Introduction

The main purpose of this work is to classify del Pezzo surfaces with log-terminal singularities of index one or two. By classification, we understand a description of the intersection graph of all exceptional curves on an appropriate (called *right*) resolution of singularities together with the subgraph of the curves which are contracted to singular points.

The final results are similar to classical results about classification of non-singular del Pezzo surfaces and use the usual finite root systems. However, the intermediate considerations use K3 surfaces and reflection groups in hyperbolic spaces.

The work is self-contained and can also serve as an introduction to del Pezzo and K3 surfaces. It is based on our paper [AN88]. See also [AN89] for a short exposition of these results.

In this work, we *consider algebraic varieties over the field* $\mathbb{C}$ *of complex numbers,* and do not mention this further.

0.1. Historical remarks and our main principle of classification of log del Pezzo surfaces of index ≤ 2

A complete algebraic surface Z with log terminal singularities is a **del Pezzo surface** if its anticanonical divisor $-K_Z$ is ample. A 2-dimensional **log terminal singularity over** $\mathbb{C}$ is a singularity which is analytically equivalent to a quotient singularity $\mathbb{C}^2/G$, where $G \subset \mathrm{GL}(2, \mathbb{C})$ is a finite subgroup. The index i of $z \in Z$ is the minimal positive integer for which the divisor iK_Z is a Cartier divisor in a neighbourhood of z.

The aim of this work is to classify del Pezzo surfaces with log terminal singularities (or simply **log del Pezzo surfaces**) of index ≤ 2.

Log del Pezzo surfaces of index ≤ 2 include classical cases of non-singular del Pezzo surfaces and log del Pezzo surfaces of index 1, i.e.

Gorenstein log del Pezzo surfaces. Let us recall some classical results about these del Pezzo surfaces.

In 1849, Cayley [Ca1849] and Salmon [Sa1849] discovered 27 lines on a non-singular cubic surface Z. Now we know that they are all exceptional curves on a non-singular del Pezzo surface Z of degree 3, and that they are crucial for its geometry. Here, the degree d of a del Pezzo surface Z is $d = (K_Z)^2$.

Classification of nonsingular del Pezzo surfaces is well known, and they are classical examples of rational surfaces (see, e.g. [Nag60, Man86, MT86]). A connection between nonsingular del Pezzo surfaces and reflection groups was noticed a long time ago. Schoutte [Sch10] noted that there is an incidence-preserving bijection between 27 lines on a smooth cubic and vertices of a certain polytope in $\mathbb{R}^6$. In modern terminology, this polytope is the convex hull of an orbit of reflection group $W(E_6)$. Coxeter [Cox28] and Du Val [DV33] noted a similar correspondence between (-1)-curves on del Pezzo surfaces of degree 2 and 1 and reflection polytopes for groups $W(E_7)$ and $W(E_8)$.

Du Val was the first to investigate the relationship between reflection groups and singular surfaces. In [DV34a] he introduced Du Val singularities. Possible singularities of cubic surfaces $Z_3 \subset \mathbb{P}^3$ were classified by Schläfli [Sc1863] and Cayley [Ca1869]. In [DV34b] Du Val found all possible configurations of Du Val singularities on the "surfaces of del Pezzo series" of degree 2 and 1, i.e. double covers $Z_2 \to \mathbb{P}^2$ ramified in a quartic and double covers $Z_1 \to Q$ over a quadratic cone ramified in an intersection of Q with a cubic. As was proved much later [Dem80, HW81], these are precisely the Gorenstein log del Pezzo surfaces of degree 2 and 1.

Du Val observed the following amazing fact: the configurations of singularities on del Pezzo surfaces Z_d of the degree d with Du Val singularities are in a one-to-one correspondence with subgroups generated by reflections (i.e. root subsystems) of a reflection group of type E_{9-d}, i.e. $E_8, E_7, E_6, D_5, A_4, A_2 + A_1$ respectively for $d = 1, \dots, 6$, with four exceptions: $8A_1$, $7A_1$, $D_4 4A_1$ for $d = 1$ and $7A_1$ for $d = 2$. (These days, we know that the prohibited cases do appear in characteristic 2.) He also noted that in some cases (for example $4A_1$ in E_8) there are two non-conjugate ways to embed a subgroup and, on the other hand, there are two distinct deformation types of surfaces.

The proof was by comparing two long lists. The reflection subgroups were conveniently classified by Coxeter [Cox34] in the same 1934 volume of Proceedings of Cambridge Philosophical Society. Du Val went through all possibilities for quartics on $\mathbb{P}^2$ and sextic curves on the quadratic cone Q and computed the singularities of the corresponding double covers Z_d,

$d = 1, 2$. The modern explanation for the fact that configurations of singularities correspond to some reflection subgroups is simple: (-2)-curves on the minimal resolution Y of a Gorenstein del Pezzo Z lie in the lattice $(K_Y)^{\perp}$ which is a root lattice of type E_{9-d}.

In the 1970s, Gorenstein del Pezzo surfaces attracted new attention in connection with deformations of elliptic singularities, see [Loo77, Pin77, BW79]. The list of possible singularities was rediscovered and reproved using modern methods, see [HW81, Ura83, BBD84, Fur86].

In addition, Demazure [Dem80], and Hidaka and Watanabe [HW81] established a fact which Du Val intuitively understood but did not prove, lacking modern definitions and tools: the minimal resolutions Y_d of Gorenstein log del Pezzo surfaces $Z_d \neq \mathbb{P}^1 \times \mathbb{P}^1$ are precisely the blowups of $9 - d$ points on $\mathbb{P}^2$ in "almost general position", and Z_d is obtained from such blowup by contracting all (-2)-curves.

In addition to clarifying, unifying and providing new results for the index 1 case, our methods are general enough to obtain similar results in the much more general case of log del Pezzo surfaces Z of index ≤ 2. Thus, we admit log terminal singularities of index 1 and index 2 as well. Classification of the much larger class of log del Pezzo surfaces of index ≤ 2 (together with the described above classical index-1 case) is the subject of our work.

By classification, we understand a description of the dual graphs of all exceptional curves (i. e. irreducible with negative self-intersection) on an appropriate resolution of singularities $\sigma : Y \to Z$, together with the subset of curves contracted by σ. We call σ the *right resolution.* See Section 0.3 below for the precise definition. For Gorenstein, i. e. of index 1 singularities, σ is simply the minimal resolution.

Thus, in the principle of classification we follow the classical discovery by Cayley [Ca1849] and Salmon [Sa1849] of 27 lines on a non-singular cubic surface which we have mentioned above.

The dual graph of exceptional curves provides complete information about the surface. Indeed, knowing the dual graph of exceptional curves on Y, we can describe all the ways to obtain Y and Z by blowing up $Y \to \overline{Y}$ from the relatively minimal rational surfaces $\overline{Y} = \mathbb{P}^2$ or $\mathbb{F}_n$, $n = 0, 2, 3 \ldots$. Images of exceptional curves on Y then give a configuration of curves on $\overline{Y}$ related with these blow ups. Vice versa, if one starts with a "similar" configuration of curves on $\overline{Y}$ and performs "similar" blowups then the resulting surface Y is guaranteed to be the right resolution of a log del Pezzo surface Z of index ≤ 2, by Theorem 3.20.

In the singular case of index 1, we add to the classical results which were described above a description of all graphs of exceptional curves on the minimal resolution of singularities. In the case of $\operatorname{Pic} Z = \mathbb{Z}$ this was done by Bindschadler, Brenton and Drucker [BBD84].

0.2. Classification of log del Pezzo surfaces of index ≤ 2 and K3 surfaces

The main method for obtaining our classification of log del Pezzo surfaces of index ≤ 2 is to reduce it to a classification of K3 surfaces with non-symplectic involution and to K3 surfaces theory. The main points of the latter are contained in [Nik80a, Nik79, Nik83, Nik84a, Nik87].

In Chapter 1, we show that on each log del Pezzo surface Z of index ≤ 2 the linear system $|-2K_Z|$ contains a nonsingular curve, and that there exists an appropriate ("right") resolution of singularities $\sigma : Y \to Z$ for which the linear system $|-2K_Y|$ contains a nonsingular divisor C (i.e. Y is a right DPN surface) such that the component of C that belongs to $\sigma^*|-2K_Z|$ has genus ≥ 2 (i.e. the DPN surface Y is of elliptic type).

In Chapter 2, following [Nik80a, Nik79, Nik83, Nik84a, Nik87], we build a general theory of DPN surfaces Y. Here, we use the fact that the double cover X of Y branched along C is a K3 surface with a non-symplectic involution θ. In this way, the classification of DPN surfaces Y and DPN pairs (Y, C) is equivalent to the classification of K3 surfaces with non-symplectic involution (X, θ). The switch to K3 surfaces is important because it is easy to describe exceptional curves on them and there are powerful tools available: the global Torelli Theorem [PS-Sh71] due to Piatetsky-Shapiro and Shafarevich, and surjectivity of the period map [Kul77] due to Vik. Kulikov.

In Chapter 3, we extend this theory to the classification of DPN surfaces Y of elliptic type, i.e. when one of the components of C has genus ≥ 2, by describing dual diagrams of exceptional curves on Y. See Theorems 3.18, 3.19 and 3.20. In Section 3.6, we give an application of this classification to a classification of curves D of degree 6 on $\mathbb{P}^2$ (and $D \in |-2K_{\mathbb{F}_n}|$ as well) with simple singularities, in the case when one of components of D has geometric genus ≥ 2.

In obtaining results of Chapters 2 and 3, a big role is played by the arithmetic of quadratic forms and by reflection groups in hyperbolic spaces which are very important in the theory of K3 surfaces. From this point of view, the success of our classification hinges mainly on the fact that we explicitly describe some hyperbolic quadratic forms and their subgroups generated by all reflections (2-elementary even hyperbolic lattices of small rank, see Theorem 3.1). These computations are also important by themselves for the arithmetic of quadratic forms.

In Chapter 4, the results of Chapters 1—3 are applied to the classification of log del Pezzo surfaces of index ≤ 2. In particular, we show that there are exactly 18 log del Pezzo surfaces of index 2 with Picard number 1. For completeness, we also included the list of the isomorphism classes in the index 1 Picard number 1 case. This list, which for the most difficult degree 1 case can be deduced from [MP86], is skipped or given with some inaccuracies in other references.

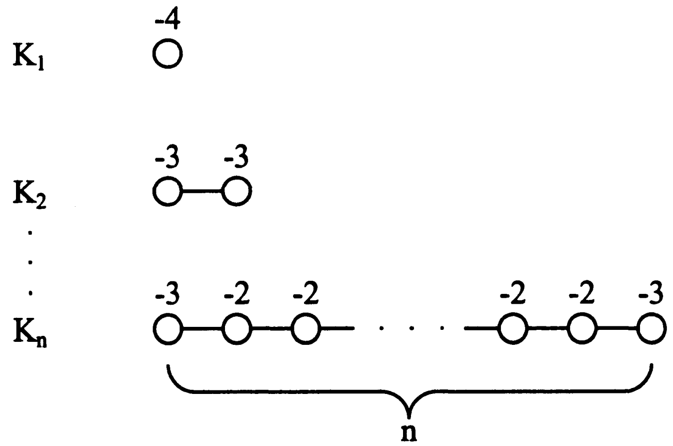
In Section 4.3, following [BBD84], we give an application of our classification to describe some rational compactifications of certain affine surfaces. In Section 4.4, we give formulae for the dimension of moduli spaces of log del Pezzo surfaces of index ≤ 2.

In Section 2.2 we review results about K3 surfaces over $\mathbb{C}$ which we use. In Appendix, for reader's convenience, we review known results about lattices, discriminant forms of lattices, non-symplectic involutions on K3 which we use (see Sections A.1— A.3). For instance, in Section A.2 we review the classification of main invariants (r, a, δ) (see below) of non-symplectic involutions on $K3$ and their geometric interpretation, which are very important in this work. In Section A.4 we give details of calculations of fundamental chambers of hyperbolic reflection groups which were skipped in the main part of the work. They are very important by themselves. Thus, except for some standard results from Algebraic Geometry (mainly about algebraic surfaces), and reflection groups and root systems, our work is more or less self-contained.

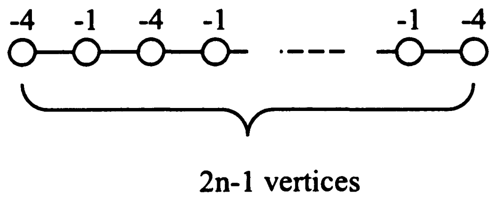
0.3. Final classification results for log del Pezzo surfaces of index ≤ 2

Below, we try to give an explicit and as elementary exposition as possible of our final results on classification of log del Pezzo surfaces of index ≤ 2. In spite of importance of K3 surfaces, in the final classification results K3 surfaces disappear, and it is possible to formulate all results in terms of only del Pezzo surfaces and their appropriate non-singular models, which are DPN surfaces.

Let Z be a log del Pezzo surface of index ≤ 2. Its singularities of index 1 are **Du Val singularities** classified by their minimal resolution of singularities. They are described by Dynkin diagrams A_n, D_n or E_n, with each vertex having weight -2. Singularities of Z of index 2 are singularities K_n which have minimal resolutions with dual graphs shown below:



To get the **right resolution** of singularities $\sigma : Y \to Z$, additionally one has to blow up all points of intersection of components in preimages of singular points K_n. Then the right resolution of a singular point K_n is described by the graph



In these graphs every vertex corresponds to an irreducible non-singular rational curve F_i with F_i^2 equal to the weight of the vertex. Two vertices are connected by an edge if $F_i \cdot F_j = 1$, and are not connected if $F_i \cdot F_j = 0$. Thus, the right resolution of singularities $\sigma : Y \to Z$ of a log del Pezzo surface of index ≤ 2 is obtained by taking the minimal resolutions of the singular points of index 1 and the right resolutions, as in the above figure, of the singular points K_n of index 2.

Our classification of log del Pezzo surfaces Z of index ≤ 2 and the corresponding DPN surfaces of elliptic type which are right resolutions of singularities of Z (i. e. they are appropriate non-singular models of the del Pezzo surfaces) is contained in Table 3 (see Section 3.5).

All cases of Table 3 are labelled by a number $1 \leq N \leq 50$. For $N = 7, 8, 9, 10, 20$ we add some letters and get cases: 7a,b, 8a–c, 9a–f, 10a–m, 20a–d. Thus, altogether, Table 3 contains

$$50 + (2-1) + (3-1) + (6-1) + (13-1) + (4-1) = 73$$

cases.

The labels $N = 1, \ldots, 50$ enumerate the so-called **main invariants** of log del Pezzo surfaces Z. They are triplets (r, a, δ) (equivalently $(k = (r-a)/2, g = (22-r-a)/2, \delta)$) where r, a, δ are integers: $r \geq 1, a \geq 0$,

$\delta \in \{0,1\}$, $g \geq 2$, $k \geq 0$. Thus, there exist exactly 50 possibilities for the main invariants (r, a, δ) (equivalently, (k, g, δ)) of log del Pezzo surfaces of index ≤ 2.

The main invariants have a very important geometric meaning. Any log del Pezzo surface Z of index ≤ 2 and its right resolution of singularities Y are rational. The number

$$r = \operatorname{rk} \operatorname{Pic} Y$$

is the Picard number of Y, i. e. $\operatorname{Pic} Y = \mathbb{Z}^r$. We prove that $|-2K_Z|$ contains a non-singular irreducible curve C_g of genus $g \geq 2$ which explains the geometric meaning of g. This is equivalent to saying that there is a curve

$$C = C_g + E_1 + \cdots + E_k \in |-2K_Y|,$$

where E_i are all exceptional curves on Y with $(E_i)^2 = -4$. (The inequality $g \geq 2$ means that Y is of **elliptic type**). All these curves E_i come from the right resolution of singularities of Z described above. Thus, the invariant k equals the number of exceptional curves on Y with square -4.

All of them are nonsingular and rational. E.g., $k = 0$ if and only if Z is Gorenstein and all of its singularities are Du Val, see Chapter 1.

Let us describe the invariant $\delta \in \{0,1\}$. The components C_g, $E_1, \ldots,$ E_k are disjoint. Since C is divisible by 2 in $\operatorname{Pic} Y$, it defines a double cover $\pi : X \to Y$ ramified in C. Let θ be the involution of the double cover. Then the set of fixed points $X^\theta = C$. Here, X is a K3 surface and

$$(1) \qquad \delta = 0 \iff X^\theta \sim 0 \mod 2 \text{ in } H_2(X, \mathbb{Z}) \iff$$

there exist signs $(\pm)_i$ for which

$$(2) \qquad \frac{1}{4} \sum_i (\pm)_i cl(C^{(i)}) \in \operatorname{Pic} Y,$$

where $C^{(i)}$ are all irreducible components (i. e. C_g, $E_1, \ldots, E_k$) of C.

As promised, our classification describes all intersection (or dual) graphs $\Gamma(Y)$ of exceptional curves on Y and also shows exceptional curves which must be contracted by $\sigma : Y \to Z$ to get the log del Pezzo surface Z of index ≤ 2 from Y. All of these graphs can be obtained from graphs Γ in the right column of Table 3 of the same main invariants (r, a, δ). Let us describe this in more details.

All exceptional curves E on Y are irreducible, non-singular, and rational. They are of three types:

(1) $E^2 = -4$, equivalently E is a component of genus 0 of $C \in |-2K_Y|$. In the graphs of Table 3 these correspond to double transparent vertices;
(2) $E^2 = -2$. In the graphs of Table 3 these correspond to black vertices;

(3) $E^2 = -1$ (the 1st kind). In the graphs of Table 3 these correspond to simple transparent vertices.

All exceptional curves E_i with $(E_i)^2 = -4$, $i = 1, \dots k$, together with all exceptional curves F of the 1st kind such that: there exist two different curves E_i, E_j, $i \neq j$, with $(E_i)^2 = (E_j)^2 = -4$ and $F \cdot E_i = F \cdot E_j = 1$, define $\operatorname{Log}\Gamma(Y) \subset \Gamma(Y)$, the **logarithmic part of surface Y.** Since $F \cdot (-2K_Y) = F \cdot C = 2$, the curves F are characterized by the property $C_g \cdot F = 0$. The logarithmic part $\operatorname{Log}\Gamma(Y)$ can be easily seen on graphs Γ of Table 3: curves E_i, $i = 1, \dots, k$, are shown as double transparent vertices, the curves F of the first kind of $\operatorname{Log}\Gamma(Y)$ are shown as simple transparent vertices connected by two edges with (always two) double transparent vertices. This part of Γ is denoted by $\operatorname{Log}\Gamma$ and is also called the **logarithmic part of graph Γ.** Thus, we have:

$$\operatorname{Log}(\Gamma(Y)) = \operatorname{Log}\Gamma \tag{3}$$

(with the same main invariants (r, a, δ)). *The logarithmic part* $\operatorname{Log}\Gamma(Y)$ *gives precisely the preimage of singular points of Z of index two.*

All exceptional curves E on Y with $E^2 = -2$ define $\operatorname{Duv}\Gamma(Y) \subset \Gamma(Y)$ of $\Gamma(Y)$, the **Du Val part of surface Y.** Its connected components are Dynkin graphs A_n, D_n or E_n and they correspond to all Du Val singularities of Z. Thus *the Du Val part* $\operatorname{Duv}\Gamma(Y) \subset \Gamma(Y)$ *gives precisely the preimage of all Du Val (i. e. of index one) singular points of Z.* For each of the graphs of Table 3, the **Du Val part of graph Γ**, is defined by all of its black vertices. We have:

$$D = \operatorname{Duv}\Gamma(Y) \subset \operatorname{Duv}\Gamma \tag{4}$$

(for the same main invariants (r, a, δ)). *Any subgraph D of* $\operatorname{Duv}\Gamma$ *can be taken.*

Let us describe the remaining part of $\Gamma(Y)$. Each graph Γ of Table 3 defines a lattice S_Y in the usual way. It is

$$S_Y = \left(\bigoplus_{v \in V(\Gamma)} \mathbb{Z}e_v \right) / \operatorname{Ker}$$

defined by the intersection pairing: $e_v^2 = -1$, if v is simple transparent, $e_v^2 = -2$, if v is black, $e_v^2 = -4$, if v is double transparent, $e_v \cdot e_{v'} = m$ if the vertices $v \neq v'$ are connected by m edges. Here, $\oplus$ means the direct sum of $\mathbb{Z}$-modules, and "Ker" denotes the kernel of this pairing. We denote $E_v = e_v \mod \operatorname{Ker}$. In all cases except the trivial cases $N = 1$ when $Y = \mathbb{P}^2$, $N = 2$ when $Y = \mathbb{F}_0$ or $\mathbb{F}_2$, $N = 3$ when $Y = \mathbb{F}_1$, and $N = 11$ when $Y = \mathbb{F}_4$, the lattice S_Y gives the Picard lattice of Y.

Thus, $\operatorname{Log}\Gamma(Y) = \operatorname{Log}\Gamma$ and $D = \operatorname{Duv}\Gamma(Y) \subset \operatorname{Duv}\Gamma$ define divisor classes $E_v, v \in V(\operatorname{Log}\Gamma(Y) \cup \operatorname{Duv}\Gamma(Y))$, of the corresponding exceptional curves on Y. Each exceptional curve E is evidently defined by its divisor class.

Black vertices $v \in V(\operatorname{Duv}\Gamma)$ define roots $E_v \in S_Y$ with $E_v^2 = -2$ and define reflections s_{E_v} in these roots which are automorphisms of S_Y such that $s_{E_v}(E_v) = -E_v$ and s_{E_v} gives identity on the orthogonal complement $E_v^\perp$ to E_v in S_Y. These reflections s_{E_v}, $v \in V(\operatorname{Duv}\Gamma)$, generate a finite Weyl group $W \subset O(S_Y)$.

The remaining part

$$\operatorname{Var}\Gamma(Y) = \Gamma(Y) - (\operatorname{Log}\Gamma(Y) \cup \operatorname{Duv}\Gamma(Y))$$

(it is called the **varying part of surface $\boldsymbol{Y}$**) is defined by

$$\operatorname{Var}\Gamma = \Gamma - (\operatorname{Duv}\Gamma \cup \operatorname{Log}\Gamma)$$

of the graph Γ of Table 3. Further, we identify exceptional curves $v \in V(\Gamma(Y))$ with their divisor classes $E_v \in S_Y$. We have the main formula

$$(5)\quad V(\operatorname{Var}\Gamma(Y)) = \{E \in W(\{E_v \mid v \in V(\operatorname{Var}\Gamma)\}) \mid E \cdot D \geq 0\} \subset S_Y$$

which describes $\operatorname{Var}\Gamma(Y)$ completely. Here $E \cdot D \geq 0$ means $E \cdot E_i \geq 0$ for any $E_i \in D$. The intersection pairing on S_Y then defines the full graph $\Gamma(Y)$ of Y. This completes the description of possible graphs $\Gamma(Y)$ of exceptional curves of log del Pezzo surfaces Z of index ≤ 2.

Thus, to find all possible graphs $\Gamma(Y)$ of exceptional curves of $\sigma : Y \to Z$, one has to choose one of the graphs Γ of Table 3 (this also defines main invariants (r, a, δ) of Y and Z), then choose a subgraph $D = \operatorname{Duv}\Gamma(Y) \subset \operatorname{Duv}\Gamma$. Then $\Gamma(Y)$ consists of D, $\operatorname{Log}\Gamma(Y) = \operatorname{Log}\Gamma$ and the remaining part $\operatorname{Var}\Gamma(Y)$ defined by the formula (5), the elements in the W-orbits of $\operatorname{Var}\Gamma$ that have non-negative intersection with the Du Val part. See Theorems 3.18, 3.19, 3.20 and 4.1. See Section 4.2 about such type of calculations in the most non-trivial case $N = 20$.

We note two important opposite cases.

Extremal case. This is the case when $D = \operatorname{Duv}\Gamma(Y) = \operatorname{Duv}\Gamma$. Then $\Gamma(Y) = \Gamma$ is completely calculated in Table 3. This case is called **extremal** and gives log del Pezzo surfaces Z with Du Val singularities of the highest rank, respectively $\operatorname{rk}\operatorname{Pic} Z = r - \#V(\operatorname{Log}\Gamma(Y)) - \#V(\operatorname{Duv}\Gamma(Y))$ is minimal for the fixed main invariants. In particular, this case delivers all the cases of minimal log del Pezzo surfaces of index ≤ 2 with $\operatorname{rk}\operatorname{Pic} Z = 1$. See Theorems 3.18, 4.2, 4.3.

No Du Val singularities. This is the case when $D = \operatorname{Duv}\Gamma(Y) = \emptyset$. Equivalently, all singularities of Z have index 2, if they exist. Then $\Gamma(Y) =$

$\operatorname{Log}\Gamma \cup \operatorname{Var}\Gamma(Y)$ where

$$V(\operatorname{Var}\Gamma(Y)) = W\left(\{E_v \mid v \in V(\operatorname{Var}\Gamma)\}\right). \tag{6}$$

Here, all the multiple cases 7a,b, 8a–c, 9a–f, 10a–m, 20a–d give the same graphs (because they have the same, equal to zero, root invariant, see below), and one can always take the cases 7a, 8a, 9a, 10a, 20a for the main invariants. This case is very similar to and includes the classical case of non-singular del Pezzo surfaces corresponding to the cases 1—10. See Theorem 4.4 about this (without Du Val singularities) case. Log del Pezzo surfaces of this case are defined by their main invariants (r, a, δ) up to deformation. The Du Val parts $\operatorname{Duv}\Gamma$ of graphs Γ of Table 3 in this case can be considered to be analogs of root systems (or Dynkin diagrams) which one usually associates to non-singular del Pezzo surfaces. Its true meaning is to give the type of the Weyl group W that describes the varying part $\operatorname{Var}(\Gamma(Y))$ from $\operatorname{Var}\Gamma$ by the formula (6). In the cases 7 — 10, 20, one can take graphs Γ of the cases 7a — 10a, 20a.

The Root invariant. It is possible that two different subgraphs $D \subset \operatorname{Duv}\Gamma$, $D \subset \operatorname{Duv}\Gamma'$ of graphs of Table 3 (with the same main invariants (r, a, δ)) give isomorphic graphs $\Gamma(Y)$ and $\Gamma(Y')$ for the corresponding right resolutions, and then they give similar log del Pezzo surfaces Z and Z' of index ≤ 2, according to our classification. The **root invariant**

$$([D], \xi) \tag{7}$$

gives the necessary and sufficient condition for this to happen.

To define the root invariant (7), we first remark that the main invariants (r, a, δ) define a unique hyperbolic (i. e. with one positive square) even 2-elementary lattice S with these invariants. Here $r = \operatorname{rk} S$, $S^*/S \cong (\mathbb{Z}/2)^a$, and $\delta = 0$, if and only if $(x^*)^2 \in \mathbb{Z}$ for any $x \in S^*$. In (7), $[D]$ is the root lattice generated by D, and $\xi : [D]/2[D] \to S^*/S$ a homomorphism preserving finite forms $(x^2)/2 \mod 2$, $x \in [D]$, and $y^2 \mod 2$, $y \in S^*$. The construction of the root invariant (7) uses the double cover $\pi : X \to Y$ by a K3 surface X (see above) with the non-symplectic involution θ. Then $S = H^2(X, \mathbb{Z})^\theta$ is the sublattice where θ^* acts as identity. The root invariant (7) is considered up to automorphisms of S and the root lattice $[D]$. See Sections 2.5 and 3.2 about this construction and a very easy criterion (the kernel H of ξ is almost equivalent to ξ) for two root invariants to be isomorphic. The root invariant was first introduced and used in [Nik84a] and [Nik87].

In practise, to calculate the root invariant of a log del Pezzo surface of index ≤ 2, one should just go from the graphs Γ of Table 3 to the equivalent graphs $\Gamma(P(\mathcal{M}^{(2,4)}))$ or $\Gamma(P(X)_+)$ of Tables 1 or 2 of exceptional curves for the K3 pairs (X, θ) (see Sections 3.2, 3.5).

Thus, two Du Val subgraphs $D \subset \operatorname{Duv}\Gamma$, $D \subset \operatorname{Duv}\Gamma'$ of graphs of Table 3 give isomorphic full graphs $\Gamma(Y)$ and $\Gamma(Y')$ of their log del Pezzo surfaces if and only if their root invariants (7) are isomorphic (see Theorem 3.5). Moreover, we constantly use the root invariant to prove existence of the corresponding K3 pairs (X, θ) and log del Pezzo surfaces Z. The main invariants (r, a, δ) and the root invariants (7) are the main tools in our classification. They are equivalent to the full graphs $\Gamma(Y)$ of exceptional curves on Y, but they are much more convenient to work with. For non-singular del Pezzo surfaces and log del Pezzo surfaces of index ≤ 2 without Du Val singularities the root invariant is zero. This is why, in these cases, we have such a simple classification as above.

See Section 4.2 about enumeration of root invariants (equivalently of graphs of exceptional curves) in the most non-trivial case $N = 20$.

Acknowledgments. We are grateful to V.A. Iskovskikh for useful discussions. The first author was supported by NSF for part of this work. The second author is grateful to Steklov Mathematical Institute, Max-Planck-Insitut für Mathematik and the University of Liverpool for hospitality.

We also would like to thank the Referees for many helpful suggestions.