CHAPTER VI

Invariance of plurigenera

§1. Background

A *deformation* (or a smooth deformation) of a compact complex manifold X is by definition a proper smooth surjective morphism $\pi\colon \mathcal{X} \to S$ of complex analytic varieties together with a point $s \in S$ such that the fiber $\mathcal{X}_s = \pi^{-1}(s)$ is isomorphic to X. The deformation is called projective if π is a projective morphism along X. A compact complex manifold is said to be *in the class* $\mathcal{C}$ if it is bimeromorphically equivalent to a compact Kähler manifold ([**18**], [**143**]). We are interested in the following:

1.1. Conjecture The m-genus $P_m(X) = \mathrm{h}^0(X, mK_X)$ is invariant under a deformation of a compact complex manifold in the class $\mathcal{C}$.

The deformation invariance of the plurigenera of compact complex surfaces was proved by Iitaka [**42**] by the classification theory of surfaces. Nakamura [**94**] gave a counterexample to the invariance in the case where X is not in the class $\mathcal{C}$. The invariance of the geometric genus $P_1(X) = p_g(X)$ for X in the class $\mathcal{C}$ is derived from the Hodge decomposition $\mathrm{H}^n(X, \mathbb{C}) = \bigoplus_{p+q=n} \mathrm{H}^q(X, \Omega_X^p)$ and the upper semi-continuity of $\mathrm{h}^q(X, \Omega_X^p)$. Levine [**75**] proved **1.1** for $m > 1$ in the case where mK_X is linearly equivalent to a reduced normal crossing divisor. Levine applied the Hodge theory to the cyclic covering branched along the divisor in order to show the existence of an infinitesimal lifting of a general section of $\mathrm{H}^0(X, mK_X)$.

A *degeneration* of compact complex manifolds is by definition a proper surjective morphism $\pi\colon \mathcal{X} \to S$ with connected fibers from a non-singular complex analytic variety into a non-singular curve that is smooth outside a given point $0 \in S$. We denote by $\mathcal{X}_t$ the scheme-theoretic fiber $\pi^{-1}(t)$. We say that a smooth fiber $\mathcal{X}_t$ ($t \neq 0$) degenerates into the special fiber $\mathcal{X}_0$. The degeneration is called projective if π is so. Let $\mathcal{X}_0 = \bigcup \Gamma_i$ be the irreducible decomposition of the special fiber. In the study of degeneration of algebraic surfaces (cf. [**15**]), the lower semi-continuity of the Kodaira dimension: $\kappa(\mathcal{X}_t) \geq \max \kappa(\Gamma_i)$ is expected to be true. However, there are counterexamples ([**108**], [**109**], [**140**], [**19**]) in the case where some Γ_i is not in the class $\mathcal{C}$. The following stronger conjecture is posed in [**98**]:

1.2. Conjecture If any irreducible component Γ_i of the special fiber $\mathcal{X}_0$ belongs to the class $\mathcal{C}$, then

$$P_m(\mathcal{X}_t) \geq \sum P_m(\Gamma_i)$$

for a smooth fiber $\mathcal{X}_t$. In particular, $\kappa(\mathcal{X}_t) \geq \max \kappa(\Gamma_i)$.

The author considered **1.2** from the viewpoint of the relative minimal model theory in [**96**], [**98**]. For a projective degeneration, **1.2** is reduced to the flip and the abundance conjectures. In the case of a projective deformation of a threefold, the existence of related flips is proved in [**73**] and hence the invariance of plurigenera follows from the abundance theorem [**84**], [**59**] for threefolds. Siu [**130**] has succeeded in proving **1.1** in the case of a projective deformation in which any fiber $\mathcal{X}_t$ is of general type: $\kappa(\mathcal{X}_t) = \dim \mathcal{X}_t$. Siu used *multiplier ideals* together with delicate arguments of L^2 properties which avoid the difficulty in showing the existence of flips. Even though the argument contains analytic methods, the essence is not so transcendental. Kawamata [**60**] gave an algebraic interpretation of Siu's argument and showed that small deformations of canonical singularities are canonical, as an application. The author's preprint [**105**] gave an algebraic modification of Siu's argument which is slightly different from that by Kawamata, and obtained the following stronger results:

- The numerical Kodaira dimension κ_σ is lower semi-continuous under a projective degeneration and is invariant under a projective deformation. In particular, a non-singular projective variety deformed to a variety of general type under a projective deformation is also of general type;
- The invariance of plurigenera P_m holds for a projective deformation in which a 'general' fiber F satisfies the abundance: $\kappa(F) = \kappa_\sigma(F)$. The lower semi-continuity of P_m holds for a projective degeneration satisfying the same assumption of abundance, for infinitely many m.
- Small deformations of terminal singularities are terminal.

In this chapter, we shall generalize slightly the results of [**105**]. As in the preprint [**105**], we need only the theory of resolution of singularities and the flattening theorem by Hironaka ([**39**], [**40**], [**41**]), the theory of linear systems, and the analytic version **II.5.12** of Kawamata–Viehweg's vanishing theorem **II.5.9** as well as the analytic version **V.3.13** of Kollár's injectivity theorem **V.3.7**.

§2. Special ideals

§2.a. Setting.

2.1. Definition Let $\pi\colon X \to S$ be a projective surjective morphism from a non-singular space and let $X = \bigsqcup X_i$ be the decomposition into connected components.

(1) A divisor L of X is called π-*effective* if $\pi_*\mathcal{O}_{X_i}(L) \neq 0$ for every i.

(2) For a π-effective divisor L, we denote by $|L|_{\text{fix}}$ the maximum effective divisor D with the property

$$\pi_*\mathcal{O}_X(L - D) = \pi_*\mathcal{O}_X(L).$$

It is so-called the *relative fixed divisor* of L over S.

2.2. Situation Let $\pi\colon V \to S$ be a projective surjective morphism from a non-singular variety with connected fibers, $X = \bigsqcup X_i$ a disjoint union of non-singular prime divisors X_i of V, and Δ an effective $\mathbb{R}$-divisor of V such that

(1) $X_i \not\subset \operatorname{Supp} \Delta$ for any i,
(2) $X \cup \operatorname{Supp} \Delta$ is a normal crossing divisor,
(3) $\ulcorner \Delta \urcorner$ is reduced or $\Delta = 0$, and
(4) $X \cap \operatorname{Supp} \llcorner \Delta \lrcorner = \emptyset$.

Let Δ_X be the effective $\mathbb{R}$-divisor $\Delta|_X$. Then $\operatorname{Supp} \Delta_X$ is a normal crossing divisor, $\llcorner \Delta_X \lrcorner = 0$, and

$$(K_V + X + \Delta)|_X = K_X + \Delta_X.$$

Moreover, we fix a $(\pi|_X)$-ample divisor A_0 of X such that $A_0 - (\dim X)H_0$ is $(\pi|_X)$-ample for a $(\pi|_X)$-very ample divisor H_0.

In §§**2** and **3**, we fix these π, V, S, Δ, $X = \sum X_i$, and Δ_X. We study analytic spaces projective over the fixed space S. However, we change S freely by its open subsets, because most statements to prove are local on S. In particular, the number of connected components of X is assumed to be finite.

2.3. Definition ($\mathbb{E}_V$, $\mathbb{E}_X$, $\mathbb{E}$, $\mathbb{E}_{\mathrm{big}}$ and $\mathcal{G}[L]$)

(1) Let $\mathbb{E}_V$ be the set of the linear equivalence classes of π-effective divisors of V.
(2) Let $\mathbb{E}_X$ be the set of the linear equivalence classes of $(\pi|_X)$-effective divisors of X.
(3) For a divisor L of V and a component X_i of X, we denote by $\mathcal{G}_i[L]$ the image of the homomorphism
$$\pi_*\mathcal{O}_V(L) \to \pi_*\mathcal{O}_{X_i}(L).$$
We also denote by $\mathcal{G}[L] \subset \bigoplus \mathcal{G}_i[L]$ the image of
$$\pi_*\mathcal{O}_V(L) \to \pi_*\mathcal{O}_X(L).$$
(4) Let $\mathbb{E}$ be the set of the linear equivalence classes of divisors L of V with $\mathcal{G}_i[L] \neq 0$ for any i.
(5) Let $\mathbb{E}_{\mathrm{big}}$ be the subset of $\mathbb{E}$ consisting of divisors L such that the meromorphic mappings
$$V \dashrightarrow \mathbb{P}_S(\pi_*\mathcal{O}_V(L)) \quad \text{and} \quad X \dashrightarrow \mathbb{P}_S(\mathcal{G}[L])$$
are both bimeromorphic mappings into their own images.

2.4. Definition (Conditions **E**, **G**, and **B**) Let L be a divisor of V and let M be a divisor of X.

(1) Let $\rho\colon W \to V$ be a bimeromorphic morphism from a non-singular variety and let D be a $(\pi \circ \rho)$-effective divisor of W. We say that W satisfies the condition **E** for D if the following two conditions are satisfied:
   - The union of the ρ-exceptional locus, the proper transform Y of X, and $\operatorname{Supp} |D|_{\mathrm{fix}}$ is a normal crossing divisor;

- $D - |D|_{\text{fix}}$ is $(\pi \circ \rho)$-free.

If $L \in \mathbb{E}_V$ and if W satisfies the condition **E** for $\rho^* L$, then we say that ρ satisfies the condition **E** for L. In this case, we write $E(L) := |\rho^* L|_{\text{fix}}$.

(2) Suppose that $M \in \mathbb{E}_X$. A bimeromorphic morphism $f\colon Y \to X$ from a non-singular space is said to satisfy the condition **G** for M if, for the divisor $G(M) := |f^* M|_{\text{fix}}$, the following two conditions are satisfied:
- The union of the f-exceptional locus and $\operatorname{Supp} G(M)$ is a normal crossing divisor;
- $f^* M - G(M)$ is $((\pi|_X) \circ f)$-free.

(3) Suppose that $L \in \mathbb{E}$. A bimeromorphic morphism $f\colon Y \to X$ from a non-singular space is said to satisfy the condition **B** for L if there is an effective divisor $B(L)$ of Y such that
- the union of the f-exceptional locus and $\operatorname{Supp} B(L)$ is a normal crossing divisor, and
- $\mathcal{O}_Y(f^* L - B(L))$ is the image of the homomorphism

$$f^* \pi^* \mathcal{G}[L] \to \mathcal{O}_Y(f^* L).$$

Convention

(1) For a bimeromorphic morphism $\rho\colon W \to V$ satisfying the condition **E** for a divisor $L \in \mathbb{E}_V$, we denote the proper transform of X by Y and the restriction of ρ by $f\colon Y \to X$.

(2) We shall write the total transform $\mu^* E(L)$ of $E(L)$ by the same symbol $E(L)$ for a bimeromorphic morphism $\mu\colon W' \to W$ such that $\rho \circ \mu$ also satisfies the condition **E** for L. Also for $G(M)$ and $B(L)$, we shall also write the total transform by the same symbol.

If $\rho\colon W \to V$ is a bimeromorphic morphism satisfying the condition **E** for L, then $f\colon Y \to X$ satisfies the condition **B** for L. Here $B(L) = E(L)|_Y$. Conversely, for any bimeromorphic morphism $f'\colon Y' \to X$ satisfying the condition **B** for L, there exist a bimeromorphic morphism $\rho\colon W \to V$ satisfying the condition **E** for L and a bimeromorphic morphism $\lambda\colon Y \to Y'$. Here we have $\lambda^* B(L) = B(L) = E(L)|_Y$.

2.5. Definition (Ideals $\mathcal{I}[M]$ and $\mathcal{J}[L]$) Let M be a divisor of X and let L be a divisor of V.

(1) $\mathcal{I}[M]$ is defined to be the ideal sheaf of X such that $\mathcal{I}[M]\mathcal{O}_X(M)$ is the image of the natural homomorphism

$$\pi^* \pi_* \mathcal{O}_X(M) \to \mathcal{O}_X(M).$$

(2) $\mathcal{J}[L]$ is defined to be the ideal sheaf of X such that $\mathcal{J}[L]\mathcal{O}_X(L)$ is the image of the natural homomorphism

$$\pi^* \mathcal{G}[L] \to \mathcal{O}_X(L).$$

For any i, $\pi_*\mathcal{O}_{X_i}(M) = 0$ if and only if $\mathcal{I}[M]|_{X_i} = 0$. If $M \in \mathbb{E}_X$ and if a bimeromorphic morphism $f\colon Y \to X$ satisfies the condition **G** for M, then

$$f^*\mathcal{I}[M]/(\mathrm{tor}) \simeq \mathcal{O}_Y(-G(M)).$$

The sheaf $\mathcal{J}[L]\mathcal{O}_X(L)$ is also the image of the composite

$$\pi^*\pi_*\mathcal{O}_V(L) \to \mathcal{O}_V(L) \to \mathcal{O}_X(L).$$

For any i, $\mathcal{G}_i[L] = 0$ if and only if $\mathcal{J}[L]|_{X_i} = 0$. Suppose that $L \in \mathbb{E}$. Then

$$f^*\mathcal{J}[L]/(\mathrm{tor}) \simeq \mathcal{O}_Y(-B(L))$$

for a bimeromorphic morphism $f\colon Y \to X$ satisfying the condition **B** for L.

2.6. Definition (Ramification divisors R_W and R_Y) Let $\rho\colon W \to V$ be a bimeromorphic morphism from a non-singular variety such that the proper transform Y of X is non-singular. In this situation, we define an $\mathbb{R}$-divisor:

$$R_W := K_W + Y - \rho^*(K_V + X + \Delta).$$

Let $f\colon Y \to X$ be a bimeromorphic morphism from a non-singular space. We define

$$R_Y := K_Y - f^*(K_X + \Delta_X).$$

Note that the $\ulcorner R_W \urcorner$ is effective on a neighborhood of $\rho^{-1}(X)$ by **II.4.4**. A prime divisor Γ of W with $\mathrm{mult}_\Gamma R_W > 0$ is ρ-exceptional. We have $R_Y = R_W|_Y$ for the proper transform Y of X in W.

2.7. Definition (Ideals $\mathcal{Q}[L, m]$, $\mathcal{I}[M, m]$, and $\mathcal{J}[L, m]$) Let L be a $\mathbb{Q}$-divisor of V, M a $\mathbb{Q}$-divisor of X, and m a positive integer with $mL \in \mathbb{E}$ and $mM \in \mathbb{E}_X$. Let $\rho\colon W \to V$ be a bimeromorphic morphism satisfying the condition **E** for mL and let $f\colon Y \to X$ be a bimeromorphic morphism satisfying the conditions **G** for mM and **B** for mL. We define the following three ideal sheaves:

$$\begin{aligned}\mathcal{Q}[L, m] &:= \rho_*\mathcal{O}_W(\ulcorner R_W - \frac{1}{m}E(mL)\urcorner),\\ \mathcal{I}[M, m] &:= f_*\mathcal{O}_Y(\ulcorner R_Y - \frac{1}{m}G(mM)\urcorner),\\ \mathcal{J}[L, m] &:= f_*\mathcal{O}_Y(\ulcorner R_Y - \frac{1}{m}B(mL)\urcorner).\end{aligned}$$

2.8. Lemma

(1) *The ideal sheaf* $\mathcal{Q}[L, m]$ *is independent of the choice of bimeromorphic morphisms* ρ *satisfying the condition* **E** *for* mL.
(2) *The ideal sheaf* $\mathcal{I}[M, m]$ *is independent of the choice of bimeromorphic morphisms* f *satisfying the condition* **G** *for* mM. *There is an inclusion* $\mathcal{I}[mM] \subset \mathcal{I}[mM, 1]$.
(3) *The ideal sheaf* $\mathcal{J}[L, m]$ *is independent of the choice of bimeromorphic morphisms* f *satisfying the condition* **B** *for* mL. *There is an inclusion* $\mathcal{J}[mL] \subset \mathcal{J}[mL, 1]$.

Proof. (1) Let $\mu\colon W' \to W$ be a bimeromorphic morphism such that $\rho \circ \mu$ satisfies the condition **E** for mL and let Y' be the proper transform of Y. Then

$$K_W + Y = \rho^*(K_V + X + \Delta) + R_W,$$
$$K_{W'} + Y' = \rho^*(K_V + X + \Delta) + R_{W'}.$$

Since any component of Y is not contained in $\operatorname{Supp} E(mL)$, we have

$$K_{W'} + Y' + \ulcorner \mu^*(R_W - \frac{1}{m}E(mL)) \urcorner \geq \mu^*(K_W + Y + \ulcorner R_W - \frac{1}{m}E(mL) \urcorner),$$

by **II.4.4**. Since

$$R_{W'} - \frac{1}{m}E(mL) = K_{W'} + Y' - \mu^*(K_W + Y) + \mu^*(R_W - \frac{1}{m}E(mL)),$$

we have

$$\ulcorner R_{W'} - \frac{1}{m}E(mL) \urcorner = K_{W'} + Y' - \mu^*(K_W + Y) + \ulcorner \mu^*(R_W - \frac{1}{m}E(mL)) \urcorner$$
$$\geq \mu^*(\ulcorner R_W - \frac{1}{m}E(mL) \urcorner).$$

Hence

$$\mu_*\mathcal{O}_{W'}(\ulcorner R_{W'} - \frac{1}{m}E(mL) \urcorner) \simeq \mathcal{O}_W(\ulcorner R_W - \frac{1}{m}E(mL) \urcorner).$$

Thus both $\mathcal{Q}[L, m]$ are identical.

(2) and (3) We can show the independence of choices by the same argument as in (1) by using **II.4.3**. The inclusions $\mathcal{I}[mM] \subset \mathcal{I}[mM, 1]$ and $\mathcal{J}[mL] \subset \mathcal{J}[mL, 1]$ are derived from the property that $\ulcorner R_Y \urcorner$ is effective. □

Convention

- For divisors L of V and M of X, we write $\mathcal{I}[L|_X + M]$ by $\mathcal{I}[L + M]$, for short. In the case $L|_X + M \in \mathbb{E}_X$, we write $G(L|_X + M)$ by $G(L + M)$.
- If $m(L|_X + M) \in \mathbb{E}_X$ for $\mathbb{Q}$-divisors L of V and M of X, we write $\mathcal{I}[L|_X + M, m]$ by $\mathcal{I}[L + M, m]$.

For a bimeromorphic morphism $\rho\colon W \to V$ satisfying the condition **E** for mL and for the proper transform Y of X, we have

$$\ulcorner R_W - \frac{1}{m}E(mL) \urcorner|_Y = \ulcorner R_Y - \frac{1}{m}B(mL) \urcorner.$$

Thus

$$\mathcal{J}[L, m] \simeq f_*\mathcal{O}_Y(\ulcorner R_W - \frac{1}{m}E(mL) \urcorner).$$

§2.b. Inclusions of ideals. We consider the following conditions for a $\mathbb{Q}$-divisor L of V:

(VI-1) $L - (K_V + X + \Delta)$ is π-nef and L is π-pseudo-effective;
(VI-2) $L-(K_V+X+\Delta)$ is π-nef and π-abundant, and $L-(K_V+X+\Delta) \succcurlyeq_\pi X$ (cf. **V.2.24**).

Note that if $L-(K_V+X+\Delta)$ is π-nef and π-abundant and if $\pi(X) \neq S$, then L satisfies (VI-2). If $L - (K_V + X + \Delta)$ is π-nef and π-big, then L satisfies (VI-2).

Let L' be another $\mathbb{Q}$-divisor of V. We consider the following conditions for the pair (L, L'):

(VI-3) $L - L' - (K_V + X + \Delta)$ is π-nef and L' is π-big;
(VI-4) $L-L'-(K_V+X+\Delta)$ is π-nef and π-abundant, and $L' \succeq_\pi X$ (cf. **V.2.24**);
(VI-5) $L - L'$ satisfies (VI-2).

2.9. Proposition *Let L' be a $\mathbb{Q}$-divisor, L a $\mathbb{Z}$-divisor of V, and let n be a positive integer with $nL' \in \mathbb{E}$ such that (L, L') satisfies one of the three conditions* (VI-3), (VI-4), *and* (VI-5). *Then*

$$\pi_*(\mathcal{J}[L', n]\mathcal{O}_X(L)) \subset \mathcal{G}[L] \subset \pi_*\mathcal{O}_X(L).$$

Suppose in addition that there exist a $\mathbb{Q}$-divisor M of X and a positive integer m satisfying the following three conditions:

(1) $mM \in \mathbb{E}_X$;
(2) $\mathcal{I}[M, m] \subset \mathcal{J}[L', n]$;
(3) $L|_X - M - (K_X + \Delta_X) - A_0$ *is* $(\pi|_X)$*-nef.*

Then $\mathcal{I}[M, m]\mathcal{O}_X(L)$ is $(\pi|_X)$-generated, $L \in \mathbb{E}$, and $\mathcal{I}[M, m] \subset \mathcal{J}[L]$.

Proof. We note that $\mathcal{J}[L', n] \subset \mathcal{J}[L', nk]$ for $k > 0$. Therefore, in the case (VI-3), we may assume that the meromorphic mapping

$$V \cdots\to \mathbb{P}_S(\pi_*\mathcal{O}_V(nL'))$$

is a bimeromorphic mapping into its image. Let $\rho\colon W \to V$ be a bimeromorphic morphism satisfying the condition **E** for nL'. In the case (VI-4), we may assume that $n\rho^*L' - E(nL') \succeq_\pi Y$. In any case, the $\mathbb{R}$-divisor

$$R_W - \frac{1}{n}E(nL') + \rho^*L - K_W - Y = \rho^*(L - L' - (K_V + X + \Delta)) + \frac{1}{n}(n\rho^*L' - E(nL'))$$

is $(\pi \circ \rho)$-nef. In the case (VI-3), the $\mathbb{R}$-divisor is also $(\pi \circ \rho)$-big and hence

$$\mathrm{R}^p(\pi \circ \rho)_*\mathcal{O}_W(\ulcorner R_W - \frac{1}{n}E(nL')\urcorner + \rho^*L - Y) = 0$$

for $p > 0$ by **II.5.12**. In the cases (VI-4) and (VI-5), the $\mathbb{R}$-divisor is $(\pi \circ \rho)$-abundant and hence

$$\mathrm{R}^p(\pi \circ \rho)_* \mathcal{O}_W(\ulcorner R_W - \frac{1}{n}E(nL')\urcorner + \rho^* L - Y) \longrightarrow \mathrm{R}^p(\pi \circ \rho)_* \mathcal{O}_W(\ulcorner R_W - \frac{1}{n}E(nL')\urcorner + \rho^* L)$$

is injective for any p by **V.3.13**. Therefore, the homomorphism

$$\pi_*(\mathcal{Q}[L', n]\mathcal{O}_V(L)) \to \pi_*(\mathcal{J}[L', n]\mathcal{O}_X(L))$$

is surjective in any case. Thus $\pi_*(\mathcal{J}[L', n]\mathcal{O}_X(L))$ is contained in $\mathcal{G}[L]$.

Let $f : Y \to X$ be a bimeromorphic morphism satisfying the condition **G** for mM and let us consider the $\mathbb{R}$-divisor

$$C := R_Y - \frac{1}{m}G(mM) + f^*(L|_X).$$

Then

$$C - K_Y - f^* A_0 = \frac{1}{m}(mf^*M - G(mM)) + f^*(L|_X - M - (K_X + \Delta_X) - A_0)$$

is $(\pi \circ f)$-nef. Therefore

$$f_*\mathcal{O}_Y(\ulcorner C \urcorner) = \mathcal{I}[M, m]\mathcal{O}_X(L)$$

is $(\pi|_X)$-generated by **V.3.19** (cf. **2.2**, **II.5.12**). Since we have the inclusion

$$\pi_*(\mathcal{J}[L', n]\mathcal{O}_X(L)) = \bigoplus \pi_*(\mathcal{J}[L', n]\mathcal{O}_{X_i}(L)) \subset \mathcal{G}[L] \subset \bigoplus \mathcal{G}_i[L],$$

$\mathcal{G}_i[L] \neq 0$ for any i and $\mathcal{I}[M, m] \subset \mathcal{J}[L]$. □

Remark In the proof above, the sheaf $\mathcal{J}[L', n]\mathcal{O}_X(L)$ for $n > 0$ with $nL' \in \mathbb{E}$ is an ω-sheaf in a relative sense of **V.3.8**.

2.10. Lemma *Let L and M be $\mathbb{Q}$-divisors of X. Assume that*

(1) *M is $(\pi|_X)$-semi-ample,*
(2) *$a(\alpha L + M) \in \mathbb{E}_X$ for some $\alpha \in \mathbb{Q}_{>0}$ and $a \in \mathbb{N}$.*

Then, for any $\beta \in \mathbb{Q}$ with $0 < \beta < \alpha$, there is a positive integer b such that

$$b(\beta L + M) \in \mathbb{E}_X \quad \text{and} \quad \mathcal{I}[\alpha L + M, a] \subset \mathcal{I}[\beta L + M, b].$$

Proof. Let n be a positive integer with $na\alpha \in \mathbb{N}$ and $b := na\alpha\beta^{-1} \in \mathbb{N}$ such that

$$(b - an)M = na(\alpha\beta^{-1} - 1)M$$

is a π-free $\mathbb{Z}$-divisor. Then $b(\beta L + M) \in \mathbb{E}_X$, since

$$b(\beta L + M) = an(\alpha L + M) + (b - an)M.$$

Let $f : Y \to X$ be a bimeromorphic morphism satisfying the conditions **G** for $a(\alpha L + M)$, **G** for $an(\alpha L + M)$, and **G** for $b(\beta L + M)$. Then we have inequalities

$$\frac{1}{a}G(a(\alpha L + M)) \geq \frac{1}{an}G(an\alpha L + anM) \geq \frac{1}{an}G(b\beta L + bM) \geq \frac{1}{b}G(b(\beta L + M)).$$

Therefore $\mathcal{I}[\alpha L + M, a] \subset \mathcal{I}[\beta L + M, b]$. □

2.11. Proposition *Let A be a π-ample divisor of V and let M be a $(\pi|_X)$-semi-ample divisor of X such that*

$$A|_X - (K_X + \Delta_X) - A_0 - M$$

is $(\pi|_X)$-nef. Let L be a divisor of V satisfying either (VI-1) *or* (VI-2)*.*

(1) *If the condition*

$$C\langle l, m\rangle : \qquad m(lL|_X + M) \in \mathbb{E}_X$$

is satisfied for positive integers l and m, then $\mathcal{I}[lL + M, m]\mathcal{O}_X(lL + A)$ is $(\pi|_X)$-generated, $lL + A \in \mathbb{E}$, and $\mathcal{I}[lL + M, m] \subset \mathcal{J}[lL + A]$.

(2) *For any $l \in \mathbb{N}$,*

$$\mathcal{I}[lL + M] \subset \mathcal{J}[lL + A].$$

Proof. (1) We shall prove by induction on l. Assume that $C\langle 1, m\rangle$ is satisfied for some $m \in \mathbb{N}$. We have $\mathcal{J}[A, k] = \mathcal{O}_X$ for some $k \in \mathbb{N}$. Hence

$$\mathcal{I}[L + M, m] \subset \mathcal{J}[A, k].$$

Then $(L + A, A)$ satisfies (VI-3) or (VI-5), and $(L + A, A, L|_X + M, m, k)$ satisfies the condition of **2.9** as (L, L', M, m, n). Thus $\mathcal{I}[L + M, m]\mathcal{O}_X(L + A)$ is $(\pi|_X)$-generated, $L + A \in \mathbb{E}$, and $\mathcal{I}[L + M, m] \subset \mathcal{J}[L + A]$. Thus (1) is true for $l = 1$.

Next we consider the case $l > 1$ and assume that (1) is true for $l - 1$. If $C\langle l, m\rangle$ is satisfied for some m, then there is a positive integer m' such that

$$m'((l-1)L|_X + M) \in \mathbb{E}_X \quad \text{and} \quad \mathcal{I}[lL + M, m] \subset \mathcal{I}[(l-1)L + M, m']$$

by **2.10**. By induction,

$$(l-1)L + A \in \mathbb{E} \quad \text{and} \quad \mathcal{I}[(l-1)L + M, m'] \subset \mathcal{J}[(l-1)L + A].$$

Therefore, we have the inclusion

$$\mathcal{I}[lL + M, m] \subset \mathcal{J}[(l-1)L + A] \subset \mathcal{J}[(l-1)L + A, 1].$$

Here $(lL + A, (l-1)L + A)$ satisfies (VI-3) or (VI-5), since $(l-1)L + A$ is π-big in the case (VI-1). Furthermore, $(lL + A, (l-1)L + A, lL|_X + M, m, 1)$ satisfies the condition of **2.9** as (L, L', M, m, n). Therefore, $\mathcal{I}[lL + M, m]\mathcal{O}_X(lL + A)$ is $(\pi|_X)$-generated, $lL + A \in \mathbb{E}$, and $\mathcal{I}[lL + M, m] \subset \mathcal{J}[lL + A]$. Thus we have proved by induction.

(2) For a connected component X_i of X, we set $\Delta^{(i)} = \Delta + (X - X_i)$. Then we may replace (X, Δ) by $(X_i, \Delta^{(i)})$ in the situation **2.2**. Moreover, the replacement does not affect the conditions (VI-1)–(VI-5). Thus we can apply (1) to the case $X = X_i$. Hence if $\mathcal{I}[lL + M]|_{X_i} \neq 0$, i.e., $(lL_X + M)|_{X_i} \in \mathbb{E}_{X_i}$, then

$$\mathcal{I}[lL + M]|_{X_i} \subset \mathcal{I}[lL|_{X_i} + M|_{X_i}, 1] \subset \mathcal{J}[lL + A]|_{X_i}.$$

Therefore,

$$\mathcal{I}[lL + M] = \bigoplus \mathcal{I}[lL + M]|_{X_i} \subset \bigoplus \mathcal{J}[lL + A]|_{X_i} = \mathcal{J}[lL + A]. \qquad \square$$

2.12. Corollary *Let L be a divisor of V such that $L|_{X_i}$ is $(\pi|_{X_i})$-pseudo-effective for some i. If L satisfies* (VI-2)*, then L is π-pseudo-effective.*

Proof. By the same replacement as above, we can apply **2.11** to the case $X = X_i$. If we choose M as a $(\pi|_X)$-ample divisor, then for any $l > 0$, $C\langle l, m\rangle$ is satisfied for some $m > 0$, since $L|_X$ is $(\pi|_X)$-pseudo-effective. Thus **2.11**-(1) implies that $\mathcal{J}[lL + A] \neq 0$ for any $l > 0$. Hence L is π-pseudo-effective. □

§3. Surjectivity of restriction maps

§3.a. Big case.

3.1. Lemma *Let L and L' be $\mathbb{Q}$-divisors of V with $\langle L\rangle \leq \Delta$, $\llcorner L|_X \lrcorner \in \mathbb{E}_X$ such that (L, L') satisfies one of the three conditions* (VI-3), (VI-4)*, and* (VI-5)*, and let n be a positive integer with $nL' \in \mathbb{E}$. Suppose that there is a bimeromorphic morphism $\rho\colon W \to V$ satisfying the condition* **E** *for nL' in which $\rho|_Y = f$ satisfies the condition* **G** *for $\llcorner L|_X \lrcorner$ and the inequality*

$$-G(\llcorner L \lrcorner) \leq \ulcorner R_W + \rho^*\langle L\rangle - \frac{1}{n}E(nL')\urcorner|_Y = \ulcorner R_Y + f^*\langle L|_X\rangle - \frac{1}{n}B(nL')\urcorner$$

holds. Then $\pi_*\mathcal{O}_V(\llcorner L \lrcorner) \to \pi_*\mathcal{O}_X(\llcorner L \lrcorner)$ is surjective.

Proof. Let Δ' be the $\mathbb{R}$-divisor $\Delta - \langle L\rangle$. By replacing Δ with Δ', we may assume that $\langle L\rangle = 0$. The inequality above implies that $\mathcal{I}[L] \subset \mathcal{J}[L', n]$. Hence, by **2.9**, we have the inclusion

$$\pi_*\mathcal{O}_X(L) = \pi_*(\mathcal{I}[L]\mathcal{O}_X(L)) \subset \mathcal{G}[L],$$

which means the expected surjectivity. □

3.2. Proposition *Let L and L' be $\mathbb{Q}$-divisors of V with $\langle L\rangle \leq \Delta$ such that (L, L') satisfies one of the three conditions* (VI-3), (VI-4)*, and* (VI-5)*. Suppose that there exist positive integers m, m^*, a $\mathbb{Z}$-divisor A of V, an effective $\mathbb{Q}$-divisor Δ^* of V, and a bimeromorphic morphism $\rho\colon W \to V$ from a non-singular variety satisfying the following conditions*:

1. *mL and m^*L' are $\mathbb{Z}$-divisors with $mL + A \in \mathbb{E}_V$, $m^*L' \in \mathbb{E}_V$;*
2. *$\mathcal{I}[mL] \subset \mathcal{J}[mL + A]$;*
3. $\operatorname{Supp}\Delta^*$ *contains no components of X and $(V\&X, \Delta + \Delta^*)$ is log-terminal along X* (cf. **II.4.8**);
4. *ρ satisfies the conditions* **E** *for $mL + A$ and* **E** *for m^*L' in which the inequality*

$$-\frac{1}{m}E(mL + A) \leq \rho^*\Delta^* - \frac{1}{m^*}E(m^*L')$$

holds.

Then $\pi_*\mathcal{O}_V(\llcorner L \lrcorner) \to \pi_*\mathcal{O}_X(\llcorner L \lrcorner)$ is surjective.

PROOF. If $\pi_*\mathcal{O}_{X_i}(\llcorner L \lrcorner) = 0$, then we can replace (Δ, X) by $(\Delta + X_i, X - X_i)$. Thus we may assume that $\llcorner L|_X \lrcorner \in \mathbb{E}_X$. Then $mL + A \in \mathbb{E}$ and $m^*L' \in \mathbb{E}$ by (2) and (4). We may assume that the restriction $\rho|_Y = f$ satisfies the conditions **G** for $\llcorner L|_X \lrcorner$ and **G** for $mL|_X$. Then (2) induces the inequalities:

$$\frac{1}{m}B(mL + A) \leq \frac{1}{m}G(mL) \leq G(\llcorner L \lrcorner) + (\rho^*\langle L \rangle)|_Y.$$

Therefore

$$-G(\llcorner L \lrcorner) \leq (\llcorner \rho^*\langle L \rangle - \frac{1}{m}E(mL + A) \lrcorner)|_Y. \tag{VI-6}$$

We have

$$\ulcorner R_W - \rho^*\Delta^* \urcorner + \llcorner \rho^*\langle L \rangle + \rho^*\Delta^* - \frac{1}{m^*}E(m^*L') \lrcorner \leq \ulcorner R_W + \rho^*\langle L \rangle - \frac{1}{m^*}E(m^*L') \urcorner, \tag{VI-7}$$

in which the inequality $\ulcorner R_W - \rho^*\Delta^* \urcorner \geq 0$ holds along $\rho^{-1}(X)$ by (3). The restriction of (VI-7) to Y, (VI-6), and the inequality in (4) induce

$$-G(\llcorner L \lrcorner) \leq \ulcorner R_W + \rho^*\langle L \rangle - \frac{1}{m^*}E(m^*L') \urcorner|_Y.$$

Thus the result follows from **3.1**. □

3.3. Lemma *Let L and L' be $\mathbb{Q}$-divisors of V with $\langle L \rangle \leq \Delta$ such that (L, L') satisfies one of the three conditions* (VI-3)*,* (VI-4)*, and* (VI-5)*. Suppose that there exist*

- *a rational number $0 < \beta < 1$, positive integers m, m', and an integer b,*
- *$\mathbb{Z}$-divisors A and D of V, and*
- *a bimeromorphic morphism $\rho: W \to V$ from a non-singular variety*

satisfying the following conditions:

(1) *mL, $m'L$, and bL' are $\mathbb{Z}$-divisors with $mL + A \in \mathbb{E}_V$, $m'L + bL' \in \mathbb{E}_V$;*
(2) *$m\beta \leq m' + b\beta$ and $L' - \beta L$ is π-semi-ample;*
(3) *$\mathcal{I}[mL] \subset \mathcal{J}[mL + A]$;*
(4) *D is an effective divisor containing no components of X and $(V\&X, \Delta + (1/m)D)$ is log-terminal along X;*
(5) *ρ satisfies the conditions **E** for $mL + A$ and **E** for $m'L + bL'$ in which the inequality*

$$-E(mL + A) \leq \rho^*D - E(m'L + bL')$$

holds.

Then $\pi_*\mathcal{O}_V(\llcorner L \lrcorner) \to \pi_*\mathcal{O}_X(\llcorner L \lrcorner)$ is surjective.

PROOF. Let k be a positive integer such that $k\beta \in \mathbb{Z}$, $k\beta L$, and kL' are $\mathbb{Z}$-divisors, and that $k(L' - \beta L)$ is a π-free $\mathbb{Z}$-divisor. We may assume that ρ satisfies the conditions **E** for $mL+A$, **E** for $m'L+bL'$, **E** for $m'k\beta L+bk\beta L'$, and **E** for $k(m'+b\beta)L'$, then we have

$$\begin{aligned}\frac{1}{m}E(m'L+bL') &\geq \frac{1}{mk\beta}E(m'k\beta L+bk\beta L') \geq \frac{1}{mk\beta}E(k(m'+b\beta)L')\\ &\geq \frac{1}{k(m'+b\beta)}E(k(m'+b\beta)L').\end{aligned}$$

Therefore, if we set $m^* := k(m'+b\beta)$ and $\Delta^* = (1/m)D$, then all the conditions of **3.2** are satisfied. □

3.4. Lemma *Let L be a π-big $\mathbb{Z}$-divisor of V such that $kL \in \mathbb{E}_{\mathrm{big}}$ for some $k \in \mathbb{N}$ and let A be a divisor of V. Then, locally over S, there exist a positive integer a with $aL \in \mathbb{E}_{\mathrm{big}}$ and an effective divisor D of V containing no components of X such that $aL \sim A+D$.*

Proof. We may assume that S is Stein and A is π-very ample, since $A+A'$ is so for some π-very ample divisor A'. For an integer a with $aL \in \mathbb{E}_{\mathrm{big}}$, let $\rho\colon W \to V$ be a bimeromorphic morphism satisfying the condition **E** for aL. Then $a\rho^*L - E(aL)$ is $(\pi\circ\rho)$-big and $(\pi\circ\rho)$-free, and $E(aL)$ contains no components of Y. Let

$$W \xrightarrow{\varphi} Z \to \mathbb{P}_S(\pi_*\mathcal{O}_V(aL))$$

be the Stein factorization of the morphism given by $a\rho^*L - E(aL)$, where φ is a bimeromorphic morphism contracting no components of Y. Here $a\rho^*L - E(aL) \sim \varphi^*H$ for a divisor H of Z, which is relatively ample over S. Now the support of the cokernel of

$$\varphi_*\mathcal{O}_W(-\rho^*A - Y_i) \to \varphi_*\mathcal{O}_W(-\rho^*A)$$

is $\varphi(Y_i)$. Hence

$$\pi_*\mathcal{O}_W(m\varphi^*H - \rho^*A - Y_i) \to \pi_*\mathcal{O}_W(m\varphi^*H - \rho^*A)$$

is not isomorphic for $m \gg 0$. Therefore, Y_i is not contained in the relative fixed part $|m\varphi^*H - \rho^*A|_{\mathrm{fix}}$. Hence there is an effective divisor D' on W such that $\operatorname{Supp} D'$ contains no components of Y and $m\varphi^*H - \rho^*A \sim D'$ for some $m > 0$. Here, the effective divisor $D := \rho_*(mE(aL)+D')$ contains no components of X and $amL \sim A+D$. □

Remark Suppose that $d = \dim V - \dim S > 0$ and that $\pi(X_i)$ is a prime divisor for any component X_i of X. Then, for a π-big divisor L of V, $kL \in \mathbb{E}_{\mathrm{big}}$ for some $k > 0$ if and only if, for any i,

$$\varlimsup_{m\to\infty} m^{-d} \operatorname{rank} \mathcal{G}_i[mL] > 0.$$

3.5. Lemma *Suppose that $d = \dim V - \dim S > 0$. Let L, C be $\mathbb{Z}$-divisors of V, Θ a prime divisor of V dominating S, and X_i a component of X with $\pi(X_i)$ being a divisor of S. Suppose that*

$$\varlimsup_{m\to\infty} m^{-d} \operatorname{rank} \mathcal{G}_i[mL+C+\Theta] > 0,$$

where $\operatorname{rank}\mathcal{G}_i[mL+C+\Theta]$ *is the rank as a torsion-free sheaf of* $\pi(X_i)$. *Then*

$$\overline{\lim_{m\to\infty}}\, m^{-d}\operatorname{rank}\mathcal{G}_i[mL+C] > 0.$$

PROOF. We consider the following commutative diagram:

$$\begin{array}{ccccc}
\mathcal{O}_V(mL+C) & \longrightarrow & \mathcal{O}_V(mL+C+\Theta) & \longrightarrow & \mathcal{O}_\Theta(mL+C+\Theta) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{O}_{X_i}(mL+C) & \longrightarrow & \mathcal{O}_{X_i}(mL+C+\Theta) & \longrightarrow & \mathcal{O}_{X_i\cap\Theta}(mL+C+\Theta).
\end{array}$$

Let $\mathcal{E}_m$ be the image of the homomorphism

$$\pi_*\mathcal{O}_V(mL+C+\Theta) \to \pi_*\mathcal{O}_\Theta(mL+C+\Theta).$$

Then this is a torsion-free sheaf of S and

$$\overline{\lim_{m\to\infty}}\, m^{-d}\operatorname{rank}\mathcal{E}_m = 0,$$

since $\operatorname{rank}\mathcal{E}_m$ is at most

$$\dim \mathrm{H}^0\left(V_s\cap\Theta, mL+C+\Theta|_{V_s\cap\Theta}\right)$$

for a general fiber $V_s = \pi^{-1}(s)$. By the commutative diagram above, we infer that there is a surjective homomorphism

$$\mathcal{E}_m\otimes\mathcal{O}_{\pi(X_i)} \twoheadrightarrow \mathcal{G}_i[mL+C+\Theta]/\mathcal{G}_i[mL+C].$$

Thus we have the expected estimate of $\operatorname{rank}\mathcal{G}_i[mL+C]$. □

3.6. Lemma *Let* Λ *be a* π*-nef and* π*-big divisor of* V. *Suppose that* X_i *is not* π*-exceptional and* $\Lambda|_{X_i}$ *is* $(\pi|_{X_i})$*-big for any* i. *Then, locally on* S, *there exist an effective divisor* D *containing no* X_i *and a positive integer* a *such that* $a\Lambda - D$ *is* π*-ample.*

PROOF. We can take a prime divisor Θ such that $\Theta - A - K_V - X_i$ is π-ample for a π-ample divisor A and for any i. Hence

$$\pi_*\mathcal{O}_V(m\Lambda - A + \Theta) \to \pi_*\mathcal{O}_{X_i}(m\Lambda - A + \Theta)$$

is surjective for any $m \geq 0$ and i by **II.5.12**. Hence, by **3.5**, $\mathcal{G}_i[a\Lambda - A] \neq 0$ for some $a > 0$ and for any i with $\pi(X_i)$ being a prime divisor. Thus there is an effective divisor $D \in |a\Lambda - A|$ containing no X_i with $\operatorname{codim}\pi(X_i) = 1$. By the same argument as **III.3.8**, we can change a and D so that any component X_i with $\pi(X_i) = S$ is not contained in $\operatorname{Supp} D$. □

3.7. Theorem *Let* L *be a* π*-pseudo-effective* $\mathbb{Z}$*-divisor of* V *such that* $L - (K_V + X + \Delta)$ *is* π*-nef. Let* Λ *be a* π*-nef and* π*-big* $\mathbb{Q}$*-divisor of* V *such that* $\Delta \geq \langle\Lambda\rangle$ *and* $k\Lambda \in \mathbb{E}_{\mathrm{big}}$ *for some* $k \in \mathbb{N}$. *Then the homomorphism*

$$\pi_*\mathcal{O}_V(lL + \llcorner\Lambda\lrcorner) \to \pi_*\mathcal{O}_X(lL + \llcorner\Lambda\lrcorner)$$

is surjective for $l \gg 0$. *If* $L|_X$ *is* $(\pi|_X)$*-pseudo-effective, then the homomorphism above is surjective for any* $l > 0$.

Remark If X_i is not π-exceptional for any i, then, by **3.6**, we can replace the condition "$k\Lambda \in \mathbb{E}_{\mathrm{big}}$ for some $k \in \mathbb{N}$" by "$\Lambda|_{X_i}$ is $(\pi|_{X_i})$-big for any i."

PROOF. If $L|_{X_i}$ is not $(\pi|_{X_i})$-pseudo-effective, then $\pi_*\mathcal{O}_{X_i}(lL+\llcorner\Lambda\lrcorner) = 0$ except for a finite number of positive integers l. Hence we can replace X with $X - X_i$ and Δ with $\Delta + X_i$. Thus we may assume that $L|_X$ is $(\pi|_X)$-pseudo-effective.

First we consider the case $l = 1$. The $\mathbb{R}$-divisor

$$L + \llcorner\Lambda\lrcorner - (K_V + X + \Delta - \langle\Lambda\rangle) = L - (K_V + X + \Delta) + \Lambda$$

is π-nef and π-big. Thus $(\Delta - \langle\Lambda\rangle, L + \llcorner\Lambda\lrcorner, 0, 1)$ satisfies the condition of **2.9** as (Δ, L, L', m). Hence

$$\pi_*\mathcal{O}_X(L + \llcorner\Lambda\lrcorner) \subset \mathcal{G}[L + \llcorner\Lambda\lrcorner].$$

Therefore we have the surjectivity for $l = 1$.

Next, we assume that $l > 1$. Let A_1 be a π-very ample divisor of V such that

$$A_1|_X - (K_X + \Delta_X) - A_0$$

is $(\pi|_X)$-nef. Let b be a positive integer with $b\Lambda$ being a $\mathbb{Z}$-divisor. Then

$$mlL + b\Lambda + 2A_1 \in \mathbb{E} \quad \text{and} \quad \mathcal{I}[mlL + b\Lambda + A_1] \subset \mathcal{J}[mlL + b\Lambda + 2A_1]$$

for any $m \in \mathbb{N}$ by **2.11**. In particular,

$$\mathcal{I}[m(lL + \Lambda)] \subset \mathcal{I}[m(lL + \Lambda) + A_1] \subset \mathcal{J}[m(lL + \Lambda) + 2A_1]$$

for $m \in b\mathbb{N}$. There is an $a \in b\mathbb{N}$ such that $(a - b)\Lambda - 4A_1$ is linearly equivalent to an effective divisor D_1 containing no components of X locally over S by **3.4**. In particular, $\Lambda - \varepsilon D_1$ is π-ample for $0 < \varepsilon \le 1/(a - b)$. There is an effective divisor D of V locally over S containing no components of X such that

$$D \sim a(lL + \Lambda) - 2A_1 = (alL + b\Lambda + 2A_1) + (a - b)\Lambda - 4A_1.$$

From the linear equivalence $(m + a)(lL + \Lambda) \sim D + m(lL + \Lambda) + 2A_1$ for $m \in b\mathbb{N}$, we infer that $(m + a)(lL + \Lambda) \in \mathbb{E}$ and the inequality

$$-E(m(lL + \Lambda) + 2A_1) \le \rho^*D - E((m + a)(lL + \Lambda))$$

holds for a bimeromorphic morphism $\rho\colon W \to V$ satisfying the conditions **E** for $m(lL + \Lambda) + 2A_1$ and **E** for $(m + a)(lL + \Lambda)$. Let ε be a positive rational number such that $l\varepsilon < 1/(a - b)$ and $(V\&X, \Delta + \varepsilon D_1)$ is log-terminal along X. We can choose m so that $(V\&X, \Delta + \varepsilon D_1 + (1/m)D)$ is log-terminal along X. Hence the condition of **3.3** is satisfied for

$$(\Delta + \varepsilon D_1, lL + \Lambda, (l - 1)L + \Lambda - \varepsilon D_1, (l - 1)/l, m, m + a, 0, 2A_1, D)$$
$$\text{as} \quad (\Delta, L, L', \beta, m, m', b, A, D).$$

Thus the surjectivity follows. □

3.8. Corollary *Let L be a $\mathbb{Z}$-divisor of V such that $L - (K_V + X + \Delta)$ is π-nef and π-big, and $k(L - (K_V + X + \Delta)) \in \mathbb{E}_{\mathrm{big}}$ for some $k \in \mathbb{N}$. Then the homomorphism $\pi_*\mathcal{O}_V(lL) \to \pi_*\mathcal{O}_X(lL)$ is surjective for any $l \in \mathbb{N}$.*

Proof. We may assume that $L|_X$ is $(\pi|_X)$-pseudo-effective. Then, by **2.12**, L is π-pseudo-effective. Locally on S, there is an effective divisor D linearly equivalent to $k(L-(K_V+X+\Delta))$ that contains no components of X by **3.4**. Let $\rho\colon W \to X$ be a bimeromorphic morphism from a non-singular variety such that the union of the ρ-exceptional locus, $\rho^{-1}(X)$, and $\rho^{-1}(\operatorname{Supp} D)$ is a normal crossing divisor. Let Y be the proper transform of X as before. Let R_+ and R_-, respectively, be the positive and the negative parts of the prime decomposition of $\ulcorner R_W \urcorner$. Then R_+ is ρ-exceptional and $\operatorname{Supp} R_- \cap \rho^{-1}(X) = \emptyset$. There is an integer $m \gg k$ such that

$$\left\langle -(R_W - \frac{1}{m}\rho^* D)\right\rangle = \left\langle \langle -R_W\rangle + \frac{1}{m}\rho^* D\right\rangle \geq \frac{1}{m}\rho^* D.$$

Then $\ulcorner R_W - (1/m)\rho^* D\urcorner = \ulcorner R_W \urcorner$. We set

$$L_W := \rho^* L + R_+, \quad \Lambda := (1/m)\rho^* D, \quad \Delta'_W := \left\langle -(R_W - \frac{1}{m}\rho^* D)\right\rangle + R_-.$$

Then

$$L_W - (K_W + Y + \Delta'_W) = \rho^*(L - (K_V + X + \Delta + \frac{1}{m}D)) \sim_{\mathbb{Q}} (\frac{1}{k} - \frac{1}{m})\rho^* D$$

is $(\pi\circ\rho)$-nef and $(\pi\circ\rho)$-big, and $\langle \Lambda\rangle = \Lambda \leq \Delta'_W$. Thus, by **3.7**,

$$\pi_*\rho_*\mathcal{O}_W(lL_W) \to \pi_*\rho_*\mathcal{O}_Y(lL_W)$$

is surjective for any $l \in \mathbb{N}$. The expected surjectivity follows from the isomorphisms $\mathcal{O}_V(lL) \simeq \rho_*\mathcal{O}_W(lL_W)$ and $\mathcal{O}_X(lL) \simeq \rho_*\mathcal{O}_Y(lL_W)$. □

3.9. Theorem *Let L be a π-big divisor of V such that $kL \in \mathbb{E}_{\text{big}}$ for some $k \in \mathbb{N}$ and $L - (K_V + X + \Delta)$ is π-nef. Then the homomorphism*

$$\pi_*\mathcal{O}_V(lL) \to \pi_*\mathcal{O}_X(lL)$$

is surjective for any integer $l > 1$. If L satisfies (VI-2) *in addition, then the homomorphism is surjective also for $l = 1$.*

Proof. In the case $l = 1$, this is derived from **2.9**, since $(L, 0, 1)$ satisfies the condition of **2.9** as (L, L', n). Suppose that $l > 1$. By **2.11**, there is a π-ample divisor A of V such that $mL + A \in \mathbb{E}$ and $\mathcal{I}[mL] \subset \mathcal{J}[mL + A]$ for any $m > 0$. By **3.4**, there exist a positive integer a and an effective divisor D of V containing no components of X such that $A + D \sim alL$. Thus, for any $m > 0$, $mlL + A, (m + a)lL \in \mathbb{E}$, and

$$-E(mlL + A) \leq \rho^* D - E((m + a)lL)$$

for a bimeromorphic morphism $\rho\colon W \to V$ satisfying the conditions **E** for $mlL + A$ and **E** for $(m + a)lL$. If m is sufficiently large, then $(V\&X, \Delta + (1/m)D)$ is log-terminal along X. Then $(lL, (l-1)L, (l-1)/l, m, m+a, 0, A, D)$ satisfies the condition of **3.3** as $(L, L', \beta, m, m', b, A, D)$. Hence the surjectivity follows. □

3.10. Theorem *Let L be a divisor of V such that L satisfies the condition* (VI-2)*. Suppose that $\pi(X_i)$ is a prime divisor of S and $L|_{X_i}$ is $(\pi|_{X_i})$-big for any component X_i. Then $\pi_*\mathcal{O}_V(lL) \to \pi_*\mathcal{O}_X(lL)$ is surjective for any $l \geq 1$.*

Proof. If π is generically finite, then this follows from **3.9**. Suppose that $d = \dim V - \dim S > 0$. We may assume that L is π-pseudo-effective. Let Θ be a π-ample prime divisor of V. Then

$$\pi_*\mathcal{O}_V(mL + \Theta) \to \pi_*\mathcal{O}_{X_i}(mL + \Theta)$$

is surjective for $m > 0$ by **3.7**. Thus $kL \in \mathbb{E}_{\mathrm{big}}$ for some k by **3.5**. Hence the condition of **3.9** is satisfied. □

Example Let $f\colon Z \to S$ be a generically finite proper surjective morphism of normal complex analytic varieties. For a Cartier divisor L, a prime divisor Γ, and for an effective $\mathbb{R}$-divisor Δ of Z, suppose that

(1) $(Z\&\Gamma, \Delta)$ is log-terminal,
(2) $L - (K_Z + \Gamma + \Delta)$ is f-nef.

Then the restriction homomorphism $f_*\mathcal{O}_Z(mL) \to f_*\mathcal{O}_\Gamma(mL)$ is surjective for any $m \geq 0$. This is shown as follows: Let $\mu\colon V \to Z$ be a bimeromorphic morphism from a non-singular variety projective over S and let X be the proper transform of Γ. We may assume that X is non-singular and there exist effective $\mathbb{R}$-divisor Δ_V and a μ-exceptional effective divisor E such that $X \cup \operatorname{Supp}\Delta_V \cup \operatorname{Supp} E$ is a normal crossing divisor, $\llcorner \Delta_V \lrcorner = 0$, and

$$K_V + X + \Delta_V = \mu^*(K_Z + \Gamma + \Delta) + E.$$

We set $L_V := \mu^*L + E$. Then $f_*\mu_*\mathcal{O}_V(mL_V) \to f_*\mu_*\mathcal{O}_X(mL_V)$ is surjective for any $m > 0$ by **3.7** (or by **3.8**, **3.9**, **3.10**). This induces the expected surjection, since $\mu_*\mathcal{O}_V(mE) \simeq \mathcal{O}_Z$ for $m \geq 0$ and Γ is normal (cf. **II.4.9**).

§3.b. Abundant case.

3.11. Situation In addition to **2.2**, we consider the commutative diagram

$$\text{(VI-8)} \qquad \begin{array}{ccc} V & \xleftarrow{\ \rho\ } & W \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ S & \xleftarrow{\ \phi\ } & Z, \end{array}$$

where the following conditions are satisfied:

(1) W and Z are non-singular;
(2) ρ is a projective bimeromorphic morphism, ϕ is a projective morphism, and φ is a fiber space;
(3) $\varphi(Y) \neq Z$;
(4) any φ-exceptional divisor is exceptional for the bimeromorphic morphism $W \to V_1$ into the normalization V_1 of the image of $(\rho, \varphi)\colon W \to V \times Z$.

3.12. Lemma *In the situation* **3.11***, let* L *be a* π*-pseudo-effective* $\mathbb{Z}$*-divisor of* V *such that*

(1) $\operatorname{Supp} N_\sigma(L; V/S)$ *does not contain any* X_i*,*
(2) $\kappa_\sigma(\rho^*L; W/Z) = \kappa(\rho^*L; W/Z) = 0$.

Let A be a π-ample divisor of V such that $mL + A \in \mathbb{E}$ for any $m > 0$ and let H be a ϕ-ample divisor of Z. Then, there exist positive integers m_0, d, k and an effective divisor D of V containing no X_i such that

$$-|mm_0L + A|_{\mathrm{fix}} \leq \rho^*D - |\rho^*(mm_0L) + \varphi^*(dH)|_{\mathrm{fix}}$$

for $m \geq k$, if S is replaced by a relatively compact open subset. In particular, if ρ satisfies the conditions **E** *for $\rho^*(mm_0L) + \varphi^*(dH)$ and* **E** *for $mm_0L + A$ for an $m \geq k$, then*

$$-E(mm_0L + A) \leq \rho^*D - E(\rho^*(mm_0L) + \varphi^*(dH))$$

and $Y_i \not\subset \operatorname{Supp} E(\rho^*(mm_0L) + \varphi^*(dH))$ for any i.

Proof. There is a $\mathbb{Q}$-divisor Ξ_0 on Z such that

$$\rho^*L \sim_{\mathbb{Q}} \varphi^*\Xi_0 + N_\sigma(\rho^*L; W/Z)$$

by **V.2.26**. Let m_0 be a positive integer such that $N := m_0N_\sigma(\rho^*L; W/Z)$ and $\Xi := m_0\Xi_0$ are $\mathbb{Z}$-divisors and the linear equivalence $\rho^*(m_0L) \sim \varphi^*\Xi + N$ holds. Note that $\operatorname{Supp} N$ contains no proper transforms Y_i. There is a positive integer k such that $\sigma_\Gamma(\rho^*A + kN; W/Z) > 0$ for any prime component Γ of $\operatorname{Supp} N$. Thus

$$\varphi_*\mathcal{O}_W(\rho^*A + kN) \to \varphi_*\mathcal{O}_W(\rho^*A + mN)$$

is isomorphic for any $m \geq k$. There is a φ-exceptional effective divisor E'' such that $\varphi_*\mathcal{O}_W(\rho^*A + kN + E'')$ is reflexive. Here, $\rho^*A + kN$ is the pullback of a Cartier divisor of V_1 and E'' is exceptional for $W \to V_1$. Thus

$$\mathcal{F} := \varphi_*\mathcal{O}_W(\rho^*A + kN)$$

is reflexive. Since we may assume that S is Stein, there exists a surjective homomorphism

$$\mathcal{O}_Z^{\oplus r} \to \mathcal{F}^\vee \otimes \mathcal{O}_Z(dH)$$

for some positive integers r and d. By taking its dual, we have an exact sequence

$$0 \to \mathcal{F} \to \mathcal{O}_Z(dH)^{\oplus r} \to \mathcal{F}' \to 0,$$

in which $\mathcal{F}'$ is torsion-free. Let $\widetilde{\mathcal{F}'}$ be the quotient $\varphi^*\mathcal{F}'/(\mathrm{tor})$ by the torsion part and let $\widetilde{\mathcal{F}}$ be the kernel of

$$\varphi^*\mathcal{O}_Z(dH) \to \widetilde{\mathcal{F}'}.$$

Then $\mathcal{F} \simeq \varphi_*\widetilde{\mathcal{F}}$ and we have a φ-exceptional effective divisor $\widehat{E}$ of W and a commutative diagram

$$\text{(VI-9)} \qquad \begin{array}{ccc} \varphi^*\mathcal{F} & \longrightarrow & \widetilde{\mathcal{F}} \\ \downarrow & & \downarrow \\ \mathcal{O}_W(\rho^*A + kN) & \longrightarrow & \mathcal{O}_W(\rho^*A + kN + \widehat{E}), \end{array}$$

where φ_* of the bottom and the right arrows are isomorphisms. We fix an integer $m \geq k$. By replacing W by a blowing-up, we may assume that the image of the homomorphism

$$\varphi^*\phi^*\phi_*\mathcal{O}_Z(m\Xi + dH) \to \mathcal{O}_W(\varphi^*(m\Xi + dH))$$

is invertible. In other words, we assume that W satisfies the condition **E** for $\varphi^*(m\Xi+dH)$. Moreover, we assume that W satisfies the condition **E** for mm_0L+A. Let Θ_m be the relative fixed divisor $|\varphi^*(m\Xi + dH)|_{\mathrm{fix}} = E(\varphi^*(m\Xi + dH))$. From the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_W(\varphi^*(m\Xi + dH) - \Theta_m)^{\oplus r} & \longrightarrow & \widetilde{\mathcal{F}'} \otimes \mathcal{O}_W(\varphi^*(m\Xi) - \Theta_m) \\ \big\downarrow & & \big\downarrow \\ \mathcal{O}_W(\varphi^*(m\Xi + dH))^{\oplus r} & \longrightarrow & \widetilde{\mathcal{F}'} \otimes \mathcal{O}_W(\varphi^*(m\Xi)), \end{array}$$

we infer that the injection

$$\phi_*\varphi_*(\widetilde{\mathcal{F}} \otimes \mathcal{O}_W(\varphi^*(m\Xi) - \Theta_m)) \to \phi_*\varphi_*(\widetilde{\mathcal{F}} \otimes \mathcal{O}_W(\varphi^*(m\Xi)))$$

is isomorphic. Therefore,

$$\pi_*\rho_*\mathcal{O}_W(\rho^*A + kN + \widehat{E} + m\varphi^*\Xi - \Theta_m) \to \pi_*\rho_*\mathcal{O}_W(\rho^*A + kN + \widehat{E} + m\varphi^*\Xi)$$

is an isomorphism by (VI-9). Since $\widehat{E}$ is ρ-exceptional, $\widehat{E} + E(mm_0L + A)$ is the relative fixed divisor of $\rho^*(mm_0L + A) + \widehat{E}$ over S. Thus we have an inequality

$$E(mm_0L + A) + \widehat{E} \geq (m - k)N + \Theta_m.$$

On the other hand, $mN + \Theta_m$ is the relative fixed divisor of $\varphi^*(m\Xi + dH) + mN \sim mm_0\rho^*L + d\varphi^*H$ and hence W satisfies the condition **E** for $\rho^*(mm_0L) + \varphi^*(dH)$. Therefore,

$$-E(mm_0L + A) \leq \widehat{E} + kN - E(\rho^*(mm_0L) + \varphi^*(dH)).$$

There is an effective divisor D on V such that $\operatorname{Supp} D$ contains no X_i and $\rho^*D \geq \widehat{E} + kN$. Thus we are done. □

3.13. Lemma *In the situation* **3.11***, suppose that any X_i is not π-exceptional. Let Λ be a π-nef and π-abundant $\mathbb{Z}$-divisor of V such that*

(1) *$\rho^*\Lambda$ is $\mathbb{Q}$-linearly equivalent to the pullback of a ϕ-nef and ϕ-big $\mathbb{Q}$-divisor of Z,*
(2) $\kappa(\Lambda|_{X_i}; X_i/\pi(X_i)) \geq \dim Z - 1 - \dim \pi(X_i)$.

Then there is an effective divisor D on W locally over S such that $\rho^*\Lambda - \varepsilon D$ for $0 < \varepsilon \ll 1$ is $\mathbb{Q}$-linearly equivalent to the pullback of a ϕ-ample $\mathbb{Q}$-divisor and $\operatorname{Supp} D$ *contains no components Y_i of Y.*

Proof. Let Ξ be the ϕ-nef and ϕ-big divisor of Z with $\rho^*\Lambda \sim_{\mathbb{Q}} \varphi^*\Xi$. Then $\Xi|_{\varphi(X_i)}$ is ϕ-big. Hence there is an effective divisor D' on Z such that $\operatorname{Supp} D'$ contains no $\varphi(X_i)$ and $\Xi - \varepsilon D'$ is ϕ-ample by **3.6**. Thus $D = \varphi^*D'$ satisfies the condition. □

3.14. Theorem *Let L be a π-pseudo-effective divisor and let Λ be a π-nef and π-abundant $\mathbb{Q}$-divisor of V with $\Delta \geq \langle \Lambda \rangle$. Suppose that*

(1) *any X_i is not π-exceptional,*
(2) *$L - (K_V + X + \Delta)$ is π-nef and π-abundant,*
(3) *$L|_X$ is $(\pi|_X)$-pseudo-effective,*
(4) *$\kappa(\Lambda; V/S) = \kappa_\sigma(kL + \Lambda; V/S)$ for some $k > 0$,*
(5) *$\kappa(\Lambda|_{X_i}; X_i/\pi(X_i)) = \kappa(\Lambda; V/S) + \dim S - \dim \pi(X_i) - 1$ for any X_i.*

Then the restriction homomorphism

$$\pi_* \mathcal{O}_V(lL + \llcorner \Lambda \lrcorner) \to \pi_* \mathcal{O}_X(lL + \llcorner \Lambda \lrcorner)$$

is surjective for any $l \geq 1$.

Proof. In the case: $\dim V = \dim S$, this is already proved in **3.7**. Thus we may assume that $\dim V > \dim S$.

Step 1 A reduction. We may replace V by a blowing-up as follows: let $\rho_1 \colon W_1 \to V$ be a projective bimeromorphic morphism from a non-singular variety such that the union of the ρ_1-exceptional locus, $\rho_1^{-1}(\operatorname{Supp} \Delta)$, and the proper transform Y_1 of X is a normal crossing divisor. Let R_+ and R_-, respectively, be the positive and the negative parts of the prime decomposition of $\ulcorner R_1 \urcorner$ for the $\mathbb{R}$-divisor $R_1 = K_{W_1} + Y_1 - \rho_1^*(K_V + X + \Delta)$. Here, R_+ is ρ_1-exceptional and $\operatorname{Supp} R_- \cap \rho_1^{-1}(X) = \emptyset$. Setting

$$L_1 := \rho_1^* L + R_+, \quad \Delta_1 := \langle -R_1 \rangle + R_-,$$

we have the equality

$$L_1 - (K_{W_1} + Y_1 + \Delta_1) = \rho^*(L - (K_V + X + \Delta))$$

and an isomorphism

$$\rho_{1*} \mathcal{O}_{W_1}(lL_1 + \llcorner \rho_1^* \Lambda \lrcorner) \simeq \mathcal{O}_V(lL + \llcorner \Lambda \lrcorner).$$

Hence we can replace $(V, X, \Delta, L, \Lambda)$ by $(W_1, Y_1, \Delta_1, L_1, \rho_1^* \Lambda)$. Therefore, we may assume that there exist a projective morphism $p \colon T \to S$ from a non-singular variety and a fiber space $\psi \colon V \to T$ over S such that Λ is $\mathbb{Q}$-linearly equivalent to the pullback of a p-nef and p-big $\mathbb{Q}$-divisor of T. Then the condition (5) is equivalent to that $\psi(X_i)$ is a prime divisor for any i. Since $\Lambda \succcurlyeq_\pi X$, $L + \Lambda$ satisfies the condition (VI-2). Thus if $l = 1$, then the surjectivity follows from **2.9**. So, we may assume $l \geq 2$.

By **3.13**, we can find an effective divisor D_1 and $\varepsilon \in \mathbb{Q}_{>0}$ such that

- $X_i \not\subset \operatorname{Supp} D_1$ for any i,
- $\Lambda - \varepsilon l D_1$ is the pullback of a ψ-ample $\mathbb{Q}$-divisor of T,
- $(V \& X, \Delta + \varepsilon D_1)$ is log-terminal along X.

Since $L - \psi^* K_T - (K_{V/T} + X + \Delta)$ is π-nef, we have

$$\kappa_\sigma(L - X + \psi^* Q; V/S) = \kappa_\sigma(L; V/T) + \dim T - \dim S$$

for an $\mathbb{R}$-divisor Q on T with $Q + K_T$ being p-big by **V.4.1**. The condition (4) implies that $\kappa_\sigma(L + \alpha\Lambda; V/S) = \kappa_\sigma(\Lambda; V/S)$ for any $\alpha > 0$. Hence, by **V.4.8**,

$$\kappa_\sigma(L; V/T) = \kappa(L; V/T) = 0.$$

By considering the flattening $\mu\colon Z \to T$ of ψ, we have the commutative diagram (VI-8) such that $\phi = p \circ \mu$.

Step 2. The case: Λ *is a* $\mathbb{Z}$*-divisor.* Let A be a π-very ample divisor of V. Applying **3.7** to $j\Lambda + A$ as Λ, we infer that $mlL + j\Lambda + A \in \mathbb{E}$ and

$$\mathcal{I}[mlL + j\Lambda] \subset \mathcal{I}[mlL + j\Lambda + A] = \mathcal{J}[mlL + j\Lambda + A]$$

for any $m \in \mathbb{N}$ and $j \in \mathbb{Z}_{\geq 0}$. Let H be a ϕ-ample divisor on Z. Applying **3.12** to $lL + j\Lambda$, we have positive integers m_0, d, k, and an effective divisor D of V containing no X_i satisfying the following conditions: If $m \geq k$ and if ρ satisfies the conditions **E** for $mm_0(lL + \Lambda)$ and **E** for $mm_0\rho^*(lL + \Lambda) + \varphi^*(dH)$, then

$$-E(mm_0(lL + \Lambda) + A) \leq \rho^* D - E(mm_0\rho^*(lL + \Lambda) + \varphi^*(dH)).$$

There exist a positive integer a and an effective divisor D' of V such that $a(\Lambda - \varepsilon l D_1) \sim \varphi^*(dH) + D'$ and $X_i \not\subset \operatorname{Supp} D'$ for any i. Then

$$E(mm_0(lL + \Lambda) + a(\Lambda - \varepsilon l D_1)) \leq E(mm_0(lL + \Lambda) + \varphi^*(dH)) + \rho^* D'$$

and thus

$$-E(mm_0(lL + \Lambda) + A) \leq \rho^*(D + D') - E(mm_0(lL + \Lambda) + a(\Lambda - \varepsilon l D_1)),$$

if ρ satisfies also the condition **E** for $mm_0(lL + \Lambda) + a(\Lambda - \varepsilon l D_1)$. We can choose $m \gg 1$ so that $(V \& X, \Delta + \varepsilon D_1 + (1/mm_0)(D + D'))$ is log-terminal. Here

$$mm_0(lL + \Lambda) + a(\Lambda - \varepsilon l D_1) = m'(lL + \Lambda) + b'((l-1)L + \Lambda - \varepsilon D_1)$$

for $m' = mm_0 - a(l-1)$ and $b' = al$. Thus we can apply **3.3** to

$$(\Delta + \varepsilon D_1, lL + \Lambda, (l-1)L + \Lambda - \varepsilon D_1, (l-1)/l, mm_0, m', b', A, D + D')$$
$$\text{as } (\Delta, L, L', \beta, m, m', b, A, D).$$

Hence, the surjectivity follows.

Step 3. General case. Let b be a positive integer with $b\Lambda$ being a $\mathbb{Z}$-divisor. We may assume that $\pi_*\mathcal{O}_{X_i}(lL + \llcorner \Lambda \lrcorner) \neq 0$ for any i. Then $\pi_*\mathcal{O}_{X_i}(m(lL + \Lambda)) \neq 0$ for any $m > 0$ divisible by b and for any i. Thus we infer that $m(lL + \Lambda) \in \mathbb{E}$ and $\mathcal{I}[m(lL + \Lambda)] = \mathcal{J}[m(lL + \Lambda)]$ by applying *Step 2* to $m\Lambda$ instead of Λ. If $m > 0$ is divisible by b, $m\varepsilon \in \mathbb{Z}$, and $\operatorname{Bs}|m(\Lambda - \varepsilon D_1)| = \emptyset$, and if ρ satisfies the conditions **E** for $m(l-1)(lL + \Lambda)$ and **E** for $ml((l-1)L + \Lambda)$, then

$$-E(m(l-1)(lL + \Lambda)) \leq m\varepsilon\rho^* D_1 - E(ml((l-1)L + \Lambda)).$$

Note that $(V \& X, \Delta + \Delta^*)$ is log-terminal for $\Delta^* := (\varepsilon/(l-1))D_1$. Then we infer that $(lL + \Lambda, (l-1)L + \Lambda, m(l-1), ml, \Delta^*, 0)$ satisfies the condition of **3.2** as $(L, L', m, m^*, \Delta^*, A)$. Thus the surjectivity follows. □

3.15. Lemma *In the situation* **3.11***, suppose that* $\dim V > \dim S$*. Let* L *be a* π*-pseudo-effective divisor of* V*,* C *a divisor of* V*,* Θ *a prime divisor of* V*, and* $X_i \subset X$ *a component of* X *satisfying the following conditions*:

(1) $\pi(X_i)$ *is a prime divisor of* S;
(2) $\kappa_\sigma(\rho^* L; W/Z) = \kappa(\rho^* L; W/Z) = 0$;
(3) $\pi(\Theta) = S$ *and* $\varphi(\Theta')$ *is a prime divisor of* Z *for the proper transform* Θ' *of* Θ *in* W;
(4)
$$\overline{\lim_{m\to\infty}}\, m^{-(\dim Z - \dim S)} \operatorname{rank} \mathcal{G}_i[mL + C + \Theta] > 0.$$

Then
$$\overline{\lim_{m\to\infty}}\, m^{-(\dim Z - \dim S)} \operatorname{rank} \mathcal{G}_i[mL + C] > 0.$$

Proof. By **V.2.26**, we may assume that $\rho^* L \sim \varphi^* \Xi + N$ for a divisor Ξ on Z and the effective divisor $N = N_\sigma(\rho^* L; W/Z)$. There exists a positive integer b such that
$$\pi_* \rho_* \mathcal{O}_W(m\varphi^* \Xi + bN + \rho^*(C + \Theta)) \to \pi_* \mathcal{O}_V(mL + C + \Theta)$$
is isomorphic for $m \geq 0$. Thus we may assume that $W = V$ and $L = \varphi^* \Xi$ for a ϕ-pseudo-effective divisor Ξ. We consider the following commutative diagram of exact sequences:
$$\begin{array}{ccccc} \mathcal{O}_V(mL + C) & \longrightarrow & \mathcal{O}_V(mL + C + \Theta) & \longrightarrow & \mathcal{O}_\Theta(mL + C + \Theta) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{O}_{X_i}(mL + C) & \longrightarrow & \mathcal{O}_{X_i}(mL + C + \Theta) & \longrightarrow & \mathcal{O}_{X_i \cap \Theta}(mL + C + \Theta). \end{array}$$
Let $\mathcal{E}_m$ be the image of the homomorphism
$$\pi_* \mathcal{O}_V(mL + C + \Theta) \to \pi_* \mathcal{O}_\Theta(mL + C + \Theta).$$
Then this is a torsion-free sheaf of S and
$$\overline{\lim_{m\to\infty}}\, m^{-(\dim Z - \dim S)} \operatorname{rank} \mathcal{E}_m = 0,$$
since
$$\operatorname{rank} \mathcal{E}_m \leq \operatorname{rank} \pi_* \mathcal{O}_\Theta(mL + C + \Theta) = \operatorname{rank} \phi_* \left(\mathcal{O}_Z(m\Xi) \otimes \varphi_* \mathcal{O}_\Theta(C + \Theta)\right).$$
By the commutative diagram above, we infer that there is a surjection
$$\mathcal{E}_m \otimes \mathcal{O}_{\pi(X_i)} \twoheadrightarrow \mathcal{G}_i[mL + C + \Theta]/\mathcal{G}_i[mL + C].$$
Thus we have the estimate of $\mathcal{G}_i[mL + C]$ by (4). □

3.16. Theorem *Let* L *be a* π*-abundant divisor of* V*. Suppose that*

(1) $\pi(X_i)$ *is a prime divisor of* S *for any* X_i*,*
(2) $L - (K_V + X + \Delta)$ *is* π*-nef and* π*-abundant,*
(3) $\kappa(L|_{X_i}; X_i/\pi(X_i)) \geq \kappa(L; V/S)$ *for any* i*.*

Then the restriction homomorphism $\pi_* \mathcal{O}_V(lL) \to \pi_* \mathcal{O}_X(lL)$ *is surjective for any* $l \geq 1$*.*

Proof. The result for the case: $l = 1$ is derived from **2.9**, since L satisfies the condition (VI-2). Thus we may assume $l > 1$. Furthermore, we may assume $\dim V - \dim S > \kappa(L; V/S)$ by **3.10**. By **V.4.2**, L is geometrically π-abundant. Thus we have a commutative diagram (VI-8) such that $\kappa(L; V/S) = \dim Z - \dim S$ and $\kappa_\sigma(\rho^* L; W/Z) = \kappa(\rho^* L; W/Z) = 0$. We may assume $W = V$ by the same argument as in *Step 1* of the proof of **3.14**. By applying **3.14** to $\Lambda = \varphi^* H$ for a ϕ-very ample divisor H on Z, we infer that

$$\pi_* \mathcal{O}_V(mL + \varphi^* H) \to \pi_* \mathcal{O}_X(mL + \varphi^* H)$$

is surjective for $m > 0$. In particular,

$$\mathcal{I}[mL] \subset \mathcal{I}[mL + \varphi^* H] = \mathcal{J}[mL + \varphi^* H].$$

The surjection and the condition (3) imply the estimate

$$\overline{\lim_{m\to\infty}}\, m^{-(\dim Z - \dim S)} \operatorname{rank} \mathcal{G}_i[mL + \varphi^* H] > 0$$

for any i. By applying **3.15** to $C = -\varphi^* H$ and a general member Θ of $|2\varphi^* H|$, we have

$$\overline{\lim_{m\to\infty}}\, m^{-(\dim Z - \dim S)} \operatorname{rank} \mathcal{G}_i[mL - \varphi^* H] > 0.$$

In particular, there exist a positive integer a and an effective divisor D such that $alL \sim D + \varphi^* H$ and $\operatorname{Supp} D$ contains no X_i. Thus $(m + a)lL \in \mathbb{E}$ for any $m > 0$. Moreover, if $\rho\colon W \to V$ is a bimeromorphic morphism satisfying the conditions **E** for $mlL + \varphi^* H$ and **E** for $(m + a)lL$, then

$$-E(mlL + \varphi^* H) \le \rho^* D - E((m + a)lL).$$

We choose m so large that $(V \& X, \Delta + (1/m)D)$ is log-terminal. Then the condition of **3.3** is satisfied for

$$(lL, (l-1)L, (l-1)/l, m, m+a, 0, \varphi^* H, D) \quad \text{as} \quad (L, L', \beta, m, m', b, A, D).$$

Hence the surjectivity follows. □

§4. Degeneration of projective varieties

In this section, we consider a projective surjective morphism $\mathcal{X} \to S$ with connected fibers from a normal complex analytic variety onto a non-singular curve, and a point $0 \in S$. Let $\mathcal{X}_s$ denote the scheme-theoretic fiber over $s \in S$ and let $\mathcal{X}_0 = \bigcup \Gamma_i$ be the irreducible decomposition of the special fiber. In this situation, after replacing S by an open neighborhood of 0, we have a bimeromorphic morphism $\nu\colon V \to \mathcal{X}$ from a non-singular variety such that

(1) the proper transform X_i of Γ_i is non-singular,
(2) X_i are disjoint to each other,
(3) the composite $\pi\colon V \to \mathcal{X} \to S$ is projective.

Note that $\pi^{-1}(s)$ is a non-singular projective model of the normal projective variety $\mathcal{X}_s$ for general $s \in S$. For a projective variety Γ with singularities, the Kodaira dimension $\kappa(\Gamma)$, the numerical Kodaira dimension $\kappa_\sigma(\Gamma)$, and the m-genus $P_m(\Gamma)$, respectively, are defined as the corresponding invariants for a non-singular model of Γ (cf. Chapter **III**, §**4.a**, and **V.2.29**).

4.1. Theorem *The numerical Kodaira dimension κ_σ is lower semi-continuous in the sense that, for a general fiber $\mathcal{X}_s$,*

$$\kappa_\sigma(\mathcal{X}_s) \geq \max \kappa_\sigma(\Gamma_i).$$

PROOF. We may assume that K_{X_i} is pseudo-effective for some i. By setting $X := \sum X_i$, $L := K_V + X$, and $\Delta := 0$, we apply results in §**2**. Then L is π-pseudo-effective by **2.12**. Therefore, for any π-ample divisor A of V and for $m \gg 0$, the restriction homomorphism

$$\pi_*\mathcal{O}_V(mL + A) \otimes \mathbb{C}(0) \to \bigoplus\nolimits_i \mathrm{H}^0(X_i, mK_{X_i} + A|_{X_i}),$$

is surjective by **3.7**. The direct image $\pi_*\mathcal{O}_V(mL+A)$ is a locally free sheaf of rank

$$\dim \mathrm{H}^0(V_s, mK_{V_s} + A|_{V_s}),$$

for a general fiber V_s of π. Thus the lower semi-continuity follows. □

As a consequence, we have:

4.2. Theorem *The numerical Kodaira dimension κ_σ is invariant under a smooth projective deformation.*

In particular, if a smooth fiber is of general type, then any other smooth fiber is also of general type.

4.3. Theorem *Let I be the set of indices i such that Γ_i is of general type. If $I \neq \emptyset$, then, for any $m > 0$,*

$$P_m(\mathcal{X}_s) \geq \sum\nolimits_{i \in I} P_m(\Gamma_i).$$

PROOF. We set $X := \sum_{i \in I} X_i$, $\Delta := 0$, and $L := K_V + X$. Now $L|_{X_i}$ is big for any i. Thus L is π-big by **4.1**. The restriction homomorphism

$$\pi_*\mathcal{O}_V(mL) \to \bigoplus\nolimits_{i \in I} \mathrm{H}^0(X_i, mK_{X_i})$$

is surjective for any $m > 0$, by **3.10**. Hence the inequality follows since $P_m(\mathcal{X}_s) = \operatorname{rank} \pi_*\mathcal{O}_V(mL)$. □

As a consequence of **4.2** and **4.3**, we have:

4.4. Theorem *The plurigenera P_m are invariant under a smooth projective deformation of an algebraic variety of general type.*

Next, we shall treat the case in which the abundance $\kappa_\sigma(\mathcal{X}_s) = \kappa(\mathcal{X}_s)$ holds for a 'general' fiber $\mathcal{X}_s$.

4.5. Theorem *Suppose that* $\kappa(\mathcal{X}_s) = \kappa_\sigma(\mathcal{X}_s)$ *for a 'general' fiber* $\mathcal{X}_s$. *Let* I *be the set of indices* i *with* $\kappa_\sigma(\Gamma_i) = \kappa(\mathcal{X}_s)$. *Then, for any* $m > 0$,

$$P_m(\mathcal{X}_s) \geq \sum\nolimits_{i \in I} P_m(\Gamma_i).$$

PROOF. We set $X := \sum_{i \in I} X_i$, $\Delta := 0$, and $L := K_V + X$, where L is π-abundant. Then the restriction homomorphism

$$\pi_* \mathcal{O}_V(mL) \to \bigoplus\nolimits_{i \in I} \mathrm{H}^0(X_i, mK_{X_i})$$

is surjective for any $m > 0$, by **3.16**. Hence the inequality follows since $P_m(\mathcal{X}_s) = \operatorname{rank} \pi_* \mathcal{O}_V(mL)$. □

4.6. Corollary *The plurigenera* P_m *are invariant under a smooth projective fibration of algebraic varieties in which the abundance* $\kappa_\sigma(\mathcal{X}_s) = \kappa(\mathcal{X}_s)$ *holds for a 'general' fiber* $\mathcal{X}_s$.

§5. Deformation of singularities

Let S be a normal variety, $\Theta \subset S$ a prime divisor, and $\pi\colon V \to S$ a projective bimeromorphic morphism from a non-singular variety such that the proper transform X of Θ is non-singular. Then, by **3.9**, the homomorphism

$$\pi_* \mathcal{O}_V(m(K_V + X)) \to \pi_* \mathcal{O}_X(mK_X) \tag{VI-10}$$

is surjective for any $m > 0$. Furthermore, if A is a π-ample divisor of V, then

$$\pi_* \mathcal{O}_V(m(K_V + X) + A) \to \pi_* \mathcal{O}_X(mK_X + A) \tag{VI-11}$$

is also surjective for $m > 0$ by **3.7**.

Let Δ be an effective $\mathbb{R}$-divisor of S whose support does not contain Θ. Suppose that

(1) $K_S + \Theta + \Delta$ is $\mathbb{R}$-Cartier,
(2) $\llcorner \Delta \lrcorner = 0$,
(3) Θ is normal,
(4) the union of $\pi^{-1}(\operatorname{Supp} \Delta \cup \Theta)$ and the π-exceptional locus is a normal crossing divisor.

For the $\mathbb{R}$-divisor

$$R := K_V + X - \pi^*(K_S + \Theta + \Delta),$$

we set $\Delta_\Theta := -(\pi|_X)_*(R|_X)$. Then we have

$$R|_X - K_X = -(\pi|_X)^*(K_\Theta + \Delta_\Theta) \quad \text{and} \quad (K_S + \Theta + \Delta)|_\Theta \sim_{\mathbb{R}} K_\Theta + \Delta_\Theta.$$

The following result is known as the inversion of adjunction (cf. [**132**], [**74**]):

5.1. Proposition *If* (Θ, Δ_Θ) *is log-terminal, then* $(S \& \Theta, \Delta)$ *is log-terminal along* Θ (cf. **II.4.8**).

PROOF. It is enough to show $\ulcorner R \urcorner \geq 0$ over a neighborhood of Θ. Since $R - X - K_V$ is π-nef, we have the surjection

$$\pi_* \mathcal{O}_V(\ulcorner R \urcorner) \twoheadrightarrow \pi_* \mathcal{O}_X(\ulcorner R \urcorner)$$

by the vanishing theorem **II.5.12**. By assumption, $\ulcorner R \urcorner$ is a π-exceptional divisor and $\ulcorner R|_X \urcorner$ is an effective $(\pi|_X)$-exceptional divisor. Therefore, for the natural injection

$$\pi_* \mathcal{O}_V(\ulcorner R \urcorner) \hookrightarrow \pi_* \mathcal{O}_V \simeq \mathcal{O}_S,$$

the tensor product

$$\pi_* \mathcal{O}_V(\ulcorner R \urcorner) \otimes \mathcal{O}_\Theta \to \mathcal{O}_\Theta$$

is surjective. Therefore, $\pi_* \mathcal{O}_V(\ulcorner R \urcorner) \hookrightarrow \mathcal{O}_S$ is isomorphic along Θ. Thus $\ulcorner R \urcorner \geq 0$ over Θ. □

By using (VI-10) and (VI-11), we have the following inversions of adjunction.

5.2. Theorem *Let S be a normal variety and let Θ be a prime divisor. Suppose that $K_S + \Theta$ is $\mathbb{Q}$-Cartier and Θ is Cartier in codimension two in S.*

(1) *If Θ has only canonical singularities, then $S\&\Theta$ is canonical along Θ.*
(2) *If Θ has only terminal singularities, then $S\&\Theta$ is terminal along Θ.*

PROOF. (1) Let m be a positive integer such that $m(K_S + \Theta)$ is Cartier. By assumption,

$$\mathcal{O}_\Theta(m(K_S + \Theta)) \simeq \mathcal{O}_\Theta(mK_\Theta) \simeq \pi_* \mathcal{O}_X(mK_X).$$

Since (VI-10) is surjective, the homomorphism

$$\pi_* \mathcal{O}_V(m(K_V + X)) \otimes \mathcal{O}_\Theta \to \mathcal{O}_S(m(K_S + \Theta)) \otimes \mathcal{O}_\Theta$$

is also surjective. Hence $\pi_* \mathcal{O}_V(m(K_V + X)) \simeq \mathcal{O}_S(m(K_S + \Theta))$ along Θ. Therefore $S\&\Theta$ is canonical along Θ.

(2) For the bimeromorphic morphism $\pi\colon V \to S$, we may assume that there is an effective divisor E such that

- $-E$ is π-ample,
- $\operatorname{Supp} E$ is the π-exceptional locus,
- $X \cap \operatorname{Supp} E$ is also $(\pi|_X)$-exceptional.

Thus the homomorphism

$$\pi_* \mathcal{O}_V(m(K_V + X) - E) \to \pi_* \mathcal{O}_X(mK_X - E|_X)$$

is of the form (VI-11) and hence is surjective for any $m > 0$. There is a positive integer m such that $m(K_S + \Theta)$ is Cartier, $\mathcal{O}_\Theta(m(K_S + \Theta)) \simeq \mathcal{O}_\Theta(mK_\Theta)$, and $\pi_* \mathcal{O}_X(mK_X - E|_X) \simeq \mathcal{O}_\Theta(mK_\Theta)$. Thus the homomorphism

$$\pi_* \mathcal{O}_V(m(K_V + X) - E) \otimes \mathcal{O}_\Theta \to \mathcal{O}_S(m(K_S + \Theta)) \otimes \mathcal{O}_\Theta$$

is surjective. Hence $\pi_* \mathcal{O}_V(m(K_V + X) - E) \simeq \mathcal{O}_S(m(K_S + \Theta))$ along Θ. Therefore $S\&\Theta$ is terminal along Θ. □

5.3. Corollary

(1) *Small deformations of canonical singularities are canonical* ([**60**], cf. [**61**, 7-2-4]).
(2) *Small deformations of terminal singularities are terminal.*

Proof. In the situation above, suppose that Θ is a Cartier divisor of S and that Θ is a normal variety with only canonical singularities. The complement $S^\circ \subset S$ of $\operatorname{Sing}\Theta$ is non-singular. Let $j\colon S^\circ \hookrightarrow S$ be the immersion and let m be a positive integer with mK_Θ being Cartier. We have a commutative diagram

$$\begin{array}{ccccc} \pi_*\mathcal{O}_V(m(K_V+X)) & \longrightarrow & \mathcal{O}_S(m(K_S+\Theta)) & = & j_*\mathcal{O}_{S^\circ}(m(K_S+\Theta)) \\ \downarrow & & & & \downarrow \\ \pi_*\mathcal{O}_X(mK_X) & = & \mathcal{O}_\Theta(mK_\Theta) & = & j_*(\mathcal{O}_\Theta(mK_\Theta)|_{S^\circ}). \end{array}$$

The left vertical arrow is just (VI-10) and is surjective. Hence

$$\mathcal{O}_S(m(K_S+\Theta)) \otimes \mathcal{O}_\Theta \to \mathcal{O}_\Theta(mK_\Theta)$$

is surjective and moreover is an isomorphism, since Θ is Cartier (cf. **II.2.2**-(2)). Therefore, mK_S is Cartier along Θ. By **5.2**, S has only canonical singularities or only terminal singularities according as Θ has so. □

5.4. Definition (Knöller [**65**]) Let (X,P) be a normal isolated singularity. For $m \in \mathbb{N}$ and for a resolution of the singularity $\mu\colon Y \to X$, the *m-genus* γ_m is defined by

$$\gamma_m(X,P) := \operatorname{length} \mathcal{O}_X(mK_X)_P/\mu_*\mathcal{O}_Y(mK_Y)_P.$$

This is independent of the choice of resolutions.

Ishii [**44**] has proved the following theorem under some assumption [**44**, 1.9]. However the assumption is satisfied since (VI-10) is surjective.

5.5. Theorem *The m-genus γ_m is upper semi-continuous under a flat deformation in the following sense: let $f\colon S \to T$ be a flat morphism into an open neighborhood $T \subset \mathbb{C}$ of the origin 0 such that the central fiber $f^{-1}(0) = S_0$ is scheme-theoretically a normal variety with only one singular point P. Then there is an open neighborhood $U \subset S$ of P such that the inequality*

$$\gamma_m(S_0,P) \geq \sum\nolimits_{Q \in \operatorname{Sing} S_t \cap U} \gamma_m(S_t,Q)$$

holds for any other fiber $S_t = f^{-1}(t)$.

Proof. We write $\Theta = S_0$ and use the same notation as before. Let $\mathcal{C}_m$ be the cokernel of the natural injection

$$\pi_*\mathcal{O}_V(m(K_V+X)) \to \mathcal{O}_S(m(K_S+\Theta)).$$

Then $\operatorname{Supp}\mathcal{C}_m$ is finite over a neighborhood of $0 \in T$. By replacing T, we may assume that $\operatorname{Supp}\mathcal{C}_m$ is finite over T and $f_*\mathcal{C}_m$ is a coherent $\mathcal{O}_T$-module. Then

$$\operatorname{rank}_{\mathcal{O}_T} f_*\mathcal{C}_m = \sum\nolimits_{Q\in S_t} \gamma_m(S_t, Q) \quad \text{for } t \neq 0, \quad \text{and}$$

$$\operatorname{length}_{\mathcal{O}_{\Theta,P}}(\mathcal{C}_m \otimes \mathcal{O}_\Theta)_P = \dim f_*\mathcal{C}_m \otimes \mathbb{C}(0) \geq \operatorname{rank}_{\mathcal{O}_T} f_*\mathcal{C}_m.$$

In the commutative diagram

$$\begin{array}{ccc} \pi_*\mathcal{O}_V(m(K_V + X)) \otimes \mathcal{O}_\Theta & \longrightarrow & \mathcal{O}_S(m(K_S + \Theta)) \otimes \mathcal{O}_\Theta \\ \downarrow & & \downarrow \\ \pi_*\mathcal{O}_X(mK_X) & \longrightarrow & \mathcal{O}_\Theta(mK_\Theta), \end{array}$$

the left vertical arrow of is surjective. The right vertical arrow is injective, since Θ is normal and Cartier. Therefore, we have an injection

$$\mathcal{C}_m \otimes \mathcal{O}_\Theta \hookrightarrow \mathcal{O}_\Theta(mK_\Theta)/\pi_*\mathcal{O}_X(mK_X),$$

which induces the upper semi-continuity of γ_m. $\square$