$X = M$ CONJECTURE

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CONTENTS

1. INTRODUCTION

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra of rank n. Let $Y(\mathfrak{g})$ be the Yangian associated to $\mathfrak{g}$. In 1990 Kirillov and Reshetikhin [KR1] conjectured the existence of a family of finite dimensional $Y(\mathfrak{g})$-modules $W_s^{(r)}$ ($r \in \{1, 2, \dots, n\}, s \in \mathbb{Z}_{\geq 1}$) whose tensor product decomposes into $\mathfrak{g}$-modules by the following multiplicity formula.

$$[\bigotimes_j W_{s_j}^{(r_j)} : V_\lambda] = \sum_{\{m_i^{(a)}\}} \prod_{\substack{1 \leq a \leq n \\ i \geq 1}} \binom{p_i^{(a)} + m_i^{(a)}}{m_i^{(a)}} \tag{1.1}$$

Here V_λ is the irreducible highest weight $\mathfrak{g}$-module with highest weight λ and the LHS is the multiplicity of V_λ in the $\mathfrak{g}$-module $\bigotimes_j W_{s_j}^{(r_j)}$. In the RHS $\binom{p+m}{m}$ stands for the binomial coefficient. All other notations will be explained in section 5.2. This conjecture is now proven for any simple Lie algebra $\mathfrak{g}$ in the "weak" version (see the explanation after Conjecture 5.6), by combining several results [HKOTY, KNT, N, H]. (See also [C, KNS].)

In 1999 we introduced a "fermionic formula" M as a suitable q-analog of the RHS of (1.1) [HKOTY]. And we began to seek the LHS that equates it. In a subsequent paper [HKOTT] we defined it as a 1D sum X and extend the equality to twisted affine Lie algebra cases. The purpose of this note is to explain how to define X and to persuade you to believe the so called "$X = M$ conjecture." We note here that the combinatorics of M and some strategies for proof of the $X = M$ conjecture is discussed in Schilling's contribution of this volume.

2. Crystals

2.1. **Definition.** In this subsection we recall the definition of crystal bases. For more details along with the definition of $U_q(\mathfrak{g})$, refer to [Ka1].

Let $\mathfrak{g}$ be a symmetrizable Kac-Moody Lie algebra and let M be a $U_q(\mathfrak{g})$-module. M is said to be **integrable** if

- $M = \oplus_{\lambda \in P} M_\lambda$,
- $\dim M_\lambda < \infty$ for any λ, and
- for any i, M is a union of finite-dimensional $U_q(\mathfrak{g}_i)$-modules.

Here P is the weight lattice of $\mathfrak{g}$, M_λ is the weight space of M of weight λ and $U_q(\mathfrak{g}_i)$ is the subalgebra generated by Chevalley generators e_i and f_i. If M is integrable, we have

$$M = \bigoplus_{0 \le n \le \langle h_i, \lambda \rangle} f_i^{(n)}(\mathrm{Ker}\ e_i \cap M_\lambda).$$

Note that we use the following notations: $[m]_i = (q_i^m - q_i^{-m})/(q_i - q_i^{-1})$, $[n]_i! = \prod_{m=1}^n [m]_i$, $f_i^{(n)} = f_i^n/[n]_i!$ with $q_i = q^{(\alpha_i,\alpha_i)}$, where $(\ ,\ )$ is an invariant bilinear form on P. We define the endomorphisms $\tilde{e}_i, \tilde{f}_i$ of M by

$$\tilde{f}_i(f_i^{(n)}u) = f_i^{(n+1)}u \quad \text{and} \quad \tilde{e}_i(f_i^{(n)}u) = f_i^{(n-1)}u$$

for $u \in \mathrm{Ker}\ e_i \cap M_\lambda$ with $0 \le n \le \langle h_i, \lambda \rangle$. These are called **Kashiwara operators**. (Kashiwara called them modified Chevalley generators.)

Roughly speaking, a crystal basis of a $U_q(\mathfrak{g})$-module M is a basis of M "at $q = 0$". Let us look at the definition more precisely. Let A be the subring of $\mathbb{Q}(q)$ consisting of rational functions without poles at $q = 0$. Let M be an integrable $U_q(\mathfrak{g})$-module.

Definition 2.1. A pair (L, B) is called a **crystal basis** of M if it satisfies the following 6 conditions:

(2.1) L is a free sub-A-module of M such that $M \simeq \mathbb{Q}(q) \otimes_A L$,

(2.2) B is a basis of the $\mathbb{Q}$-vector space L/qL,

(2.3) $\tilde{e}_i L \subset L$ and $\tilde{f}_i L \subset L$ for any i.

By (2.3) $\tilde{e}_i$ and $\tilde{f}_i$ act on L/qL.

(2.4) $\tilde{e}_i B \subset B \cup \{0\}$ and $\tilde{f}_i B \subset B \cup \{0\}$.

(2.5) $L = \oplus_{\lambda \in P} L_\lambda$ and $B = \sqcup_{\lambda \in P} B_\lambda$

where $L_\lambda = L \cap M_\lambda$ and $B_\lambda = B \cap (L_\lambda / qL_\lambda)$.

(2.6) For $b, b' \in B, b' = \tilde{f}_i b$ if and only if $\tilde{e}_i b' = b$.

If (2.6) holds, we draw an arrow as

$$b \xrightarrow{i} b'.$$

By this B gets endowed with the structure of colored oriented graph, which we call the **crystal graph**. Note that we often say B is a crystal basis rather than (L, B). Standard notations are in order. For $b \in B$ we set

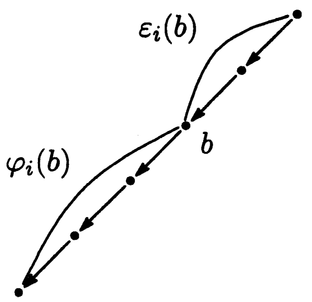
$$(2.7) \qquad \varepsilon_i(b) = \max\{m \in \mathbb{Z}_{\geq 0} \mid \tilde{e}_i^m b \neq 0\}, \qquad \varphi_i(b) = \max\{m \in \mathbb{Z}_{\geq 0} \mid \tilde{f}_i^m b \neq 0\},$$

$$(2.8) \qquad \varepsilon(b) = \sum_i \varepsilon_i(b)\Lambda_i, \qquad \varphi(b) = \sum_i \varphi_i(b)\Lambda_i,$$

$$(2.9) \qquad wt\, b = \varphi(b) - \varepsilon(b).$$

Here $\{\Lambda_i\}$ stands for the set of fundamental weights of the Kac-Moody Lie algebra $\mathfrak{g}$. The sequence of arrows penetrating a crystal element b is called the i-string through b. An i-string is illustrated in Figure 1 together with the meaning of $\varepsilon_i(b)$ and $\varphi_i(b)$ in the crystal graph.

FIGURE 1. i-string



The crystal basis behaves nicely under the tensor product. Let (L_j, B_j) be the crystal basis of an integrable $U_q(\mathfrak{g})$-module M_j $(j = 1, 2)$. Set $L = L_1 \otimes_A L_2$ and $B = \{b_1 \otimes b_2 \mid b_j \in B_j (j = 1, 2)\}$. Then (L, B) is a crystal basis of $M_1 \otimes M_2$. Moreover, the action of Kashiwara operators becomes very simple as

$$(2.10) \qquad \tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$$

$$(2.11) \qquad \tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases}$$

Here $0 \otimes b$ and $b \otimes 0$ are understood to be 0. We denote this B by $B_1 \otimes B_2$. This rule can be depicted as in Figure 2. ε_i, φ_i and wt are given by

$$(2.12) \qquad \varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1), \varepsilon_i(b_1) + \varepsilon_i(b_2) - \varphi_i(b_1)),$$

$$(2.13) \qquad \varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_2), \varphi_i(b_1) + \varphi_i(b_2) - \varepsilon_i(b_2)),$$

$$(2.14) \qquad wt\,(b_1 \otimes b_2) = wt\, b_1 + wt\, b_2.$$

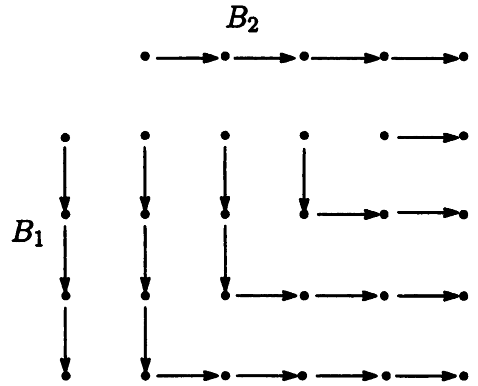
In order to compute the action of $\tilde{e}_i, \tilde{f}_i$ on multiple tensor products, it is convenient to use the rule called **signature rule**. Let $b = b_1 \otimes b_2 \otimes \cdots \otimes b_m$ be an element of the tensor product of crystals $B_1 \otimes B_2 \otimes \cdots \otimes B_m$. One wishes to find the indices j, j' such that

$$\tilde{e}_i(b_1 \otimes \cdots \otimes b_m) = b_1 \otimes \cdots \otimes \tilde{e}_i b_j \otimes \cdots \otimes b_m,$$
$$\tilde{f}_i(b_1 \otimes \cdots \otimes b_m) = b_1 \otimes \cdots \otimes \tilde{f}_i b_{j'} \otimes \cdots \otimes b_m.$$

To do it, we introduce (i-)signature by

$$\overbrace{- \cdots -}^{\varepsilon_i(b_1)}\overbrace{+ \cdots +}^{\varphi_i(b_1)}\overbrace{- \cdots -}^{\varepsilon_i(b_2)}\overbrace{+ \cdots +}^{\varphi_i(b_2)} \cdots\cdots \overbrace{- \cdots -}^{\varepsilon_i(b_m)}\overbrace{+ \cdots +}^{\varphi_i(b_m)}.$$

FIGURE 2. Tensor product rule



We then reduce the signature by deleting the adjacent $+-$ pair successively. Eventually we obtain a reduced signature of the following form.

$$- - \cdots - + + \cdots +$$

Then the action of $\tilde{e}_i$ (resp. $\tilde{f}_i$) corresponds to changing the rightmost $-$ to $+$ (resp. leftmost $+$ to $-$). If there is no $-$ (resp. $+$) in the signature, then the action of $\tilde{e}_i$ (resp. $\tilde{f}_i$) should be set to 0. The value of $\varepsilon_i(b)$ (resp. $\varphi_i(b)$) is given by the number of $-$ (resp. $+$) in the reduced signature.

Example 2.2. Consider an element $b_1 \otimes b_2 \otimes b_3$ of the 3 fold tensor product $B_1 \otimes B_2 \otimes B_3$. Suppose $\varepsilon_i(b_1) = 1, \varphi_i(b_1) = 3, \varepsilon_i(b_2) = 1, \varphi_i(b_2) = 1, \varepsilon_i(b_3) = 2, \varphi_i(b_3) = 1$. Then the signature and reduced one read

$$\begin{array}{lccccc} \text{sig} & -++ & \cdot & -+ & \cdot & --+ \\ \text{red sig} & - & \cdot & & \cdot & +. \end{array}$$

Thus we have

$$\tilde{e}_i(b_1 \otimes b_2 \otimes b_3) = \tilde{e}_i b_1 \otimes b_2 \otimes b_3,$$

$$\tilde{f}_i(b_1 \otimes b_2 \otimes b_3) = b_1 \otimes b_2 \otimes \tilde{f}_i b_3.$$

2.2. **Conjecture and known cases.** It is known that for a symmetrizable Kac-Moody algebra the integrable highest weight $U_q(\mathfrak{g})$-module $V(\lambda)$ with highest weight λ (or even the lower part $U_q^-(\mathfrak{g})$ of $U_q(\mathfrak{g})$ generated by f_i) has a crystal basis [Ka1]. The crystal basis $(L(\lambda), B(\lambda))$ of $V(\lambda)$ is given as follows. $L(\lambda)$ is the sub-A-module generated by the vectors $\tilde{f}_{i_1} \cdots \tilde{f}_{i_l} u_\lambda$, where u_λ is the highest weight vector of $V(\lambda)$. $B(\lambda)$ is the subset of $L(\lambda)/qL(\lambda)$ consisting of the nonzero vectors of the form $\tilde{f}_{i_1} \cdots \tilde{f}_{i_l} u_\lambda \bmod qL(\lambda)$.

Let us restrict ourselves to the case when $\mathfrak{g}$ is an affine Lie algebra. Let $U_q'(\mathfrak{g})$ be the subalgebra of $U_q(\mathfrak{g})$ defined by dropping q^d (d is the degree operator) from the generators. Then $U_q'(\mathfrak{g})$ has $P_{cl} := P/\mathbb{Z}a_0^{-1}\delta$ as the weight lattice (see the next subsection for details), and one can consider a $U_q'(\mathfrak{g})$-module which is finite dimensional. As shown in next remark, a finite dimensional $U_q'(\mathfrak{g})$-module does not always have a crystal basis.

Remark 2.3. Consider $U_q(\mathfrak{gl}_n)$. (See e.g. [J] for the definition.) Irreducible finite dimensional modules are parametrized by partitions with at most n parts. By using an algebra homomorphism from $U_q'(\widehat{\mathfrak{sl}}_n)$ to $U_q(\mathfrak{gl}_n)$ [J], any $U_q'(\mathfrak{gl}_n)$-module admits the $U_q(\widehat{\mathfrak{sl}}_n)$-module structure. Such $U_q'(\widehat{\mathfrak{sl}}_3)$-module corresponding to the partition $(2,1)$ does not have a crystal basis.

So we have the following fundamental problem.

Problem: *Find all irreducible finite dimensional modules having crystal bases.*

As a first step toward solving this problem we propose

Conjecture 2.4. *KR module $W_s^{(r)}$ has a crystal basis. (We denote it by $B^{r,s}$.)*

We call this crystal **KR crystal**, although its existence is still conjectural in general. However, there are some settled cases, which we summarize below.

(i) $\mathfrak{g} = A_n^{(1)}$

As a $U_q(A_n)$-module, $W_s^{(r)}$ is isomorphic to the irreducible module corresponding to the partition (s^r). The crystal $B^{r,s}$ for any r, s is given in [KMN2]. $B^{r,s}$ is identified with the set of semistandard tableaux of shape (s^r) with letters from $\{1, 2, \dots, n+1\}$. The action of $\tilde{e}_i, \tilde{f}_i$ for $i \neq 0$ agrees with the one in [KN] for type A_n. For $i = 0$ it is given by $\tilde{e}_0 = \mathrm{pr}^{-1} \circ \tilde{e}_1 \circ \mathrm{pr}, \tilde{f}_0 = \mathrm{pr}^{-1} \circ \tilde{f}_1 \circ \mathrm{pr}$ using the promotion operator pr that represents the symmetry of $B^{r,s}$ corresponding to the rotation of the Dynkin diagram of type $A_n^{(1)}$ [Sh]. We will look at it again in section 3.2.

(ii) $\mathfrak{g}$: non exceptional, $B^{1,s}$

In [KMN2] $W_s^{(1)}$ is constructed by the fusion construction from $W_1^{(1)}$ for all non exceptional affine Lie algebra $\mathfrak{g}$ except type $C_n^{(1)}$. The module $W_s^{(1)}$ so constructed is shown to have a crystal $B^{1,s}$ by giving a "polarization." In [KKM] $B^{1,s}$ is identified with the set of lattice points included in some convex polytope, and the action of $\tilde{e}_i, \tilde{f}_i$ is also given explicitly. This paper contains the $C_n^{(1)}$ case, although their B_s should be understood as $B^{1,2s}$ in our notation. We call such a representation of crystal elements **coordinate representation**. We call this crystal $B^{1,s}$ **KKM crystal** and look at it again in section 3.3.

(iii) $\mathfrak{g}$: any type, $B^{r,1}$

In [Ka2] Kashiwara constructed a finite dimensional $U_q'(\mathfrak{g})$-module $W(\varpi_r)$ for any affine Lie algebra $\mathfrak{g}$ as a quotient of an extremal weight module, notion introduced by him as a generalization of the integrable highest weight modules. He has also shown that $W(\varpi_r)$ has a crystal basis, which we denote by $B^{r,1}$. Naito and Sagaki [NS] realized it as the set of Lakshmibai-Seshadri paths and the action of $\tilde{e}_i, \tilde{f}_i$ is given by Littelmann's root operator. However, it is not still clear whether $B^{r,1}$ has the desired decomposition as a $U_q(\mathring{\mathfrak{g}})$-module given by the fermionic formula except for simple cases.

(iv) Other cases

There are other works. [KMN2] also treated spin representation cases: $(C_n^{(1)}; B^{n,s})$, $(D_n^{(1)}; B^{n-1,s}, B^{n-1,s}), (D_{n+1}^{(2)}; B^{n,s})$. [BFKL] treated for any affine Lie algebra $\mathfrak{g}$ the case when the highest component of the corresponding finite dimensional $U_q'(\mathfrak{g})$-module by the decomposition as a $U_q(\mathring{\mathfrak{g}})$-module is the adjoint representation. See also

[Ko] for $(B_n^{(1)}; B^{r,1+\delta_{rn}}), (C_n^{(1)}; B^{r,2}(r \neq n))), (D_n^{(1)}; B^{r,1}(r \neq n-1, n))$,
[JMO] for $(C_n^{(1)}; B^{r,1}), (A_{2n}^{(2)}; B^{r,1}), (A_{2n-1}^{(2)\dagger}; B^{r,1})$,
[SSt] for $(D_n^{(1)}; B^{2,s})$,
[Ya] for $(G_2^{(1)}; B^{1,s})$,
[KMOY] for $(D_4^{(3)}; B^{1,s})$.

2.3. **Combinatorial R and energy function.** Let us recall the classical weight lattice P_{cl}. Let a_0 be the positive integer such that $\delta - a_0\alpha_0 \in \sum_{i\in I_0}\mathbb{Z}_{\geq 0}\alpha_i$. Namely, a_0 is the 0-th Kac label. $a_0 = 1$ except when $\mathfrak{g} = A^{(2)}_{2n}$, in which case $a_0 = 2$. We take the weight lattice

$$P = \sum_{i\in I}\mathbb{Z}\Lambda_i \oplus \mathbb{Z}a_0^{-1}\delta,$$

and consider the projection

$$cl : P \longrightarrow P_{cl} = P/\mathbb{Z}a_0^{-1}\delta.$$

We take a map $af : P_{cl} \to P$ satisfying

$$cl \circ af = id,$$
$$(af \circ cl)(\alpha_i) = \alpha_i \quad \text{for } i \in I_0.$$

Then we have $(af \circ cl)(\alpha_0) = \alpha_0 - a_0^{-1}\delta$. Note that the above conditions do not determine af uniquely. We fix it by imposing $(af \circ cl)(\Lambda_i) = \Lambda_i$. By this one can also consider $P_{cl} \subset P$.

Let V be a finite dimensional $U'_q(\mathfrak{g})$-module. V is P_{cl}-weighted. Set

(2.15) $$V_z = \mathbb{Q}(q)[z, z^{-1}] \otimes_{\mathbb{Q}(q)} V.$$

It should not be confused with a weight space. V_z is endowed with the $U_q(\mathfrak{g})$-module structure by

(2.16) $$e_i = z^{\delta_{i0}} \otimes e_i, \quad f_i = z^{-\delta_{i0}} \otimes f_i,$$

(2.17) $$wt(z^d \otimes v) = af(\lambda) + da_0^{-1}\delta \text{ for a weight vector } v \text{ of weight } \lambda \in P_{cl}.$$

Thus V_z becomes a P-weighted module.

Let V, V' be finite dimensional $U'_q(\mathfrak{g})$-modules. It is known that there exists a nonzero $U_q(\mathfrak{g})$-linear map

(2.18) $$R_{VV'}(x,y) : V_x \otimes V'_y \longrightarrow V'_y \otimes V_x$$

called the ***R*-matrix**. The existence follows from that of the universal R-matrix for $U_q(\mathfrak{g})$.

Proposition 2.5. *Suppose $V \otimes V'$ is an irreducible $U'_q(\mathfrak{g})$-module. Then we have*

(1) *Up to a multiple of $\mathbb{Q}(q)[x, x^{-1}, y, y^{-1}]$, $R_{VV'}(x,y)$ is unique and depends only on x/y.*

Hereafter we write $R_{VV'}(x/y)$ instead of $R_{VV'}(x,y)$.

(2) *$R_{V'V}(y/x)R_{VV'}(x/y) = f(x/y)id$ for some $f(x/y) \in \mathbb{Q}(q)[x/y, y/x]$.*

Suppose further that $V \otimes V' \otimes V''$ is an irreducible $U'_q(\mathfrak{g})$-module.

(3) *The* **Yang-Baxter equation** *holds on* $\mathrm{End}_{\mathbb{Q}(q)}(V \otimes V' \otimes V'')$.

$$(R_{V'V''}(y/z) \otimes 1)(1 \otimes R_{VV''}(x/z))(R_{VV'}(x/y) \otimes 1)$$
$$= (1 \otimes R_{VV'}(x/y))(R_{VV''}(x/z) \otimes 1)(1 \otimes R_{V'V''}(y/z))$$

We wish to consider the $q \to 0$ limit of the R-matrix. For this purpose, we consider the following conditions for a $U'_q(\mathfrak{g})$-module V.

- V has a crystal basis (L, B),
- there exists $\lambda \in P_{cl}$ such that

(2.19) $$\dim V_\lambda = 1 \text{ and } \dim V_\mu = 0 \text{ for } \mu \in (\lambda + \mathring{Q}_+) \setminus \{\lambda\}.$$

Here $\mathring{Q}_+ = \sum_{i \in I_0} \mathbb{Z}_{\geq 0}\alpha_i$. Call a nonzero vector $u_0 \in V_\lambda$ **dominant extremal** vector.

Suppose $U_q'(\mathfrak{g})$-modules V, V' satisfy the above conditions and let $(L, B), (L', B')$ be their crystal bases. We assume

$$B \otimes B' \text{ is connected.}$$

Note that it implies that both B and B' are connected. For a positive integer m, let $[m]$ denote the set $\{0, 1, \ldots, m-1\}$. For a finite set S, let $\sharp S$ denote its cardinality. Take a basis $\{u_j\}$ $(j \in [\sharp B])$ of L (resp. $\{u'_j\}$ $(j \in [\sharp B'])$ of L') in such a way that $u_j \bmod qL \in B$ (resp. $u'_j \bmod qL' \in B'$) and u_0 (resp. u'_0) is a dominant extremal vector. Let us normalize the R-matrix by

$$R_{VV'}(x/y)(u_0 \otimes u'_0) = g(x/y)(u'_0 \otimes u_0) \tag{2.20}$$

for some $g(x/y) \in A[x/y, y/x]$ such that $g_0(x/y) = g(x/y)|_{q=0} \neq 0$. With the above setup, we have the following proposition.

Proposition 2.6 ([KMN1]). *Let $r_{j,k}^{l,m}(x/y)$ be the entry of the R-matrix defined by*

$$R_{VV'}(x/y)(u_j \otimes u'_k) = \sum_{l,m} r_{j,k}^{l,m}(x/y)(u'_l \otimes u_m). \tag{2.21}$$

Then there exists a bijection

$$\iota : [\sharp B] \times [\sharp B'] \longrightarrow [\sharp B'] \times [\sharp B]$$

and an integer valued function H such that

$$r_{j,k}^{l,m}(x/y)\Big|_{q=0} = \begin{cases} (x/y)^{-H(u_j \otimes u'_k)} g_0(x/y) & \text{if } (l,m) = \iota(j,k), \\ 0 & \text{otherwise.} \end{cases} \tag{2.22}$$

Proof. Set $R = R_{VV'}(x/y)$ and

$$\tilde{L} = (L \otimes_A L')[x/y, y/x], \quad \tilde{\tilde{L}} = (L' \otimes_A L)[x/y, y/x].$$

First we prove

$$R\tilde{L} \subset \tilde{\tilde{L}}. \tag{2.23}$$

Note that R commutes with $\tilde{e}_i, \tilde{f}_i$. Since $B \otimes B'$ is connected, we see the map

$$R^{-1}(\tilde{\tilde{L}}) \cap \tilde{L} / R^{-1}(\tilde{\tilde{L}}) \cap q\tilde{L} \longrightarrow \tilde{L}/q\tilde{L}$$

is surjective. Hence $\tilde{L} \subset R^{-1}(\tilde{\tilde{L}}) + q\tilde{L}$ and we have

$$R\tilde{L} \subset \tilde{\tilde{L}} + qR\tilde{L}.$$

By (2.20) $R\tilde{L} \subset \tilde{\tilde{L}}$ holds on the weight space of weight $wt\, u_0 + wt\, u'_0$. By applying $\tilde{e}_i, \tilde{f}_i$ we obtain $R\tilde{L} \subset \tilde{\tilde{L}}$.

Once we know (2.23), we can take the $q \to 0$ limit of (2.21). (2.22) follows from (2.20) and the commutativity of R with $\tilde{e}_i, \tilde{f}_i$. A recursion relation for H, which shows H is integer valued, will be given shortly. □

Rewriting Proposition 2.6 as

$$g_0(x/y)^{-1} R_{VV'}(x/y)\big|_{q=0} (x^d u_j \otimes y^{d'} u'_k) = y^{d'+H} u'_{j'} \otimes x^{d-H} u_{k'}$$

where $(j', k') = \iota(j, k)$ and $H = H(u_j \otimes u'_k)$, we are led to introduce a new notion for the R-matrix at $q = 0$. For a crystal B as above, define

$$\mathit{Aff}(B) = \{b[d] \mid b \in B, d \in \mathbb{Z}\}$$

and call it the **affinization** of B. In view of (2.16), the action of $\tilde{e}_i, \tilde{f}_i$ on $\mathit{Aff}(B)$ is defined by

$$\tilde{e}_i \cdot b[d] = (\tilde{e}_i b)[d + \delta_{i0}], \quad \tilde{f}_i \cdot b[d] = (\tilde{f}_i b)[d - \delta_{i0}].$$

Then the **combinatorial** R for $B \otimes B'$ is a map

$$R : \mathit{Aff}(B) \otimes \mathit{Aff}(B') \longrightarrow \mathit{Aff}(B') \otimes \mathit{Aff}(B)$$

sending $b[d] \otimes b'[d']$ to $\tilde{b}'[d' + H] \otimes \tilde{b}[d - H]$, where $b \otimes b'$ and $\tilde{b}' \otimes \tilde{b}$ are related under the bijection ι and $H = H(b \otimes b')$. We call the bijection ι **(crystal) isomorphism** and the function H **(local) energy function**. Proposition 2.6 assures that such a map R exists.

Let us examine how R is determined combinatorially. First, recall R commutes with $\tilde{e}_i, \tilde{f}_i$. By the isomorphism, $u_0 \otimes u_0'$ should be sent to $u_0' \otimes u_0$, since there is no other crystal element with the same weight by (2.19). For the other elements, the images are determined inductively using $\iota \circ \tilde{e}_i = \tilde{e}_i \circ \iota$, $\iota \circ \tilde{f}_i = \tilde{f}_i \circ \iota$, since $B \otimes B'$ is connected. Next we consider the energy function H. Suppose $\tilde{e}_i(b \otimes b') \neq 0$. Applying $\tilde{e}_i$ on both sides of $\iota(b \otimes b') = \tilde{b}' \otimes \tilde{b}$ forces us to consider the following 4 cases:

$$\begin{aligned}
&\text{(LL)} \quad \iota(\tilde{e}_i b \otimes b') = \tilde{e}_i \tilde{b}' \otimes \tilde{b},\\
&\text{(LR)} \quad \iota(\tilde{e}_i b \otimes b') = \tilde{b}' \otimes \tilde{e}_i \tilde{b},\\
&\text{(RL)} \quad \iota(b \otimes \tilde{e}_i b') = \tilde{e}_i \tilde{b}' \otimes \tilde{b},\\
&\text{(RR)} \quad \iota(b \otimes \tilde{e}_i b') = \tilde{b}' \otimes \tilde{e}_i \tilde{b}.
\end{aligned}$$

Suppose (LL) occurs with $i = 0$. We have

$$\begin{aligned}
R\tilde{e}_0(b[d] \otimes b'[d']) &= R((\tilde{e}_0 b)[d+1] \otimes b'[d'])) = (\tilde{e}_0 \tilde{b}')[d' + H'] \otimes \tilde{b}[d + 1 - H'],\\
\tilde{e}_0 R(b[d] \otimes b'[d']) &= \tilde{e}_0(\tilde{b}'[d' + H] \otimes \tilde{b}[d - H]) = (\tilde{e}_0 \tilde{b}')[d' + H + 1] \otimes \tilde{b}[d - H],
\end{aligned}$$

where $H = H(b \otimes b'), H' = H(\tilde{e}_0(b \otimes b'))$. Since $R\tilde{e}_0 = \tilde{e}_0 R$, we obtain $H(\tilde{e}_0(b \otimes b')) = H(b \otimes b') + 1$. Similar calculations show that for any i and $b \otimes b' \in B \otimes B'$ such that $\tilde{e}_i(b \otimes b') \neq 0$,

$$H(\tilde{e}_i(b \otimes b')) = \begin{cases} H(b \otimes b') + 1 & \text{if } i = 0 \text{ and case (LL) holds,}\\ H(b \otimes b') - 1 & \text{if } i = 0 \text{ and case (RR) holds,}\\ H(b \otimes b') & \text{otherwise.} \end{cases} \tag{2.24}$$

This relation determines the value of H up to additive constant. Similarly, for any i and $b \otimes b' \in B \otimes B'$ such that $\tilde{f}_i(b \otimes b') \neq 0$,

$$H(\tilde{f}_i(b \otimes b')) = \begin{cases} H(b \otimes b') - 1 & \text{if } i = 0 \text{ and case (LL) holds,}\\ H(b \otimes b') + 1 & \text{if } i = 0 \text{ and case (RR) holds,}\\ H(b \otimes b') & \text{otherwise.} \end{cases} \tag{2.25}$$

Of course, (LL) or (RR) in this case should be decided according to the application of $\tilde{f}_0$.

We finish this subsection with the following proposition, which follows from Proposition 2.5.

Proposition 2.7. (1) $R_{B'B} R_{BB'} = id$.

(2) *The Yang-Baxter equation holds on* $\mathit{Aff}(B) \otimes \mathit{Aff}(B') \otimes \mathit{Aff}(B'')$.

$$\begin{aligned}
&(R_{B'B''} \otimes 1)(1 \otimes R_{BB''})(R_{BB'} \otimes 1)\\
&\qquad = (1 \otimes R_{BB'})(R_{BB''} \otimes 1)(1 \otimes R_{B'B''})
\end{aligned}$$

Note that by (2) we have $H_{BB'}(b \otimes b') = H_{B'B}(\tilde{b}' \otimes \tilde{b})$ if $\iota(b \otimes b') = \tilde{b}' \otimes \tilde{b}$.

The above proposition implies the following. Let $\mathfrak{S}_m$ be the m-th symmetric group. Let s_i be the simple reflection which interchanges i and $i+1$, and let $l(w)$ be the length of $w \in \mathfrak{S}_m$. Suppose the combinatorial R $R : \mathit{Aff}(B_i) \otimes \mathit{Aff}(B_j) \to \mathit{Aff}(B_j) \otimes \mathit{Aff}(B_i)$ is known for any (i,j) such that $1 \le i, j \le m, i \ne j$. Then for any $w \in \mathfrak{S}$ we can construct a map $R_w : \mathit{Aff}(B_1) \otimes \cdots \otimes \mathit{Aff}(B_m) \to \mathit{Aff}(B_{w(1)}) \otimes \cdots \otimes \mathit{Aff}(B_{w(m)})$ by

$$
\begin{aligned}
R_1 &= 1,\\
R_{s_i} &= \underbrace{1 \otimes \cdots \otimes 1}_{i-1} \otimes R \otimes \underbrace{1 \otimes \cdots \otimes 1}_{m-i-1},\\
R_{ws_i} &= R_{s_i} \circ R_w \text{ for } w \text{ such that } l(ws_i) = l(w) + 1.
\end{aligned}
$$

3. Examples of crystals

3.1. **Type** A $B^{1,s}$. In this subsection we will look at the crystal $B^{1,s}$ for type $A_{n-1}^{(1)}$ in detail.

As a set the crystal $B^{1,s}$ is identified with the set of semistandard tableaux of shape (s) with letters from $\{1, 2, \ldots, n\}$ (tableau representation). Letting x_i be the number of letter i in the tableau, we have an alternative description of $B^{1,s}$ (coordinate representation).

$$B^{1,s} = \{x = (x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{Z}_{\ge 0}, x_1 + x_2 + \cdots + x_n = s\}.$$

The action of $\tilde{e}_i, \tilde{f}_i$ $(i = 0, 1, \ldots, n-1)$ reads

$$
\begin{aligned}
\tilde{e}_i x &= (x_1, \ldots, x_i + 1, x_{i+1} - 1, \ldots, x_n) \text{ for } i \ne 0,\\
\tilde{e}_0 x &= (x_1 - 1, x_2, \ldots, x_{n-1}, x_n + 1),\\
\tilde{f}_i x &= (x_1, \ldots, x_i - 1, x_{i+1} + 1, \ldots, x_n) \text{ for } i \ne 0,\\
\tilde{f}_0 x &= (x_1 + 1, x_2, \ldots, x_{n-1}, x_n - 1).
\end{aligned}
$$

If a negative entry appears upon application, we should understand it is 0. $\varepsilon_i(x), \varphi_i(x)$ $(i = 0, 1, \ldots, n-1)$ are given by

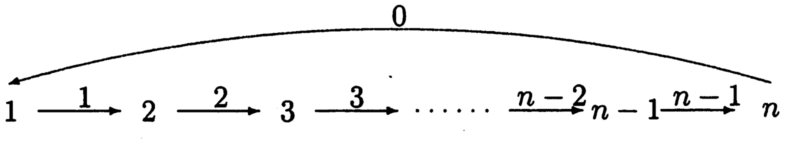
$$\varepsilon_i(x) = x_{i+1}, \quad \varphi_i(x) = x_i. \tag{3.1}$$

Here x_0 should be understood as x_n.

Here follows some examples of crystal graphs.

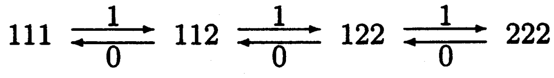
Example 3.1. (1) n: arbitrary

$B^{1,1}$:



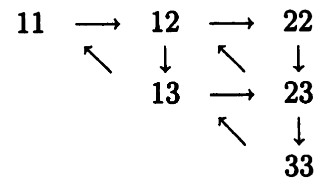
(2) $n = 2$

$B^{1,3}$:



(3) $n = 3$

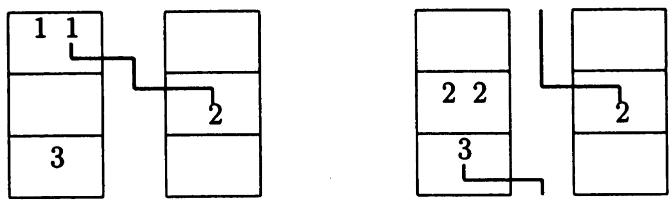
$B^{1,2}$:



Here the colors of the arrows $\longrightarrow, \downarrow, \nwarrow$ are $1, 2, 0$, respectively.

Explicit rules to calculate the crystal isomorphism $B^{1,s} \otimes B^{1,s'} \to B^{1,s'} \otimes B^{1,s}$ and the corresponding energy function are obtained in [NY]. We assume $s \geq s'$. Let $b \otimes b'$ be an element in $B^{1,s} \otimes B^{1,s'}$ in tableau representation such as $b = \nu_1 \cdots \nu_s$ and $b' = \nu'_1 \cdots \nu'_{s'}$. Let x_i (resp. y_i) be the number of i in b (resp. b'). We represent $b \otimes b'$ by the two column diagram. Each column has n rows, enumerated as 1 to n from the top to the bottom. We put x_i (resp. y_i) letters i in the i-th row of the left (resp. right) column. The rule to obtain the isomorphism and the energy function H is as follows.

(1) Pick any letter, say i, in the right column and connect it with a letter j in the left column by a line. The partner j is chosen so that $j = \max\{\nu_k \mid \nu_k < i\}$. If there is no such j, we return to the bottom and the partner j is chosen so that $j = \max\{\nu_k\}$. In the latter case, we call such a pair or line "winding".



unwinding winding

(2) Repeat the procedure (1) for the remaining unconnected letters $(s' - 1)$-times.
(3) The isomorphism is obtained by sliding the remaining $(s - s')$ unpaired letters in the left column to the right.
(4) The value of the energy function is the number of the "winding" pairs.

Note that we normalized H so that we have $\min H = 0$. When $s' = 1$, the rule simplifies to the following:

If there exists k such that $\nu_k < \nu'$, then

$$\begin{aligned} &\nu_1 \cdots \nu_s \otimes \nu' \mapsto \nu_j \otimes \nu_1 \cdots \overset{j}{\nu'} \cdots \nu_s, \\ &H(\nu_1 \cdots \nu_s \otimes \nu') = 0, \end{aligned} \tag{3.2}$$

where $j = \max\{k \mid \nu_k < \nu'\}$. Otherwise,

$$\begin{aligned} &\nu_1 \cdots \nu_s \otimes \nu' \mapsto \nu_s \otimes \nu' \nu_1 \cdots \nu_{s-1}, \\ &H(\nu_1 \cdots \nu_s \otimes \nu') = 1. \end{aligned} \tag{3.3}$$

Example 3.2. $s = 4, s' = 2$

$$\begin{aligned} 1123 \otimes 23 &\simeq 12 \otimes 1233 && H = 0 \\ 1123 \otimes 12 &\simeq 13 \otimes 1122 && H = 1 \\ 2344 \otimes 12 &\simeq 44 \otimes 1223 && H = 2 \end{aligned}$$

Here $b \otimes b' \simeq \tilde{b}' \otimes \tilde{b}$ means that they corresponds to each other under the isomorphism $\iota : B^{1,s} \otimes B^{1,s'} \to B^{1,s'} \otimes B^{1,s}$.

There is also a piecewise linear formula for the isomorphism and energy function in this case. Let $x = (x_1, \ldots, x_n)$ (resp. $y = (y_1, \ldots, y_n)$) be a coordinate representation for $b \in B^{1,s}$ (resp. $b' \in B^{1,s'}$) and suppose $\iota(b \otimes b') = \tilde{b}' \otimes \tilde{b}$. Then the coordinate x' (resp. y') of $\tilde{b}$ (resp. $\tilde{b}'$) is given by

$$x'_i = x_i + Q_i(x,y) - Q_{i-1}(x,y), \quad y'_i = y_i + Q_{i-1}(x,y) - Q_i(x,y) \tag{3.4}$$

for $1 \le i \le n$ with

$$Q_i(x,y) = \min\{\sum_{j=1}^{k-1} x_{i+j} + \sum_{j=k+1}^{n} y_{i+j} \mid 1 \le k \le n\}.$$

Here the indices of x, y are to be understood in $\mathbb{Z}/n\mathbb{Z}$. The energy function is given by $H(b \otimes b') = -Q_0(x,y)$. This H is so normalized that we have $\min H = -\min(s, s')$. We note that this formula can be used irrespective of whether s is greater than s' or not. Intriguingly, this formula is first obtained in [HHIKTT] through studies of the box-ball system, a kind of cellular automaton with solitons.

3.2. **Type** A $B^{r,s}$. In this subsection, we investigate the crystal structure of $B^{r,s}$ for arbitrary r, s and the combinatorial R for $B^{r,s} \otimes B^{r',s'}$. Our reference is [Sh]. We also use fundamental operations in tableau combinatorics, such as "jeu de taquin" or "bumping algorithm." See e.g. [F] for the details.

Our $U'_q(A^{(1)}_{n-1})$-crystal $B^{r,s}$ $(1 \le r \le n-1, s \in \mathbb{Z}_{>0})$ is, as a set, identified with the set of semistandard tableaux of rectangular shape (s^r) with letters from $\{1, 2, \ldots, n\}$. For an element t of $B^{r,s}$, let t_{ij} denote the letter in the i-th row and j-th column of t. We first describe the action of $\tilde{e}_i, \tilde{f}_i$ for $i = 1, 2, \ldots, n-1$. For this purpose, let us define the **Japanese reading word** of t by

$$J(t) = w^{(s)} w^{(s-1)} \cdots w^{(1)}, \quad w^{(j)} = t_{1j} t_{2j} \cdots t_{rj} \ (j = 1, 2, \ldots, s).$$

We then regard $J(t)$ as an element of $(B^{1,1})^{\otimes(rs)}$. Namely, each letter is considered to be an element of $B^{1,1}$. The action of $\tilde{e}_i, \tilde{f}_i$ is given by applying the signature rule in section 2.1 to $J(t)$. Note that for the i-signature, letter i (resp. $i+1$) corresponds to $+$ (resp. $-$).

Example 3.3. Let $n = 7$. Take an element

$$t = \begin{array}{ccccc} 1&1&2&3&4 \\ 2&3&3&5&5 \\ 3&4&5&6&7 \end{array}$$

of $B^{3,5}$. We are to apply $\tilde{e}_1, \tilde{f}_1$. $J(t)$ and its 1-signature is given by

$$\begin{array}{lcccccccccccccccc} J(t) & = & 4 & 5 & 7 & 3 & 5 & 6 & 2 & 3 & 5 & 1 & 3 & 4 & 1 & 2 & 3 \\ \text{1-sig} & & & & & & & & - & & & + & & & + & - & \end{array}.$$

Since the reduced signature is $-+$, the action of $\tilde{e}_1$ (resp. $\tilde{f}_1$) corresponds to changing the 1st 2 (resp. 1st 1) to 1 (resp. 2) in $J(t)$. Therefore, we have

$$\tilde{e}_1 t = \begin{array}{ccccc} 1&1&1&3&4 \\ 2&3&3&5&5 \\ 3&4&5&6&7 \end{array}, \quad \tilde{f}_1 t = \begin{array}{ccccc} 1&2&2&3&4 \\ 2&3&3&5&5 \\ 3&4&5&6&7 \end{array}.$$

Similarly,

$$\tilde{e}_2 t = \begin{array}{ccccc} 1&1&2&3&4 \\ 2&2&3&5&5 \\ 3&4&5&6&7 \end{array}, \quad \tilde{f}_2 t = 0.$$

Next we consider the action of $\tilde{e}_0, \tilde{f}_0$. Instead of giving the rule directly, we introduce the **promotion operator** pr satisfying

$$\tilde{e}_0 = \mathrm{pr}^{-1} \circ \tilde{e}_1 \circ \mathrm{pr}, \quad \tilde{f}_0 = \mathrm{pr}^{-1} \circ \tilde{f}_1 \circ \mathrm{pr}. \tag{3.5}$$

For a tableau t, $\mathrm{pr}(t)$ is given by performing the **jeu de taquin**. We illustrate it by example.

Example 3.4. Set $n = 7$. t is the same as in the previous example. We delete the letter $7(= n)$ from t and perform jeu de taquin as follows.

$$\begin{array}{c} 11234 \\ 23355 \\ 3456\cdot \end{array} \rightarrow \begin{array}{c} 11234 \\ 23355 \\ 345\cdot 6 \end{array} \rightarrow \begin{array}{c} 11234 \\ 233\cdot 5 \\ 34556 \end{array} \rightarrow \begin{array}{c} 112\cdot 4 \\ 23335 \\ 34556 \end{array}$$

$$\rightarrow \begin{array}{c} 11\cdot 24 \\ 23335 \\ 34556 \end{array} \rightarrow \begin{array}{c} 1\cdot 124 \\ 23335 \\ 34556 \end{array} \rightarrow \begin{array}{c} \cdot 1124 \\ 23335 \\ 34556 \end{array}$$

We then replace $\cdot$ with 0 and add 1 to all the letters. That's the tableau $\mathrm{pr}(t)$, which is

$$\begin{array}{c} 12235 \\ 34446 \\ 45667 \end{array}$$

for the present example. If there are more than two 7's, we do it from the leftmost one. From (3.5) we have

$$\tilde{e}_0 t = \begin{array}{c} 12334 \\ 23555 \\ 34677 \end{array}, \quad \tilde{f}_0 t = \begin{array}{c} 11124 \\ 23335 \\ 34556 \end{array}.$$

In order to describe the combinatorial R explicitly, we need to explain Schensted's **bumping algorithm**. The bumping algorithm is defined for a pair of tableau t and single word u and depicted as $t \leftarrow u$. First, let us consider the case where t is a one-row tableau. If t is empty, $t \leftarrow u$ is defined to be the tableau u with one node. Otherwise, let $t = t_{11}t_{12}\cdots t_{1m}$ and look at

$$t_{11}t_{12}\cdots t_{1m} \leftarrow u.$$

If $t_{1m} \le u$, then define

$$t \leftarrow u = t_{11}t_{12}\cdots t_{1m}u$$

and the algorithm stops (case (a)). Otherwise, set $i_1 = \min\{i \mid t_{1i} > u\}$ and define

$$t \leftarrow u = t_{11}\cdots t_{1i_1-1}ut_{1i_1+1}\cdots t_{1m}$$

and we have the single word t_{1i_1} bumped out from t (case (b)). Now suppose we have a tableau t of l rows and let t_i be the i-th row of t. The bumping algorithm $t \leftarrow u$ proceeds as follows. Set $t'_1 = t_1 \leftarrow u$. If case (a) occurs in this case, the algorithm stops. Otherwise, let u_1 be the letter bumped out and set $t'_2 = t_2 \leftarrow u_1$. We again divide into the two cases. The algorithm proceeds until it stops. If case (b) still occurs in the deepest row, we append the empty row below it.

Example 3.5. Let

$$t = \begin{array}{l} 112344 \\ 23345 \\ 34 \end{array}$$

and $u = 2$. The bumping algorithm proceeds as follows.

$$\begin{array}{llllll} 1\,1\,2\,\mathbf{3}\,4\,4 & \leftarrow\ 2 & 1\,1\,2\,2\,4\,4 & & 1\,1\,2\,2\,4\,4 & \\ 2\,3\,3\,4\,5 & & 2\,3\,3\,\mathbf{4}\,5 & \leftarrow\ 3 & 2\,3\,3\,3\,5 & \\ 3\,4 & & 3\,4 & & 3\,4 & \leftarrow\ 4 \end{array}$$

And we have the answer.

$$\begin{array}{l} 1\,1\,2\,2\,4\,4 \\ 2\,3\,3\,3\,5 \\ 3\,4\,4 \end{array}$$

For a tableau $t \in B^{r,s}$ we define the **row word** $row(t)$ by

$$row(t) = t_r t_{r-1} \cdots t_1 \quad t_i = t_{i1} t_{i2} \cdots t_{is} \quad (i = 1, 2, \ldots, r).$$

Let t be a tableau and $w = u_1 u_2 \cdots u_l$ a word of length l. Let $t \leftarrow w$ be a tableau obtained by applying the bumping algorithm for a single word u_j successively as

$$(\cdots((t \leftarrow u_1) \leftarrow u_2) \leftarrow \cdots) \leftarrow u_l.$$

Then we have the following proposition to obtain the combinatorial R for $B^{r,s} \otimes B^{r',s'}$.

Proposition 3.6 ([Sh]). *$t \otimes t'$ is mapped to $\tilde{t}' \otimes \tilde{t}$ by the crystal isomorphism*

$$B^{r,s} \otimes B^{r',s'} \longrightarrow B^{r',s'} \otimes B^{r,s},$$

if and only if

$$t' \leftarrow row(t) = \tilde{t} \leftarrow row(\tilde{t}').$$

Moreover, the energy function $H(t \otimes t')$ is given by the number of nodes in the shape of $t' \leftarrow row(t)$ that are strictly east of the $\max(s, s')$*-th column.*

Note that the decomposition of $B^{r,s} \otimes B^{r',s'}$ into $U_q(A_{n-1})$-crystals is multiplicity free. From this fact, it follows that for a given pair $t \otimes t'$ we can determine $\tilde{t}', \tilde{t}$ uniquely. To explain the algorithm of computing $\tilde{t}', \tilde{t}$, we prepare terminology. Let θ be a skew tableau, that is, set-theoretical difference of a Young diagram from a larger one with letters in each node. Let τ be the shape of θ. θ is called a vertical m-strip if $|\tau| = m$ and $\tau_i \leq 1$ for any $i \geq 1$. The algorithm to obtain $\tilde{t}', \tilde{t}$ is given as follows. Let p be the tableau obtained by the bumping algorithm $t' \leftarrow row(t)$. We attach an integer from 1 to $r's'$ to each node of the skew tableau $p - p'$, where p' is the NW part of p whose shape is (s^r). The integers should be labeled in the following manner. Let θ_1 be the rightmost vertical r'-strip in $p - p'$ as upper as possible. We attach integers 1 through r' from lower nodes. Remove θ_1 from $p - p'$ and define the vertical r'-strip θ_2 in a similar manner. Continue it until we finish attaching all integers up to $r's'$. Next we apply the reverse bumping algorithm according to the order of the labeling. Namely, we find a word u_1 and a tableau p_1 whose shape is (shape of p)-(node of label 1), such that $p_1 \leftarrow u_1 = p$. (Note that such a pair (p_1, u_1) is unique.) We repeat this procedure to obtain u_2 and p_2 by replacing p and the node of label 1 with p_1 and the node of label 2 and continue until we arrive at a tableau of shape (s^r). Then we have

$$\tilde{t}' = ((\cdots(\phi \leftarrow u_{r's'}) \leftarrow \cdots) \leftarrow u_2) \leftarrow u_1 \quad \text{and} \quad \tilde{t} = p_{r's'}.$$

Example 3.7. (1) Let $t = \begin{array}{l} 1\,1\,2 \\ 2\,3\,3 \end{array}$, $t' = \begin{array}{l} 1\,2 \\ 4\,4 \end{array}$. Then we have

$$p = t' \leftarrow row(t) = \begin{array}{l} 1\ 1\ 1\ 2_4 3_2 \\ 2\ 2\ 3 \\ 4_3 4_1 \end{array} \quad \text{and } H = 2.$$

Here the subscripts in p are the labels. Since $p_4 = \begin{matrix}1\,1\,2\\2\,4\,4\end{matrix}$ and $u_4u_3u_2u_1 = 3132$, we get

$$\begin{matrix}1\,1\,2\\2\,3\,3\end{matrix} \otimes \begin{matrix}1\,2\\4\,4\end{matrix} \simeq \begin{matrix}1\,2\\3\,3\end{matrix} \otimes \begin{matrix}1\,1\,2\\2\,4\,4\end{matrix}.$$

(2) Let $t = \begin{matrix}1\,2\,4\,4\\2\,4\,5\,6\end{matrix}, t' = \begin{matrix}2\,2\\3\,4\\4\,6\end{matrix}$. Then we have

$$p = t' \leftarrow row(t) = \begin{matrix}1\;2\;2\;2\;4_6\,4_3\\2\;4\;4\;5\;6_2\\3_5\,6_1\\4_4\end{matrix} \quad \text{and } H = 3.$$

Since $p_6 = \begin{matrix}2\,2\,2\,3\\4\,4\,4\,6\end{matrix}$ and $u_6u_5u_4u_3u_2u_1 = 421654$, we get

$$\begin{matrix}1\,2\,4\,4\\2\,4\,5\,6\end{matrix} \otimes \begin{matrix}2\,2\\3\,4\\4\,6\end{matrix} \simeq \begin{matrix}1\,4\\2\,5\\4\,6\end{matrix} \otimes \begin{matrix}2\,2\,2\,3\\4\,4\,4\,6\end{matrix}.$$

3.3. **KKM crystals.** In this subsection, we recall KKM crystals for all non exceptional affine algebras except type A. (Type A has been dealt with already.) In our notation, it is denoted by $B^{1,s}$. Our basic reference is [KKM].

Affine Lie algebras we consider here are

$$\mathfrak{g} = B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, A_{2n}^{(2)}, A_{2n-1}^{(2)}, D_{n+1}^{(2)}. \tag{3.6}$$

Looking at their Dynkin diagrams near the 0-node, we divide these algebras into 3 types:

(i) Type $\boxminus$: $B_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}$,

(ii) Type $\square\!\square$: $C_n^{(1)}$,

(iii) Type $\square$: $A_{2n}^{(2)}, D_{n+1}^{(2)}$.

Let $\mathring{\mathfrak{g}}$ be the finite dimensional simple Lie algebra whose Dynkin diagram is obtained by removing the 0-node from that of $\mathfrak{g}$. According to the order in (3.6), we have

$$\mathring{\mathfrak{g}} = B_n, C_n, D_n, C_n, C_n, B_n.$$

As a $U_q(\mathring{\mathfrak{g}})$-crystal, the KKM crystal $B^{1,s}$ decomposes as follows.

$$B^{1,s} \simeq \begin{cases} B(s\overline{\Lambda}_1) & \text{for type } \boxminus, \\ B(s\overline{\Lambda}_1) \oplus B((s-2)\overline{\Lambda}_1) \oplus \cdots \oplus (B(\overline{\Lambda}_1) \text{ or } B(0)) & \text{for type } \square\!\square, \\ B(s\overline{\Lambda}_1) \oplus B((s-1)\overline{\Lambda}_1) \oplus \cdots \oplus B(0) & \text{for type } \square. \end{cases}$$

Here $\overline{\Lambda}_1$ stands for the first fundamental weight of $\mathring{\mathfrak{g}}$. The irreducible $\mathring{\mathfrak{g}}$-module with this highest weight is the vector representation.

Remark 3.8. The division into the cases (i)-(iii) corresponds to the decomposition of KR crystals into $U_q(\mathring{\mathfrak{g}})$-crystals. Take $B^{r,s}$ from type $\boxminus$. Assume $s \neq n$ (and $s \neq n-1$ for $D_n^{(1)}$). $B^{r,s}$ corresponds to the $r \times s$ rectangular diagram. Irreducible components as $U_q(\mathring{\mathfrak{g}})$-crystal are obtained by removing $\boxminus$ from the $r \times s$ rectangle. The rule is similar to the other 2 cases. See Appendix A of [HKOTY] and of [HKOTT] for details.

We list the coordinate representation of the KKM crystal $B^{1,s}$ and its crystal structure for all non exceptional affine algebras except type A. The crystal structure with respect to the color 0 (resp. n (and $n-1$ for $D_n^{(1)}$)) is determined by the types $\boxminus, \square\!\square, \square$ (resp. $\mathring{\mathfrak{g}}$). It is common to all cases for the other colors. If $\tilde{e}_i b \neq B^{1,s}$ (resp. $\tilde{f}_i b \neq B^{1,s}$) for $b \in B^{1,s}$, we understand $\tilde{e}_i b = 0$ (resp. $\tilde{f}_i b = 0$). We define $(a)_+ = \max(a, 0)$.

(i) Type $\boxminus$

- $B_n^{(1)}$

$$B^{1,s} = \{b = (x_1, \dots, x_n, x_0, \bar{x}_n, \dots, \bar{x}_1) \mid x_0 = 0 \text{ or } 1,$$
$$x_i, \bar{x}_i \in \mathbb{Z}_{\geq 0}, \sum_{i=1}^{n} (x_i + \bar{x}_i) + x_0 = s\}.$$

$$\tilde{e}_0 b = \begin{cases} (x_1, x_2 - 1, \dots, \bar{x}_2, \bar{x}_1 + 1) & \text{if } x_2 > \bar{x}_2, \\ (x_1 - 1, x_2, \dots, \bar{x}_2 + 1, \bar{x}_1) & \text{if } x_2 \leq \bar{x}_2, \end{cases}$$
$$\tilde{e}_i b = \begin{cases} (x_1, \dots, x_i + 1, x_{i+1} - 1, \dots, \bar{x}_1) & \text{if } x_{i+1} > \bar{x}_{i+1}, \\ (x_1, \dots, \bar{x}_{i+1} + 1, \bar{x}_i - 1, \dots, \bar{x}_1) & \text{if } x_{i+1} \leq \bar{x}_{i+1}, \end{cases}$$
$$\tilde{e}_n b = \begin{cases} (x_1, \dots, x_n, x_0 + 1, \bar{x}_n - 1, \dots, \bar{x}_1) & \text{if } x_0 = 0, \\ (x_1, \dots, x_n + 1, x_0 - 1, \bar{x}_n, \dots, \bar{x}_1) & \text{if } x_0 = 1, \end{cases}$$

$$\tilde{f}_0 b = \begin{cases} (x_1, x_2 + 1, \dots, \bar{x}_2, \bar{x}_1 - 1) & \text{if } x_2 \geq \bar{x}_2, \\ (x_1 + 1, x_2, \dots, \bar{x}_2 - 1, \bar{x}_1) & \text{if } x_2 < \bar{x}_2, \end{cases}$$
$$\tilde{f}_i b = \begin{cases} (x_1, \dots, x_i - 1, x_{i+1} + 1, \dots, \bar{x}_1) & \text{if } x_{i+1} \geq \bar{x}_{i+1}, \\ (x_1, \dots, \bar{x}_{i+1} - 1, \bar{x}_i + 1, \dots, \bar{x}_1) & \text{if } x_{i+1} < \bar{x}_{i+1}, \end{cases}$$
$$\tilde{f}_n b = \begin{cases} (x_1, \dots, x_n - 1, x_0 + 1, \bar{x}_n, \dots, \bar{x}_1) & \text{if } x_0 = 0, \\ (x_1, \dots, x_n, x_0 - 1, \bar{x}_n + 1, \dots, \bar{x}_1) & \text{if } x_0 = 1, \end{cases}$$

$$\varepsilon_0(b) = x_1 + (x_2 - \bar{x}_2)_+, \quad \varphi_0(b) = \bar{x}_1 + (\bar{x}_2 - x_2)_+,$$
$$\varepsilon_i(b) = \bar{x}_i + (x_{i+1} - \bar{x}_{i+1})_+, \quad \varphi_i(b) = x_i + (\bar{x}_{i+1} - x_{i+1})_+,$$
$$\varepsilon_n(b) = 2\bar{x}_n + x_0, \quad \varphi_n(b) = 2x_n + x_0$$

for $i = 1, \dots, n-1$.

- $D_n^{(1)}$

$$B^{1,s} = \{b = (x_1, \dots, x_n, \bar{x}_n, \dots, \bar{x}_1) \mid x_i, \bar{x}_i \in \mathbb{Z}_{\geq 0},$$
$$x_n \text{ or } \bar{x}_n = 0, \sum_{i=1}^{n} (x_i + \bar{x}_i) = s\}.$$

$$\tilde{e}_0 b = \begin{cases} (x_1, x_2 - 1, \dots, \bar{x}_2, \bar{x}_1 + 1) & \text{if } x_2 > \bar{x}_2, \\ (x_1 - 1, x_2, \dots, \bar{x}_2 + 1, \bar{x}_1) & \text{if } x_2 \leq \bar{x}_2, \end{cases}$$
$$\tilde{e}_i b = \begin{cases} (x_1, \dots, x_i + 1, x_{i+1} - 1, \dots, \bar{x}_1) & \text{if } x_{i+1} > \bar{x}_{i+1}, \\ (x_1, \dots, \bar{x}_{i+1} + 1, \bar{x}_i - 1, \dots, \bar{x}_1) & \text{if } x_{i+1} \leq \bar{x}_{i+1}, \end{cases}$$
$$\tilde{e}_{n-1} b = \begin{cases} (x_1, \dots, x_{n-1} + 1, x_n - 1, \bar{x}_n, \dots, \bar{x}_1) & \text{if } x_n > 0, \bar{x}_n = 0, \\ (x_1, \dots, x_n, \bar{x}_n + 1, \bar{x}_{n-1} - 1, \bar{x}_n, \dots, \bar{x}_1) & \text{if } x_n = 0, \bar{x}_n \geq 0, \end{cases}$$
$$\tilde{e}_n b = \begin{cases} (x_1, \dots, x_n + 1, \bar{x}_n, \bar{x}_{n-1} - 1, \dots, \bar{x}_1) & \text{if } x_n \geq 0, \bar{x}_n = 0, \\ (x_1, \dots, x_{n-1} + 1, x_n, \bar{x}_n - 1, \dots, \bar{x}_1) & \text{if } x_n = 0, \bar{x}_n > 0, \end{cases}$$

$$\tilde{f}_0 b = \begin{cases} (x_1, x_2+1, \dots, \bar{x}_2, \bar{x}_1 - 1) \text{ if } x_2 \geq \bar{x}_2, \\ (x_1+1, x_2, \dots, \bar{x}_2 - 1, \bar{x}_1) \text{ if } x_2 < \bar{x}_2, \end{cases}$$
$$\tilde{f}_i b = \begin{cases} (x_1, \dots, x_i - 1, x_{i+1}+1, \dots, \bar{x}_1) \text{ if } x_{i+1} \geq \bar{x}_{i+1}, \\ (x_1, \dots, \bar{x}_{i+1} - 1, \bar{x}_i + 1, \dots, \bar{x}_1) \text{ if } x_{i+1} < \bar{x}_{i+1}, \end{cases}$$
$$\tilde{f}_{n-1} b = \begin{cases} (x_1, \dots, x_{n-1} - 1, x_n + 1, \bar{x}_n, \dots, \bar{x}_1) \text{ if } x_n \geq 0, \bar{x}_n = 0, \\ (x_1, \dots, x_n, \bar{x}_n - 1, \bar{x}_{n-1} + 1, \bar{x}_n, \dots, \bar{x}_1) \text{ if } x_n = 0, \bar{x}_n > 0, \end{cases}$$
$$\tilde{f}_n b = \begin{cases} (x_1, \dots, x_n - 1, \bar{x}_n, \bar{x}_{n-1} + 1, \dots, \bar{x}_1) \text{ if } x_n > 0, \bar{x}_n = 0, \\ (x_1, \dots, x_{n-1} - 1, x_n, \bar{x}_n + 1, \dots, \bar{x}_1) \text{ if } x_n = 0, \bar{x}_n \geq 0, \end{cases}$$

$$\begin{aligned} \varepsilon_0(b) &= x_1 + (x_2 - \bar{x}_2)_+, \quad \varphi_0(b) = \bar{x}_1 + (\bar{x}_2 - x_2)_+, \\ \varepsilon_i(b) &= \bar{x}_i + (x_{i+1} - \bar{x}_{i+1})_+, \quad \varphi_i(b) = x_i + (\bar{x}_{i+1} - x_{i+1})_+, \\ \varepsilon_{n-1}(b) &= \bar{x}_{n-1} + x_n, \quad \varphi_{n-1}(b) = x_{n-1} + \bar{x}_n, \\ \varepsilon_n(b) &= \bar{x}_{n-1} + \bar{x}_n, \quad \varphi_n(b) = x_{n-1} + x_n \end{aligned}$$

for $i = 1, \dots, n-2$.

- $A^{(2)}_{2n-1}$

$$B^{1,s} = \{b = (x_1, \dots, x_n, \bar{x}_n, \dots, \bar{x}_1) \mid x_i, \bar{x}_i \in \mathbb{Z}_{\geq 0}, \sum_{i=1}^{n} (x_i + \bar{x}_i) = s\}.$$

$$\tilde{e}_0 b = \begin{cases} (x_1, x_2 - 1, \dots, \bar{x}_2, \bar{x}_1 + 1) \text{ if } x_2 > \bar{x}_2, \\ (x_1 - 1, x_2, \dots, \bar{x}_2 + 1, \bar{x}_1) \text{ if } x_2 \leq \bar{x}_2, \end{cases}$$
$$\tilde{e}_i b = \begin{cases} (x_1, \dots, x_i + 1, x_{i+1} - 1, \dots, \bar{x}_1) \text{ if } x_{i+1} > \bar{x}_{i+1}, \\ (x_1, \dots, \bar{x}_{i+1} + 1, \bar{x}_i - 1, \dots, \bar{x}_1) \text{ if } x_{i+1} \leq \bar{x}_{i+1}, \end{cases}$$
$$\tilde{e}_n b = (x_1, \dots, x_n + 1, \bar{x}_n - 1, \dots, \bar{x}_1),$$

$$\tilde{f}_0 b = \begin{cases} (x_1, x_2 + 1, \dots, \bar{x}_2, \bar{x}_1 - 1) \text{ if } x_2 \geq \bar{x}_2, \\ (x_1 + 1, x_2, \dots, \bar{x}_2 - 1, \bar{x}_1) \text{ if } x_2 < \bar{x}_2, \end{cases}$$
$$\tilde{f}_i b = \begin{cases} (x_1, \dots, x_i - 1, x_{i+1} + 1, \dots, \bar{x}_1) \text{ if } x_{i+1} \geq \bar{x}_{i+1}, \\ (x_1, \dots, \bar{x}_{i+1} - 1, \bar{x}_i + 1, \dots, \bar{x}_1) \text{ if } x_{i+1} < \bar{x}_{i+1}, \end{cases}$$
$$\tilde{f}_n b = (x_1, \dots, x_n - 1, \bar{x}_n + 1, \dots, \bar{x}_1),$$

$$\begin{aligned} \varepsilon_0(b) &= x_1 + (x_2 - \bar{x}_2)_+, \quad \varphi_0(b) = \bar{x}_1 + (\bar{x}_2 - x_2)_+, \\ \varepsilon_i(b) &= \bar{x}_i + (x_{i+1} - \bar{x}_{i+1})_+, \quad \varphi_i(b) = x_i + (\bar{x}_{i+1} - x_{i+1})_+, \\ \varepsilon_n(b) &= \bar{x}_n, \quad \varphi_n(b) = x_n \end{aligned}$$

for $i = 1, \dots, n-1$.

(ii) Type ▭

- $C^{(1)}_n$

$$B^{1,s} = \{b = (x_1, \dots, x_n, \bar{x}_n, \dots, \bar{x}_1) \mid x_i, \bar{x}_i \in \mathbb{Z}_{\geq 0}, \\ \sum_{i=1}^{n} (x_i + \bar{x}_i) \leq s, \sum_{i=1}^{n} (x_i + \bar{x}_i) \equiv s \ (\mathrm{mod}\ 2)\}.$$

$$\tilde{e}_0 b = \begin{cases} (x_1 - 2, x_2, \dots, \bar{x}_2, \bar{x}_1) \text{ if } x_1 \geq \bar{x}_1 + 2, \\ (x_1 - 1, x_2, \dots, \bar{x}_2 + 1, \bar{x}_1) \text{ if } x_1 = \bar{x}_1 + 1, \\ (x_1, x_2, \dots, \bar{x}_2, \bar{x}_1 + 2) \text{ if } x_1 \leq \bar{x}_1, \end{cases}$$

$$\tilde{e}_i b = \begin{cases} (x_1, \dots, x_i + 1, x_{i+1} - 1, \dots, \bar{x}_1) \text{ if } x_{i+1} > \bar{x}_{i+1}, \\ (x_1, \dots, \bar{x}_{i+1} + 1, \bar{x}_i - 1, \dots, \bar{x}_1) \text{ if } x_{i+1} \leq \bar{x}_{i+1}, \end{cases}$$

$$\tilde{e}_n b = (x_1, \dots, x_n + 1, \bar{x}_n - 1, \dots, \bar{x}_1),$$

$$\tilde{f}_0 b = \begin{cases} (x_1 + 2, x_2, \dots, \bar{x}_2, \bar{x}_1) \text{ if } x_1 \geq \bar{x}_1, \\ (x_1 + 1, x_2, \dots, \bar{x}_2, \bar{x}_1 - 1) \text{ if } x_1 = \bar{x}_1 - 1, \\ (x_1, x_2, \dots, \bar{x}_2, \bar{x}_1 - 2) \text{ if } x_1 = \bar{x}_1 - 2, \end{cases}$$

$$\tilde{f}_i b = \begin{cases} (x_1, \dots, x_i - 1, x_{i+1} + 1, \dots, \bar{x}_1) \text{ if } x_{i+1} \geq \bar{x}_{i+1}, \\ (x_1, \dots, \bar{x}_{i+1} - 1, \bar{x}_i + 1, \dots, \bar{x}_1) \text{ if } x_{i+1} < \bar{x}_{i+1}, \end{cases}$$

$$\tilde{f}_n b = (x_1, \dots, x_n - 1, \bar{x}_n + 1, \dots, \bar{x}_1),$$

$$\varepsilon_0(b) = \frac{s - \sum_{i=1}^n (x_i + \bar{x}_i)}{2} + (x_1 - \bar{x}_1)_+,$$
$$\varphi_0(b) = \frac{s - \sum_{i=1}^n (x_i + \bar{x}_i)}{2} + (\bar{x}_1 - x_1)_+,$$
$$\varepsilon_i(b) = \bar{x}_i + (x_{i+1} - \bar{x}_{i+1})_+, \quad \varphi_i(b) = x_i + (\bar{x}_{i+1} - x_{i+1})_+,$$
$$\varepsilon_n(b) = \bar{x}_n, \quad \varphi_n(b) = x_n$$

for $i = 1, \dots, n-1$.

(iii) Type $\square$

- $A_{2n}^{(2)}$

$$B^{1,s} = \{b = (x_1, \dots, x_n, \bar{x}_n, \dots, \bar{x}_1) \mid x_i, \bar{x}_i \in \mathbb{Z}_{\geq 0}, \sum_{i=1}^n (x_i + \bar{x}_i) \leq s\}.$$

$$\tilde{e}_0 b = \begin{cases} (x_1 - 1, x_2, \dots, \bar{x}_2, \bar{x}_1) \text{ if } x_1 > \bar{x}_1, \\ (x_1, x_2, \dots, \bar{x}_2, \bar{x}_1 + 1) \text{ if } x_1 \leq \bar{x}_1, \end{cases}$$

$$\tilde{e}_i b = \begin{cases} (x_1, \dots, x_i + 1, x_{i+1} - 1, \dots, \bar{x}_1) \text{ if } x_{i+1} > \bar{x}_{i+1}, \\ (x_1, \dots, \bar{x}_{i+1} + 1, \bar{x}_i - 1, \dots, \bar{x}_1) \text{ if } x_{i+1} \leq \bar{x}_{i+1}, \end{cases}$$

$$\tilde{e}_n b = (x_1, \dots, x_n + 1, \bar{x}_n - 1, \dots, \bar{x}_1),$$

$$\tilde{f}_0 b = \begin{cases} (x_1 + 1, x_2, \dots, \bar{x}_2, \bar{x}_1) \text{ if } x_1 \geq \bar{x}_1, \\ (x_1, x_2, \dots, \bar{x}_2, \bar{x}_1 - 1) \text{ if } x_1 < \bar{x}_1, \end{cases}$$

$$\tilde{f}_i b = \begin{cases} (x_1, \dots, x_i - 1, x_{i+1} + 1, \dots, \bar{x}_1) \text{ if } x_{i+1} \geq \bar{x}_{i+1}, \\ (x_1, \dots, \bar{x}_{i+1} - 1, \bar{x}_i + 1, \dots, \bar{x}_1) \text{ if } x_{i+1} < \bar{x}_{i+1}, \end{cases}$$

$$\tilde{f}_n b = (x_1, \dots, x_n - 1, \bar{x}_n + 1, \dots, \bar{x}_1),$$

$$\varepsilon_0(b) = s - \sum_{i=1}^{n}(x_i + \bar{x}_i) + 2(x_1 - \bar{x}_1)_+,$$
$$\varphi_0(b) = s - \sum_{i=1}^{n}(x_i + \bar{x}_i) + 2(\bar{x}_1 - x_1)_+,$$
$$\varepsilon_i(b) = \bar{x}_i + (x_{i+1} - \bar{x}_{i+1})_+, \quad \varphi_i(b) = x_i + (\bar{x}_{i+1} - x_{i+1})_+,$$
$$\varepsilon_n(b) = \bar{x}_n, \quad \varphi_n(b) = x_n$$

for $i = 1, \dots, n-1$.

- $D_{n+1}^{(2)}$

$$B^{1,s} = \{b = (x_1, \dots, x_n, x_0, \bar{x}_n, \dots, \bar{x}_1) \mid x_0 = 0 \text{ or } 1, \quad x_i, \bar{x}_i \in \mathbb{Z}_{\geq 0}, \sum_{i=1}^{n}(x_i + \bar{x}_i) + x_0 \leq s\}.$$

$$\tilde{e}_0 b = \begin{cases} (x_1 - 1, x_2, \dots, \bar{x}_2, \bar{x}_1) \text{ if } x_1 > \bar{x}_1, \\ (x_1, x_2, \dots, \bar{x}_2, \bar{x}_1 + 1) \text{ if } x_1 \leq \bar{x}_1, \end{cases}$$
$$\tilde{e}_i b = \begin{cases} (x_1, \dots, x_i + 1, x_{i+1} - 1, \dots, \bar{x}_1) \text{ if } x_{i+1} > \bar{x}_{i+1}, \\ (x_1, \dots, \bar{x}_{i+1} + 1, \bar{x}_i - 1, \dots, \bar{x}_1) \text{ if } x_{i+1} \leq \bar{x}_{i+1}, \end{cases}$$
$$\tilde{e}_n b = \begin{cases} (x_1, \dots, x_n, x_0 + 1, \bar{x}_n - 1, \dots, \bar{x}_1) \text{ if } x_0 = 0, \\ (x_1, \dots, x_n + 1, x_0 - 1, \bar{x}_n, \dots, \bar{x}_1) \text{ if } x_0 = 1, \end{cases}$$

$$\tilde{f}_0 b = \begin{cases} (x_1 + 1, x_2, \dots, \bar{x}_2, \bar{x}_1) \text{ if } x_1 \geq \bar{x}_1, \\ (x_1, x_2, \dots, \bar{x}_2, \bar{x}_1 - 1) \text{ if } x_1 < \bar{x}_1, \end{cases}$$
$$\tilde{f}_i b = \begin{cases} (x_1, \dots, x_i - 1, x_{i+1} + 1, \dots, \bar{x}_1) \text{ if } x_{i+1} \geq \bar{x}_{i+1}, \\ (x_1, \dots, \bar{x}_{i+1} - 1, \bar{x}_i + 1, \dots, \bar{x}_1) \text{ if } x_{i+1} < \bar{x}_{i+1}, \end{cases}$$
$$\tilde{f}_n b = \begin{cases} (x_1, \dots, x_n - 1, x_0 + 1, \bar{x}_n, \dots, \bar{x}_1) \text{ if } x_0 = 0, \\ (x_1, \dots, x_n, x_0 - 1, \bar{x}_n + 1, \dots, \bar{x}_1) \text{ if } x_0 = 1, \end{cases}$$

$$\varepsilon_0(b) = s - \sum_{i=1}^{n}(x_i + \bar{x}_i) - x_0 + 2(x_1 - \bar{x}_1)_+,$$
$$\varphi_0(b) = s - \sum_{i=1}^{n}(x_i + \bar{x}_i) - x_0 + 2(\bar{x}_1 - x_1)_+,$$
$$\varepsilon_i(b) = \bar{x}_i + (x_{i+1} - \bar{x}_{i+1})_+, \quad \varphi_i(b) = x_i + (\bar{x}_{i+1} - x_{i+1})_+,$$
$$\varepsilon_n(b) = 2\bar{x}_n + x_0, \quad \varphi_n(b) = 2x_n + x_0$$

for $i = 1, \dots, n-1$.

We finish this subsection by mentioning the known results on the combinatorial R for the tensor product of KKM crystals $B^{1,s} \otimes B^{1,s'}$. For the homogeneous case ($s = s'$), the isomorphism is the identity. For the energy function a piecewise linear formula is known in [KKM]. For the inhomogeneous case, a combinatorial method, as in the previous subsection, is given in [HKOT1, HKOT2] for computing the combinatorial R. For type $D_n^{(1)}$, there is also an approach from "geometric crystal" that gives a piecewise linear formula for the combinatorial R as (3.4). See [KOTY].

4. Paths and 1D sums

In this section, we introduce a set of paths built upon KR crystals. We see that it is isomorphic to the crystal basis of an integrable highest weight $U_q(\mathfrak{g})$-module or a tensor product of them. We then define 1D sums as a generating function of the set of paths with "energy" statistic.

4.1. Perfect crystal.

Let B be a KR crystal. We define the **level** of B by

$$lev B = \min\{\langle c, \varepsilon(b)\rangle \mid b \in B\} \in \mathbb{Z}_{\geq 0}.$$

Note that $\langle c, \varepsilon(b)\rangle = \langle c, \varphi(b)\rangle$ for any $b \in B$. We set

$$B_{\min} = \{b \in B \mid \langle c, \varepsilon(b)\rangle = lev B\}$$

and call an element of $B_{\min}$ **minimal**.

The notion of perfect crystal was introduced in [KMN1]. Instead of rewriting the original definition (see Definition 4.6.1 of [KMN1]), we only give the most important postulate, since we expect any KR crystal satisfies the other conditions. Let us define the following two subsets of P_{cl}.

$$P_{cl}^+ = \{\lambda \in P_{cl} \mid \langle h_i, \lambda\rangle \geq 0 \text{ for any } i\},$$
$$(P_{cl}^+)_l = \{\lambda \in P_{cl}^+ \mid \langle c, \lambda\rangle = l\}.$$

B is said to be **perfect** of level l, if $lev B = l$ and both ε and φ are bijections from $B_{\min}$ to $(P_{cl}^+)_l$. In this case, we can define a map $\sigma = \varepsilon \circ \varphi^{-1}$ from $(P_{cl}^+)_l$ to itself. We call it the **associated automorphism** of B.

Let us examine the perfectness of the crystals listed previously. For the crystal $B^{1,s}$ in section 3.1 for $A_{n-1}^{(1)}$, we see by (3.1) that every element is minimal and ε, φ are bijections to $(P_{cl}^+)_s$. So $B^{1,s}$ is perfect of level s. The associated automorphism is given by $\sigma(\sum_i m_i\Lambda_i) = \sum_i m_i\Lambda_{i-1}$ $(\Lambda_{-1} = \Lambda_{n-1})$. Similarly, $B^{r,s}$ in section 3.2 is also perfect of level s for any r $(1 \leq r \leq n-1)$. To describe the minimal elements of $B^{r,s}$, let $x_{ij}(b)$ denote the number of letter j in the i-th row of an $r \times s$ tableau b in $B^{r,s}$ and set $r' = n - r$. Note that $1 \leq i \leq r, i \leq j \leq i + r'$. Then b is minimal, if and only if $x_{ij}(b) = x_{i-1\, j-1}(b)$ for any i, j such that $i + 1 \leq j \leq i + r' - 1$. For such b, ε_i and φ_i are given by

$$\varepsilon_i(b) = \varphi_{i+r}(b) = \begin{cases} x_{rr}(b) & (i = 0), \\ x_{1\, i+1}(b) & (1 \leq i \leq r'), \\ x_{i-r'+1\, i+1}(b) - x_{i-r'\, i}(b) & (r' + 1 \leq i \leq n-1), \end{cases}$$

and the associated automorphism is $\sigma(\sum_i m_i\Lambda_i) = \sum_i m_i\Lambda_{i-r}$. Here the suffix of $\varphi_{i+r}(b)$ or Λ_{i-r} should be considered as in $\mathbb{Z}/n\mathbb{Z}$.

Next let us examine the KKM crystals in section 3.3. Stating our conclusion first, the KKM crystal $B^{1,s}$ is perfect of level s except the $C_n^{(1)}$ case. For $C_n^{(1)}$, $B^{1,s}$ is of level $s/2$ and perfect if s is even, and of level $(s+1)/2$ and not perfect if s is odd. We list below minimal elements of $B^{1,s}$ and the associated automorphism when it is perfect.

(i) Type $\mathfrak{B}$ In all cases, $B^{1,s}$ is perfect of level s.

• $B_n^{(1)}$

$$(B^{1,s})_{\min} = \{(m_1,\dots,m_n,m_0,\bar{m}_n,\dots,\bar{m}_1) \mid m_i,\bar{m}_1 \geq 0, m_0 = 0 \text{ or } 1,$$
$$m_1 + \bar{m}_1 + 2\sum_{i=2}^{n} m_i + m_0 = s\},$$

$$\sigma(\sum_{i=0}^{n} k_i\Lambda_i) = k_1\Lambda_0 + k_0\Lambda_1 + \sum_{i=2}^{n} k_i\Lambda_i.$$

• $D_n^{(1)}$

$$(B^{1,s})_{\min} = \{(m_1,\dots,m_{n-1},m_n,\bar{m}_n,m_{n-1},\dots,\bar{m}_1) \mid m_i,\bar{m}_1,\bar{m}_n \geq 0,$$
$$m_n = 0 \text{ or } \bar{m}_n = 0, m_1 + \bar{m}_1 + 2\sum_{i=2}^{n-1} m_i + m_n + \bar{m}_n = s\},$$

$$\sigma(\sum_{i=0}^{n} k_i\Lambda_i) = k_1\Lambda_0 + k_0\Lambda_1 + \sum_{i=2}^{n-2} k_i\Lambda_i + k_n\Lambda_{n-1} + k_{n-1}\Lambda_n.$$

• $A_{2n-1}^{(2)}$

$$(B^{1,s})_{\min} = \{(m_1,\dots,m_n,m_n,\dots,\bar{m}_1) \mid m_i,\bar{m}_1 \geq 0, m_0 = 0 \text{ or } 1,$$
$$m_1 + \bar{m}_1 + 2\sum_{i=2}^{n} m_i = s\},$$

$$\sigma(\sum_{i=0}^{n} k_i\Lambda_i) = k_1\Lambda_0 + k_0\Lambda_1 + \sum_{i=2}^{n} k_i\Lambda_i.$$

(ii) Type ▭▭

• $C_n^{(1)}$ $\quad lev B^{1,s} = s/2$ (s : even), $(s+1)/2$ (s : odd).

$$(B^{1,s})_{\min} = \begin{cases} \{(m_1,\dots,m_n,m_n,\dots,m_1) \mid m_i \geq 0, \sum_{i=1}^{n} m_i = s/2\} \text{ if } s \text{ is even,} \\ \{(m_1,\dots,m_k+1,\dots,m_n,m_n,\dots,m_1), \\ \quad (m_1,\dots,m_n,m_n,\dots,m_k+1,\dots,m_1) \mid \\ \qquad m_i \geq 0, 1 \leq k \leq n, \sum_{i=1}^{n} m_i = (s-1)/2\} \text{ if } s \text{ is odd.} \end{cases}$$

$B^{1,s}$ is perfect if s is even, and not perfect if s is odd. When it is perfect, the associated automorphism is the identity.

(iii) Type ▭ In all cases, $B^{1,s}$ is perfect of level s.

• $A_{2n}^{(2)}$

$$(B^{1,s})_{\min} = \{(m_1,\dots,m_n,m_n,\dots,m_1) \mid m_i \geq 0, \sum_{i=1}^{n} m_i = s/2\},$$

$$\sigma = \text{id}.$$

- $D^{(2)}_{n+1}$

$$(B^{1,s})_{\min} = \{(m_1,\dots,m_n,m_0,m_n,\dots,m_1) \mid m_i \geq 0, m_0 = 0 \text{ or } 1, \; 2\sum_{i=1}^{n} m_i + m_0 = s\},$$

$$\sigma = \mathrm{id}.$$

We have a conjecture on the level and perfectness of the KR crystal $B^{r,s}$. For $i \in I_0$ set

$$t_i = \begin{cases} \frac{2}{(\alpha_i|\alpha_i)} & \text{if } \mathfrak{g} \text{ is non twisted,} \\ 1 & \text{if } \mathfrak{g} \text{ is twisted.} \end{cases} \tag{4.1}$$

Here $(\cdot \mid \cdot)$ is the invariant bilinear form on P so normalized that the square length of the long root is 2.

Conjecture 4.1. (1) *lev* $B^{r,s}$ *is given by* $\lceil s/t_r \rceil$, *where* $\lceil m \rceil$ *stands for the smallest integer such that* $m \leq \lceil m \rceil$.
(2) $B^{r,s}$ *is perfect if and only if* s/t_r *is an integer.*

In particular, if $\mathfrak{g}$ is of ADE type or twisted, we conjecture any $B^{r,s}$ is perfect.

4.2. **Set of paths.** Let B be a KR crystal. Consider the semi-infinite tensor product of B

$$\cdots \otimes B \otimes \cdots \otimes B \otimes B.$$

An element $\mathbf{p} = \cdots \otimes \mathbf{b}_j \otimes \cdots \otimes \mathbf{b}_2 \otimes \mathbf{b}_1$ of this tensor product is called a **reference path** if $\varphi(\mathbf{b}_{j+1}) = \varepsilon(\mathbf{b}_j)$ holds for any $j \geq 1$. Set

$$\begin{aligned} \mathcal{P}(\mathbf{p}, B) &= \{p = \cdots \otimes b_j \otimes \cdots \otimes b_2 \otimes b_1 \mid b_j = \mathbf{b}_j \text{ for sufficiently large } j\} \\ wt\, p &= \varphi(\mathbf{b}_1) + \sum_{j=1}^{\infty} af(wt\, b_j - wt\, \mathbf{b}_j) - E(p) a_0^{-1}\delta, \\ E(p) &= \sum_{j=1}^{\infty} j(H(b_{j+1} \otimes b_j) - H(\mathbf{b}_{j+1} \otimes \mathbf{b}_j)). \end{aligned}$$

An element of $\mathcal{P}(\mathbf{p}, B)$ is called a **path** and $E(p)$ its **energy**. $\mathcal{P}(\mathbf{p}, B)$ has the structure of P-weighted crystal. In fact, it can be shown ([HKKOT] Theorem 3.7) that $\mathcal{P}(\mathbf{p}, B)$ is isomorphic to a direct sum of crystals of integrable highest weight $U_q(\mathfrak{g})$-modules. It may have infinitely many components.

Now let us assume that B is perfect of level l and take $\lambda \in (P^+_{cl})_l$. Then there exists a unique reference path

$$\mathbf{p}^{(\lambda)} = \cdots \otimes \mathbf{b}_j \otimes \cdots \otimes \mathbf{b}_2 \otimes \mathbf{b}_1$$

such that $\varphi(\mathbf{b}_1) = \lambda$. In fact, $\mathbf{b}_j$ is determined by $\varphi(\mathbf{b}_j) = \sigma^{j-1}\lambda$. Recall that $B(\lambda)$ is the crystal basis of the integrable highest weight $U_q(\mathfrak{g})$-modules with highest weight λ.

Theorem 4.2 ([KMN1]). *We have the following isomorphism of crystals.*

$$\mathcal{P}(\mathbf{p}^{(\lambda)}, B) \simeq B(\lambda)$$

Under the isomorphism, $\mathbf{p}^{(\lambda)}$ *corresponds to the highest weight vector of* $B(\lambda)$.

In [KMN1] $\mathbf{p}^{(\lambda)}$ is called the **ground state** path, since its energy $E(\mathbf{p}^{(\lambda)})$ is equal to 0 (minimum).

The action of $\tilde{e}_i, \tilde{f}_i$ on $\mathcal{P}(\mathbf{p}^{(\lambda)}, B)$ can be computed by using the signature rule explained in section 2.1. See the following example.

Example 4.3. Let B be the perfect crystal of level 2 for $\mathfrak{g} = A_2^{(1)}$ in Example 3.1 (2) and let $\lambda = \Lambda_0 + \Lambda_1$. Then the reference path is given by

$$\mathbf{p}^{(\Lambda_0+\Lambda_1)} = \cdots \otimes 23 \otimes 13 \otimes 12 \otimes 23 \otimes 13.$$

Consider a path

$$p = \cdots \otimes 23 \otimes 23 \otimes 11 \otimes 12 \otimes 23.$$

We wish to calculate $\tilde{e}_1 p$ and $\tilde{f}_1 p$. Take the minimal L such that $b_j = \mathbf{b}_j$ for any $j \geq L$, where b_j (resp. $\mathbf{b}_j$) is the j-th tensor component of p (resp. $\mathbf{p}^{(\Lambda_0+\Lambda_1)}$). (In effect, the minimality of L is not necessary.) In general, the i-signature for p is given by

$$+^{\varphi_i(b_L)} \cdot -^{\varepsilon_i(b_{L-1})} +^{\varphi_i(b_{L-1})} \cdots\cdots -^{\varepsilon_i(b_1)} +^{\varphi_i(b_1)} .$$

In our case, setting $i = 1, L = 5$ we have the 1-signature for p as

$$\cdot \otimes + \otimes + + \otimes - + \otimes -.$$

We then reduce the signature and obtain

$$\cdot \otimes + \otimes + \otimes \cdot \otimes \cdot.$$

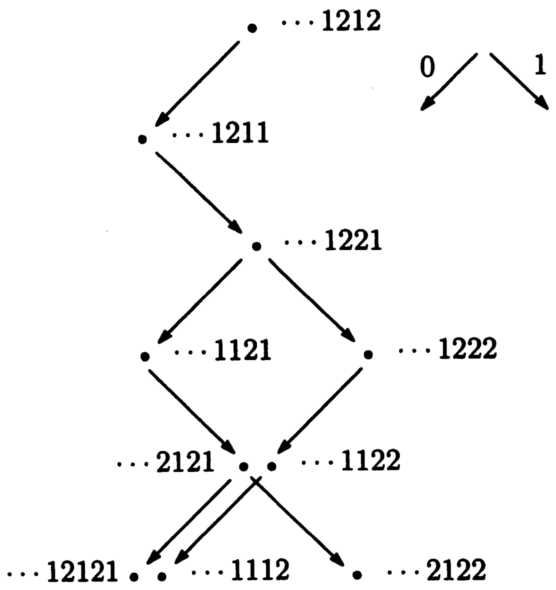
By the rule, we have

$$\begin{aligned} \tilde{e}_1 p &= \cdots \otimes b_5 \otimes \tilde{e}_1 b_4 \otimes b_3 \otimes b_2 \otimes b_1 \\ &= \cdots \otimes 23 \otimes \; 13 \; \otimes 11 \otimes 12 \otimes 23. \end{aligned}$$

Similarly, $\tilde{f}_1 p = 0$.

Figure 3 shows the path realization $\mathcal{P}(\mathbf{p}^{(\Lambda_0)}, B^{1,1})$ of the crystal graph $B(\Lambda_0)$ for $A_1^{(1)}$.

FIGURE 3. Path realization of $B(\Lambda_0)$ for $A_1^{(1)}$



There is a generalization of Theorem 4.2.

Theorem 4.4 ([OSS1]). *Let B_i $(i = 1, 2, \ldots, m)$ be a perfect crystal of level l_i satisfying some technical conditions. Suppose $l_1 \geq l_2 \geq \cdots \geq l_m$ and set $l_{m+1} = 0$. For any sequence of dominant integral weights $(\lambda_1, \ldots, \lambda_m)$ such that $\lambda_i \in (P^+_{cl})_{l_i - l_{i+1}}$, there exists a reference path $\mathbf{p}^{(\lambda_1,\ldots,\lambda_m)}$ and we have the following isomorphism of crystals.*

$$\mathcal{P}(\mathbf{p}^{(\lambda_1,\ldots,\lambda_m)}, B_1 \otimes \cdots \otimes B_m) \simeq B(\lambda_1) \otimes \cdots \otimes B(\lambda_m)$$

Under the isomorphism, $\mathbf{p}^{(\lambda_1,\ldots,\lambda_m)}$ corresponds to $u_{\lambda_1} \otimes \cdots \otimes u_{\lambda_m}$, where u_{λ_i} is the highest weight vector of $B(\lambda_i)$.

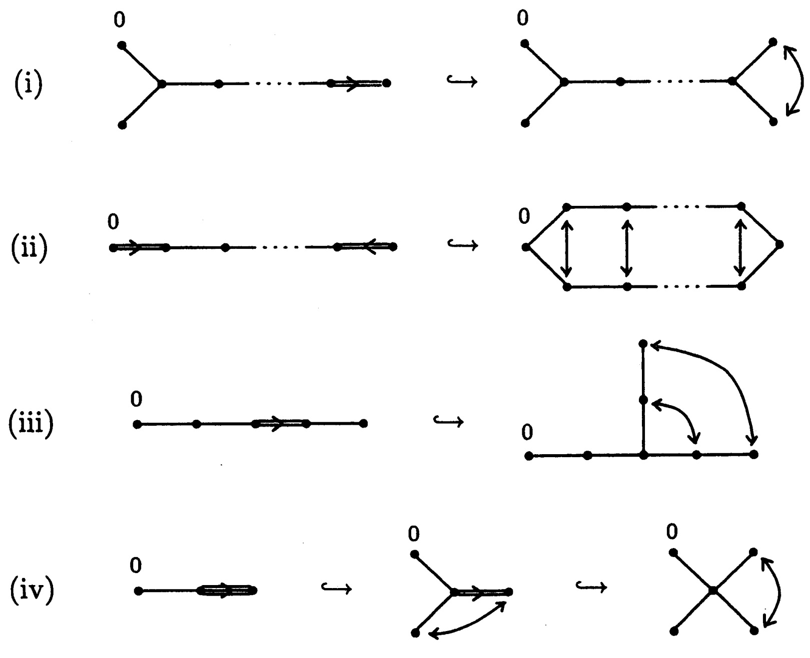
For the "technical conditions" and construction of the reference path $\mathbf{p}^{(\lambda_1,\ldots,\lambda_m)}$, see [OSS1].

One may wonder what happens if some B_i's are not perfect. Even in such a situation, we still have a conjecture. To state it, we prepare notations. Set $t_{\max} = \max_{r \in I_0} t_r$. Note that $t_{\max} > 1$ if and only if there exists a KR crystal which is not perfect. (See Conjecture 4.1.) Later in this subsection we only consider an affine Lie algebra $\mathfrak{g}$ such that $t_{\max} > 1$. For such $\mathfrak{g}$ we define $\mathfrak{g}^{(i)}$ $(1 \leq i < t_{\max})$ by

(i) $\mathfrak{g} = B_n^{(1)} \Longrightarrow \mathfrak{g}^{(1)} = D_{n+1}^{(1)}$,
(ii) $\mathfrak{g} = C_n^{(1)} \Longrightarrow \mathfrak{g}^{(1)} = A_{2n-1}^{(1)}$,
(iii) $\mathfrak{g} = F_4^{(1)} \Longrightarrow \mathfrak{g}^{(1)} = E_6^{(1)}$,
(iv) $\mathfrak{g} = G_2^{(1)} \Longrightarrow \mathfrak{g}^{(1)} = B_3^{(1)}, \mathfrak{g}^{(2)} = D_4^{(1)}$.

Notice that there are embeddings of affine Lie algebras related to the foldings of their Dynkin diagrams as shown in Figure 4.

FIGURE 4. Embeddings of affine Lie algebras



Now consider the integrable $\mathfrak{g}^{(i)}$(not $U_q(\mathfrak{g}^{(i)})$!)-module $V^{(i)}(\lambda)$ with highest weight λ. Through the embedding one can decompose $V^{(i)}(\lambda)$ into irreducible integrable $\mathfrak{g}$-modules as

$$V^{(i)}(\lambda) = \bigoplus_{\mu} V(\mu)^{\otimes n_\mu}.$$

where μ runs through the weight lattice of $\mathfrak{g}$ and n_μ denotes the multiplicity. We then define the $U_q(\mathfrak{g})$-crystal $B^{(i)}(\lambda)$ by

$$B^{(i)}(\lambda)=\bigoplus_{\mu} B(\mu)^{\otimes n_\mu}.$$

Recall that $B(\mu)$ stands for the crystal basis of the integrable $U_q(\mathfrak{g})$-module with highest weight μ. Then our conjecture is stated as follows.

Conjecture 4.5. *Let B_j $(j=1,2,\dots,m)$ be KR crystals. Associated to it one can determine integers m_i $(0\le i<t_{\max})$ and $l_1>l_2>\dots>l_{m_0}>l_{m_0+1}=0$. Let $\hat\lambda$ be a datum*

$$\hat\lambda=\{\lambda_j,\lambda^{(i)}_{k_i}\mid 1\le j\le m_0, 1\le k_i\le m_i \text{ for } 1\le i<t_{\max}\}$$

such that

$$\lambda_j \text{ is a level } l_j-l_{j+1} \text{ dominant integral weight of } \mathfrak{g},$$
$$\lambda^{(i)}_{k_i} \text{ is a level } 1 \text{ dominant integral weight of } \mathfrak{g}^{(i)}.$$

Then for any such datum $\hat\lambda$, there exists a reference path $\mathbf{p}^{(\hat\lambda)}$ and we have the following isomorphism of crystals.

$$\mathcal{P}(\mathbf{p}^{(\hat\lambda)},B_1\otimes\dots\otimes B_m)\simeq\bigotimes_{1\le j\le m_0}B(\lambda_j)\otimes\bigotimes_{\substack{1\le i<t_{\max}\\1\le k_i\le m_i}}B^{(i)}(\lambda^{(i)}_{k_i})$$

See Conjecture 3.6 of [HKOTT] for more details. We give typical examples below.

Example 4.6. (1) Let $\mathfrak{g}=C^{(1)}_n, B=B^{1,s}$ (s:odd). Then $\mathfrak{g}^{(1)}=A^{(1)}_{2n-1}$. Set $l=(s+1)/2$. Let λ (resp. μ) be a level $l-1$ (resp. 1) dominant integral weight of $\mathfrak{g}$ (resp. $\mathfrak{g}^{(1)}$). Then the statement of Conjecture 4.5 in this case is as follows. For any λ,μ there exists a reference path $\mathbf{p}^{(\lambda,\mu)}$ such that

$$\mathcal{P}(\mathbf{p}^{(\lambda,\mu)},B^{1,s})\simeq B(\lambda)\otimes B^{(1)}(\mu).$$

This has been proven in [HKKOT].

(2) Let $\mathfrak{g}=G^{(1)}_2, B=B^{2,s}$ $(s/3\notin\mathbb{Z})$. Then $\mathfrak{g}^{(1)}=B^{(1)}_3$. Let l be an integer such that $s/3<l<s/3+1$. Then we have a similar statement to (1). If we replace $B^{2,s}$ $(s/3\notin\mathbb{Z})$ with $B=B^{2,s}\otimes B^{2,s'}$ $((s+1)/3=(s'+2)/3=:l\in\mathbb{Z})$, we have again a similar statement by replacing $\mathfrak{g}^{(1)}=B^{(1)}_3$ with $\mathfrak{g}^{(2)}=D^{(1)}_4$. These are still open problems.

4.3. **1D sum as a truncated character.** In this subsection we define a one dimensional (1D) sum. It was introduced along the studies of solvable lattice models by Baxter's corner transfer matrix method [B]. There are three kinds of 1D sums: unrestricted, classically restricted and level restricted ones. Here we only consider classically restricted 1D sum, since it is exactly the one which is related to "fermionic formula."

Let B be a tensor product of KR crystals and consider a set of paths $\mathcal{P}(\mathbf{p},B)$ with a suitable reference path $\mathbf{p}$. We introduce a filtration

$$\mathcal{P}_0\hookrightarrow\mathcal{P}_1\hookrightarrow\dots\hookrightarrow\mathcal{P}_L\hookrightarrow\cdots$$

by setting

$$\mathcal{P}_L=\mathcal{P}_L(\mathbf{p},B)=\{p=\dots\otimes b_j\otimes\dots\otimes b_1\in\mathcal{P}(\mathbf{p},B)\mid b_j=\mathbf{b}_j \text{ if } j\ge L+1\}.$$

Here we have set $\mathbf{p}=\dots\otimes\mathbf{b}_j\otimes\dots\otimes\mathbf{b}_1$. We also define the set of classically restricted paths by

$$\mathcal{P}^{cl\ res}(\mathbf{p},B)=\{p\in\mathcal{P}(\mathbf{p},B)\mid \tilde e_ip=0 \text{ for any } i\ne0\}.$$

Its elements consist of highest weight vectors as $U_q(\mathring{\mathfrak{g}})$-crystals. Set

$$\mathcal{P}_L^{cl\,res} = \mathcal{P}_L^{cl\,res}(\boldsymbol{p}, B) = \mathcal{P}^{cl\,res}(\boldsymbol{p}, B) \cap \mathcal{P}_L(\boldsymbol{p}, B).$$

Let λ be a dominant integral weight of $\mathring{\mathfrak{g}}$. We define a **(classically restricted) 1D sum** by

$$X_L(\boldsymbol{p}, B, \lambda) = \sum_{\substack{p \in \mathcal{P}_L^{cl\,res} \\ wt\,p \equiv \lambda \bmod \delta, \Lambda_0}} q^{E(p)}.$$

Recall that $E(p)$ is the energy of p. Apparently, it is a polynomial in q with nonnegative coefficients. As we have mentioned in the previous subsection, the set of paths $\mathcal{P}(\boldsymbol{p}, B)$ for an appropriate reference path $\boldsymbol{p}$ is isomorphic to a tensor product of crystal bases of integrable $U_q(\mathfrak{g})$-modules (Theorem 4.4 and Conjecture 4.5). Let us consider such a situation and let $\mathcal{V}$ denote the $\mathfrak{g}$-module corresponding to its RHS. Namely, we set

$$\begin{aligned}\mathcal{V} &= V(\lambda_1) \otimes \cdots \otimes V(\lambda_m) \text{ for Theorem 4.4,} \\ &= \bigotimes_{1 \le j \le m_0} V(\lambda_j) \otimes \bigotimes_{\substack{1 \le i < t_{\max} \\ 1 \le k_i \le m_i}} V^{(i)}(\lambda_{k_i}^{(i)}) \text{ for Conjecture 4.5.}\end{aligned}$$

Then the following theorem is immediate.

Theorem 4.7.

$$\lim_{L \to \infty} X_L(\boldsymbol{p}, B, \lambda) = \sum_{d \ge 0} (\dim \mathcal{H}_{\lambda + l\Lambda_0 - d\delta}) q^{a_0 d}$$

Here l is the level of $\mathcal{V}$ and $\mathcal{H}_\mu$ is defined by

$$\mathcal{H}_\mu = \{v \in \mathcal{V} \mid e_i v = 0 \textit{ for any } i \neq 0, wt\,v = \mu\}.$$

The RHS is called a branching function for the coset $\mathfrak{g}/\mathring{\mathfrak{g}}$.

5. $X = M$

In this final section we introduce a special 1D sum X and a fermionic formula M. We then state the so called "$X = M$ conjecture" and summarize known results.

5.1. **Definition of X.** Let B_i $(1 \le i \le m)$ be crystals as treated in section 2.3. Namely, we assume for any i, j $(1 \le i, j \le m)$ there exists a combinatorial R $R_{B_i B_j}$ and they satisfy Proposition 2.7. Set $B = B_1 \otimes \cdots \otimes B_m$. Let $b_1 \otimes \cdots \otimes b_m$ be an element of B. For i, j such that $1 \le i, j \le m$, we define $b_j^{(i)} \in B_j$ as follows. First, set $b_j^{(j)} = b_j$. If $i < j$, define it by

$$\begin{array}{ccccc} B_i \otimes \cdots \otimes B_{j-1} \otimes B_j & \xrightarrow{\sim} & B_i \otimes \cdots \otimes B_j \otimes B_{j-1} & \xrightarrow{\sim} & \cdots \\ b_i \otimes \cdots \otimes b_{j-1} \otimes b_j & \mapsto & b_i \otimes \cdots \otimes b_j^{(j-1)} \otimes b'_{j-1} & \mapsto & \cdots \\ & & \cdots & \xrightarrow{\sim} & B_j \otimes B_i \otimes \cdots \otimes B_{j-1} \\ & & \cdots & \mapsto & b_j^{(i)} \otimes b'_i \otimes \cdots \otimes b'_{j-1}. \end{array}$$

If $i > j$, define it by

$$\begin{array}{ccccc} B_j \otimes B_{j+1} \otimes \cdots \otimes B_i & \xrightarrow{\sim} & B_{j+1} \otimes B_j \otimes \cdots \otimes B_i & \xrightarrow{\sim} & \cdots \\ b_j \otimes b_{j+1} \otimes \cdots \otimes b_i & \mapsto & b''_{j+1} \otimes b_j^{(j+1)} \otimes \cdots \otimes b_i & \mapsto & \cdots \\ & & \cdots & \xrightarrow{\sim} & B_{j+1} \otimes \cdots \otimes B_i \otimes B_j \\ & & \cdots & \mapsto & b''_{j+1} \otimes \cdots \otimes b''_i \otimes b_j^{(i)}. \end{array}$$

Now let $b = b_1 \otimes \cdots \otimes b_m, \mathbf{b} = \mathbf{b}_1 \otimes \cdots \otimes \mathbf{b}_m$ be elements of B. We introduce a function $D_{\mathbf{b}} : B \to \mathbb{Z}$ by

$$D_{\mathbf{b}}(b) = \sum_{j=1}^{m} H_{B_j B_j}(b_j^{(m)} \otimes b_j^{(1)}) + \sum_{1\le i<j\le m} H_{B_i B_j}(b_i \otimes b_j^{(i+1)}). \tag{5.1}$$

Remark 5.1. If B_i is connected as a $U_q(\mathring{\mathfrak{g}})$-crystal for any i, the first term of (5.1) becomes independent of an element of B. To see this, recall the following property of crystals. Let B_1, B_2 be crystals and let $b_1 \in B_1, b_2 \in B_2$. Then

$$\tilde{e}_i(b_1 \otimes b_2) = 0 \text{ implies } \tilde{e}_i b_1 = 0.$$

Thus for $b = b_1 \otimes \cdots \otimes b_m \in B, \tilde{e}_i b = 0$ for $i \in I_0$ implies $\tilde{e}_i b_j^{(1)} = 0$ for $i \in I_0$ and $j = 1, \ldots, m$. Since there is only one element b such that $\tilde{e}_i b = 0$ for $i \in I_0$ in each B_j, the claim is correct.

We prepare two lemmas.

Lemma 5.2. *Set $B_{<m} = B_1 \otimes \cdots \otimes B_{m-1}$, then we have*

$$H_{B_{<m}B_m}((b_1 \otimes \cdots \otimes b_{m-1}) \otimes b_m) = \sum_{1\le j\le m-1} H_{B_i B_m}(b_j \otimes b_m^{(j+1)}).$$

Proof. The formula follows from the fact that the combinatorial R $R : \mathit{Aff}(B_{<m}) \otimes \mathit{Aff}(B_m) \to \mathit{Aff}(B_m) \otimes \mathit{Aff}(B_{<m})$ is given by the composition of fundamental ones as

$$R = R_1 \circ R_2 \circ \cdots \circ R_{m-1},$$

where R_j is the combinatorial R that interchanges $\mathit{Aff}(B_j)$ and $\mathit{Aff}(B_m)$ acting on the j-th and $(j+1)$-th components of

$$\mathit{Aff}(B_1) \otimes \cdots \otimes \mathit{Aff}(B_j) \otimes \mathit{Aff}(B_m) \otimes \mathit{Aff}(B_{j+1}) \otimes \cdots \otimes \mathit{Aff}(B_{m-1}).$$

□

Lemma 5.3. *Set $B = B_1 \otimes B_2$, then we have*

$$\begin{aligned} H_{BB}((b_1 \otimes b_2) \otimes (b'_1 \otimes b'_2)) &= H_{B_1B_2}(b_1 \otimes b_2) + H_{B_1B_1}(\tilde{b}_1 \otimes b'_1) \\ &+ H_{B_2B_2}(b_2 \otimes \tilde{b}'_2) + H_{B_1B_2}(b'_1 \otimes b'_2). \end{aligned}$$

Here $\tilde{b}_1, \tilde{b}'_2$ are defined as

$$\begin{array}{ccc} B_1 \otimes B_2 & \stackrel{\sim}{\to} & B_2 \otimes B_1 \\ b_1 \otimes b_2 & \mapsto & \tilde{b}_2 \otimes \tilde{b}_1 \\ b'_1 \otimes b'_2 & \mapsto & \tilde{b}'_2 \otimes \tilde{b}'_1. \end{array}$$

Proof. By identifying $b_1 \otimes b_2$ with $\tilde{b}_2 \otimes \tilde{b}_1$ under $B_1 \otimes B_2 \simeq B_2 \otimes B_1$, we see the LHS is equal to

$$H_{B_2\otimes B_1\ B_1\otimes B_2}((\tilde{b}_2 \otimes \tilde{b}_1) \otimes (b'_1 \otimes b'_2)).$$

Noting that the combinatorial R for interchanging $\mathit{Aff}(B_2 \otimes B_1)$ and $\mathit{Aff}(B_1 \otimes B_2)$ are given by $R_2 \circ R_3 \circ R_1 \circ R_2$ on

$$\mathit{Aff}(B_2) \otimes \mathit{Aff}(B_1) \otimes \mathit{Aff}(B_1) \otimes \mathit{Aff}(B_2),$$

one obtains the RHS. □

Proposition 5.4. *Set $B = B_1 \otimes \cdots \otimes B_m$. Let $\mathbf{b} = \mathbf{b}_1 \otimes \cdots \otimes \mathbf{b}_m, b = b_1 \otimes \cdots \otimes b_m \in B$. Then $H_{BB}(\mathbf{b} \otimes b) - D_{\mathbf{b}}(b)$ is independent of b.*

Proof. We prove by induction on m. When $m = 1$, the difference is 0 and the statement is ok. Next, set $B_{<m} = B_1 \otimes \cdots \otimes B_{m-1}, \boldsymbol{b}_{<m} = \boldsymbol{b}_1 \otimes \cdots \otimes \boldsymbol{b}_{m-1}, b_{<m} = b_1 \otimes \cdots \otimes b_{m-1}$. By Lemma 5.3 we have

$$\begin{aligned} H_{BB}(\boldsymbol{b} \otimes b) &= H_{B_{<m}B_m}(\boldsymbol{b}_{<m} \otimes \boldsymbol{b}_m) + H_{B_{<m}B_{<m}}(\tilde{\boldsymbol{b}} \otimes b_{<m}) \\ &\quad + H_{B_m B_m}(\boldsymbol{b}_m \otimes b_m^{(1)}) + H_{B_{<m}B_m}(b_{<m} \otimes b_m), \end{aligned} \tag{5.2}$$

where $\tilde{\boldsymbol{b}}$ is the 2nd component of the image of $\boldsymbol{b}_{<m} \otimes \boldsymbol{b}_m$ under the crystal isomorphism $B_{<m} \otimes B_m \to B_m \otimes B_{<m}$. By induction hypothesis we know that $H_{B_{<m}B_{<m}}(\tilde{\boldsymbol{b}} \otimes b_{<m}) - D_{\tilde{\boldsymbol{b}}}(b_{<m})$ is independent of $b_{<m}$. Write $\tilde{\boldsymbol{b}} = \tilde{\boldsymbol{b}}_1 \otimes \cdots \otimes \tilde{\boldsymbol{b}}_{m-1} \in B_{<m}$. Note that $\boldsymbol{b}_j^{(m)} = \tilde{\boldsymbol{b}}_j^{(m-1)}$ for $1 \le j \le m-1$ by the Yang-Baxter equation. Using Lemma 5.2 for the last term of the RHS of (5.2) one obtains the desired result. □

We now consider a special case of the 1D sum defined in section 4.3. Let B be a tensor product of KR crystals, i.e., $B = B_1 \otimes \cdots \otimes B_m$ and each B_j is a KR crystal. Set $l_j = lev B_j$ and $l_{\max} = \max_{1 \le j \le m} l_j$. We take a reference path $\boldsymbol{p} = \cdots \otimes \boldsymbol{b}_2 \otimes \boldsymbol{b}_1$ satisfying the following conditions.

(i) $\varphi(\boldsymbol{b}_2) = l_{\max}\Lambda_0$.
(ii) $\varphi(\boldsymbol{c}_j^{(m)}) = l_j\Lambda_0$ for $1 \le j \le m$.

Here we have set $\boldsymbol{b}_2 = \boldsymbol{c}_1 \otimes \cdots \otimes \boldsymbol{c}_m$ ($\boldsymbol{c}_j \in B_j$). We do not go into details on the existence of such a reference path. In all known cases it does exist. Moreover, for a known KR crystal B, an element $b \in B$ such that $\varphi(b) = (lev B)\Lambda_0$ is unique. From (i) one can show that for $p \in \mathcal{P}_1(\boldsymbol{p}, B)$, $\tilde{e}_i p = 0$ $(i \neq 0)$ implies $\tilde{e}_i b = 0$ $(i \neq 0)$, where $b \in B$ is the 1st component of p. Therefore, in this case, using Proposition 5.4 one can rewrite $X_1(\boldsymbol{p}, B, \lambda)$ up to a power of q as follows.

$$X(B, \lambda) = \sum_{\substack{b \in B \\ \tilde{e}_i b = 0\ (i \neq 0),\ wt\, b = \lambda}} q^{D(b)}, \tag{5.3}$$

where

$$D(b) = D_{\boldsymbol{b}_2}(b) = \sum_{j=1}^{m} H_{B_j B_j}(b_j^{\natural} \otimes b_j^{(1)}) + \sum_{1 \le i < j \le m} H_{B_i B_j}(b_i \otimes b_j^{(i+1)}).$$

Here we have set $b = b_1 \otimes \cdots \otimes b_m$ ($b_j \in B_j$) and $b_j^{\natural}$ is the unique element of B_j such that $\varphi(b_j^{\natural}) = (lev B_j)\Lambda_0$. Suppose $B_j = B^{r_j, s_j}$. Then it is easy to see that

$$X(B, \lambda)|_{q=1} = [\bigotimes_{j=1}^{m} W_{s_j}^{(r_j)} : V_\lambda]. \tag{5.4}$$

Here the RHS means the multiplicity of the irreducible highest weight $U_q(\mathring{\mathfrak{g}})$-module V_λ with highest weight λ in $\bigotimes_{j=1}^{m} W_{s_j}^{(r_j)}$. Thus one can consider (5.3) as a q-analog of the multiplicity of the RHS of (5.4).

5.2. **Fermionic formula M.** We first review the q-binomial coefficient. For $m \in \mathbb{Z}_{\ge 0}$ and $p \in \mathbb{Z}$, we set

$$\begin{bmatrix} p+m \\ m \end{bmatrix}_q = \frac{(q^{p+1}; q)_m}{(q; q)_m},$$

where $(x; q)_m = \prod_{j=0}^{m-1}(1 - xq^j)$. It is a polynomial in q with nonnegative coefficients for $p \in \mathbb{Z}_{\ge 0}$, vanishes for $-m \le p \le -1$ and is equal to $(-q^{p+(m+1)/2})^m \begin{bmatrix} -p-1 \\ m \end{bmatrix}_q$ for

$p \le -m-1$. In the $q \to 1$ limit it becomes the usual binomial coefficient $\left(\begin{array}{c} p+m \\ m \end{array}\right) = (p+1)(p+2)\cdots(p+m)/m!$.

We prepare some more notations. We set

$$\overline{P} = \sum_{i \in I_0} \mathbb{Z}\overline{\Lambda}_i, \quad \overline{P}^+ = \sum_{i \in I_0} \mathbb{Z}_{\ge 0}\overline{\Lambda}_i.$$

Here $\overline{\Lambda}_i = \Lambda_i - \langle \Lambda_i, c\rangle \Lambda_0$ is the classical part of Λ_i [Kac]. We regard $\overline{P}$ as the weight lattice of $\mathring{\mathfrak{g}}(\subset \mathfrak{g})$. Let $\mathfrak{g}^\vee$ be the affine Lie algebra whose Cartan matrix is the transpose of that of $\mathfrak{g}$. Recall the definition of t_i (4.1). We define $t_i^\vee$ for $\mathfrak{g}$ to be t_i for $\mathfrak{g}^\vee$. $t_i^\vee = 1$ for $i \in I_0$ for any non twisted affine Lie algebra.

Let ν be a datum $\{\nu_i^{(a)} \in \mathbb{Z}_{\ge 0} \mid a \in I_0, i \ge 1\}$ such that $\sum_{a,i} \nu_i^{(a)} < \infty$. Let λ be an element of $\overline{P}$. Assume $\mathfrak{g} \ne A_{2n}^{(2)}$ for notational simplicity. We define the **fermionic formula** $M(\nu, \lambda)$ by

$$M(\nu,\lambda) = \sum_m q^{c(m)} \prod_{a \in I_0, i \ge 1} \left[\begin{array}{c} p_i^{(a)} + m_i^{(a)} \\ m_i^{(a)} \end{array}\right]_{q_a}, \tag{5.5}$$

$$c(m) = \frac{1}{2} \sum_{a,b \in I_0, i,j \ge 1} (\alpha_a|\alpha_b) \min(t_b i, t_a j) m_i^{(a)} m_j^{(b)} - \sum_{a \in I_0} t_a^\vee \sum_{i,j \ge 1} \min(i,j) \nu_i^{(a)} m_j^{(a)}, \tag{5.6}$$

$$p_i^{(a)} = \sum_{j \ge 1} \nu_j^{(a)} \min(i,j) - \frac{1}{t_a^\vee} \sum_{b \in I_0, j \ge 1} (\alpha_a|\alpha_b) \min(t_b i, t_a j) m_j^{(b)}, \tag{5.7}$$

where q_a is given by

$$q_a = q^{t_a^\vee}.$$

The sum $\sum_m$ is taken over all $(m_i^{(a)} \in \mathbb{Z}_{\ge 0} \mid a \in I_0, i \ge 1)$ satisfying

$$p_i^{(a)} \ge 0 \text{ for any } a \in I_0, i \ge 1, \tag{5.8}$$

$$\textstyle\sum_{a \in I_0, i \ge 1} i m_i^{(a)} \alpha_a = \sum_{a \in I_0, i \ge 1} i \nu_i^{(a)} \overline{\Lambda}_a - \lambda. \tag{5.9}$$

Fundamental properties of M are given in next proposition.

Proposition 5.5. (1) *For a datum ν and $\lambda \in \overline{P}$, there exists a positive integer N such that for any $m = (m_i^{(a)})$ satisfying (5.8) and (5.9) we have $m_i^{(a)} = 0$ for $a \in I_0, i \ge N$.*

(2) $M(\nu,\lambda) \in \mathbb{Z}_{\ge 0}[q^{-1}]$.

(3) $M(\nu,\lambda) = 0$ *unless* $\lambda \in \left(\sum_{a \in I_0, i \ge 1} i\nu_i^{(a)} \overline{\Lambda}_a - \sum_{a \in I_0} \mathbb{Z}_{\ge 0} \alpha_a\right) \cap \overline{P}^+$.

Proof. (1) The constraint (5.9) fixes $\sum_i i m_i^{(a)}$ for any a.
(2) From (1) one sees that M is a Laurent polynomial in q with nonnegative coefficients. To see that it is a polynomial in q^{-1}, it suffices to note the following equality.

$$M(\nu,\lambda;q^{-1}) = \sum_m q^{cc(m)} \prod_{a \in I_0, i \ge 1} \left[\begin{array}{c} p_i^{(a)} + m_i^{(a)} \\ m_i^{(a)} \end{array}\right]_{q_a},$$

$$cc(m) = \frac{1}{2} \sum_{a,b \in I_0, i,j \ge 1} (\alpha_a|\alpha_b) \min(t_b i, t_a j) m_i^{(a)} m_j^{(b)}.$$

(3) $\lambda \in \sum_{a,i} i\nu_i^{(a)}\overline{\Lambda}_a - \sum_a \mathbb{Z}_{\geq 0}\alpha_a$ is clear from (5.9). To show $\lambda \in \overline{P}^+$, it suffices to check that $\langle h_a, \lambda\rangle \geq 0$ for any a. From (5.9) we have

$$\langle h_a, \lambda\rangle = \sum_i i\nu_i^{(a)} - \sum_{b,j} jm_j^{(b)}\langle h_a, \alpha_b\rangle.$$

Now from (5.7)

$$\lim_{i\to\infty} p_i^{(a)} = \sum_j j\nu_j^{(a)} - \sum_{b,j} \frac{t_a}{t_a^\vee}(\alpha_a|\alpha_b)jm_j^{(b)}.$$

Since $h_a = \frac{t_a}{t_a^\vee}\alpha_a$ (see [Kac]), we have the desired result by (5.8). □

We introduce another fermionic formula $\tilde{M}$. It is defined by the formula (5.5-5.7). The only difference from M is that we do not take the constraint (5.8) into account. Namely, we allow $p_i^{(a)}$ to be negative. Proposition 5.5 (1) is valid also for $\tilde{M}$. It is easy to see that $\tilde{M}$ is a Laurent polynomial in q.

5.3. **Conjecture and known results.** We state the $X = M$ **conjecture**.

Conjecture 5.6. *For a datum $\boldsymbol{\nu}$, set $B = \bigotimes_{a\in I_0, i\geq 1}(B^{a,i})^{\otimes \nu_i^{(a)}}$. Let $\lambda \in \overline{P}^+$. Then there exists an integer $n(\boldsymbol{\nu})$ and we have*

(1) $X(B,\lambda) = q^{n(\boldsymbol{\nu})}M(\boldsymbol{\nu},\lambda)$,
(2) $X(B,\lambda) = q^{n(\boldsymbol{\nu})}\tilde{M}(\boldsymbol{\nu},\lambda)$.

(2) is called the "weak" version. (1) and (2) imply the following. Let us call m "unwanted" if $p_i^{(a)}(m) < 0$ for some (a,i). Then the contribution in $\tilde{M}$ from unwanted m cancel out. Note also that if $\lambda \in \overline{P}\setminus\overline{P}^+$, then both sides of (1) are 0. However, $\tilde{M}$ can be non zero even when $\lambda \in \overline{P}\setminus\overline{P}^+$. See Conjecture 4.3 of [HKOTT] for details.

We finish this note by mentioning known results related to the conjecture (1). Firstly, let $\mathfrak{g}$ be of type A. Assume that $\nu_i^{(a)} = 0$ for $a > 1, i \geq 1$ for the datum $\boldsymbol{\nu} = (\nu_i^{(a)})$. It is exactly the case when both X and M agree with the **Kostka polynomial** $K_{\lambda\mu}(q)$. For the Kostka polynomial λ and μ stands for the partitions of the same number of nodes. The correspondence of the partition λ in $M(\boldsymbol{\nu},\lambda)$ is given in the well known manner. Namely, to the partition $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_l)$ $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l \geq \lambda_{l+1} = 0)$ we associate $\sum_{j=1}^l(\lambda_j - \lambda_{j+1})\overline{\Lambda}_j \in \overline{P}^+$. The correspondence of μ and $\boldsymbol{\nu}$ is given by $\nu_i^{(1)}$ =the number of times i occurs in μ. The equality $M(\boldsymbol{\nu},\lambda) = K_{\lambda\mu}$ (up to a power of q) was proven in [KKR, KR2] by constructing an explicit bijection between the combinatorial objects on both sides, tableaux and "rigged configuration." The equality $X(B,\lambda) = K_{\lambda\mu}$ was proven in [NY] by showing that D in (5.1) (see also Remark 5.1) is related to Lascoux-Schützenberger's "charge." For type A, the result is extended to the most general case by [KSS]. See Schilling's contribution of this volume. Apart from type A, not so much is known. If B is a tensor product of the simplest KKM crystal $B^{1,1}$, $X = M$ is proven in [OSS2]. The result is extended to general KKM crystals in [SS] by using "virtual crystal" technique [OSS3]. See also [Sch].

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