.h).f$ for $h : x \to y$ and $g : y \to z$ and $f \in E$ with $\pi(f) = x$.

A *right G-space* is the left G^{op}-space obtained by switching the source and target maps of G_1.

4.4 Differentiable structures on orbifolds

Now, we go back to the original definition of orbifolds using charts.

Let $\mathcal{O}$ be an orbifold. We are given a smooth structure on each $(\tilde{U}, G, \phi)$; i.e., $\tilde{U}$ is given a smooth structure and G is a finite group with a smooth action on it. All embeddings in the atlas are smooth. Then M is given a *smooth* structure under embeddings. Given a chart $(\tilde{U}, G, \phi)$, we define *the space of smooth forms* to be the space of smooth forms in $\tilde{U}$ invariant under the G-action. A *smooth form* on the orbifold is the collection of smooth forms on all model open sets of the charts so that they match under embeddings and the local group actions.

This enables one to define the space $\Lambda^p(\mathcal{O}), p \geq 0$ of smooth p-forms on $\mathcal{O}$ and the boundary operators, which are defined as usual since one can define boundary operators on the model neighborhoods. Let $H^p(\mathcal{O})$ denote the p-th de Rham cohomology of $\mathcal{O}$. Let $H^q_c(\mathcal{O}), q \geq 0$, denote the q-th de Rham cohomology of $\mathcal{O}$ defined from compactly supported smooth forms.

A *smooth simplex* defined from a simplex Δ to an orbifold $\mathcal{O}$ is simply a smooth map. One can define an integral of a differential form with respect to a smooth singular simplex into a model neighborhood by lifting to the model neighborhood by Theorem 4.1.3. A smooth singular simplex may have different lifts to model neighborhoods; however, the integral itself is well-defined. (One needs to look at the currents in the inverse image of the simplex.) This can be extended to any smooth simplex using partition of unity and barycentric subdivisions of the simplex. Given a locally finite covering of $\mathcal{O}$, we can define a smooth partition of unity (in the same way as in the manifold case). (See for example the book [Munkres (1991)].)

- We refine to obtain a cover by open sets whose closures are invariant compact subsets.
- The idea is to find a smooth function on each chart which vanishes outside the invariant compact subsets.
- The images of compact subsets can be chosen to cover $\mathcal{O}$.
- Thus, these functions become functions on $\mathcal{O}$ which sum to a positive valued function.
- We divide by the sum.

An orbifold $\mathcal{O}$ is *orientable* if one can choose an atlas of charts where $\tilde{U}$ is given an orientation with G acting in an orientation-preserving manner and each embedding of charts to another charts is orientation-preserving. For example, a reflection about a hypersurface is excluded and hence silvered boundary is excluded. (However, one can use densities or forms of odd degrees to replace n-forms and can integrate when $\mathcal{O}$ is not orientable. See the book [de Rham (1984)].)

An n-form ω can be integrated on an orientable orbifold $\mathcal{O}$: Let $(\tilde{U}, G, \phi)$ is a model triple for a model neighborhood U of $\mathcal{O}$ and let ω' denote the n-form on $\tilde{U}$ representing ω. Then the integral of ω on U is defined as

$$\int_U \omega = \frac{1}{|G|} \int_{\tilde{U}} \omega'$$

where $|G|$ is the order of G. Then for any n-form, the integral upon $\tilde{\mathcal{O}}$ can be integrated by using a partition of unity.

The Poincaré duality pairing: For an orientable orbifold $\mathcal{O}$,

$$\int : H^p(\mathcal{O}) \otimes H_c^{n-p}(\mathcal{O}) \to \mathbb{R}$$

is given by sending (ω, η) for a closed p-form ω and a closed and compactly supported $(n-p)$-form η to $\int_{\mathcal{O}} \omega \wedge \eta$. This is a nondegenerate bilinear form when $\mathcal{O}$ is a closed orientable orbifold. Adem, Leida, and Ruan (2007) prove this.

4.4.1 *Bundles over orbifolds*

An *orbifold-bundle* (or V-bundle) E over an orbifold $\mathcal{O}$ is given by a smooth orbifold E and a smooth map $\pi : E \to \mathcal{O}$ with the following properties:

- Let F be a smooth manifold with a Lie Group $\mathbf{G}$ acting on it smoothly.
- A pair of defining families $\mathcal{F}$ for $\mathcal{O}$ and $\mathcal{F}'$ for E so that a model triple (U, K, ϕ) of $\mathcal{O}$ corresponds to a model triple (U^*, K^*, ϕ^*) so that $U^* = U \times F$ and $\pi \circ \phi^* = \phi \circ \pi_1$ where $\pi_1 : U^* \to U$ is the projection to the first factor.

- Given $(U, K, \phi), (U^*, K^*, \phi^*)$, and $(U', K', \phi), (U'^{,*}, K'^{,*}, \phi'^{,*})$, we require that there is a one-to-one correspondence of embeddings $\lambda : (U, K, \phi) \to (U', K', \phi)$ and
$$\lambda^* : (U^*, K^*, \phi^*) \to (U'^{,*}, K'^{,*}, \phi'^{,*})$$
where $\lambda^*(p, q) = (\lambda(p), g_\lambda(p)q)$ for $(p, q) \in U^* = U \times F$ with $g_\lambda(p) \in \mathbf{G}$.
- We have
$$g_{\mu\circ\lambda}(p) = g_\mu(\lambda(p)) \circ g_\lambda(p) \tag{4.1}$$
for embeddings
$$(U, K, \phi) \xrightarrow{\lambda} (U', K', \phi') \xrightarrow{\mu} (U'', K'', \phi'').$$
- If $F = \mathbf{G}$, then this is a principal orbifold bundle (with a right $\mathbf{G}$-action).

Notice that by the one-to-one correspondence property of the third item, there is an isomorphism $K \to K^*$ given by sending $\sigma \in K$ to $\sigma^* \in K^*$ defined by
$$\sigma^*(\tilde{p}, q) = (\sigma(\tilde{p}), g_\sigma(\tilde{p})q), \tilde{p} \in U.$$

Conversely, the above data are enough to construct an orbifold-bundle as we can verify that the quotient space of the collection of sets of form $U \times F$ by the identification map is still Hausdorff and second-countable and hence an orbifold.

4.4.1.1 *Principal bundles using the groupoids language.*

Finally, using the groupoid language, we can define the principal bundles. See the article [Moerdijk (2002)] and the book [Adem, Leida, and Ruan (2007)] for details.

A *principal L-bundle* for a Lie group L over a Lie groupoid G is a G-space P with a right action $P \times L \to P$ which makes $\pi : P \to G_0$ into a principal L-bundle over the manifold G_0 and is compatible with the G-action in the sense that $g.(p.l) = (g.p).l$ for $p \in P, l \in L$ and an arrow $g : x \to y$.

4.4.2 *Tangent bundles and tensor bundles*

Given the orbifold $\mathcal{O}$, we build a *tangent orbifold-bundle* $T(\mathcal{O})$ by taking $F = \mathbb{R}^n$, $\mathbf{G} = \mathbb{GL}(n, \mathbb{R})$, and $g_\lambda(p)$ to be the Jacobian of λ at p for each embedding $\lambda : (U, K, \phi) \to (U', K', \phi)$ as above. We can build any tensor bundles in this way by letting $F = T^r_s(\mathbb{R}^n)$ and $\mathbf{G} = \mathbb{GL}(n, \mathbb{R})$ and $g_\lambda(p)$ be the induced map $T^r_s(\mathbb{R}^n) \to T^r_s(\mathbb{R}^n)$ of λ at p.

A *reduction* of a Lie group $\mathbf{G}$ to a subgroup H means an injective homomorphism $H \to \mathbf{G}$ which induces a bundle morphism of the principal bundle with the Lie group H to the principal bundle with the Lie group $\mathbf{G}$.

A frame bundle is obtained by taking F to be $F_n(\mathbb{R}^n)$ the space of frames in $\mathbb{R}^n$, $\mathbf{G}$ to be $\mathbb{GL}(n, \mathbb{R})$, and $g_\lambda(p)$ to be the induced map $F_n(\mathbb{R}^n) \to F_n(\mathbb{R}^n)$ of λ at p.

An *affine frame bundle* is given by taking $F = A(\mathbb{R}^n)$ the space of affine frames and $\mathbf{G} = \mathbb{A}(\mathbb{R}^n)$, the Lie group of affine autormorphisms. An *affine tangent bundle* is given by taking $F = \mathbb{R}^n$ with the same Lie group.

An orthogonal frame bundle is a reduction of the frame bundle to $\mathbb{O}(n, \mathbb{R})$: Orthogonal frame bundles can be built in this way. We let $F = O_n(\mathbb{R}^n)$ the space of orthonormal frames and let $\mathbf{G} = \mathbb{O}(n, \mathbb{R})$ and choose $g_\lambda(p)$ be a map $O_n(\mathbb{R}^n) \to O_n(\mathbb{R}^n)$ corresponding to each λ at p.

Let $\mathbf{G}$ be a Lie group with a Lie algebra $\mathfrak{g}$. Given a principal bundle P, one defines a *connection* to be an assignment of an equivariant connection on every model triple (U^*, K^*, ϕ^*) corresponding to a model triple (U, K, ϕ) of $\mathcal{O}$ which form a collection that are consistently defined under the embeddings. The *curvature* is also defined as the $\mathfrak{g}$-valued 2-form on $\mathcal{O}$ which comes from the curvature of each orbifold chart.

A *linear connection* is a connection on a frame bundle or a tangent bundle with Lie group $\mathbb{GL}(n, \mathbb{R})$. An *affine connection* is a connection on an affine frame bundle or an affine tangent bundle with the Lie group $\mathbb{A}(\mathbb{R}^n)$. Given an affine connection on an affine tangent bundle, a *geodesic* is defined as a smooth map from an open arc to $\mathcal{O}$ so that in each chart it lifts to a geodesic under the connection.

A *Riemannian metric* on an orbifold is given by an equivariant Riemannian metric on each chart which matches up under embeddings or simply as a smooth section of symmetric covariant tensor bundle $ST^2(\mathcal{O})$ whose image lie in the positive definite forms. A Riemannian metric can be built using a partition of unity again from any given Riemannian metrics on charts.

The group $\mathbb{O}(n, \mathbb{R}) \cdot \mathbb{R}^n$ is the group of rigid motions on $\mathbb{R}^n$. We can also replace the group $\mathbb{A}(\mathbb{R}^n)$ with $\mathbb{O}(n, \mathbb{R}) \cdot \mathbb{R}^n$ by reduction of the group. This corresponds to choosing a section to $ST^2(\mathcal{O})$. Then the connections on the reduced affine bundles are also called *affine connections*. (As usual, an $\mathbb{O}(n, \mathbb{R})$-connection of a tangent bundle or a frame bundle is also considered an affine connection since we can always construct a canonical affine connection from a linear connection by the Levi-Civita constructions. The set of geodesics does not change here. See the reasoning in [Kobayashi and Nomizu (1997)] that can be directly generalized to the orbifold setting.)

Finally, one defines an exponential map $\exp : T(\mathcal{O}) \to \mathcal{O}$: one defines the exponential map using the linear or affine connection on each model neighborhood and then patching up the consistent results.

Lemma 4.4.1. *Given an orbifold $\mathcal{O}$ with boundary, we can give a Riemannian metric on $\mathcal{O}$ so that boundary components are totally geodesic.*

Proof. Let x be a point of $\partial\mathcal{O}$. Then we find a model triple (U, G, ϕ) of x. We obtain a reflection r_F fixing ∂U and we form a finite group L_U isomorphic to $\mathbb{Z}_2$ generated by these. Then let $U' = \bigcup_{g \in L_U} g(U)$ is an invariant open set in $\mathbb{R}^n$ generated by G and L_U. We find an invariant Riemannian metric g_U on U'.

Now, we cover $\mathcal{O}$ by a locally finite covering by model open sets U_i with models $(\tilde{U}_i, G_i, \phi_i)$. Let U_i' be obtained as above by taking the union under the reflections in faces of U_i. Obtain Riemannian metric g_{U_i} for each U_i. We use a partition of unity to obtain a Riemannian metric μ on $\mathcal{O}$. This induces a new Riemannian metric g'_{U_i} on U_i'.

Let O' be an open n-orbifold containing $\mathcal{O}$ and a tubular neighborhood N of $\partial\mathcal{O}$. This can be obtained by taking open model open sets instead of half-open ones in $\mathbb{R}^n$. Extend the metric μ to O'.

For each component F of $\partial\mathcal{O}$, we find a reflection r_F defined on a tubular neighborhood of F in O' given by sending points of distance r on a geodesic perpendicular to F to its opposite point on the geodesic with same distance. Then we form the Riemannian metric $(r_F^*\mu + \mu)/2$. We use a partition of unity so that we have a Riemannian metric, on $\mathcal{O}$, that is invariant under r_F in a smaller tubular neighborhood of F in O'. Then F is totally geodesic this metric as in the note [Francis (2010)]. (See also Lemma 4.1.7.) $\square$

An *isotopy* $F : Y_1 \times I \to Y_2$ for two orbifolds Y_1 and Y_2 is an orbifold-map such that for each $t \in I$ where I is an interval, F restricts to a diffeomorphism of $Y_1 \times \{t\}$ into suborbifolds of Y_2. (We will often consider codimension-zero suborbifolds.)

Let $\mathcal{O}$ be an n-orbifold with boundary. A *neatly embedded suborbifold* is a suborbifold A of $\mathcal{O}$ such that $\partial A = \partial\mathcal{O} \cap A$ or $\partial A = \emptyset$ and $A \cap \partial\mathcal{O} = \emptyset$. (See Section 1.4 of the book [Hirsch (1976)].) In this case, we can make A perpendicular to $\partial\mathcal{O}$ by an isotopy from the inclusion map of A. Basically, we make the inverse image of A in the model open sets be perpendicular to boundary and then we use averaging of the defining functions of A and use partition of unity to build an isotopied suborbifold in $\mathcal{O}$ and the defining functions in the models C^2-close to the original ones. Finally, we show that we can achieve this by an isotopy generated by the vector fields.

A *normal vector* of a suborbifold $\mathcal{O}_1$ at a point x in $\mathcal{O}$ is an equivalence class of a vector v in the tangent space of model neighborhood (U, G, ϕ) with a chart ϕ at a point $\tilde{x}$ corresponding to x and perpendicular to the tangent vectors of the inverse image of $\mathcal{O}_1$ in U under ϕ.

Let Σ be an i-dimensional neat suborbifold of $\mathcal{O}$ for $i < n$. Denote by $N(\Sigma)$ the space of normal vectors of Σ. The exponential map is a diffeomorphism from

$$N_\epsilon(\Sigma) := \{v \in N_x(\Sigma) |\, x \in \Sigma, ||v|| < \epsilon(x)\}$$

to its image provided $\epsilon : \Sigma \to (0, \infty)$ is a sufficiently small valued function. The proof is entirely similar to those in the Riemannian manifold theory and we omit these. (See Sections 4.5 and 4.6 of the book [Hirsch (1976)].) The image is said to be a *tubular-neighborhood* of Σ. (Here we use the total geodesic properties and orthogonality of boundary components of $\mathcal{O}$ meeting the suborbifold.)

Since we understand the normal bundle of Σ, the orbifold structure of a tubular-neighborhood can be understood as an orbifold-bundle over Σ where the fiber over $x \in \Sigma$ can be described as D^{n-i}/G_x for an $(n-i)$-disk D^{n-i} and G_x is a finite group.

If Σ is a boundary component, then we define $N^+(\Sigma)$ to be the set of vectors pointing inside. Each boundary component Σ of an orbifold $\mathcal{O}$ has a *collar*, i.e., a neighborhood diffeomorphic to $\Sigma \times [0, 1)$. Using the exponential map from the normal bundle $N^+(\Sigma)$ to $\mathcal{O}$, and taking the image of vectors of length $< \epsilon(x)$ for some small valued function $\epsilon : \Sigma \to \mathbb{R}^+$, we obtain a *collar*.

4.4.3 *The existence of a locally finite good covering*

Recall Definition 4.1.

Proposition 4.4.2. *Let $\mathcal{O}$ be an orbifold with boundary. Then there exists a good covering.*

Proof. First, give a Riemannian metric on $\mathcal{O}$ where the boundary suborbifolds are totally geodesic. Each point has an orbifold chart with an orthogonal action. Now choose a sufficiently small ball in the model neighborhood centered at the origin so that it has a convexity property. (That is, any path in a model open set can be homotopied into a geodesic.) (See Chapter 3 of the book [Do Carmo (1992)].) Find a locally finite subcollection. Then the intersection set of any finite collection is still convex and hence has cells as finite coverings. ☐

4.4.4 *Silvering the boundary components*

In fact, we can fully generalize the results in Section 4.2.3:

Proposition 4.4.3. *Let $\mathcal{O}$ be an n-dimensional orbifold with a boundary component Σ. Then we can obtain an orbifold $\mathcal{O}'$ with the same underlying space and every point of Σ is now singular with generic manifold points becoming a silvered point.*

Proof. The proof of Lemma 4.4.1 contains the proof. Basically, we add the reflections to the groups of the model triples with small image open sets. ☐

4.4.5 *The Gauss-Bonnet theorem*

Let $\mathcal{O}$ be an orbifold with the underlying space X. We will show shortly that X admits a finite smooth triangulation so that the interior of each simplex lies in the connected set of singular points with locally constant local groups in Theorem 4.5.4.

We define the Euler characteristic to be

$$\chi(\mathcal{O}) = \sum_k (-1)^{\dim s_k} \frac{1}{N_{s_k}}$$

where s_k denotes the open kth-cell in the triangulation and N_{s_k} the order of the local group.

Theorem 4.4.4 (Satake). *Let M be a closed orbifold of even dimension m with a Riemannian metric. Then*

$$(2/O_m)\int_M K d\mu = \chi(M),$$

where K is the Pfaffian of the curvature form, $d\mu$ is the volume measure of M, and O_m is the volume of the standard unit m-sphere.

The proof essentially following that of Chern for manifolds is given by Satake (1957). Here Satake's work only allows for codimension ≥ 2 singularities. We see that by doubling M, the theorem holds. (See Section 4.6.1.2 for details on doubling.) Thus, the theorem holds for M by divisions by 2 by Proposition 5.1.3.

4.5 Triangulation of smooth orbifolds

In general, a smooth orbifold has a smooth topological stratification and a smooth triangulation so that each open cell is contained in a single stratum. A smooth topological stratification satisfying certain weak conditions admits a triangulation. We now show that the stratification of an orbifold by orbit types satisfies this condition. We mainly follow pp. 37–38 and pp. 126–127 of the book [Verona (1984)]. (See also the article [Moerdijk and Pronk (1999)].)

We denote by $\mathbb{R}_+$ the subset $\{x \in \mathbb{R} | x \geq 0\}$. A *manifold M with corner* is a topological manifold with boundary with atlas of charts to $\mathbb{R}^n_+ = \{(x_1, \dots, x_n) | x_1 \geq 0, \dots, x_n \geq 0\}$ with smooth transition maps. Each point of M has a neighborhood with a chart to an open subset diffeomorphic to $\mathbb{R}^i_+ \times \mathbb{R}^{n-i}$ for a minimal i, $0 \leq i \leq n$. Such a point is said to be of *corank* i. A set of points of corank 0 is the set of interior points of M and the set of points of corank ≥ 1 is the set of boundary points of M to be denoted by ∂M.

Let M be a manifold with corners and let ∂M be the boundary of M. A *face* of M is a closure of a component of the set of corank 1 in ∂M. It itself is a cornered manifold B in ∂M with an embedding $F_B : U_B \to B \times \mathbb{R}_+$ for an open neighborhood U_B of B, called a *collar* of B, where $F_B(x) = (x, 0)$ for al $x \in B$.

4.5.1 *Triangulation of the stratified spaces*

Let X, Y be two subsets of a topological space A with $X \cap Y = \emptyset$. If $X \subset \mathrm{Cl}(Y)$, then we write $X < Y$. We say $X \leq Y$ if $X = Y$ or $X < Y$.

A *face* of a topological space A is a closed subset of A with a smooth embedding $F_B : U_B \to B \times \mathbb{R}_+$ for a neighborhood U_B of B sending B to $B \times \{0\}$. F_B and U_B are said to be the *collar* and the *collar neighborhood*. We write $F_B = (p_B, r_B)$ where $p_B : U_B \to B$ and $r_B : U_B \to \mathbb{R}_+$ are smooth functions.

A Hausdorff, locally compact, paracompact space with a countable basis is said to be a *nice* space. Let A be a nice topological space and $X \subset A$ be a locally closed set. A *tube* T_X of X is a neighborhood of X in A with a retraction $\pi_X : T_X \to X$ and a function $\rho_X : T_X \to \mathbb{R}$ such that $\rho_X^{-1}(0) = X$.

Given positive valued functions $\epsilon, \delta : X \to \mathbb{R}$ with $0 \le \epsilon < \delta$, we define

$$X \times (\epsilon, \delta) = \{(x,t) \in X \times \mathbb{R} | \epsilon(x) < t < \delta(x)\}$$

with obvious extensions to closed interval cases.

Define $T_X^\epsilon = \{a \in T_X | \rho_X(a) < \epsilon(\pi_X(a))\}$ for a function $\epsilon : X \to \mathbb{R}_+$ where $\epsilon > 0$. If $X \subset U \subset A$ for an open U, then $T_X^\epsilon \subset U$ for some ϵ. $(\pi_X, \rho_X)|T_X^\epsilon$ is a proper map into

$$X \times [0, \epsilon) = \{(x,t) | x \in X, 0 \le t < \epsilon(x)\}$$

by choosing sufficiently small ϵ.

An *abstract stratification* $\mathcal{A}$ consists of

(i) a nice space A and
(ii) a locally finite family $\mathcal{A}$ of locally closed connected subsets A' (strata) of A so that A is a disjoint union of $\mathcal{A}$ whose members are smooth manifolds
(iii) a family of tubes of the strata $\{\tau_X = (T_X, \pi_X, \rho_X) : X \in \mathcal{A}\}$.
(iv) a family of closed subsets $\mathcal{A}^*$ of A called *faces*

satisfying the following properties:

- If $X, Y \in \mathcal{A}$ with $X \cap \mathrm{Cl}(Y) \neq \emptyset$, then we have $X \le Y$.
- For any face $A_i \in \mathcal{A}^*$, there exists an open neighborhood U_{A_i} and a homeomorphism $F_{A_i} : U_{A_i} \to A_i \times \mathbb{R}_+$ onto an open subset so that
  - $F_{A_i}(a) = (a, 0), a \in A_i$
  - for any $X \in \mathcal{A}$, if $X \cap A_i \neq \emptyset$, then

    $$F_{A_i}(X \cap U_{A_i}) \subset (X \cap A_i) \times \mathbb{R}_+$$

    for a collar F_{A_i} of A_i. We define

    $$p_{A_i} : U_{A_i} \to A_i \text{ and } r_{A_i} : U_{A_i} \to \mathbb{R}_+ \text{ by}$$

    $$F_{A_i}(a) = (p_{A_i}(a), r_{A_i}(a)) \text{ for } a \in U_{A_i}.$$

- Each stratum $X \in \mathcal{A}$ is a manifold with faces $X_i := X \cap A_i, A_i \in \mathcal{A}^*$ with collars

  $$F_{X_i} = F_{A_i} | X \cap U_{A_i} : U_{X_i} = X \cap U_{A_i} \to X_i \times \mathbb{R}_+$$

  whenever $X \cap A_i \neq \emptyset$.
- For $X \in \mathcal{A}$ and $A_i \in \mathcal{A}^*$, we have $\pi_X^{-1}(X_i) = A_i \cap T_X$ and

  $$F_{X_i} \circ \pi_X = ((\pi_X | T_X \cap A_i) \times \mathrm{I}_{\mathbb{R}_+}) \circ F_{A_i}$$

  in an open neighborhood of X_i provided $X_i \neq \emptyset$.

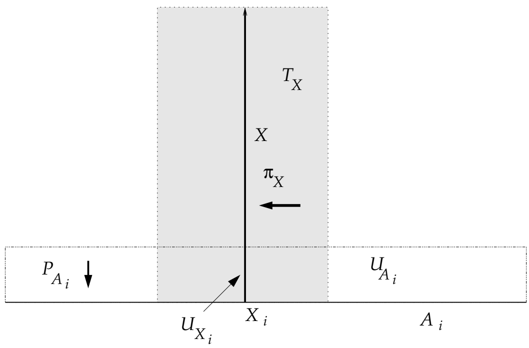

Fig. 4.6 An illustration of tubes and faces and so on.

- For any $X \in \mathcal{A}$, X has $\epsilon_X : X \to \mathbb{R}_+$, $\epsilon_X > 0$, so that $T_X^{\epsilon_X} \cap Y \neq \emptyset$ for $Y \in \mathcal{A}$ implies $X < Y$ and
$$(\pi_X, \rho_X) : T_X^{\epsilon_X} \cap Y \to X \times (0, \epsilon_X)$$
is a smooth submersion.
- For any $X, Y \in \mathcal{A}$, $X \subset \mathrm{Cl}(Y)$, there exist positive functions ϵ_X defined on X and ϵ_Y defined on Y satisfying the statement that
$$a \in T_X^{\epsilon_X} \cap T_Y^{\epsilon_Y} \text{ implies}$$
$$\pi_Y(a) \in T_X, \pi_X(\pi_Y(a)) = \pi_X(a) \text{ and } \rho_X(\pi_Y(a)) = \rho_X(a).$$
- For any $X \in \mathcal{A}$, and $A_i \in \mathcal{A}^*$, we have $\rho_X = \rho_X \circ p_{A_i}$ in a neighborhood of A_i.

The *dimension of a stratum* is the dimension as a manifold. If $X \subset \mathrm{Cl}(Y)$ for strata X and Y, then the dimension of X is strictly less than that of Y. The *depth* of a stratified space is the maximal cardinality of collections of form $\{X_1, \dots, X_n\}$ of strata X_i satisfying $X_i < X_{i+1}$ for $i = 1, \dots, n-1$. Note that the dimensions strictly increase in the chain. The maximal dimensional strata are open manifolds where the tubes are identical with themselves. A stratification has a finite depth if the maximal dimension of the strata is finite.

A *triangulation* of a topological space A consists of a pair (K, ϕ) where K is a countable locally finite simplicial complex and $\phi : |K| \to A$ for a geometric realization $|K|$ of K is a homeomorphism.

A *relative manifold* (with corners) is a pair of topological spaces $(V, \delta V)$ so that δV is a closed subset of V and $V - \delta V$ is a manifold with corners. A triangulation (K, ϕ) of a relative manifold $(V, \delta V)$ is *smooth* if K contains a subcomplex δK so that $\phi(\delta K) = \delta V$ and for any simplex σ of K, the restriction ϕ to $|\sigma| - |\delta K|$ is smooth and for each $x \in |K| - |\delta K|$ the differential $D\phi_x$ of ϕ at x is injective.

A *smooth triangulation* of an abstract stratification $\mathcal{A}$ is a triangulation (K, ϕ) of A satisfying the condition that for each stratum X, there is a subcomplex K_X so that $K_X, \phi|K_X$ is a smooth triangulation of $(\mathrm{Cl}(X), \mathrm{Cl}(X) - X)$.

Theorem 4.5.1 (Verona). *Let X be a nice space, and let $\mathcal{A}$ be an abstract stratification of X of finite depth. Then there exists a smooth triangulation of $\mathcal{A}$.*

4.5.2 *Orbifolds as stratified spaces*

Lemma 4.5.2. *Let V be a Euclidean vector space or $\mathbb{R}^i \times \mathbb{R}_+^{n-i}$ for a fixed $i = 0, 1, \dots, n$. Let G be a finite group effectively acting on V orthogonally preserving each face of $\mathbb{R}^i \times \mathbb{R}_+^{n-i}$.*

- *The fixed-point set of a linear finite group G action is a closed subspace of V.*
- *The subset $F_{G'}$ of points fixed exactly by a subgroup G' of G is a vector subspace with a finite number of closed subspaces removed. $F_{G'}$ is dense open in the subspace of fixed points of G'.*
- *F_G and $F_{G'}$ are orthogonal to faces of $\mathbb{R}^i \times \mathbb{R}_+^{n-i}$.*
- *For distinct subgroups G' and G'', $F_{G'}$ and $F_{G''}$ are disjoint.*
- *If $G'' \subset G'$ properly, then $F_{G'}$ is in the closure of $F_{G''}$.*

Proof. The first item is clear.

The second item follows from the fact that the fixed-point set of any subgroup is a subspace. One has to remove subspaces fixed by a larger group from inside. The third item and the fourth items are also clear. The final item follows from the second item. $\square$

To prove our result, we will use the results from Section 4.4. (This is strictly for convenience, and we will need simple results in exponential maps.)

First, let G_x be a nontrivial local subgroup of a point x of an orbifold $\mathcal{O}$. Then the set of points with local groups locally conjugate to G_x forms a locally closed connected manifold by the existence of linear charts and Lemma 4.5.2.

Thus, the underlying space X of $\mathcal{O}$ is a disjoint union of connected submanifolds determined by the local topological conjugacy classes of the local groups. Let us call the collection of connected submanifolds $\mathcal{A}$. Since X is a nice topological space, the set $\mathcal{A}$ forms a stratification:

Suppose $X \cap \mathrm{Cl}(Y) \neq \emptyset$ for two strata X, Y. Given the local linear chart U for $x \in X$, we see that the stabilizer G_x corresponding to x is the maximal local group in the chart. Then $X \cap U \subset \mathrm{Cl}(Y) \cap U$ for each linear chart neighborhood U of x. Hence $X \subset \mathrm{Cl}(Y)$.

We need a slight generalization of orbifolds with boundary. Recall $\mathbb{R}_+$ is the space of nonnegative real numbers and $\mathbb{R}_+^n$ the Cartesian product. We define an *orbifold with corners* as an orbifold $\mathcal{O}$ with the following properties:

- Each point has a model (U, G, ψ) where U is an open subset of $\mathbb{R}_+^n$ and G is a finite group acting on it that acts on each face of $\mathbb{R}_+^n$ that U meets.

- In the manifold cases, we define the corank of a point of $\mathcal{O}$ as the corank in the models.
- We define the *face* as a subset of $\partial\mathcal{O}$ as the closure of a component of the set of corank 1 in $\partial\mathcal{O}$ and it is required to a face of the underlying space $|\mathcal{O}|$.

(Recall Remark 4.2.5 also.)

The following is a direct generalization of Lemma 4.4.1 with an almost identical proof.

Lemma 4.5.3. *Given an orbifold $\mathcal{O}$ with corners, we can give a Riemannian metric on $\mathcal{O}$ so that*

- *faces are totally geodesic and they are perpendicular to each other when they meet at codimension-two subspaces.*
- *each stratum X of $\mathcal{O}$ is a totally geodesic manifold with faces in $\partial\mathcal{O}$ and perpendicular to faces in $\partial\mathcal{O}$ and is neatly embedded with a collar about $X \cap F$ for every face F of $\partial\mathcal{O}$.*

Proof. Let x be a point of $\partial\mathcal{O}$. Then we find a model triple (U, G, ϕ) of x. For each face F of U, we obtain a reflection r_F: actually the Euclidean one will do, and we form a finite group L_U generated by these. We require that the reflections always commute with one another. Then let $U' = \bigcup_{g \in L_U} g(U)$ is an invariant open set in $\mathbb{R}^n$ generated by G and L_U. We find an invariant Riemannian metric g_U on U'.

Now, we cover $\mathcal{O}$ by a locally finite covering by model open sets U_i with models $(\tilde{U}_i, G_i, \phi_i)$. Let U'_i be obtained as above by taking the union under the reflections in faces of U_i. Obtain a Riemannian metric g_{U_i} for each U_i. We use a partition of unity to obtain a Riemannian metric μ on $\mathcal{O}$. This induces a new Riemannian metric g'_{U_i} on U'_i. Also, every pair of intersecting faces of $\mathcal{O}$ are orthogonal to each other.

Let O' be an open n-orbifold containing $\mathcal{O}$. Extend the metric μ to O'.

Take a face F of $\partial\mathcal{O}$. We find a reflection r_F defined on a tubular neighborhood of F in O' given by sending points of distance r on a geodesic perpendicular to F to its opposite point on the geodesic with same distance. (We might need to define this on an ambient manifold containing F and extending F slightly.) Then we form the Riemannian metric $(r_F^*\mu + \mu)/2$. We use a partition of unity so that we have a Riemannian metric on $\mathcal{O}$ which is invariant under r_F in a smaller tubular neighborhood of F in O' bounded by some extensions of other faces. Then F is totally geodesic in this metric and still perpendicular to other faces. (See the note [Francis (2010)].)

Using the reflection r_F for the new metric, perhaps a little changed now, we can silver F by taking a small tubular N_F neighborhood of F in O' bounded by some extensions of other faces and define charts by using charts of points of F with images in N_F and adding r_F to the group.

We now do an induction process and we can silver every face of $\partial\mathcal{O}$ since the set of faces is locally finite. Now it is clear that the faces are all totally geodesic and orthogonal when they meet.

Each model neighborhood (U, G) of $\mathcal{O}$ has an invariant Riemannian metric induced from that of $\mathcal{O}$. Then since G acts on U intersected with faces of $\mathbb{R}^i \times \mathbb{R}^{n-i}_+$, it follows that each fixed point set of a subgroup of G is a submanifold A perpendicular to faces of U.

Since a subgroup of G fixes each point of A, it follows that A is totally geodesic. Thus, each stratum of $\mathcal{O}$ is totally geodesic.

Using the exponential map from the normal vector bundle of each face F of U, in this case using normal vectors in one direction, we obtain an ϵ-collar of F for a positive valued function $\epsilon : F \to \mathbb{R}$. We obtain a collar of the image of F in U/G. Since X is totally geodesic, the collar restricts to $X \cap F$ and we obtain an ϵ-collar of $X \cap F$. By patching together, we see that each i-dimensional stratum X has a collar about $X \cap F$ for each face F. □

Now we move to the main theorem of this section.

Theorem 4.5.4. *Let $\mathcal{O}$ be an n-orbifold with corners. Each singularity x of an orbifold $\mathcal{O}$ with a local group G_x always lies in a submanifold of points whose local groups are locally conjugate to G_x. Then the collection of such submanifolds with the nonsingular components forms an abstract stratification of the underlying space of the orbifold $\mathcal{O}$ with corner. Therefore, $\mathcal{O}$ with the stratification is smoothly triangulated.*

Proof.
First, we put a Riemannian metric with totally geodesic faces by Lemma 4.5.3. We let $\mathcal{A}^*$ be the set of totally geodesic faces of $\partial\mathcal{O}$. Cover $\mathcal{O}$ by locally finite linear models.

Suppose that $\mathcal{O}$ has only codimension-one strata. Then the result is clear.

As an induction hypothesis, suppose that we proved the result when orbifolds have only codimension i strata.

Let $\mathcal{O}$ have a mutually disjoint collection of codimension-$(i+1)$ strata $Y_1, Y_2, \dots$ but no higher codimension ones. Since Y_i is relatively closed and with no lower dimensional stratum in its closure, it follows that Y_i is a properly embedded manifold with $\partial Y_i \subset \partial\mathcal{O}$. In fact, $Y = \bigcup_{i=1,2,\dots} Y_i$ is a properly embedded manifold. Since Y_i is in a stratum of conjugate local groups, it follows that Y_i is a neat suborbifold of $\mathcal{O}$. Hence, Y_i has a tubular neighborhood. (See Section 4.4.2.)

Define a smooth positive valued function ρ_{Y_i} for each Y_i so that $\rho_{Y_i}^{-1}(0) = Y_i$ and define each tubular neighborhood $T^j_{Y_i}$ as $\rho_{Y_i}^{-1}([0, \epsilon^i_j))$ for some small positive valued functions $\epsilon^i_1, \epsilon^i_2 : Y_i \to \mathbb{R}$, $0 < \epsilon^i_1 < \epsilon^i_2$ so that the tubular neighborhoods are mutually disjoint for fixed j. We assume that $\epsilon^i_2 = 2\epsilon^i_1$. We may assume that $T^j_{Y_i}$ are tubular neighborhoods of Y_i formed by exponential maps of the normal bundles of Y_i. Let $U^j = \bigcup_{i=1,2,\dots} T^j_{Y_i}$. Define $\pi_X : U^2 \to Y$ by the nearest point projection. Define a foliation $\mathcal{F}$ on $U^2 - Y \to Y$ by inverse images of points under π_X. (We need to choose sufficiently small ϵ^i_j.)

We define a map and a graph of $t\epsilon^i_2$, $0 < t < 1$:

$$(\pi_{Y_i}, \rho_{Y_i}) : T^j_{Y_i} \to Y_i \times \mathbb{R}_+ \text{ and } G_{t\epsilon^i_2} := \{(y, t\epsilon^i_2(y)) | y \in Y_i)\}.$$

Define Σ_t be the union of the inverse image of $G_{t\epsilon^i_2}$ for $0 < t < 1$ under (π_{Y_i}, ρ_{Y_i}) for $i = 1, 2, \dots$.

The orbifold $\mathcal{O} - U^1$ is an orbifold with corner and codimension i strata only but with new faces in the boundary. (Note that collars can be obtained by the tubular neighborhoods.) Hence by induction, we can form π_X, T_X, ρ_X for each stratum X in it satisfying the abstract stratification conditions. Let t_0 satisfy $1/2 = \epsilon^i_1/\epsilon^i_2 < t_0 < 1$. Let U' be the open submanifold of U^2 containing X bounded by Σ_{t_0}. Consider $\mathcal{O} - U'$. Now, we radially extend the open set T_X, the stratum X itself, and these maps. T_X is extended by taking $T_X \cap \Sigma_{t_0}$ and isotopying them into $\Sigma_{t'}$ for $0 < t' < t_0$ preserving the leaves of $\mathcal{F}$ and similarly for X. For each t, $0 < t < t_0$, we define $\pi_X : \Sigma_t \to X \cap \Sigma_t$ and $\rho_X : \Sigma_t \to \mathbb{R}^+$ by conjugating the map ρ_X and π_X on Σ_{t_0} by a diffeomorphism and so on for strata X of $\mathcal{O}$ other than Y_is. Now the smoothness of π_X and ρ_X is obtained by smoothing operations that preserve Σ_ts. (We use the coordinates where Σ_t are defined by a coordinate function.) For each face A_i meeting X, we can extend the maps p_{A_i} and r_{A_i} similarly.

Hence, it follows that $\mathcal{O}$ has an abstract stratification.

Given an orbifold $\mathcal{O}$, one can remove tubular neighborhoods of the union of singular loci of dimension-zero forming another orbifold $\mathcal{O}_1$ and removing tubular neighborhoods of the union of singular loci of dimension-one and so on. Therefore, we see that we can build $\mathcal{O}$ starting from a manifold and adding tubular neighborhoods of strata of codimension $n-1, n-2, \dots, 2, 1$. At each step, of course, we obtain orbifolds with corners.

The conditions in Section 4.5.1 are satisfied with our choices. This proves that $\mathcal{O}$ has an abstract stratification. Finally, we obtain the smooth triangulation by Theorem 4.5.1. □

4.6 Covering spaces of orbifolds

Let X be an orbifold. Let X' be an orbifold with a smooth map $p : X' \to X$ so that for each point x of X, there is a connected model (U, G, ϕ) and the inverse image of $\phi(U)$ is a union of open sets $U_i, i \in I$ for an index set I with models isomorphic to (U, G'_i, π_i) where π_i is equivalent to the quotient map $q_i : U \to U/G'_i$ and G'_i is a subgroup of G so that the following diagram commutes for each $i \in I$

$$\begin{tikzcd} U \arrow[rr, two heads, "\pi_i"] \arrow[d, two heads, "q_i"'] & & U_i \arrow[d, "p"] \\ U/G'_i \arrow[urr, "\hat{\pi}_i", "\cong"'] \arrow[r, "q'_i"'] & U/G \arrow[r, "\hat{\phi}", "\cong"'] & \phi(U) \end{tikzcd} \tag{4.2}$$

where q'_i is the quotient map, $\hat{\phi}$ is the induced map of ϕ and $\hat{\pi}_i$ is the induced map of π_i.

Then we say that $p : X' \to X$ is a *covering* and X' is a *covering orbifold* of X. Usually, we will require the underlying spaces $|X|$ and $|X'|$ to be connected unless we mention otherwise.

We can see it as an orbifold bundle over X with discrete fibers. We can choose the fibers to be acted upon by a discrete group G (usually on the right), and hence a principal G-bundle.

Given two covering orbifolds $p_1 : X_1 \to X$ and $p_2 : X_2 \to X$, we define a *covering morphism* to be a smooth orbifold map $f : X_1 \to X_2$ so that $p_2 \circ f = p_1$. A covering automorphism group of a covering $p : X' \to X$ is a group of diffeomorphisms γ satisfying

$$\begin{array}{ccc} X' & \xrightarrow{\gamma} & X' \\ p\downarrow & & p\downarrow \\ X & = & X. \end{array}$$

An element is called a *covering automorphism* or a *deck transformation.* A *regular covering* is a covering where the deck transformation group acts transitively on the fibers. Sometimes, this is called a *Galois covering* and the covering automorphism group is called a *Galois group* or a *deck transformation group.*

4.6.1 ***The fiber product construction by Thurston***

Let us first review the fiber product constructions for the ordinary covering space theory.

Let Y be a connected manifold, and $\tilde{Y}$ a regular covering map $\tilde{p}$ with the covering automorphism group Γ. Let $\Gamma_i, i \in I$ for an index set I be a sequence of subgroups of Γ, and let $p_i : \tilde{Y}/\Gamma_i \to Y$ be the sequence of induced covering maps.

- The projection $\tilde{p}_i : \tilde{Y} \times (\Gamma_i \backslash \Gamma) \to \tilde{Y}$ induces a covering
$$\hat{p}_i : (\tilde{Y} \times (\Gamma_i \backslash \Gamma))/\Gamma \to \tilde{Y}/\Gamma = Y$$
where Γ acts by
$$\gamma(\tilde{x}, \Gamma_i \gamma_i) = (\gamma(\tilde{x}), \Gamma_i \gamma_i \gamma^{-1})$$
- This map is equivalent to $p_i : \tilde{Y}/\Gamma_i \to Y$ since Γ acts transitively on the set of components of $\tilde{Y} \times (\Gamma_i \backslash \Gamma)$.

- We now define the *fiber-product* $\tilde{Y} \times (\prod_{i \in I} \Gamma_i \backslash \Gamma) \to \tilde{Y}$ of $\tilde{p}_i$ for $i \in I$. Define the left-action of Γ by
$$\gamma(\tilde{x}, (\Gamma_i \gamma_i)_{i \in I}) = (\gamma(\tilde{x}), (\Gamma_i \gamma_i \gamma^{-1})), \gamma \in \Gamma.$$
By taking quotients of both sides by Γ, we obtain that the *fiber-product* of $p_i : \tilde{Y}/\Gamma_i \to Y$, $i \in I$ is isomorphic to
$$p^f : Y^f := (\tilde{Y} \times \prod_{i \in I} \Gamma_i \backslash \Gamma)/\Gamma \to \tilde{Y}/\Gamma = Y.$$
(This construction gives us coverings with perhaps many components.)
- For each $i \in I$, there is a covering map
$$p_i^f : Y^f \to \tilde{Y}/\Gamma_i \text{ satisfying } p_i \circ p_i^f = p^f \tag{4.3}$$
induced from the projection
$$\tilde{Y} \times \prod_{i \in I} \Gamma_i \backslash \Gamma \to \tilde{Y} \times (\Gamma_i \backslash \Gamma)$$
(This is the "categorical" universal property we need.)

4.6.1.1 *The fiber product of orbifolds*

Let Y be a connected orbifold. We can let Γ be a discrete group acting on an orbifold $\tilde{Y}$ properly discontinuously but maybe not freely. $Y = \tilde{Y}/\Gamma$ is said to be an *orbifold quotient* of $\tilde{Y}$ and Y is said to be *developable* or *good* if $\tilde{Y}$ is a manifold.

In the above example, we can let Γ be a discrete group acting on an orbifold $\tilde{Y}$ properly discontinuously but possibly not freely. Let Γ_i for each $i \in I$ be a subgroup and $p_i : \tilde{Y}/\Gamma_i \to Y$ be the covering map for each $i \in I$ where I is an index set. $p^f : Y^f \to Y$ is again defined to be the *fiber product* of orbifold maps $p_i : \tilde{Y}/\Gamma_i \to Y$. Moreover, p^f has the *universal property* for the collection $p_i, i \in I$ that there is a covering $p_i^f : \tilde{Y} \to \tilde{Y}/\Gamma_i$ for each i so that $p^f = p_i^f \circ p_i$.

4.6.1.2 *The doubling orbifolds*

A *mirror point* or *silvered point* is a singular point with the stabilizer group $\mathbb{Z}_2$ acting as a reflection group. One can double an orbifold M with mirror points so that mirror points disappear.

- Let V_i for $i \in I$ be the neighborhoods of M with charts (U_i, G_i, ϕ_i), where I is an index set.
- Define new charts $(U_i \times \{-1, 1\}, G_i, \phi_i^*)$ where G_i acts by
$$g(x, l) = (g(x), s(g)l)$$
where $s(g)$ is 1 if g is orientation-preserving and -1 if not and ϕ_i^* is the quotient map to $U_i \times \{-1, 1\}/G_i$.

- For each embedding $i : (W, H, \psi) \to (U_i, G_i, \phi_i)$, we define a lift

$$(W \times \{-1, 1\}, H, \psi^*) \to (U_i \times \{-1, 1\}, G_i, \phi_i^*).$$

  These define the gluing maps.
- The result of the quotiening by the gluing maps is the doubled orbifold and the local group actions are orientation preserving. (We just need to verify that the topology is second-countable and Hausdorff.)
- The result double-covers the original orbifold with Galois group or the covering automorphism group isomorphic to $\mathbb{Z}_2$.

Proposition 4.6.1. *A doubled orbifold has no reflection with a hypersurface fixed set. Hence the set of regular points is dense open and locally-path-connected and path-connected.*

Proof. Since there is no orientation reversing element in the local group, the first statement is clear. If there is no reflection, then the singularity is of codimension two or greater and hence the set of regular points is dense open and path-connected locally. Thus, the second statement follows. □

For example, if we double a cell with a corner-reflector, it becomes a cell with a cone-point.

4.6.2 *Universal covering orbifolds by fiber-products*

Let Y be a connected orbifold. A *base point* of a covering is a regular point of the cover mapping to a regular base point of the covered orbifold. A *universal cover* of Y is an orbifold $\tilde{Y}$ so that for any covering orbifold Y' of Y and base points y^* of $\tilde{Y}$ and y' of Y' mapping to a base point y of Y, there exists a covering map $p : \tilde{Y} \to Y'$ satisfying $p(y^*) = y'$.

As some examples, we state without justifications:

- Clearly, manifolds are orbifolds. Manifold coverings provide examples.
- A *tear-drop* is a sphere with one cone-point of order n. Let Y be a tear-drop orbifold with a cone-point of order n. Then this cannot be covered by any other type of an orbifold and hence is a universal cover of itself. (See Section 4.7.1.3 and Theorem 4.7.4.)
- A sphere Y with two cone-points of orders p and q which are relatively prime is a universal cover of itself. (See Section 4.7.1.3 and Theorem 4.7.4.)
- Choose a cyclic action of Y of order m fixing the cone-points. Then $Y/\mathbb{Z}_m$ is an orbifold with two cone-points of order pm and qm, and Y is the universal cover of $Y/\mathbb{Z}_m$.

We will now show that the universal covering orbifold exists by using fiber-product constructions. For this, we need to discuss elementary neighborhoods.

An *elementary* neighborhood for a covering $p : Y' \to Y$ is an open subset $\phi(U)$ with a model triple (U, G, ϕ) so that the situation in equation 4.2 is satisfied.

We can take the model open set in the chart to be one so that U in the model triple (U, G, ϕ) is a cell. Then such an open set is elementary as we can see from below.

4.6.2.1 *Fiber-products for D^n/G_i*

Let D^n be a cell, i.e., a contractible manifold homeomorphic to a convex subset of $\mathbb{R}^n_+$, with possibly nonempty boundary. Suppose that V is an orbifold D^n/G for a finite group G acting effectively. We deduce that

- We can show that any covering of D^n/G is equivalent to D^n/G_1 for a subgroup G_1 of G. (See Proposition 7 in the article [Choi (2004)].)
- Given two covering orbifolds D^n/G_1 and D^n/G_2 for subgroups G_1 and G_2 of G, one can induce a covering morphism $D^n/G_1 \to D^n/G_2$ by $g \in G$ so that $gG_1g^{-1} \subset G_2$.
- The covering morphism is in one-to-one correspondence with the double cosets of form G_2gG_1 for g such that $gG_1g^{-1} \subset G_2$.
- The covering automorphism group of D^n/G_1 for a subgroup G_1 of G is given by $N(G_1)/G_1$ where $N(G_1)$ is the normalizer of G_1 in G.

(For the detailed proofs of these elementary facts, see the article [Choi (2004)].)

Given a collection of coverings $p_i : D^n/G_i \to D^n/G$ for $i \in I$ for a collection I, $G_i \subset G$, and an n-cell D^n, we form a fiber-product.

$$V^f = (D^n \times \prod_{i \in I} G_i \backslash G)/G \to D^n/G.$$

If we choose all subgroups G_i of G, then any covering D^n/G_i of D^n/G is covered by V^f induced by projection to G_i-factor by Section 4.6.1.1. This is the universal property we seek.

4.6.2.2 *The construction of the fiber-product of a collection of covering orbifolds*

Let $Y_i, i \in I$ be a collection of the orbifold-coverings of Y. We cover Y by elementary neighborhoods V_j for $j \in J$ for an index set J forming a good cover. Now fix j. We take components of $p_i^{-1}(V_j)$ each of which is equivalent to a disjoint union of V/G_k for some finite group G_k where V is a convex open subset of $\mathbb{R}^n_+$. Fix j. We take one component of $p_i^{-1}(V_j)$ for each i and form one fiber product. Then we are left with a disjoint union of fiber products indexed by the choice of components of $p_i^{-1}(V_j)$ for each i. Over regular points of V_j, this is the ordinary fiber-product. Now, we wish to patch these up using embeddings. Let $U \to V_j \cap V_k$ be an embedding. We can assume $U = V_j \cap V_k$ which has a convex cell as a cover.

- We form the fiber product $p_U : U^f \to U$ of $p_i^{-1}(U)$, $i \in I$ and form the fiber product $p_{V_j} : V_j^f \to V_j$ and $p_{V_k} : V_k^f \to V_k$.
- $U_j = p_{V_j}^{-1}(U)$ in V_j^f is identifiable with U^f since the fiber-product construction of U_j in V_j^f is identical with one in U^f with just different labeling.
- Similarly, $U_k = p_{V_k}^{-1}(U)$ in V_k^f is identifiable with U^f.
- Thus, each component of the fiber-products can be identified with another one by the natural maps of form $U_j \to U_k$.

By patching, we obtain a covering Y^f of Y with the covering map p^f. Note that Y^f is not necessarily connected. But each component of Y^f is Hausdorff and second-countable and hence is an orbifold.

Let $\tilde{Y}$ be a component of Y^f. Also for any cover (Y_i, y_i), there is a covering morphism $q_i : \tilde{Y} \to Y_i$ with $q_i(y^*) = y_i$ and so that $p_i \circ q_i = p^f$: the basic reason is that for each component of $p_i^{-1}(U)$ for an elementary neighborhood U of p_i in Y, there is a map from a component of $p^{f,-1}(U)$ mapping to it by Section 4.6.1.1 and we can patch these maps together: We show the consistent definition of this map by considering chains of intersecting open components of sets of form $p^{f,-1}(U)$ for an elementary neighbrhood U in Y. Basically, if three such open sets intersect, then we can show that the map is consistently defined. This is similar to the way one obtains developing maps for geometric structures (see Section 6.1.2). (See the bottom of page 178 of the article [Choi (2004)] also.)

4.6.2.3 *Thurston's example of a fiber product*

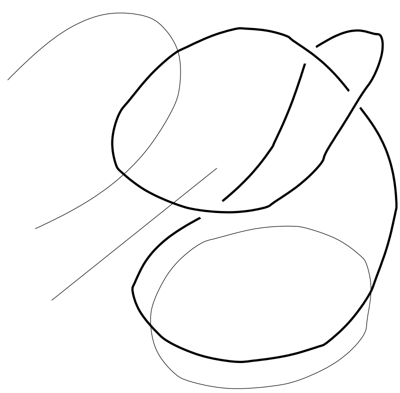

Fig. 4.7 The fiber product of two two-fold covers of the interval I with silvered endpoints by a circle and interval I with silvered endpoints. It is convenient to visualize a cylinder over the bottom circle parallel to the z-axis and the sheet parallel to the y-axis passing the curved arc in the left. The circle is almost on the intersection.

Let I be the unit interval. Make two endpoints into silvered points. Then $I_1 = I$ is double-covered by $\mathbf{S}^1$ with the deck transformation group $\mathbb{Z}_2$. Let p_1 denote the covering map. $I_2 = I$ is also covered by I by a map $x \mapsto 2x$ for $x \in [0, 1/2]$ and $x \mapsto 2-2x$ for $x \in [1/2, 1]$. Let p_2 denote this covering map. Then we determine the fiber product of p_1 and p_2: Cover I by $A_1 = [0, \epsilon), A_2 = (\epsilon/2, 1-\epsilon/2), A_3 = (\epsilon, 1]$ for $0 < \epsilon < 1/4$.

- $p_1^{-1}(A_1)$ is an open interval and $p_2^{-1}(A_1)$ is a union of two half-open intervals. The fiber-product is a union of two copies of open intervals.
- Over A_2, the fiber product is a union of four copies of open intervals.
- Over A_3, the fiber product is a union of two copies of open intervals.
- By pasting considerations, we obtain a circle mapping 4-1 almost everywhere to I. This could be a long process.

4.6.2.4 *The construction of the universal cover*

Consider the collection Y_i, $i \in I$, of all covers of an orbifold Y. We take each one Y_i with a different choice of a base point y_i over a fixed regular point y of Y. These all are regular points. We take a fiber product of (Y_i, y_i), $i \in I$ and we take a connected component $\tilde{Y}$ containing a base point y^*. Let $\tilde{p}$ denote the restriction of the fiber-product map p^f to $\tilde{Y}$. Hence, $\tilde{Y}$ is a universal cover.

Proposition 4.6.2. *Let Y be a connected orbifold. The universal cover $\tilde{Y}$ of an orbifold Y has an open dense connected set of regular points. Any covering automorphism $\phi : \tilde{Y} \to \tilde{Y}$ that fixes a regular point is the identity map.*

Proof. A universal cover has a morphism to a double Y^2 of the orbifold. Any point mapping to a regular point is also regular. The set of such points is also dense and open and locally path connected. Since the subspace $Y^{2,r}$ of regular points of Y^2 is connected and the set of singular points is at least of codimension 2, the first part follows.

Let $\tilde{Y}^r$ denote the inverse image of the subspace $Y^{2,r}$. Then $\tilde{Y}^r$ is connected and is a covering in the ordinary sense of topology. If ϕ fixes a regular point of $\tilde{Y}$, then it fixes the points of an open model neighborhood. By density, ϕ fixes a point of $\tilde{Y}^r$. If ϕ fixes a point in $\tilde{Y}^r$, then it is the identity on $\tilde{Y}^r$. Since $\tilde{Y}^r$ is dense, ϕ is the identity. □

Theorem 4.6.3. *Let Y be a connected orbifold. The universal cover of an orbifold Y is unique up to covering orbifold-isomorphisms by the universality property.*

Proof. If (Y', y') is another universal cover, then it arises in the list of covers and hence there is a covering morphism $q : \tilde{Y} \to Y'$ with $q(y^*) = y'$. Conversely, we have a morphism $p' : Y' \to \tilde{Y}$ with $p'(y') = y^*$. We obtain a morphism $p' \circ q : \tilde{Y} \to \tilde{Y}$ fixing y^*. By Proposition 4.6.2, $p' \circ q$ is the identity. Similarly, so is $q \circ p'$. □

4.6.2.5 *Properties of the universal cover*

The group of covering automorphisms of the universal cover $\tilde{Y}$ is called a *fundamental group* and is denoted by $\pi_1(Y)$, which is well-defined up to isomorphism by Theorem 4.6.3. (This will be known more accurately as a *Galois-group* in Section 4.7.)

Proposition 4.6.4. *Let $\tilde{Y}$ be a universal cover of an orbifold Y with the covering map $\tilde{p}$.*

- *The deck transformation group $\pi_1(Y)$ of $\tilde{Y}$ acts transitively on fibers of $\tilde{p}^{-1}(x)$ for each x in Y.*
- *$\tilde{p}$ induces a diffeomorphism $\tilde{Y}/\pi_1(Y) \to Y$.*
- *For a subgroup Γ of $\pi_1(Y)$, $\tilde{Y}/\Gamma$ is a covering of Y with the induced covering map from $\tilde{p}$.*
- *Any covering of Y is of form $\tilde{Y}/\Gamma$ for a subgroup Γ of $\pi_1(Y)$.*
- *The set of isomorphism classes of coverings of Y is in one-to-one correspondence with the set of conjugacy classes of subgroups of $\pi_1(Y)$.*

Proof. Let y be a regular base-point of Y. We change the base point of $\tilde{Y}$ to any point z of $\tilde{p}^{-1}(y)$. Then there always is a morphism $q : (\tilde{Y}, y^*) \to (\tilde{Y}, z)$. We find an inverse to q by finding $t = q^{-1}(y^*)$. Then there exists a morphism $q' : (\tilde{Y}, y^*) \to (\tilde{Y}, t)$. Hence, $q \circ q'(y^*) = y^*$. Thus, q' is the inverse and q is a covering automorphism by Proposition 4.6.2. Thus, $\pi_1(Y)$ acts transitively on $\tilde{p}^{-1}(y)$.

Given a point x, we find a path γ in Y with endpoints x and y so that its local lifts to the model neighborhoods have nonzero derivative vectors everywhere. Then each lift to a model open set is unique up to the model group action. Thus, γ lifts to a smooth curve in $\tilde{Y}$ with endpoints a point of $\tilde{p}^{-1}(x)$ and $\tilde{p}^{-1}(y^*)$. In fact the lift is unique up to the choice of the starting point in $\tilde{p}^{-1}(x)$. We see that $\pi_1(Y)$ also acts transitively on the set of lifts. Since we can find a lift starting from any point of $\tilde{p}^{-1}(x)$, we see that $\pi_1(Y)$ acts transitively on $\tilde{p}^{-1}(y)$ for any $y \in Y$.

We see that the quotient orbifold $\tilde{Y}/\pi_1(Y)$ is clearly in a one-to-one correspondence with Y. The charts are also compatible.

We omit the proof of the third item.

For a covering $Y' \to Y$, there is a covering morphism $p' : \tilde{Y} \to Y'$. Now, Y' is actually of form the quotient orbifold $\tilde{Y}/\Gamma$ for a subgroup Γ of $\pi_1(Y)$: Suppose that two regular points p and q of $\tilde{Y}$ go to the same regular point q' of Y' and hence to a point q'' of Y. Then there exists a deck transformation γ so that $\gamma(p) = q$. By considering an elementary neighborhood U of p in Y and the components C_1, $p \in C_1$ and C_2, $q \in C_2$, of its inverse images in $\tilde{Y}$. Consider also the component V of its inverse image in Y' containing q'. Then C_1 and C_2 cover V respectively. This can be seen by a path-lifting argument using curves as above. Clearly, $\gamma(C_1) \subset C_2$ and $\gamma^{-1}(C_2) \subset C_1$ since these are path-components of the inverse image of U. Thus, we have $\gamma(C_1) = C_2$. Since p' sends C_1 and C_2 into V, it follows that every pair of points $(x, \gamma(x))$ of $x \in C_1$ go to a point of V under p'. From this, it follows that every pair of points $(x, \gamma(x))$ for $x \in \tilde{Y}$ go to a point in Y' under p'. If p and q are not regular, then we can find nearby regular points that go to a same point in Y' by lifting a path.

Let Γ be the subset of elements γ of $\pi_1(Y)$ so that the pairs $(x, \gamma(x))$ for all $x \in \tilde{Y}$ are identified under p'. Then Γ is clearly a subgroup. Moreover, if $\gamma \in \pi_1(Y)$ is so that x and $\gamma(x)$ are identified to a point of Y' under p', then $\gamma \in \Gamma$ by the above argument. Hence, it follows that Y' is the quotient orbifold $\tilde{Y}/\Gamma'$.

Given two coverings $Y_1 \to Y$ and $Y_2 \to Y$, we see that an isomorphism $f : Y_1 \to Y_2$ lifts to a diffeomorphism $\tilde{Y} \to \tilde{Y}$. We choose an automorphism fixing y^* by multiplying by an element of $\pi_1(Y)$. By restricting to the regular part, we see that the morphism is the identity map and f is induced by an element of $\pi_1(Y)$. Since Y_1 can be identified with $\tilde{Y}/\Gamma_1$ and Y_2 with $\tilde{Y}/\Gamma_2$, it follows that Γ_1 and Γ_2 are conjugate. The converse is also simple. □

Let Γ be a subgroup of $\pi_1(Y)$. Given the quotient space $\tilde{Y}/\Gamma$, one deduces that an element γ of $\pi_1(Y)$ represents a covering isomorphism $\tilde{Y}/\Gamma \to \tilde{Y}/\Gamma$ if and only if $\gamma\Gamma = \Gamma\gamma$. Thus, γ is in the normalizer $N(\Gamma)$. Conversely, each covering automorphism of $\tilde{Y}/\Gamma \to Y$ lifts to an element $\gamma \in \pi_1(Y)$. Given a covering $\tilde{Y}/\Gamma \to Y$, we determine that the group of covering automorphisms is $N(\Gamma)/\Gamma$. Therefore, a covering is regular or Galois if and only if Γ is a normal subgroup of $\pi_1(Y)$. (These proofs are identical with the ordinary covering-space theory.)

A *good orbifold* is an orbifold with a cover that is a manifold. A *very good orbifold* is an orbifold with a finite cover that is a manifold. A good orbifold has a symply connected manifold as a universal covering space: it has a covering space that is a manifold and the universal covering orbifold must cover this manifold and hence the universal covering space has to be a manifold.

4.6.2.6 *Induced homomorphisms of the fundamental group*

Given two orbifolds Y_1 and Y_2 and an orbifold-diffeomorphism $g : Y_1 \to Y_2$, we obtain that the lift to the universal covers $\tilde{Y}_1$ and $\tilde{Y}_2$ is also an orbifold-diffeomorphism. Furthermore, if the lift value is determined at a point, then the lift is unique.

Proposition 4.6.5. *Let Y_1 and Y_2 be connected orbifolds of same dimension. An isotopy $f_t : Y_1 \to Y_2$ for $t \in [0, 1]$ of orbifold-diffeomorphisms lifts to an isotopy in the universal covering orbifold $\tilde{f}_t : \tilde{Y}_1 \to \tilde{Y}_2$ for each $t \in I$ unique up to a choice of $\tilde{f}_0(y)$.*

Proof. We consider regular parts and model neighborhoods where the lifts clearly exist uniquely for each t. The map $t \mapsto f_t(y)$ for a regular base point y of Y is a path in Y. Then $f_t(y)$ is regular for all $t \in I$. This lifts to a smooth path $\tilde{\gamma} : t \mapsto p^{-1}(f_t(y))$. Since f_t is an orbifold diffeomorphism, there is a lifting diffeomorphism $\tilde{f}_t : \tilde{Y} \to \tilde{Y}$ for each t determined up to post-composing with the deck transformations. By post-composing with elements of $\pi_1(Y)$ if necessary, we can make sure that a lift $\tilde{f}_t : \tilde{Y} \to \tilde{Y}$ satisfies $\tilde{f}_t(y) = \tilde{\gamma}(t)$ for each t. Now, we can verify that $\tilde{f}_t$ forms an isotopy.

□

Given an orbifold-diffeomorphism $f : Y \to Z$ which lifts to a diffeomorphism $\tilde{f} : \tilde{Y} \to \tilde{Z}$, we obtain a homomorphism $\tilde{f}_* : \pi_1(Y) \to \pi_1(Z)$: for each $\gamma \in \pi_1(Y)$, there exists a unique $\delta \in \pi_1(Z)$, so that $\tilde{f} \circ \gamma = \delta \circ \tilde{f}$. If g is isotopic to f and so is its lift $\tilde{g}$ to $\tilde{f}$, then it follows that $\tilde{g}_* = \tilde{f}_*$. (Note that we can define f_* for orbifold-diffeomorphisms only. When f is not a diffeomorphism, we need also the information on the local lifts as well to describe the map using the path-approach below. We will not attempt this in this book.)

Finally notice that if Y_1 is an open suborbifold of Y_2, then we can define a homomorphism $\tilde{\iota}_* : \pi_1(Y_1) \to \pi_1(Y_2)$ where $\tilde{\iota}$ is the lift $\tilde{Y}_1 \to \tilde{Y}_2$ of the inclusion map $\iota : Y_1 \to Y_2$.

Using the path-approach of Haeflger, we obtain a more general result for this. (See Section 4.7.1.2.)

4.7 The path-approach to the universal covering spaces following Haefliger

We will now study the path-approach to the fundamental groups and the universal covering spaces following Bridson and Haefliger (1999). Thus, we see that the ordinary covering theory for topological spaces and the covering theory for orbifolds are very much alike.

4.7.1 *$\mathcal{G}$-paths*

We generalize the notion of paths in the topological spaces to one of those on groupoids: Given an étale groupoid X with the space of arrows $\mathcal{G}$ and the space of objects X_0, we define a *$\mathcal{G}$-path* c to be an object $(g_0, c_1, g_1, \dots, c_k, g_k)$ with a subdivision $a = t_0 \leq t_1 \leq \cdots \leq t_k = b$ of interval $[a, b]$ consisting of

- continuous maps $c_i : [t_{i-1}, t_i] \to X_0$
- elements $g_i \in X_1$ so that $s(g_i) = c_{i+1}(t_i)$ for $i = 0, 1, \dots, k-1$ and $t(g_i) = c_i(t_i)$ for $i = 1, \dots, k$.

The initial point is $t(g_0)$ and the terminal point is $s(g_k)$. We will normally require that the orbit space $|X|$ of X is connected. That is, the underlying space is connected since the orbit space is homeomorphic to the underlying space. (See Example 4.3.)

The three operations define an equivalence relation:

- Subdivision: Add new division point t'_i in $[t_i, t_{i+1}]$ and $g'_i = \mathrm{I}_{c_i(t'_i)}$ and replacing c_i with c'_i, g'_i, c''_i where c'_i, c''_i are restrictions to $[t_i, t'_i]$ and $[t'_i, t_{i+1}]$.
- Adjoining: We reverse the subdivision process.
- Replacement: replace c with $c' = (g'_0, c'_1, g'_1, \dots, c'_k, g'_k)$ as follows. For each i choose continuous map $h_i : [t_{i-1}, t_i] \to X_1$ so that $s(h_i(t)) = c_i(t)$ and define $c'_i(t) = t(h_i(t))$ and $g'_i = h_i(t_i) g_i h^{-1}_{i+1}(t_i)$ for $i = 1, \dots, k-1$ and $g'_0 = g_0 h_1^{-1}(t_0)$ and $g'_k = h_k(t_k) g_k$.

All paths are defined on $[0,1]$ from now on. Given two $\mathcal{G}$-paths $c = (g_0, c_1, \dots, c_k, g_k)$ over $0 = t_0 \leq t_1 \leq \cdots \leq t_k = 1$ and $c' = (g'_0, c'_1, \dots, c'_{k'}, g'_{k'})$ over $0 = t'_0 \leq t'_1 \leq \cdots \leq t'_{k'} = 1$ such that the terminal point of c equals the initial point of c', we define the *composition* $c * c'$ to be the $\mathcal{G}$-path $c'' = (g''_0, c''_1, \dots, c''_{k+k'}, g''_{k+k'})$ so that

- $t''_i = t_i/2$ for $i = 0, \dots, k$ and $t''_i = 1/2 + t'_{i-k}/2$ for $i = k+1, \dots, k+k'$;
- $c''_i(t) = c_i(2t)$ for $i = 1, \dots, k$ and $c''_i(t) = c'_{i-k}(2t-1)$ for $i = k+1, \dots, k+k'$; and
- $g''_i = g_i$ for $i = 0, \dots, k-1$ and $g''_k = g_k g'_0, g''_i = g'_{i-k}$ for $i = k+1, \dots, k+k'$.

The inverse c^{-1} is $(g'_0, c'_1, \dots, c'_k, g'_k)$ over the subdivision where $t'_i = 1 - t_i$ so that $g'_i = g^{-1}_{k-i}$ and $c'_i(t) = c_{k-i+1}(1-t)$.

4.7.1.1 *Homotopies of $\mathcal{G}$-paths*

There are two types of homotopies:

- Equivalences
- An *elementary homotopy* is a family of $\mathcal{G}$-paths $c^s = (g^s_0, c^s_1, \dots, c^s_k, g^s_k)$ over the subdivision $0 = t^s_0 \leq t^s_1 \leq \cdots \leq t^s_k = 1$ so that each of t^s_k, g^s_i, c^s_i depends continuously on s. We require that $t(g^s_0)$ and $s(g^s_k)$ are to be constant independent of s as usual for a homotopy of paths.

Two $\mathcal{G}$-paths a and b are *homotopic* if there is a sequence of $\mathcal{G}$-paths $a = a_1, a_2, \dots, a_n = b$ so that a_i and a_{i+1} are either equivalent or there is an elementary homotopy between them.

A homotopy class of c is denoted $[c]$. $[c * c']$ is well-defined in the homotopy classes $[c]$ and $[c']$. Hence, we define $[c] * [c'] = [c * c']$.

We have the associativity $[c * (c' * c'')] = [(c * c') * c'']$.

The constant path e_x at x is given as $(\mathrm{I}_x, x, \mathrm{I}_x)$. Then $[c * c^{-1}] = [e_x]$ if the initial point of c is x and $[c^{-1} * c] = [e_y]$ if the terminal point of c is y. Thus, $[c]^{-1} = [c^{-1}]$.

We can show easily that the homotopy classes of paths form a *fundamental groupoid.*

4.7.1.2 *The fundamental group $\pi_1(X, x_0)$*

A *loop* is a $\mathcal{G}$-path with the identical initial and terminal points. The *fundamental group* $\pi_1(X, x_0)$ based at $x_0 \in X_0$ is the group of homotopy classes of loops based at x_0. (We will require x_0 to be a regular point.) The associativity, identity and inverse properties are proven above.

Let X be an open suborbifold of Y. Then the inclusion map $f : X \to Y$ induces a homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ where $f(x_0)$ is regular also. In fact, f could be any homomorphism of groupoids. Hence f could be an orbifold map $X \to Y$ so that for each point $x \in X$, there exists a model triple (U, G, ϕ) and a model triple (V, H, ψ) of $f(x) \in Y$ so that f lifts to a map $\tilde{f} : U \to V$ unique up to the action

$$\tilde{f} \mapsto h \circ f \circ g \text{ for } h \in H, g \in G.$$

Therefore, a covering map will induce the homomorphism.

Theorem 4.7.1 (Seifert-Van Kampen). *Let X be an orbifold with the space of objects X_0 and the space of arrows $\mathcal{G}$. Assume that the space $|X|$ of orbits is connected. Let $X_0 = U \cup V$ where U and V are open and $U \cap V = W$. Assume that the groupoid restrictions $\mathcal{G}_U$, $\mathcal{G}_V$, $\mathcal{G}_W$ to U, V, W are connected. And let $x_0 \in W$. Then $\pi_1(X, x_0)$ is isomorphic to the quotient group of the free product $\pi_1(\mathcal{G}_U, x_0) * \pi_1(\mathcal{G}_V, x_0)$ by the normal subgroup generated by $j_U(\gamma)j_W(\gamma^{-1})$ for $\gamma \in \pi_1(\mathcal{G}_W, x_0)$ for the induced homomorphism $j_U : \pi_1(\mathcal{G}_W, x_0) \to \pi_1(\mathcal{G}_U, x_0)$ and the induced homomorphism $j_V : \pi_1(\mathcal{G}_W, x_0) \to \pi_1(\mathcal{G}_V, x_0)$.*

Here a groupoid restriction $\mathcal{G}_U$ means restricting the space of objects to U and the space of arrows to those arrows with tails and sources in U.

In a more set theoretic language, this means: Let X be an orbifold so that $|X| = U \cup V$ for two open subsets U and V and let $W = U \cap V$ be a connected open set. Let $x_0 \in W$ and $\hat{U}$, $\hat{V}$, and $\hat{V}$ denote the induced orbifolds. Then $\pi_1(X, x_0)$ is isomorphic to the quotient group of $\pi_1(\hat{U}, x_0) * \pi_1(\hat{V}, x_0)$ by the normal subgroup generated by $j_U(\gamma)j_V(\gamma^{-1})$ for $\gamma \in \pi_1(\hat{W}, x_0)$ for the induced homomorphism $j_U : \pi_1(\hat{W}, x_0) \to \pi_1(\hat{U}, x_0)$ and the one $j_V : \pi_1(\hat{W}, x_0) \to \pi_1(\hat{V}, x_0)$.

The proof is omitted but is remarkably similar to the elementary topology proof using dividing homotopies into small ones mapping into model-neighborhoods. This is an exercise in Chapter III$\mathcal{G}$ in the book [Bridson and Haefliger (1999)].

4.7.1.3 *Examples*

- Consider a tear-drop orbifold. We remove a small disk about the cone-point. The remainder is a disk and has a trivial fundamental group. The disk about the cone-point has the fundamental group isomorphic to the cyclic group of order n by equation 4.4. By the Van-Kampen theorem, a tear-drop has the trivial fundamental group.

- Similarly, we can show that a sphere Y with two cone-points of relatively prime orders p and q has a trivial fundamental group: Here, we remove two disjoint disks around the singularities and the Van-Kampen theorem to prove this.
- Let a discrete group Γ act on a connected manifold X_0 properly discontinuously. Then (Γ, X_0) has an orbifold structure. (See 4.3.2.1.) Let x_0 be a point with trivial stabilizer subgroup. Let g_γ denote the arrow in X_1 with starting point x_0 and the end point $\gamma(x_0)$ for $\gamma \in \Gamma$. Any loop in this groupoid is equivalent to a $\mathcal{G}$-path $(\mathrm{I}_{x_0}, c, g_\gamma)$ so that $\gamma(x_0) = c(1)$ and $c(0) = x_0$ by joining all paths in $c = (g_0, c_1, g_1, \dots, c_k, g_k)$ into a single path, i.e., by changing g_0 to 1 and c_1 to $\gamma_0^{-1} \circ c_1$ where γ_0 is the deck transformation corresponding to g_0, and g_1 to 1 and c_2 to $\gamma_0^{-1} \circ \gamma_1^{-1} \circ c_2$ where γ_1 corresponds to g_1 and so on and joining these paths. Thus, there is an exact sequence for a base point $x_0 \in X_0$:
$$1 \to \pi_1(X_0, x_0) \to \pi_1((\Gamma, X_0), x_0) \to \Gamma \to 1 \tag{4.4}$$
given by sending $[(\mathrm{I}_x, c, g_\gamma)]$ to γ. That is, $\pi_1((\Gamma, X_0), x_0)$ is an extension of Γ by $\pi_1(X_0, x_0)$. (See Example 3.7 in Chapter III.$\mathcal{G}$ in the book [Bridson and Haefliger (1999)].)
- A 2-orbifold that is a disk with an arc silvered has the fundamental group isomorphic to $\mathbb{Z}_2$: A disk with a group action generated by a reflection about an arc covers it. Thus, the result follows from equation 4.4.
- An annulus A with one boundary component silvered has a fundamental group isomorphic to $\mathbb{Z} \times \mathbb{Z}_2$ since our orbifold is covered by an annulus A_1 by an action of $\mathbb{Z}_2$ which fixes the middle circle of the annulus. There exists a section from $\mathbb{Z}_2$ to $\pi_1(A)$ given by a path γ going to the silvered arc and returning to the base point. Clearly, γ^2 is trivial.
- Consider a 2-orbifold with cone-points which is boundaryless and with no silvered point. One can cover the cone points by sufficiently small disks and we can cut out the disks. Then the Van-Kampen theorem enables one to compute the fundamental group. (See Theorem 5.1.1.)
- Suppose that a two-dimensional orbifold has boundary and silvered points. Then remove open-ball neighborhoods of the cone-points and corner-reflector points. The fundamental group of the remaining part can be computed by the Van-Kampen theorem by considering open neighborhoods of silvered boundary arcs. Finally, adding the open-ball neighborhoods, we compute the fundamental group again using the Van-Kampen theorem.

The last item implies

Corollary 4.7.2. *Let Σ be a compact 2-dimensional orbifold. Then $\pi_1(\Sigma, x_0)$ is finitely presented for any regular point x_0.*

In fact, compact n-orbifolds have finitely presented fundamental groups but we omit the proof that is a higher-dimensional generalization. The fundamental group of a three-dimensional orbifold can be computed similarly using the Van Kampen theorem. However, we need the detailed knowledge of the structure of 3-orbifolds as can be found in the book [Thurston (1977)] and some papers such as [Dunbar (1988)].

4.7.2 *Covering spaces and the fundamental group*

One can build the theory of covering spaces using the fundamental group. We review the relationship of the homotopy group of $\mathcal{G}$-paths to covering spaces first. (Here, we will only consider orbifolds with connected underlying space.)

Let us be given a covering $X' \to X$ for two orbifolds X and X'. For every $\mathcal{G}$-path c in X, there is a lift $\mathcal{G}$-path in X'. If we assign the initial point, the lift is unique. If c' is homotopic to c, then the lift of c' is also homotopic to the lift of c provided the initial points are the same. From this it follows that the induced homomorphism $\pi_1(X', x_0') \to \pi_1(X, x_0)$ is injective.

Moreover, the following familiar proposition holds:

Proposition 4.7.3. *Let $p : X' \to X$ be a covering of an orbifold X with a based point x_0 and let $p'' : X'' \to X$ be another one. Let X' have a base point x_0' going to x_0 under p and X'' has one x_0'' going to x_0 under p''. Then*
$$p''_*(\pi_1(X'', x_0'')) \subset p_*(\pi_1(X', x_0'))$$
if and only if there is a covering map $X'' \to X'$ sending x_0'' to x_0'.

Proof. This is proved using paths as in the covering theory in topology.

□

A map from a simply connected orbifold to an orbifold lifts to a cover. The lift is unique if the base-point lift is assigned. Thus, a simply connected cover of an orbifold covers any cover of the given orbifold. From this, we can show that the fiber-product construction is symply connected.

Two simply connected coverings of an orbifold are isomorphic and if base-points are given, we can find an isomorphism preserving the base-points.

Theorem 4.7.4. *A symply connected covering of an orbifold X is a universal cover (Galois-covering) with the Galois-group isomorphic to $\pi_1(X, x_0)$.*

Proof. Consider $p^{-1}(x_0)$. Choose a base-point $\tilde{x}_0$ in it. Given a point of $p^{-1}(x_0)$, we connected it with $\tilde{x}_0$ by a path. Since the paths map to the elements of the fundamental group, the Galois-group acts transitively on $p^{-1}(x)$. Hence the Galois-group is isomorphic to the fundamental group. □

Corollary 4.7.5. *An orbifold-covering $(X', x_0') \to (X, x_0)$ is Galois (regular) if and only if the image of $\pi_1(X', x_0')$ in $\pi_1(X, x_0)$ is normal.*

Proof. Again, Proposition 4.7.3 implies this. □

4.7.2.1 *The existence of the universal cover using the path-approach*

The construction follows that of the ordinary covering space theory. This is included in Exercise 3.20 in Chapter III$\mathcal{G}$ in the book [Bridson and Haefliger (1999)]. Let X be an orbifold with the space of arrows X_1 and the space of objects X_0.

- Let $\hat{X}$ be the set of homotopy classes $[c]$ of $\mathcal{G}$-paths in X with a fixed starting point x_0.
- We define a topology on $\hat{X}$ by open set $U_{[c]}$ that is the set of paths ending at a symply connected open subset U of X_0 with the homotopy class of $c*d$ for a path d in U.
- Define a map $\hat{X} \to X$ sending $[c]$ to its endpoint other than x_0.
- Define a map $\hat{X} \times X_1 \to \hat{X}$ given by $([c], g) \to [c * g]$. This defines a right X_1-action on $\hat{X}$. This makes $\hat{X}$ into a bundle over X.
- Define a left action of $\pi_1(X, x_0)$ on $\hat{X}$ given by $[c] * [c'] = [c * c']$ for $[c'] \in \pi_1(X, x_0)$. This is transitive on fibers.
- We show that $\hat{X}$ is a simply connected orbifold.

4.8 Notes

For compact group actions, see the books [Bredon (1972); Hsiang (1975)]. Good references for triangulation under group actions are articles [Illman (1978, 1983)]. For triangulation of stratified spaces, and hence orbifolds, see the articles [Goresky (1978); Johnson (1983); Verona (1984); Weinberger (1994)]. The work [Verona (1984)] is most self-contained. For general introduction to the orbifold theory, see Chapter 5 of the book [Thurston (1977)] and the article [Matsumoto and Montesinos-Amilibia (1991)]. The original papers [Satake (1956, 1957)] are also very readable. Adem, Leida, and Ruan (2007) and Bridson and Haefliger (1999) treat orbifolds as groupoids. Read the articles [Moerdijk (2002); Moerdijk and Pronk (1997)] for this approach in detail. Haefliger (1990) and Chapter 13 of the book [Ratcliffe (2006)] treat the path approaches to the covering spaces. Chapter 5 in the book [Thurston (1977)] and the article Choi (2004) have contents on the covering space theory using fiber products.

We do not study general maps or morphisms between orbifolds and induced bundles. This is related to defining the notion of suborbifold as well. Perhaps one should view orbifolds as 2-categories as Lerman (2010) has done.