## 4. Fundamental groups

In this section we give a review of some background material relating to fundamental groups and covering spaces. Since these serve primarily as illustrative examples, our presentation will be informal, and we omit proofs. We will describe some consequences for free groups.

### 4.1. Definition of fundamental groups.

The fundamental group is an invariant of a topological space. We are not interested here in "pathological" examples. We will generally assume that our spaces are reasonably "nice". For example, manifolds and simplicial complexes are all "nice", and in practice that is all we care about.

Let $X$ be a topological space. Fix a "basepoint" $p \in X$. A *loop* based at $p$ is a path $\alpha : [0,1] \longrightarrow X$ with $\alpha(0) = \alpha(1) = p$. Two such loops, $\alpha, \beta$, are *homotopic* if one can be deformed to the other through other loops; more precisely, if there is a map $F : [0,1]^2 \longrightarrow X$ with $F(t,0) = \alpha(t)$, $F(t,1) = \beta(t)$ and $F(0,u) = F(1,u) = p$ for all $t, u \in [0,1]$. This defines an equivalence relation on the set of paths. Write $[\alpha]$ for the homotopy class of $\alpha$.

Given loops $\alpha, \beta$, write $\alpha * \beta$ for the path that goes around $\alpha$ (twice as fast) then around $\beta$ (i.e. $\alpha * \beta(t)$ is $\alpha(2t)$ for $t \leq 1/2$ and $\beta(2t-1)$ for $t \geq 1/2$) (Figure 4a). Write $[\alpha][\beta] = [\alpha * \beta]$.

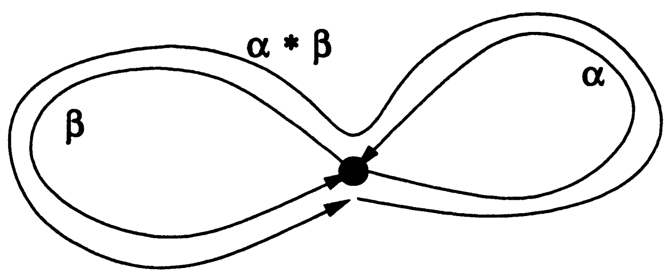


Figure 4a.

**Exercise:** This is well-defined and gives the set of homotopy classes of loops based at $p$ the structure of a group.

**Definition :** We call this group the *fundamental group* of $X$ (based at $p$).

It is denoted by $\pi_1(X, p)$.

The following is not hard to see:

**Fact:** If the points $p, q \in X$ are connected by a path, then $\pi_1(X, p) \cong \pi_1(X, q)$.

To define the isomorphism, given a loop based at $p$, we can obtain a loop based at $q$ by following the path from $q$ to $p$, then going around the loop, and then following the same path back to $q$. We leave the details as an exercise.

**Definition :** A space $X$ is *path connected* if any two points are connected by a path.

Note that any path-connected space is connected, and any "nice" connected space will also be path-connected. (There are counterexamples to the latter statement, but these are not nice.)

Thus if $X$ is path-connected, then the fundamental group is well-defined up to isomorphism. It is denoted $\pi_1(X)$. Clearly homeomorphic spaces have isomorphic fundamental groups.

**Examples.**

(1) $\pi_1(\text{point}) = \{1\}$.

(2) $\pi_1(S^1) = \mathbf{Z}$.

(3) $\pi_1(S^1 \times [0, 1]) = \mathbf{Z}$.

(4) $\pi_1(S^1 \times \mathbf{R}) = \mathbf{Z}$.

(5) $\pi_1(\text{torus}) = \mathbf{Z}^2$. This is generated by two (homotopy classes of) loops, $a, b$, on the torus that cross just once at $p$ (Figure 4b). It is easily seen that $aba^{-1}b^{-1} = 1$. This gives us a presentation of $\mathbf{Z}^2$. Of course, a lot more work would be needed to prove directly that

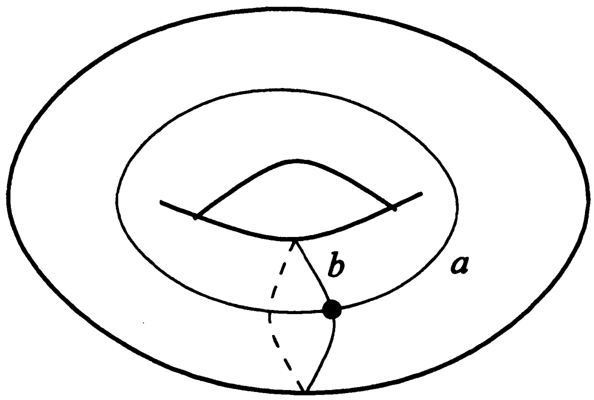


Figure 4b.

this is actually the fundamental group. This will follow from a result stated below.

(6) $\pi_1$(figure of eight) $= F_2$. The two generators correspond to the two loops. This time there is no relation.

(7) More generally the fundamental group of a wedge of circles is free. (A "wedge" is constructed by gluing together a collection of spaces at a single point.) For a finite wedge of $n$ circles, we get $F_n$. A wedge of 5 circles is illustrated in Figure 4c. A "figure of eight" is a wedge of 2 circles.

**Definition :** A space $X$ is *simply connected* if it is path-connected and $\pi_1(X)$ is trivial.

This means that every closed path bounds a disc. More precisely, if $D$ is the unit disc in $\mathbf{R}^2$ then any map $f : \partial D \longrightarrow X$ extends to a map $f : D \longrightarrow X$.

**Examples:** A point is simply connected! So is $\mathbf{R}^n$ for any $n$. So is any tree. So is the $n$-sphere for any $n \geq 2$.

Suppose $X$ is "nice" and $Y \subseteq X$ is a "nice" closed simply connected subset. Then it turns out that the fundamenal group is un-

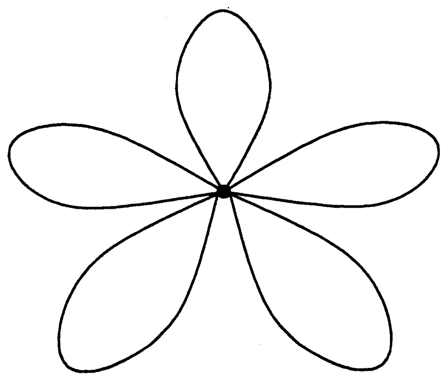

Figure 4c.

changed by collapsing $Y$ to a point.

For example, if $X$ is a graph, and $Y$ is a subtree (a subgraph that is a tree) then we can collapse $Y$ to a single vertex and get another graph. If we take $Y$ to be a maximal tree, we get a wedge of circles (Figure 4d).

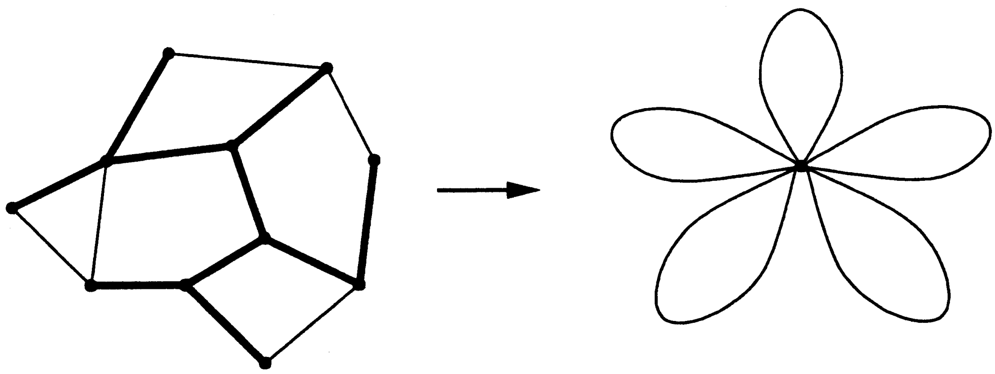

Figure 4d.

Since a maximal tree always exists (if you believe the Axiom of Choice in the case of an infinite graph) we can deduce the following facts:

**Facts:**

(1) The fundamental group of a graph is free.

(2) The fundamental group of a finite graph is $F_n$ for some $n$.

## 4.2. Covering spaces.

Covering spaces give an alternative viewpoint on fundamental groups for nice spaces. Indeed the discussion here could serve to give an equivalent definition of fundamental group, suited to our purposes. We shall not approach the subject systematically here. We just say enough to give the general ideas needed for subsequent applications.

Suppose that a group $\Gamma$ acts properly discontinuously on a proper space $X$. For our purposes here, we can take $X$ to be a proper geodesic space and assume that $\Gamma$ acts by isometry. We shall also assume the action is free (i.e. no element of $\Gamma$ has any fixed points.) This is yet another usage of "free". Here it means there are no fixed points, i.e. if $gx = x$ for some $x \in X$, then $g = 1$. We can form the quotient space $X/\Gamma$. The space $X$, together with the quotient map to $X/\Gamma$ is a (particular) example of a "covering space" as we discuss below.

Of particular interest is the case where $X$ is simply connected. In this case, we have the following:

**Fact:** In the above situation, $\pi_1(X/\Gamma) \cong \Gamma$.

The isomorphism can be seen by fixing some $p \in X$. Given $g \in \Gamma$, connect $p$ to $gp$ by some path $\alpha$ in $X$. (Since $X$ is simply connected, it doesn't really matter which one.) This projects to a loop in $X/\Gamma$ with a fixed basepoint, and hence determines an element of $\pi_1(X/\Gamma)$. This gives a homomorphism from $\Gamma$ to $\pi_1(X/\Gamma)$, which turns out to be an isomorphism.

Conversely, given a nice space, $Y$, one can construct a simply connected space $X$, and free p.d. action of $\Gamma = \pi_1(Y)$ on $X$ such that $Y = X/\Gamma$. Then, $X$ is called the *universal cover* of $Y$. (It is well-defined up to homeomorphism.) It is often denoted $\tilde{Y}$.

**Examples.**
(1) $\Gamma = \mathbf{Z}$. $X = \mathbf{R}$, $Y = S^1$.
(2) $\Gamma = \mathbf{Z}$, $X = \mathbf{R} \times [0,1]$, $Y = S^1 \times [0,1]$.
(3) $\Gamma = \mathbf{Z}$, $X = \mathbf{R}^2$, $Y = S^1 \times \mathbf{R}$.
(4) $\Gamma = \mathbf{Z}^2$, $X = \mathbf{R}^2$, $Y = S^1 \times S^1$ is the torus.
(5) $\Gamma = F_n$, $X = T_{2n}$, $Y$ is a wedge of $n$ circles. We are taking the action of $F_n$ on its Cayley graph, $T_{2n}$, which here happens to be the universal cover.

Note that if $G \leq \Gamma$ is a subgroup, we also get a natural map from $Z = X/G$ to $Y = X/\Gamma$. This is a more general example of a "covering space". Formally we say that a map $p : Z \longrightarrow Y$ is a *covering map* if every point $y \in Y$ has a neighbourhood $U$ such that if we restrict $p$ to any connected component of $p^{-1}U$ we get a homeomorphism of this component to $U$. In this situation, $Z$ is called a *covering space*. We will not worry too much about this formal definition here. The following examples illustrate the essential points:

(1) Consider the action of $\mathbf{Z}$ on $\mathbf{R}$, and the subgroup $n\mathbf{Z} \leq \mathbf{Z}$. We get a covering $\mathbf{R}/n\mathbf{Z} \longrightarrow \mathbf{R}/\mathbf{Z}$. This is a map from the circle to itself wrapping around $n$ times.
(2) We get a covering map of the cylinder to the torus, $\mathbf{R}^2/\mathbf{Z} \longrightarrow \mathbf{R}^2/\mathbf{Z}^2$.

The main point to note is that the fundamental group of a cover is a subgroup of the fundamental group of the quotient. If both spaces are compact, then the subgroup will have finite index. These statements can be thought of in terms of the actions on the universal cover.

**Exercise.** If $G$ happens to be normal in $\Gamma$, then there is a natural action of the group $\Gamma/G$ on $Z$, and $Y$ can be naturally identified as the quotient of $Z$ by this action. The covering space $Z \longrightarrow Y$ is then the quotient map.

If $G$ is not normal, then the cover will not arise from a group action; so the notion of a covering space is more general than that of a free p.d. group action. An example described at the end of this section illustrates this.

## 4.3. Applications to free groups.

As well as being useful to illustrate later results, these constructions have implications for free groups.

**Theorem 4.1** *Any subgroup of a free group is free.*

**Proof :** Suppose $F$ is free, and $G \leq F$. Its universal cover, $X$, is a tree. (It will only be a proper space if $F$ is finitely generated, but that doesn't really matter here.) Now $G$ acts on $T$ and $T/G$ is a graph. By the earlier discussion, $G \cong \pi_1(T/G)$ is free. ◊

This is a good example of a result that is relatively easy by topological/geometric means, but quite hard to prove by direct combinatorial means.

We note that $G$ need not be f.g. even if $F$ is. As an example consider the $F_2 = \langle a, b\rangle$ and let $G = \langle \{b^n ab^{-n} \mid n \in \mathbf{Z}\}\rangle$. In this case, the covering space, $K = T_4/G$, is the real line with a loop attached to each integer point (Figure 4e).

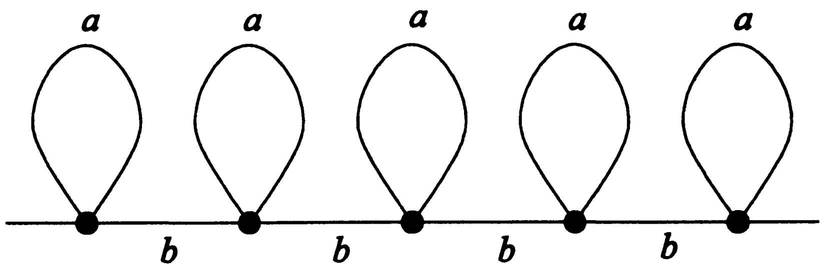


Figure 4e.

Collapsing the real line to a point we get an infinite wedge of circles, and so $G$ is free on an infinite set, and so cannot be finitely generated (this was an exercise in Section 1). In fact, $\{b^n ab^{-n} \mid n \in$

$\mathbf{Z}\}$ is a free generating set. This is one of the simplest examples of an infinitely generated subgroup of a finitely generated group.

**Exercise.** $G = \langle\langle a \rangle\rangle$. In particular, $G \lhd F$. In fact, $G$ corresponds to the set of words in $a, a^{-1}, b, b^{-1}$ with the same number of $b$'s and $b^{-1}$'s. Writing $J = F/G$, we have $J \cong \mathbf{Z}$. Now $J$ acts by translation on the graph $K$, and the quotient graph, $K/J$, is a "figure of eight", which is naturally identified with $T_4/F$.
The map $K \longrightarrow K/J$ is another example of a covering space.

The subgroup $H = \langle a, bab^{-1}, b^2ab^{-2}, \ldots \rangle$ is also infinitely generated (but not normal). In this case, the covering space $T_4/H$ is a bit more complicated, and the covering map to the figure of eight does not arise from a group action.

**Theorem 4.2** *If $p, q \geq 2$, then $F_p \approx F_q$.*

**Proof :** Let $K_n$ be the graph obtained by taking the circle, $\mathbf{R}/n\mathbf{Z}$, and attaching a loop at each point of $\mathbf{Z}/n\mathbf{Z}$ — that is $n$ additional circles (see Figure 4f, where $n = 5$).

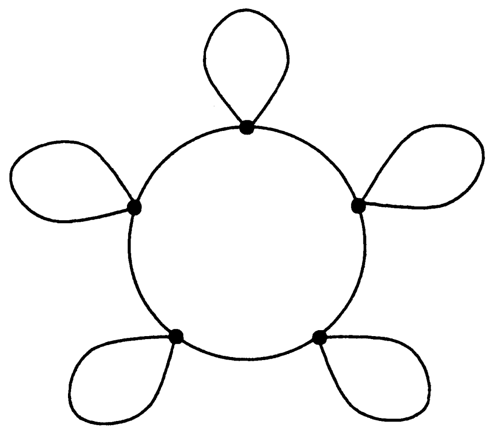

Figure 4f.

We can collapse down a maximal subtree of $K_n$ to give us a wedge of $n+1$ circles. Thus $\pi_1(K_n) = F_{n+1}$. (The universal cover of $K_n$ is $T_4$.) We also note that for any $m \in \mathbf{N}$, $K_{mn}$ is a cover of $K_n$.

Now given $p, q \geq 2$, set $r = pq - p - q + 2 = (p-1)(q-1)+1$, and note that $K_{r-1}$ covers both $K_{p-1}$ and $K_{q-1}$. Since these are all compact, we see that $F_r$ is a finite index subgroup of both $F_p$ and $F_q$. ♢

This proves something we commented on earlier, namely that two f.g. free groups are q.i. if and only if they are commensurable. There are three classes: $F_0 = \{1\}$, $F_1 = \mathbf{Z}$, and $F_n$ for $n \geq 2$.