CHAPTER 4

The Rigidity Theorem and Infinitesimal Derivatives

In this chapter, we will introduce infinitesimal derivatives of secondary classes after Heitsch [42]. Deformations of foliations (and pseudogroup structures) are discussed by Kodaira [53], Kodaira–Spencer, [54], Heitsch [40], Duchamp–Kalka [26], Girbau–Haefliger–Sundararaman [31], Girbau–Nicolau [32], et al. It will be shown that complex secondary classes determined by the image of $H^*(\mathrm{WU}_{q+1})$ under the natural mapping to $H^*(\mathrm{WU}_q)$ are rigid under actual and infinitesimal deformations. In particular, the Godbillon–Vey class is shown to be rigid in the category of transversely holomorphic foliations. On the other hand, classes in $H^*(\mathrm{WU}_q)$ which admit continuous deformations are called variable classes. The imaginary part of the Bott class is one of the variable classes. Heitsch introduced in [42] the infinitesimal derivatives for *cocycles* in WU_q which represent variable classes of lowest degree. In the same paper, the infinitesimal derivatives for any *classes* in $H^*(\mathrm{WO}_q)$ were also introduced. The most of this section will be devoted to completing Heitsch's construction by defining the infinitesimal derivatives for any *classes* in $H^*(\mathrm{WU}_q)$. The construction seems known for specialists, indeed, the most of the definitions and the proofs are only small modifications of Heitsch's in [42] using notions in [26]. However, we give the details for completeness and for their importance.

Throughout the construction, corresponding steps or statements in [42] are referred so far as possible.

4.1. Definitions and Statements of Results

In what follows, S will denote parameter spaces of deformations. Usually S is assumed to be an analytic space, which is not necessarily reduced, with a distinguished point 0. If smooth deformations are considered, then S is assumed to be an open neighborhood of the origin of a finite dimensional Euclidean space.

DEFINITION 4.1.1. A *family of transversely holomorphic foliations* $\{\mathcal{F}_s\}$ on M parametrized by $s \in S$ is given by the following data.

1) An open covering $\{U_i\}$ of M.
2) A family of submersions $\{\varphi_{i,s} \colon U_i \to \mathbb{C}^q\}$ such that $\mathcal{F}_s$ is locally given by the fibers of $\varphi_{i,s}$.
3) A family $\{\gamma_{ji,s}\}$ of local biholomorphic diffeomorphisms of $\mathbb{C}^q$ such that

$$\varphi_{j,s} = \gamma_{ji,s} \circ \varphi_{i,s}.$$

The family $\{\mathcal{F}_s\}$ is smooth (resp. holomorphic) if $\varphi_{i,s}$ and $\gamma_{ji,s}$ are of class C^∞ (resp. holomorphic) in s.

DEFINITION 4.1.2. If $\mathcal{F}$ is a transversely holomorphic foliation, then an *actual deformation* of $\mathcal{F}$ is a family $\{\mathcal{F}_s\}$ as in Definition 4.1.1 such that $\mathcal{F}_0 = \mathcal{F}$. If the family is smooth (resp. holomorphic), the deformation is said to be *smooth* (resp. *holomorphic*).

An actual deformation induces an infinitesimal deformation. See Section 4.3.

REMARK 4.1.3. If $\{\mathcal{F}_s\}$ is a smooth actual deformation of $\mathcal{F}$, then $\{\mathcal{F}_s\}$ is a smooth family of foliations so that we may assume that $Q(\mathcal{F}_s)$ are isomorphic on a neighborhood of $0 \in S$.

Certain type of deformations will be of interest.

DEFINITION 4.1.4. Let $\{\mathcal{F}_s\}$ be an actual deformation of a transversely holomorphic foliation.

1) If there exists a smooth family of diffeomorphisms which conjugate each $\mathcal{F}_s$ to $\mathcal{F}_0$, then $\{\mathcal{F}_s\}$ is said to be a *deformation preserving the diffeomorphism type*.
2) If $\mathcal{F}_s$ are identical to $\mathcal{F}$ as real foliations, then the family $\{\mathcal{F}_s\}$ is said to be a *deformation of transverse holomorphic structures*.

There is a natural mapping from $H^*(\mathrm{WU}_{q+1})$ to $H^*(\mathrm{WU}_q)$ induced by the standard inclusion of $\mathbb{C}^q$ into $\mathbb{C}^{q+1}$.

DEFINITION 4.1.5 (cf. [39]). Let ρ be the DGA-homomorphism from WU_{q+1} to WU_q defined by the following formulae:

$$\rho(\widetilde{u}_i) = \begin{cases} \widetilde{u}_i & \text{if } i \neq q+1 \\ 0 & \text{if } i = q+1 \end{cases},$$

$$\rho(v_i) = \begin{cases} v_i & \text{if } i \neq q+1 \\ 0 & \text{if } i = q+1 \end{cases}, \quad \rho(\overline{v}_i) = \begin{cases} \overline{v}_i & \text{if } i \neq q+1 \\ 0 & \text{if } i = q+1 \end{cases}.$$

We denote by ρ_* the induced homomorphism from $H^*(\mathrm{WU}_{q+1})$ to $H^*(\mathrm{WU}_q)$.

The following theorem has been well-known for specialists.

THEOREM B1. *The secondary classes defined by $H^*(\mathrm{WU}_q)$ are rigid under smooth deformations if they belong to the image of ρ_*. More precisely, if $\{\mathcal{F}_s\}$ is a smooth family of transversely holomorphic foliations of complex codimension q and if ω is an element of $\rho_*(H^*(\mathrm{WU}_{q+1}))$, then $\omega(\mathcal{F}_s) \in H^*(M)$ is independent of s.*

Infinitesimal deformations of a transversely holomorphic foliation $\mathcal{F}$ are elements of $H^1(M;\Theta_{\mathcal{F}})$ (see Definition 4.3.5 for details). Infinitesimal derivatives of elements of $H^*(\mathrm{WU}_q)$ are given by the mapping

$$D.(\,\cdot\,)\colon H^1(M;\Theta_{\mathcal{F}}) \times H^*(\mathrm{WU}_q) \to H^*(M;\mathbb{C})$$

in Definition 4.3.13. It will be shown that a smooth family $\{\mathcal{F}_s\}$ as above naturally determines an infinitesimal derivative $\beta \in H^1(M;\Theta_{\mathcal{F}})$ such that $D_\beta(\omega) = \left.\dfrac{\partial}{\partial s}\omega(\mathcal{F}_s)\right|_{s=0}$ for $\omega \in H^*(\mathrm{WU}_q)$ (Theorem 4.3.28). The infinitesimal version of Theorem B1 is as follows.

THEOREM B2. *The image of* $H^1(M;\Theta_{\mathcal{F}}) \times (\rho_* H^*(\mathrm{WU}_{q+1}))$ *under the above mapping* $D.(\,\cdot\,)$ *is trivial.*

Theorems B1 and B2 are shown in Section 4.3. The most important consequence of these theorems is the following

THEOREM B. *The Godbillon–Vey class is rigid under both smooth and infinitesimal deformations in the category of transversely holomorphic foliations.*

PROOF OF THEOREM B. Let q be the codimension of the foliations. Then

$$\mathrm{GV}_{2q} = \frac{(2q)!}{q!\,q!}\sqrt{-1}\,\widetilde{u}_1 v_1^q \overline{v}_1^q$$

in $H^{4q+1}(\mathrm{WU}_q)$ by Theorem 2.1. On the other hand,

$$\xi_{q+1}\cdot \mathrm{ch}_1^{q-1} = \sqrt{-1}\,\widetilde{u}_1(v_1^{q+1}\overline{v}_1^{q-1} + v_1^q\overline{v}_1^q + v_1^{q-1}\overline{v}_1^{q+1})$$

in $H^{4q+1}(\mathrm{WU}_{q+1})$, where ξ_{q+1} is defined in Definition 1.2.1. Therefore

$$\mathrm{GV}_{2q} = \rho_*\left(\frac{(2q)!}{q!\,q!}\xi_{q+1}\cdot \mathrm{ch}_1^{q-1}\right)$$

in $H^{4q+1}(\mathrm{WU}_q)$. □

The following corollary is a consequence of Theorem B and Theorem 2.1.

COROLLARY 4.1.6. *If* $\{\mathcal{F}_s\}$ *is a smooth family of transversely holomorphic foliations of codimension* q, *then the product of* $\mathrm{ch}_1(\mathcal{F}_0)^q$ *and* $\frac{d}{ds}\xi(\mathcal{F}_s)$ *is identically equal to zero. Similarly, if* β *is an infinitesimal deformation of* $\mathcal{F}$, *then* $D_\beta\xi_q(\mathcal{F})\,\mathrm{ch}_1(\mathcal{F})^q = 0$ *holds, where* $D_\beta\xi_q(\mathcal{F})$ *denotes the infinitesimal derivative of* ξ_q *with respect to* β.

PROOF. Since $\mathrm{ch}_1(\mathcal{F}_s)^q$ is independent of s, we have

$$\left(\frac{d}{ds}\xi(\mathcal{F}_s)\right)\mathrm{ch}_1(\mathcal{F}_0)^q = \frac{q!\,q!}{(2q)!}\frac{d}{ds}\,\mathrm{GV}_{2q}(\mathcal{F}_s) = 0.$$

The second claim holds for the same reason. □

There are alternative proofs of Theorem B and Corollary 4.1.6. See Corollary 4.3.30 and Theorem 5.14.

Let $\{\mathcal{F}_s\}$ be a smooth family of transversely holomorphic foliations of codimension q, and assume that $\mathrm{GV}_{2q}(\mathcal{F}_s)$ is non-trivial. Then $\mathrm{ch}_1(\mathcal{F}_0)^q$ is non-trivial by Theorem 2.1. If the mapping $\cup\, \mathrm{ch}_1(\mathcal{F}_0)^q \colon H^*(M;\mathbb{C}) \to H^{*+2q}(M;\mathbb{C})$ is injective, then $\dfrac{d}{ds}\xi_q(\mathcal{F}_s)$ is trivial because $\dfrac{d}{ds}\mathrm{GV}_{2q}(\mathcal{F}_s) = \dfrac{(2q)!}{q!\,q!}\dfrac{d}{ds}\xi_q(\mathcal{F}_s)\,\mathrm{ch}_1(\mathcal{F}_0)^q = 0$ by Theorem B1. This implies that the class ξ_q is in fact rigid in such a case. In fact, so far as we know, if $\{\mathcal{F}_s\}$ is a continuous family such that $\xi_q(\mathcal{F}_s)$ vary continuously, then $\mathrm{ch}_1(\mathcal{F}_s)$ are always trivial. In this line, we have the following

QUESTION 4.1.7. *Is there a smooth family of transversely holomorphic foliations for which the imaginary part of the Bott class varies continuously and the first Chern class of the complex normal bundle is non-trivial? How about infinitesimal deformations?*

4.2. Rigidity under Smooth Deformations

The calculations in this section are used to prove Theorem B1 and also to construct infinitesimal derivative in the next section. We begin with some definitions.

DEFINITION 4.2.1 ([39]). Let $\{\mathcal{F}_s\}$ be a smooth deformation of transversely holomorphic foliations. We define differential forms Δ_f and V as follows. As the complex normal bundles of the foliations remain isomorphic, denote them by Q and consider the same unitary connection θ_0 for some Hermitian metric on Q. Let $\{\theta_1^s\}$ be a smooth family of complex Bott connections on Q, namely, assume that each θ_1^s is a Bott connection for $\mathcal{F}_s$ and $\{\theta_1^s\}$ is smooth as a family of connections. Let ψ_s be the derivative of θ_1^s with respect to s, namely, $\psi_s = \dfrac{\partial}{\partial s}\theta_1^s$. Let f be a homogeneous polynomial of degree $2k$ in v_i and $\overline{v}_j$. We set $\theta_t^s = t\theta_1^s + (1-t)\theta_0$ and denote by Ω_t^s its curvature, and set

$$\Delta_f(\theta_1^s,\theta_0) = k\int_0^1 f(\theta_1^s-\theta_0,\Omega_t^s,\dots,\Omega_t^s)\,dt,$$

$$V_f(\theta_1^s,\theta_0) = \int_0^1 tf(\psi_s,\theta_1^s-\theta_0,\Omega_t^s,\dots,\Omega_t^s)\,dt.$$

The following formulae are shown in [39, Theorem 1]:

$$\frac{\partial}{\partial s}(\Delta_f(\theta_1^s,\theta_0)) = k(k-1)\,dV_f(\theta_1^s,\theta_0) + kf(\psi_s,\Omega_1^s,\dots,\Omega_1^s), \tag{4.2.2a}$$

$$\frac{\partial}{\partial s}\,d(\Delta_f(\theta_1^s,\theta_0)) = \frac{\partial}{\partial s}f(\Omega_1^s,\dots,\Omega_1^s) = k\,df(\psi_s,\Omega_1^s,\dots,\Omega_1^s), \tag{4.2.2b}$$

where Ω_1^s denotes the curvature form of the connection θ_1^s and the the exterior derivative is considered only on M, namely, along the fibers of $M\times\mathbb{R}\to\mathbb{R}$.

The following auxiliary definition will be convenient.

Definition 4.2.3. Set $\widetilde{\mathrm{WU}}_q = \bigwedge[\widetilde{u}_1,\dots,\widetilde{u}_q]\otimes\mathbb{C}[v_1,\dots,v_q]\otimes\mathbb{C}[\overline{v}_1,\dots,\overline{v}_q]$ and equip $\widetilde{\mathrm{WU}}_q$ with a differential $\widetilde{d}$ by requiring $\widetilde{d}\widetilde{u}_i = v_i-\overline{v}_i$ and $\widetilde{d}v_i=\widetilde{d}\overline{v}_i=0$. Let $\widetilde{\mathcal{I}}_q$ be the ideal of $\widetilde{\mathrm{WU}}_q$ generated by cochains of the form $\widetilde{u}_Iv_J\overline{v}_K$ with $|J|>q$ or $|K|>q$. Note that $\mathrm{WU}_q = \widetilde{\mathrm{WU}}_q/\widetilde{\mathcal{I}}_q$. If φ is a cochain in WU_q, then its lift $\widetilde{\varphi}$ in $\widetilde{\mathrm{WU}}_q$ is said to be a natural lift if $\widetilde{\varphi}$ is a linear combination of cochains of the form $\widetilde{u}_Iv_J\overline{v}_K$ with $|J|\le q$ and $|K|\le q$.

It is easy to verify the relation $\widetilde{d}\circ\widetilde{d}=0$. The DGA $(\widetilde{\mathrm{WU}}_q,\widetilde{d})$ is obtained from WU_q by forgetting the Bott vanishing. Note that $\widetilde{d}\widetilde{d}\widetilde{\varphi}$ is exactly equal to 0 for any $\widetilde{\varphi}\in\widetilde{\mathrm{WU}}_q$. This simple property is frequently used in what follows.

The following differential form is significant.

Definition 4.2.4. Let θ^u be a unitary connection and θ a Bott connection on $Q(\mathcal{F})$, respectively. Let θ' be a derivative of a family of Bott connections or an infinitesimal derivative of a Bott connection which will be introduced in Definition 4.3.9, or a certain matrix valued function which will appear in proving Theorem 4.3.18. For $\widetilde{\varphi}\in\widetilde{\mathrm{WU}}_q$, we define a differential form $\Delta\widetilde{\varphi}(\theta^u,\theta,\theta')$ as follows. First, if $\widetilde{\varphi}=\widetilde{u}_Iv_J\overline{v}_K$, then we set

$$\delta(\widetilde{\varphi})(\theta^u,\theta,\theta') = (|J|+|K|)v_J\overline{v}_K(\theta',\Omega)\widetilde{u}_I(\theta,\theta^u),$$

where Ω is the curvature form of θ. We set

$$\Delta\widetilde{\varphi}(\theta^u,\theta,\theta') = \delta(\widetilde{d}\widetilde{\varphi})(\theta^u,\theta,\theta').$$

We extend δ and Δ to the whole $\widetilde{\mathrm{WU}}_q$ by linearity.

If $\widetilde{\varphi} = \widetilde{u}_I v_J \overline{v}_K \in \widetilde{\mathrm{WU}}_q$ and $I = \{i_1, \dots, i_t\}$ with $i_1 < i_2 < \cdots < i_t$, then

$$\begin{aligned}&\Delta\widetilde{\varphi}(\theta^u, \theta, \theta')\\&= \sum_l (-1)^{l-1}(|J| + |K| + i_l)\left(v_J \overline{v}_K (v_{i_l} - \overline{v}_{i_l})\right)(\theta', \Omega)\widetilde{u}_{I(l)}(\theta, \theta^u),\end{aligned}$$

where $I(l) = I \setminus \{i_l\}$.

The following is easy (see Lemma A.6 for the first formula).

Lemma 4.2.5. *We have the following formulae*:

$$(|J| + |K|)(v_J v_K)(\theta', \Omega) = |J|\, v_J(\theta', \Omega) v_K(\Omega) + |K|\, v_J(\Omega) v_K(\theta', \Omega), \tag{4.2.5a}$$

$$\begin{cases} v_J(\theta', \Omega) = 0 & \text{as differential forms if } |J| > q+1, \\ \overline{v}_K(\theta', \Omega) = 0 & \text{as differential forms if } |K| > q+1. \end{cases} \tag{4.2.5b}$$

Proposition 4.2.6. *If $\varphi \in \mathrm{WU}_q$ is a cocycle, then $\dfrac{\partial}{\partial s}\chi_s(\varphi)$ is represented by $\Delta\widetilde{\varphi}(\theta_0, \theta_1^s, \psi_s)$, where $\widetilde{\varphi}$ is any lift of φ to $\widetilde{\mathrm{WU}}_q$.*

Proof. In the proof, we will make use of the following notations, namely, $\widetilde{u}_i(\theta_1^s, \theta_0)$, $v_j(\Omega_1^s)$ and $\overline{v}_k(\Omega_1^s)$ are simply denoted by $\widetilde{u}_i(s)$, $v_j(s)$ and $\overline{v}_k(s)$, respectively. The differential form $v_j(\psi_s, \Omega_1^s)$ is denoted by $w_j(s)$, and $\overline{v}_k(\psi_s, \Omega_1^s)$ is denoted by $\overline{w}_k(s)$. We denote $V_{v_i}(\theta_1^s, \theta_0)$ and $\overline{V}_{v_i}(\theta_1^s, \theta_0)$ simply by V_i and $\overline{V}_i$, respectively. Finally, we set $\widetilde{V}_i = V_i - \overline{V}_i$ and $\widetilde{w}_i(s) = w_i(s) - \overline{w}_i(s)$. Under these notations, $\dfrac{\partial}{\partial s}\widetilde{u}_i(s) = i(i-1)\, d\widetilde{V}_i + i(w_i(s) - \overline{w}_i(s)) = i(i-1)\, d\widetilde{V}_i + i\widetilde{w}_i$.

Let φ be a cocycle in WU_q. We will compute $\dfrac{\partial}{\partial s}\chi_s(\varphi)$. For each i, where $1 \le i \le q$, there are elements α_i and β_i of WU_q which do not involve $\widetilde{u}_i$ and such that $\varphi = \widetilde{u}_i\alpha_i + \beta_i$. Note that α_i is closed because φ is closed. Let $\dfrac{\partial}{\partial s_i}$ be the differential operator obtained by applying $\dfrac{\partial}{\partial s}$ only to $\widetilde{u}_i(\theta_1^s, \theta_0)$, $v_i(\theta_1^s)$ and $\overline{v}_i(\theta_1^s)$. Then $\dfrac{\partial}{\partial s}$ is decomposed as $\dfrac{\partial}{\partial s} = \dfrac{\partial}{\partial s_1} + \cdots + \dfrac{\partial}{\partial s_q}$. In order to compute $\dfrac{\partial}{\partial s_i}\chi_s(\varphi)$, we write $\alpha_i = \sum_{j,k} v_i^j \overline{v}_i^k a_{j,k}^i$ and $\beta_i = \sum_{j,k} v_i^j \overline{v}_i^k b_{j,k}^i$ so that neither $a_{j,k}^i$ nor $b_{j,k}^i$ involves v_i and $\overline{v}_i$. Then,

$$\begin{aligned}
\frac{\partial}{\partial s_i}\chi_s(\varphi) &= \frac{\partial}{\partial s_i}\chi_s(\widetilde{u}_i\alpha_i + \beta_i) \\
&= \sum_{j,k} \big(i(i-1)\, d\widetilde{V}_i + i\widetilde{w}_i(s)\big) v_i^j(s)\overline{v}_i^k(s) a_{j,k}^i(s) \\
&\quad + \sum_{j,k} ij\widetilde{u}_i(s) v_i^{j-1}(s)\, dw_i(s)\overline{v}_i^k(s) a_{j,k}^i(s) \\
&\quad + \sum_{j,k} ik\widetilde{u}_i(s) v_i^j(s)\overline{v}_i^{k-1}(s)\, d\overline{w}_i(s) a_{j,k}^i(s) \\
&\quad + \sum_{j,k} ijv_i^{j-1}(s)\, dw_i(s)\overline{v}_i^k(s) b_{j,k}^i(s) \\
&\quad + \sum_{j,k} ikv_i^j(s)\overline{v}_i^{k-1}(s)\, d\overline{w}_i(s) b_{j,k}^i(s).
\end{aligned}$$

The first term is equal to

$$i(i-1)\, d\widetilde{V}_i\alpha_i(s) + \sum_{j,k} i\widetilde{w}_i(s) v_i^j(s)\overline{v}_i^k(s) a_{j,k}^i(s).$$

Note that $d\widetilde{V}_i\alpha_i(s) = d(\widetilde{V}_i\alpha_i(s))$ because α_i is closed. The second term is cohomologous to

$$\begin{aligned}
&\sum_{j,k} ij(v_i(s) - \overline{v}_i(s)) v_i^{j-1}(s) w_i(s)\overline{v}_i^k(s) a_{j,k}^i(s) \\
&\quad + \sum_{j,k} ij\widetilde{u}_i(s) v_i^{j-1}(s) w_i(s)\overline{v}_i^k(s)\, da_{j,k}^i(s),
\end{aligned}$$

which is equal to

$$\begin{aligned}
&\sum_{j,k} ijv_i^j(s)\overline{v}_i^k(s) w_i(s) a_{j,k}^i(s) - \sum_{j,k} ijv_i^{j-1}(s)\overline{v}_i^k w_i(s) a_{j,k-1}^i(s) \\
&\quad - \sum_{j,k} ijv_i^{j-1}(s) w_i(s)\overline{v}_i^k(s)\widetilde{u}_i(s)\, da_{j,k}^i(s),
\end{aligned}$$

where $a_{j,-1}^i$ is understood to be zero. Similarly, the third term is cohomologous to

$$\begin{aligned}
&- \sum_{j,k} ikv_i^j(s)\overline{v}_i^k(s)\overline{w}_i(s) a_{j,k}^i(s) + \sum_{j,k} ikv_i^j(s)\overline{v}_i^{k-1}\overline{w}_i(s) a_{j-1,k}^i(s) \\
&\quad - \sum_{j,k} ikv_i^j(s)\overline{v}_i^{k-1}(s) w_i(s)\widetilde{u}_i(s)\, da_{j,k}^i(s),
\end{aligned}$$

where $a^i_{-1,k} = 0$. The fourth and fifth terms are respectively cohomologous to

$$\sum_{j,k} ijv_i^{j-1}(s)\overline{v}_i^k(s)w_i(s)\,db^i_{j,k}(s),$$

and

$$\sum_{j,k} ikv_i^j(s)\overline{v}_i^{k-1}(s)\overline{w}_i(s)\,db^i_{j,k}(s).$$

Hence we have the following equalities modulo exact terms, namely,

$$\begin{aligned}
&\frac{\partial}{\partial s_i}\chi_s(\varphi) \\
&= \sum_{j,k} i(j+1)v_i^j(s)\overline{v}_i^k(s)w_i(s)a^i_{j,k}(s) - \sum_{j,k} i(k+1)v_i^j(s)\overline{v}_i^k(s)\overline{w}_i(s)a^i_{j,k}(s) \\
&\quad - \sum_{j,k} ijv_i^{j-1}(s)\overline{v}_i^k w_i(s)a^i_{j,k-1}(s) + \sum_{j,k} ikv_i^j(s)\overline{v}_i^{k-1}\overline{w}_i(s)a^i_{j-1,k}(s) \\
&\quad - \sum_{j,k} ijv_i^{j-1}(s)w_i(s)\overline{v}_i^k(s)\widetilde{u}_i(s)\,da^i_{j,k}(s) - \sum_{j,k} ikv_i^j(s)\overline{v}_i^{k-1}(s)w_i(s)\widetilde{u}_i(s)\,da^i_{j,k}(s) \\
&\quad + \sum_{j,k} ijv_i^{j-1}(s)\overline{v}_i^k(s)w_i(s)\,db^i_{j,k}(s) + \sum_{j,k} ikv_i^j(s)\overline{v}_i^{k-1}(s)\overline{w}_i(s)\,db^i_{j,k}(s) \\
&= \sum_{j,k} ijv_i^{j-1}(s)\overline{v}_i^k w_i(s)\left(a^i_{j-1,k}(s) - a^i_{j,k-1}(s) + db^i_{j,k}(s) - \widetilde{u}_i(s)\,da^i_{j,k}(s)\right) \\
&\quad + \sum_{j,k} ikv_i^j(s)\overline{v}_i^{k-1}\overline{w}_i(s)\left(-a^i_{j,k-1}(s) + a^i_{j-1,k}(s) + db^i_{j,k}(s) - \widetilde{u}_i(s)\,da^i_{j,k}(s)\right).
\end{aligned}$$

On the other hand, if $\widetilde{\varphi}$ is the natural lift of φ, then one has

$$\begin{aligned}
\widetilde{d}\widetilde{\varphi} &= ((v_i - \overline{v}_i)\alpha_i - \widetilde{u}_i d\alpha_i + d\beta_i) \\
&= \sum_{j,k}(v_i - \overline{v}_i)v_i^j\overline{v}_i^k a^i_{j,k} + \sum_{j,k} v_i^j\overline{v}_i^k\,db^i_{j,k} - \sum_{j,k}\widetilde{u}_i v_i^j\overline{v}_i^k\,da^i_{j,k} \\
&= \sum_{j,k} v_i^j\overline{v}_i^k(a^i_{j-1,k} - a^i_{j,k-1} + db^i_{j,k} - \widetilde{u}_i\,da^i_{j,k}).
\end{aligned}$$

From (4.2.5a), we see that Proposition 4.2.6 holds if $\widetilde{\varphi}$ a natural lift. In order to show the proposition for general choices of $\widetilde{\varphi}$, it suffices to show that $\Delta(\widetilde{d}\widetilde{\alpha} + \widetilde{\beta})(\theta_0, \theta_1^s, \psi_s)$ is exact if $\widetilde{\alpha} \in \widetilde{\mathrm{WU}}_q$ and if $\widetilde{\beta} \in \widetilde{\mathcal{I}}_q$. One has $\Delta(\widetilde{d}\widetilde{\alpha})(\theta_0, \theta_1^s, \psi_s) = \delta(\widetilde{d}(\widetilde{d}\widetilde{\alpha}))(\theta_0, \theta_1^s, \psi_s) = 0$. Let $\widetilde{\beta} = \widetilde{u}_I v_J \overline{v}_K$, where $|J| > q$. If $I = \varnothing$, then

$\Delta(v_J\overline{v}_K)(\theta_0,\theta_1^s,\psi_s)=0$ because $\widetilde{d}(v_J\overline{v}_K)=0$. If $I\neq\varnothing$, then the following equality holds, namely,

$$\begin{aligned}
&\Delta(\widetilde{u}_I v_J\overline{v}_K)(\theta_0,\theta_1^s,\psi_s)\\
&=\sum_l(-1)^{l-1}(|J|+|K|+i_l)v_J\overline{v}_K(v_{i_l}-\overline{v}_{i_l})(\psi_s,\Omega^s)\widetilde{u}_{I(l)}(\theta_1^s,\theta_0)\\
&=-\sum_l(-1)^{l-1}\,|J|\,v_J(\psi_s,\Omega^s)\overline{v}_K\overline{v}_{i_l}(\Omega^s)\widetilde{u}_{I(l)}(\theta_1^s,\theta_0)\\
&=d\big(\,|J|\,v_J(\psi_s,\Omega^s)\overline{v}_K(\Omega^s)\widetilde{u}_I(\theta_1^s,\theta_0)\big),
\end{aligned}$$

where the second equality holds because $v_J(\Omega^s)=0$ and $v_J(\psi_s,\Omega^s)v_{i_l}(\Omega^s)=0$ by the Bott vanishing. The last equality follows from (4.2.2b) and $dv_J(\psi_s,\Omega^s)=0$. Finally, $\dfrac{\partial}{\partial s}\chi_s(\varphi)$ is closed as $\chi_s(\varphi)$ is closed independent of s. □

Proof of Theorem B1. Let φ be a cocycle in WU_{q+1} and let $\widetilde{\varphi}$ be any lift of φ to $\widetilde{\mathrm{WU}}_{q+1}$. Then $\widetilde{d}\widetilde{\varphi}$ is a linear combination of the monomials of the form $\widetilde{u}_I v_J\overline{v}_K$ with $|J|>q+1$ or $|K|>q+1$. Hence $\Delta(\rho\widetilde{\varphi})(\theta_0,\theta_1^s,\varphi_s)$ identically vanishes by (4.2.5b). □

Compared with the real case, the space $H^*(\mathrm{WU}_q)$ and the cokernel of ρ_* are rather complicated. For example, we have the following.

Proposition 4.2.7 (cf. [5, Theorem 1.8]). *In the lower codimensional cases, the cokernel of ρ_* is described as follows*:

$q=1$: *A basis for* $\operatorname{coker}\rho_*$ *is* $\{\widetilde{u}_1(v_1+\overline{v}_1)\}$.

$q=2$: *A basis for* $\operatorname{coker}\rho_*$ *consists of* $v_1+\overline{v}_1$, $v_1^2+v_2+2v_1\overline{v}_1+\overline{v}_1^2+\overline{v}_2$ *and the classes in* $H^*(\mathrm{WU}_2)$ *of degree* 5, 10 *or* 12, *namely, the classes in Table 4.2.1, where the numbers in the left column stand for the degree of the classes in the same row.*

Example 1.1.6 of Bott shows that the secondary classes of the lowest degree can vary. We do not know if the classes of higher degree can vary.

5	$\widetilde{u}_1(v_1^2+v_1\overline{v}_1+\overline{v}_1^2),\ \widetilde{u}_1(v_2+\overline{v}_2)+\widetilde{u}_2(v_1+\overline{v}_1)$
10	$\widetilde{u}_1\widetilde{u}_2v_1\overline{v}_1(v_1+\overline{v}_1)$
12	$\widetilde{u}_1\widetilde{u}_2v_1^2\overline{v}_1^2,\ \widetilde{u}_1\widetilde{u}_2v_1^2\overline{v}_2,\ \widetilde{u}_1\widetilde{u}_2v_2\overline{v}_1^2,\ \widetilde{u}_1\widetilde{u}_2v_2\overline{v}_2$

TABLE 4.2.1. A part of basis for coker ρ_*, where $q = 2$.

4.3. Infinitesimal Deformations, Infinitesimal Derivatives and Rigidity under Infinitesimal Deformations

We first recall that $T_{\mathbb{C}}M = TM \otimes \mathbb{C}$ and E is the complex vector bundle locally spanned by $T\mathcal{F}$ and the transverse antiholomorphic vectors $\dfrac{\partial}{\partial \bar{z}_i}$. Then, $Q(\mathcal{F}) = T_{\mathbb{C}}M/E$ (Definition 1.1.4). The space of C^∞ sections of $\bigwedge^* E^* \otimes Q(\mathcal{F})$ is denoted by $\Gamma^\infty(\bigwedge^* E^* \otimes Q(\mathcal{F}))$.

DEFINITION 4.3.1 ([42, 1.4], [26]). Let ∇ be a Bott connection on $Q(\mathcal{F})$. We define a derivation $d_\nabla : \Gamma^\infty(\bigwedge^p E^* \otimes Q(\mathcal{F})) \to \Gamma^\infty(\bigwedge^{p+1} E^* \otimes Q(\mathcal{F}))$ by

$$\begin{aligned}
&d_\nabla \sigma(X_0, \ldots, X_p)\\
&= \sum_{0\le i\le p} (-1)^i \nabla_{X_i} \sigma(X_0, \ldots, \widehat{X_i}, \ldots, X_p)\\
&\quad + \sum_{0\le i<j\le p} (-1)^{i+j} \sigma([X_i, X_j], X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_p),
\end{aligned}$$

where $\sigma \in \Gamma^\infty(\bigwedge^p E^* \otimes Q(\mathcal{F}))$, $X_i \in \Gamma^\infty(E)$ and the symbol '$\widehat{\ }$' means omission.

A section σ of $Q(\mathcal{F})$ is said to be *foliated and transversely holomorphic* if $\mathcal{L}_X\sigma = 0$ for $X \in E$, where $\mathcal{L}_X$ denotes the Lie derivative with respect to X. In other words, σ is foliated and transversely holomorphic if σ is locally constant along the leaves and transversely holomorphic.

DEFINITION 4.3.2. Let $\Theta_{\mathcal{F}}$ be the sheaf of germs of foliated transversely holomorphic vector fields.

The following fact, which is relevant in studying infinitesimal deformations, can be found in the proof of Theorem 1.27 of [26].

LEMMA 4.3.3. *Let* $\Theta_{\mathcal{F}}$ *be the sheaf of germs of foliated transversely holomorphic vector fields. Then* $d_\triangledown \circ d_\triangledown = 0$, *and*

$$0 \longrightarrow \Theta_{\mathcal{F}} \longrightarrow \varGamma^\infty(\textstyle\bigwedge^0 E^* \otimes Q(\mathcal{F})) \xrightarrow{d_\triangledown} \varGamma^\infty(\textstyle\bigwedge^1 E^* \otimes Q(\mathcal{F})) \xrightarrow{d_\triangledown} \cdots$$

is a resolution of $\Theta_{\mathcal{F}}$.

We have the following.

THEOREM 4.3.4 ([26, Theorem 1.27]). $H^*(M;\Theta_{\mathcal{F}})$ *is isomorphic to the cohomology of* $(\varGamma^\infty(\bigwedge^* E^* \otimes Q(\mathcal{F})), d_\triangledown)$. *Moreover,* $H^*(M;\Theta_{\mathcal{F}})$ *is of finite dimension.*

DEFINITION 4.3.5. An *infinitesimal deformation* of $\mathcal{F}$ is by definition an element of $H^1(M;\Theta_{\mathcal{F}})$.

See also Definitions 4.3.21 and 4.3.26.

In what follows, we follow the conventions in [42] but we will work on $Q(\mathcal{F})$ instead of $Q(\mathcal{F})^*$.

Let P be the principal bundle associated with $Q(\mathcal{F})$, and let $\pi\colon P \to M$ the projection. If $\alpha \in P$, then α is a linear isomorphism from $\mathbb{C}^q$ to $Q(\mathcal{F})_{\pi(\alpha)}$.

DEFINITION 4.3.6. If $X \in T_\alpha P$, then we set $\omega(X) = \alpha^{-1}(\pi_* X)$. The differential form ω is called the *canonical form*. The i-th component of $\omega(X)$ is denoted by $\omega^i(X)$, and each ω^i is regarded as a 1-form.

Let θ be the connection form of a connection on P induced from any Bott connection on $Q(\mathcal{F})$. Then $d\omega = -\theta \wedge \omega$, where the sign is opposite when compared with [42]. This is because we work on $Q(\mathcal{F})$. Let $\Omega = d\theta + \theta \wedge \theta$ be the curvature form. Then $\Omega \wedge \omega = 0$.

Let σ be a section of $\bigwedge^* E^* \otimes Q(\mathcal{F})$. We can regard σ as a section of $\bigwedge^* T^*_{\mathbb{C}} M \otimes Q(\mathcal{F})$ by arbitrarily extend it. Then, by pulling back to P and considering the horizontal lifts, we can regard σ as a section, say $\widetilde{\sigma}$, of $\bigwedge^* P^* \otimes Q(\widehat{\mathcal{F}})$, where $\widehat{\mathcal{F}} = \pi^*\mathcal{F}$ is the lift of $\mathcal{F}$ to P. Finally, we can regard $\widetilde{\sigma}$ as a $\mathbb{C}^q$-valued differential form on

P by composing it with the canonical form ω. A section of $\bigwedge^* P^* \otimes Q(\widehat{\mathcal{F}})$ is *always* considered as a $\mathbb{C}^q$-valued differential form in this way, and represented in columns. Conversely, a section $\widetilde{\sigma}$ projects down to a section σ of $\bigwedge^* T^*_{\mathbb{C}} M \otimes Q(\mathcal{F})$ if and only if

1) $\widetilde{\sigma}$ is horizontal, that is, $\widetilde{\sigma}(X_1, \ldots, X_k) = 0$ if $\pi_*(X_i) = 0$ for some i,
2) $R_g^* \widetilde{\sigma} = g^{-1} \widetilde{\sigma}$ for $g \in \mathrm{GL}(q; \mathbb{C})$, where R_g is the right action of $\mathrm{GL}(q; \mathbb{C})$ on P.

In what follows, $\widetilde{\sigma}$ is also denoted by σ by abuse of notation.

Let $\beta \in H^1(M; \Theta_{\mathcal{F}})$ and let σ be a representative of β. Such a σ is a section of $E^* \otimes Q(\mathcal{F})$ such that $d_\nabla \sigma' = 0$. We denote again by σ the $\mathbb{C}^q$-valued 1-form on P obtained in the above manner. Then, σ satisfies the above conditions 1) and 2), and

3) $d\sigma + \theta \wedge \sigma = 0$ when restricted to $\pi^* E$.

Let $\mathcal{I}(\omega)$ be the ideal generated by $\omega^1, \ldots, \omega^q$ in the space of differential forms on P. The condition 3) is equivalent to $d\sigma + \theta \wedge \sigma \in \mathcal{I}(\omega)$.

Definition 4.3.7 ([42, Definition 3.8]). Let β be an element of $H^1(M; \Theta_{\mathcal{F}})$ and let σ be a representative of β as a $\mathbb{C}^q$-valued 1-form on P. The *infinitesimal derivative of the canonical form ω with respect to σ*, denoted by ω', is defined by

$$\omega' = -\sigma.$$

It follows from the condition 3) above that $d\omega' + \theta \wedge \omega' \in \mathcal{I}(\omega)$. Let θ' be a $\mathfrak{gl}_q\mathbb{C}$-valued 1-form on P such that

$$d\omega' + \theta \wedge \omega' = -\theta' \wedge \omega. \tag{4.3.8}$$

The infinitesimal derivative of a Bott connection is defined as follows.

Definition 4.3.9 ([42, Definition 3.10]). Any $\mathfrak{gl}_q\mathbb{C}$-valued 1-form θ' on P satisfying (4.3.8) is called an *infinitesimal derivative of θ with respect to σ*.

If θ'_0 and θ'_1 are two infinitesimal derivatives of θ with respect to σ, then $(\theta'_1 - \theta'_0) \wedge \omega = 0$. Hence

$$(\theta'_1 - \theta'_0)^i_j = \sum_k \lambda^i_{j,k} \omega^k \tag{4.3.10}$$

for some $\mathbb{C}$-valued functions $\lambda^i_{j,k}$ on P satisfying $\lambda^i_{j,k} = \lambda^i_{k,j}$.

LEMMA 4.3.11 ([42, Lemma 2.12]). *If θ' is an infinitesimal derivative of θ, then*

1) *θ' is horizontal,*
2) *θ' is tensorial of type* ad *modulo ω, namely, $R_g^*\theta' - g^{-1}\theta' g \in \mathcal{I}(\omega)$.*

PROOF. Let $u \in P$. If $X \in T_uP$ satisfies $\pi_*X = 0$, then $\omega(X) = 0$, where ω is the canonical form. As ω' is horizontal, one has also $\omega'(X) = 0$. We extend X to an equivariant vector field on P and denote the extension again by X. Note that X is vertical because $\pi_*X_u = 0$. Let Y_j, $j = 1, \dots, q$, be vector fields on P which are equivariant under the right action of $\mathrm{GL}(q;\mathbb{C})$, and such that $\omega^k((Y_k)_u) = 1$ and $\omega^k((Y_j)_u) = 0$ if $j \neq k$. We set $\alpha = \omega' - A\omega$, where A is a matrix valued function defined by setting $A = (\omega'(Y_1) \ \cdots \ \omega'(Y_q))$. Then α is horizontal. Note that $X_uA = 0$ because the both Y_j and ω' are equivariant and X is vertical. Then $\alpha(Y_u) = 0$ and $\omega(Y_u)$ is the identity matrix. One has by (4.3.8)

$$\begin{aligned} d\alpha &= -\theta \wedge \omega' - \theta' \wedge \omega - dA \wedge \omega + A\theta \wedge \omega \\ &= -\theta \wedge \alpha - \theta' \wedge \omega - dA \wedge \omega + A\theta \wedge \omega - \theta \wedge A\omega. \end{aligned}$$

On the other hand, since $\pi_*[X,Y]_u = 0$, we have $d\alpha(X_u, Y_u) = X\alpha(Y)_u - Y\alpha(X)_u - \alpha([X,Y]_u) = 0$. It follows that

$$\begin{aligned} \theta'(X_u) &= \theta'(X_u)\omega(Y_u) \\ &= (\theta' \wedge \omega)(X_u, Y_u) \\ &= (-\theta \wedge \alpha - dA \wedge \omega + A\theta \wedge \omega - \theta \wedge A\omega)(X_u, Y_u) \\ &= -dA(X_u) + A\theta(X_u) - \theta(X_u)A. \end{aligned}$$

Since $dA(X_u) = X_uA = 0$, it suffices to show that $\theta(X_u) = 0$. This follows from the equalities $\theta(X_u) = \theta(X_u)\omega(Y_u) = (\theta \wedge \omega)(X_u, Y_u) = -d\omega(X_u, Y_u)$ and $d\omega(X_u, Y_u) = X\omega(Y)_u - Y\omega(X)_u - \omega([X,Y]_u) = 0$. In order to show 2), note that $R_g^*\theta = g^{-1}\theta g$, $R_g^*\omega = g^{-1}\omega$ and $R_g^*\omega' = g^{-1}\omega'$. Applying R_g^* to (4.3.8), we see that

$$-R_g^*\theta' \wedge g^{-1}\omega = g^{-1}d\omega' + g^{-1}\theta g \wedge g^{-1}\omega' = -g^{-1}\theta' g \wedge g^{-1}\omega,$$

from which 2) follows. ☐

Definition 4.3.12. Let $\widetilde{\varphi} \in \widetilde{\mathrm{WU}}_q$ be a lift of a cocycle φ in WU_q, $\beta \in H^1(M;\Theta_{\mathcal{F}})$ and σ a representative of β. Let θ^u be a unitary connection for some Hermitian metric on $Q(\mathcal{F})$ and θ a Bott connection. Let Ω be the curvature form of θ, and let θ' be an infinitesimal derivative of θ with respect to σ. Under these assumptions, we define a differential form on P by

$$D_\sigma(\widetilde{\varphi}) = \Delta\widetilde{\varphi}(\theta^u, \theta, \theta'),$$

where the right hand side is as in Definition 4.2.4.

We will show in Lemma 4.3.15 and Theorem 4.3.18 that $D_\sigma(\widetilde{\varphi})$ projects down to a closed form on M, and that its cohomology class depends only on $[\varphi] \in H^*(\mathrm{WU}_q)$ and $\beta \in H^1(M;\Theta_{\mathcal{F}})$. Then the following definition is justified.

Definition 4.3.13. For $f \in H^*(\mathrm{WU}_q)$ and $\beta \in H^1(M;\Theta_{\mathcal{F}})$, the *infinitesimal derivative of f with respect to β* is defined as follows. Let φ and σ be representatives of f and β, respectively. Set then $D_\beta(f) = [D_\sigma(\widetilde{\varphi})]$, where $\widetilde{\varphi}$ is any lift of φ to $\widetilde{\mathrm{WU}}_q$.

Remark 4.3.14. If

$$\varphi = (\widetilde{u}_{i_1} v_{i_2} \cdots v_{i_k}) + (\overline{v}_{i_1} \widetilde{u}_{i_2} v_{i_3} \cdots v_{i_k}) + \cdots + (\overline{v}_{i_1} \cdots \overline{v}_{i_{k-1}} \widetilde{u}_{i_k}),$$

then φ is denoted by hc_I in [42, Definition 3.14], and $D_\beta(\widetilde{\varphi})$ coincides with the original definition. Moreover, if we begin with cocycles of the form $h_I c_J \in \mathrm{WO}_q$ and repeat the same construction, then the same differential forms appeared in [42] are obtained by following Definition 4.3.13. In this sense, the formula in Definition 4.3.13 is a complex version of (2.15) in [42].

Proof of Theorem B2. Once infinitesimal derivatives are seen to be well-defined, the theorem follows from Definition 4.3.12 by using (4.2.5b). □

We come back to verify that infinitesimal derivatives are well-defined.

Lemma 4.3.15 ([42, Theorem 3.17]). *The differential form $D_\sigma(\widetilde{\varphi})$ in Definition 4.3.12 projects down to a well-defined closed form on M which depends on σ, θ, θ^u and the choice of the lift $\widetilde{\varphi}$.*

Proof.

Claim 1. *$D_\sigma(\widetilde{\varphi})$ is independent of the choice of θ'.*

Let θ'_0 and θ'_1 be infinitesimal derivatives of θ with respect to σ and let g be a monomial in $v_1, \ldots, v_q$ and $\overline{v}_1, \ldots, \overline{v}_q$. Since $\theta'_1 - \theta'_0 = \lambda\omega$ by (4.3.10), $g(\theta'_1, \Omega) - g(\theta'_0, \Omega) = g(\lambda\omega, \Omega)$. As $\widetilde{\varphi}$ is a lift of a cocycle, $\widetilde{d}\widetilde{\varphi}$ is a linear combination of cochains in $\widetilde{\mathcal{I}}_q$. It follows that $\Delta\widetilde{\varphi}(\theta^u, \theta, \theta'_1) - \Delta\widetilde{\varphi}(\theta^u, \theta, \theta'_0) \in \mathcal{I}(\omega)^{q+1} \cup \overline{\mathcal{I}(\omega)}^{q+1} = \{0\}$.

Claim 2. *$D_\sigma(\widetilde{\varphi})$ projects down to M.*

It suffices to show that $v_J\overline{v}_K(v_{i_l} - \overline{v}_{i_l})(\theta', \Omega)$ projects down to M. We have

$$\begin{aligned}
&R_g^*\left(v_J\overline{v}_K(v_{i_l} - \overline{v}_{i_l})(\theta', \Omega)\right) - v_J\overline{v}_K(v_{i_l} - \overline{v}_{i_l})(\theta', \Omega)\\
=&\, v_J\overline{v}_K(v_{i_l} - \overline{v}_{i_l})(R_g^*\theta', g^{-1}\Omega g) - v_J\overline{v}_K(v_{i_l} - \overline{v}_{i_l})(g^{-1}\theta' g, g^{-1}\Omega g)\\
=&\, v_J\overline{v}_K(v_{i_l} - \overline{v}_{i_l})(R_g^*\theta' - g^{-1}\theta' g, g^{-1}\Omega g).
\end{aligned}$$

It follows that $R_g^*\Delta\widetilde{\varphi}(\theta^u, \theta, \theta') = \Delta\widetilde{\varphi}(\theta^u, \theta, \theta')$ from Lemma 4.3.11 ii) and an argument as in the proof of Claim 1.

Claim 3. *$D_\sigma(\widetilde{\varphi})$ is closed.*

Note that $\widetilde{d}\widetilde{\varphi}$ is a linear combination of cochains of the form $\widetilde{u}_I v_J\overline{v}_K$ with $|J| > q$ or $|K| > q$. Since $D_\sigma(\widetilde{\varphi}) = \Delta\widetilde{\varphi}(\theta^u, \theta, \theta') = \delta(\widetilde{d}\widetilde{\varphi})(\theta^u, \theta, \theta')$ and since $\widetilde{d}(\widetilde{d}\widetilde{\varphi}) = 0$, $D_\sigma(\widetilde{\varphi})$ is closed by the Lemma 4.3.17 below. □

The following differential forms are convenient.

Definition 4.3.16. Let θ_0^u and θ_1^u be unitary connections, not necessarily with respect to the same Hermitian metric, and let $\widetilde{u}_I v_J\overline{v}_K \in \widetilde{\mathrm{WU}}_q$. We decompose $I = I_1 \cup I_2$ so that I_1 consists only of indices less than or equal to i, and I_2 consists only of indices greater than i. We set then $\widetilde{u}_I^{(i)}(\theta, \theta_0^u, \theta_1^u) = \widetilde{u}_{I_1}(\theta, \theta_0^u)\widetilde{u}_{I_2}(\theta, \theta_1^u)$, and

$$\delta_i(\widetilde{u}_I v_J\overline{v}_K)(\theta_0^u, \theta_1^u, \theta, \theta') = (|J| + |K|)v_J\overline{v}_K(\theta', \Omega)\widetilde{u}_I^{(i)}(\theta, \theta_0^u, \theta_1^u).$$

We extend δ_i to the whole $\widetilde{\mathrm{WU}}_q$ by linearity.

The proof of Lemma 4.3.15 is completed by the following lemma.

LEMMA 4.3.17. *Let* $\widetilde{\varphi} \in \widetilde{\mathrm{WU}}_q$. *If* $\widetilde{d}\widetilde{\varphi} = 0$ *and if* $\widetilde{\varphi} \in \widetilde{\mathcal{I}}_q$, *then* $\delta_i(\widetilde{\varphi})(\theta_0^u, \theta_1^u, \theta, \theta')$ *is closed. In particular,* $v_J\overline{v}_K(\theta', \Omega)$ *is closed if* $|J| > q$ *or* $|K| > q$.

PROOF. We first assume that $|J| > q$ and show that $v_J(\theta', \Omega)$ is closed. We write $\Omega_j^i = \sum_k \mathit{\Gamma}_{j,k}^i \wedge \omega^k$ and set $\Omega_j'^i = \sum_k \mathit{\Gamma}_{j,k}^i \wedge \omega'^k$. Then $\Omega \wedge \omega' = -\Omega' \wedge \omega$. On the other hand, by using (4.3.8) and the equalities $\Omega = d\theta + \theta \wedge \theta$, $d\omega = -\theta \wedge \omega$ and (4.3.8), we obtain

$$-\Omega \wedge \omega' = (d\theta' + [\theta, \theta']) \wedge \omega,$$

where $[\theta, \theta'] = \theta \wedge \theta' + \theta' \wedge \theta$. Hence $v_J(d\theta' + [\theta, \theta'], \Omega) = v_J(\Omega', \Omega)$.

Let $\mathcal{I}_s(\omega)$ be the ideal of differential forms on P generated by $\omega^1 + s\omega'^1, \ldots, \omega^q + s\omega'^q$. Then, for any s, we have $\mathcal{I}_s(\omega)^{q+1} = \{0\}$. If we set $\Omega(s) = \Omega + s\Omega'$, then $\Omega(s) \in \mathcal{I}_s(\omega)$ because $(\Omega(s))_j^i = \sum_k \mathit{\Gamma}_{j,k}^i \wedge (\omega^k + s\omega'^k)$. Since $v_J(\Omega(s))$ is identically zero, we have the following equality:

$$\begin{aligned}
d(v_J(\theta', \Omega)) &= v_J(d\theta', \Omega) - (|J| - 1)v_J(\theta', d\Omega, \Omega) \\
&= v_J(d\theta', \Omega) + (|J| - 1)v_J(\theta', [\theta, \Omega], \Omega) \\
&= v_J(d\theta' + [\theta, \theta'], \Omega) \\
&= v_J(\Omega', \Omega) \\
&= \frac{1}{|J|}\frac{\partial}{\partial s} v_J(\Omega(s))\bigg|_{s=0} \\
&= 0.
\end{aligned}$$

On the other hand, by (4.2.5b),

$$(|J| + |K|)v_J\overline{v}_K(\theta', \Omega) = |J|\, v_J(\theta', \Omega)\overline{v}_K(\Omega) + |K|\, v_J(\Omega)\overline{v}_K(\theta', \Omega).$$

Hence $v_J\overline{v}_K(\theta', \Omega)$ is closed. Similarly, $v_J\overline{v}_K(\theta', \Omega)$ is also closed if $|K| > q$.

Assume now that $\widetilde{\varphi} = \sum_t x_t \widetilde{u}_{I_t} v_{J_t} \overline{v}_{K_t}$, where $x_t \in \mathbb{C}$. We may furthermore assume that the numbers of elements of I_t are constant, which is denoted by $\#I$. If

$\#I = 0$, then $\delta_i(\widetilde{\varphi})(\theta_0^u, \theta_1^u, \theta, \theta')$ is already shown to be closed. If $\#I > 0$, then we have

$$\begin{aligned} &d(\delta_i(\widetilde{\varphi})(\theta_0^u, \theta_1^u, \theta, \theta')) \\ &= \sum_{t,l} (-1)^l x_t(|J_t| + |K_t|) v_{J_t} \overline{v}_{K_t}(\theta', \Omega)(v_{i_l} - \overline{v}_{i_l})(\Omega) \widetilde{u}^{(i)}_{I_t(l)}(\theta, \theta_0^u, \theta_1^u). \end{aligned}$$

Since $|J_t| > q$ or $|K_t| > q$, we have

$$v_{J_t}(\theta', \Omega) v_{i_l}(\Omega) \overline{v}_{K_t}(\Omega) = v_{J_t}(\Omega) \overline{v}_{K_t}(\theta', \Omega) \overline{v}_{i_l}(\Omega) = 0.$$

Hence

$$\begin{aligned} d(\delta_i(\widetilde{\varphi})(\theta_0^u, \theta_1^u, \theta, \theta')) = &\sum_{t,l} -(-1)^l x_t \, |J_t| \, v_{J_t}(\theta', \Omega) \overline{v}_{K_t}(\Omega) \overline{v}_{i_l}(\Omega) \widetilde{u}^{(i)}_{I_t(l)}(\theta^u, \theta) \\ &+ \sum_{t,l} (-1)^l x_t \, |K_t| \, v_{J_t}(\Omega) v_{i_l}(\Omega) \overline{v}_{K_t}(\theta', \Omega) \widetilde{u}^{(i)}_{I_t(l)}(\theta^u, \theta). \end{aligned}$$

Now by (4.2.5a) and (4.2.5b),

$$(|J_t| + |K_t| + i_l) v_{J_t} \overline{v}_{K_t} \overline{v}_{i_l}(\theta', \Omega) = |J_t| \, v_{J_t}(\theta', \Omega) \overline{v}_{K_t} \overline{v}_{i_l}(\Omega),$$

and

$$(|J_t| + |K_t| + i_l) v_{J_t} v_{i_l} \overline{v}_{K_t}(\theta', \Omega) = |K_t| \, v_{J_t} v_{i_l}(\Omega) \overline{v}_{K_t}(\theta', \Omega).$$

Therefore,

$$\begin{aligned} d(\delta_i(\widetilde{\varphi})(\theta_0^u, \theta_1^u, \theta, \theta')) = &\sum_{t,l} -(-1)^l x_t (|J_t| + |K_t| + i_l) v_{J_t} \overline{v}_{K_t} \overline{v}_{i_l}(\theta', \Omega) \widetilde{u}^{(i)}_{I_t(l)}(\theta^u, \theta) \\ &+ \sum_{t,l} (-1)^l x_t (|J_t| + |K_t| + i_l) v_{J_t} v_{i_l} \overline{v}_{K_t}(\theta', \Omega) \widetilde{u}^{(i)}_{I_t(l)}(\theta^u, \theta) \\ = &\, \delta_i(\widetilde{d}\widetilde{\varphi})(\theta_0^u, \theta_1^u, \theta, \theta') \\ = &\, 0 \end{aligned}$$

because $\widetilde{d}\widetilde{\varphi} = 0$. This completes the proof of Lemma 4.3.17. □

If $f \in H^*(\mathrm{WU}_q)$ and $\beta \in H^1(M; \Theta_{\mathcal{F}})$, then we have the following

THEOREM 4.3.18 (cf. [42, Theorem 3.17]). *Let φ and σ be representatives of f and β, respectively, and let $\widetilde{\varphi}$ be any lift of φ to $\widetilde{\mathrm{WU}}_q$. Then the cohomology class $[D_\sigma(\widetilde{\varphi})]$ is independent of the choice of representatives and lifts.*

Proof. Let θ^u, θ, θ', Ω be as in Definition 4.3.12.

Claim 1. $[D_\sigma(\widetilde{\varphi})]$ *is independent of the choice of the Bott connection* θ.

Let θ_0 and θ_1 be Bott connections and choose their infinitesimal derivatives θ'_0 and θ'_1 with respect to σ. Note that $D_\sigma(\widetilde{\varphi})$ is independent of the choice of infinitesimal derivatives by Lemma 4.3.15. We set $\theta_t = \theta_0 + t(\theta_1 - \theta_0)$. Then θ_t is also a Bott connection and $\theta'_t = \theta'_0 + t(\theta'_1 - \theta'_0)$ is an an infinitesimal derivative of θ_t. Let Ω_t be the connection form of θ_t. We will show that $\dfrac{\partial}{\partial t}\Delta\widetilde{\varphi}(\theta^u, \theta_t, \theta'_t)$ is exact. First we consider the case where the $\widetilde{d\widetilde{\varphi}}$ does not involve any $\widetilde{u}_i$. Note that $\Delta\widetilde{\varphi}(\theta^u, \theta_t, \theta'_t)$ is calculated by evaluating $\widetilde{d\widetilde{\varphi}} \in \widetilde{\mathcal{I}}_q$. One has

$$\begin{aligned} d(v_J\overline{v}_K(\theta'_t, \theta_1 - \theta_0, \Omega_t)) &= v_J\overline{v}_K(d\theta'_t, \theta_1 - \theta_0, \Omega_t) - v_J\overline{v}_K(\theta'_t, d\theta_1 - d\theta_0, \Omega_t) \\ &\quad - (|J| + |K| - 2)v_J\overline{v}_K(\theta'_t, \theta_1 - \theta_0, [\theta_t, \Omega_t], \Omega_t) \\ &= v_J\overline{v}_K(d\theta'_t + [\theta_t, \theta'_t], \theta_1 - \theta_0, \Omega_t) \\ &\quad - v_J\overline{v}_K(\theta'_t, d\theta_1 - d\theta_0 + [\theta_t, \theta_1 - \theta_0], \Omega_t). \end{aligned}$$

Note that each of the differential forms in the above equality projects down to M. On the other hand,

$$\begin{aligned} &\frac{\partial}{\partial t} v_J\overline{v}_K(\theta'_t, \Omega_t) \\ &= v_J\overline{v}_K(\theta'_1 - \theta'_0, \Omega_t) + (|J| + |K| - 1)v_J\overline{v}_K(\theta'_t, d(\theta_1 - \theta_0) + [\theta_t, \theta_1 - \theta_0], \Omega_t). \end{aligned}$$

Hence

$$\begin{aligned} &\frac{\partial}{\partial t} v_J\overline{v}_K(\theta'_t, \Omega_t) + (|J| + |K| - 1)\, d(v_J\overline{v}_K(\theta'_t, \theta_1 - \theta_0, \Omega_t)) \\ &= v_J\overline{v}_K(\theta'_1 - \theta'_0, \Omega_t) + (|J| + |K| - 1)v_J\overline{v}_K(d\theta'_t + [\theta_t, \theta'_t], \theta_1 - \theta_0, \Omega_t). \end{aligned}$$

As in the proof of Lemma 4.3.17, we write $(\Omega_t)^i_j = \sum_k \mathit{\Gamma}^i_{j,k} \wedge \omega^k$ and set $(\Omega'_t)^i_j = \sum_k \mathit{\Gamma}^i_{j,k} \wedge \omega'^k$. Then $\Omega'_t \wedge \omega = (d\theta'_t + [\theta_t, \theta'_t]) \wedge \omega$. Since $\theta_0 \wedge \omega = \theta_1 \wedge \omega = -d\omega$, $(\theta_1 - \theta_0) \wedge \omega = 0$. Hence $(\theta_1 - \theta_0)^i_j = \sum_k \lambda^i_{j,k}\omega^k$ for some $\lambda^i_{j,k}$. Now by (4.3.8), we have $(\theta'_1 - \theta'_0) \wedge \omega = -(\lambda\omega) \wedge \omega' = (\lambda\omega') \wedge \omega$. If we set $\Omega(s,t) = \Omega_t + s\Omega'_t$, $\theta(s) = (\theta_1 - \theta_0) + s\lambda\omega'$ and $\mathcal{I}_s(\omega) = \mathcal{I}(\omega^1 + s\omega'^1, \dots, \omega^q + s\omega'^q)$, then $\Omega(s,t), \theta(s) \in \mathcal{I}_s(\omega)$.

Therefore, $v_J\overline{v}_K(\theta(s),\Omega(s,t)) = 0$ if $|J| > q$ or $|K| > q$. Differentiating with respect to s and setting $s = 0$, we obtain

$$v_J\overline{v}_K(\lambda\omega',\Omega_t) + (|J|+|K|-1)v_J\overline{v}_K(\theta_1-\theta_0,\Omega_t',\Omega_t) = 0.$$

As the left hand side is equal to $v_J\overline{v}_K((\theta_1'-\theta_0',\Omega_t) + (|J|+|K|-1)v_J\overline{v}_K(\theta_1-\theta_0, d\theta_t' + [\theta_t,\theta_t'],\Omega_t)$,

$$\frac{\partial}{\partial t}v_J\overline{v}_K(\theta_t',\Omega_t) = -(|J|+|K|-1)\,d(v_J\overline{v}_K(\theta_t',\theta_1-\theta_0,\Omega_t))$$

if $|J| > q$ or $|K| > q$.

Suppose now that $\widetilde{d}\widetilde{\varphi}$ involves some of $\widetilde{u}_i$'s. We write $\widetilde{d}\widetilde{\varphi} = \sum_i x_i v_{J_i}\overline{v}_{K_i}\widetilde{u}_{I_i}$, where $|J_i| > q$ or $|K_i| > q$, and $x_i \in \mathbb{C}$. Then by definition,

$$\Delta\widetilde{\varphi}(\theta^u,\theta_t,\theta_t') = \sum_i x_i(|J_i|+|K_i|)v_{J_i}\overline{v}_{K_i}(\theta_t',\Omega_t)\widetilde{u}_{I_i}(\theta_t,\theta^u).$$

Hence

$$\begin{aligned}
&\frac{\partial}{\partial t}\Delta\widetilde{\varphi}(\theta^u,\theta_t,\theta_t') \\
&= -\sum_i x_i(|J_i|+|K_i|)(|J_i|+|K_i|-1)\,d(v_{J_i}\overline{v}_{K_i}(\theta_t',\theta_1-\theta_0,\Omega_t))\widetilde{u}_{I_i}(\theta_t,\theta^u) \\
&\quad + \sum_{i,l} x_i(|J_i|+|K_i|)v_{J_i}\overline{v}_{K_i}(\theta_t',\Omega_t)(-1)^{l-1}i_l(i_l-1)\,d\widetilde{V}_{i_l}(\theta_t,\theta^u)\widetilde{u}_{I_i(l)}(\theta_t,\theta^u) \\
&\quad + \sum_{i,l} x_i(|J_i|+|K_i|)v_{J_i}\overline{v}_{K_i}(\theta_t',\Omega_t)(-1)^{l-1}i_l\widetilde{v}_{i_l}(\theta_1-\theta_0,\Omega_t)\widetilde{u}_{I_i(l)}(\theta_t,\theta^u),
\end{aligned}$$

where $I_i(l) = I_i \setminus \{i_l\}$. If we fix an integer k and rewrite $\widetilde{d}\widetilde{\varphi}$ as $\widetilde{d}\widetilde{\varphi} = \widetilde{u}_k\alpha_k + \beta_k$ so that α_k and β_k do not involve $\widetilde{u}_k$, then $\widetilde{d}(\widetilde{d}\widetilde{\varphi}) = 0$ implies that $\widetilde{d}\alpha_k = 0$. Hence

$$\begin{aligned}
&\sum_{\substack{i,l\\ i_l=k}} x_i(|J_i|+|K_i|)v_{J_i}\overline{v}_{K_i}(\theta_t',\Omega_t)(-1)^{l-1}i_l(i_l-1)\,d\widetilde{V}_{i_l}(\theta_t,\theta^u)\widetilde{u}_{I_i(l)}(\theta_t,\theta^u) \\
&= k(k-1)\,d(\widetilde{V}_k(\theta_t,\theta^u)\,\delta(\alpha_k)(\theta^u,\theta_t,\theta_t'))
\end{aligned}$$

because $\delta(\alpha_k)(\theta^u,\theta_t,\theta'_t)$ is closed by Lemma 4.3.17. Hence $\dfrac{\partial}{\partial t}\Delta\widetilde{\varphi}(\theta^u,\theta_t,\theta'_t)$ is cohomologous to R, where

$$\begin{aligned}R = &-\sum_i x_i(|J_i|+|K_i|)(|J_i|+|K_i|-1)\,d(v_{J_i}\overline{v}_{K_i}(\theta'_t,\theta_1-\theta_0,\Omega_t))\widetilde{u}_{I_i}(\theta_t,\theta^u)\\ &+\sum_{i,l} x_i(|J_i|+|K_i|)v_{J_i}\overline{v}_{K_i}(\theta'_t,\Omega_t)(-1)^{l-1}i_l\widetilde{v}_{i_l}(\theta_1-\theta_0,\Omega_t)\widetilde{u}_{I_i(l)}(\theta_t,\theta^u).\end{aligned}$$

It suffices to show that R is exact. This is indeed done as follows, namely, by (A.6b) we have the following equality:

$$\begin{aligned}&-(|J_i|+|K_i|)(|J_i|+|K_i|-1)\,d(v_{J_i}\overline{v}_{K_i}(\theta'_t,\theta_1-\theta_0,\Omega_t))\widetilde{u}_{I_i}(\theta_t,\theta^u)\\ \equiv&(|J_i|+|K_i|)(|J_i|+|K_i|-1)v_{J_i}\overline{v}_{K_i}(\theta'_t,\theta_1-\theta_0,\Omega_t)\,d\widetilde{u}_{I_i}(\theta_t,\theta^u)\\ =&\sum_l(|J_i|+|K_i|)(|J_i|+|K_i|-1)v_{J_i}\overline{v}_{K_i}(\theta'_t,\theta_1-\theta_0,\Omega_t)(-1)^{l-1}(v_{i_l}-\overline{v}_{i_l})(\Omega_t)\widetilde{u}_{I_i(l)}(\theta_t,\theta^u)\\ =&-\sum_l|J_i|(|J_i|-1)v_{J_i}(\theta'_t,\theta_1-\theta_0,\Omega_t)\overline{v}_{K_i}(\Omega_t)(-1)^{l-1}\overline{v}_{i_l}(\Omega_t)\widetilde{u}_{I_i(l)}(\theta_t,\theta^u)\\ &-\sum_l|J_i||K_i|v_{J_i}(\theta'_t,\Omega_t)\overline{v}_{K_i}(\theta_1-\theta_0,\Omega_t)(-1)^{l-1}\overline{v}_{i_l}(\Omega_t)\widetilde{u}_{I_i(l)}(\theta_t,\theta^u)\\ &-\sum_l|J_i||K_i|v_{J_i}(\theta_1-\theta_0,\Omega_t)v_{i_l}(\Omega_t)\overline{v}_{K_i}(\theta'_t,\Omega_t)(-1)^{l-1}\widetilde{u}_{I_i(l)}(\theta_t,\theta^u)\\ &+\sum_l|K_i|(|K_i|-1)v_{J_i}(\Omega_t)v_{i_l}(\Omega_t)\overline{v}_{K_i}(\theta'_t,\theta_1-\theta_0,\Omega_t)(-1)^{l-1}\widetilde{u}_{I_i(l)}(\theta_t,\theta^u),\end{aligned}$$

where the symbol '$\equiv$' means that the equality holds modulo exact forms. On the other hand, we have

$$\begin{aligned}&(|J_i|+|K_i|)v_{J_i}\overline{v}_{K_i}(\theta'_t,\Omega_t)(-1)^{l-1}i_l\widetilde{v}_{i_l}(\theta_1-\theta_0,\Omega_t)\widetilde{u}_{I_i(l)}(\theta_t,\theta^u)\\ =&-|J_i|\,v_{J_i}(\theta'_t,\Omega_t)\overline{v}_{K_i}(\Omega_t)(-1)^{l-1}i_l\overline{v}_{i_l}(\theta_1-\theta_0,\Omega_t)\widetilde{u}_{I_i(l)}(\theta_t,\theta^u)\\ &-|K_i|\,v_{J_i}(\Omega_t)v_{i_l}(\theta_1-\theta_0,\Omega_t)\overline{v}_{K_i}(\theta'_t,\Omega_t)(-1)^{l-1}i_l\widetilde{u}_{I_i(l)}(\theta_t,\theta^u).\end{aligned}$$

Therefore, we have

$$\begin{aligned}R\equiv&-\sum_{i,l}x_i\,|J_i|\,(|J_i|-1)v_{J_i}(\theta'_t,\theta_1-\theta_0,\Omega_t)\overline{v}_{K_i}(\Omega_t)\overline{v}_{i_l}(\Omega_t)(-1)^{l-1}\widetilde{u}_{I_i(l)}(\theta_t,\theta^u)\\ &-\sum_{i,l}x_i\,|J_i|\,(|K_i|+i_l)v_{J_i}(\theta'_t,\Omega_t)\overline{v}_{K_i}\overline{v}_{i_l}(\theta_1-\theta_0,\Omega_t)(-1)^{l-1}\widetilde{u}_{I_i(l)}(\theta_t,\theta^u)\\ &-\sum_{i,l}x_i(|J_i|+i_l)\,|K_i|\,v_{J_i}v_{i_l}(\theta_1-\theta_0,\Omega_t)\overline{v}_{K_i}(\theta'_t,\Omega_t)(-1)^{l-1}\widetilde{u}_{I_i(l)}(\theta_t,\theta^u)\\ &+\sum_{i,l}x_i\,|K_i|\,(|K_i|-1)v_{J_i}(\Omega_t)v_{i_l}(\Omega_t)\overline{v}_{K_i}(\theta'_t,\theta_1-\theta_0,\Omega_t)(-1)^{l-1}\widetilde{u}_{I_i(l)}(\theta_t,\theta^u).\end{aligned}$$

Let R' be the right hand side of the above equality. Then by (A.6b),

$$\begin{aligned}&(|J_i|+|K_i|+i_l)v_{J_i}v_{i_l}\overline{v}_{K_i}(\theta'_t,\theta_1-\theta_0,\Omega_t)\\&=-(|J_i|+i_l)\,|K_i|\,v_{J_i}v_{i_l}(\theta_1-\theta_0,\Omega_t)\overline{v}_{K_i}(\theta'_t,\Omega_t)\\&\quad+|K_i|\,(|K_i|-1)v_{J_i}v_{i_l}(\Omega_t)\overline{v}_{K_i}(\theta'_t,\theta_1-\theta_0,\Omega_t),\end{aligned}$$

and

$$\begin{aligned}&(|J_i|+|K_i|+i_l)v_{J_i}\overline{v}_{K_i}\overline{v}_{i_l}(\theta'_t,\theta_1-\theta_0,\Omega_t)\\&=|J_i|\,(|J_i|-1)v_{J_i}(\theta'_t,\theta_1-\theta_0,\Omega_t)\overline{v}_{K_i}\overline{v}_{i_l}(\Omega_t)\\&\quad+|J_i|\,(|K_i|+i_l)v_{J_i}(\theta'_t,\Omega_t)\overline{v}_{K_i}\overline{v}_{i_l}(\theta_1-\theta_0,\Omega_t).\end{aligned}$$

It follows that

$$\begin{aligned}R'&=\sum_{i,l}x_i(|J_i|+|K_i|+i_l)v_{J_i}v_{i_l}\overline{v}_{K_i}(\theta'_t,\theta_1-\theta_0,\Omega_t)(-1)^{l-1}\widetilde{u}_{I_i(l)}(\theta_t,\theta^u)\\&\quad-\sum_{i,l}x_i(|J_i|+|K_i|+i_l)v_{J_i}\overline{v}_{K_i}\overline{v}_{i_l}(\theta'_t,\theta_1-\theta_0,\Omega_t)(-1)^{l-1}\widetilde{u}_{I_i(l)}(\theta_t,\theta^u)\\&=\delta(\widetilde{d}\widetilde{d}\widetilde{\varphi})(\theta^u,\theta_t,\theta'_t)\\&=0.\end{aligned}$$

Claim 2. *$[D_\sigma(\widetilde{\varphi})]$ is independent of the choice of the unitary connection θ^u.*

We first show that $\widetilde{u}_i(\theta,\theta^u_1)-\widetilde{u}_i(\theta,\theta^u_0)=d\widetilde{V}'_i$ for some differential form $\widetilde{V}'_i$ if θ^u_0 and θ^u_1 are unitary connections. Suppose that θ^u_0 and θ^u_1 are unitary connections for a fixed Hermitian metric on $Q(\mathcal{F})$. Let $f=\widetilde{v}_i=v_i-\overline{v}_i$, $\theta^s_1=\theta+s(\theta^u_1-\theta)$ and $\theta_0=\theta^u_0$. Then we substitute them into (4.2.2a) and integrate it with respect to s. We obtain

$$\Delta_{\widetilde{v}_i}(\theta^u_1,\theta^u_0)-\Delta_{\widetilde{v}_i}(\theta,\theta^u_0)=k(k-1)\,dW_{\widetilde{v}_i}(\theta^u_1,\theta^u_0)+\Delta_{\widetilde{v}_i}(\theta^u_1,\theta),$$

where $W_{\widetilde{v}_i}(\theta^u_1,\theta^u_0)=\displaystyle\int_0^1 V_{\widetilde{v}_i}(\theta^s_1,\theta^u_0)\,ds$. Hence

$$\Delta_{\widetilde{v}_i}(\theta,\theta^u_1)-\Delta_{\widetilde{v}_i}(\theta,\theta^u_0)+\Delta_{\widetilde{v}_i}(\theta^u_1,\theta^u_0)=k(k-1)\,dW_{\widetilde{v}_i}(\theta^u_1,\theta^u_0).\tag{4.3.19}$$

We set $\theta^u_t = \theta^u_0 + t(\theta^u_1 - \theta^u_0)$. Then by (4.2.2a),

$$\frac{\partial}{\partial t}\Delta_{\widetilde{v}_i}(\theta^u_t, \theta^u_0) = k(k-1)\, dV_{\widetilde{v}_i}(\theta^u_t, \theta_0) + k\widetilde{v}_i(\theta^u_1 - \theta^u_0, \Omega^u_t, \dots, \Omega^u_t).$$

Since θ^u_0 and θ^u_1 are unitary and since $V_{\widetilde{v}_i} = \widetilde{V}_i$,

$$\frac{\partial}{\partial t}\Delta_{\widetilde{v}_i}(\theta^u_t, \theta^u_0) = k(k-1)\, d\widetilde{V}_i(\theta^u_t, \theta_0).$$

Hence $\widetilde{u}_i(\theta, \theta^u_1) - \widetilde{u}_i(\theta, \theta^u_0)$ is exact if θ^u_0 and θ^u_1 are unitary connections for a fixed Hermitian metric.

Let now h_0 and h_1 be Hermitian metrics on $Q(\mathcal{F})$ and let θ^u_0 and θ^u_1 be unitary connections for h_0 and h_1, respectively. The equality (4.3.19) is still valid so that it suffices to show that $\Delta_f(\theta^u_1, \theta^u_0)$ is exact if $f = \widetilde{v}_i = v_i - \overline{v}_i$. We denote by ι_t the natural isomorphism from M to $M \times \{t\}$ and by π the projection from $M \times \mathbb{R}$ to $\mathbb{R}$. We consider then the foliation $\mathcal{F} \times \mathbb{R}$ of $M \times \mathbb{R}$ whose leaves are given by $L \times \mathbb{R}$, where L is a leaf of $\mathcal{F}$. Let $\widetilde{\theta^u}$ be a unitary connection on $Q(\mathcal{F} \times \mathbb{R})$ for some Hermitian metric such that $\theta^u_t = \theta^u_0$ for $t \le 0$ and $\theta^u_t = \theta^u_1$ for $t \ge 1$, where $\theta^u_t = \iota^*_t \widetilde{\theta^u}$. We write $\Delta_{\widetilde{v}_i}(\widetilde{\theta}^u, \pi^*\theta^u_0) = \lambda + \mu \wedge dt$ so that λ and μ do not involve dt. Then we define a differential form $\widetilde{V}'_i(\theta^u_1, \theta^u_0)$ on M by setting

$$\widetilde{V}'_i(\theta^u_1, \theta^u_0) = -\int_0^1 \mu\, dt.$$

We have $d\widetilde{V}'_i(\theta^u_1, \theta^u_0) = \Delta_{\widetilde{v}_i}(\theta^u_1, \theta^u_0)$, which can be shown as follows. First,

$$d_{M\times\mathbb{R}}\Delta_{\widetilde{v}_i}(\widetilde{\theta}^u, \pi^*\theta_0) = \bigl(v_i(\widetilde{\theta}^u) - \pi^* v_i(\theta^u_0)\bigr) - \bigl(\overline{v}_i(\widetilde{\theta^u}) - \pi^*\overline{v}_i(\theta^u_0)\bigr) = 0.$$

Hence $\dfrac{\partial \lambda}{\partial t} + d_M \mu = 0$, where d_M denotes the exterior derivative along the fiber of $\pi\colon M \times \mathbb{R} \to \mathbb{R}$. On the other hand,

$$d\widetilde{V}'_i(\theta^u_1, \theta^u_0) = -\int_0^1 d_M\mu\, dt = \int_0^1 \frac{\partial\lambda}{\partial t}\, dt = \lambda(1) - \lambda(0)$$

and $\lambda(t) = \iota^*_t\lambda = \iota^*_t\Delta_{\widetilde{v}_i}(\widetilde{\theta}^u, \pi^*\theta_0) = \Delta_{\widetilde{v}_i}(\theta^u_t, \theta_0)$. Finally, $\lambda(1) = \Delta_{\widetilde{v}_i}(\theta^u_1, \theta^u_0)$ and $\lambda(0) = \Delta_{\widetilde{v}_i}(\theta^u_0, \theta^u_0) = 0$.

Let $\widetilde{\varphi} \in \widetilde{\mathrm{WU}}_q$ be the natural lift of φ. Let α_i and β_i be such that $\widetilde{d}\widetilde{\varphi} = \widetilde{u}_i\alpha_i + \beta_i$ and that α_i and β_i do not involve $\widetilde{u}_i$. Then

$$\Delta\widetilde{\varphi}(\theta_1^u, \theta, \theta') = \delta_0(\widetilde{d}\widetilde{\varphi})(\theta_0^u, \theta_1^u, \theta, \theta')$$

and

$$\Delta\widetilde{\varphi}(\theta_1^u, \theta, \theta') = \delta_q(\widetilde{d}\widetilde{\varphi})(\theta_0^u, \theta_1^u, \theta, \theta').$$

Hence it suffices to show that $\delta_{k-1}(\widetilde{d}\widetilde{\varphi})(\theta_0^u, \theta_1^u, \theta, \theta')$ and $\delta_k(\widetilde{d}\widetilde{\varphi})(\theta_0^u, \theta_1^u, \theta, \theta')$ are cohomologous for each k. Since β_k does not involve $\widetilde{u}_k$,

$$\begin{aligned}
&\delta_{k-1}(\widetilde{d}\widetilde{\varphi})(\theta_0^u, \theta_1^u, \theta, \theta') \\
&= -\widetilde{u}_k(\theta, \theta_1^u)\,\delta_{k-1}(\alpha_k)(\theta_0^u, \theta_1^u, \theta, \theta') + \delta_{k-1}(\beta_k)(\theta_0^u, \theta_1^u, \theta, \theta') \\
&= -\widetilde{u}_k(\theta, \theta_1^u)\,\delta_{k-1}(\alpha_k)(\theta_0^u, \theta_1^u, \theta, \theta') + \delta_k(\beta_k)(\theta_0^u, \theta_1^u, \theta, \theta').
\end{aligned}$$

On the other hand, $\widetilde{d}\widetilde{\alpha}_i = 0$ because $\widetilde{d}\widetilde{d}\widetilde{\varphi} = 0$. It follows that

$$d\delta_{k-1}(\alpha_k)(\theta_0^u, \theta_1^u, \theta, \theta') = 0$$

by Lemma 4.3.17. Hence

$$\begin{aligned}
&\delta_{k-1}(\widetilde{d}\widetilde{\varphi})(\theta_0^u, \theta_1^u, \theta, \theta') + d(\widetilde{V}_k'\delta_{k-1}(\alpha_k)(\theta_0^u, \theta_1^u, \theta, \theta')) \\
&= -\widetilde{u}_k(\theta, \theta_0^u)\,\delta_{k-1}(\alpha_k)(\theta_0^u, \theta_1^u, \theta, \theta') + \delta_k(\beta_k)(\theta_0^u, \theta_1^u, \theta, \theta') \\
&= -\widetilde{u}_k(\theta, \theta_0^u)\,\delta_k(\alpha_k)(\theta_0^u, \theta_1^u, \theta, \theta') + \delta_k(\beta_k)(\theta_0^u, \theta_1^u, \theta, \theta') \\
&= \delta_k(\widetilde{d}\widetilde{\varphi})(\theta_0^u, \theta_1^u, \theta, \theta')
\end{aligned}$$

because α_k does not involve $\widetilde{u}_k$.

Claim 3. $[D_\sigma(\widetilde{\varphi})]$ *is independent of the choice of representative of* β.

We recall that representatives of β are by definition sections of $E^* \otimes Q(\mathcal{F})$. They are considered as $\mathbb{C}^q$-valued 1-forms on P after arbitrarily extended to sections of $T_{\mathbb{C}}^*M \otimes Q(\mathcal{F})$ and then lifted to P.

We first show that $[D_\sigma(\widetilde{\varphi})]$ is independent of such extensions as above. Suppose that σ_0 and σ_1 are representatives of β and assume that $\sigma_0 = \sigma_1$ when restricted to π^*E, where $\pi\colon P \to M$ is the projection. Then $\sigma_1 - \sigma_0 = \mu\omega$ for some $\mathfrak{gl}(q;\mathbb{C})$-valued function μ. Let θ_0' and θ_1' be infinitesimal derivatives of θ with respect to σ_0 and σ_1, respectively. Then by (4.3.8),

$$(\theta_1' - \theta_0') \wedge \omega = d(\sigma_1 - \sigma_0) + \theta \wedge (\sigma_1 - \sigma_0) = (d\mu + [\theta, \mu]) \wedge \omega.$$

Hence

$$\begin{aligned} v_J \overline{v}_K(\theta_1', \Omega) - v_J \overline{v}_K(\theta_0', \Omega) &= v_J \overline{v}_K(d\mu + [\theta, \mu], \Omega) \\ &= v_J \overline{v}_K(d\mu, \Omega) + (|J| + |K| - 1) v_J \overline{v}_K(\mu, -[\theta, \Omega]) \\ &= v_J \overline{v}_K(d\mu, \Omega) + (|J| + |K| - 1) v_J \overline{v}_K(\mu, d\Omega) \\ &= d(v_J \overline{v}_K(\mu, \Omega)). \end{aligned}$$

Let $\widetilde{u}_I v_J \overline{v}_K$ be an element of $\widetilde{\mathrm{WU}}_q$ such that $|J| > q$. Then

$$\begin{aligned} &(|J| + |K|) v_J \overline{v}_K(\theta_1', \Omega) \widetilde{u}_I(\theta, \theta^u) - v_J \overline{v}_K(\theta_0', \Omega) \widetilde{u}_I(\theta, \theta^u) \\ =& d((|J| + |K|) v_J \overline{v}_K(\mu, \Omega)) \widetilde{u}_I(\theta, \theta^u) \\ =& d(|J|\, v_J(\mu, \Omega) \overline{v}_K(\Omega)) \widetilde{u}_I(\theta, \theta^u) \\ \equiv& -|J|\, v_J(\mu, \Omega) \overline{v}_K(\Omega) d\widetilde{u}_I(\theta, \theta^u) \\ =& -|J| \sum_t (-1)^{t-1} v_J(\mu, \Omega) \overline{v}_K(\Omega) (v_{i_t} - \overline{v}_{i_t})(\Omega) \widetilde{u}_{I(t)}(\theta, \theta^u) \\ =& |J| \sum_t (-1)^{t-1} v_J(\mu, \Omega) \overline{v}_K(\Omega) \overline{v}_{i_t}(\Omega) \widetilde{u}_{I(t)}(\theta, \theta^u) \\ =& (|J| + |K| + i_t) \sum_t (-1)^{t-1} v_J \overline{v}_K \overline{v}_{i_t}(\mu, \Omega) \widetilde{u}_{I(t)}(\theta, \theta^u) \\ =& \delta(\widetilde{d}(\widetilde{u}_I v_J \overline{v}_K))(\theta^u, \theta, \mu). \end{aligned}$$

Similarly, if $|K| > q$ then we have

$$(|J| + |K|) v_J \overline{v}_K(\theta_1', \Omega) \widetilde{u}_I(\theta, \theta^u) - v_J \overline{v}_K(\theta_0', \Omega) \widetilde{u}_I(\theta, \theta^u) \equiv \delta(\widetilde{d}(\widetilde{u}_I v_J \overline{v}_K))(\theta^u, \theta, \mu).$$

Hence

$$\delta(\widetilde{d}\widetilde{\varphi})(\theta^u, \theta, \theta_1') - \delta(\widetilde{d}\widetilde{\varphi})(\theta^u, \theta, \theta_0') \equiv \delta(\widetilde{d}\widetilde{d}\widetilde{\varphi})(\theta^u, \theta, \mu) = 0.$$

Since $D_{\sigma_0 + \sigma_1}(\widetilde{\varphi}) = D_{\sigma_0}(\widetilde{\varphi}) + D_{\sigma_1}(\widetilde{\varphi})$, the proof of Claim 3 is completed if we show that $D_\sigma(\widetilde{\varphi})$ is exact for any section σ which corresponds to $d_\nabla \gamma$, where γ is a section of $E^* \otimes Q(\mathcal{F})$ and d_∇ is as in Definition 4.3.1. By the definition, such a σ can be obtained as follows. We choose an extension Y of γ to $T_{\mathbb{C}}M$ and let $\widehat{Y}$ be its horizontal lift. Let g be a function on P defined by $g(u) = \omega(\widehat{Y}_u)$. Then $dg + \theta g$ can be chosen as σ, and we have by definition $\omega' = -dg - \theta g$. An infinitesimal derivative θ' with respect to σ is by definition a $\mathfrak{gl}(q;\mathbb{C})$-valued 1-form satisfying $\theta' \wedge \omega = -d\omega' - \theta \wedge \omega'$. The right hand side is now equal to $d\theta g - \theta \wedge dg + \theta \wedge dg + \theta \wedge \theta g = \Omega g$. If $\{\mathit{\Gamma}_k\}$ is a family of $\mathfrak{gl}(q;\mathbb{C})$-valued 1-forms such that $\Omega = \sum\limits_k \mathit{\Gamma}_k \wedge \omega^k$, then $\Omega g = \sum\limits_k \mathit{\Gamma}_k g \wedge \omega^k$. Note that if we write $\mathit{\Gamma}_k = (\mathit{\Gamma}^i_{j,k})$, then $\mathit{\Gamma}^i_{j,k} = \mathit{\Gamma}^i_{k,j}$. Hence $\left(\sum\limits_k \mathit{\Gamma}_k \omega^k(\widehat{Y})\right) \wedge \omega = \left(\sum\limits_k \mathit{\Gamma}_k g^k\right) \wedge \omega = \sum\limits_j \mathit{\Gamma}_j g \wedge \omega^j = \Omega g$ and $\left(\sum\limits_k \mathit{\Gamma}_k(\widehat{Y})\omega^k\right) \wedge \omega = 0$. Hence by setting $\theta' = -i_{\widehat{Y}}\Omega$, we see that

$$\theta' \wedge \omega = \sum_k (\mathit{\Gamma}_k(\widehat{Y})\omega)\omega^k + \left(\sum_k \mathit{\Gamma}_k \omega^k(\widehat{Y})\right) \wedge \omega = \Omega g.$$

Therefore, for this choice of θ',

$$v_J \overline{v}_K(\theta', \Omega) = -\frac{1}{|J| + |K|} i_{\widehat{Y}} v_J \overline{v}_K(\Omega) = 0$$

if $|J| > q$ or $|K| > q$. Hence $\delta(\widetilde{d}\widetilde{\varphi})(\theta^u, \theta, \theta') = 0$ if φ is closed in WU_q.

Claim 4. *$[D_\sigma(\widetilde{\varphi})]$ is independent of the choice of φ and its lift $\widetilde{\varphi}$.*

It suffices to show that $D_\sigma(\widetilde{d}\widetilde{\varphi} + \alpha)$ is exact, where $\widetilde{\varphi} \in \widetilde{\mathrm{WU}}_q$ and $\alpha \in \widetilde{\mathcal{I}}_q$. First, $D_\sigma(\widetilde{d}\widetilde{\varphi}) = 0$ because $\widetilde{d}(\widetilde{d}\widetilde{\varphi}) = 0$. In order to show that $D_\sigma(\alpha)$ is exact for $\alpha \in \widetilde{\mathcal{I}}_q$, we first show the claim for $\alpha = \widetilde{u}_I v_J \overline{v}_K$ with $|J| > q$. If I is empty, then $\widetilde{d}\alpha = 0$ so that $D_\sigma(\alpha) = \Delta\alpha(\theta^u, \theta, \theta') = 0$. If I is non-empty, then by using the equalities $v_J(\Omega) = 0$ and $v_J(\theta', \Omega)v_{i_l}(\Omega) = 0$, we have

$$\begin{aligned} D_\sigma(\alpha) &= \Delta\alpha(\theta^u, \theta, \theta') \\ &= \sum_l (-1)^{l-1}(|J| + |K| + i_l)\left(v_J \overline{v}_K(v_{i_l} - \overline{v}_{i_l})\right)(\theta', \Omega)\widetilde{u}_{I(l)}(\theta^u, \theta) \\ &= \sum_l (-1)^l \, |J| \, v_J(\theta', \Omega)\overline{v}_K(\Omega)\overline{v}_{i_l}(\Omega)\widetilde{u}_{I(l)}(\theta^u, \theta) \\ &= d(|J| \, v_J(\theta', \Omega)\overline{v}_K(\Omega)\widetilde{u}_I(\theta^u, \theta)), \end{aligned}$$

where the last equality holds because $v_J(\theta', \Omega)$ is closed by Lemma 4.3.17. Similarly, $D_\sigma(\alpha)$ is exact if $|K| > q$.

This completes the proof of Theorem 4.3.18. □

Finally we show that the infinitesimal derivative of secondary classes coincide with the actual derivative when there is an actual deformation realizing the infinitesimal derivative.

An actual deformation induces an infinitesimal derivative as follows. We express by '˙' the derivative with respect to s at $0 \in S$, where $0 \in S$ is a distinguished point. We have

$$\dot{\varphi}_{j,s} = \dot{\gamma}_{ji,s} \circ \varphi_{i,0} + \frac{\partial \gamma_{ji,0}}{\partial z_i} \dot{\varphi}_{i,s}, \tag{4.3.20}$$

where $\dot{\gamma}_{ji,s}$ is regarded as a holomorphic vector field on an open set of $\mathbb{C}^q$.

Definition 4.3.21 ([53], [31]). The *infinitesimal deformation associated with* $\{\mathcal{F}_s\}$ is an element of $H^1(M; \Theta_{\mathcal{F}})$, where $\mathcal{F} = \mathcal{F}_0$, represented by the 1-cocycle of which the value on $U_i \cap U_j$ is the vector field

$$(\varphi_{j,0})^* \dot{\gamma}_{ji,s} \in \Theta_{\mathcal{F}}|_{U_i \cap U_j}.$$

Note that $\Theta_{\mathcal{F}}|_{U_i \cap U_j}$ is the pull-back of $\Theta_{T_{j,i}}$ by $\varphi_{j,0}$, where $T_{j,i}$ is an open subset of $\varphi_{j,0}(U_j)$.

Definition 4.3.22 ([40, Definition 2.7]). Let $\{\mathcal{F}_s\}$ be a smooth deformation of transversely holomorphic foliations of M, and let π_s be the projection from $T_{\mathbb{C}}M$ to $Q(\mathcal{F}_s)$. We fix a Hermitian metric on $T_{\mathbb{C}}M$. Assuming that s is small if necessary, we can find, by using the metric, a smooth family of splittings $T_{\mathbb{C}}M = E_s \oplus \nu_s$, where $\nu_s \cong Q(\mathcal{F}_s)$. Let π'_s be the projection from $T_{\mathbb{C}}M$ to ν_s. The infinitesimal deformation σ associated with $\{\mathcal{F}_s\}$ is the smooth section σ of $E_0^* \otimes Q(\mathcal{F}_0)$ defined by

$$\sigma(X) = -\pi_0 \left(\frac{\partial}{\partial s} \pi'_s(X) \bigg|_{s=0} \right).$$

Lemma 4.3.23 ([40, Lemma 2.8]). *σ is independent of the choice of the splitting.*

Proof. We will give an essentially the same as the one in [40] but slightly different proof. It suffices to work on a foliation chart. Let $\{e_1, \dots, e_q\}$ be a local frame of $Q(\mathcal{F}_0)$, Fix a splitting as above and let $\{e'_1, \dots, e'_q\}$ be the lift of $\{e_1, \dots, e_q\}$ to $T_{\mathbb{C}}M$. We may assume that there is a smooth family of frames $\{e'_1(s), \dots, e'_q(s)\}$ of ν_s such that $e'_i(0) = e'_i$, $i = 1, \dots, q$. If $X \in E_0$, then $\pi'_s(X) = \sum_{i=1}^q f_i(X,s)e'_i(s)$ holds for some functions f_i. Since $0 = \pi'_0(X) = \sum_{i=1}^q f_i(X,0)e'_i$, we have $f_i(X,0) = 0$ for any i. Hence

$$\begin{aligned}\frac{\partial}{\partial s}\pi'_s(X)\bigg|_{s=0} &= \sum_{i=1}^q \frac{\partial f_i}{\partial s}(X,s)\bigg|_{s=0} e'_i(0) + \sum_{i=1}^q f_i(X,0)\frac{\partial e'_i}{\partial s}(0) \\ &= \sum_{i=1}^q \frac{\partial f_i}{\partial s}(X,s)\bigg|_{s=0} e'_i(0).\end{aligned}$$

Therefore

$$\pi_0\left(\frac{\partial}{\partial s}\pi'_s(X)\bigg|_{s=0}\right) = \sum_{i=1}^q \frac{\partial f_i}{\partial s}(X,s)\bigg|_{s=0} e_i(0).$$

Let $T_{\mathbb{C}}M = E_s \oplus \nu'_s$ be another splitting and let $\{e''_1(s), \dots, e''_q(s)\}$ be the family of frames of ν'_s such that $\pi_s e''_i(s) = \pi_s e'_i(s) \in Q(\mathcal{F}_s)$. If we denote by π''_s the projection to ν'_s, then $\pi''_s(X) = \sum_{i=1}^q f_i(X,s)e''_i(s)$. In other words, f_i's are independent of the choice of splitting. Hence so is σ. $\square$

Lemma 4.3.24 ([40, Corollary 2.11]). $d_\nabla \sigma = 0$.

Proof. Let $X, Y \in E_0$. Then, $\nabla_X Z = \pi_0[X, \widetilde{Z}]$ for $Z \in Q(\mathcal{F})$, where $\widetilde{Z}$ is any lift of Z to $T_{\mathbb{C}}M$. Hence

$$\begin{aligned}d_\nabla\sigma(X,Y) &= \nabla_X\sigma(Y) - \nabla_Y\sigma(X) - \sigma([X,Y]) \\ &= \pi_0([X, \widetilde{\sigma(Y)}]) - \pi_0([Y, \widetilde{\sigma(X)}]) - \sigma([X,Y]), \\ &= \pi_0\left(\left[X, -\pi'_0\frac{\partial}{\partial s}\pi'_s(Y)\bigg|_{s=0}\right]\right) - \pi_0\left(\left[Y, -\pi'_0\frac{\partial}{\partial s}\pi'_s(X)\bigg|_{s=0}\right]\right) \\ &\quad + \pi_0\left(\frac{\partial}{\partial s}\pi'_s[X,Y]\bigg|_{s=0}\right).\end{aligned}$$

If $v \in E_s$, then $\pi'_s \frac{\partial}{\partial s}\pi'_s(v) = \frac{\partial}{\partial s}\pi'_s(v)$. Indeed, $\pi'_s \circ \pi'_s = \pi'_s$ implies that

$$\left(\frac{\partial}{\partial s}\pi'_s\right)\pi'_s + \pi'_s\left(\frac{\partial}{\partial s}\pi'_s\right) = \frac{\partial}{\partial s}\pi'_s.$$

Hence $\frac{\partial}{\partial s}\pi'_s(v) \in \nu_s$. Therefore

$$\pi'_0\left(\left[X, -\pi'_0\frac{\partial}{\partial s}\pi'_s(Y)\Big|_{s=0}\right]\right) = -\pi'_0\left(\left[X, \frac{\partial}{\partial s}\pi'_s(Y)\Big|_{s=0}\right]\right).$$

Similarly,

$$\pi'_0\left(\left[Y, -\pi'_0\frac{\partial}{\partial s}\pi'_s(X)\Big|_{s=0}\right]\right) = -\pi'_0\left(\left[Y, \frac{\partial}{\partial s}\pi'_s(X)\Big|_{s=0}\right]\right).$$

On the other hand,

$$\begin{aligned} &\frac{\partial}{\partial s}\pi'_s\left[X - \pi'_s(X), Y - \pi'_s(Y)\right]\Big|_{s=0} \\ &= \frac{\partial}{\partial s}\pi'_s\Big|_{s=0}\left[X - \pi'_0(X), Y - \pi'_0(Y)\right] \\ &\quad + \pi'_0\left[-\frac{\partial}{\partial s}\pi'_s(X)\Big|_{s=0}, Y - \pi'_0(Y)\right] + \pi'_0\left[X - \pi'_0(X), -\frac{\partial}{\partial s}\pi'_s(Y)\Big|_{s=0}\right] \\ &= \frac{\partial}{\partial s}\pi'_s\Big|_{s=0}[X,Y] - \pi'_0\left[\frac{\partial}{\partial s}\pi'_s(X)\Big|_{s=0}, Y\right] - \pi'_0\left[X, \frac{\partial}{\partial s}\pi'_s(Y)\Big|_{s=0}\right] \end{aligned}$$

because $X, Y \in E$. Therefore $d_\nabla\sigma(X,Y) = \frac{\partial}{\partial s}\pi'_s\left[X - \pi'_s(X), Y - \pi'_s(Y)\right]\Big|_{s=0}$. Since $X - \pi'_s(X), Y - \pi'_s(Y) \in E_s$ and E_s is integrable, $d_\nabla\sigma(X,Y) = 0$. □

Remark 4.3.25. If E_s are not necessarily integrable, $d_\nabla\sigma$ is called the integrability tensor in [40].

Definition 4.3.26. Let $\{\mathcal{F}_s\}$ be a smooth family of transversely holomorphic foliations of M and let σ be as above. The element $[\sigma]$ in $H^1(M;\Theta_{\mathcal{F}})$ is also called the *infinitesimal deformation associated with* $\{\mathcal{F}_s\}$.

Given a smooth deformation of $\mathcal{F}$, two infinitesimal deformations are defined. By [40, Theorem 2.5] and [26, Theorem 1.27] (cf. Lemma 4.3.3), we have the following.

Theorem 4.3.27. *The infinitesimal deformations defined in Definitions 4.3.21 and 4.3.26 coincide each other.*

PROOF. We regard $\{\dot{\varphi}_{i,s}\}$ as a family of sections of $Q(\mathcal{F})$. From (4.3.20), we have $\dot{\varphi}_{j,s} - \dot{\varphi}_{i,s} = \varphi_{j,0}^*\dot{\gamma}_{ji,s}$. If we define a section s of $E_0^* \otimes Q(\mathcal{F}_0)$ by $s = -d_\nabla \dot{\varphi}_{i,s}$ on U_i, then s is well-defined again by (4.3.20). It is easy to see that s coincides with σ in Definition 4.3.22. □

THEOREM 4.3.28 ([42, Theorem 3.23]). *Let $\{\mathcal{F}_s\}_{s\in\mathbb{R}}$ be a differentiable family of transversely holomorphic foliations of M, of complex codimension q. If $\beta \in H^1(M;\Theta_{\mathcal{F}})$ is the infinitesimal deformation of $\mathcal{F}_0$ associated with $\{\mathcal{F}_s\}$, then*

$$D_\beta(f) = \frac{\partial}{\partial s} f(\mathcal{F}_s)\bigg|_{s=0}$$

for $f \in H^*(\mathrm{WU}_q)$.

PROOF. Let P_s be the principal bundle associated with $Q(\mathcal{F}_s)$. We may assume that s is small so that P_s is canonically isomorphic to P_0. Hence there are families of canonical forms ω_s and complex Bott connections θ_s on $Q(\mathcal{F}_s)$ such that $d\omega_s = -\theta_s \wedge \omega_s$. If we set $\dot{\omega}_s = \frac{\partial}{\partial s}\omega_s\Big|_{s=0}$ and $\dot{\theta}_s = \frac{\partial}{\partial s}\theta_s\Big|_{s=0}$, then

$$d\dot{\omega}_s = -\dot{\theta}_s \wedge \omega_s - \theta_s \wedge \dot{\omega}_s.$$

On the other hand, if σ is the infinitesimal deformation associated with $\{\mathcal{F}_s\}$, then a 1-form $\widehat{\sigma}$ on P representing σ is given as follows. Let $\widehat{\mathcal{F}_s}$ be the pull-back of $\mathcal{F}_s$ by the projection to M. Let $\omega_s = {}^t(\omega_s^1, \dots, \omega_s^q)$ be the canonical form on $Q(\widehat{\mathcal{F}_s})$. Then

$$\widehat{\sigma}(\widehat{X}) = -\pi_0\left(\frac{\partial}{\partial s}(\omega_s^1(X)\widetilde{e}_1(s) + \cdots + \omega_s^q(X)\widetilde{e}_q(s))\bigg|_{s=0}\right),$$

where $\widetilde{e}_i(s)$, $i = 1, \dots, q$, are defined as in the proof of Lemma 4.3.23. Since $\frac{\partial}{\partial s}\widetilde{e}_i(s)\Big|_{s=0}$ belongs to the kernel of π_0, one has

$$\begin{aligned}\widehat{\sigma}(\widehat{X}) &= -\pi_0\left(\frac{\partial}{\partial s}\omega_s^1(X)\bigg|_{s=0}\widetilde{e}_1(0) + \cdots + \frac{\partial}{\partial s}\omega_s^q(X)\bigg|_{s=0}\widetilde{e}_q(0)\right)\\ &= -\frac{\partial}{\partial s}\omega_s^1(X)\bigg|_{s=0} e_1 - \cdots - \frac{\partial}{\partial s}\omega_s^q(X)\bigg|_{s=0} e_q\\ &= -\dot{\omega}(\widehat{X}).\end{aligned}$$

It follows that $\dot{\theta}_s$ can be chosen as an infinitesimal derivative of θ_0 with respect to σ. Therefore Theorem 4.3.28 follows from Proposition 4.2.6. □

The Bott class is known to vary continuously. Hence its infinitesimal derivative is of interest. The above construction gives the infinitesimal derivative of the imaginary part of the Bott class. If $K_{\mathcal{F}}$ is trivial, then the infinitesimal derivative of the Bott class including the real part is constructed by Heitsch [42]. It is still possible to define the derivative without the triviality of $K_{\mathcal{F}}$, and the derivative is an element of $H^{2q+1}(M;\mathbb{C})$. Indeed, we have the following

THEOREM 4.3.29 ([10, Theorems 2.14 and 2.19]). *Let $\mu \in H^1(M;\Theta_{\mathcal{F}})$ and σ be a representative of μ. Let θ be a Bott connection and θ' be an infinitesimal derivative of θ with respect to σ. Then, the infinitesimal derivative of the Bott class is represented by $(-2\pi\sqrt{-1})^{q+1}(q+1)\theta' \wedge (d\theta)^q$.*

We denote by $D_\mu B_q(\mathcal{F})$ the infinitesimal derivative of the Bott class. We have $D_\mu \xi_q(\mathcal{F}) = -2\,\mathrm{Im}\, D_\mu B_q(\mathcal{F})$. It is known that $D_\mu B_q(\mathcal{F})$ can be represented in terms of the projective Schwarzian derivatives in the Čech–de Rham complex ([57] for $q = 1$, [10, Theorem 4.10] for arbitrary q).

Let $I_q(\mathcal{F})$ be the space of differential forms on open sets of M which are locally of the form $\omega \wedge dz^1 \wedge \cdots \wedge dz^q$. It follows from Theorem 4.3.29 that $D_\mu B_q(\mathcal{F})$ can be represented by an element of $I_q(\mathcal{F})$. Hence we have the following

COROLLARY 4.3.30 (cf. [10, Corollary 4.16]). *Let J be an index set as in Notation 1.1.11. Let $\mathrm{ch}_J(\mathcal{F}) = \chi^{\mathbb{C}}_{\mathcal{F}}(v_J)$, where $\chi^{\mathbb{C}}_{\mathcal{F}}$ is the characteristic mapping (Definition 1.1.17). If $J \neq \varnothing$, then $D_\mu B_q(\mathcal{F})\,\mathrm{ch}_J(\mathcal{F})$ is trivial. In particular, $D_\mu B_q(\mathcal{F})\,\mathrm{ch}_1(\mathcal{F})^k$ and $D_\mu \xi_q(\mathcal{F})\,\mathrm{ch}_1(\mathcal{F})^k$ are trivial if $k > 0$.*

PROOF. The class $\mathrm{ch}_J(\mathcal{F})$ is represented by an element of $I_1(\mathcal{F})$. Hence the first part follows from the Bott vanishing theorem. By setting $J = (k, 0, \dots, 0)$, we see that $D_\mu B_q(\mathcal{F})\,\mathrm{ch}_1(\mathcal{F})^k$ is trivial if $k > 0$. Since $\mathrm{ch}_1(\mathcal{F}) \in H^2(M;\mathbb{R})$, we have $\overline{D_\mu B_q(\mathcal{F})}\,\mathrm{ch}_1(\mathcal{F})^k = \overline{D_\mu B_q(\mathcal{F})\,\mathrm{ch}_1(\mathcal{F})^k}$. The last part follows from this equality. □

Note that Corollary 4.3.30 gives an alternative proof of Theorem B2.