Chapter 2. Existence and properties of the measure $\mathbf{W}^{(2)}$.

We shall now establish a number of results similar to those of Chapter 1, but this time $(X_t,\ t \geq 0)$ is a 2-dimensional Brownian motion.

2.1. Existence of $\mathbf{W}^{(2)}$.

2.1.1 Notations and Feynman-Kac penalisations in two dimensions.

$\big(\Omega = \mathcal{C}(\mathbb{R}_+ \to \mathbb{C}), (X_t,\ \mathcal{F}_t)_{t\geq 0},\ W_z^{(2)}(z \in \mathbb{C})\big)$ denotes the two dimensional canonical Brownian motion, which takes its values in $\mathbb{C}$. We write $W^{(2)}$ for $W_0^{(2)}$. $\mathcal{I}$ denotes here the set of positive Radon measures on $\mathbb{C}$ admitting a density q with compact support and such that $\displaystyle\int q(x)dx > 0$. Define :

$$A_t^{(q)} := \int_0^t q(X_s)ds \tag{2.1.1}$$

Here is the analogue in dimension 2 of Theorem 1.1.1. A proof of this Theorem (in dimension 2) is found in [RVY, VI].

Theorem 2.1.1. *Let $q \in \mathcal{I}$ and, for every $t \geq 0$ and $z \in \mathbb{C}$:*

$$W_{z,t}^{(2,q)} := \frac{\exp\big(-\frac{1}{2}A_t^{(q)}\big)}{Z_{z,t}^{(2,q)}} \cdot W_z^{(2)} \tag{2.1.2}$$

with

$$Z_{z,t}^{(2,q)} := W_z^{(2)}\left(\exp -\frac{1}{2}A_t^{(q)}\right) \tag{2.1.3}$$

1) *For every $s \geq 0$ and $\Gamma_s \in b(\mathcal{F}_s)$:*
$W_{z,t}^{(2,q)}(\Gamma_s)$ admits a limit $W_{z,\infty}^{(2,q)}(\Gamma_s)$ as $t \to \infty$:

$$W_{z,t}^{(2,q)}(\Gamma_s) \underset{t\to\infty}{\longrightarrow} W_{z,\infty}^{(2,q)}(\Gamma_s) \tag{2.1.4}$$

2) *$W_{z,\infty}^{(2,q)}$ is a probability on $(\Omega, \mathcal{F}_\infty)$ such that :*

$$W_{z,\infty}^{(2,q)}|_{\mathcal{F}_s} = M_s^{(2,q)} \cdot W_z^{(2)}|_{\mathcal{F}_s}$$

where $(M_s^{(2,q)},\ s \geq 0)$ is the $\big((\mathcal{F}_s,\ s \geq 0),\ W_z^{(2)}\big)$ martingale defined by :

$$M_s^{(2,q)} = \frac{\varphi_q(X_s)}{\varphi_q(z)}\ \exp\left(-\frac{1}{2}A_s^{(q)}\right) \tag{2.1.5}$$

3) *The function $\varphi_q : \mathbb{C} \to \mathbb{R}_+$ featured in (2.1.5) is strictly positive, continuous and satisfies :*

$$\varphi_q(z) \underset{|z|\to\infty}{\sim} \frac{1}{\pi}\ \log\big(|z|\big) \tag{2.1.6}$$

It may be defined via one or the other of the following descriptions :
i) φ_q is the unique solution of the Sturm-Liouville equation :

$$\Delta\varphi = q \cdot \varphi \quad \textit{(in the sense of Schwartz distributions)}$$

which satisfies the limiting condition :

$$|z|\frac{\partial\varphi}{\partial r}\,(z)\underset{r\to\infty}{\longrightarrow}\frac{1}{\pi}\qquad\big(r=|z|\big)\tag{2.1.7}$$

ii) $$\frac{1}{2\pi}(\log\ t)\,W_z^{(2)}\left(\exp\left(-\frac{1}{2}\,A_t^{(q)}\right)\right)\underset{t\to\infty}{\longrightarrow}\varphi_q(z)\tag{2.1.8}$$

4) *Under the family of probabilities* $(W_{z,\infty}^{(2,q)}, z\in\mathbb{C})$*, the canonical process* $(X_t,\ t\geq 0)$ *is a transient diffusion. More precisely, there exists a* $\big(\Omega,(\mathcal{F}_t,\ t\geq 0),\ W_{z,\infty}^{(2,q)}\big)$ *Brownian motion* $(B_t,\ t\geq 0)$ *valued in* $\mathbb{C}$ *and starting from 0 such that :*

$$X_t=z+B_t+\int_0^t\frac{\nabla\varphi_q}{\varphi_q}(X_s)ds\tag{2.1.9}$$

2.1.2 Existence of the measure $\mathbf{W}^{(2)}$.

Theorem 2.1.2. *There exists on* $\big(\Omega=\mathcal{C}(\mathbb{R}_+\to\mathbb{C}),\ \mathcal{F}_\infty\big)$ *a* σ*-finite and positive measure* $\mathbf{W}^{(2)}$ *(with infinite total mass) such that, for every* $q\in\mathcal{I}$ *:*

$$\mathbf{W}^{(2)}=\varphi_q(0)\ \exp\left(+\frac{1}{2}\,A_\infty^{(q)}\right)\cdot W_\infty^{(2,q)}\tag{2.1.10}$$

In other terms, the RHS of (2.1.10) does not depend on $q\in\mathcal{I}$*.*
In fact, just as in the case of dimension 1, we show for every $z\in\mathbb{C}$, the existence of a measure $\mathbf{W}_z^{(2)}$, this measure being defined by :

$$\mathbf{W}_z^{(2)}\big(F(X_s,\ s\geq 0)\big)=\mathbf{W}^{(2)}\big(F(z+X_s,\ s\geq 0)\big)\tag{2.1.11}$$

Proof of Theorem 2.1.2.
It consists in showing that $\varphi_q(0)\ \exp\left(+\frac{1}{2}\,A_\infty^{(q)}\right)\cdot W_\infty^{(2,q)}$ does not depend on q. The proof is quite similar to that of Theorem 1.1.2. It hinges upon :

- $\varphi_q(z)>0$ for every $q\in\mathcal{I}$ and $z\in\mathbb{C}$;
- $\dfrac{\varphi_{q_1}(z)}{\varphi_{q_2}(z)}\underset{|z|\to\infty}{\longrightarrow}1$ for every q_1 and $q_2\in\mathcal{I}$;
- $\varphi_q(z)\underset{|z|\to\infty}{\longrightarrow}+\infty$ and the $(W_{z,\infty}^{(2,q)},\ z\in\mathbb{C})$ process $(X_t,\ t\geq 0)$ is transient.

These properties follow from Theorem 2.1.1. We also note, just as we did in Lemma 1.1.3 :

$$W_{z,\infty}^{(2,q)}\left(\exp+\frac{\lambda}{2}\,A_\infty^{(q)}\right)\ <\ \infty\quad\text{if}\ \ \lambda<1\tag{2.1.12}$$

$$W_{z,\infty}^{(2,q)}\left(\exp+\frac{\lambda}{2}\,A_\infty^{(q)}\right)\ =\ \infty\quad\text{if}\ \ \lambda\geq 1\tag{2.1.13}$$

These two properties show that $\mathbf{W}^{(2)}$ is well defined via (2.1.10) $\big($since $A_\infty^{(q)}<\infty\ \ W_\infty^{(2,q)}$ a.s.$\big)$ and that $\mathbf{W}^{(2)}$ has infinite total mass ; it is σ-finite on $(\Omega,\mathcal{F}_\infty)$ and it is such that

$\mathbf{W}^{(2)}(\Gamma_t) = 0$ or $+\infty$ for any $\Gamma_t \in b^+(\mathcal{F}_t)$ depending whether $W^{(2)}(\Gamma_t)$ is equal to 0 or is strictly positive.

2.2 Properties of $\mathbf{W}^{(2)}$.

2.2.1 Some notation.

We shall now prepare for Theorem 2.2.1 - which plays for $\mathbf{W}^{(2)}$ a similar role as Theorem 1.1.5 for $\mathbf{W}$. However, in order to prepare for Theorem 2.2.1, we need the following notation :

i) Denote by C the unit circle in $\mathbb{C}$:

$$C = \{z \in \mathbb{C} \; ; \; |z| = 1\} \tag{2.2.1}$$

and $(L_t^{(C)}, \; t \geq 0)$ the (continuous) local time process on C, which may be defined as :

$$L_t^{(C)} := \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi\varepsilon} \int_0^t 1_{C_\varepsilon}(X_s) ds \tag{2.2.2}$$

where

$$C_\varepsilon = \{z \in \mathbb{C} \; ; \; 1 - \varepsilon \leq |z| \leq 1 + \varepsilon\}$$

so that, a.s. if q_0 denotes the uniform probability on C :

$$\int_C f(z) \, q_0(dz) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) d\theta \tag{2.2.3}$$

we have :

$$(L_t^{(C)}, \; t \geq 0) = (A_t^{(q_0)}, \; t \geq 0) \tag{2.2.4}$$

In other terms, $(L_t^{(C)}, \; t \geq 0)$ is the additive functional which admits q_0 as Revuz's measure (see [Rev]). We denote by $(\tau_l^{(C)}, \; l \geq 0)$ the right continuous inverse of $(L_t^{(C)}, \; t \geq 0)$:

$$\tau_l^{(C)} := \inf\{t \geq 0 \; ; \; L_t^{(C)} > l\}, \qquad l \geq 0 \tag{2.2.5}$$

and we denote by $W_0^{(2, \tau_l^{(C)})}$ the law of a 2-dimensional Brownian motion starting from 0, considered up to $\tau_l^{(C)}$.

ii) We denote by $P_1^{(2,\log)}$ the law of the process $(R_t, \; t \geq 0)$ which solves the stochastic differential equation :

$$R_t = 1 + \beta_t + \int_0^t \frac{ds}{R_s} \left(\frac{1}{2} + \frac{1}{\log R_s} \right) \tag{2.2.6}$$

where $(\beta_t, \; t \geq 0)$ is a 1-dimensional Brownian motion starting from 0. We note that the process $(R_t, \; t \geq 0)$ starts from 1 and that $P(R_t > 1 \text{ for every } t > 0) = 1$.

We adopted the notation $P_1^{(2,\log)}$ to indicate :

a) that this process R starts from 1 ;

b) that it "differs at infinity from a 2-dimensional Bessel process" by the presence of the term $\dfrac{1}{\log R_s}$, in the drift part of equation (2.2.6).

iii) Here is another description of the process $(R_t,\ t \geq 0)$ defined by (2.2.6) :

$$(\log\ R_t,\ t \geq 0) \overset{\text{(law)}}{=} (\rho_{H_t},\ t \geq 0) \tag{2.2.7}$$

with :

- $(\rho_u,\ u \geq 0)$ a 3-dimensional Bessel process starting from 0 ;
- $H_t := \displaystyle\int_0^t \frac{ds}{R_s^2}$ (2.2.8)

We prove (2.2.7).

We apply Itô's formula to the process (R_t) solution of (2.2.6) and we obtain :

$$\log\ R_t = \int_0^t \frac{d\beta_s}{R_s} + \int_0^t \frac{ds}{R_s^2 \cdot \log\ R_s} \tag{2.2.9}$$

We denote by $(\nu_h,\ h \geq 0)$ the inverse of the process $(H_t,\ t \geq 0)$ and we replace t by ν_h in (2.2.9). Thus :

$$\log\ R_{\nu_h} = \int_0^{\nu_h} \frac{d\beta_s}{R_s} + \int_0^{\nu_h} \frac{ds}{R_s^2\ \log\ R_s} \tag{2.2.10}$$

$$= \widetilde{\beta}_h + \int_0^h \frac{du}{\log\ R_{\nu_u}} \tag{2.2.11}$$

after the change of variable $s = \nu_u$ and with $(\widetilde{\beta}_h,\ h \geq 0) := \left(\displaystyle\int_0^{\nu_h} \frac{d\beta_s}{R_s^2},\ h \geq 0\right)$, which is a 1-dimensional Brownian motion since this - local - martingale admits as bracket $\left(\displaystyle\int_0^{\nu_h} \frac{ds}{R_s^2} = H_{\nu_h} = h,\ h \geq 0\right)$. Hence, from (2.2.11) $(\log\ R_{\nu_h},\ h \geq 0)$ is a 3-dimensional Bessel process starting from 0.

iv) Let now $(\alpha_t,\ t \geq 0)$ be another 1-dimensional Brownian motion, independent from $(\beta_t,\ t \geq 0)$ (hence independent from $(R_t,\ t \geq 0)$). We define the law $W^{(2,\tau_l^{(C)})} \circ \widetilde{P}_1^{(2,\log)}$ as the law of the 2-dimensional process $(Y_t,\ t \geq 0)$ satisfying to :

a) $(Y_t,\ t \leq \tau_l^{(C)})$ is a 2-dimensional Brownian motion starting from 0 and stopped in $\tau_l^{(C)}$; its law, from point *i)*, is $W^{(2,\tau_l^{(C)})}$. Here, $\tau_l^{(C)}$ is the right-continuous inverse of $(L_t^{(C)}, t \geq 0)$, the local time on C of the process $(Y_t, t \geq 0)$.

b) after $\tau_l^{(C)}$, the process $(Y_{\tau_l^{(C)}+t}\ t \geq 0)$ writes :

$$Y_{\tau_l^{(C)}+t} := R_t \cdot e^{i\alpha_{H_t}} \qquad (t \geq 0) \tag{2.2.12}$$

where :

- the law of the process $(R_t,\ t \geq 0)$ is $P_1^{(2,\,\log)}$
- $(\alpha_t,\ t \geq 0)$ is a 1-dimensional Brownian motion starting from α_0, with $e^{i\alpha_0} = Y_{\tau_l^{(C)}}$ (we note that $Y_{\tau_l^{(C)}} \in C$)
- $H_t = \displaystyle\int_0^t \frac{ds}{R_s^2}$

c) $(\alpha_t,\ t \geq 0)$ and $(\beta_t,\ t \geq 0)$, the driving Brownian motion of $(R_t,\ t \geq 0)$ (see (2.2.6)) are, conditionally on α_0, independent from the process $(Y_t,\ t \leq \tau_l^{(C)})$.
Formula (2.2.7) - the second description of $(R_t,\ t \geq 0)$ - permits to write (2.2.12) in another form :

$$Y_{\tau_l^{(C)}+t} = \exp(\rho_u + i\alpha_u)|_{u=H_t} \qquad (t \geq 0) \tag{2.2.13}$$

where $(\rho_u,\ u \geq 0)$ is a 3-dimensional Bessel process starting from 0 and $H_t = \displaystyle\int_0^t \frac{ds}{R_s^2}$.

2.2.2 Description of the canonical process $(X_t,\ t \geq 0)$ under $W_\infty^{(2,q_0)}$.
In order to describe the measure $\mathbf{W}^{(2)}$, we shall use the formula :

$$\mathbf{W}^{(2)} = \varphi_{q_0}(0)(e^{\frac{1}{2}L_\infty^{(C)}}) \cdot W_\infty^{(2,q_0)} \tag{2.2.14}$$

This is formula (2.1.10), with $q = q_0$ (in fact, we use here a slight extension of (2.1.10) since q_0 is not absolutely continuous with respect to Lebesgue measure on $\mathbb{C}$). We now need to study the probability $W_\infty^{(2,q_0)}$. This is the aim of the following Theorem :
Theorem 2.2.1. *With the notation of Theorem 2.1.1 :*

1)
$$\begin{aligned} \varphi_{q_0}(z) &= 2 + \frac{1}{\pi}\log|z| \qquad \text{if } |z| \geq 1 \\ &= 2 \qquad\qquad\qquad \text{if } |z| \leq 1 \end{aligned} \tag{2.2.15}$$

and $(M_s^{(q_0)},\ s \geq 0)$ *is the martingale defined by :*

$$M_s^{(q_0)} = \frac{\varphi_{q_0}(X_s)}{\varphi_{q_0}(0)}\exp\left(-\frac{1}{2}L_s^{(C)}\right) \tag{2.2.16}$$

$$= 1 + \frac{1}{\varphi_q(0)}\int_0^s < \nabla\varphi_{q_0}(X_u),\ dX_u > e^{-\frac{1}{2}L_u^{(C)}} \tag{2.2.17}$$

2) *Let* $g_C := \sup\{t \geq 0\ ;\ X_t \in C\}$. *Then* g_C *is* $W_\infty^{(2,q_0)}$ *a.s. finite and the r.v.* $L_\infty^{(C)}(= L_{g_C}^{(C)})$ *admits as density* $f_{L_\infty^{(C)}}^{W_\infty^{(2,q_0)}}$ *with :*

$$f_{L_\infty^{(C)}}^{W_\infty^{(2,q_0)}}(l) = \frac{1}{2}e^{-\frac{l}{2}}1_{[0,\infty[}(l) \tag{2.2.18}$$

3) *Under the probability* $W_\infty^{(2,q_0)}$ *:*
i) Conditionally on X_{g_C}, $(X_s,\ s \leq g_C)$ *and* $(X_{g_C+s},\ s \geq 0)$ *are independent*
ii) The law of the process $(X_{g_C+s},\ s \geq 0)$ *is* $\widetilde{P}_1^{(2,\log)}$ (*defined in point 2.2.1, iv)*)
iii) Conditionally on $L_{g_C}^{(C)} = l$, *the process* $(X_s,\ s \leq g_C)$ *is a 2-dimensional Brownian process stopped at* $\tau_l^{(C)}$, *and its law, from point 2.2.1 i), is* $W_0^{(2,\tau_l^{(C)})}$.
In other terms :
iv)
$$W_\infty^{(2,q_0)} = \frac{1}{2}\int_0^\infty e^{-\frac{l}{2}}dl\big(W_0^{(2,\tau_l^{(C)})} \circ \widetilde{P}_1^{(2,\log)}\big) \tag{2.2.19}$$

We note, in particular, that $X_{\tau_l^{(C)}}$ under $W_\infty^{(2,q_0)}$ is uniformly distributed on C.

Proof of Theorem 2.2.1.

In dimension 1, this Theorem is, essentially, proven in $\big([\text{RVY, II}]\big)$. The only item which really differs from those of Theorem 8 in [RVY, II] is point 3, *ii)*. We shall emphasize the corresponding arguments.

We prove point 3, *ii)*.

We first recall and adapt to dimension 2 the notation and results of [RVY, II].

i) Let $(\mathcal{G}_t,\ t\geq 0)$ be the smallest filtration containing $(\mathcal{F}_t,\ t\geq 0)$ and such that g_C is a $(\mathcal{G}_t,\ t\geq 0)$ stopping time. Then, there exists a $\big((\mathcal{G}_t,\ t\geq 0),\ W_\infty^{2,q_0}\big)$ 2-dimensional Brownian motion $(B_t,\ t\geq 0)$ such that :

$$X_t = B_t + \int_{t\wedge g_C}^t \frac{n_u}{M_u^{(q_0)}-\underline{M}_u^{(q_0)}}\ du \tag{2.2.20}$$

with :

• $n_u := e^{-\frac{1}{2}L_u^{(C)}}\cdot\frac{\nabla\varphi_{q_0}(X_u)}{\varphi_{q_0}(0)}$ (2.2.21)

• $M_u^{(q_0)}$ is defined by (2.2.16) and :

$$\underline{M}_u^{(q_0)} := \inf_{s\leq u} M_s^{(q_0)} \tag{2.2.22}$$

ii) The function $\varphi_{q_0}(z) = 2+\frac{1}{\pi}\ \log|z|$ (for $|z|\geq 1$) $\big($see (2.2.15)$\big)$ is increasing in $|z|$. On the other hand, for $u\geq g_C$, $L_u^{(q_0)} = L_{g_C}^{(C)}$. Thus :

$$\begin{aligned}\underline{M}_u^{(q_0)} &= M_{g_C}^{(q_0)} = \frac{\varphi_{q_0}(X_{g_C})}{\varphi_{q_0}(0)}\ e^{-\frac{1}{2}L_{g_C}^C}\\ &= e^{-\frac{1}{2}L_{g_C}^{(C)}}\end{aligned} \tag{2.2.23}$$

$\big($from (2.2.15) and since $X_{g_C}\in C\big)$.

iii) Gathering (2.2.20), (2.2.21) and (2.2.23), we obtain :

$$\begin{aligned}X_t &= B_t + \int_{t\wedge g_C}^t du\ \frac{\nabla\varphi_{q_0}(X_u)\,e^{-\frac{1}{2}L_{g_C}^{(C)}}}{\varphi_{q_0}(X_u)\,e^{-\frac{1}{2}L_{g_C}^{(C)}} - 2e^{-\frac{1}{2}L_{g_C}^{(C)}}},\\ &= B_t + \int_{t\wedge g_C}^t \frac{\nabla\big(\log|\cdot|\big)(X_u)}{\log|X_u|}\ du \ \text{(after simplification by } e^{-\frac{1}{2}L_{g_C}^{(C)}})\end{aligned} \tag{2.2.24}$$

$\big($from (2.2.15), since $\varphi_{q_0}(X_u)-2 = \frac{1}{\pi}\ \log|X_u|$ and $\nabla\varphi_{q_0}(X_u) = \frac{1}{\pi}\big(\nabla\log|\cdot|\big)(X_u)\big)$.

iv) We now use Itô's formula to express $|X_{g_C+t}| := \widetilde{R}_t$. We obtain, from (2.2.24) :

$$\widetilde{R}_t = (\widetilde{B}_{g_C+t}-\widetilde{B}_{g_C}) + \int_0^t \frac{ds}{\widetilde{R}_s}\left(\frac{1}{2}+\frac{1}{\log\widetilde{R}_s}\right) \tag{2.2.25}$$

where $\big(\widetilde{B}_{g_C+t}-\widetilde{B}_{g_C},\ t\geq 0\big)$ is a 1-dimensional Brownian motion started at 1. Thus, from (2.2.6), the law of $\big(|X_{g_C+t}|,\ t\geq 0\big)$ is $P_1^{(2,\log)}$.

Now, operating in an analogous manner to calculate Arg (X_{g_C+t}), we obtain :

$$(X_{g_C+t},\ t\geq 0) = (R_t\,e^{i\alpha_{H_t}},\ t\geq 0) \tag{2.2.26}$$

with notation of points 2.2.1, *ii)*, *iii)* and *iv)*.

2.2.3 Another description of the measure $\mathbf{W}^{(2)}$.

We now present a description of $\mathbf{W}^{(2)}$ which is analogous, in dimension 2, to the description of $\mathbf{W}$ given by Theorem 1.1.6.

Theorem 2.2.2.

1) $\mathbf{W}^{(2)} = \int_0^\infty dl\big(W_0^{(2,\tau_l^{(C)})} \circ \widetilde{P}_1^{(2,\log)}\big)$ (2.2.27)

2) *For every* $t \geq 0$ *and* $\Gamma_t \in b(\mathcal{F}_t)$ *:*

$$\mathbf{W}^{(2)}[\Gamma_t\, 1_{g_C \leq t}] = \frac{1}{\pi}\, W^{(2)}\big[\Gamma_t\, \log^+(|X_t|)\big] \tag{2.2.28}$$

$\big($*Recall that* $g_C := \sup\{s \geq 0\,;\ X_s \in C\}\big)$

3) *i)*

$$\mathbf{W}^{(2)}(g_C \in dt) = e^{-\frac{1}{2t}}\, \frac{dt}{2\pi t} \quad (t \geq 0) \tag{2.2.29}$$

ii) Conditionally on $g_C = t$*, the law of the process* $(X_u,\ u \leq g_C)$*, under* $\mathbf{W}^{(2)}$ *is* $\Pi_0^{(2,t,U)}$*, where :*

- *U is a r.v. uniformly distributed on C ;*
- *Conditionally on* $U = u$, $\Pi_0^{(2,t,U)}$ *is the law of a 2-dimensional Brownian bridge* $(b_s^{(2,t,u)},\ 0 \leq s \leq t)$ *of length* t *such that* $b_0^{(2,t,u)} = 0$ *and* $b_t^{(2,t,u)} = u$.

iii)

$$\mathbf{W}^{(2)} = \int_0^\infty \frac{dt}{2\pi t}\, e^{-\frac{1}{2t}}\big(\Pi^{2,t,U} \circ \widetilde{P}_1^{(2,\log)}\big) \tag{2.2.30}$$

Proof of Theorem 2.2.2.

i) Point 1) is an easy consequence of (2.2.14), (2.2.19) and (2.2.18).

ii) We now show (2.2.28)

For this purpose, we use the definition (2.1.10) of $\mathbf{W}^{(2)}$ with $q = \lambda\, q_0$ (where q_0 is defined by (2.2.3), and $\lambda > 0$). We have :

$$\varphi_{\lambda\, q_0}(z) = \frac{2}{\lambda} + \frac{1}{\pi}\, \log^+(|z|) \tag{2.2.31}$$

$\big($see (2.2.15)$\big)$. Thus, for every $t \geq 0$ and $\Gamma_t \in b(\mathcal{F}_t)$:

$$\begin{aligned} W^{(2)}\left(\Gamma_t\left(\frac{2}{\lambda} + \frac{1}{\pi}\, \log^+(|X_t|)\right)\right) &= \varphi_{\lambda\, q_0}(0) W_\infty^{(2,\lambda\, q_0)}\big(\Gamma_t\, e^{\frac{\lambda}{2}\, L_t^{(C)}}\big) \\ &= \mathbf{W}^{(2)}\big(\Gamma_t\, e^{-\frac{\lambda}{2}(L_\infty^{(C)} - L_t^{(C)})}\big) \end{aligned} \tag{2.2.32}$$

We then let $\lambda \to \infty$ in (2.2.32) and note that $L_\infty^{(C)} - L_t^{(C)} > 0$ on the set $(g_C > t)$ (and equals to 0 on $g_C \leq t$). The monotone convergence Theorem implies :

$$\frac{1}{\pi}\, W^{(2)}\big(\Gamma_t\, \log^+(|X_t|)\big) = \mathbf{W}^{(2)}(\Gamma_t\, 1_{g_C \leq t})$$

This is (2.2.28). Note that we may replace t by a stopping time T in (2.2.28). We obtain :

$$\mathbf{W}^{(2)}\big(\Gamma_T\, 1_{g_C\le T<\infty}\big) = \frac{1}{\pi}\, W^{(2)}\big(\Gamma_T\, \log^+(|X_T|)1_{T<\infty}\big) \tag{2.2.33}$$

with $\Gamma_T \in b(\mathcal{F}_T)$.

Remark 2.2.3.

We deduce from (2.2.32) and (2.2.28) :

$$\begin{aligned} \frac{2}{\lambda}\, W^{(2)}(\Gamma_t) &= \mathbf{W}^{(2)}\left(\Gamma_t\, 1_{g_C>t}\, \exp\big(-\frac{\lambda}{2}\big(L_\infty^{(C)} - L_t^{(C)}\big)\big)\right) \\ &= W^{(2)}(\Gamma_t)\left(\int_0^\infty e^{-\frac{\lambda}{2}l} dl\right) \end{aligned} \tag{2.2.34}$$

and

$$\frac{1}{\pi} W^{(2)}\big(\log^+|X_t|\big) = \mathbf{W}^{(2)}(g_C \le t) = \mathbf{W}^{(2)}(L_\infty^{(C)} - L_t^{(C)} = 0) \tag{2.2.35}$$

Then, operating as in the proof of Theorem 1.1.6, point 3) *i)* $\big($see (1.1.45) and (1.1.46)$\big)$, we obtain :

i) $\mathbf{W}^{(2)}(L_\infty^{(C)} - L_t^{(C)} \in dl) = 1_{[0,\infty[}(l)dl + \frac{1}{\pi}W^{(2)}\big(\log^+(|X_t|)\big)\delta_0(dl)$ (2.2.36)

ii) Conditionally on $L_\infty^C - L_t^C = l \quad (l>0)$, $(X_u,\ u\le t)$ is, under $\mathbf{W}^{(2)}$, a 2-dimensional Brownian motion indexed by $[0,t]$.

Remark 2.2.4. We can obtain (2.2.28) in the same manner as for point 2) of Remark 1.1.9. For this purpose, we need a scale function for the $W^{(2,q_0)}$ process. The function $z \to \dfrac{1}{1+\frac{1}{\pi}\log(|z|)}$ $(|z|\ge 1)$ is an adequate choice.

iii) <u>We now prove point 3 *i)* of Theorem 2.2.2.</u>

We write (2.2.28) with $\Gamma_t \equiv 1$:

$$\mathbf{W}^{(2)}(g_C \le t) = \frac{1}{\pi}\, W^{(2)}(\log^+|X_t|) \tag{2.2.37}$$

and we differentiate (2.2.37) with respect to t. Thus :

$$\begin{aligned} \mathbf{W}^{(2)}(g_C\in dt) &= \frac{1}{\pi}\left(\frac{d}{dt}\, W^{(2)}(\log^+|X_t|)\right)\cdot dt \\ &= \frac{1}{\pi}\frac{d}{dt} W^{(2)}\left(1_{|X_1|>\frac{1}{\sqrt{t}}}\left(\log\sqrt{t} - \log\frac{1}{|X_1|}\right)\right)\cdot dt \quad \text{(by scaling)} \\ &= \frac{1}{2\pi t}\, W^{(2)}\left(\frac{|X_1|^2}{2} > \frac{1}{2t}\right) dt \\ &= \frac{1}{2\pi t}\, e^{-\frac{1}{2t}} dt \qquad (t\ge 0) \end{aligned}$$

since $\dfrac{|X_1|^2}{2}$ is a standard exponential r.v.

The end of the proof of Theorem 2.2.2 is obtained by using arguments similar to those used for Theorem 1.1.6. We note, in particular, that conditionally on X_{g_C}, $(X_{g_C+t},\ t\ge 0)$ and $(X_s,\ s\le g_C)$ are independent.

Remark 2.2.5. From (2.2.29), we deduce :

$$\mathbf{W}^{(2)}(e^{-\frac{\lambda^2}{2}g_C}) = \int_0^\infty \frac{dt}{2\pi t}\, e^{-\frac{\lambda^2}{2}t - \frac{1}{2t}} = K_0(\lambda) \tag{2.2.38}$$

where K_0 denotes the Bessel-Mc Donald function with index 0 $\big($see [L], formula 5.10.25$\big)$.

2.3. Study of the winding process under $\mathbf{W}^{(2)}$.

Formula (2.2.12) :

$$X_{g_C+t} = R_t\, e^{i\alpha_{H_t}},\ (t \geq 0)$$

which provides a representation of X after g_C under $W^{(2)}$ invites to establish for this process a theorem similar to the classical theorem of Spitzer, which we recall :

2.3.1 Spitzer's Theorem.

Theorem. (Spitzer [S])

Let $(X_t,\ t \geq 0)$ a $\mathbb{C}$ valued Brownian motion, starting from $z \neq 0$. We have :

$$X_t = |X_t|\, e^{i\alpha_{H_t}} \tag{2.3.1}$$

with :

i) $(\alpha_u,\ u \geq 0)$ a 1-dimensional Brownian motion independent from the 2-dimensional Bessel process $(|X_t|,\ t \geq 0)$ (one can also find a precise study of the winding process of planar Brownian motion in [PY1]).

ii) $$H_t = \int_0^t \frac{ds}{|X_s|^2} \tag{2.3.2}$$

Let $(\theta_t,\ t \geq 0) := (\alpha_{H_t},\ t \geq 0) = \left(\theta_0 + \operatorname{Im} \int_0^t \frac{dX_s}{X_s},\ t \geq 0\right)$ *be the winding process. Then :*

$$\frac{2\theta_t}{\log\, t} \underset{t\to\infty}{\overset{\text{(law)}}{\longrightarrow}} \Gamma \overset{\text{(law)}}{=} \alpha_{T_1(\gamma)} \tag{2.3.3}$$

In (2.3.3), $(\gamma_t,\ t \geq 0)$ is a 1-dimensional Brownian motion started from 0 and independent from $(\alpha_u,\ u \geq 0)$. and :

$$T_1(\gamma) := \inf\{s \geq 0\ ;\ \gamma_s = 1\} \tag{2.3.4}$$

iii) Consequently Γ is a standard Cauchy r.v.

2.3.2. An analogue of Spitzer's Theorem.

Now, here is the analogue of the above (Spitzer) Theorem for the process $(X_{g_C+t},\ t \geq 0)$:

Theorem 2.3.1. *Under $\widetilde{P}_1^{(2,\log)}$, the winding process $(\theta_t,\ t \geq 0) = (\alpha_{H_t},\ t \geq 0)$ satisfies :*

1) $$\frac{4}{(\log\, t)^2}\, H_t \underset{t\to\infty}{\overset{\text{(law)}}{\longrightarrow}} T_1^{(3)} \tag{2.3.5}$$

where $$T_1^{(3)} := \inf\{u\ ;\ \rho_u = 1\} \tag{2.3.6}$$

is the first hitting time of level 1 by a 3-dimensional Bessel process $(\rho_u,\ u \geq 0)$ started at 0.

2) $$\frac{2}{\log\, t}\, \theta_t \underset{t\to\infty}{\overset{\text{(law)}}{\longrightarrow}} \alpha_{T_1^{(3)}} \tag{2.3.7}$$

where $(\alpha_u,\ u \geq 0)$ is a 1-dimensional Brownian motion independent from $(\rho_u,\ u \geq 0)$.

We now recall our notation (see Section 2.2.1)

- $(R_t,\ t \geq 0)$ is the process defined in (2.2.6)
- $$H_t = \int_0^t \frac{ds}{R_s^2} \tag{2.3.8}$$
- $(\alpha_u,\ u \geq 0)$ is a 1-dimensional Brownian motion independent from $(R_t,\ t \geq 0)$

- $(\log R_t,\ t \geq 0) = (\rho_{H_t},\ t \geq 0)$ and $(\rho_u,\ u \geq 0)$ is a 3-dimensional Bessel process started at 0.

Remark 2.3.2.
1. Theorem 2.3.1 differs from Spitzer's Theorem in that T_1 has been replaced by $T_1^{(3)}$.
2. Let, for every $z \in \mathbb{C}$, $\mathbf{W}_z^{(2)}$ be defined by :

$$\mathbf{W}_z^{(2)}\big(F(X_s,\ s \geq 0)\big) := \mathbf{W}^{(2)}\big(F(z + X_s,\ s \geq 0)\big)$$

Theorem 2.3.1 then implies that, for $z \neq 0$, under $\mathbf{W}_z^{(2)}$ and conditionally on $g_C \leq a$, the winding process $(\theta_t,\ t \geq 0)$ satisfies :

$$\frac{2}{\log\ t}\ \theta_t \underset{t\to\infty}{\longrightarrow} \alpha_{T_1^{(3)}} \tag{2.3.9}$$

for all $a > 0$. This easily results from (2.3.7) and from the representation formula (2.2.6).
<u>Proof of Theorem 2.3.1.</u>
i) <u>We use the notation</u> (2.3.8). We admit for a moment that :

$$H_t - H_{T_{\sqrt{t}}(R)} \qquad \text{converges in law as } t \to \infty, \text{ with :} \tag{2.3.10}$$

$$T_{\sqrt{t}}(R) := \inf\{s \geq 0\ ;\ R_s \geq \sqrt{t}\} \tag{2.3.11}$$

and we show that (2.3.10) implies Theorem 2.3.1. Indeed, from (2.3.10), we have :

$$\frac{4}{(\log\ t)^2}\ H_t \underset{t\to\infty}{\sim} \frac{1}{(\log\ \sqrt{t})^2}\ H_{T_{\sqrt{t}}(R)} \tag{2.3.12}$$

But :

$$\frac{1}{(\log\ a)^2}\ H_{T_a(R)} = \frac{1}{(\log\ a)^2}\ T_{\log\ a}(\rho) \overset{(\text{law})}{=} T_1(\rho) \tag{2.3.13}$$

with

$$T_{\log(a)}(\rho) := \inf\{t \geq 0\ ;\ \rho_t \geq \log\ a\} \tag{2.3.14}$$

The first equality in (2.3.13) results from definitions (see point 4 of (2.3.8)) and the second from the scaling property. Thus, from (2.3.10), we deduce :

$$\frac{4}{(\log\ t)^2}\ H_t \underset{t\to\infty}{\overset{(\text{law})}{\longrightarrow}} T_1^{(3)} \tag{2.3.15}$$

and

$$\begin{aligned}
\frac{2}{\log\ t}\ \theta_t \quad &= \quad \frac{2}{\log\ t}\ \alpha_{H_t} \overset{(\text{law})}{=} \frac{2\sqrt{H_t}}{\log\ t}\ \alpha_1 \quad \text{(by scaling)} \\
&\underset{t\to\infty}{\overset{(\text{law})}{\longrightarrow}} \sqrt{T_1^{(3)}} \cdot \alpha_1 \\
&\overset{(\text{law})}{=} \alpha_{T_1^{(3)}} \qquad \text{(by scaling)}
\end{aligned}$$

which proves Theorem 2.3.1.
ii) <u>It remains to prove (2.3.10).</u>

For this purpose, we start with the following Lemma :

Lemma 2.3.3. *Let* $(R_t,\ t \geq 0)$ *be defined by (2.2.6). Then :* $\left(\frac{1}{\sqrt{t}}\, R_{tv},\ v \geq 0\right)$ *converges in law, as* $t \to \infty$*, to a 2-dimensional Bessel process starting from 0.*

Proof of Lemma 2.3.3.

From (2.2.6) we have :

$$R_t = 1 + \beta_t + \int_0^t \left(\frac{1}{2R_s} + \frac{1}{R_s \log\, R_s}\right) ds$$

Thus :

$$\frac{1}{\sqrt{t}}\, R_{tv} = \frac{1}{\sqrt{t}} + \frac{1}{\sqrt{t}}\, \beta_{tv} + \frac{1}{\sqrt{t}} \int_0^{tv} \left(\frac{1}{2R_s} + \frac{1}{R_s \log\, R_s}\right) ds \tag{2.3.16}$$

Denoting by $(\widetilde{\beta}_v,\ v \geq 0)$ the Brownian motion $\left(\frac{1}{\sqrt{t}}\, \beta_{tv},\ v \geq 0\right)$ and making the change of variable $s = tv$, we obtain, with $\left(\widetilde{R}_v^{(t)} = \frac{1}{\sqrt{t}}\, R_{tv},\ v \geq 0\right)$:

$$\widetilde{R}_v^{(t)} = \frac{1}{\sqrt{t}} + \widetilde{\beta}_v + \int_0^v \left(\frac{1}{2\widetilde{R}_u^{(t)}} + \frac{1}{\widetilde{R}_u^{(t)}\big(\log\, \sqrt{t} + \log \widetilde{R}_u^{(t)}\big)}\right) du \tag{2.3.17}$$

Hence, as $t \to \infty$, $(\widetilde{R}_v^{(t)},\ v \geq 0)$ converges in law to the law of the solution of the SDE :

$$\widetilde{R}_v = \widetilde{\beta}_v + \int_0^v \frac{du}{2\widetilde{R}_u}$$

i.e. to (the law of) a 2-dimensional Bessel process started at 0.

iv) <u>We may now end up the proof of (2.3.10).</u>

We have, from (2.3.8) :

$$H_{T_{\sqrt{t}}(R)} - H_t = \int_t^{T_{\sqrt{t}}(R)} \frac{du}{R_u^2} = \int_1^{\frac{1}{t}\, T_{\sqrt{t}}(R)} \frac{dv}{\left(\frac{1}{t}\, R_{vt}^2\right)}$$

after making the change of variable $u = tv$. But, from Lemma 2.3.3, $\left(\frac{1}{\sqrt{t}}\, R_{vt},\ v \geq 0\right)$ converges in law to a 2-dimensional Bessel process $\big(R_0^{(2)}(v),\ v \geq 0\big)$ starting from 0. Thus : $H_t - H_{\sqrt{t}}(R)$ converges in law, as $t \to \infty$, to

$$\int_1^{T_1(R_0^{(2)})} \frac{du}{(R_0^{(2)}(u))^2} \tag{2.3.18}$$

with $T_1(R_0^{(2)}) = \inf\{s \geq 0\ ;\ R_0^{(2)}(s) = 1\}$.

Remark 2.3.4. (An extension of Theorem 2.3.1.)

Let $(\beta_t,\ t \geq 0)$ denote a 1-dimensional Brownian motion starting at 0, $\delta > 0$ and $(R_t^{(\delta)},$ $t \geq 0)$ the solution of :

$$R_t^{(\delta)} = 1 + \beta_t + \int_0^t \left(\frac{1}{2R_s^{(\delta)}} + \frac{\delta}{R_s^{(\delta)} \log\, R_s^{(\delta)}}\right) ds \tag{2.3.19}$$

The case we have just studied is that of $\delta = 1$. Let :

$$H_t^{(\delta)} := \int_0^t \frac{ds}{(R_s^{(\delta)})^2} \tag{2.3.20}$$

and

$$\theta_t^{(\delta)} = \alpha_{H_t^{(\delta)}}$$

where $(\alpha_u,\ u \geq 0)$ is a 1-dimensional Brownian motion independent from $(\beta_t,\ t \geq 0)$.
The technique we have just developed allows to obtain :

i) $(\log R_t^{(\delta)},\ t \geq 0) = (\rho^{(2\delta+1)}_{H_t^{(\delta)}},\ t \geq 0)$ (2.3.21)

where $(\rho_u^{(2\delta+1)},\ u \geq 0)$ is a $(2\delta+1)$-dimensional Bessel process starting at 0.

ii) $\dfrac{4}{(\log t)^2}\, H_t^{(\delta)} \underset{t\to\infty}{\overset{\text{(law)}}{\longrightarrow}} T_1^{(2\delta+1)}$

where $T_1^{(2\delta+1)} := \inf\{u \geq 0\ ;\ \rho_u^{(2\delta+1)} = 1\}$.

iii) $\dfrac{2\theta_t^{(\delta)}}{\log t} = \dfrac{2\alpha^{(\delta)}_{H_t}}{\log t} \underset{t\to\infty}{\overset{\text{(law)}}{\longrightarrow}} \alpha_{T_1^{(2\delta+1)}}$ (2.3.22)

where $T_1^{(2\delta+1)}$ is independent from the 1-dimensional Brownian motion $(\alpha_u,\ u \geq 0)$.

2.4 $W^{(2)}$ martingales associated to $\mathbf{W}^{(2)}$.

Just as in Chapter 1, we associated to any r.v. $F \in L^1(\mathcal{F}_\infty, \mathbf{W})$ the $\big((\mathcal{F}_t,\ t \geq 0),\ W\big)$ martingale $\big(M_t(F),\ t \geq 0\big)$, we now associate to every r.v. $F \in L^1(\mathcal{F}_\infty, \mathbf{W}^{(2)})$ a $\big((\mathcal{F}_t,\ t \geq\ 0),\ W^{(2)}\big)$ martingale $\big(M_t^{(2)}(F),\ t \geq 0\big)$.

2.4.1 Definition of $\big(M_t^{(2)}(F),\ t \geq 0\big)$.

Theorem 2.4.1. *Let $F \in L^1\big(\Omega = \mathcal{C}(\mathbb{R}_+ \to \mathbb{C}),\ \mathcal{F}_\infty, \mathbf{W}^{(2)}\big)$. There exists a $\big((\mathcal{F}_t,\ t \geq 0),\ W^{(2)}\big)$ martingale (which is necessarily continuous) $\big(M_t^{(2)}(F),\ t \geq 0\big)$, positive if $F \geq 0$, such that :*

1) *For every $t \geq 0$ and $\Gamma_t \in b(\mathcal{F}_t)$:*

$$\mathbf{W}^{(2)}(F \cdot \Gamma_t) = W^{(2)}\big(M_t^{(2)}(F) \cdot \Gamma_t\big) \tag{2.4.1}$$

In particular, for every $t \geq 0$:

$$\mathbf{W}^{(2)}(F) = W^{(2)}\big(M_t^{(2)}(F)\big) \tag{2.4.2}$$

and, if F and G belong to $L^1_+(\mathcal{F}_\infty, \mathbf{W}^{(2)})$:

$$W^{(2)}\big(M_t^{(2)}(F) \cdot M_t^{(2)}(G)\big) = \mathbf{W}^{(2)}\big(F \cdot M_t^{(2)}(G)\big) = \mathbf{W}^{(2)}\big(M_t^{(2)}(F) \cdot G\big) \tag{2.4.3}$$

2) $M_t^{(2)}(F) = \widehat{W}^{(2)}_{X_t(\omega_t)}(F(\omega_t, \widehat{\omega}^t))$ (2.4.4)

3) $M_t^{(2)}(F) \underset{t\to\infty}{\longrightarrow} 0 \qquad W^{(2)}$ a.s. (2.4.5)

In particular, the martingale $(M_t^{(2)}(F),\ t \geq 0)$ is not uniformly integrable if $F \neq 0$.

4) *For every $q \in \mathcal{I}$:*

$$M_t^{(2)}(F) = \varphi_q(0)\, M_t^{(q)}\, W_\infty^{(2,q)}(F\, e^{\frac{1}{2}A_\infty^{(q)}} | \mathcal{F}_t) \tag{2.4.6}$$

where $M_t^{(q)}$, φ_q *and* $W_\infty^{(2,q)}$ *are defined in Theorem 2.1.1.*

The proof of Theorem 2.4.1 is, mutatis mutandis, the proof of Theorem 1.2.1. Here are some examples of martingales $(M_t^{(2)}(F),\ t \geq 0)$.

Example 2.1. Let $q \in \mathcal{I}$ and $F_q = \exp\left(-\frac{1}{2}\, A_\infty^{(q)}\right)$. We have, from (2.1.10) :

$$\mathbf{W}^{(2)}(F_q) = \varphi_q(0) \tag{2.4.7}$$

and

$$\left(M_t^{(2)}(F_q) = \varphi_q(X_t)\, \exp\left(-\frac{1}{2}\, A_t^{(q)}\right),\ t \geq 0\right) \tag{2.4.8}$$

In particular, for $q = \lambda q_0$ $\big($see (2.2.3) and (2.2.31)$\big)$:

$$M_t^{(2)}\left(\exp -\frac{\lambda}{2}\, A_\infty^{(q_0)}\right) = \left(\frac{2}{\lambda} + \frac{1}{\pi}\, \log^+\big(|X_t|\big)\right) \exp\left(-\frac{\lambda}{2}\, L_t^{(C)}\right) \tag{2.4.9}$$

Example 2.2. $\big($see [RVY, VI]$\big)$.

We write the skew-product representation of the canonical 2-dimensional Brownian motion $(X_t,\ t \geq 0)$ starting at $z \neq 0$ as :

$$X_t = |X_t| \cdot \exp(i\, \alpha_{H_t}) \tag{2.4.10}$$

where :

i) $\big(|X_t|,\ t \geq 0\big)$ is a 2-dimensional Bessel process starting at $|z|$.

ii) $H_t = \displaystyle\int_0^t \frac{ds}{|X_s|^2}$

iii) $(\alpha_u,\ u \geq 0)$ is a 1-dimensional Brownian motion, independent from $\big(|X_u|,\ u \geq 0\big)$.

Let $(\theta_t := \alpha_{H_t},\ t \geq 0)$ denote the winding process and introduce :

$$S_t^\theta := \sup_{s \leq t} \theta_s = \sup_{u \leq H_t} \alpha_u \tag{2.4.11}$$

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ Borel and integrable. Then :

$$\big(M_t^{(2)}(\varphi(S_\infty^\theta)),\ t \geq 0\big) = \left(\varphi(S_t^\theta)(S_t^\theta - \theta_t) + \int_{S_t^\theta}^\infty \varphi(y)dy, \quad t \geq 0\right) \tag{2.4.12}$$

2.4.2 A decomposition Theorem of positive $W^{(2)}$ supermartingales.

Just as in Theorem 1.2.5, we have obtained a decomposition Theorem for every $\big((\mathcal{F}_t,\ t \geq 0),$ $W\big)$ positive supermartingale, we now present a decomposition theorem for every $\big((\mathcal{F}_t,\ t \geq 0),$ $W^{(2)}\big)$ positive supermartingale.

Theorem 2.4.2. *Let* $(Z_t,\ t \geq 0)$ *denote a positive* $\big(\Omega = \mathcal{C}(\mathbb{R}_+ \to \mathbb{C}),\ (\mathcal{F}_t,\ t \geq 0),\ W^{(2)}\big)$ *supermartingale. We denote* $Z_\infty := \lim\limits_{t\to\infty} Z_t$, $W^{(2)}$ *a.s. Then :*

1) $\quad z_\infty := \lim\limits_{t\to\infty}\ \pi \dfrac{Z_t}{1 + \log^+(|X_t|)}$ *exists* $\mathbf{W}^{(2)}$ *a.s.* (2.4.13)

and : $\quad \mathbf{W}^{(2)}(z_\infty) < \infty$ (2.4.14)

2) $(Z_t,\ t \geq 0)$ *decomposes in a unique manner in the form :*

$$Z_t = M_t^{(2)}(z_\infty) + W^{(2)}(Z_\infty|\mathcal{F}_t) + \xi_t \quad (t \geq 0) \tag{2.4.15}$$

where $\big(M_t^{(2)}(z_\infty),\ t \geq 0\big)$ *and* $\big(W^{(2)}(Z_\infty|\mathcal{F}_t),\ t \geq 0\big)$ *denote two* $\big((\mathcal{F}_t,\ t \geq 0),\ W^{(2)}\big)$ *martingales and :*
$(\xi_t,\ t \geq 0)$ *is a* $\big((\mathcal{F}_t,\ t \geq 0),\ W^{(2)}\big)$ *positive supermartingale such that :*
i) $Z_\infty \in L^1_+(\mathcal{F}_\infty, W^{(2)})$, *hence* $W^{(2)}(Z_\infty|\mathcal{F}_t)$ *converges* $W^{(2)}$ *a.s. and in* $L^1(\mathcal{F}_\infty, W^{(2)})$ *towards* Z_∞.

ii) $$\frac{W(Z_\infty|\mathcal{F}_t) + \xi_t}{1 + \log^+(|X_t|)} \underset{t\to\infty}{\longrightarrow} 0 \quad \mathbf{W}^{(2)} \ \text{a.s.}$$

iii) $$M_t^{(2)}(z_\infty) + \xi_t \underset{t\to\infty}{\longrightarrow} 0 \qquad W^{(2)} \ \text{a.s.}$$

In particular, if $F \in L^1(\mathcal{F}_\infty, \mathbf{W}^{(2)})$, *then :*

$$\pi \cdot \frac{M_t(F)}{1 + \log^+(|X_t|)} \underset{t\to\infty}{\longrightarrow} F \qquad \mathbf{W}^{(2)} \ \text{a.s.} \tag{2.4.16}$$

and the map : $F \to \big(M_t^{(2)}(F),\ t \geq 0\big)$ *is injective.*

Corollary 2.4.3. *(A characterisation of martingales of the form* $\big(M_t^{(2)}(F),\ t \geq 0\big)$*. A* $\big((\mathcal{F}_t,\ t \geq 0),\ W^{(2)}\big)$ *positive martingale* $(Z_t,\ t \geq 0)$ *is equal to* $\big(M_t^{(2)}(F),\ t \geq 0\big)$ *for an* $F \in L^1(\mathcal{F}_\infty, \mathbf{W}^{(2)})$ *if and only if :*

$$Z_0 = \mathbf{W}^{(2)}\left(\lim_{t\to\infty} \pi \cdot \frac{Z_t}{1 + \log^+(|X_t|)}\right) \tag{2.4.17}$$

Note that $\lim_{t\to\infty} \dfrac{Z_t}{1 + \log^+(|X_t|)}$ exists $\mathbf{W}^{(2)}$ a.s. from (2.4.13).

Sketches of Proofs of Theorem 2.4.2 and of Corollary 2.4.3.
This proof is essentially the same as those of Theorem 1.2.5 and of Corollary 1.2.6. Two arguments need to be modified :
i) The role of the r.v. g in the proof of Theorem 1.2.5 is played here by that of the r.v. g_C.
ii) The relation (1.1.41) : $\mathbf{W}(\Gamma_t 1_{g\leq t}) = W(\Gamma_t |X_t|)$
and the limiting result :

$$\frac{\varphi_q(X_t)\ \exp(-\frac{1}{2} A_t^{(q)})}{1 + |X_t|} \underset{t\to\infty}{\longrightarrow} \exp\left(-\frac{1}{2} A_\infty^{(q)}\right) \tag{2.4.18}$$

which were used in the proof of Lemma 1.2.8 need to be replaced respectively by :

$$\mathbf{W}^{(2)}\left(\Gamma_t 1_{(g_C\leq t)}\right) = \frac{1}{\pi} W^{(2)}(\Gamma_t \log^+|X_t|)\Big).$$

(This is relation (2.2.28) of Theorem 2.2.2) and by :

$$\pi \cdot \frac{\varphi_q(X_t)\ \exp(-\frac{1}{2} A_t^{(q)})}{1 + \log^+(|X_t|)} \underset{t\to\infty}{\longrightarrow} \exp\left(-\frac{1}{2} A_\infty^{(q)}\right) \quad \mathbf{W}^{(2)} \ \text{a.s.} \tag{2.4.19}$$

The latter (2.4.19) follows easily from :

$$\pi \cdot \varphi_q(z) \underset{|z|\to\infty}{\sim} \log(|z|), \quad \text{from (2.1.6)}$$

and from : $|X_t| \underset{t\to\infty}{\longrightarrow} \infty \quad \mathbf{W}^{(2)}$ a.s.
since the canonical process under $W_\infty^{(2,q)}$ is transient.

2.4.3 A decomposition Theorem for the martingales $\big(M_t^{(2)}(F),\ t \geq 0\big)$.

A difference with the preceding subsection is that the r.v.'s F which we now consider belong to $L^1(\mathcal{F}_\infty, \mathbf{W}^{(2)})$ but are not necessarily positive. Here is the analogue, in dimension 2, of Theorem 1.2.11.

Theorem 2.4.4. *$F \in L^1(\mathcal{F}_\infty, \mathbf{W}^{(2)})$ and let $\big(M_t^{(2)}(F),\ t \geq 0\big)$ the $\big((\mathcal{F}_t,\ t \geq 0),\ W^{(2)}\big)$ martingale associated to F by Theorem 2.4.1. Let C, $(L_t^{(C)},\ t \geq 0)$ and g_C be as in Section 2.2.1, i) and Section 2.2.2. Then :*

1) *i) There exists a previsible process $\big(k_s^{(C)}(F),\ s \geq 0\big)$ which is defined $dL_s^{(C)} \cdot W^{(2)}(d\omega)$ a.s., positive if $F \geq 0$, and such that :*

$$W^{(2)}\left(\int_0^\infty \big|k_s^{(C)}(F)\big| dL_s^{(C)}\right) = \mathbf{W}^{(2)}\big(\big|k_{g_C}^{(C)}(F)\big|\big) \leq \mathbf{W}^{(2)}(|F|) < \infty \qquad (2.4.20)$$

and for every bounded previsible process $(\Phi_s,\ s \geq 0)$:

$$\mathbf{W}^{(2)}(\Phi_{g_C} \cdot F) = W^{(2)}\left(\int_0^\infty \Phi_s\, k_s^{(C)}(F)\, dL_s^{(C)}\right) \qquad (2.4.21)$$

$$= \mathbf{W}^{(2)}\big(\Phi_{g_C}\, k_{g_C}^{(C)}(F)\big) \qquad (2.4.22)$$

Thus :

$$\mathbf{W}^{(2)}\big(F|\mathcal{F}_{g_C}\big) = k_{g_C}^{(C)}(F) \qquad (2.4.23)$$

ii) $\big(k_s^{(C)}(k_{g_C}^{(C)}(F)),\ s \geq 0\big) = \big(k_s^{(C)}(F),\ s \geq 0\big)$ (2.4.24)

iii) If $(h_s,\ s \geq 0)$ is a previsible process such that : $\mathbf{W}^{(2)}\big(|h_{g_C}|\big) < \infty$,

$$\big(k_s^{(C)}(h_{g_C}),\ s \geq 0\big) = (h_s,\ s \geq 0) \quad dL_s^{(C)} \cdot W^{(2)}(d\omega) \text{ a.s.} \qquad (2.4.25)$$

2) *There exist two continuous quasimartingales $\big(\Sigma_t^{(2,C)},\ t \geq 0\big)$ and $(\Delta_t^{(2,C)},\ t \geq 0)$ such that, for every $t \geq 0$:*

$$M_t^{(2)}(F) = \Sigma_t^{(2,C)}(F) + \Delta_t^{(2,C)}(F) \qquad (2.4.26)$$

with :
i) For every $t \geq 0$ and $\Gamma_t \in b(\mathcal{F}_t)$:

$$\mathbf{W}^{(2)}(\Gamma_t\, 1_{g_C \leq t} \cdot F) = W^{(2)}\big(\Gamma_t\, \Sigma_t^{(2,C)}(F)\big) \qquad (2.4.27)$$

$$\mathbf{W}^{(2)}(\Gamma_t\, 1_{g_C > t} \cdot F) = W^{(2)}\big(\Gamma_t\, \Delta_t^{(2,C)}(F)\big) \qquad (2.4.28)$$

In particular, from (2.4.27) applied with $\widetilde{\Gamma}_t = \Gamma_t\, 1_{|X_t| \leq 1}$ and since $1_{g_C \leq t} \cdot 1_{|X_t| \leq 1} = 0$, the process $\big(\Sigma_t^{(2,C)}(F), t \geq 0\big)$ vanishes on the set $\big(|X_t| \leq 1\big)$.

ii) The Doob-Meyer decompositions of $\Sigma_t^{(2,C)}(F)$ *and* $\Delta_t^{(2,C)}(F)$ *write :*

$$\Sigma_t^{(2,C)}(F) = -M_t^{\Sigma^{(2,C)}}(F) + \int_0^t k_s^{(C)}(F)dL_s^{(C)} \tag{2.4.29}$$

$$\Delta_t^{(2,C)}(F) = M_t^{\Delta^{(2,C)}}(F) - \int_0^t k_s^{(C)}(F)dL_s^{(C)} \tag{2.4.30}$$

where $\left(M_t^{\Sigma^{(2,C)}}(F),\ t \geq 0\right)$ *and* $\left(M_t^{\Delta^{(2,C)}}(F),\ t \geq 0\right)$ *are the martingale parts of the corresponding left-hand sides. The first martingale is not uniformly integrable ; the second one is uniformly integrable. In fact, we have :*

$$M_t^{\Delta^{(2,C)}}(F) = W^{(2)}\left(\int_0^\infty k_s^{(C)}(F)dL_s^{(C)}|\mathcal{F}_t\right) \tag{2.4.31}$$

with, from (2.4.20), $\displaystyle\int_0^\infty k_s^{(C)}(F)dL_s^{(C)} \in L^1(\mathcal{F}_\infty, \mathbf{W}^{(2)})$.

iii) The "explicit formula" :

$$\Sigma_t^{(2,C)}(F) = \frac{1}{\pi}\ \log^+(|X_t|) \cdot \widehat{\widetilde{E}}^{(2,\log)}_{X_t(\omega_t)}\big(F(\omega_t, \widehat{\omega}^t)\big) \tag{2.4.32}$$

holds, where in (2.4.32) the expectation is taken with respect to $\widehat{\omega}^t$, *and the argument* ω_t *is frozen.* $\widetilde{E}^{(2,\log)}$ *denotes the expectation with respect to the law* $\widetilde{P}^{(2,\log)}$ *defined in Theorem 2.2.2. In particular :*

- $\Sigma_t^{(2,C)}$ *vanishes on* $\{t\ ;\ |X_t| \leq 1\}$, *as we already observed,*
- $$\pi\ \frac{\Sigma_t^{(2,C)}(F)}{1+\log^+(|X_t|)} \underset{t\to\infty}{\longrightarrow} F \quad \mathbf{W}^{(2)} \ a.s. \tag{2.4.33}$$

and, from (2.4.16)

$$\pi\ \frac{\Delta_t^{(2,C)}(F)}{1+\log^+(|X_t|)} \underset{t\to\infty}{\longrightarrow} 0 \quad \mathbf{W}^{(2)} \ a.s. \tag{2.4.34}$$

Corollary 2.4.5. *Let* $F \in L^1(\mathcal{F}_\infty, \mathbf{W}^{(2)})$.
One has $M_t^{(2)}(F) = 0$ *for every* $t \geq 0$ *such that* $|X_t| \leq 1$, *if and only if :*

$$k_{g_C}^{(C)}(F) = 0$$