Chapter 5

Hydrodynamic model

This chapter is devoted to the proofs of Theorems 2.1 and 2.3. Theorem 2.1 shows the unique existence and the asymptotic behavior of the time global solution to the hydrodynamic model for the large initial data. It is summarized in Theorem 2.3 that the time global solution for this model convergences to that for the energy-transport model as the parameter ε tends to zero. Theorem 2.1 is proven in Sections 5.1–5.3. The uniform estimate of the time local solution, of which existence is discussed in Appendix 6.2, is established in Section 5.1. The time global existence is proven in Sections 5.2 and 5.3. These discussions are essentially same as those for the energy-transport model in Sections 4.1–4.3. The relaxation limit of the solution thus constructed for the hydrodynamic model is studied in Section 5.4, which completes the proof of Theorem 2.3.

5.1 Uniform estimate of local solution

We show that there exists a positive constant T_*, independent of ε, such that the initial boundary value problem (2.11), (2.12) and (2.4)–(2.6) has a solution until time T_*. It is summarized in Corollary 5.3. To show it, we firstly prove the following lemma, which asserts the existence of the time local solution even though the existence time T_ε may depend on ε. Its proof is postponed until Appendix 6.2.

Lemma 5.1. *Suppose the initial data* $(\rho_0, j_0, \theta_0) \in H^2(\Omega) \times H^2(\Omega) \times H^3(\Omega)$ *and the boundary data* ρ_l*,* ρ_r *and* ϕ_r *satisfy* (2.4)*,* (2.6)*,* (2.7)*,* (2.10a)*,* (2.10b) *and* (2.13)*. Let* n *and* N *be certain positive constants satisfying*

$$\inf \rho_0,\ \inf \theta_0,\ \inf S[\rho_0, j_0, \theta_0] \geq n, \quad \|(\rho_0, j_0)(t)\|_2 + \|\theta_0\|_3 \leq N,$$

respectively. Then there exists a positive constant T_ε*, depending on* ε*,* n *and* N*, such that the initial boundary value problem* (2.11)*,* (2.12) *and* (2.4)*–*(2.6) *has a unique solution* (ρ, j, θ, ϕ)

satisfying $\rho, j \in \mathfrak{X}_2([0,T_\varepsilon])$, $\theta, \theta_x \in \mathfrak{Y}([0,T_\varepsilon])$ *and* $\phi \in C^2([0,T_\varepsilon];H^2)$ *with the conditions* (2.10a), (2.10b) *and* (2.13).

In order to apply the same argument as in Section 4 to the hydrodynamic model, we have to improve the above lemma so that we can take the existence time T_* independently of the parameter ε (see Corollary 5.3). To this end, we define $Y(T;m,M_1,M_2)$, for positive constants m, M_1 and M_2, by a set of the functions satisfying the regularity

$$\rho, j \in \mathfrak{X}_2([0,T]), \quad \theta, \theta_x \in \mathfrak{Y}([0,T]), \quad \phi \in C^2([0,T];H^2)$$

as well as the estimates

$$\inf_{x\in\Omega} \rho, \quad \inf_{x\in\Omega} \theta, \quad \inf_{x\in\Omega} S[\rho,j,\theta] \geq m, \tag{5.1a}$$

$$\frac{1}{\zeta}\|(\theta-1)(t)\|^2 + \|(\rho,\theta)(t)\|_1^2 + \int_0^t \|\theta_t(\tau)\|^2\, d\tau \leq M_1, \tag{5.1b}$$

$$\|j(t)\|_1^2 + \frac{1}{\zeta}\|\theta_x(t)\|^2 + \|(\rho_{xx},\theta_{xx})(t)\|^2 + \varepsilon\|(j_{xx},\theta_{xt},\theta_{xxx})(t)\|^2 + \varepsilon^2\|\rho_{tt}(t)\|^2 + \int_0^t \|(j_{xx},\theta_{xt})(\tau)\|^2 + \varepsilon\|(j_t,\rho_{tt},\theta_{xxt})(\tau)\|^2\, d\tau \leq M_2 \tag{5.1c}$$

for $t \in [0,T]$.

The regularities of the functions ρ and j, constructed in Lemma 5.1, is insufficient to justify the following computations. This problem is resolved by an argument with using the mollifier with respect to the time variable t. We, however, omit this argument since it is a standard manner.

Lemma 5.2. *There exist positive constants* m, M_1 *and* M_2, *depending on* $\|(\rho_0,\theta_0)\|_2$, $\|j_0\|_1$, $\|\sqrt{\varepsilon}j_{0xx}\|$, $\|\sqrt{\varepsilon}\theta_{0xxx}\|$, $\inf\rho_0$, $\inf\theta_0$ *and* $\inf S[\rho_0,j_0,\theta_0]$ *but independent of* ε, *such that if the solution* (ρ,j,θ,ϕ) *to the problem* (2.11), (2.12) *and* (2.4)–(2.6) *belongs to* $Y(T;m/2,2M_1,2M_2)$, *then* (ρ,j,θ,ϕ) *satisfies*

$$\inf_{x\in\Omega} \rho, \quad \inf_{x\in\Omega} \theta, \quad \inf_{x\in\Omega} S[\rho,j,\theta] \geq m - C[m,M_1,M_2]\sqrt{t}, \tag{5.2a}$$

$$\frac{1}{\zeta}\|(\theta-1)(t)\|^2 + \|(\rho,\theta)(t)\|_1^2 + \int_0^t \|\theta_t(\tau)\|^2\, d\tau \leq M_1 + C[m,M_1,M_2]t, \tag{5.2b}$$

$$\|j(t)\|_1^2 + \frac{1}{\zeta}\|\theta_x(t)\|^2 + \|(\rho_{xx},\theta_{xx})(t)\|^2 + \varepsilon\|(j_{xx},\theta_{xt},\theta_{xxx})(t)\|^2 + \varepsilon^2\|\rho_{tt}(t)\|^2 + \int_0^t \|(j_{xx},\theta_{xt})(\tau)\|^2 + \varepsilon\|(j_t,\rho_{tt},\theta_{xxt})(\tau)\|^2\, d\tau \leq M_2 + C[m,M_1,M_2]t \tag{5.2c}$$

for $t \in [0,T]$, *where* $c[m,M_1,M_2]$ *and* $C[m,M_1,M_2]$ *are positive constants depending on* m, M_1 *and* M_2 *but independent of* ε *and* t.

Proof. We may assume $T \leq 1$ without loss of generality. Let

$$m := \min \left\{ \inf_{x\in\Omega} \rho_0, \quad \inf_{x\in\Omega} \theta_0, \quad \inf_{x\in\Omega} S[\rho_0, j_0, \theta_0] \right\}.$$

The Schwarz and the Sobolev inequalities together with (2.11a) yield the lower bound

$$\rho(t,x) = \rho_0(x) - \int_0^t \rho_t \, d\tau \geq \inf_{x\in\Omega} \rho_0 - c \int_0^t \|j_x\|_1 \, d\tau \geq m - c[m, M_1, M_2]\sqrt{t}$$

as $(\rho, j, \theta, \phi) \in Y(T; m/2, 2M_1, 2M_2)$, Similarly, it holds that

$$\inf_{x\in\Omega} \theta, \quad \inf_{x\in\Omega} S[\rho, j, \theta] \geq m - c[m, M_1, M_2]\sqrt{t}.$$

Hence, the estimate (5.2a) holds.

In order to determine M_1 and M_2, we derive several estimates. For this purpose, differentiate the equation (2.11b) in x and use (2.11a) to get

$$\varepsilon \rho_{tt} - (S[\rho, j, \theta]\rho_x)_x + \rho_t = (\rho\theta_x)_x + 2\varepsilon \left(\frac{jj_x}{\rho} \right)_x - (\rho\phi_x)_x \,. \tag{5.3}$$

In addition, dividing the equation (2.11c) by ρ and differentiating the resultant equation in x lead to

$$\theta_{xt} + \left\{ \frac{j}{\rho}\theta_x + \frac{2}{3} \left(\frac{j}{\rho} \right)_x \theta \right\}_x - \left(\frac{2\kappa_0}{3\rho}\theta_{xx} \right)_x = \left(\frac{2}{3} - \frac{\varepsilon}{3\zeta} \right) \left(\frac{j^2}{\rho^2} \right)_x - \frac{1}{\zeta}\theta_x. \tag{5.4}$$

The elliptic estimate

$$\|\partial_t^i \phi(t)\|_2 \leq C\|\partial_t^i \rho(t)\| \tag{5.5}$$

for $i = 0, 1$ follows from the formula (2.8). Taking $\mathcal{B}^0$-, L^2- and L^2- norms of the equations (2.11b), (5.3) and (5.4), respectively, we have from (5.5)

$$\begin{aligned} &|\varepsilon j_t(t)|_0 + \|\varepsilon \rho_{tt}(t)\| + \sqrt{\varepsilon}\|\theta_{xt}(t)\| \\ &\qquad \leq C[\|(\rho, \theta)(t)\|_2, \|j(t)\|_1, \sqrt{\varepsilon}\|(j_{xx}, \theta_{xxx})(t)\|, \inf_x \rho(t)]. \end{aligned} \tag{5.6}$$

The constant M_1 is determined as follows. Multiply the equations (2.11a) and (5.3) by 2ρ and $2\rho_t$, respectively. Then sum up two resultant equalities, integrate the result by part over the domain $[0,t]\times\Omega$ and use the boundary conditions $\rho_t(t,0)=\rho_t(t,1)=0$ and (2.11a). The result is

$$\begin{aligned}\int_0^1 (\rho^2 + S[\rho,j,\theta]\rho_x^2 + \varepsilon j_x^2)(t)\,dx + \int_0^t\int_0^1 2\rho_t^2\,dxd\tau &= \int_0^1 \rho_0^2 + S[\rho_0,j_0,\theta_0]\rho_{0x}^2 + \varepsilon j_{0x}^2\,dx\\ - 2\int_0^t\int_0^1 j_x\rho - \frac{(S[\rho,j,\theta])_t}{2}\rho_x^2 &+ \left\{(\rho\phi_x)_x - (\rho\theta_x)_x - \varepsilon\left(\frac{2jj_x}{\rho}\right)_x\right\}\rho_t\,dxd\tau. \quad (5.7)\end{aligned}$$

Applying the Schwarz and the Sobolev inequalities to the right hand side of (5.7) with using (5.5) and $(\rho,j,\theta,\phi)\in Y(T;m/2,2M_1,2M_3)$, we obtain

$$\|\rho(t)\|_1^2 \le C[\|(\rho_0,j_0,\theta_0)\|_1,m] + \mu\int_0^1 \|\theta_t(\tau)\|^2\,d\tau + C[\mu,m,M_1,M_2]t, \quad (5.8)$$

where μ is an arbitrary constant to be determined. Multiplying the equation (2.11c) by $2(\theta-1-\theta_{xx}+\theta_t)/\rho$, integrating the result by part over $[0,t]\times\Omega$ and using the boundary condition (2.6) give

$$\begin{aligned}&\int_0^1 \left\{\frac{1+\zeta}{\zeta}(\theta-1)^2+\theta_x^2\right\}(t)\,dx + 2\int_0^t\int_0^1 \frac{1}{\zeta}\{(\theta-1)^2+\theta_x^2\}+\theta_t^2+\frac{2\kappa_0}{3\rho}(\theta_x^2+\theta_{xx}^2)\,dxd\tau\\ &= \int_0^1 \frac{1+\zeta}{\zeta}(\theta_0-1)^2+\theta_{0x}^2\,dx + \int_0^t\int_0^1 \frac{2\kappa_0}{3\rho^2}\rho_x\theta_x(\theta-1)+\frac{2\kappa_0}{3\rho}\theta_{xx}\theta_t\,dxd\tau\\ &\quad+\int_0^t\int_0^1 2\left\{\frac{j}{\rho}\theta_x+\frac{2}{3}\left(\frac{j}{\rho}\right)_x\theta+\left(\frac{2}{3}-\frac{\varepsilon}{3\zeta}\right)\frac{j^2}{\rho^2}\right\}(\theta-1-\theta_{xx}+\theta_t)\,dxd\tau\\ &\le C[\|\theta_0\|_1]+\mu\int_0^1\|\theta_t(\tau)\|^2\,d\tau + C[\mu,m,M_1,M_2]t, \quad (5.9)\end{aligned}$$

where last inequality is shown similarly as in the derivation of (5.8). Adding (5.8) to (5.9) and then making μ small enough lead to

$$\|(\rho,\theta)(t)\|_1^2+\frac{1}{\zeta}\|(\theta-1)(t)\|^2+\int_0^t\|\theta_t(\tau)\|^2\,d\tau \le \overline{C}_1[\|(\rho_0,j_0,\theta_0)\|_1,m]+C[m,M_1,M_2]t, \quad (5.10)$$

where $\overline{C}_1$ is a constant independent of M_1, M_2 and ε. Hence, we have the estimate (5.2b) by letting

$$M_1 := \overline{C}_1[\|(\rho_0,j_0,\theta_0)\|_1,m].$$

Next we determine M_2. To this end, dividing the equation (2.11c) by ρ and then differentiating the resultant equation and (5.3) in t give

$$\varepsilon\rho_{ttt} - (S[\rho,j,\theta]\rho_{xt})_x - \rho\theta_{xxt} + 2\varepsilon\frac{j}{\rho}\rho_{xtt} + \rho_{tt} = K_1, \tag{5.11}$$

$$\theta_{tt} - \frac{2\theta}{3\rho}\rho_{tt} - \frac{2\kappa_0}{3\rho}\theta_{xxt} + \frac{1}{\zeta}\theta_t = K_2 + K_3, \tag{5.12}$$

$$\begin{aligned}
K_1 :=& \{(S[\rho,j,\theta])_t\rho_x\}_x + \rho_t\theta_{xx} + 2\varepsilon\left(\frac{j}{\rho}\right)_t j_{xx} + \left\{2\varepsilon\left(\frac{j}{\rho}\right)_x j_x + \rho_x\theta_x - (\rho\phi_x)_x\right\}_t,\\
K_2 :=& \frac{\theta_x}{\rho}j_t + \frac{2}{3}\left(\frac{j}{\rho}\right)_x\theta_t + \frac{2\theta\rho_x}{3\rho^2}j_t + \left(\frac{2}{3} - \frac{\varepsilon}{3\zeta}\right)\left(\frac{j^2}{\rho^2}\right)_t,\\
K_3 :=& j\left(\frac{\theta_x}{\rho}\right)_t - \frac{2\theta}{3}\left(\frac{\rho_t^2}{\rho^2} - \frac{j}{\rho^2}\rho_{xt} + \frac{2j}{\rho^3}\rho_t\rho_x\right) + \left(\frac{2\kappa_0}{3\rho}\right)_t\theta_{xx}.
\end{aligned}$$

Multiply (5.11) by $\theta(2\rho_t + 4\varepsilon\rho_{tt})/\rho^2$, integrate the result by part over $[0,t]\times\Omega$ and then use the boundary conditions $\rho_t(t,0) = \rho_t(t,1) = \rho_{tt}(t,0) = \rho_{tt}(t,1) = 0$ and (2.11a). These computations yield

$$\begin{aligned}
&\int_0^1 \left\{\frac{\theta}{\rho^2}(2\varepsilon^2\rho_{tt} - 2\varepsilon\rho_{tt}j_x + j_x^2) + 2\varepsilon S[\rho,j,\theta]\frac{\theta}{\rho^2}j_{xx}^2\right\}(t)\,dx\\
&\quad + \int_0^t\int_0^1 2S[\rho,j,\theta]\frac{\theta}{\rho^2}j_{xx}^2 + \frac{2\varepsilon\theta}{\rho^2}\rho_{tt}^2\,dxd\tau - \int_0^t\int_0^1 \frac{\theta}{\rho}\theta_{xxt}(2\rho_t + 4\varepsilon\rho_{tt})\,dxd\tau\\
&= \int_0^1 \left\{\frac{\theta_0}{\rho_0^2}(2\varepsilon^2\rho_{tt} - 2\varepsilon\rho_{tt}j_{0x} + j_{0x}^2) + 2S[\rho_0,j_0,\theta_0]\frac{\theta_0}{\rho_0^2}j_{0xx}^2\right\}(0)\,dx\\
&\quad + \int_0^t\int_0^1 \left(\frac{\theta}{\rho^2}\right)_t(2\varepsilon^2\rho_{tt} - 2\varepsilon\rho_{tt}j_x + j_x^2) - S[\rho,j,\theta]\left(\frac{\theta}{\rho^2}\right)_x\rho_{xt}(2\rho_t + 4\varepsilon\rho_{tt})\\
&\qquad + 2\varepsilon\left(S[\rho,j,\theta]\frac{\theta}{\rho^2}\right)_t j_{xx}^2 + 4\varepsilon\left(\frac{j\theta}{\rho^2}\rho_t\right)_x\rho_{tt} + 4\varepsilon^2\left(\frac{j\theta}{\rho^2}\right)_x\rho_{tt}^2 + \frac{\theta}{\rho^2}K_1(2\rho_t + 4\varepsilon\rho_{tt})\,dxd\tau\\
&\le C[\|(\rho_0,\theta_0)\|_2, \|j_0\|_1, \sqrt{\varepsilon}\|(j_{0xx},\theta_{0xxx})\|, \inf_x \rho_0]\\
&\quad + \mu\int_0^1 \|(\theta_{xt}, j_{xx}, \sqrt{\varepsilon}\rho_{tt})(\tau)\|^2\,d\tau + \int_0^1 \|\theta_t(\tau)\|^2\,d\tau + C[\mu, m.M_1, M_2]t.
\end{aligned} \tag{5.13}$$

In deriving the last inequality, we have applied the Schwarz and the Sobolev inequalities with using (5.5), (5.6) and $(\rho,j,\theta,\phi) \in Y(T; m/2, 2M_1, 2M_2)$. Multiply the equations (2.11c) by $-3\theta_{xxt}/\rho$ and the equations (5.12) by $-6\varepsilon\theta_{xxt}$, respectively. Sum up these two results, integrate the resultant equality by part over $[0,T]\times\Omega$ and then use the boundary conditions

$\theta_{xt}(t,0) = \theta_{xt}(t,1) = 0$ to get

$$\begin{aligned}
&\int_0^1 \left\{ \frac{3}{2\zeta}\theta_x^2 + \frac{\kappa_0}{\rho}\theta_{xx}^2 + 3\varepsilon\theta_{xt}^2 \right\}(t)\,dx \\
&\quad + \int_0^t\int_0^1 3\theta_{xt}^2 + 3\frac{\varepsilon}{\zeta}\theta_{xt}^2 + 4\varepsilon\frac{\kappa_0}{\rho}\theta_{xxt}^2\,dxd\tau + \int_0^t\int_0^1 \frac{\theta}{\rho}\theta_{xxt}(2\rho_t + 4\varepsilon\rho_{tt})\,dxd\tau \\
&\quad = \int_0^1 \left\{ \frac{3}{2\zeta}\theta_{0x}^2 + \frac{\kappa_0}{\rho}\theta_{0xx}^2 + 3\varepsilon\theta_{xt}^2 \right\}(0)\,dx + 6\varepsilon\int_0^t\int_0^1 K_{2x}\theta_{xt} - K_3\theta_{xxt}\,dxd\tau \\
&\quad + \int_0^t\int_0^1 \left(\frac{\kappa_0}{\rho}\right)_t \theta_{xx}^2 + 3\left\{ \frac{j}{\rho}\theta_x - \frac{2j}{3\rho^2}\rho_x\theta - \left(\frac{2}{3} - \frac{\varepsilon}{3\zeta}\right)\frac{j^2}{\rho^2} \right\}_x \theta_{xt}\,dxd\tau \\
&\quad \le C[\|(\rho_0,\theta_0)\|_2, \|j_0\|_1, \sqrt{\varepsilon}\|(j_{0xx},\theta_{0xxx})\|, \inf_x \rho_0] \\
&\quad + \mu\int_0^1 \|(\theta_{xt}, j_{xx}, \sqrt{\varepsilon}\rho_{tt}, \sqrt{\varepsilon}\theta_{xxt})(\tau)\|^2\,d\tau + \int_0^1 \|\theta_t(\tau)\|^2\,d\tau + C[\mu, m.M_1, M_2]t, \qquad (5.14)
\end{aligned}$$

where the last inequality follows from the similar computation as in the derivation of (5.13). Adding (5.14) to (5.13), making μ sufficiently small and then using (5.2b), we have

$$\begin{aligned}
&\frac{1}{\zeta}\|\theta_x(t)\|^2 + \|(j_x,\theta_{xx})(t)\|^2 + \varepsilon\|(j_{xx},\theta_{xt})(t)\|^2 + \varepsilon^2\|\rho_{tt}(t)\|^2 \\
&\qquad + \int_0^t \|(j_{xx},\theta_{xt})(\tau)\| + \varepsilon\|(\rho_{tt},\theta_{xxt})(\tau)\|\,d\tau \\
&\qquad \le \overline{C}_2[\|(\rho_0,\theta_0)\|_2, \|j_0\|_1, \sqrt{\varepsilon}\|(j_{0xx},\theta_{0xxx})\|, m, M_1] + C[m.M_1, M_2]t, \qquad (5.15)
\end{aligned}$$

where $\overline{C}_2$ is a constant independent of M_2 and ε.

Multiply (2.11b) by $2j_t$ and integrate the resulting equality by part over $[0,t]\times\Omega$. Then applying the Schwarz and the Sobolev inequalities to the resultant equality with using (2.11a), (5.2b) and (5.5), we have

$$\begin{aligned}
&\int_0^1 j^2(t)\,dx + \int_0^t\int_0^1 2\varepsilon j_t^2\,dxd\tau = \int_0^1 j_0^2 + \{(\theta_0\rho_0)_x - \rho_0(\Phi[\rho_0])_x\}\,j_0\,dx \\
&\qquad - \int_0^1 \{(\theta\rho)_x - \rho\phi_x\}\,j(t)\,dx + \int_0^t\int_0^1 \{(\theta\rho)_x - \rho\phi_x\}_t\,j - \varepsilon\left(\frac{j^2}{\rho}\right)_x j_t\,dxd\tau \\
&\qquad \le C[\|(\rho_0,j_0,\theta_0)\|_1, M_1] + C[m, M_1, M_2]t + \int_0^t \|(j_{xx},\theta_{xt})(\tau)\|^2\,d\tau. \\
&\qquad + \int_0^1 \frac{1}{2}j^2(t)\,dx + \int_0^t\int_0^1 \varepsilon j_t^2\,dxd\tau. \qquad (5.16)
\end{aligned}$$

It together with (5.15) gives

$$\|j(t)\|^2 + \int_0^t \varepsilon\|j_t(\tau)\|^2\,d\tau \le \overline{C}_3[\|\rho_0,\theta_0\|_2, \|j_0\|_1, \sqrt{\varepsilon}\|(j_{0xx},\theta_{0xxx})\|, m, M_1] + C[m, M_1, M_2]t, \tag{5.17}$$

where $\overline{C}_3$ is a constant independent of M_2 and ε.

Finally, solving the equation (5.3) with respect to ρ_{xx}, taking L^2-norm and then estimating the result with using (5.2b), (5.5), (5.15) and (5.17), we have

$$\|\rho_{xx}(t)\|^2 \le \overline{C}_4[\|\rho_0,\theta_0\|_2, \|j_0\|_1, \sqrt{\varepsilon}\|(j_{0xx},\theta_{0xxx})\|, m, M_1] + C[m, M_1, M_2]t, \tag{5.18}$$

where $\overline{C}_4$ is a constant independent of M_2 and ε. Moreover, solve the equation (5.4) with respect to θ_{xxx}, take L^2-norm and then estimate the result similarly as above to get

$$\sqrt{\varepsilon}\|\theta_{xxx}(t)\|^2 \le \overline{C}_5[\|\rho_0,\theta_0\|_2, \|j_0\|_1, \sqrt{\varepsilon}\|(j_{0xx},\theta_{0xxx})\|, m, M_1] + C[m, M_1, M_2]t, \tag{5.19}$$

where $\overline{C}_5$ is a constant independent of M_2 and ε. Consequently, summing up (5.15) and (5.17)–(5.19) and then letting

$$M_2 := \overline{C}_2 + \overline{C}_3 + \overline{C}_4 + \overline{C}_5,$$

we have the desired estimate (5.2c). □

Owing to Lemmas 5.1 and 5.2, we can take the existence time of the solution (ρ, j, θ, ϕ) independently of ε. As this fact is shown similarly as in the proof of Corollary 4.5, we omit the proof.

Corollary 5.3. *Suppose the initial data* $(\rho_0, j_0, \theta_0) \in H^2(\Omega) \times H^2(\Omega) \times H^3(\Omega)$ *and the boundary data* ρ_l, ρ_r *and* ϕ_r *satisfy* (2.4), (2.6), (2.7), (2.10a), (2.10b) *and* (2.13). *Let* n_0 *and* N_0 *be certain positive constants satisfying*

$$\inf \rho_0,\ \inf \theta_0,\ \inf S[\rho_0, j_0, \theta_0] \ge n_0, \quad \|j_0\|_1 + \|(\rho_0,\theta_0)(t)\|_2 + \sqrt{\varepsilon}\|(j_{0xx},\theta_{0xxx})(t)\| \le N_0,$$

respectively. Then there exists a positive constant T^*, *depending on* n_0 *and* N_0 *but independent of* ε, *such that the initial boundary value problem* (2.11), (2.12) *and* (2.4)–(2.6) *has a unique solution* (ρ, j, θ, ϕ) *satisfying* $\rho, j \in \mathfrak{X}_2([0,T^*])$, $\theta, \theta_x \in \mathfrak{Y}([0,T^*])$ *and* $\phi \in C^2([0,T^*]; H^2)$ *with the conditions* (2.10a), (2.10b) *and* (2.13). *Moreover, it satisfies*

$$\inf_{x\in\Omega} \rho, \quad \inf_{x\in\Omega} \theta, \quad \inf_{x\in\Omega} S[\rho, j, \theta] \ge c, \tag{5.20a}$$

$$\|j(t)\|_1^2 + \frac{1}{\zeta}\|(\theta-1)(t)\|_1^2 + \|(\rho,\theta)(t)\|_2^2 + \varepsilon\|(j_{xx},\theta_{xt},\theta_{xxx})(t)\|^2 + \varepsilon^2\|\rho_{tt}(t)\|^2$$
$$+ \int_0^t \|(\theta_t, j_{xx}, \theta_{xt})(\tau)\|^2 + \varepsilon\|(j_t, \rho_{tt}, \theta_{xxt})(\tau)\|^2\,d\tau \le C, \tag{5.20b}$$

where c *and* C *are positive constants independent of* ε.

5.2 Semi-global existence of solution

The semi-global existence of the solution is established in this section. We also show in Corollary 5.8 that the perturbation $(\rho - \tilde{\rho}, j - \tilde{j}, \theta - \tilde{\theta}, \phi - \tilde{\phi})(T, x)$ at time T becomes arbitrarily small provided that T is sufficiently large (and thus ε_T is small). These proofs essentially follow from the same arguments as in Section 4.2.

For clarity, throughout this section, we write the solution for the hydrodynamic model by $(\rho^\varepsilon, j^\varepsilon, \theta^\varepsilon, \phi^\varepsilon)$ and the solution for the energy-transport model by $(\rho^0, j^0, \theta^0, \phi^0)$, respectively. The difference between $(\rho^\varepsilon, j^\varepsilon, \theta^\varepsilon, \phi^\varepsilon)$ and $(\rho^0, j^0, \theta^0, \phi^0)$ is denoted by

$$R^\varepsilon := \rho^\varepsilon - \rho^0, \quad J^\varepsilon := j^\varepsilon - j^0, \quad Q^\varepsilon := \theta^\varepsilon - \theta^0, \quad \varPhi^\varepsilon := \phi^\varepsilon - \phi^0,$$
$$L^\varepsilon(t) := \sup_{T^* \le \tau \le t} \left(\frac{1}{\zeta} \|Q^\varepsilon(\tau)\|_1 + \|J^\varepsilon(\tau)\|_1 + \|(R^\varepsilon, Q^\varepsilon)(\tau)\|_2 + \sqrt{\varepsilon} \|(j^\varepsilon_{xx}, \theta^\varepsilon_{xxx})(\tau)\| \right).$$

In this section, T^* denotes existence time defined in Corollary 5.3 with

$$n_0 := \min \{\inf \rho_0, \ \inf \theta_0, \ \inf S[\rho_0, j_0, \theta_0]\},$$
$$N_0 := \|j_0\|_1 + \|(\rho_0, \theta_0)(t)\|_2 + \sqrt{\varepsilon} \|(j_{0xx}, \theta_{0xxx})(t)\|.$$

Subtracting (2.11) from (2.14) leads to the system for $(R^\varepsilon, J^\varepsilon, Q^\varepsilon, \varPhi^\varepsilon)$

$$R^\varepsilon_t + J^\varepsilon_x = 0, \tag{5.21a}$$
$$\varepsilon j^\varepsilon_t + \varepsilon \left(\frac{(j^\varepsilon)^2}{\rho^\varepsilon} \right)_x + \theta^\varepsilon R^\varepsilon_x + \rho^\varepsilon Q^\varepsilon_x + J^\varepsilon = F_1, \tag{5.21b}$$
$$Q^\varepsilon_t - \frac{2\kappa_0}{3\rho^\varepsilon} Q^\varepsilon_{xx} + \frac{2\theta^\varepsilon}{3\rho^\varepsilon} J^\varepsilon_x + \frac{\varepsilon}{3\zeta} \frac{(j^\varepsilon)^2}{(\rho^\varepsilon)^2} + \frac{1}{\zeta} Q^\varepsilon = F_2 + F_3, \tag{5.21c}$$
$$\varPhi^\varepsilon_{xx} = R^\varepsilon, \tag{5.21d}$$

$$F_1 := \phi^0_x R^\varepsilon + \rho^\varepsilon \varPhi^\varepsilon_x - \rho^0_x Q^\varepsilon - \theta^0_x R^\varepsilon, \quad F_2 := -\frac{\theta^\varepsilon_x}{\rho^\varepsilon} J^\varepsilon - \frac{2\rho^\varepsilon_x \theta^\varepsilon}{3(\rho^\varepsilon)^2} J^\varepsilon + \frac{2}{3} \left(\frac{(j^\varepsilon)^2}{(\rho^\varepsilon)^2} - \frac{(j^0)^2}{(\rho^0)^2} \right),$$
$$F_3 := -\frac{2}{3} \left\{ \frac{j^0_x}{\rho^\varepsilon} Q^\varepsilon - \frac{j^0_x \theta^0}{\rho^\varepsilon \rho^0} R^\varepsilon + \frac{j^0_x \theta^\varepsilon}{(\rho^\varepsilon)^2} R^\varepsilon_x + \frac{j^0 \rho^0_x}{(\rho^\varepsilon)^2} Q^\varepsilon + j^0 \rho^0_x \theta^0 \left(\frac{1}{(\rho^\varepsilon)^2} - \frac{1}{(\rho^0)^2} \right) \right\}$$
$$- \frac{j^0}{\rho^\varepsilon} Q^\varepsilon_x + \frac{j^0 \theta^0_x}{\rho^\varepsilon \rho^0} R^\varepsilon_x - \frac{2\kappa_0 \theta^0_{xx}}{3\rho^\varepsilon \rho^0} R^\varepsilon.$$

From (2.4) and (2.6), we have the boundary conditions

$$R^\varepsilon(t, 0) = R^\varepsilon(t, 1) = Q^\varepsilon_x(t, 0) = Q^\varepsilon_x(t, 1) = \varPhi^\varepsilon(t, 0) = \varPhi^\varepsilon(t, 1) = 0. \tag{5.22}$$

Differentiate the equation (5.21b) with respect to x and then utilize the equations (2.11a) and (5.21a) to get

$$\varepsilon\rho^\varepsilon_{tt} - \varepsilon\left(\frac{(j^\varepsilon)^2}{\rho^\varepsilon}\right)_{xx} - \theta^\varepsilon R^\varepsilon_{xx} - \rho^\varepsilon Q^\varepsilon_{xx} + R^\varepsilon_t = -F_{1x} + F_4, \quad F_4 := \theta^\varepsilon_x R^\varepsilon_x + \rho^\varepsilon_x Q^\varepsilon_x. \tag{5.23}$$

The first two estimates in (5.68) in Lemma 5.4 show the positivity of the density and the temperature. This essential condition follows from the same property of the solution to the energy-transport model in Corollary 4.17 by assuming the difference of the solutions $L_\varepsilon(T)$ suitably small.

Lemma 5.4. *Let T be an arbitrary positive constant greater than or equal to T^*, and $(\rho^\varepsilon, j^\varepsilon, \theta^\varepsilon, \phi^\varepsilon)$ be a solution to* (2.11)*, satisfying $\rho^\varepsilon, j^\varepsilon \in \mathfrak{X}_2([0,T])$, $\theta, \theta_x \in \mathfrak{Y}([0,T])$, $\phi \in C^2([0,T] : H^2(\Omega))$ with the conditions* (2.10a), (2.10b) *and* (2.13)*. Then there exist positive constants δ_0 and ζ_0 such that if $\delta \le \delta_0$ and $\zeta \le \zeta_0$, there exist a positive constant ε_0, depending on ζ but independent of δ, such that if $L_\varepsilon(T) + \varepsilon \le \varepsilon_0$, then the estimates*

$$\inf_{x\in\Omega} \rho^\varepsilon, \quad \inf_{x\in\Omega} \theta^\varepsilon, \quad \inf_{x\in\Omega} S[\rho^\varepsilon, j^\varepsilon, \theta^\varepsilon] \ge c, \tag{5.24a}$$

$$|\phi^\varepsilon(t)|_2 + |\varepsilon j^\varepsilon_t(t)|_0 \le C, \tag{5.24b}$$

$$\frac{1}{\zeta}\|(\theta^\varepsilon - 1)(t)\|_1^2 + \|j^\varepsilon(t)\|_1^2 + \|(\rho^\varepsilon, \theta^\varepsilon)(t)\|_2^2 + \varepsilon\|(j^\varepsilon_{xx}, \theta^\varepsilon_{xxx})(t)\|^2 \le C, \tag{5.24c}$$

$$\varepsilon\|\theta^\varepsilon_{xt}(t)\|^2 + \varepsilon^2\|\rho^\varepsilon_{tt}(t)\|^2 \le C, \tag{5.24d}$$

$$\int_0^t \|(\theta^\varepsilon_t, \theta^\varepsilon_{xt}, j^\varepsilon_{xx})(\tau)\|^2 + \varepsilon\|(j^\varepsilon_t, \rho^\varepsilon_{tt})(\tau)\|^2 \, d\tau \le C(1+t) \tag{5.24e}$$

hold for an arbitrary $t \in [0,T]$, where c and C are positive constants independent of t, δ and ε.

Proof. The estimates in (5.24a) hold for $t \in [0, T^*]$ owing to Corollary 5.3. Because $L_\varepsilon(t) \le \varepsilon_0$ for suitably small ε_0, we also have (5.24a) for $t \in [T^*, T]$ with aid of (4.81a). Similarly, the estimate (5.24c) are proven for $t \in [0,T]$. Then estimating the $\mathcal{B}^0$-norm of the formula (2.8) and the equation (2.11b) gives (5.24b). Moreover, take the L^2-norm of the equation (5.3) and (5.4) with aid of (5.24c) to obtain (5.24d). The inequality (5.24e) is derived similarly as the derivations of (5.2a) and (5.2b). □

Lemmas 5.5 and 5.6 ensure that $L_\varepsilon(T)$ becomes arbitrarily small if ε is taken small enough. To show these two lemmas, we use estimates F_1, F_2, F_3 and F_4:

$$\|F_1\|_1^2 \le C\|(R^\varepsilon, Q^\varepsilon)(t)\|_1^2, \quad \|(F_3, F_4)\|^2 \le C\|(R^\varepsilon, Q^\varepsilon)(t)\|_1^2, \tag{5.25}$$

which are shown by the inequalities (4.81a), (4.81b) and (5.24a)–(5.24c) as well as the elliptic estimate

$$\|\partial_t^i \Phi^\varepsilon(t)\|_2 \le C\|\partial_t^i R^\varepsilon(t)\| \tag{5.26}$$

for $i = 0, 1$. Using the equation (5.21b), we have

$$\|F_2\|^2 \le C\|(J^\varepsilon, R^\varepsilon)(t)\|^2 \le C(\varepsilon^2 + \varepsilon^2\|j_t^\varepsilon\|^2 + \|(R^\varepsilon, Q^\varepsilon)(t)\|_1^2). \tag{5.27}$$

Lemma 5.5. *Assume that* (5.24) *hold for* $t \in [0, T]$. *Then it holds that*

$$\|(R^\varepsilon, Q^\varepsilon)(t)\|_1^2 + \int_0^t \|(R_t^\varepsilon, Q_{xx}^\varepsilon)(\tau)\|^2 + \frac{1}{\zeta}\|Q^\varepsilon(\tau)\|^2 \, d\tau \le C\left(\varepsilon + \frac{\varepsilon^2}{\zeta^2}\right) e^{\beta t}, \tag{5.28a}$$

$$\frac{1}{\zeta}\|Q^\varepsilon(t)\|^2 + \int_0^t \|(R_{xx}^\varepsilon, Q_t^\varepsilon)(\tau)\|^2 \, d\tau \le C\left(\varepsilon + \frac{\varepsilon^2}{\zeta^2}\right) e^{\beta t}, \tag{5.28b}$$

$$\|J^\varepsilon(t)\|^2 \le \|J^\varepsilon(0)\|^2 e^{-t/\varepsilon} + C\left(\varepsilon + \frac{\varepsilon^2}{\zeta^2}\right) e^{\beta t} \tag{5.28c}$$

for $t \in (0, T]$, *where* β *and* C *are positive constants independent of* t, δ *and* ε.

Proof. Multiply the equation (5.21b) by $Q^\varepsilon - 3Q_{xx}^\varepsilon/2$, integrate the resultant equality by part over $[0, t] \times \Omega$ and then use the boundary condition (5.22) to obtain

$$\begin{aligned}
&\int_0^1 \frac{1}{2}(Q^\varepsilon)^2 + \frac{3}{4}(Q_x^\varepsilon)^2 \, dx + \int_0^t\int_0^1 \frac{1}{\zeta}(Q^\varepsilon)^2 + \left(\frac{2\kappa_0}{3\rho^\varepsilon} + \frac{3}{2\zeta}\right)(Q_x^\varepsilon)^2 + \frac{\kappa_0}{\rho^\varepsilon}(Q_{xx}^\varepsilon)^2 - \frac{\theta^\varepsilon}{\rho^\varepsilon} Q_{xx}^\varepsilon J_x^\varepsilon \, dx d\tau \\
&= \int_0^t\int_0^1 \left(\frac{2\kappa_0}{3(\rho^\varepsilon)^2}\rho_x^\varepsilon Q_x^\varepsilon - \frac{2\theta^\varepsilon}{3\rho^\varepsilon} J_x^\varepsilon\right) Q^\varepsilon - \left(\frac{\varepsilon}{3\zeta}\left(\frac{j^\varepsilon}{\rho^\varepsilon}\right)^2 - F_2 - F_3\right)\left(Q^\varepsilon - \frac{3}{2}Q_{xx}^\varepsilon\right) dx d\tau \\
&\le \mu \int_0^t \|(R_t^\varepsilon, Q_{xx}^\varepsilon)(\tau)\|^2 \, d\tau + C[\mu] \int_0^t \|(Q^\varepsilon, R^\varepsilon)(\tau)\|_1^2 \, d\tau + C[\mu]\left(\varepsilon + \frac{\varepsilon^2}{\zeta^2}\right)(1 + t),
\end{aligned} \tag{5.29}$$

where μ is an arbitrary positive constant. In deriving the above inequality, we have also utilized the estimates (5.24), (5.25) and (5.27) as well as the Sobolev and the Schwarz inequalities. Multiplying the equation (5.23) by $\theta^\varepsilon R_t^\varepsilon/(\rho^\varepsilon)^2$ and integrating the resultant equality by part over $[0, t] \times \Omega$ with using the boundary condition (5.22) and the equation

(5.21a), we have

$$
\begin{aligned}
&\int_0^1 \frac{(\theta^\varepsilon)^2}{2(\rho^\varepsilon)^2}(R_x^\varepsilon)^2\,dx + \int_0^t\int_0^1 \frac{\theta^\varepsilon}{(\rho^\varepsilon)^2}(R_t^\varepsilon)^2 + \frac{\theta^\varepsilon}{\rho^\varepsilon}Q_{xx}^\varepsilon J_x^\varepsilon\,dxd\tau \\
&= \int_0^t\int_0^1 \left(\frac{(\theta^\varepsilon)^2}{2(\rho^\varepsilon)^2}\right)_t (R_x^\varepsilon)^2 - \left(\frac{(\theta^\varepsilon)^2}{(\rho^\varepsilon)^2}\right)_x R_x^\varepsilon R_t^\varepsilon \\
&\qquad\qquad - \frac{\theta^\varepsilon}{(\rho^\varepsilon)^2}\left\{\varepsilon\rho_{tt}^\varepsilon - \varepsilon\left(\frac{(j^\varepsilon)^2}{\rho^\varepsilon}\right)_{xx} + F_{1x} - F_4\right\} R_t^\varepsilon\,dxd\tau \\
&\le \mu\int_0^t \|R_t^\varepsilon(\tau)\|\,d\tau + C[\mu]\int_0^t (1+\|(\rho_t^\varepsilon,\theta_t^\varepsilon)(\tau)\|_1)\|(Q^\varepsilon,R^\varepsilon)(\tau)\|_1^2\,d\tau + C[\mu]\varepsilon(1+t),
\end{aligned}
\tag{5.30}
$$

where the last inequality follows from the similar computation as in the derivation of (5.29). Add (5.29) to (5.30), let μ sufficiently small, and then apply the Poincaré inequality $\|R^\varepsilon\| \le C\|R_x^\varepsilon\|$ to obtain that

$$
\begin{aligned}
&\|(R^\varepsilon,Q^\varepsilon)(t)\|_1^2 + \int_0^t \|(R_t^\varepsilon,Q_{xx}^\varepsilon)(\tau)\|^2\,d\tau \\
&\qquad \le C\int_0^t (1+\|(\rho_t^\varepsilon,\theta_t^\varepsilon)(\tau)\|_1)\|(Q^\varepsilon,R^\varepsilon)(\tau)\|_1^2\,d\tau + C\left(\varepsilon+\frac{\varepsilon^2}{\zeta^2}\right)(1+t).
\end{aligned}
\tag{5.31}
$$

The desired estimate (5.28a) follows from the Gronwall inequality applied to (5.31) together with (5.24e).

Solve the equation (5.23) with respect to R_{xx}^ε, take the L^2-norm and then use (5.28a) to get

$$
\int_0^t \|R_{xx}^\varepsilon(\tau)\|^2\,d\tau \le C\left(\varepsilon+\frac{\varepsilon^2}{\zeta^2}\right)e^{\beta t}. \tag{5.32}
$$

Multiply the equation (5.21b) by Q_t^ε and integrate the result by the part over $[0,t]\times\Omega$. After that, applying the Sobolev and the Schwarz inequalities with using (5.28a) yield

$$
\begin{aligned}
&\frac{1}{2\zeta}\int_0^1 (Q^\varepsilon)^2\,dx + \int_0^t\int_0^1 (Q_t^\varepsilon)^2\,dxd\tau \\
&= \int_0^t\int_0^1 \left(\frac{2\kappa_0}{3\rho^\varepsilon}Q_{xx}^\varepsilon - \frac{2\theta^\varepsilon}{3\rho^\varepsilon}J_x^\varepsilon - \frac{\varepsilon}{3\zeta}\frac{(j^\varepsilon)^2}{(\rho^\varepsilon)^2} + F_2 + F_3\right)Q_t^\varepsilon\,dxd\tau \\
&\le \frac{1}{2}\int_0^t\int_0^1 (Q_t^\varepsilon)^2\,dxd\tau + C\left(\varepsilon+\frac{\varepsilon^2}{\zeta^2}\right)(1+t).
\end{aligned}
\tag{5.33}
$$

Then adding (5.33) to (5.32), we have the estimate (5.28b).

Multiply the equation (5.21b) by $e^{t/\varepsilon}J^\varepsilon$ and integrate the result by the part over $[0,t]\times\Omega$. Then estimating the resulting equality by the Schwarz and the Sobolev inequalities as well as the estimates (4.81d), (5.24) and (5.28a), we have

$$\begin{aligned}
&\frac{\varepsilon}{2}e^{t/\varepsilon}\int_0^1 (J^\varepsilon)^2(t)\,dx+\frac{1}{2}\int_0^t\int_0^1 e^{\tau/\varepsilon}(J^\varepsilon)^2\,dxd\tau\\
&=\frac{\varepsilon}{2}\int_0^1 (J^\varepsilon)^2(0)\,dx-\int_0^t\int_0^1 e^{\tau/\varepsilon}\left\{\varepsilon j_t^0+\varepsilon\left(\frac{(j^\varepsilon)^2}{\rho^\varepsilon}\right)_x+\theta^\varepsilon R_x^\varepsilon+\rho^\varepsilon Q_x^\varepsilon-F_1\right\}J^\varepsilon\,dxd\tau\\
&\le\frac{\varepsilon}{2}\|J^\varepsilon(0)\|^2+\frac{1}{4}\int_0^t\int_0^1 e^{\tau/\varepsilon}(J^\varepsilon)^2\,dxd\tau+C\varepsilon\left(\varepsilon+\frac{\varepsilon^2}{\zeta^2}\right)e^{t/\varepsilon}e^{\beta t},
\end{aligned}\tag{5.34}$$

where we have also used $\int_0^t e^{\tau/\varepsilon}\,d\tau\le\varepsilon e^{t/\varepsilon}$. Obviously the desired estimate (5.28c) follows from (5.34). □

To apply the similar arguments as in the proof of Theorem 4.9, which shows semi-global existence for the energy-transport model, we derive the estimates for the higher order derivatives of the difference $(R^\varepsilon,J^\varepsilon,Q^\varepsilon)$. For this purpose, differentiating the equations (5.21c) and (5.23) with respect to t yields

$$Q_{tt}^\varepsilon-\frac{2\kappa_0}{3\rho^\varepsilon}Q_{xxt}^\varepsilon+\frac{2\theta^\varepsilon}{3\rho^\varepsilon}J_{xt}^\varepsilon+\frac{1}{\zeta}Q_t^\varepsilon=\left(\frac{2\kappa_0}{3\rho^\varepsilon}\right)_tQ_{xx}^\varepsilon-\left(\frac{2\theta^\varepsilon}{3\rho^\varepsilon}\right)_tJ_x^\varepsilon-\left(\frac{\varepsilon(j^\varepsilon)^2}{3\zeta(\rho^\varepsilon)^2}\right)_t+F_{2t}+F_{3t},\tag{5.35}$$

$$\varepsilon\rho_{ttt}^\varepsilon-\theta^\varepsilon R_{xxt}^\varepsilon-\rho^\varepsilon Q_{xxt}^\varepsilon+R_{tt}^\varepsilon=\varepsilon\left(\frac{(j^\varepsilon)^2}{\rho^\varepsilon}\right)_{xxt}+\theta_t^\varepsilon R_{xx}^\varepsilon+\rho_t^\varepsilon Q_{xx}^\varepsilon-F_{1xt}+F_{4t}.\tag{5.36}$$

By the estimates (4.81a), (4.81b), (5.24a)–(5.24c) and (5.26), the L^2-norm of F_{1xt}, F_{2x}, F_{3x}, F_{3t} and F_{4t} are handled as

$$\|(F_{1xt},F_{3t},F_{4t})\|\le C\left(\|(j_t^0,\rho_{tt}^0)(t)\|+\|(j_x^\varepsilon,\theta_t^\varepsilon)(t)\|_1+\|(\rho_t^0,\theta_t^0)(t)\|_2\right),\tag{5.37}$$

$$\|(F_{2x},F_{3x})\|\le C\left(\|(J^\varepsilon,R_t^\varepsilon)(t)\|+\|(R^\varepsilon,Q^\varepsilon)(t)\|_2+\|(\rho_{xt}^0,\theta_{xxx}^0)(t)\|\|(R^\varepsilon,Q^\varepsilon)(t)\|_1\right),\tag{5.38}$$

where we have also used the equations (2.11a), (2.14a) and (5.21a). Due to the estimates (4.81d), (4.81e), (5.24e), (5.28), (5.37) and (5.38), it holds that

$$\int_0^t\tau^2\|(F_{1xt},F_{3t},F_{4t})\|^2\,d\tau\le C(1+t^3),\qquad\int_0^t\tau^2\|(F_{2x},F_{3x})\|^2\,d\tau\le C\left(\varepsilon+\frac{\varepsilon^2}{\zeta^2}\right)e^{\beta t}.\tag{5.39}$$

Lemma 5.6. *Assume that* (5.24) *hold for* $t\in[0,T]$. *Then it holds that*

$$\frac{1}{\zeta}\|Q_x^\varepsilon(t)\|^2+\|(R_{xx}^\varepsilon,J_x^\varepsilon,Q_{xx}^\varepsilon)(t)\|^2+\varepsilon\|(j_{xx}^\varepsilon,\theta_{xxx}^\varepsilon)(t)\|^2\le C\left(\varepsilon+\frac{\varepsilon^2}{\zeta^2}\right)^{1/2}e^{\beta t}t^{-2}\tag{5.40}$$

for $t\in(0,T]$, *where* β *and* C *are positive constants independent of* t, δ *and* ε.

Proof. Firstly multiplying the equation (5.21c) by $-3t^2 Q^\varepsilon_{xxt}/2$ and integrating the resultant equality by part over $[0,T]\times\Omega$ with using the boundary condition (5.22), we obtain

$$t^2\int_0^1 \frac{3}{4\zeta}(Q^\varepsilon_x)^2+\frac{\kappa_0}{2\rho^\varepsilon}(Q^\varepsilon_{xx})^2\,dx+\int_0^t\!\!\int_0^1 \frac{3\tau^2}{2}(Q^\varepsilon_{xt})^2+\tau^2\frac{\theta^\varepsilon}{\rho^\varepsilon}Q^\varepsilon_{xxt}R^\varepsilon_t\,dxd\tau=I_0, \tag{5.41}$$

$$I_0:=\int_0^t\!\!\int_0^1 \frac{3\tau}{2\zeta}(Q^\varepsilon_x)^2+\left\{\frac{\kappa_0\tau}{\rho^\varepsilon}+\frac{\tau^2}{2}\left(\frac{\kappa_0}{\rho^\varepsilon}\right)_t\right\}(Q^\varepsilon_{xx})^2+\frac{\tau^2}{2}\left\{\frac{\varepsilon(j^\varepsilon)^2}{\zeta(\rho^\varepsilon)^2}-3(F_2+F_3)\right\}_x Q^\varepsilon_{xt}\,dxd\tau.$$

By the Schwarz and the Sobolev inequalities as well as the estimates (5.24a)–(5.24c), the term I_0 is estimated as

$$\begin{aligned}I_0&\le Ct\int_0^t \frac{1}{\zeta}\|Q^\varepsilon_x(\tau)\|^2+\|Q^\varepsilon_{xx}(\tau)\|^2\,d\tau+Ct^2\left(\int_0^t\|\rho^\varepsilon_t(\tau)\|_1^2\,d\tau\right)^{1/2}\left(\int_0^t\|Q^\varepsilon_{xx}(\tau)\|^2\,d\tau\right)^{1/2}\\&\quad+C\left(\int_0^t\frac{\varepsilon^2}{\zeta^2}\tau^2+\tau^2\|(F_{2x},F_{3x})\|^2\,d\tau\right)^{1/2}\left(\int_0^t\tau^2\|Q^\varepsilon_{xt}(\tau)\|^2\,d\tau\right)^{1/2}\\&\le C\left(\varepsilon+\frac{\varepsilon^2}{\zeta^2}\right)^{1/2}e^{\beta t}.\end{aligned} \tag{5.42}$$

In deriving the last inequality, we have also used (4.81d), (5.24e), (5.28a), (5.39) and $\rho^\varepsilon_{xt}=-j^\varepsilon_{xx}$.

Secondly multiply the equation (5.36) by $t^2\theta^\varepsilon R^\varepsilon_t/(\rho^\varepsilon)^2$, integrate the resultant equality by part over $[0,T]\times\Omega$ and use the convergence (4.80) to get

$$\begin{aligned}&t^2\int_0^1\frac{\varepsilon\theta^\varepsilon}{(\rho^\varepsilon)^2}R^\varepsilon_{tt}R^\varepsilon_t+\frac{\theta^\varepsilon}{2(\rho^\varepsilon)^2}(R^\varepsilon_t)^2\,dx\\&\qquad+\int_0^t\!\!\int_0^1\tau^2\frac{(\theta^\varepsilon)^2}{(\rho^\varepsilon)^2}(R^\varepsilon_{xt})^2-\frac{\varepsilon\tau^2\theta^\varepsilon}{(\rho^\varepsilon)^2}(R^\varepsilon_{tt})^2-\tau^2\frac{\theta^\varepsilon}{\rho^\varepsilon}Q^\varepsilon_{xxt}R^\varepsilon_t\,dxd\tau=I_1+I_2,\end{aligned} \tag{5.43}$$

$$I_1:=\varepsilon\int_0^t\!\!\int_0^1\left\{\frac{2\tau\theta^\varepsilon}{(\rho^\varepsilon)^2}+\tau^2\left(\frac{\theta^\varepsilon}{(\rho^\varepsilon)^2}\right)_t\right\}R^\varepsilon_{tt}R^\varepsilon_t-\frac{\tau^2\theta^\varepsilon}{(\rho^\varepsilon)^2}\rho^0_{ttt}R^\varepsilon_t-\tau^2\left(\frac{(j^\varepsilon)^2}{\rho^\varepsilon}\right)_{xt}\left(\frac{\theta^\varepsilon R^\varepsilon_t}{(\rho^\varepsilon)^2}\right)_x dxd\tau,$$

$$\begin{aligned}I_2:=\int_0^t\!\!\int_0^1&\left\{\frac{\tau\theta^\varepsilon}{(\rho^\varepsilon)^2}+\frac{\tau^2}{2}\left(\frac{\theta^\varepsilon}{(\rho^\varepsilon)^2}\right)_t\right\}(R^\varepsilon_t)^2-\tau^2\left(\frac{(\theta^\varepsilon)^2}{(\rho^\varepsilon)^2}\right)_x R^\varepsilon_{xt}R^\varepsilon_t\\&+\frac{\tau^2\theta^\varepsilon}{(\rho^\varepsilon)^2}\left(\theta^\varepsilon_t R^\varepsilon_{xx}+\rho^\varepsilon_t Q^\varepsilon_{xx}-F_{1xt}+F_{4t}\right)R^\varepsilon_t\,dxd\tau.\end{aligned}$$

Apply the Schwarz and the Sobolev inequalities to the term I_1 with using the estimates (5.24a)–(5.24c) and (5.37) to obtain

$$\begin{aligned}I_1&\le C\sqrt{\varepsilon}\int_0^t(\tau+\tau^2)\left(\|(\rho^\varepsilon_t,\theta^\varepsilon_t,\rho^0_t,\theta^0_t)(\tau)\|_1^2+\varepsilon\|(j^\varepsilon_t,\rho^\varepsilon_{tt},\rho^0_{tt})(\tau)\|^2\right)+\tau^3\|\rho^0_{ttt}(\tau)\|^2\,d\tau\\&\le C\sqrt{\varepsilon}(1+t^3).\end{aligned} \tag{5.44}$$

In deriving the last inequality in (5.44), the estimates (4.81d)–(4.81f) and (5.24e) have been also used. Using the Schwarz and the Sobolev inequalities as well as the estimates (4.81), (5.24), (5.28a) and (5.39), we estimate I_2 as

$$
\begin{aligned}
I_2 &\le C\left(\int_0^t 1+\tau^2\|(\rho_t^\varepsilon,\theta_t^\varepsilon,\rho_t^0,\theta_t^0)(\tau)\|_1^2+\tau^2\|(F_{1xt},F_{4t})\|^2\,d\tau\right)^{1/2}\left(\int_0^t \tau^2\|R_t^\varepsilon(\tau)\|^2\,d\tau\right)^{1/2}\\
&\le C\left(\varepsilon+\frac{\varepsilon^2}{\zeta^2}\right)^{1/2} e^{\beta t}. \qquad (5.45)
\end{aligned}
$$

Thirdly multiplying the equation (5.35) by $-3\varepsilon t^2 Q_{xxt}^\varepsilon$, integrating the resultant equality by part over $[0,T]\times\Omega$ and using (4.80) yield

$$
\begin{aligned}
&\varepsilon t^2\int_0^1 \frac{3}{2}(Q_{xt}^\varepsilon)^2\,dx+2\varepsilon\int_0^t\!\!\int_0^1 \frac{\kappa_0\tau^2}{\rho^\varepsilon}(Q_{xxt}^\varepsilon)^2+\frac{3}{2\zeta}\tau^2(Q_{xt}^\varepsilon)^2+\tau^2\frac{\theta^\varepsilon}{\rho^\varepsilon}Q_{xxt}^\varepsilon R_{tt}^\varepsilon\,dxd\tau\\
&=2\varepsilon\int_0^t\!\!\int_0^1 \frac{3\tau}{2}(Q_{xt}^\varepsilon)^2-\tau^2\left(\frac{\kappa_0}{\rho^\varepsilon}\right)_t Q_{xx}^\varepsilon Q_{xxt}^\varepsilon+\tau^2\left(\frac{\theta^\varepsilon}{\rho^\varepsilon}\right)_t J_x^\varepsilon Q_{xxt}^\varepsilon\\
&\quad+\frac{\varepsilon\tau^2}{2\zeta}\left(\frac{(j^\varepsilon)^2}{(\rho^\varepsilon)^2}\right)_t Q_{xxt}^\varepsilon+\frac{3\tau^2}{2}F_{3t}Q_{xxt}^\varepsilon+\tau^2\left(\frac{3\theta_x^\varepsilon}{2\rho^\varepsilon}+\frac{\rho_x^\varepsilon\theta^\varepsilon}{(\rho^\varepsilon)^2}\right)_t J^\varepsilon Q_{xxt}^\varepsilon\\
&\quad-\tau^2\left\{\left(\frac{3\theta_x^\varepsilon}{2\rho^\varepsilon}+\frac{\rho_x^\varepsilon\theta^\varepsilon}{(\rho^\varepsilon)^2}\right)J_t^\varepsilon\right\}_x Q_{xt}^\varepsilon+\tau^2\left(\frac{(j^\varepsilon)^2}{(\rho^\varepsilon)^2}-\frac{(j^0)^2}{(\rho^0)^2}\right)_{xt} Q_{xt}^\varepsilon\,dxd\tau\\
&\le C\left(\varepsilon+\frac{\varepsilon^2}{\zeta^2}\right)^{1/2} e^{\beta t}. \qquad (5.46)
\end{aligned}
$$

The last inequality follows from the similar calculation as in the derivation of (5.44).

Fourthly multiply the equation (5.36) by $2t^2\theta^\varepsilon R_{tt}^\varepsilon/(\rho^\varepsilon)^2$, integrate the resultant equality by part over $[0,T]\times\Omega$ and use the convergence (4.80) and the equality $R^\varepsilon=\rho^\varepsilon-\rho^0$. The result is

$$
\begin{aligned}
&t^2\int_0^1 \frac{\varepsilon^2\theta^\varepsilon}{(\rho^\varepsilon)^2}(R_{tt}^\varepsilon)^2+\frac{\varepsilon(\theta^\varepsilon)^2}{(\rho^\varepsilon)^2}(R_{xt}^\varepsilon)^2\,dx+2\varepsilon\int_0^t\!\!\int_0^1 \frac{\tau^2\theta^\varepsilon}{(\rho^\varepsilon)^2}(R_{tt}^\varepsilon)^2-\frac{\tau^2\theta^\varepsilon}{\rho^\varepsilon}Q_{xxt}^\varepsilon R_{tt}^\varepsilon\,dxd\tau\\
&=2\varepsilon\int_0^t\!\!\int_0^1 \varepsilon\left\{\frac{\tau\theta^\varepsilon}{(\rho^\varepsilon)^2}+\frac{\tau^2}{2}\left(\frac{\theta^\varepsilon}{(\rho^\varepsilon)^2}\right)_t\right\}(R_{tt}^\varepsilon)^2+\left\{\frac{\tau(\theta^\varepsilon)^2}{(\rho^\varepsilon)^2}+\frac{\tau^2}{2}\left(\frac{(\theta^\varepsilon)^2}{(\rho^\varepsilon)^2}\right)_t\right\}(R_{xt}^\varepsilon)^2\\
&\quad+\tau^2\left(\frac{(\theta^\varepsilon)^2}{(\rho^\varepsilon)^2}\right)_x R_{xt}^\varepsilon R_{tt}^\varepsilon+\frac{\tau^2\theta^\varepsilon}{(\rho^\varepsilon)^2}\left(\theta_t^\varepsilon R_{xx}^\varepsilon+\rho_t^\varepsilon Q_{xx}^\varepsilon-F_{1xt}+F_{4t}\right)R_{tt}^\varepsilon-\frac{\tau^2\theta^\varepsilon}{(\rho^\varepsilon)^2}\rho_{ttt}^0 R_{tt}^\varepsilon\\
&\quad-\varepsilon\left(\frac{\tau^2\theta^\varepsilon}{(\rho^\varepsilon)^2}\right)_x\left(\frac{(j^\varepsilon)^2}{\rho^\varepsilon}\right)_{xt} R_{tt}^\varepsilon+\varepsilon\frac{\tau^2\theta^\varepsilon}{(\rho^\varepsilon)^2}\left(\frac{(j^\varepsilon)^2}{\rho^\varepsilon}\right)_{xt}\rho_{xtt}^0\,dxd\tau-I_3, \qquad (5.47)\\
&I_3:=2\varepsilon^2\int_0^t\!\!\int_0^1 \frac{\tau^2\theta^\varepsilon}{(\rho^\varepsilon)^2}\left(\frac{(j^\varepsilon)^2}{\rho^\varepsilon}\right)_{xt}\rho_{xtt}^\varepsilon\,dxd\tau.
\end{aligned}
$$

Due to the integration by part, the term I_3 is rewritten as

$$I_3 = -\varepsilon^2 t^2 \int_0^1 \frac{\theta^\varepsilon (j^\varepsilon)^2}{(\rho^\varepsilon)^4} (\rho^\varepsilon_{xt})^2 \, dx + 2\varepsilon^2 \int_0^t \int_0^1 \left\{ \tau \frac{\theta^\varepsilon (j^\varepsilon)^2}{(\rho^\varepsilon)^4} + \frac{\tau^2}{2} \left(\frac{\theta^\varepsilon (j^\varepsilon)^2}{(\rho^\varepsilon)^4} \right)_t \right\} (\rho^\varepsilon_{xt})^2$$
$$+ \left(\frac{\tau^2 j^\varepsilon \theta^\varepsilon}{(\rho^\varepsilon)^2} \right)_x (\rho^\varepsilon_{tt})^2 + \tau^2 \left\{ \left(2 \left(\frac{j^\varepsilon}{\rho^\varepsilon} \right)_x j^\varepsilon_t - \left(\frac{(j^\varepsilon)^2}{(\rho^\varepsilon)^2} \right)_x \rho^\varepsilon_t \right) \frac{\theta^\varepsilon}{(\rho^\varepsilon)^2} \right\}_x \rho^\varepsilon_{tt} \, dx d\tau. \tag{5.48}$$

Substitute (5.48) in (5.47) and then estimate the right hand side of the resulting equality by the similar calculation as in the derivation of (5.44). These procedures give

$$t^2 \int_0^1 \frac{\varepsilon^2 \theta^\varepsilon}{(\rho^\varepsilon)^2} (R^\varepsilon_{tt})^2 + \frac{\varepsilon (\theta^\varepsilon)^2}{(\rho^\varepsilon)^2} (R^\varepsilon_{xt})^2 \, dx$$
$$+ 2\varepsilon \int_0^t \int_0^1 \frac{\tau^2 \theta^\varepsilon}{(\rho^\varepsilon)^2} (R^\varepsilon_{tt})^2 - \frac{\tau^2 \theta^\varepsilon}{\rho^\varepsilon} Q^\varepsilon_{xxt} R^\varepsilon_{tt} \, dx d\tau \le C\sqrt{\varepsilon}(1 + t^3). \tag{5.49}$$

Then sum up (5.41), (5.43), (5.46) and (5.49) and then substitute the estimates (5.42), (5.44) and (5.45) in the result. After that, using the equation (5.21a), we have

$$\frac{t^2}{\zeta} \|Q^\varepsilon_x(t)\|^2 + t^2 \|(J^\varepsilon_x, Q^\varepsilon_{xx}, \sqrt{\varepsilon} J^\varepsilon_{xx}, \sqrt{\varepsilon} Q^\varepsilon_{xt}, \varepsilon R^\varepsilon_{tt})(t)\|^2 \le C \left(\varepsilon + \frac{\varepsilon^2}{\zeta^2} \right)^{1/2} e^{\beta t}. \tag{5.50}$$

To complete the derivation of the desired estimate (5.40), it suffices to show the estimates of R^ε_{xx}, j^ε_{xx} and θ^ε_{xxx} in (5.40). The estimate

$$\varepsilon t^2 \|j^\varepsilon_{xx}(t)\|^2 \le C \left(\varepsilon + \frac{\varepsilon^2}{\zeta^2} \right)^{1/2} e^{\beta t} \tag{5.51}$$

immediately follows from (4.81c), (5.50) and $J^\varepsilon_{xx} = j^\varepsilon_{xx} + \rho^0_{xt}$. Solve the equation (5.23) in R^ε_{xx}, take the L^2-norm and then use $\rho^\varepsilon_{tt} = R^\varepsilon_{tt} + \rho^0_{tt}$ to get

$$t^2 \|R^\varepsilon_{xx}(t)\|^2 \le C t^2 \left(\varepsilon^2 \|(R^\varepsilon_{tt}, \rho^0_{tt}, j^\varepsilon_{xx})(t)\|^2 + \varepsilon^2 + \|(R^\varepsilon, Q^\varepsilon)(t)\|_1^2 + \|(J^\varepsilon_x, Q^\varepsilon_{xx})(t)\|^2 \right)$$
$$\le C \left(\varepsilon + \frac{\varepsilon^2}{\zeta^2} \right)^{1/2} e^{\beta t}. \tag{5.52}$$

In deriving the last inequality, we have also utilized (4.81c), (5.50) and (5.51). The estimate of θ^ε_{xxx} is derived as follows. Divide the equation (2.11c) by ρ^ε and differentiate the result in x. Multiply the resulting equation by $\varepsilon \theta^\varepsilon_{xxx}$ and integrate the result by part over Ω. Then applying the Schwarz and the Sobolev inequalities with using the estimates (4.81c), (5.24a)–(5.24c), (5.50) and (5.51), we have the inequality

$$\frac{\varepsilon t^2}{\zeta} \|\theta^\varepsilon_{xx}(t)\|^2 + \varepsilon t^2 \|\theta^\varepsilon_{xxx}(t)\|^2 \le C\varepsilon t^2 \|(j^\varepsilon_{xx}, Q^\varepsilon_{xt}, \theta^0_{xt})(t)\|^2 + C\varepsilon t^2 \le C \left(\varepsilon + \frac{\varepsilon^2}{\zeta^2} \right)^{1/2} e^{\beta t}. \tag{5.53}$$

Consequently, summing up the estimates (5.50)–(5.53) yields the desired estimate (5.40). □

Using Corollary 5.3 and Lemmas 5.4–5.6, we establish the semi-global existence of the solution to the hydrodynamic model. Once the estimates (5.28) and (5.40) are shown, it is proven by essentially same arguments as in the proof of Theorem 4.9. Hence we omit the proof.

Theorem 5.7. *Suppose the initial data* $(\rho_0, j_0, \theta_0) \in H^2(\Omega) \times H^2(\Omega) \times H^3(\Omega)$ *and the boundary data* ρ_l, ρ_r *and* ϕ_r *satisfy* (2.4), (2.6), (2.7), (2.10a), (2.10b) *and* (2.13). *Then there exist positive constants* δ_0 *and* ζ_0 *such that if* $\delta \le \delta_0$ *and* $\zeta \le \zeta_0$, *for arbitrary positive time* T, *there exist a positive constant* ε_T, *depending on* ζ *and* T *but independent of* δ, *such that if* $\varepsilon \le \varepsilon_T$, *then the initial boundary value problem* (2.11), (2.12) *and* (2.4)–(2.6) *has a unique solution* $(\rho^\varepsilon, j^\varepsilon, \theta^\varepsilon, \phi^\varepsilon)$ *verifying* $\rho^\varepsilon, j^\varepsilon \in \mathfrak{X}_2([0,T])$, $\theta^\varepsilon, \theta^\varepsilon_x \in \mathfrak{Y}([0,T])$ *and* $\phi^\varepsilon \in C^2([0,T]; H^2)$ *with the conditions* (2.10a), (2.10b) *and* (2.13). *In addition, it satisfies the estimates* (5.24), (5.28) *and* (5.40).

The next corollary is proven by using Theorem 5.7 together with the estimates (3.37), (3.38) and (4.81b). Since this proof is similar to that of Corollary 4.10, we omit the details.

Corollary 5.8. *Suppose the same assumptions as in Theorem 5.7. For an arbitrary positive number* Λ, *there exist positive constants* T_Λ *and* ε_Λ *such that if* $0 < \varepsilon \le \varepsilon_\Lambda$, *the problem* (2.11), (2.12) *and* (2.4)–(2.6) *has a unique solution* $(\rho^\varepsilon, j^\varepsilon, \theta^\varepsilon, \phi^\varepsilon)$ *verifying* $\rho^\varepsilon, j^\varepsilon \in \mathfrak{X}_2([0,T])$, $\theta^\varepsilon, \theta^\varepsilon_x \in \mathfrak{Y}([0,T])$ *and* $\phi^\varepsilon \in C^2([0,T]; H^2)$ *with the conditions* (2.10a), (2.10b) *and* (2.13). *Moreover, it satisfies the estimates* (5.24), (5.28), (5.40) *and*

$$\frac{1}{\zeta}\|(\theta^\varepsilon - \tilde{\theta}^\varepsilon_\zeta)(T_\Lambda)\|_1^2 + \|(j^\varepsilon - \tilde{j}^\varepsilon_\zeta)(T_\Lambda)\|_1^2 + \|(\rho^\varepsilon - \tilde{\rho}^\varepsilon_\zeta, \theta^\varepsilon - \tilde{\theta}^\varepsilon_\zeta)(T_\Lambda)\|_2^2 + \varepsilon\|(\{j^\varepsilon - j^\varepsilon_\zeta\}_{xx}, \{\theta^\varepsilon - \tilde{\theta}^\varepsilon_\zeta\}_{xxx})(T_\Lambda)\|^2 \le \Lambda. \quad (5.54)$$

5.3 Global existence of solution

To construct the time global solution for the hydrodynamic model with the large initial data, it suffices to show the asymptotic stability of the stationary solution with the small initial disturbance.

Theorem 5.9. *Let* $\varepsilon < \zeta$ *and* $(\tilde{\rho}, \tilde{j}, \tilde{\theta}, \tilde{\phi})$ *be the stationary solution of* (2.17). *Suppose that the initial data* $(\rho_0, j_0, \theta_0) \in H^2(\Omega) \times H^2(\Omega) \times H^3(\Omega)$ *and the boundary data* ρ_l, ρ_r *and* ϕ_r *satisfy* (2.4), (2.6), (2.7), (2.10a), (2.10b) *and* (2.13). *Then there exists a positive constant* δ_* *such that if*

$$\zeta + \delta + \frac{1}{\sqrt{\zeta}}\|\theta_0 - \tilde{\theta}\| + \|j_0 - \tilde{j}\|_1 + \|(\rho_0 - \tilde{\rho}, \theta_0 - \tilde{\theta})\|_2 + \sqrt{\varepsilon}\|(\{j_0 - \tilde{j}\}_{xx}, \{\theta_0 - \tilde{\theta}\}_{xxx})\| \le \delta_*, \quad (5.55)$$

the initial boundary value problem (2.11), (2.12) *and* (2.4)–(2.6) *has a unique solution* (ρ, j, θ, ϕ) *satisfying* $\rho,\ j \in \mathfrak{X}_2([0,\infty))$, $\theta, \theta_x \in \mathfrak{Y}([0,\infty))$, $\phi \in C^2([0,\infty); H^2(\Omega))$ *and the conditions* (2.10a), (2.10b) *and* (2.13). *Moreover, the solution* (ρ, j, θ, ϕ) *verifies the additional regularity* $\phi - \tilde{\phi} \in \mathfrak{X}_2^2([0,\infty))$ *and the decay estimate*

$$
\begin{aligned}
&\frac{1}{\zeta}\|(\theta-\tilde{\theta})(t)\|_1^2 + \|(j-\tilde{j})(t)\|_1^2 + \|(\rho-\tilde{\rho}, \theta-\tilde{\theta})(t)\|_2^2 \\
&\qquad\qquad + \varepsilon\|(\{j-\tilde{j}\}_{xx}, \{\theta-\tilde{\theta}\}_{xxx})(t)\|^2 + \|(\phi-\tilde{\phi})(t)\|_4^2 \\
&\le C\left(\frac{1}{\zeta}\|\theta_0-\tilde{\theta}\|_1^2 + \|j_0-\tilde{j}\|_1^2 + \|(\rho_0-\tilde{\rho}, \theta_0-\tilde{\theta})\|_2^2\right. \\
&\qquad\qquad \left. +\varepsilon\|(\{j_0-\tilde{j}\}_{xx}, \{\theta_0-\tilde{\theta}\}_{xxx})\|^2\right) e^{-\alpha t},
\end{aligned} \tag{5.56}
$$

where C *and* α *are positive constants independent of* δ, ε, ζ *and* t.

We begin the proof of Theorem 5.9 with rewriting the problem (2.11), (2.12) and (2.4)–(2.6) to that for the perturbation

$$
\begin{aligned}
&\psi(t,x) := \rho(t,x) - \tilde{\rho}(x), \quad \eta(t,x) := j(t,x) - \tilde{j}(x), \\
&\chi(t,x) := \theta(t,x) - \tilde{\theta}(x), \quad \sigma(t,x) := \phi(t,x) - \tilde{\phi}(x)
\end{aligned}
$$

from the stationary solution to (2.17). Divide (2.11b) by ρ and use the equation (2.11a) to get

$$
\varepsilon\left(\frac{j}{\rho}\right)_t + \frac{\varepsilon}{2}\left(\frac{j^2}{\rho^2}\right)_x + \theta\left(\log\rho\right)_x + \theta_x = \phi_x - \frac{j}{\rho}. \tag{5.57}
$$

Similarly, it follows from (2.17b) that

$$
\frac{\varepsilon}{2}\left(\frac{\tilde{j}^2}{\tilde{\rho}^2}\right)_x + \tilde{\theta}\left(\log\tilde{\rho}\right)_x + \tilde{\theta}_x = \tilde{\phi}_x - \frac{\tilde{j}}{\tilde{\rho}}. \tag{5.58}
$$

Subtracting (2.17a) from (2.11a), (5.58) from (5.57), (2.17c) from (2.11c) and (2.17d) from (2.11d), respectively, we obtain the equations for the perturbation $(\psi, \eta, \chi, \sigma)$:

$$
\psi_t + \eta_x = 0, \tag{5.59a}
$$

$$
\begin{aligned}
&\varepsilon\left(\frac{\tilde{j}+\eta}{\tilde{\rho}+\psi}\right)_t + \frac{\varepsilon}{2}\left(\frac{(\tilde{j}+\eta)^2}{(\tilde{\rho}+\psi)^2} - \frac{\tilde{j}^2}{\tilde{\rho}^2}\right)_x + \tilde{\theta}\left\{\log(\tilde{\rho}+\psi) - \log\tilde{\rho}\right\}_x \\
&\qquad\qquad + \frac{\tilde{\rho}_x+\psi_x}{\tilde{\rho}+\psi}\chi + \chi_x - \sigma_x + \frac{\tilde{j}+\eta}{\tilde{\rho}+\psi} - \frac{\tilde{j}}{\tilde{\rho}} = 0,
\end{aligned} \tag{5.59b}
$$

$$
(\tilde{\rho}+\psi)\chi_t + \frac{2}{3}(\tilde{\theta}+\chi)\eta_x - \frac{2}{3}(\tilde{\theta}+\chi)\frac{\tilde{\rho}_x+\psi_x}{\tilde{\rho}+\psi}\eta - \frac{2\kappa_0}{3}\chi_{xx} + \frac{\tilde{\rho}+\psi}{\zeta}\chi = \mathcal{G}_1 + \mathcal{G}_2, \tag{5.59c}
$$

$$
\sigma_{xx} = \psi, \tag{5.59d}
$$

$$\mathcal{G}_1 := \frac{2\tilde{j}}{3}\left(\frac{\tilde{\theta}+\chi}{\tilde{\rho}+\psi}\psi_x + \frac{\tilde{\rho}_x}{\tilde{\rho}+\psi}\chi - \frac{\tilde{\theta}\tilde{\rho}_x}{\tilde{\rho}(\tilde{\rho}+\psi)}\psi\right) - \frac{\tilde{\theta}-1}{\zeta}\psi,$$
$$\mathcal{G}_2 := -(\tilde{j}+\eta)\chi_x - \tilde{\theta}_x\eta + \left(\frac{2}{3} - \frac{\varepsilon}{3\zeta}\right)\left(\frac{2\tilde{j}+\eta}{\tilde{\rho}+\psi}\eta - \frac{\tilde{j}^2\psi}{(\tilde{\rho}+\psi)\tilde{\rho}}\right).$$

The initial and the boundary data to (5.59) are derived from (2.4)–(2.6) and (2.12) as

$$\psi(x,0) = \psi_0(x) := \rho_0(x) - \tilde{\rho}(x), \quad \eta(x,0) = \eta_0(x) := j_0(x) - \tilde{j}(x),$$
$$\chi(x,0) = \chi_0(x) := \theta_0(x) - \tilde{\theta}(x), \tag{5.60}$$
$$\psi(t,0) = \psi(t,1) = \chi(t,0) = \chi(t,1) = \sigma(t,0) = \sigma(t,1) = 0. \tag{5.61}$$

The unique existence of the time local solution (ψ,η,χ,σ) to the problem (5.59)–(5.61) follows from Theorem 3.5 and Corollary 5.3.

Lemma 5.10. *Suppose that the initial data (ψ_0,η_0,χ_0) belongs to $H^2(\Omega)\times H^2(\Omega)\times H^3(\Omega)$ and $(\tilde{\rho}+\psi_0,\tilde{j}+\eta_0,\tilde{\theta}+\chi_0)$ satisfies* (2.10a), (2.10b) *and* (2.13). *Then there exists a positive constant T^*, independent of ε, such that the initial boundary value problem* (5.59)–(5.61) *has a unique solution (ψ,η,χ,σ) satisfying $\psi,\eta\in\mathfrak{X}_2([0,T^*])$, $\chi,\chi_x\in\mathfrak{Y}([0,T^*])$ and $\sigma\in\mathfrak{X}_2^2([0,T^*])$ with the property that $(\tilde{\rho}+\psi,\tilde{j}+\eta,\tilde{\theta}+\chi)$ satisfies* (2.10a), (2.10b) *and* (2.13).

The existence of the time global solution in Theorem 5.9 follows from the continuation argument together with the local existence of the solution in Lemma 5.10 and an a-priori estimate (5.62) below. To derive the estimate (5.62), it is convenient to use notations

$$N_\varepsilon(t) := \sup_{0\le\tau\le t} n_\varepsilon(\tau), \quad n_\varepsilon(\tau) := \frac{1}{\sqrt{\zeta}}\|\chi(\tau)\|_1 + \|\eta(\tau)\|_1 + \|(\psi,\chi)(\tau)\|_2 + \sqrt{\varepsilon}\|(\eta_{xx},\chi_{xxx})(\tau)\|.$$

Proposition 5.11. *Let $T>0$ and let (ψ,η,χ,σ) be the solution to* (5.59)–(5.61), *satisfying $\psi,\eta\in\mathfrak{X}_2([0,T])$, $\chi,\chi_x\in\mathfrak{Y}([0,T])$ and $\sigma\in\mathfrak{X}_2^2([0,T])$. Then there exist positive constants δ_0 and ζ_0, independent of T, such that if $N_\varepsilon(T)+\delta\le\delta_0$ and $\varepsilon<\zeta\le\zeta_0$, then the estimate*

$$n_\varepsilon^2(t) + \|\sigma(t)\|_4^2 + \int_0^t n_\varepsilon^2(\tau) + \|\sigma(\tau)\|_4^2\,d\tau \le Cn_\varepsilon^2(0) \tag{5.62}$$

holds for $t\in[0,T]$, where C is a positive constant independent of T, δ, ζ and ε.

The proof of this proposition is derived in the several steps, which are stated in Lemma 5.12–5.16, and is completed at the end of this section.

We first derive the basic estimate (5.70) in Lemma 5.12. To this end, define an energy form $\mathcal{E}_2$, which is almost same as in [32], by

$$\mathcal{E}_2 := \frac{\varepsilon}{2\rho}(j-\tilde{j})^2 + \rho\tilde{\theta}\Psi\left(\frac{\tilde{\rho}}{\rho}\right) + \frac{1}{2}\left\{(\phi-\tilde{\phi})_x\right\}^2 + \frac{3}{2}\rho\tilde{\theta}\Psi\left(\frac{\theta}{\tilde{\theta}}\right), \tag{5.63}$$
$$\Psi(s) := s - 1 - \log s.$$

Here $\Psi(s)$ is equivalent to $|s-1|^2$ if $s \geq c > 0$. Thus, if the quantity $|(\psi,\chi)|$ is sufficiently small, $\mathcal{E}_2$ is equivalent to $|(\psi,\sqrt{\varepsilon}\eta,\chi,\sigma_x)|^2$ thanks to the estimates in (3.11). Namely,

$$c|(\psi,\sqrt{\varepsilon}\eta,\chi,\sigma_x)|^2 \leq \mathcal{E}_2 \leq C|(\psi,\sqrt{\varepsilon}\eta,\chi,\sigma_x)|^2, \tag{5.64}$$

where c and C are positive constants.

Multiply the equation (5.59b) by η. Apply the product rule for derivatives to the first term, the third and the sixth terms, respectively, Then substitute $\eta_x = -\psi_t$ and $\eta_x = -\sigma_{xxt}$ in the third and the sixth terms, respectively. These procedure yields

$$\left\{\frac{\varepsilon}{2\rho}\eta^2 + \rho\tilde{\theta}\Psi\left(\frac{\tilde{\rho}}{\rho}\right) + \frac{1}{2}(\sigma_x)^2\right\}_t + \frac{1}{\tilde{\rho}}\eta^2 + \frac{\rho_x}{\rho}\chi\eta = -\chi_x\eta + \mathcal{R}_{3x} + \mathcal{R}_4, \tag{5.65}$$
$$\mathcal{R}_3 := \sigma\sigma_{xt} + \sigma\eta - \tilde{\theta}\left\{\log\rho - \log\tilde{\rho}\right\}\eta,$$
$$\mathcal{R}_4 := -\varepsilon\frac{\eta+2\tilde{j}}{2\rho^2}\eta_x\eta - \frac{\varepsilon}{2}\left(\frac{j^2}{\rho^2} - \frac{\tilde{j}^2}{\tilde{\rho}^2}\right)_x\eta - j\left(\frac{1}{\rho} - \frac{1}{\tilde{\rho}}\right)\eta + \tilde{\theta}_x\left\{\log\rho - \log\tilde{\rho}\right\}\eta.$$

Multiply the equation (5.59c) by $3\chi/2\theta$ and apply the product rule for derivatives to the first and the fourth terms on the left hand side to obtain

$$\left\{\frac{3}{2}\rho\tilde{\theta}\Psi\left(\frac{\theta}{\tilde{\theta}}\right)\right\}_t + \frac{3\rho}{2\zeta\theta}\chi^2 + \frac{\kappa_0}{\theta}\chi_x^2 - \frac{\rho_x}{\rho}\chi\eta = -\chi\eta_x + \mathcal{R}_{5x} + \mathcal{R}_6, \tag{5.66}$$
$$\mathcal{R}_5 := \frac{\kappa_0}{\theta}\chi\chi_x, \quad \mathcal{R}_6 := -\frac{3}{2}\tilde{\theta}\eta_x\Psi\left(\frac{\theta}{\tilde{\theta}}\right) + \frac{\kappa_0\theta_x}{\theta^2}\chi\chi_x + \frac{3}{2\theta}(\mathcal{G}_1+\mathcal{G}_2)\chi.$$

Adding (5.65) to (5.66), we have an equation for the energy form $\mathcal{E}_2$

$$\mathcal{E}_{2t} + \frac{1}{\tilde{\rho}}\eta^2 + \frac{3\rho}{2\zeta\theta}\chi^2 + \frac{\kappa_0}{\theta}\chi_x^2 = -(\chi\eta)_x + (\mathcal{R}_3+\mathcal{R}_5)_x + \mathcal{R}_4 + \mathcal{R}_6. \tag{5.67}$$

The estimates

$$\|\partial_t^i\sigma(t)\|_2^2 \leq C\|\partial_t^i\psi(t)\|^2 \quad \text{for} \quad i = 0,1,2, \tag{5.68}$$
$$\|\sigma_{xt}(t)\|^2 \leq \|\eta(t)\|^2 \tag{5.69}$$

follow from the same computation as in Lemma 3.3 in [30].

Lemma 5.12. *Under the same conditions as in Proposition 5.11, it holds that*

$$\|(\psi,\chi,\sigma_x)(t)\|^2+\varepsilon\|\eta(t)\|^2+\int_0^t\|(\psi,\eta,\chi_x,\sigma_x)(\tau)\|^2+\frac{1}{\zeta}\|\chi(\tau)\|^2\,d\tau$$
$$\le Cn_\varepsilon^2(0)+C(N_\varepsilon(T)+\delta+\zeta^{1/4})\int_0^t\|(\psi_x,\psi_t)(\tau)\|^2\,dx \quad (5.70)$$

for $t\in[0,T]$, where C is a positive constant independent of t, δ, ζ and ε.

Proof. We show the basic estimate (5.70) by the similar manner as in Lemma 3.6 in [32]. Integrating (5.63) over the domain Ω yields

$$\frac{d}{dt}\int_0^1\mathcal{E}_2\,dx+\int_0^1\frac{1}{\tilde\rho}\eta^2+\frac{3\rho}{2\theta}\chi^2+\frac{\kappa_0}{\theta}\chi_x^2\,dx$$
$$\le C(N_\varepsilon(T)+\delta+\zeta^{1/4})\left(\frac{1}{\zeta}\|\chi(t)\|^2+\|(\psi,\eta,\chi)(t)\|_1^2\right) \quad (5.71)$$

since the integration of $(\mathcal{R}_3+\mathcal{R}_5)_x$ is zero with aid of the boundary conditions (5.61). Here we also used the estimate

$$\int_0^1-(\chi\eta)_x+\mathcal{R}_4+\mathcal{R}_6\,dx\le C(N_\varepsilon(T)+\delta+\zeta^{1/4})\left(\frac{1}{\zeta}\|\chi(t)\|^2+\|(\psi,\eta,\chi)(t)\|_1^2\right),$$

where is derived from similar manner as in the derivation of (4.52) and (4.53). Moreover, multiply the equation (5.59b) by $-\sigma_x$ and integrate the result over the domain Ω. Then apply the Schwarz and the Sobolev inequalities to the resultant equality and then use (5.68), (5.69) and the mean value theorem. These computations give

$$c\|(\psi,\sigma_x)(t)\|\le\frac{d}{dt}\int_0^1\varepsilon\left(\frac{j}{\rho}-\frac{\tilde j}{\tilde\rho}\right)\sigma_x\,dx$$
$$+C\|(\eta,\chi,\chi_x)(t)\|^2+C(N_\varepsilon(T)+\delta)\|(\psi,\psi_x,\eta_x)(t)\|^2. \quad (5.72)$$

Notice that the integration in t of the first term in the left hand side of (5.72) are estimated as

$$\int_0^1\varepsilon\left(\frac{j}{\rho}-\frac{\tilde j}{\tilde\rho}\right)\sigma_x(t,x)-\varepsilon\left(\frac{j}{\rho}-\frac{\tilde j}{\tilde\rho}\right)\sigma_x(0,x)\,dx\le Cn_\varepsilon(0)+C\int_0^1\mathcal{E}_2(t)\,dx$$

with aid of (5.64) and (5.68). Hence, multiply (5.72) by α, where α is a positive constant, and then add the result to (5.71). Let α and $N_\varepsilon(T)+\delta+\zeta^{1/4}$ be sufficiently small and use $\psi_t=-\eta_x$ to obtain the desired estimate (5.70). □

We turn to the derivation of the higher order estimates. To this end, we firstly derive the estimates of the derivatives in the time variable t. Then we rewrite them in the derivatives of the spatial variable x by using the estimates

$$\frac{1}{\zeta^2}\|\chi(t)\|^2 \leq C\left(\|\chi_t(t)\|^2 + \|(\psi,\eta)(t)\|_1^2 + \|\chi(t)\|_2^2\right), \tag{5.73a}$$

$$\|\varepsilon\eta_t(t)\|^2 \leq C\|(\psi,\eta,\chi)(t)\|_1^2, \tag{5.73b}$$

$$\|\eta_x(t)\|^2 = \|\psi_t(t)\|^2, \quad \|\eta_{xt}(t)\|^2 = \|\psi_{tt}(t)\|^2, \quad \|\eta_{xx}(t)\|^2 = \|\psi_{xt}(t)\|^2, \tag{5.73c}$$

$$cn_\varepsilon^2(t) \leq \left(\left\|\left(\psi, \frac{\chi}{\sqrt{\zeta}}\right)(t)\right\|_1^2 + \|(\eta, \psi_t, \chi_{xx}, \sqrt{\varepsilon}\psi_{xt}, \sqrt{\varepsilon}\chi_{xt}, \varepsilon\psi_{tt})(t)\|^2\right) \leq Cn_\varepsilon^2(t), \tag{5.73d}$$

$$\|\sigma(t)\|_4^2 \leq C\|\psi(t)\|_2^2. \tag{5.73e}$$

The above estimates are shown by using the equation (5.59). Precisely, owing to (3.11), (3.25) and the Sobolev inequality, the estimate (5.73a) follows from the equation (5.59c); (5.73b) follows from (5.59b); (5.73c) follows from (5.59a); (5.73d) follows from (5.59b) and (5.59c); (5.73e) follows from (5.59d).

Differentiate (2.1b) with respect to x and multiply the result by $1/\rho$. Similarly, differentiate (2.17b) with respect to x and multiply the result by $1/\tilde{\rho}$. Taking a difference of the two resultant equalities and substituting the equations (5.59a) and (5.59d) in the result, we have

$$\frac{\varepsilon}{\rho}\psi_{tt} - \left\{S[\rho,j,\theta]\frac{\psi_x}{\rho}\right\}_x + \psi + \frac{\psi_t}{\rho} - \chi_{xx} = -2\varepsilon\frac{j}{\rho^2}\psi_{xt} + \mathcal{F}_1 + \mathcal{F}_2, \tag{5.74}$$

$$\begin{aligned}
\mathcal{F}_1 :=& \frac{2\varepsilon}{\rho^2}\psi_t^2 + 4\varepsilon\frac{j\rho_x}{\rho^3}\psi_t + 2\varepsilon\frac{j^2}{\rho^4}(\rho+\tilde{\rho})_x\psi_x + \varepsilon\left(\frac{j^2}{\rho^3}\right)_x\psi_x - \frac{\theta_x - 2\tilde{\theta}_x}{\rho}\psi_x + \left(\frac{\theta}{\rho^2}\rho_x - \frac{\phi_x}{\rho}\right)\psi_x, \\
\mathcal{F}_2 :=& 2\varepsilon\frac{\tilde{\rho}_x^2}{\rho^4}(j+\tilde{j})\eta - 2\varepsilon\tilde{j}^2\tilde{\rho}_x^2\frac{(\tilde{\rho}+\psi)^4 - \tilde{\rho}^4}{\tilde{\rho}^4\rho^4} - \varepsilon\frac{\tilde{\rho}_{xx}}{\rho^3}(j+\tilde{j})\eta + \varepsilon\tilde{j}^2\tilde{\rho}_{xx}\frac{(\tilde{\rho}+\psi)^3 - \tilde{\rho}^3}{\tilde{\rho}^3\rho^3} \\
&+ \frac{\tilde{\rho}_{xx}}{\rho}\chi - \frac{\tilde{\theta}\tilde{\rho}_{xx}}{\tilde{\rho}\rho}\psi + \frac{2\rho_x}{\rho}\chi_x - \frac{2\tilde{\rho}_x\tilde{\theta}_x}{\rho\tilde{\rho}}\psi - \frac{\tilde{\rho}_x}{\rho}\sigma_x + \frac{\tilde{\phi}_x\tilde{\rho}_x}{\rho\tilde{\rho}}\psi.
\end{aligned}$$

Note that the estimates

$$\|\mathcal{F}_1\| \leq C(N_\varepsilon(T)+\delta)\|(\psi_x,\psi_t)(t)\|, \quad \|\mathcal{F}_2\| \leq C\|(\psi,\eta,\chi,\chi_x)(t)\| \tag{5.75}$$

follow from the Sobolev inequality as well as the estimates (3.11), (3.25), (5.68) and (5.73c). In this calculation, the estimate

$$\left|\frac{\theta}{\rho^2}\rho_x - \frac{\phi_x}{\rho}\right| = \left|\varepsilon\frac{j_t}{\rho^2} + \varepsilon\left(\frac{j^2}{\rho}\right)_x\frac{1}{\rho^2} + \frac{\theta_x}{\rho} + \frac{j}{\rho^2}\right| \leq C(N_\varepsilon(T)+\delta) \tag{5.76}$$

has been utilized to handled the last term in $\mathcal{F}_1$. Similarly, $\mathcal{G}_1$ and $\mathcal{G}_2$ in the equation (5.59c) are estimated as

$$\|(\mathcal{G}_1, \mathcal{G}_2)\|_i \le C(N_\varepsilon(T) + \delta)\|(\psi, \eta, \chi, \psi_x, \chi_x)\|_i \tag{5.77}$$

for $i = 0, 1$.

Lemma 5.13. *Under the same assumptions as in Proposition 5.11, it holds that*

$$\begin{aligned}
&\|(\psi_x, \chi_x)(t)\|^2 + \varepsilon\|\psi_t(t)\|^2 + \int_0^t \frac{1}{\zeta}\|\chi_x(\tau)\|^2 + \|(\psi_t, \psi_x, \chi_{xx})(\tau)\|^2\,d\tau \\
&\le Cn_\varepsilon^2(0) + C\int_0^t \frac{1}{\zeta}\|\chi(\tau)\|^2 + \|(\psi, \eta, \chi_x)(\tau)\|^2\,d\tau + C(N_\varepsilon(T) + \delta + \zeta^{1/2})\int_0^t \|\chi_t(\tau)\|^2\,d\tau
\end{aligned} \tag{5.78}$$

for $t \in [0, T]$, where C is a positive constant independent of T, δ, ζ and ε.

Proof. Multiply the equation (5.74) by $\psi + 2\psi_t$, integrate the result by part over the domain Ω and then use the boundary conditions (5.61) to obtain

$$\begin{aligned}
&\frac{d}{dt}\int_0^1 \frac{1}{\rho}\left(\varepsilon\psi_t^2 + \varepsilon\psi\psi_t + \frac{\psi^2}{2}\right) + S[\rho, j, \theta]\frac{\psi_x^2}{\rho} + \psi^2\,dx \\
&\qquad + \int_0^1 \frac{\psi_t^2}{\rho} + S[\rho, j, \theta]\frac{\psi_x^2}{\rho} + \psi^2\,dx - \int_0^1 2\chi_{xx}\psi_t\,dx = \mathcal{I}_1 + \mathcal{I}_2,
\end{aligned} \tag{5.79}$$

$$\begin{aligned}
\mathcal{I}_1 &:= \int_0^1 \left(\frac{1}{\rho}\right)_t\left(\varepsilon\psi_t^2 + \varepsilon\psi\psi_t + \frac{1}{2}\psi^2\right) + \varepsilon\left(\frac{2j}{\rho^2}\psi\right)_x \psi_t + \varepsilon\left(\frac{2j}{\rho^2}\right)_x \psi_t^2 \\
&\qquad + \left(\frac{S[\rho, j, \theta]}{\rho}\right)_t \psi_x^2 + \mathcal{F}_1(\psi + 2\psi_t)\,dx, \\
\mathcal{I}_2 &:= -\int_0^1 \chi_x\psi_x - \mathcal{F}_2(\psi + 2\psi_t)\,dx.
\end{aligned}$$

We estimate the integration $\mathcal{I}_1$, by applying the Schwarz and the Sobolev inequalities with using (3.11), (3.25), (5.73b), (5.73c) and (5.75), as

$$|\mathcal{I}_1| \le C(N_\varepsilon(T) + \delta)\|(\eta, \psi, \chi, \psi_x, \psi_t, \chi_x, \chi_t)(t)\|^2. \tag{5.80}$$

By the Schwarz inequality and (5.75), the estimate

$$|\mathcal{I}_2| \le \mu\|(\psi_x, \psi_t)(t)\|^2 + C[\mu]\|(\eta, \psi, \chi, \chi_x)(t)\|^2 \tag{5.81}$$

holds, where μ is an arbitrary positive constant to be determined.

On the other hand, multiplying the equation(5.59d) by $-3\chi_{xx}$ and integrate the resulting equality by part over the domain Ω give

$$\frac{d}{dt}\int_0^1 \frac{3\rho}{2}\chi_x^2\,dx + \int_0^1 \frac{3\rho}{\zeta}\chi_x^2 + 2\kappa_0\chi_{xx}^2\,dx + \int_0^1 2\chi_{xx}\psi_t\,dx = \mathcal{I}_3 + \mathcal{I}_4, \tag{5.82}$$
$$\mathcal{I}_3 := \int_0^1 \frac{3}{2}\psi_t\chi_x^2 - 2(\theta-1)\psi_t\chi_{xx} - 3(\mathcal{G}_1+\mathcal{G}_2)\chi_{xx}\,dx,$$
$$\mathcal{I}_4 := -\int_0^1 3\rho_x\chi_t\chi_x + \frac{3\rho_x}{\zeta}\chi\chi_x + \frac{\theta\rho_x}{\rho}\eta\chi_{xx}\,dx.$$

By the application of the Schwarz and the Sobolev inequalities with using (3.11), (3.25), (5.73c) and (5.77), the integration $\mathcal{I}_3$ is estimated as

$$|\mathcal{I}_3| \le C(N_\varepsilon(T)+\delta)\|(\eta,\psi,\chi,\psi_x,\psi_t,\chi_x,\chi_{xx})(t)\|^2. \tag{5.83}$$

Similarly, it holds that

$$|\mathcal{I}_4| \le \mu\left(\frac{1}{\zeta}\|\chi_x(t)\|^2 + \|\chi_{xx}(t)\|^2\right) + C[\mu]\left(\frac{1}{\zeta}\|\chi(t)\|^2 + \|(\eta,\psi,\chi_x)(t)\|^2 + \zeta\|\chi_t(t)\|^2\right). \tag{5.84}$$

Finally, substitute the estimates (5.80) and (5.81) in (5.79) as well as the estimates (5.83) and (5.84) in (5.82), respectively. Then sum up the both results, taking μ and $N_\varepsilon(T)+\delta$ small enough and integrating the resultant inequality in t, we have the desired estimate (5.78) with aid of the subsonic condition (3.25b). □

Lemma 5.14. *Under the same assumptions as in Proposition 5.11, it holds that*

$$\frac{1}{\zeta}\|\chi(t)\|^2 + \int_0^t \|\chi_t(\tau)\|^2\,d\tau \le n_\varepsilon^2(0) + C\int_0^t \|(\psi,\eta,\psi_x,\psi_t)(\tau)\|^2 + \|\chi(\tau)\|_2^2\,d\tau \tag{5.85}$$

for $t\in[0,T]$, where C is a positive constant independent of T, δ, ζ and ε.

Proof. Multiplying the equation (5.59c) by χ_t/ρ and integrating resulting equality by part over the domain Ω lead to

$$\frac{d}{dt}\int_0^1 \frac{1}{2\zeta}\chi^2\,dx + \int_0^1 \chi_t^2\,dx = \int_0^1\left(\frac{2\theta\rho_x}{3\rho}\eta - \frac{2}{3}\theta\eta_x + \frac{2\kappa_0}{3}\chi_{xx} + \mathcal{G}_1+\mathcal{G}_2\right)\frac{\chi_t}{\rho}\,dx$$
$$\le \frac{1}{2}\|\chi_t(t)\|^2 + C(\|(\psi,\eta,\psi_x,\psi_t)(t)\|^2 + \|\chi(t)\|_2^2). \tag{5.86}$$

In deriving the last inequality, we have used the estimate (5.77) as well as the Schwarz and the Sobolev inequalities. Hence, integration of (5.86) in t yield the estimate (5.85). □

Differentiating the equations (5.59c) and (5.74) in t yields

$$\frac{\varepsilon}{\rho}\psi_{ttt} - \left\{S[\rho,j,\theta]\frac{\psi_{xt}}{\rho}\right\}_x + \psi_t + \frac{\psi_{tt}}{\rho} - \chi_{xxt} = -2\varepsilon\frac{j}{\rho^2}\psi_{xtt} + \mathcal{F}_{1t} + \mathcal{F}_{2t} + \mathcal{F}_3, \tag{5.87}$$

$$\rho\chi_{tt} + \frac{2\theta}{3}\eta_{xt} - \frac{2\theta\rho_x}{3\rho}\eta_t - \frac{2\kappa_0}{3}\chi_{xxt} + \frac{\rho}{\zeta}\chi_t = \mathcal{G}_{1t} + \mathcal{G}_{2t} + \mathcal{G}_3, \tag{5.88}$$

$$\mathcal{F}_3 := -\left(\frac{1}{\rho}\right)_t(\varepsilon\psi_{tt} + \psi_t) + \left\{\left(\frac{S[\rho,j,\theta]}{\rho}\right)_t\psi_x\right\}_x - 2\varepsilon\left(\frac{j}{\rho^2}\right)_t\psi_{xt},$$

$$\mathcal{G}_3 := -\psi_t\chi_t + \frac{2}{3}\chi_t\eta_x + \left(\frac{2\theta\rho_x}{3\rho}\right)_t\eta - \frac{\psi_t}{\zeta}\chi.$$

Using the Sobolev inequality as well as the estimates (3.11), (3.25), (5.68) and (5.73a)–(5.73d), we have

$$\|(\mathcal{F}_{1t},\mathcal{F}_{2t},\mathcal{F}_3)\| \le C\left(\|\eta(t)\| + \|(\psi,\chi,\psi_t,\chi_t)(t)\|_1 + \sqrt{\varepsilon}\|\psi_{tt}(t)\|\right), \tag{5.89}$$

$$\begin{aligned}\sqrt{\varepsilon}\|(\mathcal{G}_{1t},\mathcal{G}_3)\| + \varepsilon\|\mathcal{G}_{2t}\|_1 \le{}& C(N_\varepsilon(T) + \delta + \sqrt{\varepsilon})(\|(\psi_{xt},\chi_{xt})(t)\| + \sqrt{\varepsilon}\|(\psi_{tt},\chi_{xxt})(t)\|)\\ &+ C\|(\psi,\eta,\psi_x,\psi_t)(t)\| + C\|\chi(t)\|_2.\end{aligned} \tag{5.90}$$

Lemma 5.15. *Under the same assumptions as in Proposition 5.11, it holds that*

$$\begin{aligned}&\frac{1}{\zeta}\|\chi_x(t)\|^2 + \|(\psi_t,\chi_{xx})(t)\|^2 + \varepsilon\|(\psi_{xt},\chi_{xt})(t)\|^2 + \varepsilon^2\|\psi_{tt}(t)\|^2\\ &\qquad + \int_0^t \|(\psi_{xt},\chi_{xt})(\tau)\|^2 + \varepsilon\|(\psi_{tt},\chi_{xxt})(\tau)\|^2\,d\tau\\ &\qquad \le Cn_\varepsilon^2(0) + C\int_0^t \frac{1}{\zeta}\|\chi(\tau)\|_1^2 + \|(\psi,\eta,\psi_x,\psi_t,\chi_t,\chi_{xx})(\tau)\|^2\,d\tau\end{aligned} \tag{5.91}$$

for $t \in [0,T]$, where C is a positive constant independent of T, δ, ζ and ε.

Proof. Multiplying the equation (5.87) by $\psi_t + 2\varepsilon\psi_{tt}$, integrating the resultant equality by part over the domain Ω and then using the boundary condition (5.61), we obtain

$$\begin{aligned}&\frac{d}{dt}\int_0^1 \frac{1}{\rho}\left(\varepsilon^2\psi_{tt}^2 + \varepsilon\psi_t\psi_{tt} + \frac{\psi_t^2}{2}\right) + \varepsilon S[\rho,j,\theta]\frac{\psi_{xt}^2}{\rho} + \varepsilon\psi_t^2\,dx\\ &\quad + \int_0^1 \frac{\varepsilon}{\rho}\psi_{tt}^2 + S[\rho,j,\theta]\frac{\psi_{xt}^2}{\rho} + \psi_t^2\,dx - \int_0^1 \chi_{xxt}\psi_t + 2\varepsilon\chi_{xxt}\psi_{tt}\,dx = \mathcal{H}_1,\end{aligned} \tag{5.92}$$

$$\begin{aligned}\mathcal{H}_1 := \int_0^1 &\left(\frac{1}{\rho}\right)_t\left(\varepsilon^2\psi_{tt}^2 + \varepsilon\psi_t\psi_{tt} + \frac{1}{2}\psi_t^2\right) + \varepsilon\left(\frac{2j}{\rho^2}\psi_t\right)_x\psi_{tt} + \varepsilon^2\left(\frac{2j}{\rho^2}\right)_x\psi_{tt}^2\\ &+ \varepsilon\left(\frac{S[\rho,j,\theta]}{\rho}\right)_t\psi_{xt}^2 + (\mathcal{F}_{1t} + \mathcal{F}_{2t} + \mathcal{F}_3)(\psi_t + 2\varepsilon\psi_{tt})\,dx.\end{aligned}$$

We multiply the equation (5.59c) by $-3\chi_{xxt}/2$, integrate the resultant equality by part over the domain Ω and then use the equation (5.59a). The resultant equality is

$$\frac{d}{dt}\int_0^1 \frac{3\rho}{4\zeta}\chi_x^2 + \frac{\kappa_0}{2}\chi_{xx}^2\,dx + \int_0^1 \frac{3\rho}{2}\chi_{xt}^2\,dx + \int_0^1 \chi_{xxt}\psi_t\,dx = \mathcal{H}_2, \tag{5.93}$$

$$\mathcal{H}_2 := \frac{3}{2}\int_0^1 \left(\frac{2}{3}(\theta-1)\eta_x - \frac{2\theta\rho_x}{3\rho}\eta - \mathcal{G}_1 - \mathcal{G}_2\right)_x \chi_{xt} - \rho_x\chi_t\chi_{xt} + \frac{\psi_t}{2\zeta}\chi_x^2 - \frac{\rho_x}{\zeta}\chi\chi_{xt}\,dx.$$

Furthermore, multiplying the equation (5.88) by $-3\varepsilon\chi_{xxt}$ and integrating the resultant equality by part over the domain Ω yield

$$\frac{d}{dt}\int_0^1 \frac{3\varepsilon}{2}\rho\chi_{xt}^2\,dx + \int_0^1 2\varepsilon\kappa_0\chi_{xxt}^2 + \frac{3\varepsilon\rho}{\zeta}\chi_{xt}^2\,dx + \int_0^1 2\varepsilon\chi_{xxt}\psi_{tt}\,dx = \mathcal{H}_3, \tag{5.94}$$

$$\begin{aligned}\mathcal{H}_3 := \varepsilon\int_0^1 &\frac{3}{2}\psi_t\chi_{xt}^2 - 2(\theta-1)\psi_{tt}\chi_{xxt} - 3(\mathcal{G}_{1t}+\mathcal{G}_3)\chi_{xxt} + \left(3\mathcal{G}_{2t} + 2\frac{\theta\rho_x}{\rho}\eta_t\right)_x \chi_{xt} \\ &- \frac{3\rho_x}{\zeta}\chi_t\chi_{xt} - 3\rho_x\chi_{tt}\chi_{xt}\,dx.\end{aligned}$$

In order to estimate $\mathcal{H}_3$, by using the equation (5.88), we rewrite the term $\rho_x\chi_{tt}\chi_{xt}$ as

$$-\rho_x\chi_{tt}\chi_{xt} = \frac{\rho_x}{\rho}\left(\frac{2\theta}{3}\eta_{xt} - \frac{2\theta\rho_x}{3\rho}\eta_t - \frac{2\kappa_0}{3}\chi_{xxt} + \frac{\rho}{\zeta}\chi_t - \mathcal{G}_{1t} - \mathcal{G}_{2t} - \mathcal{G}_3\right)\chi_{xt}.$$

Then applying the Schwarz and the Sobolev inequalities to $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ with using the estimates (3.11), (3.25), (5.73a)–(5.73d), (5.77), (5.89) and (5.90), we have

$$\begin{aligned}|(\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3)| &\le \{\mu + C(N_\varepsilon(T)+\delta+\sqrt{\varepsilon})\}\|(\chi_{xt},\psi_{xt},\sqrt{\varepsilon}\psi_{tt},\sqrt{\varepsilon}\chi_{xxt})(t)\|^2 \\ &\quad + C[\mu]\left(\frac{1}{\zeta}\|\chi(t)\|_1^2 + \|(\psi,\eta,\psi_x,\psi_t,\chi_t,\chi_{xx})(\tau)\|^2\right). \end{aligned}\tag{5.95}$$

Finally, sum up the equalities (5.92)–(5.94) and substitute the estimate (5.95) in the resultant equality. Successively take μ and $N_\varepsilon(T)+\delta+\sqrt{\varepsilon}$ small enough and integrate the result in t. These procedures give the desired estimate (5.91). □

Lemma 5.16. *Under the same conditions as in Proposition 5.11, it holds that*

$$\begin{aligned}&\|\eta(t)\|^2 + \int_0^t \varepsilon\|\eta_t(\tau)\|^2\,d\tau \\ &\qquad\le Cn_\varepsilon^2(0) + C\|(\psi,\chi)(t)\|_1^2 + C\int_0^t \|\eta(\tau)\|^2 + \|(\psi,\psi_t,\chi_t)(\tau)\|_1^2\,d\tau \end{aligned}\tag{5.96}$$

for $t\in[0,T]$, where C is a positive constant independent of T, δ, ζ and ε.

Proof. Subtract (2.17b) from (2.11b), multiply the resultant equation by η_t and integrate the result by part over the domain Ω. Then apply the Schwarz and the Sobolev inequalities to the resultant equality. The result is

$$\begin{aligned}
&\frac{d}{dt}\int_0^1 \frac{\eta^2}{2}\,dx + \int_0^1 \varepsilon\eta_t^2\,dx \\
&\qquad \le \frac{d}{dt}\int_0^1 \{\rho\sigma_x + \tilde{\phi}_x\psi\}\eta - \{\theta\psi + \tilde{\rho}\chi\}_x\eta\,dx + C\|\eta(t)\|^2 + C\|(\psi,\psi_t,\chi_t)(t)\|_1^2, \quad (5.97)
\end{aligned}$$

of which integration in t yields the estimate (5.96). □

We derive the a-priori estimate (5.62) in Proposition 5.11 and complete the proof of Theorem 5.9.

Proofs of Proposition 5.11 and Theorem 5.9. Rewrite (5.70), (5.78), (5.85), (5.91) and (5.96), which are estimates of the derivatives in t, to those in x by using (5.73d), we have the a-priori estimate (5.62). It complete the proof of Proposition 5.11.

The continuation argument with the time local existence in Corollary 5.10 and the a-priori estimate (5.62) yields the time global existence for the small initial disturbance, which is asserted in Theorem 5.9.

The decay estimate (5.56) follows from the similar manner as in the proof of Theorem 4.11. Precisely, multiply (5.72) by β, (5.79) and (5.82) by β^2, (5.86) by β^3, (5.92)–(5.94) by β^4 and (4.62) by β^5, respectively, where $\beta \in (0,1]$. Sum up these results and (5.71) and then substitute the estimates (5.80), (5.81), (5.83), (5.84) and (5.95) in $\mathcal{I}_i$ and $\mathcal{H}_i$. Successively letting μ small enough gives an ordinary differential inequality

$$\frac{d}{dt}E(t) + c_1 D(t) \le C_1\beta D(t) + C(N_\zeta(T) + \delta + \zeta^{1/4} + \varepsilon^{1/2})D(t), \quad (5.98)$$

$$\begin{aligned}
E(t) := \int_0^1 &\mathcal{E}_2 - \beta\varepsilon\left(\frac{j}{\rho} - \frac{\tilde{j}}{\tilde{\rho}}\right)\sigma_x + \frac{\beta^2}{\rho}\left(\varepsilon\psi_t^2 + \varepsilon\psi\psi_t + \frac{\psi^2}{2}\right) + \beta^2 S[\rho,j,\theta]\frac{\psi_x^2}{\rho} + \beta^2\psi^2 \\
&+ \beta^2\frac{3\rho}{2}\chi_x^2 + \frac{\beta^3}{2\zeta}\chi^2 + \frac{\beta^4}{\rho}\left(\varepsilon^2\psi_{tt}^2 + \varepsilon\psi_t\psi_{tt} + \frac{\psi_t^2}{2}\right) + \beta^4\varepsilon S[\rho,j,\theta]\frac{\psi_{xt}^2}{\rho} + \beta^4\varepsilon\psi_t^2 \\
&+ \beta^4\frac{3\rho}{4\zeta}\chi_x^2 + \beta^4\frac{\kappa_0}{2}\chi_{xx}^2 + \beta^5\frac{\eta^2}{2} - \beta^5\{\rho\sigma_x + \tilde{\phi}_x\psi\}\eta + \beta^5\{\theta\psi + \tilde{\rho}\chi\}_x\eta\,dx,
\end{aligned}$$

$$\begin{aligned}
D(t) :=& \|(\eta,\chi_x)(t)\|^2 + \frac{1}{\zeta}\|\chi(t)\|^2 + \beta\|(\psi,\sigma_x)(t)\|^2 + \beta^2\|(\psi_t,\psi_x,\chi_{xx})(t)\| \\
&+ \frac{\beta^2}{\zeta}\|\chi_x(t)\|^2 + \beta^3\|\chi_t(t)\|^2 + \beta^4\|(\psi_{xt},\chi_{xt})(t)\|^2 + \beta^4\varepsilon\|(\psi_{tt},\chi_{xxt})(t)\|.
\end{aligned}$$

For the suitably small β, we see from (5.73d) that $E(t)$ is equivalent to $n_\varepsilon(t)$:

$$cn_\varepsilon(t) \le E(t) \le Cn_\varepsilon(t), \tag{5.99}$$

where c and C are positive constants. Take β so small that (5.99) and $c_1 - C_1\beta > 0$ hold. Moreover, let $N_\zeta(T) + \delta + \zeta^{1/4} + \varepsilon^{1/2}$ sufficiently small in (5.98) and substitute $cE(t) \le D(t)$, which holds for a suitably chosen small positive constant c. Consequently, we obtain an ordinary differential inequality

$$\frac{d}{dt}E(t) + \alpha E(t) \le 0, \tag{5.100}$$

where α is a positive constant. Solving (5.100) and using (5.99), we have

$$n_\varepsilon(t) \le n_\varepsilon(0)e^{-\alpha t}.$$

The above inequality together with (5.73e) yields the decay estimate (5.56). □

We are now at the position to complete the proof of Theorem 2.1 which asserts the asymptotic stability of the stationary solution for the hydrodynamic model with the large initial data. As it is done by the same discussions as in the proof of Theorem 4.2, we only state a brief sketch here.

Proof of Theorem 2.1. Take the constant Λ in Corollary 5.8 so small that the assumption (5.55) in Theorem 5.9 holds. By applying Theorem 5.9 with regarding T_Λ as initial time and $(\rho, j, \theta)(T_\Lambda)$ as the initial data, we establish the existence of the solution (ρ, j, θ, ϕ) globally in time. The decay estimate (2.21) immediately follows from (5.56). □

5.4 Momentum and energy relaxation limits

We justify the relaxation limits of the time global solution for the hydrodynamic model. We show the estimates (2.22)–(2.24), which complete the proof of Theorem 2.3. Once Theorem 2.3 is proven, Corollary 2.5 follows from Theorem 2.4. Note that the time global solution for the energy-transport model and the hydrodynamic models have been already constructed in Sections 4.3 and 5.3, respectively.

Proof of Theorem 2.3. We firstly show that the estimates (5.28) and (5.40) hold for $t \in [0, \infty)$. It suffices to prove the assumption (5.24) in Lemmas 5.5 and 5.6 holds for $t \in [0, \infty)$. Let Λ be the constant fixed in the proof of Theorem 2.1 and thus T_Λ is corresponding existence time. Corollary 5.8 means that the solution $(\rho^\varepsilon, j^\varepsilon, \theta^\varepsilon, \phi^\varepsilon)$ verifies (5.24) for arbitrary time $t \in [0, T_\Lambda]$. Hence, it is sufficient to show that the time global solution, for initial data $(\rho, j, \theta)(T_\Lambda)$ at initial time T_Λ constructed in Theorem 5.9, verifies (5.24). The estimate (5.24c) follows from the decay estimate (5.56). The solution $(\rho^\varepsilon, j^\varepsilon, \theta^\varepsilon, \phi^\varepsilon)$ satisfies (5.24a) for $t \in [0, T_{\Lambda'}]$. On the other hand, it converges to the stationary solution $(\tilde{\rho}^\varepsilon, \tilde{j}^\varepsilon, \tilde{\theta}^\varepsilon, \tilde{\phi}^\varepsilon)$, which satisfies (5.24a), as t tends to infinity. Hence, taking Λ' small, which is equivalent to taking $T_{\Lambda'}$ large, $(\rho^\varepsilon, j^\varepsilon, \theta^\varepsilon, \phi^\varepsilon)$ verifies (5.24a) for $t \in [T_{\Lambda'}, \infty)$. Moreover, the other estimates in (5.24) follow from the similar discussion as in the proof of Lemma 5.4. Hence, thanks to Lemmas 5.5 and 5.6, the estimates (5.28) and (5.40) hold for $t \in [0, \infty)$.

By applying these estimates, we show the estimates (2.22)–(2.24). Let $\lambda \in (0, 1/2)$ and then define a constant

$$T_1 := \frac{1}{\beta} \log \{\varepsilon + (\varepsilon/\zeta)^2\}^{-\lambda}.$$

The estimates (5.28) and (5.40) mean that

$$\|(R^\varepsilon, Q^\varepsilon)(t)\|_1^2 \le C\{\varepsilon + (\varepsilon/\zeta)^2\} e^{\beta T_1} \le C\{\varepsilon + (\varepsilon/\zeta)^2\}^{1-\lambda}, \tag{5.101a}$$

$$\|J^\varepsilon(t)\|^2 \le \|J^\varepsilon(0)\|^2 e^{-t/\varepsilon} + C\{\varepsilon + (\varepsilon/\zeta)^2\}^{1-\lambda}, \tag{5.101b}$$

$$\|(R^\varepsilon_{xx}, J^\varepsilon_x, Q^\varepsilon_{xx})(t)\|^2 \le C\{\varepsilon + (\varepsilon/\zeta)^2\}^{(1/2)-\lambda} t^{-2} \tag{5.101c}$$

hold for $0 < t \le T_1$. On the other hand, we have the estimate

$$\begin{aligned} \|R^\varepsilon(t)\|_2^2 &\le C\|(\rho^\varepsilon - \tilde{\rho}^\varepsilon)(t)\|_2^2 + C\|(\rho^0 - \tilde{\rho}^0)(t)\|_2^2 + C\|(\tilde{\rho}^\varepsilon - \tilde{\rho}^0)(t)\|_2^2 \\ &\le C\left(e^{-\alpha T_1} + \varepsilon + (\varepsilon/\zeta)^2\right) \le C\left(\{\varepsilon + (\varepsilon/\zeta)^2\}^{\alpha\lambda/\beta} + \varepsilon + (\varepsilon/\zeta)^2\right) \end{aligned} \tag{5.102}$$

for $t \ge T_1$ with aid of the estimates (2.21), (3.37), (3.38) and (4.81b). Similarly, it hold that

$$\|J^\varepsilon(t)\|_1^2 + \|Q^\varepsilon(t)\|_2^2 \le C\left(\{\varepsilon + (\varepsilon/\zeta)^2\}^{\alpha\lambda/\beta} + \varepsilon + (\varepsilon/\zeta)^2\right). \tag{5.103}$$

Letting $\gamma := \min\{(1/2) - \lambda, \alpha\lambda/\beta\}$, we have the inequalities

$$\|(R^\varepsilon, Q^\varepsilon)(t)\|_1^2 \le C\{\varepsilon + (\varepsilon/\zeta)^2\}^\gamma, \tag{5.104a}$$

$$\|J^\varepsilon(t)\|^2 \le \|J^\varepsilon(0)\|^2 e^{-t/\varepsilon} + C\{\varepsilon + (\varepsilon/\zeta)^2\}^\gamma, \tag{5.104b}$$

$$\|(R^\varepsilon_{xx}, J^\varepsilon_x, Q^\varepsilon_{xx})(t)\|^2 \le C(1 + t^{-1})\{\varepsilon + (\varepsilon/\zeta)^2\}^\gamma \tag{5.104c}$$

owing to the estimates (5.101)–(5.103). The inequalities in (5.104) together with the elliptic estimate $\|\Phi^\varepsilon(t)\|_4 \le C\|R^\varepsilon(t)\|_2$ yield the desired estimate (2.22)–(2.24). □

Remark 5.17. *The constants C and γ in* (5.104) *are taken independently of ζ for the special initial data $\theta_0 = 1$. It is shown by the same procedure in this chapter (also see Remark 4.18). This fact shows Remark 2.6.*