Chapter 2

Main estimates

We will now turn to analysing the conditions under which we can obtain $L^p - L^q$ decay estimates for the general m^{th} order linear, constant coefficient, strictly hyperbolic Cauchy problem

$$\begin{cases} L(D_t, D_x) \equiv D_t^m u + \displaystyle\sum_{j=1}^{m} P_j(D_x) D_t^{m-j} u + \sum_{l=0}^{m-1} \sum_{|\alpha|+r=l} c_{\alpha,r} D_x^\alpha D_t^r u = 0,\ t > 0, \\ D_t^l u(0,x) = f_l(x) \in C_0^\infty(\mathbb{R}^n), \quad l = 0, \dots, m-1,\ x \in \mathbb{R}^n\,. \end{cases} \tag{2.0.1}$$

Results of this section will show how different behaviours of the characteristic roots $\tau_1(\xi), \dots, \tau_m(\xi)$ affect the rate of decay that can be obtained. As in the introduction, the symbol $P_j(\xi)$ of $P_j(D_x)$ is a homogeneous polynomial of order j, and the $c_{\alpha,r}$ are constants. The differential operator in the first line of (2.0.1) will be denoted by $L(D_t, D_x)$ and its symbol by $L(\tau, \xi)$. The principal part of L is denoted by L_m. Thus, $L_m(\tau, \xi)$ is a homogeneous polynomial of order m. In the subsequent analysis, ideally, of course, we would like to have conditions on the lower order terms for different rates of decay; in Section 8 we shall give some results in this direction. For now, though, we concentrate on conditions on the characteristic roots.

First of all, it is natural to impose the stability condition, namely that for all $\xi \in \mathbb{R}^n$ we have

$$\operatorname{Im} \tau_k(\xi) \geq 0 \quad \text{for } k = 1, \dots, m\,; \tag{2.0.2}$$

this is equivalent to requiring the characteristic polynomial of the operator to be stable at all points $\xi \in \mathbb{R}^n$, and thus cannot be expected to be lifted. In fact, certain microlocal decay estimates are possible even without this condition if the supports of the Fourier transforms of the Cauchy data are contained in the set where condition (2.0.2) holds. However, this restriction is only technical so we may assume (2.0.2) without great loss of generality since otherwise no time decay of solution can be expected.

Also, it is sensible to divide the considerations of how characteristic roots behave into two parts: their behaviour for large values of $|\xi|$ and for bounded values of $|\xi|$. These two cases are then subdivided further; in particular the following are the key properties to consider:

- multiplicities of roots (this only occurs in the case of bounded frequencies $|\xi|$);
- whether roots lie on the real axis or are separated from it;
- behaviour as $|\xi| \to \infty$ (only in the case of large $|\xi|$);
- how roots meet the real axis (if they do);
- properties of the Hessian of the root, $\operatorname{Hess} \tau_k(\xi)$;
- a convexity-type condition, as in the case of homogeneous roots (Section 1.2).

For some frequencies away from multiplicities we can actually establish independently interesting estimates for the corresponding oscillatory integrals that contribute to the solution. Around multiplicities we need to take extra care of the structure of solutions. This will be done by dividing the frequencies into zones each of which will give a certain decay rate. Combined together they will yield the total decay rate for solution to (2.0.1). Several theorems below will deal with integrals of the form

$$\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi)\chi(\xi)d\xi, \tag{2.0.3}$$

which appear in representations of solutions to Cauchy problem (2.0.1) as kernels of propagators, where $a(\xi)$ is a suitable amplitude and $\chi(\xi)$ is a cut-off to a corresponding zone, which may be bounded or unbounded. Solution to the Cauchy problem (2.0.1) can be written in the form

$$u(t,x) = \sum_{j=0}^{m-1} E_j(t) f_j(x),$$

where propagators $E_j(t)$ are defined by

$$E_j(t)f(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} \Big(\sum_{k=1}^{m} e^{i\tau_k(\xi)t} A_j^k(t,\xi) \Big) \chi(\xi) \widehat{f}(\xi)\, d\xi\,, \tag{2.0.4}$$

with suitable amplitudes $A_j^k(t,\xi)$. In the areas where roots are simple, phases and amplitudes are smooth, and we can analyse the sum (2.0.4) termwise, reducing the analysis to integrals of the form (2.0.3). In the case of multiple characteristics we will group terms in (2.0.4) in a special way to obtain suitable decay estimates. Below we will give results for decay rates dependent on the different qualitative behaviours of the characteristic roots.

2.1 Away from the real axis: exponential decay

We begin by looking at the zone where roots are separated from the real axis. If the roots are smooth, we can analyse solution (2.0.4) termwise:

Theorem 2.1.1. *Let $\tau : U \to \mathbb{C}$ be a smooth function, $U \subset \mathbb{R}^n$ open. Let $a \in S^{-\mu}_{1,0}(U)$, i.e. assume that $a = a(\xi) \in C^\infty(U)$ satisfies $|\partial_\xi^\alpha a(\xi)| \leq C_\alpha(1+|\xi|)^{-\mu-|\alpha|}$, for all $\xi \in U$ and all multi-indices α. Let $\chi \in S^0_{1,0}(\mathbb{R}^n)$ be such that $\chi = 0$ outside U. Assume further that:*

(i) *there exists $\delta > 0$ such that $\operatorname{Im}\tau(\xi) \geq \delta$ for all $\xi \in U$;*

(ii) *$|\tau(\xi)| \leq C(1+|\xi|)$ for all $\xi \in U$.*

Then for all $t \geq 0$ we have

$$\left\| D_t^r D_x^\alpha \Big(\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi)\chi(\xi)\widehat{f}(\xi)\, d\xi \Big) \right\|_{L^q(\mathbb{R}^n_x)} \leq Ce^{-\delta t}\|f\|_{W_p^{N_p+|\alpha|+r-\mu}}, \tag{2.1.1}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $1<p\leq 2$, $N_p \geq n\big(\frac{1}{p}-\frac{1}{q}\big)$, $r \geq 0$, α a multi-index and $f \in C_0^\infty(\mathbb{R}^n)$. If $p=1$, we take $N_1 > n$.

Moreover, let us assume that equation $L(\tau,\xi)=0$ has only simple roots $\tau_k(\xi)$ which satisfy condition (i) *above, in the open set $U \subset \mathbb{R}^n$, for all $k = 1,\ldots,m$. Then solution u to (2.0.1) satisfies*

$$||D_t^r D_x^\alpha \chi(D) u(t,\cdot)||_{L^q(\mathbb{R}^n_x)} \leq Ce^{-\delta t}\sum_{l=0}^{m-1} ||f_l||_{W_p^{N_p+|\alpha|+r-l}}, \tag{2.1.2}$$

where $1 \leq p \leq 2$, $\frac{1}{p}+\frac{1}{q}=1$, and N_p, r, α are as above.

The proof of Theorem 2.1.1 will be given in Sections 6.4 and 6.10. Note also that if we omit assumption (ii) in Theorem 2.1.1, estimate (2.1.1) with $r=0$ still holds. In the case of (2.1.2), it can be shown (see Proposition 3.2.4) that characteristic roots of operator $L(D_t,D_x)$ in (2.0.1) satisfy (ii).

We also note, that we may have different norms on the right hand side of (2.1.2). For example, we will show in Section 6.4, that under conditions of Theorem 2.1.1 we also have the following estimate:

$$||D_t^r D_x^\alpha \chi(D) u(t,\cdot)||_{L^q(\mathbb{R}^n_x)} \le C e^{-\delta t} \sum_{l=0}^{m-1} ||f_l||_{W_2^{N_q'+|\alpha|+r-l}}, \tag{2.1.3}$$

where $1 < p \le 2$, $\frac{1}{p}+\frac{1}{q}=1$, $N_q' \ge \frac{n}{2}(\frac{1}{p}-\frac{1}{q})$, and $N_\infty' > \frac{n}{2}$ for $p=1$. Estimate (2.1.3) will follow from (6.4.1) and Proposition 6.4.1 by interpolation. In turn, interpolating between (2.1.2) and (2.1.3), we can obtain similar $L^p - L^q$ estimates for all intermediate p and q.

To be able to derive time decay in the case of multiple roots, we will group terms in (2.0.4) in the following way. Assume that roots $\tau_1(\xi), \ldots, \tau_L(\xi)$ coincide on a set contained in some $\mathcal{M}$, that is $\mathcal{M} \supset \{\tau_1(\xi) = \cdots = \tau_L(\xi)\}$. For $\varepsilon > 0$, we define $\mathcal{M}^\varepsilon := \{\xi \in \mathbb{R}^n : \operatorname{dist}(\xi, \mathcal{M}) < \varepsilon\}$. Choose $\varepsilon > 0$ so that these roots $\tau_1(\xi), \ldots, \tau_L(\xi)$ do not intersect with any of the other roots $\tau_{L+1}(\xi), \ldots, \tau_m(\xi)$ in $\mathcal{M}^\varepsilon$. If different numbers of roots intersect in different sets, we can apply the following theorem to such sets one by one. We note that by the strict hyperbolicity $\mathcal{M}^\varepsilon$ is bounded. Here we will estimate the sum

$$\int_{\mathcal{M}^\varepsilon} e^{ix\cdot\xi} \Big(\sum_{k=1}^{L} e^{i\tau_k(\xi)t} A_j^k(t,\xi) \Big) \chi(\xi) \widehat{f}(\xi)\, d\xi\,. \tag{2.1.4}$$

Theorem 2.1.2. *Let the sum (2.0.4) be the solution to the Cauchy problem (2.0.1). Assume that roots $\tau_1(\xi), \ldots, \tau_L(\xi)$ coincide in a set contained in $\mathcal{M}$ and do not intersect other roots in the set $\mathcal{M}^\varepsilon$. Let $\chi \in C_0^\infty(\mathcal{M}^\varepsilon)$. Assume that there exists $\delta > 0$ such that $\operatorname{Im} \tau_k(\xi) \ge \delta$ for all $\xi \in \mathcal{M}^\varepsilon$ and $k = 1, \ldots, L$.*

Then for all $t \ge 0$ we have

$$\Big\| D_t^r D_x^\alpha \Big(\int_{\mathcal{M}^\varepsilon} e^{ix\cdot\xi} \Big(\sum_{k=1}^{L} e^{i\tau_k(\xi)t} A_j^k(t,\xi) \Big) \chi(\xi) \widehat{f}(\xi)\, dx \Big) \Big\|_{L^q(\mathbb{R}^n_x)} \le C(1+t)^{L-1} e^{-\delta t} \|f\|_{L^p}\,,$$

where $\frac{1}{p}+\frac{1}{q}=1$, $1 \le p \le 2$.

Thus, if characteristic roots are separated from the real axis on the support of some $\chi \in C_0^\infty(\mathbb{R}^n)$, we can separate the solution (2.0.4) into groups of multiple roots for which the $L^p - L^q$ norms still decay exponentially as stated in Theorem 2.1.2. We also note that since $\mathcal{M}^\epsilon$ is bounded, assumption (ii) of Theorem 2.1.1 is automatically satisfied and, therefore, it is omitted in the formulation of Theorem 2.1.2. Theorem 2.1.2 will be proved in Section 7.2.

2.2 Roots with non-degeneracies

The following case that we consider is the one of roots satisfying certain non-degeneracy conditions. These may be conditions on the Hessian, convexity conditions, or simply the information on the index of the corresponding level surfaces. In this section we will give the corresponding statements. We always assume the stability condition (2.0.2) but no longer assume that roots are separated from the real axis.

First we state the result for phases with the non-degenerate Hessian. The behavior depends on critical points ξ^0 with $\nabla\tau(\xi^0) = 0$ and the behavior of the Hessian at such points. As usual, we say that the critical point ξ^0 is non-degenerate if the Hessian $\mathrm{Hess}\,\tau(\xi^0)$ is non-degenerate.

Theorem 2.2.1. *Let $U \subset \mathbb{R}^n$ be a bounded open set, and let $\tau : U \to \mathbb{C}$ be smooth and such that $\mathrm{Im}\,\tau(\xi) \geq 0$ for all $\xi \in U$. Assume that there are some constants C_0 and M such that $|\det \mathrm{Hess}\,\tau(\xi)| \geq C_0(1 + |\xi|)^{-M}$ for all $\xi \in U$. Let $\chi \in S^0_{1,0}(\mathbb{R}^n)$ be such that $\chi = 0$ outside U and let $a \in S^{-\mu}_{1,0}(U)$.*

Assume that τ has only one non-degenerate critical point in U, and that U is sufficiently small. Then there is a constant $C > 0$ independent of the position of U such that for all $t \geq 0$ we have

$$\left|\left|\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi)\chi(\xi)\widehat{f}(\xi)d\xi\right|\right|_{L^q(\mathbb{R}^n_x)} \leq C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}||f||_{W^{N_p}_p}, \quad (2.2.1)$$

with $1 \leq p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $N_p = \frac{M}{2}(\frac{1}{p} - \frac{1}{q}) - \mu$.

For example, the case of the Klein–Gordon equation corresponds to $M = n + 2$ in this theorem. If we work with a fixed bounded set U, the $||f||_{W^{N_p}_p}$ norm on the right hand side of (2.2.1) can be replaced by $||f||_{L^p}$. However, since we may also want to have estimate (2.2.1) uniform over such U (allowing it to move to infinity while remaining to be of the same size), we have the Sobolev norm in (2.2.1). From this point of view, we assume that a behaves as a symbol in U – the meaning is that if the symbolic constants here are uniform over the position of U, then also the constant in (2.2.1) is uniform over such a and U.

The condition that critical points are isolated and therefore can be localised by different sets U may follow from certain properties of τ and will be discussed in Section 6.5, in particular see Lemma 6.5.2 and remarks after it. If, in addition, we take the size of U uniform, say of volume bounded by one, then constant C in (2.2.1) is also uniform over all such sets U. We may also assume that if ξ^0 is a critical point of τ, then $\mathrm{Im}\,\tau(\xi^0) = 0$. Otherwise we would have $\mathrm{Im}\,\tau(\xi^0) > 0$ and so Theorem 2.1.1 would actually give the exponential decay rate. The proof of this theorem is based on the stationary phase method and will be given in Section 6.5. If we apply different versions of the stationary phase method under different conditions, we can reach different conclusions here. For example, we also have:

Theorem 2.2.2. *Let $U \subset \mathbb{R}^n$ be a bounded open and let $\tau : U \to \mathbb{C}$ be smooth and such that $\operatorname{Im} \tau(\xi) \geq 0$ for all $\xi \in U$. Let $\chi \in S^0_{1,0}(\mathbb{R}^n)$ be such that $\chi = 0$ outside U and let $a \in S^{-\mu}_{1,0}(U)$. Assume that τ has only one critical point ξ^0 in U, and that U is sufficiently small.*

Suppose that there are constants $C_0, M > 0$ independent of the size and position of U and of ξ^0, with the following conditions. Suppose that

$$\operatorname{rank} \operatorname{Hess} \tau(\xi^0) = k,$$

that this rank is attained on an $k \times k$ submatrix $A(\xi^0)$ and that

$$|\det A(\xi^0)| \geq C_0(1 + |\xi^0|)^{-M}.$$

Then for all $t \geq 0$ we have

$$\left|\left| \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi)\chi(\xi)\widehat{f}(\xi)d\xi \right|\right|_{L^q(\mathbb{R}^n_x)} \leq C(1+t)^{-\frac{k}{2}(\frac{1}{p}-\frac{1}{q})} ||f||_{W^{N_p}_p},$$

with $1 \leq p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $N_p = \frac{M}{2}(\frac{1}{p} - \frac{1}{q}) - \mu$.

The proof of this theorem is similar to the proof of Theorem 2.2.1 once we restrict to the set of k variables (possibly after a suitable change) on which the rank of the Hessian is attained on $A(\xi^0)$.

This result can be improved dependent on further properties of $A(\xi^0)$. For example, if $\operatorname{rank} A(\xi^0) = n-1$ and this is attained on variables $\xi_1, \ldots, \xi_{n-1}$, the analysis reduces to the behaviour of the oscillatory integral with respect to ξ_n. If the l-th derivative of the phase with respect to ξ_n is non-zero, we get an additional decay by $t^{-1/l}$. This follows from the stationary phase method, see, for example Hörmander [Hör83a, Section 7.7], or from an appropriate use of van der Corput lemma. We will not formulate further statements here since they are quite straightforward.

The next theorem is an estimate of oscillatory integrals with real-valued phases under convexity condition. It will be shown in Proposition 3.2.4 (see also Proposition 6.8.2) that for large ξ characteristic roots of the Cauchy problem (2.0.1) satisfy assumptions of these theorems given below, if the homogeneous roots of the principal part satisfy them. The convexity condition is weaker than (but does not contain) the condition that the Hessian of τ is positive definite and the result can be compared with Theorem 2.2.1, dependent on suitable properties of roots.

Let us first give the necessary definitions. Given a smooth function $\tau : \mathbb{R}^n \to \mathbb{R}$ and $\lambda \in \mathbb{R}$, set

$$\Sigma_\lambda \equiv \Sigma_\lambda(\tau) := \{\xi \in \mathbb{R}^n : \tau(\xi) = \lambda\} .$$

In the case where $\tau(\xi)$ is homogeneous of order 1 and $\tau \in C^\infty(\mathbb{R}^n \backslash 0)$, we will also write $\Sigma_\tau := \Sigma_1(\tau)$—for such τ, we then have $\Sigma_\lambda(\tau) = \lambda\Sigma_\tau$. There should be no confusion in this notation since we always reserve letters ϕ, τ for phases and λ for the real number.

Definition 2.2.3. *A smooth function $\tau : \mathbb{R}^n \to \mathbb{R}$ is said to satisfy the* convexity condition *if surface Σ_λ is convex for each $\lambda \in \mathbb{R}$. Note that the empty set and the point set are considered to be convex.*

If the Gaussian curvatures of Σ_λ never vanish, Σ_λ is automatically convex (the converse is not true). This curvature condition corresponds to the case $k = n-1$ in Theorem 2.2.2. Another important notion is that of the *maximal order of contact* of a hypersurface, similar to the one in Section 1.2:

Definition 2.2.4. *Let Σ be a hypersurface in $\mathbb{R}^n$ (i.e. a manifold of dimension $n-1$); let $\sigma \in \Sigma$, and denote the tangent plane at σ by T_σ. Now let P be a 2–dimensional plane containing the normal to Σ at σ and denote the order of the contact between the line $T_\sigma \cap P$ and the curve $\Sigma \cap P$ by $\gamma(\Sigma; \sigma, P)$. Then set*

$$\gamma(\Sigma) := \sup_{\sigma \in \Sigma} \sup_{P} \gamma(\Sigma; \sigma, P) .$$

Examples 2.2.5.

(a) $\gamma(\mathbb{S}^n) = 2$, as $\gamma(\mathbb{S}^n; \sigma, P) = 2$ for all $\sigma \in \mathbb{S}^n$ and all planes P containing σ and the origin.

(b) If $\varphi_l(\xi)$ is a characteristic root of an m^{th} order homogeneous strictly hyperbolic constant coefficient operator, then $\gamma(\Sigma_{\varphi_l}) \leq m$; see [Sug96] for a proof of this.

Now we can formulate the corresponding theorem.

Theorem 2.2.6. *Suppose $\tau : \mathbb{R}^n \to \mathbb{R}$ satisfies the convexity condition and let $\chi \in C^\infty(\mathbb{R}^n)$; furthermore, on* $\operatorname{supp} \chi$, *we assume:*

(i) *for all multi-indices α there exists a constant $C_\alpha > 0$ such that*

$$|\partial_\xi^\alpha \tau(\xi)| \leq C_\alpha (1 + |\xi|)^{1-|\alpha|};$$

(ii) *there exist constants* $M, C > 0$ *such that for all* $|\xi| \geq M$ *we have* $|\tau(\xi)| \geq C|\xi|$;

(iii) *there exists a constant* $C > 0$ *such that* $|\partial_\omega \tau(\lambda\omega)| \geq C$ *for all* $\omega \in \mathbb{S}^{n-1}$, $\lambda > 0$; *in particular,* $|\nabla\tau(\xi)| \geq C$ *for all* $\xi \in \mathbb{R}^n \setminus \{0\}$;

(iv) *there exists a constant* $R_1 > 0$ *such that, for all* $\lambda > 0$,

$$\frac{1}{\lambda}\Sigma_\lambda(\tau) \equiv \frac{1}{\lambda}\{\xi \in \mathbb{R}^n : \tau(\xi) = \lambda\} \subset B_{R_1}(0)\,.$$

Also, set $\gamma := \sup_{\lambda>0} \gamma(\Sigma_\lambda(\tau))$ *and assume this is finite. Let* $a_j = a_j(\xi) \in S^{-j}_{1,0}$ *be a symbol of order* $-j$ *of type* $(1,0)$ *on* $\mathbb{R}^n$. *Then for all* $t \geq 0$ *we have the estimate*

$$\left\| \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi)\chi(\xi)\widehat{f}(\xi)\,d\xi \right\|_{L^q(\mathbb{R}^n_x)} \leq C(1+t)^{-\frac{n-1}{\gamma}\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{W_p^{N_{p,j,t}}}\,, \tag{2.2.2}$$

where $\frac{1}{p}+\frac{1}{q} = 1$, $1 < p \leq 2$, *and the Sobolev order satisfies* $N_{p,j,t} \geq n(\frac{1}{p}-\frac{1}{q})-j$ *for* $0 \leq t < 1$, *and* $N_{p,j,t} \geq \left(n - \frac{n-1}{\gamma}\right)(\frac{1}{p}-\frac{1}{q}) - j$ *for* $t \geq 1$.

Theorem 2.2.6 will be proved in Section 6.6, where estimate (2.2.2) will follow by interpolation from the $L^2 - L^2$ estimate combined with $L^1 - L^\infty$ cases given in (6.6.1) for small t, and in (6.6.6) for large t. See those estimates also for the case of $p = 1$ in estimate (2.2.2). The estimate for large times will follow from Theorem 4.3.1, which gives the L^∞-estimate for the kernel of (2.2.2). As another consequence of Theorem 4.3.1, we will also have the following estimate:

Corollary 2.2.7. *Under conditions of Theorem 2.2.6 with* $\chi \equiv 1$, *assume that* $a \in C_0^\infty(\mathbb{R}^n)$. *Then for all* $x \in \mathbb{R}^n$ *and* $t \geq 0$ *we have the estimate*

$$\left| \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi)\,d\xi \right| \leq C(1+t)^{-\frac{n-1}{\gamma}}\,. \tag{2.2.3}$$

In Proposition 3.2.4 we show that properties (i)–(iv) of Theorem 2.2.6 are satisfied for characteristic roots of $L(D_t, D_x)$ in (2.0.1), while in Lemma 6.6.2 we will show that the index γ is also finite, both for large frequencies.

Now we turn to the case without convexity. As in the case of the homogeneous operators (see Introduction, Section 1.2) we introduce an analog of the order of contact also in the case where the convexity condition does not hold.

Definition 2.2.8. *Let Σ be a hypersurface in $\mathbb{R}^n$; set*

$$\gamma_0(\Sigma) := \sup_{\sigma\in\Sigma} \inf_{P} \gamma(\Sigma;\sigma,P) \leq \gamma(\Sigma),$$

where $\gamma(\Sigma;\sigma,P)$ is as in Definition 2.2.4.

Remark 2.2.9.

(a) When $n=2$, $\gamma_0(\Sigma)=\gamma(\Sigma)$;

(b) If $p(\xi)$ is a polynomial of order m, $\Sigma=\{\xi\in\mathbb{R}^n : p(\xi)=0\}$ is compact, and $\nabla p(\xi)\neq 0$ on Σ, then $\gamma_0(\Sigma)\leq\gamma(\Sigma)\leq m$; this is useful when applying the result below to hyperbolic differential equations and is proved in [Sug96].

Theorem 2.2.10. *Suppose $\tau:\mathbb{R}^n\to\mathbb{R}$ is a smooth function. Let $\chi\in C^\infty(\mathbb{R}^n)$; furthermore, on $\operatorname{supp}\chi$, we assume:*

(i) *for all multi-indices α there exist constants $C_\alpha>0$ such that*

$$|\partial_\xi^\alpha \tau(\xi)| \leq C_\alpha(1+|\xi|)^{1-|\alpha|};$$

(ii) *there exist constants $M,C>0$ such that for all $|\xi|\geq M$ we have $|\tau(\xi)|\geq C|\xi|$;*

(iii) *there exists a constant $C>0$ such that $|\partial_\omega\tau(\lambda\omega)|\geq C$ for all $\omega\in\mathbb{S}^{n-1}$ and $\lambda>0$;*

(iv) *there exists a constant $R_1>0$ such that, for all $\lambda>0$,*

$$\frac{1}{\lambda}\{\xi\in\mathbb{R}^n : \tau(\xi)=\lambda\}\subset B_{R_1}(0)\,.$$

Set $\gamma_0 := \sup_{\lambda>0}\gamma_0(\Sigma_\lambda(\tau))$ and assume it is finite. Let $a_j=a_j(\xi)\in S^{-j}_{1,0}$ be a symbol of order $-j$ of type $(1,0)$ on $\mathbb{R}^n$. Then for all $t\geq 0$ we have the estimate

$$\left\|\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi)\chi(\xi)\widehat{f}(\xi)\,d\xi\right\|_{L^q(\mathbb{R}^n_x)} \leq C(1+t)^{-\frac{1}{\gamma_0}\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_{W_p^{N_{p,j,t}}},$$

where $\frac{1}{p}+\frac{1}{q}=1$, $1<p\leq 2$, and the Sobolev order satisfies $N_{p,j,t}\geq n(\frac{1}{p}-\frac{1}{q})-j$ for $0\leq t<1$, and $N_{p,j,t}\geq\left(n-\frac{1}{\gamma_0}\right)(\frac{1}{p}-\frac{1}{q})-j$ for $t\geq 1$.

The proof of Theorem 2.2.10 will be given in Section 6.7. As in the convex case, as a consequence of estimates for the kernel on Theorem 5.1.2, we also have the following statement:

Corollary 2.2.11. *Under conditions of Theorem 2.2.10 with $\chi \equiv 1$, assume that $a \in C_0^\infty(\mathbb{R}^n)$. Then for all $x \in \mathbb{R}^n$ and $t \geq 0$ we have the estimate for the kernel:*

$$\left| \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi)\, d\xi \right| \leq C(1+t)^{-\frac{1}{\gamma_0}} .$$

Again, in Proposition 3.2.4 we show that properties (i)–(iv) of Theorem 2.2.10 are satisfied for characteristic roots of $L(D_t, D_x)$ in (2.0.1), while in Lemma 6.7.1 we will show that the index γ_0 is also finite, both for large frequencies.

As a corollary and an example of these theorems, we get the following possibilities of decay for parts of solutions with roots on the axis. We can use a cut-off function χ to microlocalise around points with different qualitative behaviour (hence we also do not have to worry about Sobolev orders).

Corollary 2.2.12. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $\tau : \Omega \to \mathbb{R}$ be a smooth real valued function. Let $\chi \in C_0^\infty(\Omega)$. Let us make the following choices of $K(t)$, depending on which of the following conditions are satisfied on* $\operatorname{supp}\chi$.

(1) *If* $\det \operatorname{Hess} \tau(\xi) \neq 0$ *for all $\xi \in \Omega$, we set $K(t) = (1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}$.*

(2) *If* $\operatorname{rank} \operatorname{Hess} \tau(\xi) = n-1$ *for all $\xi \in \Omega$, we set $K(t) = (1+t)^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{q})}$.*

(3) *If τ satisfies the convexity condition with index γ, we set $K(t) = (1+t)^{-\frac{n-1}{\gamma}(\frac{1}{p}-\frac{1}{q})}$.*

(4) *If τ does not satisfy the convexity condition but has non-convex index γ_0, we set $K(t) = (1+t)^{-\frac{1}{\gamma_0}(\frac{1}{p}-\frac{1}{q})}$.*

Assume in each case that other assumptions of the corresponding Theorems 2.2.1–2.2.10 *are satisfied. Let $1 \leq p \leq 2$, $\frac{1}{p}+\frac{1}{q}=1$. Then for all $t \geq 0$ we have*

$$\left|\left| \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi)\chi(\xi)\widehat{f}(\xi) d\xi \right|\right|_{L^q(\mathbb{R}^n_x)} \leq CK(t)||f||_{L^p(\mathbb{R}^n)}.$$

We note that no derivatives appear in the L^p–norm of f because the support of χ is bounded. In general, there are different ways to ensure the convexity condition for τ. Thus, we can say that the principal part L_m of operator $L(D_t, D_x)$ in (2.0.1) satisfies the convexity condition if all Hessians $\varphi_l''(\xi)$, $l = 1, \ldots, m$, are semi–definite for all $\xi \neq 0$. In this case it was shown by Sugimoto in [Sug94] that there exists a linear function $\alpha(\xi)$ such that $\widetilde{\varphi}_l = \varphi_l + \alpha$ have convex level sets $\Sigma(\widetilde{\varphi}_l)$, and we have $\gamma(\Sigma(\widetilde{\varphi}_l)) \leq 2\left[\frac{m}{2}\right]$. For large frequencies, perturbation arguments imply that the same must be true for $\Sigma_\lambda(\tau_l)$, for sufficiently large λ. If we now assume that $\Sigma_\lambda(\tau_l)$ are also convex for small λ, then τ_l will satisfy the convexity conditions. Alternatively, if they do not satisfy the convexity condition for small λ, we can cut-off this regions and analyse the decay rates by other methods developed in this paper.

2.3 Roots meeting the real axis

In this section we will present the results for characteristic roots (or phase functions) in the upper complex plane near the real axis, that become real at some point or in some set.

For $\mathcal{M} \subset \mathbb{R}^n$, denote $\mathcal{M}^\varepsilon = \{\xi \in \mathbb{R}^n : \mathrm{dist}(\xi, \mathcal{M}) < \varepsilon\}$ as before. The largest number $\nu \in \mathbb{N}$ such that $\mathrm{meas}(\mathcal{M}^\varepsilon) \leq C\varepsilon^\nu$ for all sufficiently small $\varepsilon > 0$, will be denoted by $\mathrm{codim}\, \mathcal{M}$, and we will call it the codimension of $\mathcal{M}$.

We will say that *the root τ_k meets the real axis at ξ^0 with order s_k* if $\mathrm{Im}\, \tau_k(\xi^0) = 0$ and if there exists a constant $c_0 > 0$ such that

$$c_0|\xi - \xi^0|^{s_k} \leq \mathrm{Im}\, \tau_k(\xi)\,,$$

for all ξ sufficiently near ξ^0. Here we may recall that in (2.0.2) we already assumed $\mathrm{Im}\, \tau_k(\xi) \geq 0$ for all ξ.

More generally, if the root τ_k meets the axis on the set

$$Z_k = \{\xi \in \mathbb{R}^n : \mathrm{Im}\, \tau_k(\xi) = 0\}\,,$$

we will say that *it meets the axis with order s* if

$$c_0\, \mathrm{dist}(\xi, Z_k)^s \leq \mathrm{Im}\, \tau_k(\xi)\,.$$

We will localise around each connected component of Z_k, e.g. around each point of Z_k, if it is a union of isolated points. As usual, when we talk about multiple roots intersecting in a set $\mathcal{M}$, we adopt the terminology introduced in Section 2.1. Since we are dealing with strictly hyperbolic equations, roots can meet each other only for bounded frequencies, so we may assume that set $\mathcal{M}$ is bounded.

Theorem 2.3.1. *Assume that the characteristic roots $\tau_1(\xi), \ldots, \tau_L(\xi)$ intersect in the C^1 set $\mathcal{M}$ of codimension ℓ. Assume also that they meet the real axis in $\mathcal{M}$ with the finite orders $\leq s$, i.e. that*

$$c_0\, \mathrm{dist}(\xi, \mathcal{M})^s \leq \mathrm{Im}\, \tau_k(\xi)\,,$$

for some $c_0 > 0$ and all $k = 1, \ldots, L$. Assume that (2.0.4) is the solution of the Cauchy problem (2.0.1) and we look at its part (2.1.4). Let $\chi \in C_0^\infty(\mathcal{M}^\varepsilon)$ for sufficiently small $\varepsilon > 0$. Then for all $t \geq 0$ we have

$$\Big\| D_t^r D_x^\alpha \Big(\int_{\mathcal{M}^\varepsilon} e^{ix\cdot\xi} \Big(\sum_{k=1}^{L} e^{i\tau_k(\xi)t} A_j^k(t,\xi) \Big) \chi(\xi) \widehat{f}(\xi)\, d\xi \Big) \Big\|_{L^q(\mathbb{R}^n_x)} \leq C(1+t)^{-\frac{\ell}{s}\left(\frac{1}{p}-\frac{1}{q}\right)+L-1} \|f\|_{L^p}, \quad (2.3.1)$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $1 \leq p \leq 2$.

We assume $\varepsilon > 0$ to be small enough to make sure that the type of behaviour assumed in the theorem is the only one that takes place in $\mathcal{M}^\varepsilon$. In the complement of $\mathcal{M}^\varepsilon$ we may use other theorems to analyse the decay rate. Moreover, we assume that set $\mathcal{M}$ is C^1. In fact, it is usually Lipschitz, so in order to avoid to go into depth about its structure and existence of almost everywhere differentiable coordinate systems, we make the technical C^1 assumption. The proof of Theorem 2.3.1 will be given in Section 7.3.

Let us now give a special case of this theorem where simple roots meet the axis at a point, so that we have $L = 1$ and $\ell = n$. The following statement is also global in frequency, so we have the result in Sobolev spaces.

Theorem 2.3.2. *Consider the m^{th} order strictly hyperbolic Cauchy problem (2.0.1) for operator $L(D_t, D_x)$, with initial data $f_j \in W_p^{N_p+|\alpha|+r-j}$, for $j = 0, \ldots, m-1$, where $1 \leq p \leq 2$ and $2 \leq q \leq \infty$ are such that $\frac{1}{p} + \frac{1}{q} = 1$, $r \geq 0$ and α is a multi-index. We assume that the Sobolev index N_p satisfies $N_p \geq n(\frac{1}{p} - \frac{1}{q})$ for $1 < p \leq 2$ and $N_1 > n$ for $p = 1$.*

Assume that the characteristic roots $\tau_1(\xi), \ldots, \tau_m(\xi)$ of $L(\tau, \xi) = 0$ satisfy $\operatorname{Im} \tau_k \geq 0$ for all k, and also the following conditions:

(H1) *for all $k = 1, \ldots, m$, we have*

$$\liminf_{|\xi|\to\infty} \operatorname{Im} \tau_k(\xi) > 0\,;$$

(H2) *for each $\xi^0 \in \mathbb{R}^n$ there is at most one index k for which $\operatorname{Im} \tau_k(\xi^0) = 0$ and there exists a constant $c > 0$ such that*

$$|\xi - \xi^0|^s \leq c \operatorname{Im} \tau_k(\xi),$$

for ξ in some neighbourhood of ξ^0. Assume also that there are finitely many points ξ^0 with $\operatorname{Im} \tau_k(\xi^0) = 0$.

Then the solution $u = u(t,x)$ to Cauchy problem (2.0.1) satisfies the following estimate for all $t \geq 0$:

$$\|D_t^r D_x^\alpha u(t,\cdot)\|_{L^q} \leq C_{\alpha,r}(1+t)^{-\frac{n}{s}(\frac{1}{p}-\frac{1}{q})} \sum_{j=0}^{m-1} \|f_j\|_{W_p^{N_p+|\alpha|+r-j}}. \tag{2.3.2}$$

Theorem 2.3.2 is proved in Section 6.11, where we will also give microlocal versions of this result around points ξ^0 from hypothesis (H2). In the complement of such points, we have roots separated from the real axis, so we get the exponential decay from Theorems 2.1.1 and 2.1.2. Moreover, in the exponential decay zone we may have different versions of the estimate, for example we can use estimate (2.1.3) there instead of (2.1.2). As a special case, such estimate together with (2.3.4) below (used with $s = s_1 = 2$), we improve the indices in Sobolev spaces over L^2 for the dissipative wave equation in (1.1.5) and (1.1.6) compared to [Mat77].

If conditions of Theorem 2.3.2 hold only with $\xi^0 = 0$, namely if $\operatorname{Im}\tau_k(\xi^0) = 0$ implies $\xi^0 = 0$, we will call the polynomial $L(\tau,\xi)$ *strongly stable.* Such polynomials will be discussed in more detail in applications in Section 8.5. Now we will give some improvements of (2.3.2) under additional assumptions on the roots:

Remark 2.3.3. *The order of time decay in Theorem 2.3.2 may be improved in the following cases, if we make additional assumptions. If, in addition, we assume that $\operatorname{Im}\tau_k(\xi^0) = 0$ in (H2) implies that $\xi^0 = 0$, then we actually get the estimate*

$$\left\|D_t^r D_x^\alpha u(t,\cdot)\right\|_{L^q(\mathbb{R}^n_x)} \leq C(1+t)^{-\frac{n}{s}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{|\alpha|}{2}} \sum_{j=0}^{m-1} \|f_j\|_{W_p^{N_p+|\alpha|+r-j}},$$

where here and further in this remark N_p is as in Theorem 2.3.2.

Now, assume further that for all ξ^0 in (H2) we also have the estimate

$$|\tau_k(\xi)| \leq c_1|\xi-\xi^0|^{s_1}, \tag{2.3.3}$$

with some constant $c_1 > 0$, for all ξ sufficiently close to ξ^0.

If we have that $\operatorname{Im}\tau_k(\xi^0) = 0$ in (H2) implies that we have (2.3.3) around such ξ^0, then we actually get

$$\left\|D_t^r D_x^\alpha u(t,\cdot)\right\|_{L^q(\mathbb{R}^n_x)} \leq C(1+t)^{-\left(\frac{n}{s}\right)\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{rs_1}{s}} \sum_{j=0}^{m-1} \|f_j\|_{W_p^{N_p+|\alpha|+r-j}}.$$

And finally, assume that for all ξ^0 such that $\operatorname{Im}\tau_k(\xi^0) = 0$ in (H2), we also have $\xi^0 = 0$ and (2.3.3) around such ξ^0. Then we actually get

$$\left\| D_t^r D_x^\alpha u(t,\cdot) \right\|_{L^q(\mathbb{R}^n_x)} \leq C(1+t)^{-\frac{n}{s}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{|\alpha|}{s}-\frac{rs_1}{s}} \sum_{j=0}^{m-1} \|f_j\|_{W_p^{N_p+|\alpha|+r-j}} . \quad (2.3.4)$$

Estimate (2.3.4) with $s = s_1 = 2$ gives the decay estimate for the dissipative wave equation in (1.1.5). The proof of this remark is given in Remark 6.11.3.

Moreover, there are other possibilities of *multiple roots intersecting each other while lying entirely on the real axis.* For example, this is the case for the wave equation or for more general equations with homogeneous symbols, when several roots meet at the origin. In this case roots always lie on the real axis, but they become irregular at the point of multiplicity, which is the origin for homogeneous roots. In the case when lower order terms are presents, characteristics roots are not homogeneous in general, so we can not eliminate time from the estimates as was done in Section 1.2. It means that we have to look at the structure of such multiple points by making cut-offs around them and studying their structure in more detail. In particular, there is an interaction between low frequencies and large times, which does not take place for homogeneous symbols. The detailed discussion of this topic and corresponding decay rates will be determined in Section 7.4.

2.4 Application to the Cauchy problem

Putting together theorems from previous sections we obtain the following conclusion about solutions to the Cauchy problem (2.0.1). We will first formulate the following general result collecting statements of previous sections, and then will explain how this result can be used.

Theorem 2.4.1. *Suppose $u = u(t,x)$ is the solution of the m^{th} order linear, constant coefficient, strictly hyperbolic Cauchy problem (2.0.1). Denote the characteristic roots of the operator by $\tau_1(\xi), \ldots, \tau_m(\xi)$, and assume that $\operatorname{Im}\tau_k(\xi) \geq 0$ for all $k = 1, \ldots, n$, and all $\xi \in \mathbb{R}^n$.*

We introduce two functions, $K^{(l)}(t)$ and $K^{(b)}(t)$, which take values as follows:

I. *Consider the behaviour of each characteristic root, $\tau_k(\xi)$, in the region $|\xi| \geq M$, where M is a large enough real number. The following table gives values for the function $K_k^{(l)}(t)$ corresponding to possible properties* *of $\tau_k(\xi)$; if $\tau_k(\xi)$ satisfies more than one, then take $K_k^{(l)}(t)$ to be function that decays the slowest as $t \to \infty$.*

Location of $\tau_k(\xi)$	Additional Property	$K_k^{(l)}(t)$
away from real axis		$e^{-\delta t}$, some $\delta > 0$
on real axis	$\det \operatorname{Hess} \tau_k(\xi) \neq 0$	$(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}$
	$\operatorname{rank} \operatorname{Hess} \tau_k(\xi) = n-1$	$(1+t)^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{q})}$
	convexity condition γ	$(1+t)^{-\frac{n-1}{\gamma}(\frac{1}{p}-\frac{1}{q})}$
	no convexity condition, γ_0	$(1+t)^{-\frac{1}{\gamma_0}}$

Then take $K^{(l)}(t) = \max_{k=1\dots,n} K_k^{(l)}(t)$.

II. *Consider the behaviour of the characteristic roots in the bounded region $|\xi| \leq M$; again, take $K^{(b)}(t)$ to be the maximum (slowest decaying) function for which there are roots satisfying the conditions in the following table:*

Location of Root(s)	Properties	$K^{(\mathrm{b})}(t)$
away from axis	no multiplicities	$e^{-\delta t}$, some $\delta > 0$
	L roots coinciding	$(1+t)^L e^{-\delta t}$
on axis,	$\det \operatorname{Hess} \tau_k(\xi) \neq 0$	$(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}$
no multiplicities *	convexity condition γ	$(1+t)^{-\frac{n-1}{\gamma}(\frac{1}{p}-\frac{1}{q})}$
	no convexity condition, γ_0	$(1+t)^{-\frac{1}{\gamma_0}(\frac{1}{p}-\frac{1}{q})}$
on axis, multiplicities*,**	L roots coincide on set of codimension ℓ	$(1+t)^{L-1-\ell}$
meeting axis with finite order s	L roots coincide on set of codimension ℓ	$(1+t)^{L-1-\frac{\ell}{s}(\frac{1}{p}-\frac{1}{q})}$

* *These two cases of roots lying on the real axis require some additional regularity assumptions; see corresponding microlocal statements for details.*
** *This is the $L^1 - L^\infty$ rate in a shrinking region; see Proposition 7.4.3 for details. For different types of L^2 estimates see Section 7.4, and then interpolate.*

Then, with $K(t) = \max\left(K^{(b)}(t), K^{(l)}(t)\right)$, the following estimate holds:

$$\|D_x^\alpha D_t^r u(t,\cdot)\|_{L^q} \leq C_{\alpha,r} K(t) \sum_{l=0}^{m-1} \|f_l\|_{W_p^{N_p - l}},$$

where $1 \leq p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, and $N_p = N_p(\alpha, r)$ is a constant depending on p, α and r.

The scheme of the proof of this theorem and precise relations to microlocal theorems of previous sections will be given in Section 2.5. However, let us now briefly explain how to understand this theorem. Since the decay rates do depend on the behaviour of characteristic roots in different regions and theorems from previous sections determine the corresponding rates, in Theorem 2.4.1 we single out properties which determine the final decay rate. Since the same characteristic root, say τ_k, may exhibit different properties in different regions, we look at the corresponding rates $K^{(\mathrm{b})}(t), K^{(\mathrm{l})}(t)$ under each possible condition and then take the slowest one for the final answer. The value of the Sobolev index $N_p = N_p(\alpha, r)$ depends on the regions as well, and it can be found from microlocal statements of previous sections for each region.

In conditions of Part I of the theorem, it can be shown by the perturbation arguments that only three cases are possible for large ξ, namely, the characteristic root may be uniformly separated from the real axis, it may lie on the axis, or it may converge to the real axis at infinity. If, for example, the root lies on the axis and, in addition, it satisfies the convexity condition with index γ, we get the corresponding decay rate $K^{(\mathrm{l})}(t) = (1+t)^{-\frac{n-1}{\gamma}(\frac{1}{p}-\frac{1}{q})}$. Indices γ and γ_0 in the tables are defined as the maximum of the corresponding indices $\gamma(\Sigma_\lambda)$ and $\gamma_0(\Sigma_\lambda)$, respectively, where $\Sigma_\lambda = \{\xi : \tau_k(\xi) = \lambda\}$, over all k and over all λ, for which ξ lies in the corresponding region. At present, we do not have examples of characteristic roots tending to the real axis for large frequencies while remaining in the open upper half of the complex plane, so we do not give any estimates for this case in Theorem 2.4.1. However, in Section 6.8 we will still discuss what happens in this case.

The statement in Part II is more involved since we may have multiple roots intersecting on rather irregular sets. The number L of coinciding roots corresponds to the number of roots which actually contribute to the loss of regularity. For example, operator $(\partial_t^2 - \Delta)(\partial_t^2 - 2\Delta)$ would have $L = 2$ for both pairs of roots $\pm|\xi|$ and $\pm\sqrt{2}|\xi|$, intersecting at the origin. Meeting the axis with finite order s means that we have the estimate

$$\operatorname{dist}(\xi, Z_k)^s \leq c|\operatorname{Im}\tau_k(\xi)| \tag{2.4.1}$$

for all the intersecting roots, where $Z_k = \{\xi : \operatorname{Im}\tau_k(\xi) = 0\}$. In Part II of Theorem 2.4.1, the condition that L roots meet the axis with finite order s on a set of codimension ℓ means that all these estimates hold and that there is a (C^1) set $\mathcal{M}$ of codimension ℓ such that $Z_k \subset \mathcal{M}$ for all corresponding k (see Theorem 2.3.1 for details). In Theorem 2.3.2 we discuss the special case of a single root τ_k meeting the axis at a point ξ_0 with order s, which means that $\operatorname{Im}\tau_k(\xi_0) = 0$ and that we have the estimate $|\xi - \xi_0|^s \leq c|\operatorname{Im}\tau_k(\xi)|$. In fact, under certain conditions an improvement in this part of the estimates is possible, see Theorem 2.3.2 and Remark 2.3.3.

In Part II of the theorem, condition ** is formulated in the region of the size decreasing with time: if we have L multiple roots which coincide on the real axis on a set $\mathcal{M}$ of codimension ℓ, we have an estimate

$$|u(t,x)| \leq C(1+t)^{L-1-\ell} \sum_{l=0}^{m-1} \|f_l\|_{L^1}, \tag{2.4.2}$$

if we cut off the Fourier transforms of the Cauchy data to the ϵ-neighbourhood $\mathcal{M}^\epsilon$ of $\mathcal{M}$ with $\epsilon = 1/t$. Here we may relax the definition of the intersection above and say that if L roots coincide in a set $\mathcal{M}$, then they coincide on a set of codimension ℓ if the measure of the ϵ-neighborhood $\mathcal{M}^\epsilon$ of $\mathcal{M}$ satisfies $|\mathcal{M}^\epsilon| \leq C\epsilon^\ell$ for small $\epsilon > 0$; here $\mathcal{M}^\epsilon = \{\xi \in \mathbb{R}^n : \operatorname{dist}(\xi, \mathcal{M}) \leq \epsilon\}$. The estimate (2.4.2) follows from the procedure described in Section 7.1 of the resolution of multiple roots, and details and proof of estimate (2.4.2) are given in Section 7.4, especially in Proposition 7.4.3.

We can then combine this with the remaining cases outside of this neighborhood, where it is possible to establish decay by different arguments. In particular, this is the case of homogeneous equations with roots intersecting at the origin. However, one sometimes needs to introduce special norms to handle L^2-estimates around the multiplicities. Details of this are given in the L^2 part of Section 7.4, in particular in Proposition 7.4.2. Finally, in the case of a simple root we may set $L = 1$, and $\ell = n$, if it meets the axis at a point.

2.5 Schematic of method

Let us briefly explain some ideas behind the reduction of Theorem 2.4.1 to the proceeding theorems. The realisation of the steps below will be done in Sections 6 and 7.

Step 1: Representation of the solution.

Using the Fourier transform in x, this reduces the problem to studying time-dependent oscillatory integrals, at least for frequencies with no multiplicities. In the case near multiplicities we will introduce a special procedure to deal with them in Section 7.

Step 2: Division of the integral.

We reduce the problem to several microlocal cases using suitable cut-off functions. The problem is divided into studying the behaviour of the characteristic roots in three regions of the phase space—large $|\xi|$, bounded $|\xi|$ away from multiplicities of roots and bounded $|\xi|$ in a neighbourhood of multiplicities.

Step 3: Interpolation reduces problem to finding $L^1 - L^\infty$ and $L^2 - L^2$ estimates.

Step 4: Large $|\xi|$:

- root separated from the real axis (Theorem 2.1.1);
- root lying on the real axis (Theorems 2.2.2–2.2.10).

Step 5: Bounded $|\xi|$, away from multiplicities:

- root away from the real axis (Theorem 2.1.1);
- root meeting the real axis with finite order (Theorem 2.3.2);
- root lying on the real axis (Theorems 2.2.2–2.2.10).

Step 6: Bounded $|\xi|$, around multiplicities of roots:

- all intersecting roots away from the real axis (Theorem 2.1.2);
- all intersecting roots lie on the real axis around the multiplicity (Section 7.4);
- all intersecting roots meet the real axis with finite order (Theorem 2.3.1);
- one or more of the roots meets the real axis with infinite order (similar to Theorems 2.2.2–2.2.10).

2.6 Strichartz estimates and nonlinear problems

Let us denote by $\kappa_{p,q}(L(D_t, D_x))$ the time decay rate for the Cauchy problem (2.0.1), so that function $K(t)$ from Theorem 2.4.1 satisfies $K(t) \simeq t^{-\kappa_{p,q}(L)}$ for large t. Thus, for polynomial decay rates, we have

$$\kappa_{p,q}(L) = -\lim_{t\to\infty} \frac{\ln K(t)}{\ln t}. \tag{2.6.1}$$

We will also abbreviate the important case $\kappa(L) = \kappa_{1,\infty}(L)$ since by interpolation we have $\kappa_{p,p'} = \kappa_{2,2}\frac{2}{p'} + \kappa_{1,\infty}(\frac{1}{p} - \frac{1}{p'})$, $1 \le p \le 2$. These indices $\kappa(L)$ and $\kappa_{p,p'}(L)$ of operator $L(D_t, D_x)$ will be responsible for the decay rate in the Strichartz estimates for solutions to (2.0.1), and for the subsequent well-posedness properties of the corresponding semilinear equation which are discussed below.

In order to present an application to nonlinear problems let us first consider the inhomogeneous equation

$$\begin{cases} L(D_t, D_x)u = f, \quad t > 0, \\ D_t^l u(0,x) = 0, \quad l = 0, \dots, m-1, \ x \in \mathbb{R}^n, \end{cases} \tag{2.6.2}$$

with $L(D_t, D_x)$ as in (1.0.1). By the Duhamel's formula the solution can be expressed as

$$u(t) = \int_0^t E_{m-1}(t-s) f(s) ds, \tag{2.6.3}$$

where E_{m-1} is given in (2.0.4). Let $\kappa = \kappa_{p,p'}(L)$ be the time decay rate of operator L, determined by Theorem 2.4.1 and given in (2.6.1). Then Theorem 2.4.1 implies that we have estimate

$$||E_{m-1}(t)g||_{W^s_{p'}} \le C(1+t)^{-\kappa} ||g||_{W^s_p}.$$

Together with (2.6.3) this implies

$$||u(t)||_{W^s_{p'}(\mathbb{R}^n_x)} \le C \int_0^t (t-s)^{-\kappa} ||f(s)||_{W^s_p} ds \le C|t|^{-\kappa} * ||f(t)||_{W^s_p}.$$

By the Hardy–Littlewood–Sobolev theorem this is $L^q(\mathbb{R}) - L^{q'}(\mathbb{R})$ bounded if $1 < q < 2$ and $1 - \kappa = \frac{1}{q} - \frac{1}{q'}$. Therefore, this implies the following Strichartz estimate:

Theorem 2.6.1. *Let $\kappa_{p,p'}$ be the time decay rate of the operator $L(D_t, D_x)$ in the Cauchy problem (2.6.2). Let $1 < p, q < 2$ be such that $1/p + 1/p' = 1/q + 1/q' = 1$ and $1/q - 1/q' = 1 - \kappa_{p,p'}$. Let $s \in \mathbb{R}$. Then there is a constant C such that the solution u to the Cauchy problem (2.6.2) satisfies*

$$||u||_{L^{q'}(\mathbb{R}_t, W^s_{p'}(\mathbb{R}^n_x))} \le C ||f||_{L^q(\mathbb{R}_t, W^s_p(\mathbb{R}^n_x))},$$

for all data right hand side $f = f(t,x)$.

By the standard iteration method we obtain the well-posedness result for the following semilinear equation

$$\begin{cases} L(D_t, D_x)u = F(t,x,u), \quad t > 0, \\ D_t^l u(0,x) = f_l(x), \quad l = 0, \dots, m-1, \ x \in \mathbb{R}^n. \end{cases} \tag{2.6.4}$$

Theorem 2.6.2. *Let* $\kappa_{p,p'}$ *be the time decay index of the operator* $L(D_t, D_x)$ *in the Cauchy problem (2.6.4). Let* p, q *be such that* $1/p+1/p' = 1/q+1/q' = 1$ *and* $1/q - 1/q' = 1 - \kappa_{p,p'}$*. Let* $s \in \mathbb{R}$*.*

Assume that for any $v \in L^{q'}(\mathbb{R}_t, W^s_{p'}(\mathbb{R}^n_x))$*, the nonlinear term satisfies* $F(t,x,v) \in L^q(\mathbb{R}_t, W^s_p(\mathbb{R}^n_x))$*. Moreover, assume that for every* $\varepsilon > 0$ *there exists a decomposition* $-\infty = t_0 < t_1 < \cdots < t_k = +\infty$ *such that the estimates*

$$||F(t,x,u) - F(t,x,v)||_{L^q(I_j, W^s_p(\mathbb{R}^n_x))} \leq \varepsilon ||u-v||_{L^{q'}(I_j, W^s_{p'}(\mathbb{R}^n_x))}$$

hold for the intervals $I_j = (t_j, t_{j+1})$*,* $j = 0, \ldots, k-1$*.*

Finally, assume that the solution of the corresponding homogeneous Cauchy problem is in the space $L^{q'}(\mathbb{R}_t, W^s_{p'}(\mathbb{R}^n_x))$*.*

Then the semilinear Cauchy problem (2.6.4) has a unique solution in the space $L^{q'}(\mathbb{R}_t, W^s_{p'}(\mathbb{R}^n_x))$*.*