CHAPTER 5

Irrationality and dynamics

The set of values of scl on all conjugacy classes in all finitely presented groups is a countable set. It is natural to try to characterize this set of real numbers, and to understand what kinds of arithmetic constraints exist on the values of scl in certain classes of groups.

As discussed in Chapter 4, the Rationality Theorem (i.e. Theorem 4.24) shows that for free groups (and more generally, for PQL groups) the scl norm is rational, and in particular, scl takes on values in $\mathbb{Q}$ in free groups. More generally, we saw that the unit ball of the scl norm on $B_1^H(F)$ is a rational polyhedron, and discussed the relationship of this example to the (polyhedral) Thurston norm on H_2 of an atoroidal irreducible 3-manifold.

It is natural to ask for which groups G the stable commutator length is rational on $[G, G]$. In fact, Gromov ([**99**], 6.C) explicitly asked whether scl is always rational, or at least algebraic, in general finitely presented groups. In the next section we describe an unexpected and elegant example due to Dongping Zhuang [**205**] of a finitely presented group in which the stable commutator length achieves *transcendental values*, thus answering Gromov's question in the negative.

There are two essential ingredients in Zhuang's examples: the groups he considers are *transformation groups* (i.e. groups of automorphisms of some geometric object), and they have an *arithmetic* origin. It is a general phenomenon, observed explicitly by Burger–Monod, Carter–Keller–Paige (as exposed by Dave Witte-Morris) and others, that (especially arithmetic) lattices in higher rank Lie groups generally admit no (nontrivial) quasimorphisms. On the other hand, such groups sometimes have nontrivial 2-dimensional bounded cohomology classes, which typically have a symplectic (or "causal") origin, which can be detected dynamically by realizing the groups as transformation groups. A central extension of such a group admits a nontrivial, but *finite dimensional* space of homogeneous quasimorphisms, and one may compute scl on such a group directly by Bavard duality, relating scl to dynamics.

In § 5.1 we discuss Zhuang's examples, which in some ways are the most elementary. In § 5.2 we discuss lattices in higher rank Lie groups from several different perspectives, eventually concentrating on lattices in symplectic groups as the most interesting examples. Finally, in § 5.3, we discuss some nonlinear generalizations of these ideas, which leads to the construction of quasimorphisms on braid groups and certain (low-dimensional) groups of area-preserving diffeomorphisms of surfaces. References for this chapter include [**28, 192, 205, 33, 34, 53, 159, 7, 86, 87**].

5.1. Stein–Thompson groups

In 1965, Richard Thompson [**195**] defined three groups $F \subset T \subset V$. Two of these (the groups T and V) were the first examples of finitely-presented, infinite simple groups. They can be defined as transformation groups (i.e. as groups of homeomorphisms of certain topological spaces): F is a group of homeomorphisms of an interval, T is a group of homeomorphisms of a circle, and V is a group of homeomorphisms of a Cantor set. Our interest in this section is on the groups F and T, and their generalizations. A basic reference for Thompson's groups is [**52**].

Definition 5.1. F is the group of orientation-preserving piecewise-linear (hereafter PL) homeomorphisms of the closed unit interval that are differentiable except at finitely many dyadic rational numbers (i.e. numbers of the form $p/2^q$ for integers p, q), and such that away from these discontinuities, the derivative is locally constant, and is equal to a power of 2.

T is the group of orientation-preserving PL homeomorphisms of the unit circle S^1 (thought of as $\mathbb{R}/\mathbb{Z}$) that maps dyadic rationals to dyadic rationals, has derivatives that are discontinuous at finitely many dyadic rationals, and are elsewhere equal to powers of 2.

Remark 5.2. All three groups can be defined as groups of *rotations* (in the sense of computer science) of infinite trivalent trees. In the case of F, the tree is rooted and planar; in the case of T, the tree is planar; in the case of V, the tree is neither rooted nor planar. See e.g. [**52**] § 2 or [**189**].

In this section we are interested in generalizations of the groups F and T due to Melanie Stein [**192**].

Definition 5.3. Let P be a multiplicative subgroup of the positive real numbers, and let A be a $\mathbb{Z}P$-submodule of the reals with $P \cdot A = A$. Choose a positive number $l \in A$. Define $F(l, A, P)$ to be the group of PL homeomorphisms of the interval $[0, l]$ taking $A \cap [0, l]$ to itself, whose derivatives have finitely many singularities in A, and take values in P.

Similarly, define $T(l, A, P)$ to be the group of PL homeomorphisms of the circle $\mathbb{R}/\langle l \rangle$ taking $A/\langle l \rangle$ to itself, whose derivatives have finitely many singularities in A, and take values in P.

Informally, we say that elements of $F(l, A, P)$ or $T(l, A, P)$ have *breakpoints* in A, and *slopes* in P.

Example 5.4. In this notation, Thompson's groups F and T are $F(1, \mathbb{Z}[\frac{1}{2}], \langle 2 \rangle)$ and $T(1, \mathbb{Z}[\frac{1}{2}], \langle 2 \rangle)$ respectively.

Stein showed in [**192**], following published and unpublished work of Brown [**28**], that for $l \in \mathbb{Z}$, for $A = \mathbb{Z}[1/n_1 n_2 \cdots n_k]$ and for $P = \langle n_1, \cdots, n_k \rangle$, the groups $F(l, A, P)$ and $T(l, A, P)$ are finitely presented, and in fact FP_∞ (i.e. there is a $K(G, 1)$ for these groups with only finitely many cells in each dimension). The method of proof is to explicitly find such a $K(G, 1)$. This is done by finding an action of these groups on suitable (explicitly described) contractible cubical complexes, such that the quotient complexes are homotopy equivalent to complexes with only finitely many cells in each dimension.

Example 5.5. A presentation for Thompson's group F is

$$F = \langle A, B \mid [AB^{-1}, A^{-1}BA], [AB^{-1}, A^{-2}BA^2] \rangle$$

A presentation for T is

$$\begin{aligned} T = \langle A, B, C \mid & [AB^{-1}, A^{-1}BA], [AB^{-1}, A^{-2}BA^2], C^{-1}B(A^{-1}CB), \\ & ((A^{-1}CB)(A^{-1}BA))^{-1}B(A^{-2}CB^2), (CA)^{-1}(A^{-1}CB)^2, C^3 \rangle \end{aligned}$$

These presentations are not terribly useful in practice, except that they do indicate algebraically how F is included as a subgroup of T. See [**52**], § 3 and § 5.

Zhuang's examples are central extensions of $T(l, A, P)$ for certain A and P as above. The remainder of this section is taken more or less verbatim from [**205**].

5.1.1. Factorization lemma. With notation as above, let $IP * A$ denote the submodule of A generated by elements of the form $(1-p)a$ where $a \in A$ and $p \in P$. In the sequel we sometimes abbreviate $T(l, A, P)$ by T for the sake of legibility (but T used in this sense should *not* be confused with Thompson's T).

LEMMA 5.6 (Stein [**192**]). *There is a natural homomorphism*

$$\nu : T(l, A, P) \to A/\langle IP * A, l\rangle$$

defined by $\nu(f) = f(a) - a$ *for* $f \in T$ *and* $a \in [0, l] \cap A$. *If* B *denotes the kernel of* ν, *then* $B' = T''$, *the second commutator subgroup of* T.

We use the following criterion of Bieri–Strebel (a proof appears in the appendix to [**192**]):

LEMMA 5.7 (Bieri–Strebel [**14**]). *Let* $a, c, a', c' \in A$ *with* $a < c, a' < c'$. *There is a* PL *homeomorphism of* $\mathbb{R}$, *with slopes in* P *and finitely many singularities in* A, *mapping* $[a, c]$ *onto* $[a', c']$ *iff* $c' - a'$ *is congruent to* $c - a$ *modulo* $IP * A$.

Lemma 5.6 and Lemma 5.7 together let one construct elements of T with desired properties. Let $f \in B$ be arbitrary. Zhuang proves the following factorization lemma.

LEMMA 5.8 (Zhuang [**205**], Lem. 3.4). *For any* $f \in B$ *there is a factorization* $f = g_1 g_2$ *in* B *where* g_1 *and* g_2 *both fix nonempty open arcs.*

PROOF. Note that any element which fixes a nonempty open arc fixes some point a in A, and is therefore in B by Lemma 5.6.

Let $f \in B$ be arbitrary. Choose points $a < b < a_1 < b_1 < c < d \in [0, l] \cap A$ such that $f([a, b]) = [a_1, b_1]$. Since $a_1 - a, b_1 - b \in IP * A$ (by the definition of B), Lemma 5.7 implies that there are PL homeomorphisms h_1, h_2 with slopes in P and singularities in A, sending $[b, c]$ to $[b_1, c]$ and $[d, a]$ to $[d, a_1]$ respectively. Now define

$$g = \begin{cases} f \text{ if } x \in [a, b] \\ h_1 \text{ if } x \in [b, c] \\ \mathrm{id} \text{ if } x \in [c, d] \\ h_2 \text{ if } x \in [d, a] \end{cases}$$

Set $g_1 = fg^{-1}$ and $g_2 = g$. Then $f = g_1 g_2$, and both g_1 and g_2 fix nonempty open arcs. □

REMARK 5.9. Factorization or "fragmentation" lemmas, together with Mayer–Vietoris and Künneth formulae, are generally the key to computing the (bounded co-) homology of transformation groups. Such techniques are used pervasively in the theory of foliations; see e.g. Tsuboi's survey [**199**].

For each $\theta \in IP * A$ the rotation R_θ is in B. The set of such θ is dense in $[0, l]$. So for $i = 1, 2$, let g_i be as in Lemma 5.8, and choose θ_i so that $R_{\theta_i} \in B$, and $h_i := R_{\theta_i} g_i R_{\theta_i}^{-1}$ has support contained in $(0, l)$.

5.1.2. Calculation of commutator subgroup. Let $F(l, A, P)$ denote the subgroup of $T(l, A, P)$ fixing 0. We abbreviate $F(l, A, P)$ by F, and think of F as a group of PL homeomorphisms of the interval $[0, l]$. Notice that $F \subset B$. There is a natural homomorphism

$$\rho : F \to P \times P$$

defined by $\rho(f) = (f'(0+), f'(l-))$; i.e. the image of ρ is the pair of elements of P consisting of the derivative of f at 0 from the right, and the derivative of f at l from the left. Let $B_1 = \ker \rho$. Note that $h_1, h_2 \in B_1$, since their support is contained strictly in the interior of $[0, l]$.

THEOREM 5.10 (Stein [**192**]). *With notation as above, the commutator subgroup B_1' is simple, and $B_1' = F'$.*

On the other hand, one has the following theorem of Brown (see [**192**] for a proof):

THEOREM 5.11 (Brown). *With notation as above, there is an isomorphism*

$$H_*(F) \cong H_*(B_1) \otimes H_*(P \times P)$$

We now specialize to the case that $l = 1, A = \mathbb{Z}[\frac{1}{pq}], P = \langle p, q \rangle$. Here p and q are arbitrary integers which form a basis for $\langle p, q \rangle$ (this is satisfied for example if p and q are distinct primes). We write $T_{p,q}, F_{p,q}$ for $T(l, A, P), F(l, A, P)$ in this case.

In [**192**], Stein explicitly calculates the homology of such $F_{p,q}$.

LEMMA 5.12 (Stein, [**192**] Thm. 4.7). *With notation as above, $H_1(F_{p,q})$ is free Abelian with rank $2(d+1)$ where d is the greatest common divisor of $p-1$ and $q-1$.*

If $d = 1$ (for instance if $p = 2, q = 3$), Lemma 5.12 implies that $H_1(F_{p,q}) = \mathbb{Z}^4 = H_1(P \times P)$. Theorem 5.11 therefore implies that $H_1(B_1) = 1$ and therefore $B_1 = B_1' = F_{p,q}'$. By Lemma 5.8 and the definition of the h_i, we see that every element of B can be written as a product of conjugates of commutators in $B_1 \subset B$. In particular, B is perfect.

By Lemma 5.6, $B = B' = T_{p,q}''$. Since $T_{p,q}' \subset B$ (because B is the kernel of ν, which is a map from T to an Abelian group) we get $B = T_{p,q}'$. Furthermore, when $l = 1$ and $d = 1$, the submodule $\langle IP * A, 1 \rangle$ is actually equal to A, so ν is the zero map. Hence $T_{p,q}$ is perfect in this case.

5.1.3. Calculation of scl**.** The final ingredient we need is the following:

THEOREM 5.13 (Calegari [**41**], Thm. A). *Let G be a subgroup of* $\mathrm{PL}^+(I)$*. Then* scl *vanishes on* $[G, G]$*.*

PROOF. Let $g \in [G, G]$, and let H be a finitely generated subgroup so that $g \in [H, H]$. The fixed point set of any element of $\mathrm{PL}^+(I)$ is a finite union of points and closed intervals, so the same is true for the common fixed point set of a finitely generated group. Let $\mathrm{fix}(H)$ denote this common fixed point set, and enumerate the (finitely many) complementary open intervals as $I_1, I_2, \cdots, I_m$.

For each interval I_j there is a homomorphism $\rho_j : H \to \mathbb{R} \oplus \mathbb{R}$ defined by $\rho_j(h) = (\log dh^+(I_j^-), \log dh^-(I_j^+))$ where I_j^+ denotes the positive endpoint of the interval I_j, and I_j^- denotes the negative endpoint, and dh^+, dh^- denotes derivative from the right and from the left respectively. Let $\rho : H \to \mathbb{R}^{2m}$ be the direct sum of these homomorphisms, and let H_0 denote the kernel. Suppose $h \in [H_0, H_0]$, and let

K be a finitely generated subgroup of H_0 with $h \in [K, K]$. Then $\mathrm{fix}(K)$ contains a neighborhood of each endpoint of each interval I_j, so there are closed intervals I'_j contained in the interior of the I_j such that the support of K is contained in the union of the I'_j.

For each closed interval J contained in the interior of some I_i, there is $j \in H$ with $j(J) \cap J = \emptyset$. By replacing j by its inverse if necessary, there is such a j which moves J to the right. We assume by induction that for any set of intervals J_i closed in the interior of each I_i, there is $j \in H$ with $j(J_i) \cap J_i = \emptyset$ for all $1 \le i \le r$. Let k satisfy $k(J_{r+1}) \cap J_{r+1} = \emptyset$ and k moves J_{r+1} to the right. Let J'_i be the smallest closed interval in the interior of I_i containing $J_i \cup k(J_i)$. By the induction hypothesis there is $j' \in H$ with $j'(J'_i) \cap J'_i = \emptyset$ for $1 \le i \le r$. Replacing j' by its inverse if necessary, we may further assume that j' moves the leftmost point of J_{r+1} to the right. Then $j'k(J_i) \cap J_i = 0$ for $1 \le i \le r+1$. It follows that we can find a single element $j \in H$ such that $j(I'_i) \cap I'_i = \emptyset$ for all i simultaneously.

For any n there is an injection $\Delta_n : K \to H$ defined by

$$\Delta_n(c) = \prod_{i=0}^{n} c^{j^i}$$

where j is as above, and the superscript denotes conjugation. Define

$$h' = \prod_{i=0}^{n} (h^{i+1})^{j^i}$$

Then $[h', j] = \Delta_n(h)(h^{-n-1})^{j^{n+1}}$. On the other hand, if $h = [a_1, b_1][a_2, b_2] \cdots [a_s, b_s]$ with $a_i, b_i \in K$ then $\Delta_n(h) = [\Delta_n(a_1), \Delta_n(b_1)] \cdots [\Delta_n(a_s), \Delta_n(b_s)]$. It follows that $\mathrm{cl}(h^{n+1}) \le s+1$ in H and therefore $\mathrm{scl}(h) = 0$, also in H. Since $h \in [H_0, H_0]$ was arbitrary, it follows that scl in H vanishes identically on $[H_0, H_0]$. On the other hand, $H/[H_0, H_0]$ is two-step solvable, and therefore amenable. Since scl vanishes in the commutator subgroup of an amenable group, for every element $g \in [H, H]$ there is a power n such that

$$g^n = [a_1, b_1] \cdots [a_s, b_s]c$$

where s/n is as small as we like, and $c \in [H_0, H_0]$. If ϕ is a homogeneous quasimorphism on H of defect 1, then ϕ vanishes on c, and therefore has value $\le 2s$ on g^n. Hence $\mathrm{scl}(g) = 0$ in H, and therefore also in G. Since $g \in [G, G]$ was arbitrary, the theorem is proved. □

Remark 5.14. Notice the use of the Münchhausen trick (i.e. Example 3.66) in the construction of Δ_n.

We are now in a position to determine scl in $T_{p,q}$.

Lemma 5.15 (Zhuang [**205**], Lem. 3.8). *Let $T_{p,q}$ be as above where $d = \gcd(p-1, q-1) = 1$. Then* scl *vanishes on $T'_{p,q} = T_{p,q}$.*

Proof. Let ϕ be a homogeneous quasimorphism on $T_{p,q}$, and let $f \in T_{p,q}$ be arbitrary. By Lemma 5.8 we can write $f = g_1 g_2$ and $h_i = R_{\theta_i} g_i R_{\theta_i}^{-1}$ where each $h_i \in B_1$. Since B_1 is a perfect subgroup of $\mathrm{PL}^+(I)$, Theorem 5.13 implies that $\mathrm{scl}(h_i) = 0$ in B_1. Note that ϕ restricts to a homogeneous quasimorphism on B_1, and therefore by Bavard's Duality Theorem 2.70 we have $\phi(h_i) = 0$, and therefore $\phi(g_i) = 0$. But $f = g_1 g_2$, so

$$|\phi(f)| \le D(\phi)$$

by the definition of the defect. Since f was arbitrary, ϕ is uniformly bounded on $T_{p,q}$. A bounded homogeneous quasimorphism is identically zero. Since ϕ was arbitrary, scl is identically zero on $T_{p,q}$ by another application of Bavard's Duality Theorem. □

There is a natural central extension

$$0 \to \mathbb{Z} \to \widehat{T}_{p,q} \to T_{p,q} \to 0$$

where $\widehat{T}_{p,q}$ is the subgroup of $\mathrm{Homeo}^+(\mathbb{R})$ which cover elements of $T_{p,q}$ under the covering projection $\mathbb{R} \to S^1$. Note that $\widehat{T}_{p,q}$ is finitely presented, since $T_{p,q}$ is. The class of this central extension is the *Euler class* of the natural action of $T_{p,q}$ on S^1. Since $\mathbb{Z}$ is amenable, Theorem 2.49 shows that the exact sequence induces an isomorphism $H_b^2(T_{p,q};\mathbb{R}) \to H_b^2(\widehat{T}_{p,q};\mathbb{R})$.

On the other hand, by construction, the kernel of the map in ordinary cohomology $H^2(T_{p,q};\mathbb{R}) \to H^2(\widehat{T}_{p,q};\mathbb{R})$ is 1-dimensional, generated by the Euler class. The usual five term exact sequence in cohomology for an extension (i.e. the Hochschild–Serre sequence; see § 1.1.6) implies that $H^1(\widehat{T}_{p,q};\mathbb{R})$ vanishes. By Theorem 2.50 the space $Q(\widehat{T}_{p,q})$ is 1-dimensional, and generated by rotation number, as in § 2.3.3. As in Proposition 2.92, $D(\mathrm{rot}) = 1$. By Bavard's Duality Theorem we have the following:

THEOREM 5.16 (Zhuang [**205**], Thm. 3.9). *With notation as above, and for p, q satisfying $\gcd(p-1, q-1) = 1$, for any element $f \in \widehat{T}_{p,q}$ there is an equality*

$$\mathrm{scl}(f) = \frac{|\mathrm{rot}(f)|}{2}$$

We will see more examples of such an intimate relationship between scl and dynamics in the sequel.

5.1.4. Rotation numbers in Stein–Thompson groups. Rotation numbers in Stein–Thompson groups have been well-studied by Isabelle Liousse [**137**]. She proves the following:

THEOREM 5.17 (Liousse [**137**], Thm. 2.C′). *Any number of the form $\frac{\log\alpha}{\log\beta}$ mod $\mathbb{Z}$ where $\alpha, \beta \in \langle p, q\rangle$ can be realized as the rotation number of an element of the group $T(d, \mathbb{Z}[\frac{1}{pq}], \langle p, q\rangle)$ where $d = \gcd(p-1, q-1)$.*

For concreteness, take $p = 2, q = 3$. An example is the following:

EXAMPLE 5.18 (Liousse [**137**]). Define $a \in T_{2,3}$ by

$$a = \begin{cases} \frac{2}{3}x + \frac{2}{3} \text{ if } x \in [0, \frac{1}{2}] \\ \frac{4}{3}x - \frac{2}{3} \text{ if } x \in [\frac{1}{2}, 1] \end{cases}$$

Then any lift $\widehat{a}$ of a to $\widehat{T}_{2,3}$ has rotation number $\frac{\log 3}{\log 2}$ (mod $\mathbb{Z}$), and consequently $\mathrm{scl}(\widehat{a})$ is irrational in $\widehat{T}_{2,3}$. In fact, scl in this case is transcendental, by the celebrated theorem of Gelfond and Schneider ([**89**],[**183**]). The graph of a is illustrated in Figure 5.1.

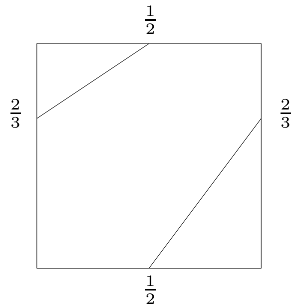

FIGURE 5.1. Graph of the homeomorphism $a \in T_{2,3}$

The map $h : x \to 2 - 2^{1-x}$ for $x \in [0,1]$ conjugates a to a rigid rotation by $\log 3/\log 2$. This example is very closely related to examples studied also by Boshernitzan [**18**]. For a full discussion, and an explanation of this and related phenomena, see Liousse [**137**], § 3.

COROLLARY 5.19 (Zhuang [**205**]). *There exists a finitely presented group containing elements with transcendental* scl.

This answers in the negative question (c) in Gromov [**99**], page 142.

REMARK 5.20. Work of Ghys–Sergiescu [**92**] already shows that the classical Thompson group T is uniformly perfect, and therefore its central extension $\widehat{T}$ satisfies $\dim(Q(\widehat{T})) = 1$, spanned by rotation number. However, [**92**] show that every element of T has a periodic point in S^1, and therefore rotation number (and consequently scl) is rational in $\widehat{T}$. In any case, $\widehat{T}$ is an example of a finitely presented group whose scl spectrum is exactly equal to the non-negative rational numbers.

5.2. Groups with few quasimorphisms

The examples in § 5.1 suggest that it is fruitful to study examples of groups with H_b^2 finite dimensional. If G is a finitely presented group with scl identically zero, then $H_b^2(G)$ injects into the finite dimensional space $H^2(G)$ by Theorem 2.50. If $\widehat{G}$ is a central extension of G, then $Q(\widehat{G})$ is finite dimensional, and scl in $\widehat{G}$ can be computed by Bavard duality. The Stein–Thompson groups discussed in § 5.1 are examples of this kind. It is psychologically useful to think of such groups as "lattices" (in a certain sense) in the group of PL homeomorphisms of S^1. Thinking of these groups in this way connects them to a wider class of examples which we now discuss.

5.2.1. Higher rank lattices.

The main references for this section are [**33, 34**] and [**66**]. Using tools from the theory of continuous bounded cohomology (see [**157**]), Burger–Monod show that the natural map from bounded cohomology to ordinary cohomology in dimension 2 is injective for a large class of important groups, namely lattices in higher rank Lie groups.

The main theorems of [**33, 34**] are stated in very general terms; we state these theorems for lattices in real Lie groups, for simplicity. First we recall some definitions.

DEFINITION 5.21. Let G be a closed subgroup of $\mathrm{SL}(m,\mathbb{R})$ for some m. A closed, connected subgroup T of G is a *torus* if T is diagonalizable over $\mathbb{C}$; i.e. if there is $g \in \mathrm{GL}(m,\mathbb{C})$ such that $g^{-1}Tg$ consists entirely of diagonal matrices. A torus T in G is $\mathbb{R}$-*split* if T is diagonalizable over $\mathbb{R}$; i.e. if there is $g \in \mathrm{GL}(m, \mathbb{R})$ such that $g^{-1}Tg$ consists entirely of diagonal matrices.

EXAMPLE 5.22. The subgroup $\mathrm{SO}(2, \mathbb{R})$ in $\mathrm{SL}(2, \mathbb{R})$ is a torus, but not an $\mathbb{R}$-split torus, since the eigenvalues of most elements are not real. On the other hand, the subgroup consisting of matrices of the form $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ where $\lambda \in \mathbb{R}^*$ is a (maximal) $\mathbb{R}$-split torus.

For G a real Lie group (not necessarily a matrix group), a closed, connected subgroup T is an $\mathbb{R}$-split torus if for every $x \in T$, the conjugation action of x on the Lie algebra of G is diagonalizable, with all real eigenvalues.

DEFINITION 5.23. Let G be a real Lie group. The *real rank* of G, denoted $\mathrm{rank}_{\mathbb{R}} G$, is the dimension of any maximal $\mathbb{R}$-split torus of G.

DEFINITION 5.24. A Lie group is said to be *simple* if it has no nontrivial, closed, proper, normal subgroups, and is not Abelian. It is *almost simple* if the only closed, proper, normal subgroups are finite.

REMARK 5.25. With this definition, the Lie group $\mathrm{SL}(2, \mathbb{R})$ is almost simple, since the only closed proper normal subgroup is the center $\pm\mathrm{id}$, but its universal cover $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ is not almost simple, since its center is $\mathbb{Z}$.

A *lattice* Γ in a Lie group G is a discrete subgroup such that $\Gamma \backslash G$ has finite volume. A lattice is *uniform* (or *cocompact*) if $\Gamma \backslash G$ is compact, and *nonuniform* otherwise. A lattice Γ in a Lie group which is a nontrivial product $G = \prod_a G_a$ is *irreducible* if the projection of Γ to each proper product of factors is dense.

The following

THEOREM 5.26 (Burger–Monod [**34**], Thm. 21, Cor. 24). *Let Γ be an irreducible lattice in a finite product $G = \prod_a G_a$ where G_a are connected, almost-simple noncompact real Lie groups. If*

$$\sum_{a \in A} \mathrm{rank}_{\mathbb{R}} G_a \geq 2$$

then $H_b^2(\Gamma; \mathbb{R}) \to H^2(\Gamma; \mathbb{R})$ is injective.

REMARK 5.27. When Γ as above is uniform, this is contained in Theorem 1.1 from [**33**].

EXAMPLE 5.28. As an example we can take $G = \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$. There is a well-known construction of lattices in $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ using quaternion algebras, which we now describe. A standard reference for this material is Vignéras [**202**].

Let F be a number field (i.e. a finite algebraic extension of $\mathbb{Q}$), all of whose embeddings in $\mathbb{C}$ are contained in the real numbers. Such a field is said to be *totally real* and can be obtained, for instance, by taking a polynomial with rational coefficients all of whose roots are real, and adjoining to $\mathbb{Q}$ all of these roots. A *quaternion algebra* A over F is an algebra which as a group is a 4-dimensional vector space over F generated by elements $1, i, j, k$ with an associative and distributive multiplication law satisfying $i^2 = a, j^2 = b, k = ij = -ji$ for some $a, b \in F$. Such an algebra is typically denoted

$$A = \left(\frac{a, b}{F}\right)$$

A (Galois) embedding of F into $\mathbb{R}$ induces an inclusion of A into a quaternion algebra over $\mathbb{R}$. The only two such algebras, up to isomorphism are the matrix algebra $M_2(\mathbb{R})$, and the ring of Hamilton's quaternions $\mathbb{H}$. An embedding $\sigma : F \to \mathbb{R}$ is *ramified* if $A \otimes_{\sigma F} \mathbb{R} \cong \mathbb{H}$. Let $\mathcal{O}_F$ denote the ring of algebraic integers in F. It is finitely generated over $\mathbb{Z}$. An *order* $\mathcal{O}$ in A is a subring of A containing 1 that generates A over F, and is a finitely generated $\mathcal{O}_F$-module. If $x = x_0+x_1i+x_2j+x_3k$ is an arbitrary element of A, where the $x_i \in F$, the *norm* of x is $x_0^2-x_1^2a-x_2^2b+x_3^2ab$ and the *trace* is $2x_0$. The norm is a multiplicative homormorphism from A to F. If $\mathcal{O}$ is an order in A, the elements $\mathcal{O}^1$ of norm 1 are a group under multiplication.

Suppose that A is ramified at all but exactly two real embeddings of F. Consider the diagonal embedding

$$\rho : A \to M_2(\mathbb{R}) \times M_2(\mathbb{R}) \times \mathbb{H} \times \cdots \times \mathbb{H}$$

where each term is the embedding of A into $A \otimes_{\sigma_i F} \mathbb{R}$ associated to an embedding $\sigma_i : F \to \mathbb{R}$.

Theorem 5.29. *With notation as above, the image $\Gamma := \rho(\mathcal{O}^1)$ is an irreducible lattice in the product*

$$\rho(\mathcal{O}^1) \subset \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2) \times \cdots \times \mathrm{SU}(2)$$

Moreover, if the degree of F is at least 3*, the lattice Γ is uniform.*

See e.g. [**202**] for a proof. Since the SU(2) factors are all compact, the image of Γ in $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ is also a lattice.

If $x \in A$ and $\sigma : F \to \mathbb{R}$ is an unramified embedding inducing $\rho_\sigma : A \to M_2(\mathbb{R})$, the trace of the matrix $\rho_\sigma(x)$ is equal to the image under σ of the trace of x. In particular, these traces are *algebraic* numbers, contained in $\sigma(F)$. If $x \in \mathcal{O}^1$ and $g = \sigma(x) \in \mathrm{SL}(2,\mathbb{R})$, we can think of $\mathrm{SL}(2,\mathbb{R})$ acting on a circle, factoring through $\mathrm{SL}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R})$. The rotation number of g under this action is

$$\mathrm{rot}(g) = \frac{\cos^{-1}(\mathrm{trace}(g)/2)}{\pi}$$

mod $\mathbb{Z}$, providing $|\mathrm{trace}(g)| \le 2$. By Gelfond–Schneider, these rotation numbers are transcendental when they are not rational. Moreover, they are rational for only finitely many conjugacy classes in $\mathcal{O}^1$.

Let Γ be such a lattice, and consider the preimage $\widehat{\Gamma}$ in $\mathrm{SL}(2,\mathbb{R}) \times \widetilde{\mathrm{SL}}(2,\mathbb{R})$. The group Γ is finitely presented, since it has a compact fundamental domain for its action on the contractible space $\mathbb{H}^2 \times \mathbb{H}^2$. Since $\widehat{\Gamma}$ is a central $\mathbb{Z}$ extension, it is also finitely presented. As in § 5.1 the group $Q(\widehat{\Gamma})$ is one dimensional, generated by rotation number on the $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ factor. Hence for $g \in \widehat{\Gamma}$, $\mathrm{scl}(g) = |\mathrm{rot}(g)|/2$. As observed above, many of these numbers are transcendental.

5.2.2. Bounded generation. For many specific (mostly nonuniform) lattices, the conclusion of Theorem 5.26 can be obtained directly by quite different methods.

Definition 5.30. A group G is *boundedly generated* by a symmetric subset $H = H^{-1}$ if every element of G can be written as a product $h_1h_2\cdots h_n$ where each $h_i \in H$.

For this definition to be useful, the subset H should be small compared to G. The prototypical example of a boundedly generated group is $\mathrm{SL}(n,\mathbb{Z})$ where $n \ge 3$, or more generally $\mathrm{SL}(n, \mathcal{O})$ where $\mathcal{O}$ is the ring of integers of a number field (this fact is due to Carter–Keller). We do not state their theorem in full generality.

Definition 5.31. For $n \geq 3$ and $i \neq j \leq n$ the *elementary matrix* e_{ij} is the element of $\mathrm{SL}(n, \mathbb{Z})$ having 1's down the diagonal and in the ij location, and 0's elsewhere. An *elementary matrix* more generally is a power e_{ij}^m of some e_{ij}.

Theorem 5.32 (Carter–Keller [**53**]). *The group* $\mathrm{SL}(n, \mathbb{Z})$ *for* $n \geq 3$ *is boundedly generated by* elementary matrices. *In other words, there is a uniform bound* $N(n)$ *such that every element* $g \in \mathrm{SL}(n, \mathbb{Z})$ *can be written as a product of at most* N *elementary matrices.*

Example 5.33 ($\mathrm{SL}(n, \mathbb{Z})$ for $n \geq 3$). The stable commutator length vanishes identically on $\mathrm{SL}(n, \mathbb{Z})$ for $n \geq 3$. For, there is an identity

$$e_{ij}^n = [e_{ik}^n, e_{kj}]$$

provided i, j, k are distinct (which can be verified by direct calculation), and therefore $\mathrm{cl}(e_{ij}^n) = 1$ for all e_{ij} and all nonzero n. Since every $g \in \mathrm{SL}(n, \mathbb{Z})$ can be written as a product of a bounded number of powers of the e_{ij}, it follows that cl is uniformly bounded on $\mathrm{SL}(n, \mathbb{Z})$ and therefore scl vanishes identically.

In unpublished work, Carter–Keller and E. Paige extended these results considerably; Dave Witte-Morris [**159**] has obtained a very nice proof of their results using the Compactness Theorem of first-order logic. A special case of particular relevance is the following:

Theorem 5.34 (Carter–Keller–Paige [**159**] Thm. 6.1). *Let A be the ring of integers in a number field K (i.e. a finite algebraic extension of $\mathbb{Q}$) containing infinitely many units. Let T be an element of* $\mathrm{SL}(2, A)$ *which is not a scalar matrix (i.e. not of the form $\lambda \cdot \mathrm{id}$). Then* $\mathrm{SL}(2, A)$ *has a finite index normal subgroup which is boundedly generated by conjugates of T.*

Remark 5.35. If A is the ring of integers in a number field K, and A has only finitely many units, then K must be either $\mathbb{Q}$ or $\mathbb{Q}(\sqrt{-d})$ for some positive integer d. Every other A as above satisfies the hypothesis of the theorem.

Remark 5.36. The hypotheses of this theorem are equivalent to the property that $\mathrm{SL}(2, A)$ is isomorphic to an irreducible lattice in a higher rank semisimple Lie group. So the conclusion that scl vanishes identically also follows from Theorem 5.26.

Example 5.37. Let $A = \mathbb{Z}[\sqrt{2}]$, the ring obtained from $\mathbb{Z}$ by adjoining $\sqrt{2}$. Then $\Gamma = \mathrm{SL}(2, A)$ is boundedly generated by conjugates of $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Since $H_1(\mathrm{SL}(2, \mathbb{Z}); \mathbb{Z}) = \mathbb{Z}/6\mathbb{Z}$, the matrix T has a power which is a product of commutators in $\mathrm{SL}(2, \mathbb{Z})$, hence also in Γ. Let $H < \Gamma$ be a finite index normal subgroup of Γ which is boundedly generated by conjugates of T. Then cl is uniformly bounded on H, and therefore scl vanishes identically on H. Since H is finite index on Γ, every element of Γ has a power which is contained in H, hence scl vanishes identically on all of Γ.

The inclusion $\mathbb{Z}[\sqrt{2}] \to \mathbb{R}$ induces an inclusion of Γ into $\mathrm{SL}(2, \mathbb{R})$ whose image is dense. Let $\widehat{\Gamma}$ be the preimage in $\widetilde{\mathrm{SL}}(2, \mathbb{R})$. As in § 5.1 we conclude that $Q(\widehat{\Gamma})$ is one-dimensional, spanned by rotation number. Hence in $\widehat{\Gamma}$ we have $\mathrm{scl}(g) = |\mathrm{rot}(g)|/2$. By Gelfond–Schneider, these values are transcendental when they are not rational.

EXAMPLE 5.38. Another example, due to Liehl [**136**], says that $\mathrm{SL}(2,\mathbb{Z}[1/2])$ is boundedly generated by elementary matrices. As above, $Q(\Gamma) = 0$, and $Q(\widehat{\Gamma})$ is one-dimensional, generated by rotation number, where Γ denotes $\mathrm{SL}(2,\mathbb{Z}[1/2])$ and $\widehat{\Gamma}$ its central extension. An element of Γ with trace 2^{-n} has transcendental rotation number when n is positive.

REMARK 5.39. If G is boundedly generated, so is a central extension $\widehat{G}$. Thus there are many examples of finitely presented groups which are boundedly generated, but for which $Q(\widehat{G})$ is nontrivial. This observation is made by Monod–Rémy in an appendix to [**143**]. They also observe that many of the groups G and $\widehat{G}$ furthermore have Kazhdan's property (T).

5.2.3. Symplectic groups. One class of Lie groups deserving special attention are the *symplectic groups*. As remarked earlier, there are two main sources of quasimorphisms. The first source, hyperbolic geometry, was studied systematically in Chapter 3. The second source is symplectic geometry (or more generally, causal or ordered structures); we turn to this subject in this section and the next. Basic references for symplectic geometry and topology are [**151**] and [**112**]. The material and exposition in this section borrows heavily from Barge–Ghys [**7**].

Given a vector space V (over $\mathbb{R}$ for simplicity), let V^* denote its dual. The *nth exterior product* $\Lambda^n V^*$, whose elements are called *n-forms on* V is the vector space generated by terms $v_1 \wedge v_2 \wedge \cdots \wedge v_n$ with the $v_i \in V^*$, which is linear in each factor separately, and subject to the relation that interchanging the order of two adjacent factors is multiplication by -1. With this notation, $\Lambda^1 V^* = V^*$, and we make the convention that $\Lambda^0 V^* = \mathbb{R}$. The sum $\oplus_i \Lambda^i V^*$ is a graded algebra, where multiplication is given by

$$v_1 \wedge \cdots \wedge v_n \times u_1 \wedge \cdots \wedge u_m = v_1 \wedge \cdots \wedge v_n \wedge u_1 \wedge \cdots \wedge u_m$$

and extended by linearity. If $x \in \Lambda^i V^*$ and $y \in \Lambda^j V^*$, then by counting signs, one sees that $xy = (-1)^{ij} yx$. If the dimension of V^* is m, then the dimension of $\Lambda^i V^*$ is equal to $\binom{m}{i}$. Hence $\Lambda^m V^* \cong \mathbb{R}$, and $\Lambda^i V^* = 0$ for all $i > m$.

DEFINITION 5.40. If V has dimension $2n$, a form $\omega \in \Lambda^2 V^*$ is *symplectic* if $\omega \wedge \omega \wedge \cdots \wedge \omega \neq 0$ for any r-fold product, where $r \leq n$. Equivalently, $\omega^n \neq 0 \in \Lambda^{2n} V^*$.

If G acts on V linearly, there is an induced action on V^* by the formula

$$g(v)(g(u)) = v(u)$$

for all $v \in V^*$ and $u \in V$. This lets us define a diagonal action of G on each $\Lambda^i V^*$ given by the formula

$$g(v_1 \wedge \cdots \wedge v_n) = g(v_1) \wedge \cdots \wedge g(v_n)$$

and extended by linearity.

DEFINITION 5.41. Let V be a vector space and $\omega \in \Lambda^2 V^*$ a symplectic form. The *symplectic group of* V, ω, denoted $\mathrm{Sp}(V,\omega)$, is the subgroup of $\mathrm{GL}(V)$ which fixes ω.

REMARK 5.42. When V has even dimension, the action of $\mathrm{GL}(V)$ on $\Lambda^2 V^*$ has a unique open dense orbit which consists exactly of the set of all symplectic elements of $\Lambda^2 V^*$. It follows that any two groups $\mathrm{Sp}(V,\omega)$ and $\mathrm{Sp}(V,\omega')$ are conjugate as subgroups of $\mathrm{GL}(V)$, and their isomorphism class depends only on the dimension of V.

A vector space with an inner product may be identified with its dual. On $\mathbb{R}^{2n}$ with orthonormal basis $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n$ there is a "standard" symplectic element given by the formula

$$\omega = x_1 \wedge y_1 + x_2 \wedge y_2 + \cdots + x_n \wedge y_n$$

Using the orthonormal basis to identify $\mathbb{R}^{2n}$ with its dual, this defines a symplectic form on $\mathbb{R}^{2n}$.

The symplectic group of $\mathbb{R}^{2n}$ with respect to ω is usually called *the symplectic group*, and denoted $\mathrm{Sp}(2n, \mathbb{R})$. If J denotes the $2n \times 2n$ matrix whose four $n \times n$ blocks have the form

$$J = \begin{pmatrix} 0 & \mathrm{id} \\ -\mathrm{id} & 0 \end{pmatrix}$$

then $\mathrm{Sp}(2n, \mathbb{R})$ is the group of matrices A for which $A^T J A = J$.

Let $\mathrm{U}(n)$ denote the *unitary group*, i.e. the group of $n \times n$ complex matrices which preserve the standard Hermitian inner product on $\mathbb{C}^n$. If we think of $\mathbb{R}^{2n}$ as the underlying real vector space of $\mathbb{C}^n$, then the inclusion $M_n(\mathbb{C}) \to M_{2n}(\mathbb{R})$ realizes $\mathrm{U}(n)$ as a compact subgroup of $\mathrm{Sp}(2n, \mathbb{R})$. In fact, $\mathrm{U}(n)$ is a *maximal* compact subgroup, and the coset space $X := \mathrm{Sp}(2n, \mathbb{R})/\mathrm{U}(n)$ admits an $\mathrm{Sp}(2n, \mathbb{R})$-invariant Riemannian metric of non-positive curvature. The space X is usually called the *Siegel upper half-space*, and has several equivalent descriptions. One well-known description says that X is the space of $n \times n$ complex symmetric matrices whose imaginary part is positive definite. If $n = 1$, this is the set of complex numbers with positive imaginary part, which is the upper half-space model of the (ordinary) hyperbolic plane.

Since X is non-positively curved and complete, it is contractible, so the inclusion $\mathrm{U}(n) \to \mathrm{Sp}(2n, \mathbb{R})$ is a homotopy equivalence. The group $\mathrm{U}(n)$ acts transitively on the unit sphere S^{2n-1} in $\mathbb{C}^n$, with stabilizer $\mathrm{U}(n-1)$, so there is a fibration

$$\mathrm{U}(n-1) \to \mathrm{U}(n) \to S^{2n-1}$$

By the homotopy exact sequence of a fibration, it follows that $\pi_1(\mathrm{U}(n)) = \mathbb{Z}$, generated by the inclusion $S^1 = \mathrm{U}(1) \to \mathrm{U}(n)$, and therefore $\pi_1(\mathrm{Sp}(2n, \mathbb{R})) = \mathbb{Z}$.

Let $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ denote the universal covering group. In the case $n = 1$ this is just $\widetilde{\mathrm{SL}}(2, \mathbb{R})$.

A closed differential 2-form ω on a manifold M^{2n} of dimension $2n$ is *symplectic* if the $2n$-form ω^n is nonzero at every point. It turns out that there is a natural symplectic form ω on the Siegel upper half-space X which is invariant under $\mathrm{Sp}(2n, \mathbb{R})$. If Γ is a (torsion-free) lattice in $\mathrm{Sp}(2n, \mathbb{R})$, then ω descends to a symplectic form on X/Γ. If Γ is cocompact, the cohomology class $[\omega] \in H^2(X/\Gamma) = H^2(\Gamma)$ is nonzero, since the integral of the top power of ω over X/Γ is nonzero. In fact, it turns out that the class of $[\omega]$ is in the image of $H^2_b(\Gamma)$. Moreover, Domic and Toledo [**66**] calculate the norm of this class, and show that it is equal to $n\pi$.

If we let $\widehat{\Gamma}$ denote the preimage of Γ in $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$, then $[\omega]$ pulls back to a class $[\tilde{\omega}]$ in $H^2_b(\widehat{\Gamma})$ whose image in $H^2(\widehat{\Gamma})$ is trivial, and therefore comes from a homogeneous quasimorphism ρ, which we normalize by scaling to have $D(\rho) = n$. Evidently, in the case $n = 1$, the quasimorphism ρ is just rotation number. Barge–Ghys [**7**] call this quasimorphism the *symplectic rotation number*. In fact, since the form ω is invariant under the action of $\mathrm{Sp}(2n, \mathbb{R})$ on X, there is a well-defined homogeneous quasimorphism ρ defined on the entire group $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$, and $\rho \in Q(\widehat{\Gamma})$ is just pulled back by inclusion.

Barge–Ghys give an explicit description for this quasimorphism, as follows.

DEFINITION 5.43. A subspace π of $\mathbb{R}^{2n}$ of real dimension n is *Lagrangian* if the symplectic form ω restricts to zero on π. That is, if $\omega(u,v) = 0$ for all $u, v \in \pi$.

A subspace π of $\mathbb{R}^{2n}$ of real dimension n is Lagrangian if and only if it is totally real when considered as a subspace of $\mathbb{C}^n$. It follows that the subgroup $\mathrm{U}(n)$ of $\mathrm{Sp}(2n,\mathbb{R})$ acts transitively on the space Λ_n of Lagrangian subspaces of $\mathbb{R}^{2n}$, with stabilizer the subgroup $\mathrm{O}(n,\mathbb{R})$. In other words, there is an isomorphism $\mathrm{U}(n)/\mathrm{O}(n,\mathbb{R}) = \Lambda_n$ as principal $\mathrm{U}(n)$-spaces. Note that we are thinking here of $\mathrm{O}(n,\mathbb{R})$ firstly as a subgroup of $\mathrm{GL}(n,\mathbb{C})$ by the inclusion $\mathbb{R} \to \mathbb{C}$, and then secondly as a subgroup of $\mathrm{Sp}(2n,\mathbb{R})$ by the inclusion $\mathrm{GL}(n,\mathbb{C}) \to \mathrm{GL}(2n,\mathbb{R})$ coming from the identification of $\mathbb{C}^n$ with $\mathbb{R}^{2n}$.

Let $* \in \Lambda_n$ be some basepoint, for example corresponding to the Lagrangian subspace $\mathbb{R}^n$ in $\mathbb{C}^n$. For each $g \in \mathrm{Sp}(2n,\mathbb{R})$, there is a unique coset $u(g)\mathrm{O}(n,\mathbb{R}) \in \mathrm{U}(n)/\mathrm{O}(n,\mathbb{R})$ with $g(*) = u(g)(*)$ for any element of the coset. The homomorphism $\det^2 : \mathrm{U}(n) \to S^1$ factors through the quotient $\mathrm{U}(n)/\mathrm{O}(n,\mathbb{R})$, and defines a function $\det^2 : \mathrm{Sp}(2n,\mathbb{R}) \to S^1$. This map is a double covering, restricted to the subgroup $S^1 = \mathrm{U}(1)$, so we get a covering map

$$\mu : \widetilde{\mathrm{Sp}}(2n,\mathbb{R}) \to \mathbb{R}$$

which turns out to be a quasimorphism.

The quasimorphism μ is not homogeneous. However, Barge–Ghys derive a formula for its homogenization ρ, at least mod $\mathbb{Z}$. To state their theorem we must first recall some standard facts about the spectrum of a symplectic matrix. Let $A \in \mathrm{Sp}(2n,\mathbb{R})$ and suppose for simplicity that A is diagonalizable over $\mathbb{C}$. The spectrum of A (i.e. the set of complex eigenvalues with multiplicity) is invariant under conjugation, since A is a real matrix. Moreover, it is invariant with respect to inversion in the unit circle in $\mathbb{C}$. Hence if λ is an eigenvalue, then $\lambda, \lambda^{-1}, \overline{\lambda}, \overline{\lambda}^{-1}$ are all eigenvalues. The case that λ is real or on the unit circle is naturally rather special. It turns out that eigenvalues λ which are not on the unit circle do not contribute to ρ.

Suppose A is diagonalizable over $\mathbb{C}$, and H is the subspace of $\mathbb{R}^{2n}$ of dimension $2k$ spanned by the 2×2 Jordan blocks of A (over $\mathbb{R}$) corresponding to pairs of complex eigenvalues $\lambda, \overline{\lambda}$ with λ on the unit circle. Then H is a symplectic subspace of $\mathbb{R}^{2n}$, and the restriction of A to H is orthogonal, and therefore unitary; hence $A|_H$ is conjugate in the symplectic group to a unitary matrix $B \in \mathrm{U}(k)$. The complex eigenvalues of B are called the *proper values* of A of absolute value 1.

Barge–Ghys' theorem gives a formula for ρ in terms of the proper values of absolute value 1.

THEOREM 5.44 (Barge–Ghys [**7**], Thm. 2.10). *Let g be an element of $\mathrm{Sp}(2n,\mathbb{R})$, and let $\lambda_1, \cdots, \lambda_k$ be the proper values of g of absolute value 1, listed with multiplicity. Then*

$$\rho(g) = \frac{1}{\pi}\sum \arg(\lambda_i) \pmod{\mathbb{Z}}$$

REMARK 5.45. If we deform a matrix in $\mathrm{Sp}(2n,\mathbb{R})$ so that some set $\{\lambda, \lambda^{-1}, \overline{\lambda}, \overline{\lambda}^{-1}\}$ of eigenvalues is deformed onto the unit circle, one obtains for the deformed matrix two proper values of absolute value one, which are equal to λ and $\overline{\lambda}$ respectively, and therefore the sum of their arguments vanishes. This explains why ρ is continuous on $\mathrm{Sp}(2n,\mathbb{R})$, which is otherwise not obvious.

Remark 5.46. If we think of $\mathbb{R}^{2n}$ with its standard symplectic form as a product of n copies of $\mathbb{R}^2$ with its standard symplectic form, we get a natural inclusion

$$\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \times \cdots \times \mathrm{SL}(2,\mathbb{R}) \to \mathrm{Sp}(2n,\mathbb{R})$$

The symplectic rotation number restricts to Poincaré's rotation number on each $\mathrm{SL}(2,\mathbb{R})$ factor, and is equal to the sum of rotation numbers on the factors on the image of the product of $\mathrm{SL}(2,\mathbb{R})$'s.

Theorem 5.26 shows that $H_b^2(\Gamma)$ includes into $H^2(\Gamma)$, when n is at least 2. Since the defect of ρ is n, there is a formula

$$\mathrm{scl}(g) = |\rho(g)|/2n = \frac{1}{2n\pi}\sum \arg(\lambda_i) \pmod{\frac{1}{2n}\mathbb{Z}}$$

for $g \in \widehat{\Gamma}$. Lattices in $\mathrm{Sp}(2n,\mathbb{R})$ for n at least 2 have algebraic entries. Hence by Gelfond–Schneider, scl is transcendental on $\widehat{\Gamma}$ when it is irrational.

Obviously the examples above can be generalized tremendously. However in every case, the irrational values of scl obtained appear to be transcendental. Hence we pose the following question.

Question 5.47. *Is there a finitely presented group G in which* scl *takes on an irrational value that is algebraic?*

More generally, one can ask for a complete characterization of the values of scl that can occur in finitely presented groups.

Question 5.48. *What real numbers are values of* scl *on elements in finitely presented groups?*

This seems like a difficult question.

5.2.4. Causal structures and quasimorphisms. In this section we give a more topological definition of the symplectic rotation quasimorphism ρ defined in § 5.2.3 which "explains" the integral value of $D(\rho)$. The construction makes use of the *causal* structure on Λ_n. This point of view is particularly explicit in [**3**]. Also compare [**54**].

Definition 5.49. Let V be a real vector space. A *cone* C in V is a subset of the form $\mathbb{R}\cdot K$ where K is compact and convex with nonempty interior, and disjoint from the origin. A vector $v \in V$ is *timelike* if it is in the interior of C, is *lightlike* if it is in the frontier of C, and is *spacelike* otherwise.

Example 5.50. Let V be an $(n+1)$-dimensional real vector space, and $q : V \times V \to \mathbb{R}$ a symmetric bilinear pairing of signature $(n,1)$ (i.e. with n positive eigenvalues and one negative eigenvalue). The set of vectors v with $q(v,v) \le 0$ is a cone in V.

If M is a smooth manifold, a *cone field* is a continuously varying choice of cone in the tangent space at each point. The set of timelike vectors at a point has two components; a *causal structure* on M is a cone field together with a continuously varying choice of one of these components (the *positive* cone) at each point. Two points p, q are *causally connected*, and we write $p \prec q$, if there is a nontrivial smooth curve from p to q whose tangent vector at every point is positive and timelike. The relation $\prec$ is transitive (but *not* typically reflexive or symmetric). A causal structure is *recurrent* if $p \prec q$ for all p and q.

REMARK 5.51. Some authors use the notational convention that $p \prec q$ means either that $p = q$ or that p is causally connected to q in the sense above. We denote this instead by $p \preceq q$.

Let M be a closed manifold which admits a recurrent causal structure, and let S be a non-separating codimension one submanifold whose tangent space is spacelike. Then S is essential in homology, and is dual to an element of $H^1(M;\mathbb{Z})$. Let M' denote the infinite cyclic cover of M dual to S. The causal structure on M lifts to one on M' (where it is no longer recurrent).

Let $C^+(M)$ denote the group of diffeomorphisms of M which preserve the causal structure, and $C^+(M')^{\mathbb{Z}}$ the preimage of this group in $\text{Homeo}^+(M')$. There is a central extension

$$0 \to \mathbb{Z} \to C^+(M')^{\mathbb{Z}} \to C^+(M) \to 0$$

where $\mathbb{Z}$ is the deck group. We write the action of the deck group on points in M' by $p \to p + n$.

For any $p, q \in M'$, define $d(p, q)$ to be the greatest integer $n \in \mathbb{Z}$ such that $p \prec q - n$. Pick a basepoint $*$ in M', and for any $\alpha \in C^+(M')^{\mathbb{Z}}$, define $\phi(\alpha) = d(*, \alpha(*))$ and $\rho(\alpha) = \lim_{n\to\infty} \phi(\alpha^n)/n$. Since the causal structure on M is recurrent, there is a least positive integer w such that any two points p and q are contained in a closed timelike curve which intersects S at most w times.

LEMMA 5.52. *The function ϕ as above is a quasimorphism, and ρ is its homogenization. Moreover, the defect of ρ is at most w.*

PROOF. For any α there is equality $\phi(\alpha - \phi(\alpha)) = 0$. Let α, β be arbitrary, and denote $\alpha' = \alpha - \phi(\alpha)$ and $\beta' = \beta - \phi(\beta)$. Then $* \prec \alpha'(*) \prec * + w$ and similarly for $\beta'(*)$. We calculate

$$* \prec \alpha'(*) \prec \alpha'\beta'(*) \prec \alpha'(* + w) \prec * + 2w$$

and therefore

$$|\phi(\alpha\beta) - \phi(\alpha) - \phi(\beta)| = |\phi(\alpha'\beta')| \leq 2w$$

This shows that ϕ is a quasimorphism; evidently ρ is its homogenization.

To estimate the defect of ρ we repeat the argument of Lemma 2.41. For any $p \in M'$ and any elements $\alpha, \beta \in C^+(M')^{\mathbb{Z}}$, after multiplying by elements of the center if necessary, we can assume

$$p \preceq \alpha(p) \preceq \alpha\beta(p) \prec \alpha(p + w) \prec p + 2w$$

$$p \preceq \beta(p) \preceq \beta\alpha(p) \prec \beta(p + w) \prec p + 2w$$

Set $q = \beta\alpha(p)$. Then $p \preceq q \prec p + 2w$ and therefore

$$q - 2w \prec p \preceq \alpha\beta(p) = [\alpha, \beta](q) \prec p + 2w \preceq q + 2w$$

Since p was arbitrary, so was q, and we have shown that $q - 2w \prec [\alpha, \beta](q) \prec q + 2w$ for any q and any commutator $[\alpha, \beta]$.

It follows that if γ is a product of m commutators, then $|\rho(\gamma)| \leq 2w(m + 1)$. Taking m large, the argument of Lemma 2.24 shows $D(\rho) \leq w$. □

REMARK 5.53. Essentially the same construction is described in [**54**], § 7–8.

Causal structures arise naturally in certain contexts.

EXAMPLE 5.54. Let $\mathfrak{G}$ be a simple Lie algebra with Lie group G. An $\mathrm{Ad}(G)$-invariant cone in $\mathfrak{G}$ exponentiates to a G-invariant cone field on G. This determines a causal structure either on G or on a double cover, which is invariant under the action of the group on itself. Let K be a maximal compact subgroup of G, with Lie algebra $\mathfrak{k}$. It turns out (Paneitz [**165**], Cor. 3.2) that there is an $\mathrm{Ad}(G)$-invariant cone in $\mathfrak{G}$ if and only if $\mathfrak{k}$ has nontrivial center.

If $G = \mathrm{Sp}(2n, \mathbb{R})$, then

$$\mathfrak{G} = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \text{ where } A, B, C \text{ are } n \times n \text{ blocks, and } B, C \text{ are symmetric}$$

$$\mathfrak{k} = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \text{ where } A \text{ is skew, and } B \text{ is symmetric}$$

The center of $\mathfrak{k}$ is nontrivial, and spanned by the matrix $\left(\begin{smallmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{smallmatrix}\right)$. If ω denotes the standard (and $\mathrm{Ad}(G)$-invariant) symplectic form on $\mathbb{R}^{2n}$, define C to be the cone of vectors $X \in \mathfrak{G}$ for which $\omega(\mathrm{ad}(X)v, v) \geq 0$ for all $v \in \mathbb{R}^{2n}$. This is nonempty and invariant, and defines a (recurrent) causal structure on $\mathrm{Sp}(2n, \mathbb{R})$.

EXAMPLE 5.55. Let $G = \mathrm{SO}(n, 2)$, the group of linear automorphisms of $\mathbb{R}^{n+2}$ which preserve the quadratic form $q(x) = x_1^2 + \cdots + x_n^2 - x_{n+1}^2 - x_{n+2}^2$. Let H be the hyperboloid of vectors x for which $q(x) = -1$. Then G acts transitively on H. At a point $x \in H$, the tangent space T_xH is naturally isomorphic to the orthogonal subspace of $\mathbb{R}^{n+2}$ to x with respect to the form q. Since $q(x) = -1$, the restriction of q to this subspace has signature $(n, 1)$, and therefore G preserves a cone field on H as in Example 5.50. There is a subgroup $\mathrm{SO}_0(n, 2)$ of index 2 which preserves the orientation on the cone field, and therefore a causal structure on H.

When $n = 1$, the group $\mathrm{SO}(1, 2)$ is isomorphic to $\mathrm{PSL}(2, \mathbb{R})$, the group of isometries of the hyperbolic plane. In the Klein (projective) model, the hyperbolic plane is identified with the interior of a round disk D in $\mathbb{RP}^2$, and the exterior $\mathbb{RP}^2 - D$ (which is homeomorphic to an open Möbius strip) is equal to $H/\pm 1$ where H is as in Example 5.55. If p is a point in $\mathbb{RP}^2 - D$, there are two straight lines through p which are tangent to ∂D. The cone at p is the set of tangents to straight lines through p which do not intersect D. A smooth curve in $\mathbb{RP}^2 - D$ is timelike if every tangent line to the curve is disjoint from D. Evidently, the causal structure on H is recurrent; in fact, one sees that any two points in H are contained in a closed timelike loop with winding number at most 2. By rotational symmetry, it follows that the same is true for arbitrary $n \geq 2$ and therefore one obtains a homogeneous quasimorphism on the universal covering group $\widetilde{\mathrm{SO}}_0(n, 2)$ with defect at most 2. When $n \geq 2$, this estimate can be seen to be sharp by an explicit construction (compare with Domic–Toledo [**66**] and [**55**]).

Causal structures on noncompact manifolds often extend to causal structures on certain natural boundaries. A *symmetric bounded domain* is a complex symmetric space that is isomorphic to a bounded domain in $\mathbb{C}^n$ for some n. It is *irreducible* if its universal cover is not a nontrivial direct product of symmetric spaces. By a theorem of Harish-Chandra, every irreducible complex symmetric space of noncompact type is bounded. An irreducible symmetric bounded domain is said to be of *tube type* if it is isomorphic to a domain of the form $V + i\Omega$ where $\Omega \subset V$ is a proper open cone in the real vector space V.

A realization of a bounded symmetric domain defines a natural compactification. The group G of holomorphic automorphisms of the domain extends to the compactification, and the *Shilov boundary* is the unique closed G-orbit in the compactification. It is known (see e.g. [**120**], § 5) that the Shilov boundary of a symmetric bounded domain of tube type admits a natural causal structure.

EXAMPLE 5.56. The Siegel upper half-space $\mathrm{Sp}(2n,\mathbb{R})/\mathrm{U}(n)$ is a symmetric bounded domain of tube type. Its Shilov boundary is the space Λ_n of Lagrangians in $\mathbb{R}^{2n}$.

The causal structure on Λ_n can be given a very geometric definition, as observed by Arnold [**3**]. If π is a Lagrangian subspace of $\mathbb{R}^{2n}$ (and therefore corresponds to a point in Λ_n) the *train* of π is the set of Lagrangian subspaces of $\mathbb{R}^{2n}$ which are not transverse to π.

Fix a Lagrangian π and a transverse Lagrangian σ, and let π_t be a 1-parameter family of Lagrangians with $\pi_0 = \pi$. For small t, the Lagrangians π_t and σ are still transverse, and span $\mathbb{R}^{2n}$. For each $v \in \mathbb{R}^{2n}$ and each such t, there is a unique decomposition $v = v(\pi_t) + v(\sigma)$ where $v(\pi_t) \in \pi_t$ and $v(\sigma) \in \sigma$ (note that for a fixed v, the vector $v(\sigma)$ typically depends on t). Define a 1-parameter family of bilinear forms q_t on $\mathbb{R}^{2n}$ by the formula

$$q_t(v,w) = \omega(v(\pi_t), w(\sigma))$$

where ω is the symplectic form. In this way, a tangent vector $\pi_0' := \frac{d}{dt}\big|_0 \pi_t$ to π determines a *symmetric* bilinear form $q_0' := \frac{d}{dt}\big|_0 q_t$ which vanishes identically on σ, and can be thought of as a symmetric bilinear form on π. The map $\pi_0' \to q_0'$ is an isomorphism from the tangent space $T_\pi\Lambda_n$ to the space of symmetric bilinear forms on π (to see this, observe that it is linear and injective, and is surjective by a dimension count, since both $\mathrm{U}(n)/\mathrm{O}(n)$ and the space of symmetric $n \times n$ matrices have dimension $n(n+1)/2$). Note that q_0' is degenerate precisely along the subspace $\pi_0' \cap \pi$. Hence the tangent cone to the train at π corresponds precisely to the degenerate bilinear forms. Exponentiating, we see that in a neighborhood of π, the train separates Λ_n into chambers, corresponding to nondegenerate quadratic forms on $\mathbb{R}^n$ of a fixed signature. The *positive cone* corresponds (infinitesimally) to positive definite quadratic forms on π.

EXAMPLE 5.57. The space $\Lambda_2 = \mathrm{U}(2)/\mathrm{O}(2)$ is diffeomorphic to the nonorientable sphere bundle over S^1. Fix co-ordinates $\Lambda_2 = S^2 \times [0,1]/\sim$ where $(\theta, 0) \sim (-\theta, 1)$. Fix a basepoint $*$ to be the north pole of the sphere $S^2 \times 0$ in these co-ordinates. The train of $*$ intersects each sphere $S^2 \times t$ in a circle of constant latitude which decreases monotonically with t, until it converges to the south pole in $S^2 \times 1$ (which is identified with $*$ by the holonomy map).

EXAMPLE 5.58. Let $G = \mathrm{SO}(n,2)$, and recall the notation from Example 5.55. The projectivization of the cone $q = 0$ is an S^{n-1} bundle over S^1 that we denote by E (this bundle is twisted by the antipodal map, so E is topologically a product if and only if n is even). Then E is a Shilov boundary for G. In the projectivization, E divides $\mathbb{RP}^{n+1}$ into two components, one of which is $H/\pm 1$. The cone field on H limits to a cone field on E, where the cone at a point $e \in E$ is the set of tangent lines to E which point into $H/\pm 1$. The group $\mathrm{SO}_0(n,2)$ preserves the causal structure on E. When $n = 1$, E is a circle, which can be thought of as the circle at infinity of the hyperbolic plane. When $n = 2$, E is a torus, and the cone structure determines a pair of transverse foliations on this torus by circles. $\mathrm{SO}_0(2,2)$ acts on the leaf spaces of these foliations (which are themselves circles) by projective transformations, exhibiting the exceptional 2-fold covering $\mathrm{SO}_0(2,2) \to \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$. When $n = 3$, E is a twisted S^2 bundle over S^1, and is equal to the space Λ_2 as described in Example 5.57; this reflects the exceptional isomorphism $\mathrm{SO}_0(3,2) = \mathrm{Sp}(4,\mathbb{R})/\pm 1$.

Causal structures become very rigid in high (≥ 3) (real) dimensions. For example, one has the following:

THEOREM 5.59 (Kaneyuki [**120**], Thm. 6.2). *Let D be an irreducible symmetric bounded domain of tube type, and $G(D)$ the group of holomorphic automorphisms of D. Let S be the Shilov boundary of D with its natural causal structure. Let $C^+(S)$ be the group of causal homeomorphisms of S. Suppose (complex)* $\dim(D) > 1$. *Then $C^+(S) = G(D)$.*

5.3. Braid groups and transformation groups

5.3.1. Braid groups.

DEFINITION 5.60. The *braid group B_n* on n strands is generated by elements σ_i for $i = 1, 2, \cdots, n-1$ and relations $[\sigma_i, \sigma_j] = 1$ when $|i - j| \neq 1$, and $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$.

These groups were introduced by Emil Artin in 1925 [**5**].

A word in the generators is represented pictorially by a projection of a tangle of n arcs running between two parallel vertical lines, where no arc has any vertical tangencies. Braids are composed by "gluing" pictures; see Figure 5.2. A generator

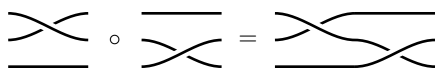

FIGURE 5.2. Braids are represented by pictures; composition is performed by gluing adjacent pictures. This picture illustrates the composition of σ_1 with σ_2^{-1} in B_3.

σ_i is represented by a crossing, where the ith strand crosses over the $(i+1)$st strand, and σ_i^{-1} is represented by a crossing where the $(i+1)$st strand crosses over the ith strand. Equivalence in B_n corresponds to equivalence of pictures up to "isotopy". The relation $[\sigma_i, \sigma_j] = 1$ when $|i - j| \neq 1$ corresponds to the fact that crossings on disjoint pairs of strands can be performed in either order. The group law $\sigma_i^{-1}\sigma_i = \sigma_i\sigma_i^{-1} = \mathrm{id}$ corresponds to the Reidemeister 2 move on diagrams, and the relation $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ corresponds to the Reidemeister 3 move on diagrams; see Figure 5.3.

Another way to think of B_n is as a mapping class group. A diagram of a braid can be thought of as a tangle in a product $D^2 \times [0,1]$ transverse to the foliation by vertical disks. In this way, an element in B_n determines a loop in the configuration space of distinct n-tuples of points in the disk. Isotopy of braids corresponds to

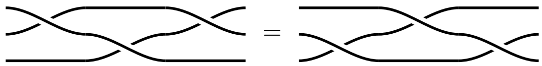

FIGURE 5.3. The relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ in the braid group corresponds to the Reidemeister 3 move on diagrams.

homotopy of loops, so B_n can be thought of as the fundamental group of the space of distinct n-tuples in D^2. Equivalently, B_n is just the mapping class group rel. boundary of a disk with n punctures. Braid groups, as examples of mapping class groups, admit a very large space of homogeneous quasimorphisms, by the construction described in § 3.5.

Gambaudo–Ghys [**87**] use symplectic geometry to define some quite different quasimorphisms. Many interesting representations of B_n can be derived from their geometric description as mapping class groups. Let D_n denote the disk with n points removed. There is an isomorphism $\pi_1(D_n) \to F_n$, the free group on n generators, and the generators may be taken to be loops, each of which winds around one puncture. Let $\epsilon : \pi_1(D_n) \to \mathbb{Z}$ take each generator to 1. This homomorphism defines a cyclic cover $\widetilde{D}_n$, whose first homology $H_1(\widetilde{D}_n)$ can be thought of as a $\mathbb{Z}[q, q^{-1}]$-module, where q generates the deck group of the covering. The first homology group is free as a module of rank $(n-1)$. If e_i is a based loop in D_n winding positively once around the ith puncture, the loops $\alpha_i := e_{i+1}e_i^{-1}$ for $1 \le i \le (n-1)$ all lift to $\widetilde{D}_n$, and freely generated $H_1(\widetilde{D}_n)$ as a $\mathbb{Z}[q, q^{-1}]$-module.

If we fix some basepoint $p \in D_n$, every braid $\psi \in B_n$ is represented by a homeomorphism which fixes p, and is covered by a unique homeomorphism $\widetilde{\psi}$ of $\widetilde{D}_n$ which fixes the preimages of p pointwise. Hence there is an induced action of B_n on $H_1(\widetilde{D}_n)$ by $\mathbb{Z}[q, q^{-1}]$-module automorphisms, and thereby a representation $\beta : B_n \to \mathrm{GL}(n-1, \mathbb{Z}[q, q^{-1}])$. This representation is called the *Burau representation*. See e.g. [**15**] for an elegant geometric interpretation of this action, and [**16**] as a general reference for braid groups. As matrices, this representation has the form

$$\sigma_1 \to \begin{pmatrix} -q^{-1} & q^{-1} \\ 0 & 1 \end{pmatrix} \oplus \mathrm{Id}_{n-3}, \quad \sigma_{n-1} \to \mathrm{Id}_{n-3} \oplus \begin{pmatrix} 1 & 0 \\ 1 & -q^{-1} \end{pmatrix},$$

and

$$\sigma_i \to \mathrm{Id}_{i-2} \oplus \begin{pmatrix} 1 & 0 & 0 \\ 1 & -q^{-1} & q^{-1} \\ 0 & 0 & 1 \end{pmatrix} \oplus \mathrm{Id}_{n-i-2} \text{ for } 1 < i < n-1$$

where the notation $A \oplus B$ stands for the block matrix $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$.

REMARK 5.61. Several different conventions exist in the literature, depending on whether one takes σ_i or σ_i^{-1} as the generators of B_n, and whether one studies the action on homology or cohomology.

Squier [**190**] showed that the image of the Burau representation is unitary, in the following sense. It turns out that there is a nonsingular matrix J_0 defined over $\mathbb{Z}[q, q^{-1}]$ such that for each $w \in B_n$, one has $\beta(w)^* J_0 \beta(w) = J_0$ (here $*$ is the conjugate transpose, where conjugation interchanges q with q^{-1}). In fact, over $\mathbb{Z}[s, s^{-1}]$ where $s^2 = q$, a change of basis replaces J_0 by a matrix J satisfying $J^* = J$.

If $\sigma : \mathbb{Z}[s, s^{-1}] \to \mathbb{C}$ takes s to an element of norm 1, the matrix $J(s)$ is Hermitian (in the usual sense) and one obtains a representation $\beta_\sigma : B_n \to \mathrm{U}(J)$, the unitary group of the form J. If J is nondegenerate, its imaginary part is a nondegenerate antisymmetric form, and one therefore obtains a representation $\beta_\sigma : B_n \to \mathrm{Sp}(2n-2;\mathbb{R})$. It turns out that the forms J are degenerate exactly when s is a $(2n)$th root of unity different from ± 1 (so that q is an nth root of unity different from 1). When s is sufficiently close to 1, the form J is positive definite. Each time q crosses an nth root of unity, the number of positive eigenvalues changes by -1. So when q is specialized to an mth root of unity with $m < n$ and m, n coprime, the form is nondegenerate, the signature is indefinite, and the image of β_σ in $\mathrm{Sp}(2n-2;\mathbb{R})$ typically has noncompact closure.

Another way to obtain these representations of B_n is by using surface topology. For each m, let $D_{n,m}$ be the surface obtained by taking an m-fold branched cover of the disk over n points. The induced action of B_n on $D_{n,m}$ is well-defined up to homotopy, and we get a representation on the vector space $H_1(D_{n,m}, \partial D_{n,m}; \mathbb{R})$. The deck group $\mathbb{Z}/m\mathbb{Z}$ acts on $D_{n,m}$. If ω is an mth root of unity, the ω-eigenspace of this action is real, and B_n-invariant. There is thus an action of B_n on the invariant vector space $H_1(D_{n,m}, \partial D_{n,m}; \mathbb{R})_\omega$. It turns out this representation is isomorphic to the Burau representation evaluated at $q = \omega$ (see e.g. [**87**], Prop. 2.2). The ordinary intersection pairing on H_1 is nondegenerate on this subspace when n and m are coprime, and one sees in another way the symplectic structure.

Remark 5.62. When n and m are not coprime, the imaginary part of J is degenerate on a subspace, and one obtains a symplectic action of B_n on the quotient by this subspace.

The cohomology of classical braid groups was computed by Arnold [**1**] (also see [**201**], Thm. 4.1). He showed the following:

Theorem 5.63 (Arnold [**1**]). *For $n \geq 2$, there are isomorphisms $H^0(B_n;\mathbb{Z}) = H^1(B_n;\mathbb{Z}) = \mathbb{Z}$. Otherwise, $H^i(B_n;\mathbb{Z})$ is finite when $i \geq 2$ and zero when $i \geq n$.*

We are concerned with the case $i = 2$. Theorem 5.63 says that $H^2(B_n;\mathbb{Z})$ is torsion. Consequently, each representation $\beta_\sigma : B_n \to \mathrm{Sp}(2n-2,\mathbb{R})$ defines a quasimorphism ρ on B_n (well-defined up to elements of H^1), whose coboundary is the pullback of the generator of $H_b^2(\mathrm{Sp}(2n-2))$ under β_σ^*.

Example 5.64. The braid group B_3 is discussed in Example 4.33. In the special case of B_3, the image of the Burau representation evaluated at -1 is equal to $\mathrm{SL}(2,\mathbb{Z})$, and ρ is the rotation quasimorphism coming from the action of $\mathrm{PSL}(2,\mathbb{Z})$ on S^1. A slightly different normalization of this quasimorphism is sometimes called the *Rademacher function* on $\mathrm{SL}(2,\mathbb{Z})$; see § 4 of [**87**], and § 6.1.7.

Example 5.65. The Burau representation of B_4 evaluated at $\omega = e^{2\pi i/3}$ is 3 (complex) dimensional, and has matrix entries in the discrete subring $\mathbb{Z}[\omega]$ of $\mathbb{C}$. The form J has signature $(1,2)$. Projectivizing, one obtains a discrete representation of B_4 into $\mathrm{PU}(1,2)$, the group of isometries of the complex hyperbolic plane. One may therefore obtain interesting de Rham quasimorphisms on B_4, as in § 2.3.1.

5.3.2. Area-preserving diffeomorphisms of surfaces. Gambaudo–Ghys [**86**] showed how to use quasimorphisms on discrete groups to obtain nontrivial quasimorphisms on certain transformation groups.

Similar ideas appeared earlier in work of Arnold [**4**], Ruelle [**181**], Gambaudo–Sullivan–Tresser [**88**] and others. A given (continuous) dynamical system is approximated (in some sense) by a discrete combinatorial model. Associated to the discrete approximation is some numerical invariant, which can then be integrated over the degrees of freedom of the continuous system. For this integration to make sense and have useful properties, the continuous dynamical system must be (at least) measure preserving, and of sufficient regularity that the integral converges.

The case presenting the fewest technical details is that of a group of area-preserving diffeomorphisms of a (finite area) surface.

Definition 5.66. For any surface S, let $\mathrm{Diff}^\infty(S, \partial S, \mathrm{area})$ (or omit the ∂S in the notation if S has no boundary) denote the group of diffeomorphisms of S, fixed pointwise on the boundary, that preserve the (standard) area form, and let $\mathrm{Diff}_0^\infty(S, \partial S, \mathrm{area})$ denote the subgroup of such diffeomorphisms isotopic to the identity.

There is an exact sequence

$$\mathrm{Diff}_0^\infty(S, \partial S, \mathrm{area}) \to \mathrm{Diff}^\infty(S, \partial S, \mathrm{area}) \to \mathrm{MCG}(S, \partial S)$$

Quasimorphisms on mapping class groups can be pulled back to $\mathrm{Diff}^\infty(S, \partial S, \mathrm{area})$. Therefore we focus on the construction of quasimorphisms on $\mathrm{Diff}_0^\infty(S, \partial S, \mathrm{area})$. A key case to consider is $S = D$, the closed unit disk.

Definition 5.67. Fix some n, and let μ be a quasimorphism on B_n. Fix n distinct points x_i^0 in D for $1 \le i \le n$. Given $g \in \mathrm{Diff}_0^\infty(D, \partial D, \mathrm{area})$, let g_t be an isotopy from id to g. For a generic ordered n-tuple of distinct points $x_1, \cdots, x_n$ in D, let $\gamma(g; x_1, \cdots, x_n) \in B_n$ be the braid obtained by first moving the x_i^0 in a straight line to the x_i, then composing with the isotopy g_t from x_i to $g(x_i)$, then finally moving the $g(x_i)$ in a straight line back to the x_i.

Now define

$$\Phi_\mu(g) = \int_{D \times \cdots \times D} \mu(\gamma(g; x_1, \cdots, x_n)) d\mathrm{area}(x_1) \times \cdots \times d\mathrm{area}(x_n)$$

and $\overline{\Phi}_\mu(g) = \lim_{n \to \infty} \frac{1}{n} \Phi_\mu(g^n)$.

Lemma 5.68. *For any quasimorphism μ on B_n, the function $\overline{\Phi}_\mu$ is a homogeneous quasimorphism on $\mathrm{Diff}_0^\infty(D, \partial D, \mathrm{area})$.*

Proof. For any two diffeomorphisms g, h, and generic $x_1, \cdots, x_n$ there is equality

$$\gamma(gh; x_1, \cdots, x_n) = \gamma(h; x_1, \cdots, x_n) \cdot \gamma(g; h(x_1), \cdots, h(x_n))$$

in B_n. Homogenizing removes the dependence on the choice of x_i^0. Integrating over $D \times \cdots \times D$ and using the fact that μ is a quasimorphism, we obtain the desired result. $\square$

Example 5.69. In case $n = 2$, the group B_2 is isomorphic to $\mathbb{Z}$ and we can take μ to be an isomorphism. In this case, the resulting function $\overline{\Phi}_\mu$ is a *homomorphism* from $\mathrm{Diff}_0^\infty(D, \partial D, \mathrm{area})$ to $\mathbb{R}$, which is equal (after normalization) to the well-known *Calabi homomorphism*.

Calabi [**38**] constructed an invariant for any symplectic diffeomorphism with compact support of a symplectic manifold without boundary. Calabi's construction can be translated into the case of area-preserving diffeomorphisms of the disk as follows. Let θ be a 1-form on D whose exterior derivative $d\theta$ is the area form. If g is an area-preserving diffeomorphism of D fixing ∂D pointwise, then $g^*\theta - \theta$ is closed, and there is a function f on D satisfying $df = g^*\theta - \theta$. The function f is unique up to addition of a constant; normalize f so that it is zero on ∂D. Calabi's homomorphism is defined by the formula

$$\Psi(g) = \int_D f d\theta$$

Changing θ to $\theta' = \theta + dh$ changes f to $f' = f + (h - h^g)$; since g is area-preserving, the integral of $(h - h^g)$ is zero, so Ψ does not depend on the choice of θ.

If g_1 and g_2 are two diffeomorphisms, then

$$(g_1g_2)^*\theta - \theta = g_2^* g_1^* \theta - g_1^*\theta + g_1^*\theta - \theta$$

so $\Psi(g_1g_2) = \Psi(g_1) + \Psi(g_2)$. The interpretation of Calabi's homomorphism as an "average braiding number" of pairs of points in the disk is due to Fathi (unpublished); see [**85**].

To define quasimorphisms on $\mathrm{Diff}_0^\infty(S^2, \mathrm{area})$, we need to construct quasimorphisms on $\widehat{B}_n$, the braid group of n-points in the sphere. One way to construct such quasimorphisms is to think of $\widehat{B}_n$ as the mapping class group of a sphere with n punctures, and use the methods of § 3.5, for instance Theorem 3.74. Another, more explicit method is to use the relationship between $\widehat{B}_n$ and B_{n-1}. By thinking of the disk as the once-punctured sphere, one sees that there is a homomorphism $B_{n-1} \to \widehat{B}_n$. The kernel of this map is $\mathbb{Z}$, generated by a "full twist" of all strands; and the image has finite index in $\widehat{B}_n$, and contains the kernel of the permutation map from $\widehat{B}_n$ to the symmetric group S_n. For example, $\widehat{B}_4$ contains the free group F_2 with finite index, and therefore admits an infinite dimensional family of homogeneous quasimorphisms.

Given a (homogeneous) quasimorphism μ on $\widehat{B}_n$, we can construct a homogeneous quasimorphism $\overline{\Phi}_\mu$ on $\mathrm{Diff}_0^\infty(S^2, \mathrm{area})$ as in Definition 5.67. In a similar way Gambaudo–Ghys show ([**86**], Theorem 1.2) that for every closed oriented surface S there exist an infinite dimensional space of homogeneous quasimorphisms on $\mathrm{Diff}_0^\infty(S, \mathrm{area})$.

5.3.3. Higher genus. When S has higher genus, one can construct quasimorphisms on $\mathrm{Diff}_0^\infty(S, \mathrm{area})$ from a hyperbolic structure on S, by a variation of the construction of de Rham quasimorphisms in § 2.3.1.

Definition 5.70 (de Rham quasimorphism). Let S be a closed surface with $\chi(S) < 0$. Fix a hyperbolic structure on S, and let α be a 1-form on S. Given $f \in \mathrm{Diff}_0^\infty(S, \mathrm{area})$, let f_t be an isotopy from id to f. For each $x \in S$, define $\gamma(x, f)$ to be the unique geodesic in S from x to $f(x)$ in the relative homotopy class of the path $f_t(x)$. Then define

$$\phi_\alpha(f) = \int_S \left(\int_{\gamma(x,f)} \alpha \right) d\mathrm{area}$$

Lemma 5.71. *The function ϕ_α is a quasimorphism on $\mathrm{Diff}_0^\infty(S, \mathrm{area})$ with defect at most $\|d\alpha\|\pi \cdot \mathrm{area}(S)$.*

PROOF. For any point x and any two elements f, g there is a geodesic triangle with edges $\gamma(x, f)$, $\gamma(f(x), g)$, and $\gamma(x, gf)$. By Stokes' theorem,

$$\left| \int_{\gamma(x,f)} \alpha + \int_{\gamma(f(x),g)} \alpha - \int_{\gamma(x,gf)} \alpha \right| \leq \|d\alpha\|\pi$$

Now integrate over $x \in S$, and use the fact that f is area-preserving to change variables in the second term on the left hand side. One obtains the estimate

$$|\phi_\alpha(f) + \phi_\alpha(g) - \phi_\alpha(gf)| \leq \|d\alpha\|\pi \cdot \text{area}(S)$$

as claimed. □

The homogenizations of ϕ_α are typically nontrivial, and generate an infinite dimensional subspace of Q. When α is a closed 1-form, ϕ_α depends only on the cohomology class $[\alpha] \in H^1(S)$, and is evidently equal to the *flux homomorphism* (Poincaré) dual to $[\alpha]$.

EXAMPLE 5.72 (Ruelle's rotation number [**181**]). The same method does not work directly on $\text{Diff}_0^\infty(T^2, \text{area})$. Nevertheless, Ruelle showed how to define a "rotation quasimorphism" on this group as follows. First, trivialize the tangent bundle; for example, we can choose a Euclidean metric on T^2, and use the flat connection to trivialize the bundle. Given $x \in T^2$ and $f \in \text{Diff}_0^\infty(T^2, \text{area})$, choose an isotopy f_t from id to f. Given a point x, the trivialization lets us canonically identify tangent spaces $T_{f_t(x)}$ and T_x, so we can think of df_t as a path in $\text{GL}(T_x)$. Projectivizing gives a path in $\text{PSL}(T_x)$; lifting to $\widetilde{\text{SL}}(T_x)$ and composing with the rotation quasimorphism defines a number $\rho(x, f)$. A different but homotopic path f_t' determines a homotopic path in $\text{PSL}(T_x)$. Since $\pi_1(\text{Diff}_0^\infty(T^2, \text{area}))$ is generated by loops of translations, $\rho(x, f)$ does not depend on any choices. Now define

$$R(f) = \int_{T^2} \rho(x, f) d\text{area}$$

Similar arguments to those above show that R is a (nontrivial) quasimorphism.

REMARK 5.73. If G is a subgroup of $\text{Diff}_0^\infty(T^2)$ and μ is any G-invariant probability measure on T^2, there is a Ruelle quasimorphism R_μ on G. Similar constructions also make sense on groups of Hamiltonian symplectomorphisms (or on their universal covers) of certain symplectic manifolds.

REMARK 5.74. There is a section from $\text{SL}(2, \mathbb{Z})$ to $\text{Diff}^\infty(T^2, \text{area})$ whose image consists of the linear automorphisms of T^2 fixing a basepoint. This group acts by conjugation on $\text{Diff}_0^\infty(T^2, \text{area})$, and the Ruelle quasimorphism is constant on orbits. Consequently, the Ruelle quasimorphism admits an extension to all of $\text{Diff}^\infty(T^2, \text{area})$.

Also see work of Py, e.g. [**173, 172, 174**] and Entov–Polterovich [**75**] for many more examples of quasimorphisms on various transformation groups.

5.3.4. C^0 case. The material in this section is taken from [**76**].

The quasimorphisms discussed in § 5.3.2 and § 5.3.3 are evidently continuous in the C^1 topology, and therefore extend continuously to quasimorphisms on groups of the form $\text{Diff}_0^1(S, \text{area})$. If a quasimorphism on $\text{Diff}_0^1(S, \text{area})$ is continuous in the C^0 topology, it extends to $\text{Homeo}_0(S, \text{area})$; this property is more delicate.

The following characterization of continuous quasimorphisms on topological groups is due to Shtern:

THEOREM 5.75 (Shtern [**188**], Thm. 1). *Let G be a topological group. A homogeneous quasimorphism ϕ on G is continuous if and only if it is bounded on some neighborhood of* id.

PROOF. One direction follows from the definition of continuity. Conversely, suppose there is a neighborhood U of id and a constant C so that $|\phi(k)| \le C$ for $k \in U$. For any $g \in G$ and $n \in \mathbb{N}$, define $U(g,n)$ to be the set of $h \in G$ such that $h^n = g^n k$ for some $k \in U$. Evidently, $U(g,n)$ is a neighborhood of g. Moreover, if $h \in U(g,n)$, then by homogeneity,

$$|\phi(h) - \phi(g)| = \frac{1}{n}|\phi(h^n) - \phi(g^n) - \phi(k) + \phi(k)|$$

where $k = g^{-n}h^n$. Since $k \in U$, there is an estimate $|\phi(k)| \le C$. Hence one can estimate

$$|\phi(h) - \phi(g)| \le \frac{1}{n}(D(\phi) + C)$$

Taking n large shows that $\phi(h) \to \phi(g)$ as $h \to g$, so ϕ is continuous. □

Using this characterization, Entov–Polterovich–Py derive the following theorem in the context of transformation groups. Given a surface S, let $\mathrm{Ham}(S, \mathrm{area})$ denote the subgroup of $\mathrm{Diff}_0^\infty(S, \mathrm{area})$ consisting of Hamiltonian diffeomorphisms (i.e. those in the kernel of every flux homomorphism).

THEOREM 5.76 (Entov–Polterovich–Py). *Let ϕ be a homogeneous quasimorphism on* $\mathrm{Ham}(S, \mathrm{area})$. *Then ϕ is continuous in the C^0 topology if and only if there is some positive constant a so that if $D \to S$ is any embedded disk of area at most a, then ϕ vanishes identically on the subgroup $G(D)$ of elements supported in D.*

PROOF. We give the sketch of a proof; for details, see [**76**]. Suppose ϕ is continuous, and let U be a neighborhood of id (in the C^0 topology) for which there is a constant C as in the conclusion of Theorem 5.75. If D_0 is sufficiently small in diameter, then $G(D_0) \subset U$, and therefore ϕ is bounded on $G(D_0)$. But since ϕ is a homogeneous quasimorphism, and $G(D_0)$ is a group, ϕ must vanish identically on $G(D_0)$. Now, if D is any other disk with $\mathrm{area}(D) \le \mathrm{area}(D_0)$, there is an area-preserving Hamiltonian isotopy from D to D_0. Hence $G(D_0)$ and $G(D)$ are conjugate, and the conclusion follows.

Conversely, suppose there is a positive constant a with the desired properties. There is a neighborhood U of the identity so that S can be covered with finitely many disks D_i for $i \le N$, each of area at most a, so that any $f \in U$ can be written as a product $f = g_1 g_2 \cdots g_N$ where the support of each g_i is contained in D_i (and therefore $g_i \in G(D_i)$). Since ϕ vanishes identically on each $G(D_i)$, the value of ϕ on f is bounded by $(N-1)D(\phi)$, and therefore ϕ is continuous, by Theorem 5.75. □

A homogeneous quasimorphism on $\mathrm{Diff}_0^\infty(S, \mathrm{area})$, continuous on $\mathrm{Ham}(S, \mathrm{area})$, and linear on every one-parameter subgroup, is continuous in the C^0 topology, and therefore extends to $\mathrm{Homeo}_0(S, \mathrm{area})$.

REMARK 5.77. The most delicate aspect of Theorem 5.76 is the fragmentation lemma (i.e. to show that one can express a Hamiltonian diffeomorphism sufficiently C^0 close to the identity as a product of boundedly many diffeomorphisms supported in small disks). This depends on work of Le Roux [**133**]. Note that the assumption that the diffeomorphism be Hamiltonian is essential.

Example 5.78. When the genus of S is large, the homogenizations of the de Rham quasimorphisms (Definition 5.70) vanish on $G(D)$ for *any* embedded disk D. Hence they are continuous in the C^0 topology, and extend to quasimorphisms on $\mathrm{Homeo}_0(S, \mathrm{area})$.

It is still unknown whether $\mathrm{Homeo}_0(S^2, \mathrm{area})$ admits any nontrivial quasimorphism.

Remark 5.79. The study of quasimorphisms on (mostly 2-dimensional) transformation groups is an active and fertile area. In addition to the work of Entov–Polterovich [**75**] and Gambaudo–Ghys referred to above, we mention only the survey [**169**] by Polterovich, and [**175**] by Py, discussing relations of this material to Zimmer's program.