CHAPTER 1

Definitions of Transversely Holomorphic Foliations and Complex Secondary Classes

1.1. Basic Notions

In this monograph, foliations are assumed to be regular (without singularities) unless otherwise mentioned.

DEFINITION 1.1.1. Let M be a manifold without boundary. A decomposition of M into immersed submanifolds $\{L_\alpha\}_{\alpha\in A}$, called *leaves*, is a *foliation* of M if there is an integer q and an atlas $\{U_\lambda\}_{\lambda\in\Lambda}$ of M which satisfy the following conditions:

1) For each λ, there is a submersion $f_\lambda\colon U_\lambda \to \mathbb{R}^q$ such that each connected component of $L_\alpha \cap U_\lambda$ is a connected component of a fiber of f_λ.
2) Let $\varphi_{\mu\lambda}$ be the transition function from U_λ to U_μ. Then, there exists a diffeomorphism $\gamma_{\mu\lambda}\colon p_\lambda(U_\lambda \cap U_\mu) \to p_\mu(U_\lambda \cap U_\mu)$ such that $\gamma_{\mu\lambda} \circ f_\lambda = f_\mu \circ \varphi_{\mu\lambda}$.

Such an atlas is called a *foliation atlas*. The integer q is called the (real) *codimension* of the foliation.

REMARK 1.1.2.

1) We may assume that fibers of f_λ are homeomorphic, and U_λ is homeomorphic to $V_\lambda \times B_\lambda$ in a way such that f_λ is the projection to the second factor, where $B_\lambda = f_\lambda(U_\lambda)$ and V_λ is the fiber of f_λ.
2) If we assume that each f_λ is only continuous and that $\gamma_{\nu\mu}\gamma_{\mu\lambda} = \gamma_{\nu\lambda}$ for any $\lambda, \mu, \nu \in \Lambda$, then structures as above is called *Γ_q-structures*, where Γ_q denotes the pseudogroup of local diffeomorphisms of $\mathbb{R}^q$.

A foliation is said to be transversely of class C^r, $r = 0, 1, \ldots, \infty, \omega$, if every $\gamma_{\mu\lambda}$ as in 2) of Definition 1.1.1 is of class C^r. Transversely holomorphic foliations can be also considered in a similar manner. A precise definition is as follows.

DEFINITION 1.1.3. Let M be a manifold without boundary and let $\mathcal{F}$ be a foliation of real codimension $2q$ of M. $\mathcal{F}$ is *transversely holomorphic* if there is a foliation atlas $\{U_\lambda\}$ such that

1) the image of f_λ is contained in $\mathbb{C}^q$ for any λ,
2) each $\gamma_{\mu\lambda}$ is a biholomorphic local diffeomorphism, where a biholomorphic local diffeomorphism is by definition a biholomorphic diffeomorphism from an open subset of $\mathbb{C}^q$ to an open subset of $\mathbb{C}^q$.

The integer q is called the *complex codimension* of $\mathcal{F}$ and denoted by $\operatorname{codim}_{\mathbb{C}}\mathcal{F}$.

If each f_λ is supposed only to be a continuous function, then structures as above is called $\Gamma_q^{\mathbb{C}}$*-structures*, where $\Gamma_q^{\mathbb{C}}$ denotes the pseudogroup of biholomorphic local diffeomorphisms of $\mathbb{C}^q$ (cf. [36]).

There are some relevant vector bundles associated with transversely holomorphic foliations.

DEFINITION 1.1.4. Let $T\mathcal{F}$ be the subbundle of TM spanned by vectors tangent to the leaves of $\mathcal{F}$. Let $T_{\mathbb{C}}M = TM \otimes \mathbb{C}$ and let E be the subbundle of $T_{\mathbb{C}}M$ locally spanned over $\mathbb{C}$ by vectors tangent to the leaves and transversal antiholomorphic vectors $\dfrac{\partial}{\partial \bar{z}^1}, \ldots, \dfrac{\partial}{\partial \bar{z}^q}$. The integrability condition for $\mathcal{F}$ ensures that E is well-defined. The quotient bundle $Q(\mathcal{F}) = T_{\mathbb{C}}M/E$ is called the *complex normal bundle* of $\mathcal{F}$, and the complex line bundle $K_{\mathcal{F}} = \bigwedge^q Q(\mathcal{F})^*$ is called the *canonical line bundle* of $\mathcal{F}$. The quotient bundle $Q_{\mathbb{R}}(\mathcal{F}) = TM/T\mathcal{F}$ is called the (*real*) *normal bundle* of $\mathcal{F}$. We have $Q(\mathcal{F}) \oplus \overline{Q(\mathcal{F})} \cong Q_{\mathbb{R}}(\mathcal{F}) \otimes \mathbb{C}$.

If $K_{\mathcal{F}}$ is trivial, then the *Bott class* is defined as follows [19]. We fix a trivialization ω of $K_{\mathcal{F}}$. Then ω is naturally a $\mathbb{C}$-valued q-form. It follows from the integrability of $\mathcal{F}$ and its transversal complex structure that there is a $\mathbb{C}$-valued

1-form η such that $d\omega = 2\pi\sqrt{-1}\eta \wedge \omega$ (cf. [26]). The differential form $\eta \wedge (d\eta)^q$ is closed by the Bott vanishing theorem (Theorem 1.1.12), and it represents a cohomology class which is independent of the choices we made [19].

DEFINITION 1.1.5. The cohomology class represented by $\eta \wedge (d\eta)^q$ is called the *Bott class* and denoted by $\mathrm{Bott}_q(\mathcal{F})$.

The Bott class is often referred as the complex Godbillon–Vey class. However, we will adopt Definition 1.1.5 because the Godbillon–Vey class and the Bott class will appear together in the following sections so that it is necessary to distinguish them. In addition, Bott introduced the Bott class already in [17] while the Godbillon–Vey class appeared in [34]. See also a comment of Bott in [18, p. 49].

The following is a fundamental example given by Bott [19] (see also [17] and [13]).

EXAMPLE 1.1.6. Let $X_\lambda = \displaystyle\sum_{i=0}^n \lambda_i z_i \frac{\partial}{\partial z_i}$ be a holomorphic vector field on $\mathbb{C}^{n+1}$, where $(z_0, \ldots, z_n)$ are the standard coordinates and each λ_i is a non-zero complex number. Assume that the origin is of Poincaré type, equivalently, assume that the convex hull of $\lambda_0, \ldots, \lambda_n$ does not contain the origin. Let $\widetilde{\mathcal{F}}_\lambda$ be the foliation of $\mathbb{C}^{n+1}$ by the orbits of X_λ. Then, $\widetilde{\mathcal{F}}_\lambda$ induces a foliation $\mathcal{F}_\lambda$ of the unit sphere S^{2n+1} because S^{2n+1} is transversal to $\widetilde{\mathcal{F}}_\lambda$. Indeed, let $\mathscr{U} = \{U_\lambda\}$ be a foliation atlas for $\widetilde{\mathcal{F}}_\lambda$ and let $\mathscr{U}' = \{U'_\lambda\}$ be a subfamily of $\mathscr{U}$ such that $U'_\lambda \cap S^{2n+1} \neq \varnothing$ for any λ and that $\displaystyle\bigcup_\lambda U'_\lambda \supset S^{2n+1}$. By taking a refinement, we may assume that $U'_\lambda \cap S^{2n+1}$ is homeomorphic to $V'_\lambda \times B'_\lambda$, where V' is an open set in $\mathbb{R}$ and B'_λ is an open ball in $\mathbb{C}^n$. Then, the transition functions are restriction of original transition functions so that we can make use of coordinates for $\widetilde{\mathcal{F}}_\lambda$ in the transverse direction in order to introduce a transverse holomorphic structure to $\mathcal{F}_\lambda$. Thus $\mathcal{F}_\lambda$ is a transversely holomorphic foliation (flow) of complex codimension n. It is known that $\mathrm{Bott}_n(\mathcal{F}_\lambda) = \dfrac{(\lambda_0 + \cdots + \lambda_n)^{n+1}}{\lambda_0 \cdots \lambda_n}[S^{2n+1}]$, where $[S^{2n+1}]$ is the fundamental class of S^{2n+1}.

Example 1.1.6 shows that the Bott class is non-trivial and admits continuous variations. See Example 5.6 for another example of the same kind.

DEFINITION 1.1.7. A connection ∇ on $Q(\mathcal{F})$ is said to be a *complex Bott connection* if ∇ satisfies

$$\nabla_X Y = \mathcal{L}_X Y$$

for any sections X of E and Y of $Q(\mathcal{F})$, where $\mathcal{L}_X$ denotes the Lie derivative with respect to X. It is equivalent to the condition $\nabla_X Y = \pi[X, \widetilde{Y}]$, where $\pi\colon T_{\mathbb{C}}M \to Q(\mathcal{F})$ is the natural projection and $\widetilde{Y}$ is any lift of Y to $T_{\mathbb{C}}M$. A connection on $K_{\mathcal{F}}$ is also called a *complex Bott connection* if it is induced from a complex Bott connection on $Q(\mathcal{F})$. Real Bott connections are defined on $Q_{\mathbb{R}}(\mathcal{F})$ in a similar way. Namely, a connection ∇ on $Q_{\mathbb{R}}(\mathcal{F})$ is said to be a *real Bott connection* if $\nabla_X Y = \pi'[X, \widetilde{Y}]$ holds for sections X of $T\mathcal{F}$ and Y of $Q_{\mathbb{R}}(\mathcal{F})$, where $\pi'\colon TM \to Q_{\mathbb{R}}(\mathcal{F})$ is the natural projection and $\widetilde{Y}$ is any lift of Y to TM.

Real Bott connections are usually called Bott connections or basic connections. In this monograph, Bott connections always mean complex Bott connections unless otherwise mentioned.

REMARK 1.1.8. The differential form η in Definition 1.1.5 is the connection form of a Bott connection on $K_{\mathcal{F}}$ with respect to ω.

The *complex secondary classes* are constructed as follows. We first remark that the Chern forms and classes will be denoted by v_i because the letter c_i usually represents a Pontrjagin form or class (cf. Definitions 1.1.18 and 1.1.19). It is also the case in this monograph.

We recall the definition of the Chern–Simons forms [22].

DEFINITION 1.1.9. Let ∇_0 and ∇_1 be connections on $Q(\mathcal{F})$ and let θ_0 and θ_1 be respective connection forms. Let f be an invariant polynomial on $\mathrm{GL}(q; \mathbb{C})$ of degree k. We set $\theta_t = (1-t)\theta_0 + t\theta_1$ and

$$\Delta_f(\theta_1, \theta_0) = \int_0^1 kf(\theta_1 - \theta_0, \Omega_t, \ldots, \Omega_t)\, dt,$$

where $\Omega_t = d\theta_t + \theta_t \wedge \theta_t$ is the curvature form of θ_t.

It is well-known that $\Delta_f(\theta_1, \theta_0)$ is well-defined. It is also well-known that $d\Delta_f(\theta_1, \theta_0) = f(\Omega_1) - f(\Omega_0)$ and $\Delta_f(\theta_0, \theta_1) = -\Delta_f(\theta_1, \theta_0)$. See Chapter 4 for additional properties of $\Delta_f(\theta_1, \theta_0)$.

DEFINITION 1.1.10. Let $\mathcal{F}$ be a transversely holomorphic foliation of M, of complex codimension q. Let ∇ be a complex Bott connection on $Q(\mathcal{F})$ and let ∇^u be a unitary connection on $Q(\mathcal{F})$ with respect to some Hermitian metric on $Q(\mathcal{F})$. We denote by θ and θ^u the connection forms of ∇ and ∇^u, respectively. Let c_i be the Chern polynomial of degree i, and

$$\begin{aligned} v_i(\Omega) &= c_i(\Omega), \\ \overline{v}_i(\Omega) &= \overline{c_i(\Omega)}, \\ \widetilde{u}_i(\theta, \theta^u) &= \Delta_{c_i}(\theta, \theta^u) - \overline{\Delta_{c_i}(\theta, \theta^u)}, \end{aligned}$$

where Ω is the curvature form of θ and $\overline{\omega}$ denotes the complex conjugate of a differential form ω.

By definition, we have

$$\det\left(tI_q - \frac{1}{2\pi\sqrt{-1}}\Omega\right) = t^q + v_1(\Omega)t^{q-1} + \cdots + v_q(\Omega),$$

where I_q is the identity matrix.

NOTATION 1.1.11. We denote by $\mathbb{C}[v_1, \ldots, v_q]$ the polynomial ring generated by $v_1, \ldots, v_q$ with coefficients in $\mathbb{C}$. Let $J = (j_1, j_2, \ldots, j_q)$, where each j_r is a non-negative integer. We set $v_J = v_1^{j_1} v_2^{j_2} \cdots v_q^{j_q}$ and $|J| = j_1 + 2j_2 + \cdots + qj_q$. If Ω is the curvature form of a connection, then set $v_J(\Omega) = v_1(\Omega)^{j_1} \cdots v_q(\Omega)^{j_q}$. Similarly, $\overline{v}_J$ and $\overline{v}_J(\Omega)$ are defined for an index set J as above. Index sets for $\overline{v}_i$'s are usually denoted by K.

The following theorem is crucial in the construction.

Theorem 1.1.12 (Bott vanishing theorem [19]). *Let* $v_J \in \mathbb{C}[v_1, \ldots, v_q]$ *be a monomial and assume that* $|J| > q$. *If* Ω *is the curvature form of a Bott connection, then* $v_J(\Omega) = 0$ *as differential forms.*

If we calculate Chern forms using Bott connections, we have Chern forms and their complex conjugates. They do not coincide in general, however, they are cohomologous. We can find natural primitives by using foliations and Chern–Simons forms. On the other hand, we have the Bott vanishing theorem. These facts lead the following definition.

Definition 1.1.13. Let WU_q be the differential graded algebra (DGA for short) defined as follows. First set the degree of v_i to be $2i$, and let $\mathcal{I}_q$ be the ideal in $\mathbb{C}[v_1, \ldots, v_q]$ generated by the monomials of degree greater than $2q$. Let $\mathbb{C}_q[v_1, \ldots, v_q] = \mathbb{C}[v_1, \ldots, v_q]/\mathcal{I}_q$ and define $\mathbb{C}_q[\overline{v}_1, \ldots, \overline{v}_q]$ by replacing v_i by $\overline{v}_i$. We set

$$\mathrm{WU}_q = \mathbb{C}_q[v_1, \ldots, v_q] \otimes \mathbb{C}_q[\overline{v}_1, \ldots, \overline{v}_q] \otimes \bigwedge[\widetilde{u}_1, \ldots, \widetilde{u}_q].$$

The exterior derivative on WU_q is defined by requiring $d\widetilde{u}_i = v_i - \overline{v}_i$ and $dv_i = d\overline{v}_i = 0$. The degree of $\widetilde{u}_i$ is set to be $2i - 1$.

If $Q(\mathcal{F})$ is trivial, then all the Chern classes are trivial. By fixing a trivialization of $Q(\mathcal{F})$, one can find a primitive u_i of v_i. Indeed, the following DGA is useful for foliations with trivialized normal bundles.

Definition 1.1.14. We set

$$\mathrm{W}_q^{\mathbb{C}} = \left(\mathbb{C}_q[v_1, \ldots, v_q] \otimes \bigwedge[u_1, u_2, \ldots, u_q]\right) \wedge \left(\mathbb{C}_q[\overline{v}_1, \ldots, \overline{v}_q] \otimes \bigwedge[\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_q]\right),$$

where the degree of u_i and $\overline{u}_i$ are $2i - 1$, and the differential is defined by requiring that $du_i = v_i$, $d\overline{u}_i = \overline{v}_i$ and $dv_i = d\overline{v}_i = 0$.

DEFINITION 1.1.15. The cohomology classes in $H^*(\mathrm{WU}_q)$ which involve $\widetilde{u}_i$'s are called *complex secondary classes.* The cohomology classes in $H^*(\mathrm{W}_q^{\mathbb{C}})$ which involve u_i's or $\overline{u}_i$'s are also called *complex secondary classes.*

Let $\mathcal{F}$ be a transversely holomorphic foliation. We fix a Hermitian metric on the complex normal bundle $Q(\mathcal{F})$ and let θ^u be a unitary connection. Let θ be a Bott connection on $Q(\mathcal{F})$ and Ω the connection form. Let f be the mapping from WU_q to the set of $\mathbb{C}$-valued differential forms such that

$$f(v_i) = v_i(\Omega),$$
$$f(\overline{v}_i) = \overline{v}_i(\Omega),$$
$$f(\widetilde{u}_i) = \widetilde{u}_i(\theta, \theta^u).$$

If $Q(\mathcal{F})$ is trivial, then we choose a trivialization, say s, and let θ^s be the flat connection determined by s. We define a mapping $\widehat{f}_s$ from $\mathrm{W}_q^{\mathbb{C}}$ to $\mathbb{C}$-valued differential forms by setting

$$\widehat{f}_s(v_i) = v_i(\Omega),$$
$$\widehat{f}_s(\overline{v}_i) = \overline{v}_i(\Omega),$$
$$\widehat{f}_s(u_i) = u_i(\theta, \theta^s),$$
$$\widehat{f}_s(\overline{u}_i) = \overline{u_i(\theta, \theta^s)}.$$

Then, we have the following.

THEOREM 1.1.16 (Bott [19]). *f induces a homomorphism $\chi_{\mathcal{F}}^{\mathbb{C}}$ from $H^*(\mathrm{WU}_q)$ to $H^*(M;\mathbb{C})$ which is independent of the choice of connections and metrics. If $Q(\mathcal{F})$ is trivial, then $\widehat{f}_s$ induces a homomorphism $\widehat{\chi}_{\mathcal{F},s}^{\mathbb{C}}$ from $H^*(\mathrm{W}_q^{\mathbb{C}})$ to $H^*(M;\mathbb{C})$ which is independent of the choice of connections and depends on the homotopy classes of the trivializations.*

DEFINITION 1.1.17. The homomorphisms $\chi_{\mathcal{F}}^{\mathbb{C}}$ and $\widehat{\chi}_{\mathcal{F},s}^{\mathbb{C}}$ in Theorem 1.1.16 are called the *characteristic mappings.* The image of $\omega \in H^*(\mathrm{WU}_q)$ under $\chi_{\mathcal{F}}^{\mathbb{C}}$ is denoted by $\chi^{\mathbb{C}}_{\mathcal{F}}(\omega)$ or by $\omega(\mathcal{F})$, and the image of $\omega \in H^*(\mathrm{W}^{\mathbb{C}}_q)$ under $\widehat{\chi}^{\mathbb{C}}_{\mathcal{F},s}$ is denoted by $\widehat{\chi}^{\mathbb{C}}_{\mathcal{F},s}(\omega)$ or by $\omega(\mathcal{F}, s)$, respectively.

Given a transversely holomorphic foliation, one can consider also real secondary classes by forgetting the transverse holomorphic structure. A DGA which plays the role of WU_q for real foliations is defined as follows. First, Pontrjagin forms instead of Chern forms appear. If we denote by c_i the i-th Pontrjagin form, then c_i is exact if i is odd. If we replace complex Bott connections by real Bott connections, then Theorem 1.1.12 holds in the same form. Hence we are led to the following definition.

Definition 1.1.18. Let $\mathbb{R}_q[c_1, \dots, c_q] = \mathbb{R}[c_1, \dots, c_q]/\mathcal{I}'_q$, where the degree of c_i is set to be $2i$ and $\mathcal{I}'_q$ is the ideal generated by monomials of degree greater than $2q$. Let q' be the largest odd integer less than $q + 1$. We set

$$\mathrm{WO}_q = \mathbb{R}_q[c_1, \dots, c_q] \otimes \bigwedge[h_1, h_3, \dots, h_{q'}],$$

where the degree of h_i is $2i - 1$ and the differential is defined by requiring $dh_i = c_i$ and $dc_i = 0$.

If the real normal bundle $Q_{\mathbb{R}}(\mathcal{F})$ is trivial, then Pontrjagin forms are exact. Hence the following DGA is suitable.

Definition 1.1.19. We set

$$\mathrm{W}_q = \mathbb{R}_q[c_1, \dots, c_q] \otimes \bigwedge[h_1, h_2, \dots, h_q],$$

where the degree of h_i is $2i - 1$ and the differential is defined by requiring $dh_i = c_i$ and $dc_i = 0$.

Definition 1.1.20. The elements of $H^*(\mathrm{WO}_q)$ and $H^*(\mathrm{W}_q)$ which involve h_i's are called *real secondary classes*.

Theorem 1.1.16 holds for real foliations as well, and characteristic mappings are also defined. Let θ and θ^m be a Bott connection and a metric connection with respect to a Riemannian metric on $Q_{\mathbb{R}}(\mathcal{F})$. We denote by Ω the curvature form of θ.

Then, we can consider a mapping g from WO_q to the set of differential forms such that

$$g(c_j) = c_j(\Omega),$$
$$g(h_i) = \Delta_{c_i}(\theta, \theta^m) \quad \text{if } i \text{ is odd.}$$

If $Q_{\mathbb{R}}(\mathcal{F})$ is trivial, then we choose a trivialization s. Let θ^s be the flat connection determined by s, and set $\widehat{g}_s(h_i) = \Delta_{c_i}(\theta, \theta^s)$ for any i. The mappings g and $\widehat{g}_s$ induce homomorphisms on the cohomology, which we denote by $\chi_{\mathcal{F}}$ and $\widehat{\chi}_{\mathcal{F},s}$, respectively. The homomorphism $\chi_{\mathcal{F}}$ is independent of the choice of connections and metrics, and $\widehat{\chi}_{\mathcal{F},s}$ is independent of the choice of Bott connections and depends on the homotopy type of s.

Definition 1.1.21. The homomorphisms $\chi_{\mathcal{F}} \colon H^*(\mathrm{WO}_q) \to H^*(M;\mathbb{R})$ and $\widehat{\chi}_{\mathcal{F},s} \colon H^*(\mathrm{W}_q) \to H^*(M;\mathbb{R})$ are called the *characteristic mappings*. The image of $\omega \in H^*(\mathrm{WO}_q)$ under $\chi_{\mathcal{F}}$ is denoted by $\chi_{\mathcal{F}}(\omega)$ or by $\omega(\mathcal{F})$, and the image of $\omega \in H^*(\mathrm{W}_q)$ under $\widehat{\chi}_{\mathcal{F},s}$ is denoted by $\widehat{\chi}_{\mathcal{F},s}(\omega)$ or by $\omega(\mathcal{F}, s)$, respectively.

For studying transversely holomorphic foliations, WO_{2q} and W_{2q} are relevant, where $(2q)' = 2q-1$. It is convenient to consider $\mathrm{WO}_{2q} \otimes \mathbb{C}$ and $\mathrm{W}_{2q} \otimes \mathbb{C}$ instead of WO_{2q} and W_{2q}. In what follows, we denote $\mathrm{WO}_{2q} \otimes \mathbb{C}$ and $\mathrm{W}_{2q} \otimes \mathbb{C}$ again by WO_{2q} and W_{2q}, respectively, and the coefficients of cohomology groups are always chosen in $\mathbb{C}$ unless otherwise stated. We also consider $\mathrm{W}_q \otimes \mathbb{C}$ instead of W_q. If we identify W_q with $\mathbb{C}_q[v_1, \dots, v_q] \otimes \bigwedge[u_1, \dots, u_q]$, then W_q is naturally a sub-DGA of $\mathrm{W}_q^{\mathbb{C}}$. It is often enough to study $H^*(\mathrm{W}_q)$ instead of $H^*(\mathrm{W}_q^{\mathbb{C}})$ when we study transversely holomorphic foliations with trivialized normal bundles.

Remark 1.1.22. The DGA's WU_q, $\mathrm{W}_q^{\mathbb{C}}$, WO_q and W_q also arise when studying the Gel'fand–Fuks cohomology (cf. [28]).

Notation 1.1.23. If $I = \{i_1, i_2, \dots, i_r\}$, where $i_1 < i_2 < \cdots < i_r$, then we set $\widetilde{u}_I = \widetilde{u}_{i_1}\widetilde{u}_{i_2}\cdots\widetilde{u}_{i_r}$ by omitting the symbol '$\wedge$'. If I is empty, then we set $\widetilde{u}_I = 1$.

We define u_I, $\overline{u}_I$ and h_I in a similar way. Classes in $H^*(\mathrm{WU}_q)$, $H^*(\mathrm{W}_q^{\mathbb{C}})$, $H^*(\mathrm{WO}_q)$ and $H^*(\mathrm{W}_q)$ are usually denoted by their representatives by abuse of notation.

Remark 1.1.24 (See [36] for details). Let $\Gamma_q^{\mathbb{C}}$ be the pseudogroup of biholomorphic local diffeomorphisms of $\mathbb{C}^q$. Transversely holomorphic foliations are $\Gamma_q^{\mathbb{C}}$-structures, and there is a classifying space, denoted by $B\Gamma_q^{\mathbb{C}}$. It is naturally equipped with a $\Gamma_q^{\mathbb{C}}$-structure, which is universal in the following sense. If $\mathcal{F}$ is a transversely holomorphic foliation of a manifold M, then there is a mapping $f\colon M \to B\Gamma_q^{\mathbb{C}}$ such that $\mathcal{F}$ is the pull-back of the $\Gamma^{\mathbb{C}}$-structure by f. The mapping f is unique up to homotopy and called the *classifying mapping* for $\mathcal{F}$. It is known that classifying spaces and secondary characteristic classes are related as follows. There is a mapping $\chi^{\mathbb{C}}\colon H^*(\mathrm{WU}_q) \to H^*(B\Gamma_q^{\mathbb{C}})$ which makes the following diagram commutative for any complex codimension-q transversely holomorphic foliation $\mathcal{F}$ of a manifold M:

$$\begin{array}{ccc} & & H^*(B\Gamma_q^{\mathbb{C}}) \\ & \overset{\chi^{\mathbb{C}}}{\nearrow} & \\ H^*(\mathrm{WU}_q) & & \Big\downarrow f^* \\ & \underset{\chi_{\mathcal{F}}^{\mathbb{C}}}{\searrow} & \\ & & H^*(M), \end{array}$$

where $f\colon M \to B\Gamma_q^{\mathbb{C}}$ is the classifying mapping for $\mathcal{F}$. The mapping $\chi^{\mathbb{C}}$ is called the *universal characteristic mapping*. Let $B\Gamma_q$ be the classifying space for real codimension-q smooth foliations and let $B\overline{\Gamma_q}$ be the homotopy fiber of the natural mapping $B\Gamma_q \to \mathrm{BGL}(q;\mathbb{R})$ given by taking the normal bundle. Similarly let $B\overline{\Gamma_q^{\mathbb{C}}}$ be the homotopy fiber of $B\Gamma_q^{\mathbb{C}} \to \mathrm{BGL}(q;\mathbb{C})$. The space $B\overline{\Gamma_q}$ is the classifying space for real codimension-q foliations with trivialized normal bundle, and $B\overline{\Gamma_q^{\mathbb{C}}}$ is the classifying space for complex codimension-q foliations with trivialized complex normal bundle.

There are universal characteristic mappings as follows:

$$\begin{aligned} \chi\colon H^*(\mathrm{WO}_q) &\longrightarrow H^*(B\Gamma_q), \\ \widehat{\chi}\colon\ H^*(\mathrm{W}_q) &\longrightarrow H^*(B\overline{\Gamma_q}), \\ \widehat{\chi}^{\mathbb{C}}\colon\ H^*(\mathrm{W}_q^{\mathbb{C}}) &\longrightarrow H^*(B\overline{\Gamma_q^{\mathbb{C}}}). \end{aligned}$$

These mappings are related as follows. There is an obvious mapping from $H^*(\mathrm{WO}_q)$ to $H^*(\mathrm{W}_q)$. It corresponds to the natural mapping from $B\overline{\Gamma_q}$ to $B\Gamma_q$. The counterparts for transversely holomorphic foliations are the natural mapping from $B\overline{\Gamma_q^{\mathbb{C}}}$ to $B\Gamma_q^{\mathbb{C}}$ and the mapping from $H^*(\mathrm{WU}_q)$ to $H^*(\mathrm{W}_q^{\mathbb{C}})$ given by sending $\widetilde{u}_i$ to $u_i - \overline{u}_i$, v_j to v_j and $\overline{v}_k$ to $\overline{v}_k$. The following diagrams are known to be commutative:

$$\begin{array}{ccc} H^*(\mathrm{WO}_{2q}) & \longrightarrow & H^*(\mathrm{W}_q) \\ \chi\big\downarrow & & \big\downarrow\widehat{\chi} \\ H^*(B\Gamma_{2q}) & \longrightarrow & H^*(B\overline{\Gamma_{2q}}), \end{array} \qquad \begin{array}{ccc} H^*(\mathrm{WU}_q) & \longrightarrow & H^*(\mathrm{W}_q^{\mathbb{C}}) \\ \chi^{\mathbb{C}}\big\downarrow & & \big\downarrow\widehat{\chi}^{\mathbb{C}} \\ H^*(B\Gamma_q^{\mathbb{C}}) & \longrightarrow & H^*(B\overline{\Gamma_q^{\mathbb{C}}}). \end{array}$$

Relations between these diagrams obtained by forgetting transverse holomorphic structures will be discussed in Chapter 2.

1.2. Godbillon–Vey Class and Bott Class

The following secondary classes are relevant [17], [19] (see also [62], [3]).

DEFINITION 1.2.1.

1) The class $h_1c_1^{2q}$ in $H^{4q+1}(\mathrm{WO}_{2q})$ is called the *Godbillon–Vey class* and denoted by GV_{2q}. The image of GV_{2q} in $H^{4q+1}(\mathrm{W}_{2q})$ is also called the *Godbillon–Vey class*.
2) The class $u_1v_1^q$ in $H^{2q+1}(\mathrm{W}_q)$ and its image in $H^{2q+1}(\mathrm{W}_q^{\mathbb{C}})$ are called the *Bott class* and denoted by Bott_q.
3) The class $\sqrt{-1}\widetilde{u}_1(v_1^q + v_1^{q-1}\overline{v}_1 + \cdots + \overline{v}_1^q)$ in $H^{2q+1}(\mathrm{WU}_q)$ and its image in $H^{2q+1}(\mathrm{W}_q^{\mathbb{C}})$ are called the *imaginary part of the Bott class* and denoted by ξ_q.

REMARK 1.2.2. If $Q(\mathcal{F})$ is trivial, the definition of the Bott class in Definition 1.1.5 coincides with the above one. The Bott class is independent of the choice of the trivializations, however, some of other secondary classes defined in terms of $H^*(\mathrm{W}_q^{\mathbb{C}})$ depend on the choice of trivializations of $Q(\mathcal{F})$.

The reason for which ξ_q is called the imaginary part of the Bott class is as follows.

LEMMA 1.2.3. *The image of ξ_q in $H^*(\mathrm{W}_q^{\mathbb{C}})$ coincides with $\sqrt{-1}(\mathrm{Bott}_q - \overline{\mathrm{Bott}_q})$.*

PROOF. The following equality holds in $\mathrm{W}_q^{\mathbb{C}}$, namely,

$$\begin{aligned}\xi_q &= \sqrt{-1}\widetilde{u}_1(v_1^q + v_1^{q-1}\overline{v}_1 + \cdots + \overline{v}_1^q)\\ &= \sqrt{-1}(u_1 v_1^q - \overline{u}_1\overline{v}_1^q)\\ &\quad + \sqrt{-1}u_1(v_1^{q-1}\overline{v}_1 + \cdots + \overline{v}_1^q) - \sqrt{-1}\overline{u}_1(v_1^q + \cdots + v_1\overline{v}_1^{q-1}).\end{aligned}$$

On the other hand,

$$\begin{aligned}&d(u_1\overline{u}_1(v_1^{q-1} + \cdots + v_1\overline{v}_1^{q-1}))\\ &= \overline{u}_1(v_1^q + \cdots + v_1\overline{v}_1^{q-1}) - u_1(v_1^{q-1}\overline{v}_1 + \cdots + \overline{v}_1^q).\end{aligned}$$

Hence the image of ξ_q is cohomologous to $\sqrt{-1}(u_1 v_1^q - \overline{u}_1\overline{v}_1^q)$ in $\mathrm{W}_q^{\mathbb{C}}$. □

In order to define the Bott class, it is enough to assume the triviality of the canonical bundle $K_{\mathcal{F}}$. Indeed, the Bott classes in Definitions 1.1.5 and 1.2.1 coincide for such foliations. If $K_{\mathcal{F}}$ is non-trivial, then the Bott class is known to be defined as an element of $H^{2q+1}(M;\mathbb{C}/\mathbb{Z})$ as follows.

THEOREM 1.2.4. *There is a well-defined element $B_q(\mathcal{F}) \in H^{2q+1}(M;\mathbb{C}/\mathbb{Z})$ with the following properties.*

1) *$B_q(\mathcal{F})$ has naturality with respect to $\mathcal{F}$.*
2) *If $K_{\mathcal{F}}$ is trivial, then $B_q(\mathcal{F})$ is the image of $\mathrm{Bott}_q(\mathcal{F})$ under the mapping $H^{2q+1}(M;\mathbb{C}) \to H^{2q+1}(M;\mathbb{C}/\mathbb{Z})$.*
3) *Let $c_1(Q(\mathcal{F}))$ be the integral first Chern class of $Q(\mathcal{F})$. Then $B_q(\mathcal{F})$ is mapped to $c_1(Q(\mathcal{F}))^{q+1}$ under the natural mapping $H^{2q+1}(M;\mathbb{C}/\mathbb{Z}) \to H^{2q+2}(M;\mathbb{Z})$.*
4) *$\xi_q(\mathcal{F}) = \sqrt{-1}(B_q(\mathcal{F}) - \overline{B_q(\mathcal{F})})$.*

The class $B_q(\mathcal{F})$ is also called the *Bott class*. The study of the real part of the Bott class is more difficult than that of the imaginary part. We refer to [7] for an explicit construction of $B_q(\mathcal{F})$ and some properties of it.

REMARK 1.2.5. Example 1.1.6 shows that several secondary classes admit continuous variations. As already remarked, the Bott class, and hence B_q and ξ_q admit continuous variations. It is known that classes such as $u_1v_J(\mathcal{F}_\lambda)$, $|J| = q$, are well-defined and can also vary continuously. Indeed, it is shown in [13] that $u_1v_J(\mathcal{F}_\lambda) = \dfrac{c_1c_J(\lambda_0,\dots,\lambda_n)}{\lambda_0\cdots\lambda_n}[S^{2n+1}]$ if $|J| = n$ by using residues, where c_1c_J denotes the Chern monomial. See also Example 5.6.

1.3. Some Known Results

It is quite important to decide if the universal characteristic mappings $\chi^{\mathbb{C}}$, $\widehat{\chi}^{\mathbb{C}}$, χ and $\widehat{\chi}$ are injective or not. It is an old open problem. There are several works on the cohomology of classifying spaces. First, Bott's examples (Example 1.1.6) imply the following

THEOREM 1.3.1. *$H^{2q+1}(B\overline{\Gamma_q^{\mathbb{C}}};\mathbb{R}) \supset \mathbb{R}^q$.*

The following result is significant.

THEOREM 1.3.2 (Adachi [1]). *$B\overline{\Gamma_q^{\mathbb{C}}}$ is q-connected.*

It is unknown if it is sharp for $q > 1$. If $q = 1$, then there is a following

THEOREM 1.3.3 (Haefliger–Sithanantham [38]). *$B\overline{\Gamma_1^{\mathbb{C}}}$ is 2-connected.*

Note that $B\overline{\Gamma_1^{\mathbb{C}}}$ is not 3-connected. Indeed, there is a foliation of S^3 of which the Bott class is non-trivial.

The following is known about $H^*(\mathrm{WU}_1)$ and $H^*(\mathrm{W}_1^{\mathbb{C}})$:

$$H^k(\mathrm{WU}_1) = \begin{cases} \mathbb{C}, & k=0,\\ \left\langle \dfrac{v_1+\overline{v}_1}{2}\right\rangle, & k=2,\\ \langle \xi_1\rangle, & k=3,\\ \langle \mathrm{GV}_2\rangle, & k=5,\\ \{0\}, & \text{otherwise,}\end{cases} \qquad H^k(\mathrm{W}_1^{\mathbb{C}}) = \begin{cases} \mathbb{C}, & k=0,\\ \langle \mathrm{Bott}_1, \overline{\mathrm{Bott}_1}\rangle, & k=3,\\ \langle \mathrm{Bott}_1\,\overline{\mathrm{Bott}_1}\rangle, & k=6,\\ \{0\}, & \text{otherwise,}\end{cases}$$

where $\langle\,\cdot\,\rangle$ denotes the vector space spanned over $\mathbb{C}$. It follows that $\chi^{\mathbb{C}}$ and $\widehat{\chi}^{\mathbb{C}}$ are injective if $q = 1$.

It is also important to determine which classes can vary independently.

DEFINITION 1.3.4. Let $h_i c_J \in \mathrm{W}_q$ and suppose that $i+|J|=2q+1$. If $J=(j_1,\dots,j_q)$, then we set $J' = \{\overbrace{1,\dots,1}^{j_1},\overbrace{2,\dots,2}^{j_2},\dots,\overbrace{q,\dots,q}^{j_q}\}$. Suppose that the smallest entry of J' is not less than i. Let $\kappa(h_i c_J) \in \mathrm{WU}_q$ be a cocycle defined by

$$\kappa(h_i c_J) = \widetilde{u}_i v_{j'_1} v_{j'_2} \cdots v_{j'_r} + \overline{v}_i \widetilde{u}_{j'_1} v_{j'_2} \cdots v_{j'_r} + \cdots + \overline{v}_i \overline{v}_{j'_1} \cdots \overline{v}_{j'_{r-1}} \widetilde{u}_{j'_r},$$

where $J' = \{j'_1,\dots,j'_r\}$ and $j'_1 \le j'_2 \le \cdots \le j'_r$.

The mapping κ induces a linear isomorphism from $H^{2q+1}(\mathrm{W}_q)$ to $H^{2q+1}(\mathrm{WU}_q)$. See Theorem 1.3.11.

From Example 5.6, we see that some secondary classes vary independently.

THEOREM 1.3.5 ([13], [62]). *Suppose that $i+|J|=2q+1$ and the smallest entry of J' is not less than i. Let $\chi^{\mathbb{C}}\colon H^{2q+1}(\mathrm{WU}_q) \to H^{2q+1}(B\Gamma_q^{\mathbb{C}};\mathbb{C})$ be the universal characteristic mapping. The elements $\chi^{\mathbb{C}}(\kappa(h_i c_J))$ all vary and vary as linearly independently as the functions* $\operatorname{Im}\dfrac{c_i c_J(A)}{\det A}$, *where $A \in M(q+1;\mathbb{C})$ and* Im *denotes the imaginary part. Let $\widehat{\chi}^{\mathbb{C}}\colon H^{2q+1}(\mathrm{W}_q^{\mathbb{C}}) \to H^{2q+1}(B\overline{\Gamma_q^{\mathbb{C}}};\mathbb{C})$ be the universal characteristic mapping. The elements $\widehat{\chi}^{\mathbb{C}}(u_i v_J)$ and $\widehat{\chi}^{\mathbb{C}}(\overline{u}_i \overline{v}_J)$ all vary and vary as linearly independently as the functions $\dfrac{c_i c_J(A)}{\det A}$ and $\dfrac{\overline{c_i c_J(A)}}{\overline{\det A}}$.*

See [41] for the case of real foliations, where the constructions are more involved.

Secondary classes as in Theorem 1.3.5 are important particularly in relation with the residue (cf. Chapter 5).

DEFINITION 1.3.6. Secondary classes in $H^*(\mathrm{WO}_q)$ or $H^*(\mathrm{W}_q)$ of the form $h_i c_J$ with $i+|J|=2q+1$ are called *residual classes*.

Related results are obtained by Hurder [46]. Let $\nu\colon B\Gamma_q^{\mathbb{C}} \to B\mathrm{U}_q$ $(\simeq B\mathrm{GL}(q;\mathbb{C}))$ be the mapping defined by the differential.

PROPOSITION 1.3.7 ([46]).

1) $\nu^*\colon H^n(B\mathrm{U}_q;\mathbb{Q}) \to H^n(B\Gamma_q^{\mathbb{C}};\mathbb{Q})$ *is injective if* $n \leq 2q$.
2) $\nu_*\colon \pi_n(B\Gamma_q^{\mathbb{C}}) \to \pi_n(B\mathrm{U}_q)$ *is an isomorphism for* $n \leq q$, *onto for* $n = q+1$ *and has finite cokernel for* $n = q+2$.

Recall that W_q is naturally a subalgebra of $\mathrm{W}_q^{\mathbb{C}}$.

THEOREM 1.3.8 ([46]). *Let* $q = 2k-2$ *with* $k > 1$. *Define a set of rigid secondary classes* $\mathscr{R} \subset H^*(\mathrm{W}_q) \subset H^*(\mathrm{W}_q^{\mathbb{C}})$ *to be*

$$\begin{aligned}\mathscr{R} = &\left\{u_1u_{i_2}\cdots u_{i_s}v_2^{k-1} \,\middle|\, 2 < i_2 < \cdots < i_s \leq q\right\}\\ &\cup \left\{u_ku_{i_2}\cdots u_{i_s}v_k \,\middle|\, k < i_2 < \cdots < i_s \leq q\right\}.\end{aligned}$$

Then the universal characteristic mapping $\widehat{\chi}^{\mathbb{C}}$ *maps* $\mathscr{R}$ *to a linearly independent subset of* $H^*(B\overline{\Gamma_q^{\mathbb{C}}})$.

We refer to [46] for further results.

In order to study secondary classes, it is relevant to determine the structure of $H^*(\mathrm{WU}_q)$, etc. The structure of $H^*(\mathrm{WO}_q)$, $H^*(\mathrm{W}_q)$ and $H^*(\mathrm{W}_q^{\mathbb{C}})$ are well-understood. Indeed, sets of bases are given by Vey as follows.

DEFINITION 1.3.9 (Vey basis [33]).

1) $\{c_J \,|\, |J| \leq q,\ j_r = 0 \text{ if } r \text{ is odd}\} \cup \{h_Ic_J \,|\, i_1 \leq j_0,\ i_1 + |J| > q\}$ is a basis for $H^*(\mathrm{WO}_q)$, where j_0 denotes the smallest odd integer with $j_{j_0} \neq 0$.
2) $\{h_Ic_J \,|\, i_1 \leq j_0',\ i_1 + |J| > q\}$ is a basis for $H^*(\mathrm{W}_q)$, where j_0' denotes the smallest integer with $j_{j_0'} \neq 0$.

On the other hand, a set of basis for $H^*(\mathrm{WU}_q)$ is known for $q \leq 3$ ([5], see also Tables 1.3.1 and 1.3.2 at the end of this section, where the numbers in the left column stand for the degree of the classes in the same row), but it seems unknown if $q \geq 4$. This is one of problems which makes the study of complex secondary classes difficult. For example, the construction of infinitesimal derivatives in Chapter 4 is much more complicated than the original one [42]. Indeed, it is necessary to construct derivatives on WU_q and to show that they induce derivatives of elements of $H^*(\mathrm{WU}_q)$. Although any explicit basis seems unknown for $H^*(\mathrm{WU}_q)$ for $q \geq 4$, there are algorithms for computing them by using spectral sequences [5], [6]. In particular, we have the following

THEOREM 1.3.10 ([6]). *WU_q is a fibration over $\mathrm{BGL}(q;\mathbb{C})$ with fiber $\mathrm{W}_q^{\mathbb{C}}$ in the sense that there is a kind of the Serre spectral sequence*

$$E_2^{p,s} \cong H^s(\mathrm{W}_q^{\mathbb{C}}) \otimes H^p(\mathrm{BGL}(q;\mathbb{C})) \Longrightarrow H^{p+s}(\mathrm{WU}_q).$$

In fact, $d_r = 0$ for $r > 2q^2 + 4q + 1$.

For real codimension-q foliations, elements of $H^{2q+1}(\mathrm{WO}_q)$ such as the Godbillon–Vey class are significant. Elements of $H^{2q+1}(\mathrm{WU}_q)$ such as the Bott class are significant for complex codimension-q foliations. In this particular degree, the following is known.

THEOREM 1.3.11 ([6]). *There is a natural isomorphism between $H^{2q+1}(\mathrm{WU}_q)$ and $H^{2q+1}(\mathrm{W}_q)$. Indeed, κ in Definition 1.3.4 induces an isomorphism κ_* from $H^{2q+1}(\mathrm{W}_q)$ to $H^{2q+1}(\mathrm{WU}_q)$. The imaginary part of the Bott class is mapped to a non-zero multiple of the Godbillon–Vey class under κ_*.*

REMARK 1.3.12. The isomorphism is related with complexifications of foliations. Although the formula for κ_* is simple, the formula for the inverse mapping is quite complicated.

REMARK 1.3.13. We will end this chapter with a remark related to a work of Fuks [28]. By using the Gel'fand–Fuks cohomology, one can show that W_q is geometrically realized as follows [28]. Let S_q be the Schubert variety of dimension $2q$ in $\mathrm{BGL}(q;\mathbb{C})$ and let X_q be the restriction of the universal $\mathrm{U}(q)$-bundle to X_q. Then $H^*(\mathrm{W}_q)$ is naturally isomorphic to $H^*(X_q)$ [28, Theorem 2.2.4]. There are also the following isomorphisms:

$$\begin{aligned} H^*(\mathrm{W}_q^{\mathbb{C}}) &\cong H^*(X_q \times X_q), \\ H^*(\mathrm{WU}_q) &\cong \quad H^*(\widetilde{X_q}), \end{aligned}$$

where $\widetilde{X_q}$ is an U(q)-bundle over $S_q \times S_q$ obtained as follows. Since $X_q \times X_q$ is a principal (U(q) × U(q))-bundle over $S_q \times S_q$, there is an U(q)-action on $X_q \times X_q$ induced from the diagonal action on the fibers on the right. The space $\widetilde{X_q}$ is the quotient. We have $H^*(S_q \times S_q) \cong \mathbb{C}_q[v_1, \ldots, v_q] \otimes \mathbb{C}_q[\overline{v}_1, \ldots, \overline{v}_q]$. It follows that the E_2-terms of the Serre spectral sequence for $\widetilde{X_q}$ are WU_q.

In this line, the proof of Theorem 1.3.11 in [6] can be read as follows. The original proof consists of two steps. First, an isomorphism between $H^*(\mathrm{WU}_q)$ with a certain cohomology of a DGA $\mathcal{WU}_q$ is established, where $\mathcal{WU}_q$ is defined by enlarging WU_q. This step corresponds to the fact that there is an isomorphism of cohomology between $\widetilde{X_q}$ and the total space Y_q of a principal (U(q)×U(q))-bundle over $S_q \times \mathrm{BGL}(q;\mathbb{C}) \times S_q$, where Y_q is defined as follows. First consider the natural principal (U(q)×U(q)×U(q))-bundle over $S_q \times \mathrm{BGL}(q;\mathbb{C}) \times S_q$. There is again a diagonal U(q)-action on fibers on the right. The quotient is Y_q. There is a mapping from Y_q to $\widetilde{X_q}$. Indeed, points of $\widetilde{X_q}$ are locally represented as $z = ((u_1, u_2, u_3), (x_1, y, x_2))$, where $u_i \in \mathrm{U}(q)$, $x_1, x_2 \in S_q$ and $y \in \mathrm{BGL}(q;\mathbb{C})$. The mapping which assigns to z the point $((u_1{u_2}^{-1}, u_3{u_2}^{-1}), x_1, x_2) \in (\mathrm{U}(q) \times \mathrm{U}(q)) \times S_q \times S_q$ induces the desired mapping. The fiber of this mapping is the universal U(q)-bundle over BGL(q; ℂ), which is contractible. Hence we have $H^*(Y_q) \cong H^*(\widetilde{X_q})$. The second step is to construct a spectral sequence with E_2-terms $H^s(\mathrm{W}_q^{\mathbb{C}}) \otimes H^p(\mathrm{BGL}(q;\mathbb{C}))$ which converges to $H^*(\mathcal{WU}_q)$. It can be seen as the Serre spectral sequence of the natural fibration $X_q \times X_q \to Y_q \to \mathrm{BGL}(q;\mathbb{C})$.

We remark that there is another fibration structure and a spectral sequence. Let $\widetilde{X_q} \to S_q$ be a mapping locally defined by $((u_1, u_2), x_1, x_2) \mapsto x_1$. The fiber is X_q so that there is a spectral sequence with E_2-terms $H^*(\mathrm{W}_q) \otimes \mathbb{C}_q[v_1, \ldots, v_q]$ which converges to $H^*(\mathrm{WU}_q)$. This spectral sequence is used in [2] for calculating $H^*(\mathrm{WU}_2)$ and $H^*(\mathrm{WU}_3)$.

2	$(v_1 + \overline{v}_1)$
4	$(v_1^2 + v_1\overline{v}_1 + \overline{v}_1^2)$, $(v_2 + \overline{v}_2)$
5	$\widetilde{u}_1(v_1^2 + v_1\overline{v}_1 + \overline{v}_1^2)$, $\widetilde{u}_2(v_1 + \overline{v}_1) + \widetilde{u}_1(v_2 + \overline{v}_2)$
7	$\widetilde{u}_1 v_1\overline{v}_1(v_1 + \overline{v}_1)$, $\widetilde{u}_2(v_1^2 + v_1\overline{v}_1 + \overline{v}_1^2) + \widetilde{u}_1(v_1\overline{v}_2 + v_2\overline{v}_1)$, $\widetilde{u}_2(v_2 + \overline{v}_2)$
9	$\widetilde{u}_1 v_1^2\overline{v}_1^2$, $\widetilde{u}_1(v_1^2\overline{v}_2 + v_2\overline{v}_1^2)$, $\widetilde{u}_1 v_2\overline{v}_2$
10	$\widetilde{u}_1\widetilde{u}_2 v_1\overline{v}_1(v_1 + \overline{v}_1)$
11	$\widetilde{u}_2 v_2\overline{v}_2$
12	$\widetilde{u}_1\widetilde{u}_2 v_1^2\overline{v}_1^2$, $\widetilde{u}_1\widetilde{u}_2 v_1^2\overline{v}_2$, $\widetilde{u}_1\widetilde{u}_2 v_2\overline{v}_1^2$, $\widetilde{u}_1\widetilde{u}_2 v_2\overline{v}_2$

TABLE 1.3.1. A basis for $H^*(\mathrm{WU}_2)$.

	$\overline{v}_1$, $\overline{v}_1^2$, $\overline{v}_2$, $\overline{v}_1^3$, $\overline{v}_1\overline{v}_2$, $\overline{v}_3$
7	$\widetilde{u}_1(v_1^3+v_1^2\overline{v}_1+v_1\overline{v}_1^2+\overline{v}_1^3)$, $\widetilde{u}_2v_1^2+\widetilde{u}_1v_1\overline{v}_2+\widetilde{u}_1\overline{v}_1\overline{v}_2$, $\widetilde{u}_2(v_2+\overline{v}_2)$, $\widetilde{u}_3v_1+\widetilde{u}_1\overline{v}_3$
9	$\widetilde{u}_1(v_1^3\overline{v}_1+v_1^2\overline{v}_1^2+v_1\overline{v}_1^3)$, $\widetilde{u}_2v_1^3+\widetilde{u}_1v_1^2\overline{v}_2+\widetilde{u}_1v_1\overline{v}_1\overline{v}_2$, $\widetilde{u}_2(v_1v_2+v_1\overline{v}_2)$, $\widetilde{u}_3v_1^2+\widetilde{u}_1v_1\overline{v}_3$, $\widetilde{u}_3v_2+\widetilde{u}_2\overline{v}_3$
11	$\widetilde{u}_1(v_1^3\overline{v}_1^2+v_1^2\overline{v}_1^3)$, $\widetilde{u}_1(v_1v_2\overline{v}_1^2+v_2\overline{v}_1^3)$, $\widetilde{u}_1(v_1+\overline{v}_1)v_2\overline{v}_2$, $\widetilde{u}_2v_2\overline{v}_2$, $\widetilde{u}_3v_1^3+\widetilde{u}_1v_1^2\overline{v}_3$, $\widetilde{u}_3v_1v_2+\widetilde{u}_1v_2\overline{v}_3$, $\widetilde{u}_3(v_3+\overline{v}_3)$
13	$\widetilde{u}_1v_1^3\overline{v}_1^3$, $\widetilde{u}_1v_1v_2\overline{v}_1^3$, $\widetilde{u}_1v_1v_2\overline{v}_1\overline{v}_2$, $\widetilde{u}_1v_1v_2\overline{v}_3$, $\widetilde{u}_1v_3\overline{v}_1^3$, $\widetilde{u}_1v_3\overline{v}_3$, $\widetilde{u}_2v_1v_2\overline{v}_2$, $\widetilde{u}_2v_2\overline{v}_3$
14	$\widetilde{u}_1\widetilde{u}_2(v_1^3\overline{v}_1^2+v_1^2\overline{v}_1^3)$, $\widetilde{u}_1\widetilde{u}_2(v_1v_2\overline{v}_1^2+v_2\overline{v}_1^3)$, $\widetilde{u}_1\widetilde{u}_2(v_1^3\overline{v}_2+v_1^2\overline{v}_1\overline{v}_2)$, $\widetilde{u}_1\widetilde{u}_2(v_1+\overline{v}_1)v_2\overline{v}_2$, $\widetilde{u}_1\widetilde{u}_3(v_1^3\overline{v}_1+v_1^2\overline{v}_1^2+v_1\overline{v}_1^3)$, $\widetilde{u}_2\widetilde{u}_3v_1^3+\widetilde{u}_1\widetilde{u}_3v_1^2\overline{v}_2+\widetilde{u}_1\widetilde{u}_3v_1\overline{v}_1\overline{v}_2-\widetilde{u}_1\widetilde{u}_2v_1^2\overline{v}_3$, $\widetilde{u}_2\widetilde{u}_3(v_1v_2+v_1\overline{v}_2)-\widetilde{u}_1\widetilde{u}_2v_2\overline{v}_3$
15	$\widetilde{u}_2v_3\overline{v}_3$
16	$\widetilde{u}_1\widetilde{u}_2v_1^3\overline{v}_1^3$, $\widetilde{u}_1\widetilde{u}_2v_1^3\overline{v}_1\overline{v}_2$, $\widetilde{u}_1\widetilde{u}_2v_1^3\overline{v}_3$, $\widetilde{u}_1\widetilde{u}_2v_1v_2\overline{v}_1^3$, $\widetilde{u}_1\widetilde{u}_2v_1v_2\overline{v}_1\overline{v}_2$, $\widetilde{u}_1\widetilde{u}_2v_1v_2\overline{v}_3$, $\widetilde{u}_1\widetilde{u}_2v_3\overline{v}_1^3$, $\widetilde{u}_1\widetilde{u}_2v_3\overline{v}_1\overline{v}_2$, $\widetilde{u}_1\widetilde{u}_2v_3\overline{v}_3$, $\widetilde{u}_1\widetilde{u}_3(v_1^3\overline{v}_1^2+v_1^2\overline{v}_1^3)$, $\widetilde{u}_1\widetilde{u}_3(v_1v_2\overline{v}_1^2+v_2\overline{v}_1^3)$, $\widetilde{u}_1\widetilde{u}_3(v_1+\overline{v}_1)v_2\overline{v}_2$, $\widetilde{u}_2\widetilde{u}_3v_2\overline{v}_2$
17	$\widetilde{u}_3v_3\overline{v}_3$
18	$\widetilde{u}_1\widetilde{u}_3v_1^3\overline{v}_1^3$, $\widetilde{u}_1\widetilde{u}_3v_1^3\overline{v}_3$, $\widetilde{u}_1\widetilde{u}_3v_1v_2\overline{v}_1^3$, $\widetilde{u}_1\widetilde{u}_3v_1v_2\overline{v}_1\overline{v}_2$, $\widetilde{u}_1\widetilde{u}_3v_1v_2\overline{v}_3$, $\widetilde{u}_1\widetilde{u}_3v_3\overline{v}_1^3$, $\widetilde{u}_1\widetilde{u}_3v_3\overline{v}_1\overline{v}_2$, $\widetilde{u}_1\widetilde{u}_3v_3\overline{v}_3$, $\widetilde{u}_2\widetilde{u}_3v_1v_2\overline{v}_2$, $\widetilde{u}_2\widetilde{u}_3v_3\overline{v}_2$, $\widetilde{u}_2\widetilde{u}_3v_2\overline{v}_3$
19	$\widetilde{u}_1\widetilde{u}_2\widetilde{u}_3(v_1^3\overline{v}_1^2+v_1^2\overline{v}_1^3)$, $\widetilde{u}_1\widetilde{u}_2\widetilde{u}_3(v_1v_2\overline{v}_1^2+v_2\overline{v}_1^3)$, $\widetilde{u}_1\widetilde{u}_2\widetilde{u}_3(v_1^3\overline{v}_2+v_1^2\overline{v}_1\overline{v}_2)$, $\widetilde{u}_1\widetilde{u}_2\widetilde{u}_3(v_1+\overline{v}_1)v_2\overline{v}_2$
20	$\widetilde{u}_2\widetilde{u}_3v_3\overline{v}_3$
21	$\widetilde{u}_1\widetilde{u}_2\widetilde{u}_3v_1^3\overline{v}_1^3$, $\widetilde{u}_1\widetilde{u}_2\widetilde{u}_3v_1^3\overline{v}_1\overline{v}_2$, $\widetilde{u}_1\widetilde{u}_2\widetilde{u}_3v_1^3\overline{v}_3$, $\widetilde{u}_1\widetilde{u}_2\widetilde{u}_3v_1v_2\overline{v}_1^3$, $\widetilde{u}_1\widetilde{u}_2\widetilde{u}_3v_1v_2\overline{v}_1\overline{v}_2$, $\widetilde{u}_1\widetilde{u}_2\widetilde{u}_3v_1v_2\overline{v}_3$, $\widetilde{u}_1\widetilde{u}_2\widetilde{u}_3v_3\overline{v}_1^3$, $\widetilde{u}_1\widetilde{u}_2\widetilde{u}_3v_3\overline{v}_1\overline{v}_2$, $\widetilde{u}_1\widetilde{u}_2\widetilde{u}_3v_3\overline{v}_3$

TABLE 1.3.2. A basis for $H^*(\mathrm{WU}_3)$.