Chapter 6

Appendix

6.1 Time local solvability of energy-transport model

We begin the detail discussion on the solvability to the problem (4.3)–(4.6) with studying the linear system of equations for an unknown function $(\hat{v}, \hat{w})$

$$\begin{pmatrix} \hat{v} \\ 3\hat{w}/2 \end{pmatrix}_t - A[v,w] \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix}_{xx} + B[v,w] \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix}_x = F[v,w], \tag{6.1}$$

$$B[v,w] := \begin{pmatrix} -e^w(v_x + w_x) & -e^w(v_x + w_x) \\ -e^w(v_x + w_x) - 5e^w w_x/2 & -e^w(v_x + w_x) - 5e^w w_x/2 - \kappa_0 e^{-v} w_x \end{pmatrix},$$

$$F[v,w] := \begin{pmatrix} -e^v + D - v_x(\Phi[e^v])_x \\ -e^v + D - (\Phi[e^v])_x\{2v_x + 7w_x/2 - e^{-w}(\Phi[e^v])_x\} - 3(1 - e^{-w})/2\zeta \end{pmatrix},$$

where A and Φ are given in (2.8) and (4.3b), respectively. The equation (6.1) is a linearization of (4.3). We prescribe the initial condition (4.4) and the boundary conditions (4.5) and (4.6).

The coefficients (v, w) in (6.1) are functions satisfying

$$v, w \in \mathfrak{Z}([0,T]) \cap \mathfrak{Y}_{loc}((0,T)) \tag{6.2}$$

and the estimates

$$\|(v - \Xi, w)(t)\|_1^2 \leq M_1, \quad \Xi(x) := (1 - x)\log \rho_l + x \log \rho_r, \tag{6.3a}$$

$$\int_0^t \|(v_t, w_t, v_{xx}, w_{xx})(\tau)\|^2 \, d\tau \leq M_2, \tag{6.3b}$$

$$t\|(v_t, w_t, v_{xx}, w_{xx})(t)\|^2 + \int_0^t \tau \|(v_{xt}, w_{xt})(\tau)\|^2 \, d\tau \leq M_3 \tag{6.3c}$$

for $t \in [0,T]$, where T, M_1, M_2 are positive constants. Hereafter, $\mathcal{X}(T; M_1, M_2, M_3)$ denotes a set of the functions satisfying (6.2) and (6.3). We often abbreviate $\mathcal{X}(T; M_1, M_2, M_3)$ by $\mathcal{X}(\,\cdot\,)$ without confusion. Note that due to (6.2) and (6.3),

$$\Phi[e^v] \in H^1(0,T;H^2), \quad \|\Phi[e^v](t)\|_2^2 \le C[M_1], \quad \int_0^t \|(\Phi[e^v])_t(\tau)\|_2^2 \, d\tau \le C[M_1, M_2]$$

holds for $t \in [0,T]$.

Lemma 6.1. *Suppose the initial data $(v_0, w_0) \in H^1(\Omega)$ and the boundary data ρ_l, ρ_r and ϕ_r satisfy* (2.4)*,* (2.6) *and* (2.7a)*. Then the initial boundary value problem* (6.1) *and* (4.4)*–*(4.6) *has a unique solution $(\hat{v}, \hat{w}) \in \mathfrak{Z}([0,T]) \cap \mathfrak{Y}_{loc}((0,T))$. Moreover, it verifies the additional regularity $(\hat{v}_t, \hat{w}_t) \in H^1_{loc}(0,T;L^2(\Omega)) \cap L^2_{loc}(0,T;H^2(\Omega))$, the convergence*

$$t\|(\hat{v}_t, \hat{w}_t, \hat{v}_{xx}, \hat{w}_{xx})(t)\|^2 \to 0 \quad \text{as} \quad t \to 0 \tag{6.4}$$

and the estimate

$$\int_0^t \tau\|(\hat{v}_{xt}, \hat{w}_{xt})(\tau)\|^2 + \tau^2\|(\hat{v}_{tt}, \hat{w}_{tt}, \hat{v}_{xxt}, \hat{w}_{xxt})(\tau)\|^2 \, d\tau \le C, \tag{6.5}$$

where C is a positive constant depending on T, M_1 and M_2.

Proof. We define a function $\hat{u} := \hat{v} - \Xi$ and rewrite the problem (6.1) and (4.4)–(4.6) as

$$\begin{pmatrix} \hat{u} \\ 3\hat{w}/2 \end{pmatrix}_t - A[v,w] \begin{pmatrix} \hat{u} \\ \hat{w} \end{pmatrix}_{xx} = -B[v,w] \begin{pmatrix} \hat{u}_x + \Xi_x \\ \hat{w}_x \end{pmatrix} + F[v,w], \tag{6.6}$$

$$\hat{u}(0,x) = u_0(x) := v_0(x) - \Xi(x), \quad \hat{w}(0,x) = w_0(x), \tag{6.7}$$

$$\hat{u}(t,0) = \hat{u}(t,1) = \hat{w}_x(t,0) = \hat{w}_x(t,1) = 0. \tag{6.8}$$

To prove Lemma 6.1, it suffices to show that the problem (6.6)–(6.8) has a unique solution $(\hat{v}, \hat{w}) \in \mathfrak{Z}([0,T]) \cap \mathfrak{Y}_{loc}((0,T))$ and that $(\hat{v},\hat{w})$ satisfies the additional regularity $(\hat{v}_t, \hat{w}_t) \in H^1_{loc}(0,T;L^2(\Omega)) \cap L^2_{loc}(0,T;H^2(\Omega))$, the convergence (6.4) and the estimate (6.5).

The solvability of the problem (6.6)–(6.8) is shown by the Galerkin method (for the details of this method, see [39, 41]). Define complete orthonormal systems $\{d_l\}_{l=1}^\infty$ and $\{e_l\}_{l=0}^\infty$ in H^1_0 and H^1, respectively, as

$$d_l(x) := \sqrt{\frac{2}{1+(l\pi)^2}} \sin l\pi x, \quad e_0(x) := 1, \quad e_l(x) := \sqrt{\frac{2}{1+(l\pi)^2}} \cos l\pi x,$$

where $l \ge 1$. Then define approximate sequences as

$$\hat{u}^n(t,x) := \sum_{l=1}^n a_l^n(t) d_l(x), \quad \hat{w}^n(t,x) := \sum_{l=1}^{n-1} b_{l-1}^n(t) e_{l-1}(x)$$

by solving a system of the ordinary differential equations for $a_l^n(t)$ and $b_{l-1}^n(t)$:

$$\int_0^1 (d_l, 0) \left\{ \left(\hat{u}_t^n, \frac{3}{2}\hat{w}_t^n \right)^\top - A[v,w]\,(\hat{u}_{xx}^n, \hat{w}_{xx}^n)^\top \right\} dx$$
$$= \int_0^1 (d_l, 0) \left\{ -B[v,w]\,(\hat{u}_x^n + \Xi_x, \hat{w}_x^n)^\top + F[v,w] \right\} dx, \qquad (6.9a)$$

$$\int_0^1 (0, e_{l-1}) \left\{ \left(\hat{u}_t^n, \frac{3}{2}\hat{w}_t^n \right)^\top - A[v,w]\,(\hat{u}_{xx}^n, \hat{w}_{xx}^n)^\top \right\} dx$$
$$= \int_0^1 (0, e_{l-1}) \left\{ -B[v,w]\,(\hat{u}_x^n + \Xi_x, \hat{w}_x^n)^\top + F[v,w] \right\} dx \qquad (6.9b)$$

with the initial condition

$$a_l^n(0) = \int_0^1 u_0 d_l + u_{0x} d_{lx}\, dx, \quad b_{l-1}^n(0) = \int_0^1 w_0 e_{l-1} + w_{0x}(e_{l-1})_x\, dx \qquad (6.10)$$

for $l = 1, 2, \cdots, n$. Note that the integrands in (6.9) are the inner products of vectors. The system of the ordinary differential equations (6.9) has a unique solution $a_l^n, b_{l-1}^n \in \mathcal{B}^1([0,T])$ owing to the standard theory of the ordinary differential equations. By the straight forward computation, $(a_l^n)_t$ and $(b_{l-1}^n)_t$ are absolutely continuous in (0,T) and satisfy $\sqrt{t}(a_l^n)_{tt}, \sqrt{t}(b_{l-1}^n)_{tt} \in L^2(0,T)$. Thus we see that $\hat{u}^n$ and $\hat{w}^n$ belong to the space $C^1([0,T]; H^2) \cap H_{loc}^2(0,T; H^2)$.

We derive the estimates of $(\hat{u}^n, \hat{w}^n)$ uniformly in n. Multiply (6.9a) by $\{1+(l\pi)^2\}a_l^n$ and (6.9b) by $\{1+(l-1)^2\pi^2\}b_{l-1}^n$ as well as sum up the resultant equalities for $l = 1, 2, \cdots, n$. Integrate the result by part with using the equalities $d_{lxx} = -(l\pi)^2 d_l$ and $e_{lxx} = -(l\pi)^2 e_l$. Then estimate the result by using the Sobolev and the Young inequalities as well as the inequalities in (6.3). These computations give

$$\frac{d}{dt}\left(\frac{1}{2}\|\hat{u}^n(t)\|_1^2 + \frac{3}{4}\|\hat{w}^n(t)\|_1^2 \right) + \int_0^1 (\hat{u}_{xx}^n, \hat{w}_{xx}^n) A[v,w] (\hat{u}_{xx}^n, \hat{w}_{xx}^n)^\top\, dx$$
$$\le \mu \|(\hat{u}_{xx}^n, \hat{w}_{xx}^n)(t)\|^2 + C[\mu](1 + \|(\hat{u}^n, \hat{w}^n)(t)\|_1^2), \qquad (6.11)$$

where μ is an arbitrary positive constant. Multiply (6.9a) by $t(l\pi)^2(a_l^n)_t$ and (6.9b) by $t(l-1)^2\pi^2(b_{l-1}^n)_t$, respectively, and then sum up the resultant equalities for $l = 1, 2, \cdots, n$. Integrate the result by part and estimate similarly as above to get

$$\frac{d}{dt}\int_0^1 \frac{t}{2}(\hat{u}_{xx}^n, \hat{w}_{xx}^n) A[v,w] (\hat{u}_{xx}^n, \hat{w}_{xx}^n)^\top\, d\tau + t\|\hat{u}_{xt}^n(t)\|^2 + \frac{3t}{2}\|\hat{w}_{xt}^n(t)\|_1^2$$
$$\le \mu t^2 \|(\hat{u}_{xxt}^n, \hat{w}_{xxt}^n)(t)\|^2 + C[\mu](1 + \|(\hat{u}^n, \hat{w}^n)(t)\|_1^2)$$
$$+ C\left(1 + t^2\|(A[v,w])_t(t)\|_1^2\right) \|(\hat{u}_{xx}^n, \hat{w}_{xx}^n)(t)\|^2. \qquad (6.12)$$

To handle $\hat{u}^n_{xxt}$ and $\hat{w}^n_{xxt}$ in the right hand side of (6.12), we differentiate the system (6.9):

$$\int_0^1 (d_l, 0) \left\{ \left(\hat{u}^n_{tt}, \frac{3}{2}\hat{w}^n_{tt} \right)^\top - A[v,w] \left(\hat{u}^n_{xxt}, \hat{w}^n_{xxt}\right)^\top \right\} dx$$

$$= \int_0^1 (d_l, 0) \left\{ (A[v,w])_t \left(\hat{u}^n_{xx}, \hat{w}^n_{xx}\right)^\top - \Big(B[v,w] \left(\hat{u}^n_x + \Xi_x, \hat{w}^n_x\right)^\top - F[v,w] \Big)_t \right\} dx, \tag{6.13a}$$

$$\int_0^1 (0, e_{l-1}) \left\{ \left(\hat{u}^n_{tt}, \frac{3}{2}\hat{w}^n_{tt} \right)^\top - A[v,w] \left(\hat{u}^n_{xxt}, \hat{w}^n_{xxt}\right)^\top \right\} dx$$

$$= \int_0^1 (0, e_{l-1}) \left\{ (A[v,w])_t \left(\hat{u}^n_{xx}, \hat{w}^n_{xx}\right)^\top - \Big(B[v,w] \left(\hat{u}^n_x + \Xi_x, \hat{w}^n_x\right)^\top - F[v,w] \Big)_t \right\} dx. \tag{6.13b}$$

Multiply (6.13a) by $t^2(l\pi)^2(a^n_l)_t$ and (6.13b) by $t^2(l-1)^2\pi^2(b^n_{l-1})_t$, and then sum up the results for $l = 1, 2, \cdots, n$. Integrating the resultant equality by part and applying the Sobolev, the Poincaré and the Young inequalities as well as (6.3), we have

$$\begin{aligned} &\frac{d}{dt}\left(\frac{t^2}{2}\|\hat{u}^n_{xt}(t)\|^2 + \frac{3t^2}{4}\|\hat{w}^n_{xt}(t)\|^2 \right) + t^2 \int_0^1 (\hat{u}^n_{xxt}, \hat{w}^n_{xxt}) A[v,w] (\hat{u}^n_{xxt}, \hat{w}^n_{xxt})^\top \, dx \\ &\le \mu t^2 \|(\hat{u}^n_{xxt}, \hat{w}^n_{xxt})(t)\|^2 + C[\mu] t \|(\hat{u}^n_{xt}, \hat{w}^n_{xt})(t)\|^2 + C[\mu] t^2 \|F_t\|^2 \\ &\quad + C[\mu] t^2 \left(\|(A[v,w])_t(t)\|_1^2 + \|(B[v,w])_t(t)\|^2 \right) \|(\hat{u}^n_{xx}, \hat{w}^n_{xx})(t)\|^2. \end{aligned} \tag{6.14}$$

Multiply (6.12) by α and (6.14) by α^2, where α is an arbitrary positive constant. Sum up the two results and (6.11), and then let μ and α are sufficiently small. Then integrate the resultant inequality over $[0, t]$ and apply the Gronwall inequality to result to get

$$\begin{aligned} &\|(\hat{u}^n, \hat{w}^n)(t)\|_1^2 + t\|(\hat{u}^n_{xx}, \hat{w}^n_{xx})(t)\|^2 + t^2 \|(\hat{u}^n_{xt}, \hat{w}^n_{xt})(t)\|^2 \\ &\qquad + \int_0^t \|(\hat{u}^n_{xx}, \hat{w}^n_{xx})(\tau)\|^2 + \tau \|(\hat{u}^n_{xt}, \hat{w}^n_{xt})(\tau)\|^2 + \tau^2 \|(\hat{u}^n_{xxt}, \hat{w}^n_{xxt})(\tau)\|^2 \, d\tau \le C, \end{aligned} \tag{6.15}$$

where C is a positive constant independent of t and n. In this computation, we have used the positivity of the matrix $A[v,w]$ and the estimates

$$\|\hat{u}^n(0)\|_1^2 \le \|u_0\|_1^2, \quad \|\hat{w}^n(0)\|_1^2 \le \|w_0\|_1^2,$$

which follow from the Bessel inequality.

Moreover, multiply (6.13a) by $(a^n_l)_t$ and (6.13b) by $(b^n_{l-1})_t$ as well as sum up the results for $l = 1, 2, \cdots, n$. Then applying the Sobolev and the Young inequalities with using (6.15) gives

$$\|(\hat{u}^n_t, \hat{w}^n_t)(t)\|^2 \le \|(\hat{u}^n_{xx}, \hat{w}^n_{xx})(t)\|^2 + C. \tag{6.16}$$

Similarly, from the system (6.13), it holds that

$$\int_0^t \tau^2 \|(\hat{u}^n_{tt}, \hat{w}^n_{tt})(\tau)\|^2 \, d\tau \le \int_0^t \tau^2 \|(\hat{u}^n_{xxt}, \hat{w}^n_{xxt})(t)\|^2 \, d\tau + C. \tag{6.17}$$

Consequently, the inequalities (6.15) and (6.16) show that the sequences $\{\hat{u}^n\}_{n=1}^\infty$ and $\{\hat{w}^n\}_{n=1}^\infty$ are bounded in $\mathfrak{Z}([0,T])$. Hence, there exist subsequences, still denoted by $\{\hat{u}^n\}_{n=1}^\infty$ and $\{\hat{w}^n\}_{n=1}^\infty$, as well as the functions $\hat{u}$ and $\hat{w}$ such that

$$\begin{aligned} &(\hat{u}^n, \hat{w}^n) \to (\hat{u}, \hat{w}) \quad \text{in} \quad C([0,T]; L^2) \quad \text{strongly}, \\ &(\hat{u}^n, \hat{w}^n) \to (\hat{u}, \hat{w}) \quad \text{in} \quad L^2(0,T; H^2) \cap H^1(0,T; L^2) \quad \text{weakly}, \end{aligned} \tag{6.18}$$

as n tends to infinity.

We show that $(\hat{u}, \hat{w}) \in C([0,T]; L^2) \cap H^1(0,T; L^2) \cap L^2(0,T; H^2)$ is a solution to the problem (6.6)–(6.8). Since $\{d_l\}_{l=1}^\infty$ and $\{e_l\}_{l=0}^\infty$ are the complete orthonormal systems in H^1_0 and H^1, $\hat{u}^n(0)$ and $\hat{w}^n(0)$ converge to $u_0 \in H^1_0$ and $w_0 \in H^1$ as n tends to ∞, respectively. Thus $\hat{u}$ and $\hat{w}$ verify the initial condition (6.7) owing to the convergence (6.18). The boundary condition (6.8) follows from $\hat{u}^n(t,0) = \hat{u}^n(t,1) = \hat{w}^n_x(t,0) = \hat{w}^n_x(t,1) = 0$ and the convergences (6.18). Passing to the limit in (6.9), we see that $(\hat{u}, \hat{w})$ satisfies the equation (6.6) in distribution sense.

We confirm that the solution $(\hat{u}, \hat{w})$ satisfies the desired properties. By the straight forward computation with using the uniform estimates (6.15)–(6.17) in n, the solution verifies the regularities $(\hat{u}, \hat{w}) \in C((0,T]; H^1) \cap \mathfrak{Y}_{loc}((0,T))$ and $(\hat{u}_t, \hat{w}_t) \in H^1_{loc}(0,T; L^2(\Omega)) \cap L^2_{loc}(0,T; H^2(\Omega))$ as well as the estimate (6.5). The convergence

$$\|(\hat{u}_x - u_{0x}, \hat{w}_x - w_{0x})(t)\|^2 + t\|(\hat{u}_t, \hat{w}_t, \hat{u}_{xx}, \hat{w}_{xx})(t)\|^2 \to 0 \quad \text{as} \quad t \to 0$$

follows from the standard theory (see [35] for example). The uniqueness is proven by the energy method. Consequently, $(\hat{u}, \hat{w})$ is the desired solution to the problem (6.6)–(6.8). □

For suitably chosen constants T, M_1, M_2 and M_3, the set $\mathcal{X}(\,\cdot\,)$ is invariant under the mapping $(v, w) \to (\hat{v}, \hat{w})$, which is defined by solving the problem (6.1) and (4.4)–(4.6). This fact is summarized in the next lemma.

Lemma 6.2. *Assume the same condition as in Lemma 6.1. Then there exist positive constants T, M_1, M_2 and M_3, such that if $(v, w) \in \mathcal{X}(\,\cdot\,)$, then the problem* (6.1) *and* (4.4)–(4.6) *admits a unique solution $(\hat{v}, \hat{w})$ in the same set $\mathcal{X}(\,\cdot\,)$.*

Proof. We firstly determine the constant M_1 by $M_1 := 2\|(v_0 - \Lambda, w_0)\|_1^2$. Take the inner product of (6.1) with the vector $(\hat{v} - \Lambda - \hat{v}_{xx}, \hat{w} - \hat{w}_{xx})$ in $L^2(0,t; L^2(\Omega))$ and apply integration by part. Then estimate the resulting equality by using (6.3a) as well as the Sobolev and the Young inequalities to get

$$\begin{aligned}\frac{1}{2}\|(\hat{v}-\Lambda)(t)\|_1^2+\frac{3}{4}\|\hat{w}(t)\|_1^2+c[M_1]\int_0^t\|(\hat{v}_{xx},\hat{w}_{xx})(\tau)\|^2\,d\tau\\ \leq\frac{1}{2}\|v_0-\Lambda\|_1^2+\frac{3}{4}\|w_0\|_1^2+C[M_1]\int_0^t\|(\hat{v}-\Lambda,\hat{w})(\tau)\|_1^2\,d\tau+C[M_1]t.\end{aligned}\tag{6.19}$$

Apply the Gronwall inequality to (6.19) and take T so small that

$$\|(\hat{v}-\Lambda,\hat{w})(t)\|_1^2\leq 2\|(v_0-\Lambda,w_0)\|_1^2=M_1\tag{6.20}$$

holds for $t\in[0,T]$.

Substituting (6.20) in (6.19) also yields that

$$\int_0^t\|(\hat{v}_{xx},\hat{w}_{xx})(\tau)\|^2\,d\tau\leq\overline{C}_1[M_1]\tag{6.21}$$

for $t\in[0,T]$. On the other hand, solve the equation (6.1) with respect to $(\hat{v}_t,\hat{w}_t)$ and then take the L^2-norm to obtain

$$\|(\hat{v}_t,\hat{w}_t)(t)\|\leq C[M_1](\|(\hat{v}_x,\hat{w}_x)(t)\|_1+1).\tag{6.22}$$

Its integration in t together with (6.20) and (6.23) immediately gives

$$\int_0^t\|(\hat{v}_t,\hat{w}_t)(\tau)\|^2\,d\tau\leq\overline{C}_2[M_1].\tag{6.23}$$

Determining $M_2:=\overline{C}_1[M_1]+\overline{C}_2[M_1]$, we see from (6.21) and (6.23) that (6.3b) holds for $t\in[0,T]$.

Finally, the constant M_3 are determined as follows. Taking the inner product of (6.1) with the vector $(-t\hat{v}_{xxt}, -t\hat{w}_{xxt})$ in $L^2(0,t;L^2(\Omega))$, we have

$$\begin{aligned}
&\frac{t}{2}\int_0^1 (\hat{v}_{xx}, \hat{w}_{xx}) A_1[v,w](\hat{v}_{xx}, \hat{w}_{xx})^\top \, dx + \int_0^t\int_0^1 \tau(\hat{v}_{xt})^2 + \frac{3}{2}\tau(\hat{w}_{xt})^2 \, dx d\tau \\
&= -t\int_0^1 (\hat{v}_{xx}, \hat{w}_{xx}) \left(B[v,w](\hat{v}_x, \hat{w}_x)^\top + F[v,w] \right) \, dx \\
&\quad + \frac{1}{2}\int_0^t\int_0^1 (\hat{v}_{xx}, \hat{w}_{xx}) A_1[v,w](\hat{v}_{xx}, \hat{w}_{xx})^\top + \tau(\hat{v}_{xx}, \hat{w}_{xx})(A_1[v,w])_t(\hat{v}_{xx}, \hat{w}_{xx})^\top \, dx d\tau \\
&\quad + \int_0^t\int_0^1 (\hat{v}_{xx}, \hat{w}_{xx}) \left\{ \left(B[v,w](\hat{v}_x, \hat{w}_x)^\top + F[v,w] \right) + \tau \left(B[v,w](\hat{v}_x, \hat{w}_x)^\top + F[v,w] \right)_t \right\} \, dx d\tau \\
&\le \mu t \|(\hat{v}_{xx}, \hat{w}_{xx})(t)\|^2 + \mu \int_0^t \tau \|(\hat{v}_{xt}, \hat{w}_{xt})(\tau)\| \, d\tau + C[M_1, M_2, \mu] + C[M_1, M_2, M_3, \mu]\sqrt{t} \\
&\quad + C[M_1]\int_0^t \left(\tau |(A_1[v,w])_t|_0^2 + \tau^{9/4} \|(B[v,w])_t\|^2 \right) \|(\hat{v}_{xx}, \hat{w}_{xx})(\tau)\|^2 \, d\tau, \qquad (6.24)
\end{aligned}$$

where μ is an arbitrary positive constant. In deriving the last inequality, we have used the Sobolev and the Young inequalities as well as (6.3), (6.20) and (6.21). Then take μ small enough, apply the Gronwall inequality to (6.24) and then take T sufficiently small subject to M_3 in (6.24), in order to get

$$t\|(\hat{v}_{xx}, \hat{w}_{xx})(t)\|^2 + \int_0^t \tau \|(\hat{v}_{xt}, \hat{w}_{xt})(\tau)\|^2 \, d\tau \le \overline{C}_3[M_1, M_2]$$

for $t \in [0,T]$, which together with (6.20) and (6.22) yields

$$t\|(\hat{v}_t, \hat{w}_t)(t)\|^2 \le \overline{C}_4[M_1, M_2].$$

Determine $M_3 := \overline{C}_3[M_1, M_2] + \overline{C}_4[M_1, M_2]$ to see the estimate (6.3c) holds for $t \in [0,T]$. Consequently, the solution $(\hat{v}, \hat{w})$ satisfies (6.3). □

The above two lemmas are used in the proof of Lemma 4.3, which asserts the unique existence of the time local solution to the non-linear problem (4.3)–(4.6).

Proof of Lemma 4.3. We define a successive approximation sequence $\{(v^n, w^n)\}_{n=0}^\infty \subset \mathfrak{Z}([0,T]) \cap \mathfrak{Y}_{loc}((0,T))$ by

$$(v^0, w^0) := (\Xi, 0)$$

and the solution to a problem

$$\begin{pmatrix} v^{n+1} \\ 3w^{n+1}/2 \end{pmatrix}_t - A[v^n, w^n] \begin{pmatrix} v^{n+1} \\ w^{n+1} \end{pmatrix}_{xx} + B[v^n, w^n] \begin{pmatrix} v^{n+1} \\ w^{n+1} \end{pmatrix}_x = F[v^n, w^n],$$
$$v^{n+1}(0,x) = v_0(x), \quad w^{n+1}(0,x) = w_0(x),$$
$$v^{n+1}(t,0) = \log \rho_l, \quad v^{n+1}(t,1) = \log \rho_r,$$
$$w_x^{n+1}(t,0) = w_x^{n+1}(t,1) = 0.$$

for $n > 0$. The sequence is well-defined and contained in $\mathcal{X}(\,\cdot\,)$ thanks to Lemmas 6.1 and 6.2. Thus (v^n, w^n) satisfies the estimates (6.3). Moreover, it apparently verifies the estimate (6.5) with the constant C independent of n. Then applying the standard energy method to the equations for $(v^n - v^{n+1}, w^n - w^{n+1})$, we see that (v^n, w^n) is the Cauchy sequence in $\mathfrak{Z}([0,T])$. Thus there exists a function $(v,w) \in \mathfrak{Z}([0,T])$ such that $(v^n, w^n) \to (v,w)$ strongly in $\mathfrak{Z}([0,T])$. In addition, apply the energy method again, to see that $(\sqrt{t}v_t^n, \sqrt{t}w_t^n)$ and $(\sqrt{t}v_{xx}^n, \sqrt{t}w_{xx}^n)$ are the Cauchy sequence in $C([0,T] : L^2(\Omega))$; $(\sqrt{t}v_{xt}^n, \sqrt{t}w_{xt}^n)$ is the Cauchy sequence in $L^2(0,T : L^2(\Omega))$. These facts together with (6.4) immediately mean that $(v,w) \in \mathfrak{Y}_{loc}((0,T))$, $(\sqrt{t}v_{xt}, \sqrt{t}w_{xt}) \in L^2(0,T : L^2(\Omega))$ and the convergence (4.9) hold. Consequently, (v,w) is the desired solution to the problem (4.3)–(4.6). □

6.2 Time local solvability of hydrodynamic model

In this section we study the unique existence of the time local solution for the initial boundary value problem (2.11), (2.12) and (2.4)–(2.6). Linearizing (2.11), we have the system for an unknown function $(\hat{\rho}, \hat{j}, \hat{\theta})$:

$$\hat{\rho}_t + \hat{j}_x = 0, \tag{6.25a}$$
$$\varepsilon \hat{j}_t + S[\rho, j, \theta]\hat{\rho}_x + 2\varepsilon \frac{j}{\rho}\hat{j}_x + \rho\hat{\theta}_x = \rho\phi_x - j, \tag{6.25b}$$
$$\rho\hat{\theta}_t + j\hat{\theta}_x + \frac{2}{3}\left(\frac{j}{\rho}\right)_x \rho\hat{\theta} - \frac{2\kappa_0}{3}\hat{\theta}_{xx} = \left(\frac{2}{3} - \frac{\varepsilon}{3\zeta}\right)\frac{j^2}{\rho} - \frac{\rho}{\zeta}(\hat{\theta} - 1), \tag{6.25c}$$
$$\phi = \Phi[\rho] \tag{6.25d}$$

with the initial data (2.12) and the boundary data (2.4)–(2.6). Here Φ in (6.25d) is given by (2.8). Suppose that the functions (ρ, j, θ) in the coefficients in (6.25) satisfy conditions

$$(\rho, j, \theta)(0,x) = (\rho_0, j_0, \theta_0), \tag{6.26}$$
$$\rho, j \in \mathfrak{X}_2([0,T]), \quad \theta, \theta_x \in \mathfrak{Y}([0,T]) \tag{6.27}$$

and inequalities

$$\inf_{x\in\Omega} \rho, \quad \inf_{x\in\Omega} \theta, \quad \inf_{x\in\Omega} S[\rho, j, \theta] \geq m, \tag{6.28a}$$

$$\|(\rho, j, \theta)(t)\|_2^2 + \|(\rho_t, j_t, \theta_t)(t)\|_1^2 + \|(\rho_{tt}, j_{tt}, \theta_{xxx})(t)\|^2 + \int_0^t \|\theta_{xxt}(\tau)\|^2 \, d\tau \leq M \tag{6.28b}$$

for an arbitrary $t \in [0, T]$, where T, m and M are positive constants. We denote by $\mathcal{Y}(T; m, M)$ a set of the functions satisfying (6.26)–(6.28b). The formula (2.8) together with the regularity (6.27) and the inequality (6.28b) imply that

$$\phi \in C^2([0, T]; H^2(\Omega)), \quad \|\partial_t^i \phi(t)\|_2 \leq C[M]$$

for $i = 0, 1, 2$ and $t \in [0, T]$. The unique solvability of the linearized system (6.25) is summarized in

Lemma 6.3. *Suppose the initial data $(\rho_0, j_0, \theta_0) \in H^2(\Omega) \times H^2(\Omega) \times H^3(\Omega)$ and the boundary data ρ_l, ρ_r and ϕ_r satisfy* (2.4), (2.6), (2.7), (2.10a), (2.10b) *and* (2.13). *Then the initial boundary value problem* (6.25), (2.12) *and* (2.4)–(2.6) *has a unique solution $(\hat{\rho}, \hat{j}, \hat{\theta})$ satisfying $\hat{\rho}, \hat{j} \in \mathfrak{X}_2([0, T])$, and $\hat{\theta}, \hat{\theta}_x \in \mathfrak{Y}([0, T])$.*

Proof. We first solve the parabolic equation (6.25c) to determine $\hat{\theta}$. Then substitute it in (6.25a) and (6.25b), and solve the resultant system with respect to $\hat{\rho}$ and $\hat{j}$. In this procedure, applying the Galerkin method similarly as the proof of Lemma 6.1, we see that the parabolic equation (6.25c) with the initial data $\hat{\theta} = \theta_0$ and the boundary data (2.5) has a unique solution $\hat{\theta}$ satisfying $\hat{\theta}, \hat{\theta}_x \in \mathfrak{Y}([0, T])$ for given functions $\rho, j \in \mathfrak{X}_2([0, T])$. Hence, it suffices to show the unique solvability of the hyperbolic equations (6.25a) and (6.25b) for given functions ρ, j, θ and $\hat{\theta}$. To this end, we consider a system

$$\mathcal{A}^0 \begin{pmatrix} v \\ \omega \end{pmatrix}_t + \mathcal{A}^1 \begin{pmatrix} v \\ \omega \end{pmatrix}_x + \mathcal{B} \begin{pmatrix} v \\ \omega \end{pmatrix} = \mathcal{F}^1 + \mathcal{F}^2, \tag{6.29}$$

$$\mathcal{A}^0 := \begin{pmatrix} S[\rho, j, \theta] & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{A}^1 := \begin{pmatrix} 0 & -S[\rho, j, \theta] \\ -S[\rho, j, \theta] & 2\varepsilon j/\rho \end{pmatrix},$$

$$\mathcal{B} := \begin{pmatrix} 0 & 0 \\ -\left(S[\rho, j, \theta]\right)_x & (2\varepsilon j/\rho)_x \end{pmatrix}, \ \mathcal{F}^1 := \begin{pmatrix} 0 \\ -\phi_{xx}\rho \end{pmatrix}, \ \mathcal{F}^2 := \begin{pmatrix} 0 \\ (\rho\hat{\theta}_x)_x - \phi_x\rho_x + j_x \end{pmatrix}$$

with the initial and the boundary conditions

$$v(0, x) = v_{0x}(x), \quad \omega(0, x) = -j_{0x}(x), \tag{6.30}$$

$$\omega(t, 0) = \omega(t, 1) = 0. \tag{6.31}$$

The system (6.29) is derived from differentiating (6.25b) with respect to x and using the equations (6.25a). Hence, if $(\hat{\rho}, \hat{j}) \in \mathfrak{X}_2([0,T])$ is a solution to the problem (6.25a), (6.25b), (2.12) and (2.4) then $(\upsilon, \omega) = (\hat{\rho}_x, \hat{\rho}_t) \in \mathfrak{X}_1([0,T])$ satisfies (6.29)–(6.31). Once solving the problem (6.29)–(6.31), we construct the solution $(\hat{\rho}, \hat{j}) \in \mathfrak{X}_2([0,T])$ to the problem (6.25a), (6.25b), (2.12), (2.4) from (υ, ω). In fact, let

$$\hat{\rho}(t,x) := \int_0^x \upsilon(t,x)\,dx + \rho_l,$$

$$\hat{j}(t,x) := \int_0^x -\omega(t,x)\,dx + \hat{j}(t,0),$$

$$\hat{j}(t,0) := \int_0^t \left\{ -S[\rho,j,\theta]\upsilon + \frac{2\varepsilon j}{\rho}\omega - \rho\hat{\theta}_x + \phi_x\rho - j \right\}(t,0)\,dt + j_0(0).$$

Then, by the straight forward computation, we see that the function $(\hat{\rho}, \hat{j}) \in \mathfrak{X}_2([0,T])$ is a desired solution to the linearized problem (6.25a), (6.25b), (2.12) and (2.4). Consequently, it suffices to show the unique solvability of the problem (6.29)–(6.31).

To solve the symmetric linear problem (6.29)–(6.31), we define approximation sequences of the symmetric matrices $\{\mathcal{A}_i^0\}_{i=0}^\infty$, $\{\mathcal{A}_i^1\}_{i=0}^\infty \subset C^2([0,T];H^2(\Omega))$ such that $\mathcal{A}_i^0$ and $\mathcal{A}_i^1$ converge to $\mathcal{A}^0$ and $\mathcal{A}^1$ strongly in $C(0,T;H^2(\Omega)) \cap C^1(0,T;H^1(\Omega))$ as i tends to infinity, respectively. Similarly take $\{\mathcal{B}_i\}_{i=0}^\infty \subset C^2([0,T];H^2(\Omega))$ such that $\mathcal{B}_i \to \mathcal{B}$ strongly in $\mathfrak{X}_1([0,T])$; $\{\mathcal{F}_i^1\}_{i=0}^\infty \subset C^1([0,T];H^1(\Omega))$ such that $\mathcal{F}_i^1 \to \mathcal{F}^1$ strongly in $C^1([0,T];L^2(\Omega))$. Define a successive approximation sequence $\{(\upsilon^i, \omega^i)\}_{i=0}^\infty$ by solutions to problems

$$\mathcal{A}_i^0 \begin{pmatrix} \upsilon^i \\ \omega^i \end{pmatrix}_t + \mathcal{A}_i^1 \begin{pmatrix} \upsilon^i \\ \omega^i \end{pmatrix}_x + \mathcal{B}_i \begin{pmatrix} \upsilon^i \\ \omega^i \end{pmatrix} = \mathcal{F}_i^1 + \mathcal{F}^2$$

with the initial data (6.30) and the boundary data (6.31) for $i = 0, 1, \cdots$. It is shown by following the proof of Theorem-A1 in [37] that the sequence $\{(\upsilon^i, \omega^i)\}_{i=0}^\infty$ is well-defined in $\mathfrak{X}_1([0,T])$. The standard energy method gives the estimates for the solution (υ^i, ω^i)

$$\|(\upsilon^i, \omega^i)(t)\|_1 + \|(\upsilon_t^i, \omega_t^i)(t)\| \le C \tag{6.32}$$

for $t \in [0,T]$, where C is a positive constant, independent of $i = 0, 1, \ldots$. Applying the energy method again to the equations for the difference $(\upsilon^i - \upsilon^j, \omega^i - \omega^j)$, we see from (6.32) that $\{(\upsilon^i, \omega^i)\}_0^\infty$ is the Cauchy sequence in $\mathfrak{X}_0([0,T])$. Hence, there exists a function $(\upsilon, \omega) \in \mathfrak{X}_0([0,T])$ such that $(\upsilon^i, \omega^i) \to (\upsilon, \omega)$ strongly in $\mathfrak{X}_0([0,T])$ as $i \to \infty$. The higher regularity $(\upsilon, \omega) \in \mathfrak{X}_1([0,T])$ follows from the standard theory for the hyperbolic equations (see [35] for example). The uniqueness of the solution (υ, ω) to the initial boundary value problem (6.29)–(6.31) immediately follows from by the standard energy method.

□

The next lemma shows that $\mathcal{Y}(T;m,M)$ is an invariant set by the mapping $(\rho,j,\theta)\to(\hat\rho,\hat j,\hat\theta)$ for suitably chosen constants T, m and M. Since it is proven similarly as in [20, 21], we omit the proof (also see the proof of Lemma 5.2).

Lemma 6.4. *There exist positive constants T, m and M such that if $(\rho,j,\theta)\in\mathcal{Y}(T;m,M)$, then the problem* (6.25), (2.12) *and* (2.4)–(2.6) *admits a unique solution $(\hat\rho,\hat j,\hat\theta)$ in the same set $\mathcal{Y}(T;m,M)$.*

Using Lemmas 6.3 and 6.4, we complete to the proof of the unique solvability of the non-linear problem (2.11), (2.12) and (2.4)–(2.6) in Lemma 5.1.

Proof of Lemma 5.1. We first define the approximation sequence $\{(\rho^n,j^n,\theta^n)\}_{n=0}^{\infty}$ by

$$(\rho^0,j^0,\theta^0):=(\rho_0,j_0,\theta_0)$$

and for $n>0$

$$\begin{gathered}\rho_t^{n+1}+j_x^{n+1}=0,\\ \varepsilon j_t^{n+1}+S[\rho^n,j^n,\theta^n]\rho_x^{n+1}+2\varepsilon\frac{j^n}{\rho^n}j_x^{n+1}+\rho^n\theta_x^{n+1}=\rho^n\phi_x^n-j^n,\\ \rho^n\theta_t^{n+1}+j^n\theta_x^{n+1}+\frac{2}{3}\left(\frac{j^n}{\rho^n}\right)_x\rho^n\theta^{n+1}-\frac{2\kappa_0}{3}\theta_{xx}^{n+1}=\left(\frac{2}{3}-\frac{\varepsilon}{3\zeta}\right)\frac{(j^n)^2}{\rho^n}-\frac{\rho^n}{\zeta}(\theta^{n+1}-1),\\ \phi^n=\Phi[\rho^n]\end{gathered}$$

with the initial and the boundary conditions

$$\begin{gathered}(\rho^{n+1},j^{n+1},\theta^{n+1})(0,x)=(\rho_0,j_0,\theta_0)(x),\\ \rho^{n+1}(t,0)=\rho_l,\quad \rho^{n+1}(t,1)=\rho_r,\\ \theta_x^{n+1}(t,0)=\theta_x^{n+1}(t,1)=0,\end{gathered}$$

where $\Phi[\,\cdot\,]$ is given by (2.8).

By virtue of Lemmas 5.1 and 5.2, the sequence $\{(\rho^n,j^n,\theta^n)\}_{n=1}^{\infty}$ is well-defined and belongs to the set $\mathcal{Y}(T;m,M)$. Hence (ρ^n,j^n,θ^n) satisfies the estimates in (6.28). Then apply the standard energy method to the system of the equations for the difference $(\rho^{n+1}-\rho^n, j^{n+1}-j^n, \theta^{n+1}-\theta^n)$. This procedure shows that $\{(\rho^n,j^n,\theta^n)\}_{n=1}^{\infty}$ is the Cauchy sequence in $\mathfrak{X}_1([0,T])\times\mathfrak{X}_1([0,T])\times\mathfrak{Y}([0,T])$. Hence, there exists a function $(\rho,j,\theta)\in\mathfrak{X}_1([0,T])\times\mathfrak{X}_1([0,T])\times\mathfrak{Y}([0,T])$ such that

$$(\rho^n,j^n,\theta^n)\to(\rho,j,\theta)\quad\text{strongly in}\quad \mathfrak{X}_1([0,T])\times\mathfrak{X}_1([0,T])\times\mathfrak{Y}([0,T]) \tag{6.33}$$

as $n\to\infty$.

The higher regularity $(\rho, j) \in \mathfrak{X}_2([0,T])$ and $\theta_x \in \mathfrak{Y}([0,T])$ is derived as follows. The estimate (6.28b), which is uniform in n, immediately means $\theta_{xxt} \in L^2(0,T;L^2(\Omega))$. By the standard theory (for example, see [35]), we see that $\theta_{xt}(t)$ is continuous in L^2 at $t=0$. On the other hand, apply the mollifier with respect to time variable t to the equation (2.11c). Apply the energy method to the equation thus obtained. Then passing the limit with using the above continuity at $t=0$, we see $\theta_{xt} \in C([0,T];L^2)$. Since these discussions are standard, we omit the details. The regularity $(\rho, j) \in \mathfrak{X}_2([0,T])$ follows from the standard theory for the hyperbolic equations (see [35]). Finally $\theta_{xxx} \in C([0,T];L^2(\Omega))$ holds by the straight forward computation with using the equation (2.11c).

Let $\phi := \Phi[\rho]$ for the function ρ thus obtained. Then we see that (ρ, j, θ, ϕ) is the desired solution to the problem (2.11), (2.12) and (2.4)–(2.6). Notice that this solution also satisfies (2.10a), (2.10b) and (2.13) owing to the convergence (6.33) and the estimate (6.28a). Consequently the proof of Lemma 5.1 is completed. □