CHAPTER II

# Preliminaries

This chapter recalls some fundamental facts for the study of complex analytic and algebraic varieties. Some of them are well-known and we include no proofs. Some new notions and terminologies are introduced for the clarification of arguments in the subsequent chapters. We review some basic properties of complex analytic varieties in §**1**. The notion of divisor and some variants are explained in §**2**. The theory of linear systems is fundamental in the subject of algebraic geometry. Iitaka's theory of $D$-dimension has its base on the study of linear systems. We generalize the theories to those applicable to $\mathbb{R}$-divisors in §**3**, by using a result in Chapter **III**. Information most essential to a variety, such as Kodaira dimension, is usually derived from the information on the canonical divisor. The singularities appearing in the minimal model program for the birational classification of algebraic varieties are all related to some properties of the canonical divisor. They are the subjects of study in §**4**. Numerical properties of ample, nef, big, and pseudo-effective for $\mathbb{R}$-divisors are discussed in §**5**. Vanishing theorems related to the Kodaira vanishing are also mentioned. In §**6**, we recall such basics as Chern classes and semi-stability, indispensable for the study of vector bundles.

## §1. Complex analytic varieties

**§1.a. General theory.** A *complex analytic space* $X$ is a locally ringed space $(X, \mathcal{O}_X)$ that is locally isomorphic to the closed subspace of an open subset $U$ of some complex affine space $\mathbb{C}^N$ defined as $X = \operatorname{Supp} \mathcal{O}_U/\mathcal{I} \subset U$ and $\mathcal{O}_X = \mathcal{O}_U/\mathcal{I}|_X$ for a coherent $\mathcal{O}_U$-ideal sheaf $\mathcal{I}$. Here $\mathcal{O}_U$ is the sheaf of germs of holomorphic functions on $U$ and a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is called *coherent* if it satisfies the following conditions:

(1) It is finitely generated locally on $X$: For any point of $X$, there exist an open neighborhood $U$ and a surjective homomorphism $\mathcal{O}_X^{\oplus k}|_U \to \mathcal{F}|_U$ for some $k \in \mathbb{N}$;

(2) For any homomorphism $\mathcal{O}_X^{\oplus l}|_U \to \mathcal{F}|_U$ over an open subset $U \subset X$, its kernel is finitely generated locally on $U$.

For a fixed complex analytic space $X$, a sheaf of $\mathcal{O}_X$-modules is called an $\mathcal{O}_X$-module, and a coherent $\mathcal{O}_X$-module is called simply a coherent sheaf. In this article, we always assume that complex analytic spaces are all Hausdorff and paracompact. We drop the words 'complex' and 'analytic' sometimes.

An analytic subspace $Z$ of $X$ is defined by a coherent $\mathcal{O}_X$-ideal sheaf $\mathcal{J}$ as $Z = \operatorname{Supp} \mathcal{O}_X/\mathcal{J}$ and $\mathcal{O}_Z := \mathcal{O}_X/\mathcal{J}|_Z$. An *analytic subset* is the support of an analytic subspace. It is also called a *Zariski-closed* subset. A *Zariski-open* subset is the complement of an analytic subset. Note that even if $V$ is a Zariski-open subset of $U$ and $U$ is a Zariski-open subset of $X$, the subset $V$ is not necessarily Zariski-open in $X$.

**Notation** Let $X$ be a complex analytic space. The assertion that a property $P$ holds for a *general* point $x \in X$ means that $P$ holds for any point $x$ contained in a Zariski-open dense subset of $X$. The assertion that $P$ holds for a 'general' point means that $P$ holds for any point $x$ contained in a countable intersection of Zariski-open dense subsets.

If $X$ is a union of two mutually distinct proper analytic subsets, then $X$ is called *reducible*. If $X$ is not reducible, it is called *irreducible*. If every local ring $\mathcal{O}_{X,x}$ is reduced, then $X$ is called *reduced*. An irreducible and reduced complex analytic space is called a *complex analytic variety*.

An *locally free sheaf* $\mathcal{E}$ of rank $r$ on a complex analytic space $X$ is a coherent $\mathcal{O}_X$-module such that $\mathcal{E} \simeq \mathcal{O}_X^{\oplus r}$ locally on $X$. The number $r$ is called the *rank* of $\mathcal{E}$ and denoted by $\operatorname{rank} \mathcal{E}$. An *invertible sheaf* is a locally free sheaf of rank one. If $\mathcal{L}$ is an invertible sheaf, then $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^\vee \simeq \mathcal{O}_X$ for the dual $\mathcal{L}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$. We define $\mathcal{L}^{\otimes(-m)}$ for $m \in \mathbb{N}$ by $(\mathcal{L}^\vee)^{\otimes m}$. The set of invertible sheaves on $X$ forms an abelian group whose product is given by the tensor-product. The group is called the *Picard group* and denoted by $\operatorname{Pic}(X)$. This is isomorphic to $\mathrm{H}^1(X, \mathcal{O}_X^\star)$ for the sheaf $\mathcal{O}_X^\star$ of germs of *invertible* (or *unit*) holomorphic functions on $X$. A locally free sheaf is called also a *vector bundle*, since it corresponds to a geometric vector bundle $\mathbb{V}(\mathcal{E})$ (cf. §**1.b**). A subsheaf $\mathcal{G} \subset \mathcal{E}$ is called a *subbundle* if $\mathcal{G}$ and $\mathcal{E}/\mathcal{G}$ are both vector bundles.

A coherent sheaf $\mathcal{F}$ on a complex analytic variety $X$ is called *torsion-free* if there is no non-zero coherent subsheaf $\mathcal{G} \subset \mathcal{F}$ with $\operatorname{Supp} \mathcal{G} \neq X$. This is the case where the local cohomology sheaf $\mathcal{H}^0_Z(\mathcal{F}) = 0$ for any proper analytic subset $Z \subset X$. If $\mathcal{F}$ is an arbitrary coherent sheaf, then there is the maximum coherent subsheaf $\mathcal{G} \subset \mathcal{F}$ with $\operatorname{Supp} \mathcal{G} \neq X$, which is called the *torsion part* of $\mathcal{F}$ and is denoted by $\mathcal{F}_{\mathrm{tor}}$. The quotient $\mathcal{F}/\mathcal{F}_{\mathrm{tor}}$ is torsion-free, which is denoted by $\mathcal{F}/(\mathrm{tor})$ for short.

A morphism $f\colon X \to Y$ of complex analytic spaces is a morphism as locally ringed spaces. It is called *proper* if $f^{-1}K$ is compact for any compact subset $K$ of $Y$. The Grauert direct image theorem (cf. [**28**], [**63**], [**14**]) states that the higher direct image sheaves $\mathrm{R}^i f_*\mathcal{F}$ for a coherent sheaf $\mathcal{F}$ are coherent for a proper morphism $f$. If $f$ is proper and $f^{-1}(y)$ is a finite set for all $y \in Y$, then $f$ is called a *finite* morphism. For any proper morphism $f\colon X \to Y$, the direct image sheaf $f_*\mathcal{O}_X$ is a coherent $\mathcal{O}_Y$-module. It defines a finite morphism $\tau\colon V \to Y$ satisfying the following conditions:

(1) there is a proper surjective morphism $g\colon X \to V$ with $f = \tau \circ g$;
(2) $\mathcal{O}_V \simeq g_*\mathcal{O}_X$.

Here $V$ is realized as $\operatorname{Specan}_Y f_*\mathcal{O}_X$ (cf. §**1.b**). By the formal function theorem, any fiber of $g$ is connected. The factorization $f = \tau \circ g$ satisfying the conditions above is unique up to isomorphisms and is called the *Stein factorization* of $f$.

The local ring $\mathcal{O}_{X,x}$ at a point $x$ of a complex analytic space $X$ is Noetherian and is a finite extension of the ring $\mathbb{C}\{z_1, z_2, \dots, z_N\}$ of convergent power series for some $N$. If $\mathcal{O}_{X,x}$ is a normal ring (an integrally closed domain), then $X$ is called *normal* at $x$. The set $X_{\mathrm{nor}}$ of points $x \in X$ with $\mathcal{O}_{X,x}$ being normal is a Zariski-open subset. If $X = X_{\mathrm{nor}}$, then $X$ is called normal. A normal complex analytic space is a disjoint union of countably many normal varieties. Suppose that $X$ is reduced. Then $X_{\mathrm{nor}}$ is dense and there is the *normalization* $\nu\colon \tilde{X} \to X$ satisfying the following properties:

(1) $\tilde{X}$ is normal;
(2) $\nu$ is finite and surjective;
(3) $\nu^{-1}X_{\mathrm{nor}}$ is a dense Zariski-open subset of $\tilde{X}$;
(4) $\nu^{-1}X_{\mathrm{nor}} \to X_{\mathrm{nor}}$ is an isomorphism.

Let $X$ be a complex analytic variety and let $f\colon Y \to X$ be a finite surjective morphism from a normal variety $Y$. Zariski's Main Theorem states that if $f^{-1}(x)$ consists of one point for a general point $x \in X$, then $f$ is isomorphic to the normalization of $X$. We can show the following property by applying Zariski's Main Theorem to the Stein factorization: *Let $f\colon Y \to X$ be a proper surjective morphism of normal varieties. If a general fiber of $f$ is connected, then $\mathcal{O}_X \simeq f_*\mathcal{O}_Y$.* A proper surjective morphism $f\colon Y \to X$ is called a *fiber space* or a *fibration* if $X$ and $Y$ are normal and $f$ has only connected fibers.

**§1.b. Spec and Proj.** Let $A$ be a finitely generated $\mathbb{C}$-algebra and let $\mathbb{C}[x] \to A$ be a surjective $\mathbb{C}$-algebra homomorphism from the polynomial ring $\mathbb{C}[x] = \mathbb{C}[x_1, x_2, \dots, x_d]$ of $d$-variables. Then the associated analytic space $\operatorname{Specan} A$ to $\operatorname{Spec} A$ is realized as a closed analytic subspace of $\mathbb{C}^d = \operatorname{Specan}\mathbb{C}[x] \simeq (\operatorname{Spec}\mathbb{C}[x])^{\mathrm{an}}$. There is also a canonical morphism $\operatorname{Specan} A \to \operatorname{Spec} A$ as locally ringed spaces. By the canonical homomorphism $A \to \mathrm{H}^0(\operatorname{Specan} A, \mathcal{O})$, we have the following universal property: let $Y$ be a complex analytic space. Then giving a morphism $Y \to \operatorname{Specan} A$ is equivalent to giving a $\mathbb{C}$-algebra homomorphism $A \to \mathrm{H}^0(Y, \mathcal{O}_Y)$.

**Example** Let $V$ be an $n$-dimensional $\mathbb{C}$-vector space and let $A$ be the symmetric algebra

$$\operatorname{Sym} V = \operatorname{Sym}^\bullet V = \bigoplus\nolimits_{d=0}^{\infty} \operatorname{Sym}^d V.$$

Then $A$ is isomorphic to the polynomial ring of $n$-variables and $\operatorname{Specan} A$ is isomorphic to the dual vector space $V^\vee = \operatorname{Hom}(V, \mathbb{C})$ as a complex analytic space. Note that $\operatorname{Sym} V \hookrightarrow \mathrm{H}^0(\operatorname{Specan} A, \mathcal{O}) = \mathrm{H}^0(V^\vee, \mathcal{O}_{V^\vee})$ is not surjective.

Let $\mathsf{X}$ be a separated scheme locally of finite type over $\operatorname{Spec}\mathbb{C}$. Then $\mathsf{X}$ is covered by open affine schemes $\operatorname{Spec} A_i$ in which $A_i$ are finitely generated over $\mathbb{C}$. We can define naturally the associated analytic space $X = \mathsf{X}^{\mathrm{an}}$ by gluing $\operatorname{Specan} A_i$. There is a canonical morphism $\epsilon\colon X \to \mathsf{X}$ as locally ringed spaces. For an $\mathcal{O}_{\mathsf{X}}$-module $\mathsf{F}$ of the scheme $\mathsf{X}$, we can associate an $\mathcal{O}_X$-module by $\mathsf{F}^{\mathrm{an}} := \epsilon^{-1}\mathsf{F} \otimes \mathcal{O}_X$. The following properties are known as GAGA [**128**]:

(1) $\mathsf{X}$ is proper over $\operatorname{Spec}\mathbb{C}$ if and only if $X = \mathsf{X}^{\mathrm{an}}$ is compact;
(2) If $X = \mathsf{X}^{\mathrm{an}}$ is compact and if $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, then $\mathcal{F} = \mathsf{F}^{\mathrm{an}}$ for a coherent $\mathcal{O}_{\mathsf{X}}$-module $\mathsf{F}$ and the natural homomorphism
$$\mathrm{H}^i(\mathsf{X}, \mathsf{F}) \longrightarrow \mathrm{H}^i(X, \mathcal{F})$$
is isomorphic for any $i$.

Let $A = \bigoplus_{d=0}^{\infty} A_d$ be a graded $\mathbb{C}$-algebra. For $k > 0$, we define $A^{(k)}$ to be the subalgebra
$$A^{(k)} = \bigoplus\nolimits_{d=0}^{\infty} A_{kd} \subset A$$
and define its grading by $A^{(k)}_d = A_{kd}$. For a homogeneous non-zero element $a \in A_d$, let $A_a = A[a^{-1}]$ be the localization of $A$ by the multiplicatively closed subset $\{a^k \mid k \geq 0\}$ and define
$$A_{(a)} := \left\{ \frac{b}{a^l} \;\Big|\; b \in A_{ld},\, l \in \mathbb{N} \right\} \subset A_a = A[a^{-1}].$$
Then the homogeneous spectrum $\mathsf{X} = \operatorname{Proj} A$ is the union of open affine subschemes $\operatorname{Spec} A_{(a)}$. Note that $(A^{(k)})_{(a^k)} = A_{(a)}$ and $(A^{(k)})_{(b)} = A_{(b)}$ for $k \in \mathbb{N}$ and for any homogeneous non-zero element $b \in A^{(k)}$. Thus we have an isomorphism $\operatorname{Proj} A \xrightarrow{\simeq} \operatorname{Proj} A^{(k)}$ which is the gluing of $\operatorname{Spec} A_{(b)} \to \operatorname{Spec} A^{(k)}_{(b)}$ for $k \mid \deg b$. Let $M = \bigoplus_{d\in\mathbb{Z}} M_d$ be a graded $A$-module. The twist $M(l)$ by an integer $l \in \mathbb{Z}$ is defined to be the module $M(l) = \bigoplus_{d\in\mathbb{Z}} M(l)_d$ with $M(l)_d = M_{l+d}$. This is also a graded $A$-module. For a non-zero element $a \in A_d$, we set
$$M_{(a)} := \left\{ \frac{m}{a^l} \;\Big|\; m \in M_{ld},\, l \in \mathbb{N} \right\} \subset M_a = M \otimes A_a.$$
Then we can associate naturally an $\mathcal{O}_{\mathsf{X}}$-module $M^{\sim}$ such that
$$\mathrm{H}^0(\operatorname{Spec} A_{(a)}, M^{\sim}) \simeq M_{(a)}.$$
The functor $M \mapsto M^{\sim}$ is exact. Note that if we set $M^{(k)} := \bigoplus_{d\in\mathbb{Z}} M_{kd}$, then $M^{\sim}$ on $\operatorname{Proj} A$ is isomorphic to $(M^{(k)})^{\sim}$ on $\operatorname{Proj} A^{(k)}$. The sheaf $A(l)^{\sim}$ is denoted by $\mathcal{O}_A(l)$. In particular, $\mathcal{O}_{A^{(k)}}(l)$ is isomorphic to $\mathcal{O}_A(kl)$. If $A$ is specified, then $\mathcal{O}_A(l)$ is denoted by $\mathcal{O}_{\mathsf{X}}(l)$. There is a natural graded $A$-linear homomorphism
$$\alpha\colon M = \bigoplus\nolimits_{l\in\mathbb{Z}} M_l \to \Gamma_*(M) := \bigoplus\nolimits_{l\in\mathbb{Z}} \mathrm{H}^0(\mathsf{X}, M(l)^{\sim}).$$
The graded algebra $A$ is called 1-*generated* (over $A_0$) if the multiplication mapping $A_1^{\otimes d} \to A_d$ is surjective for any $d > 0$.

**1.1. Lemma** (cf. [**127**], [**33**, §2], [**34**, §2]) *Suppose that $A$ is finitely generated as an $A_0$-algebra and that $M$ is finitely generated as an $A$-module. Then the following properties hold*:

(1) *There exist a positive integer* $d$ *and an integer* $k$ *such that* $A_d \otimes M_l \to M_{d+l}$ *is surjective for any* $l \geq k$;
(2) *There is a positive integer* $d$ *such that* $A^{(d)}$ *is* 1*-generated and* $A$ *is a finitely generated* $A^{(d)}$*-module*;
(3) $M^{\sim} = 0$ *if and only if there is a positive number* $k$ *such that* $M_{kl} = 0$ *for* $l \gg 0$;
(4) *The natural homomorphism* $M_l \otimes \mathcal{O}_{\mathsf{X}} \to M(l)^{\sim}$ *is surjective for* $l \gg 0$;
(5) *If* $A^{(d)}$ *is* 1*-generated, then* $\mathcal{O}_A(d)$ *is invertible and* $M^{\sim} \otimes_{\mathcal{O}_{\mathsf{X}}} \mathcal{O}_A(d) \simeq M(d)^{\sim}$;
(6) *There exists an isomorphism* $\beta\colon \Gamma_*(M)^{\sim} \xrightarrow{\simeq} M^{\sim}$ *such that* $\beta \circ \alpha^{\sim} = \mathrm{id}$ *for* $\alpha^{\sim}\colon M^{\sim} \to \Gamma_*(M)^{\sim}$;
(7) *If* $A_0$ *is Noetherian, then* $\alpha_l\colon M_l \to \mathrm{H}^0(\mathsf{X}, M(l)^{\sim})$ *is isomorphic for* $l \gg 0$.

Proof. (1) Let $a_1, a_2, \ldots, a_n$ be homogeneous elements of $A$ generating $A$ as an $A_0$-algebra and let $m_1, m_2, \ldots, m_N$ be homogeneous elements of $M$ generating $M$ as an $A$-module. We set $d_i := \deg a_i > 0$, $\mu_j := \deg m_j$, $d := \operatorname{lcm}\{d_i\}$. Furthermore, we define

$$F := \{(r_1, r_2, \ldots, r_n) \mid 0 \leq r_i < d/d_i, r_i \in \mathbb{Z}\}, \quad \text{and}$$
$$c := \max\left\{\mu_j + \sum\nolimits_{i=1}^{n} r_i d_i \;\middle|\; (r_1, r_2, \ldots, r_n) \in F, 1 \leq j \leq N\right\}.$$

Note that $\deg a_i^{d/d_i} = d$. If $\mu_j + \sum p_i d_i \geq c$ for some $p_i \in \mathbb{Z}_{\geq 0}$, then

$$(p_1 - q_1(d/d_1), p_2 - q_2(d/d_2), \ldots, p_n - q_n(d/d_n)) \in F$$

for some $q_i \in \mathbb{Z}_{\geq 0}$. Therefore, if $l \geq c - d$, then $M_{d+l} = A_d M_l$.

(2) is derived from (1).

(3) Let $\{a_i\}$ be the homogeneous generator of $A$ in the proof of (1) and set $d = \operatorname{lcm}\{\deg a_i\}$. Then $M^{\sim} = 0$ if and only if $M_{(a_i)} = 0$ for any $i$. Suppose that there is a positive integer $k$ such that $M_{kl} = 0$ for $l \gg 0$. Then $M_{(a_i)} = 0$, since we may assume $d|k$. Conversely, suppose that $M^{\sim} = 0$. If $m$ is a homogeneous element of $M^{(d)}$, then there is a positive integer $k$ such that $a_i^k m = 0$ for any $i$. Thus $M_{dl} = 0$ for $l \gg 0$ by (1).

(4) For $l \in \mathbb{N}$, let $C = \bigoplus C_n$ be the cokernel of the natural homomorphism $M_l \otimes_{A_0} A \to M(l)$ of graded A-modules. Then $C$ is finitely generated. By (1), there exist positive numbers $d$ and $k$ such that if $l \geq k$, then $C_{dn} = 0$ for any $n > 0$. Hence $C^{\sim} = 0$ by (3) and we have the expected surjection.

(5) $\mathsf{X} = \operatorname{Proj} A$ is covered by $\operatorname{Spec} A_{(a)}$ for $a \in A_d$. We have an isomorphism

$$A_{(a)} \ni \frac{b}{a^j} \longmapsto a \cdot \frac{b}{a^j} = \frac{b}{a^{j-1}} \in A(d)_{(a)}$$

for $b \in A_{dj}$. Hence $\mathcal{O}_{\mathsf{X}}(d)$ is invertible. The tensor product $M \otimes_A A(j)$ has a natural structure of graded $A$-modules and is isomorphic to $M(j)$ for any $j \in \mathbb{Z}$. We want to show the natural homomorphism

$$M^{\sim} \otimes_{\mathcal{O}_{\mathsf{X}}} A(d)^{\sim} \to (M \otimes_A A(d))^{\sim} \simeq M(d)^{\sim}$$

is isomorphic. The homomorphism on the open subset $\operatorname{Spec} A_{(a)}$ is derived from the isomorphism

$$M_{(a)} \otimes_{A_{(a)}} A(d)_{(a)} \ni \frac{m}{a^j} \otimes a \longmapsto \frac{m}{a^{j-1}} \in M(d)_{(a)},$$

where $m \in M_{dj}$. Hence, $M^{\sim} \otimes \mathcal{O}_{\mathsf{X}}(d) \simeq M(d)^{\sim}$.

(6) For $a \in A_d$, let $U_a$ be the affine open subset $\operatorname{Spec} A_{(a)}$. If $x \in \Gamma_*(M)_{(a)}$, then $x = m/a^i$ for some $m \in \mathrm{H}^0(\mathsf{X}, M(di)^{\sim})$. The restriction $m|_{U_a}$ is regarded as an element of $M(di)_{(a)}$. We can define $\beta_a \colon \Gamma_*(M)_{(a)} \to M_{(a)}$ by $x \mapsto m|_{U_a}/a^i$. If $a' \in A_{d'}$, then $\beta_a$ and $\beta_{aa'}$ commute with the restriction maps $\Gamma_*(M)_{(a)} \to \Gamma_*(M)_{(aa')}$ and $M_{(a)} \to M_{(aa')}$. Hence the homomorphism $\beta \colon \Gamma_*(M)^{\sim} \to M^{\sim}$ is defined. By construction, $\beta \circ \alpha^{\sim} = \mathrm{id}$.

Suppose that $A^{(d)}$ is 1-generated. In order to show $\beta$ is isomorphic, it is enough to show that $\beta_a$ is injective for any $a \in A_d$. Note that $\alpha(a)$ is a global section of the invertible sheaf $\mathcal{O}_A(d)$ and $U_a$ is the locus of points where $\alpha(a)$ is invertible. If the restriction $m|_{U_a}$ is zero for $m \in \mathrm{H}^0(\mathsf{X}, M^{\sim})$, then $a^k m = 0$ in $\mathrm{H}^0(\mathsf{X}, M^{\sim} \otimes \mathcal{O}_A(kd))$ for some $k > 0$, since $\mathsf{X}$ is quasi-compact. This property implies that $\beta_a$ is injective.

(7) We shall prove by applying Serre's vanishing theorem (cf. [**127**]) for ample line bundles, whose analytic analogue is explained in §**1.c** below.

*Step 1.* Suppose that $A^{(d)}$ is 1-generated for some $d > 1$ and the assertion holds for finitely generated graded $A^{(d)}$-modules $M^{(d,i)}$ for $0 \le i < d$ defined by $M^{(d,i)}_m = M_{dm+i}$ for $m \in \mathbb{Z}$. Then $\alpha \colon M_{dl+i} \to \mathrm{H}^0(\mathsf{X}, M(dl+i)^{\sim})$ is isomorphic for $l \gg 0$. Therefore, by replacing $A$ with $A^{(d)}$, we may assume that $A$ is 1-generated.

*Step 2. A reduction to the case $M = A$.* We have an exact sequence

$$\bigoplus A(q_i) \to \bigoplus A(p_j) \to M \to 0$$

of finitely generated graded $A$-modules for some finitely many integers $p_j$, $q_i$, since $A$ is Noetherian. By Serre's vanishing, this induces another exact sequence

$$\bigoplus \mathrm{H}^0(\mathsf{X}, \mathcal{O}_A(q_i + l)) \to \bigoplus \mathrm{H}^0(\mathsf{X}, \mathcal{O}_A(p_j + l)) \to \mathrm{H}^0(\mathsf{X}, M(l)^{\sim}) \to 0$$

for $l \gg 0$. Hence, we can reduce to the case $M = A$.

*Step 3. The case: $A$ is a polynomial ring over $A_0$.* Let $\mathbb{C}[x] = \mathbb{C}[x_0, x_1, \dots, x_n]$ be the polynomial ring of $(n+1)$-variables over $\mathbb{C}$. Suppose that $A$ is isomorphic to $A_0[x] = A_0 \otimes_{\mathbb{C}} \mathbb{C}[x]$ as a graded $A_0$-algebra. Then $\mathsf{X}$ is an $n$-dimensional projective space over $\operatorname{Spec} A_0$ and $\alpha$ for $M = A$ is isomorphic. This is shown by a direct calculation of $\mathrm{H}^0(\mathsf{X}, \mathcal{O}_{\mathsf{X}}(l))$.

*Step 4. General case.* There is a surjective homomorphism $A_0[x] \to A$ of graded $A_0$-algebras for some $x = (x_0, \dots, x_n)$. Thus $A$ is regarded as a finitely generated graded $A_0[x]$-module. Here, $\mathrm{H}^0(\mathsf{X}, \mathcal{O}_A(l)) \simeq \mathrm{H}^0(\operatorname{Proj} A_0[x], A(l)^{\sim})$ for $l \in \mathbb{Z}$. Thus by *Step 2* and *Step 3*, we infer that $\alpha_l$ is isomorphic for $l \gg 0$. □

Let $A$ be a graded $\mathbb{C}$-algebra generated by homogeneous elements $a_i$ in which $A_{(a_i)}$ is finitely generated as $\mathbb{C}$-algebra. Then $\mathsf{X} = \operatorname{Proj} A$ is locally of finite type over $\operatorname{Spec} \mathbb{C}$ and we can define $X = \operatorname{Projan} A$ as $(\operatorname{Proj} A)^{\mathrm{an}}$. A graded $A$-module

$M$ defines an $\mathcal{O}_{\mathsf{X}}$-module $M^\sim$ and an $\mathcal{O}_X$-module $(M^\sim)^{\mathrm{an}}$. We denote the sheaf $(\mathcal{O}_{\mathsf{X}}(l))^{\mathrm{an}}$ on $X$ by $\mathcal{O}_X(l)$.

**Example** For the symmetric algebra $\operatorname{Sym} V = \bigoplus_{d=0}^{\infty} \operatorname{Sym}^d V$ of a finite-dimensional $\mathbb{C}$-vector space $V$, we write $\mathsf{P}(V) = \operatorname{Proj} \operatorname{Sym} V$ and $\mathbb{P}(V) = \mathsf{P}(V)^{\mathrm{an}}$. These are called the *projective spaces* associated with $V$ in Grothendieck's sense. There is an isomorphism

$$\mathbb{P}(V) \simeq V^\vee \setminus \{0\}/\mathbb{C}^\star$$

of complex analytic spaces for the dual $V^\vee = \operatorname{Hom}_{\mathbb{C}}(V, \mathbb{C})$. The sheaf $\mathcal{O}_{\mathbb{P}(V)}(1) = \mathcal{O}_{\operatorname{Sym} V}(1)$ is invertible and $\mathcal{O}_{\mathbb{P}(V)}(l) \simeq \mathcal{O}_{\mathbb{P}(V)}(1)^{\otimes l}$ for $l \in \mathbb{Z}$. The sheaf $\mathcal{O}_{\mathbb{P}(V)}(1)$ is called the *tautological* invertible sheaf or the tautological line bundle. There is an isomorphism

$$\operatorname{Sym} V \simeq \bigoplus_{l \in \mathbb{Z}} \mathrm{H}^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(l)).$$

If $n + 1 = \dim V$, then $\mathbb{P}(V)$ is $n$-dimensional and is called the $n$-dimensional complex *projective space*. It is also denoted by $\mathbb{P}^n$. A complex analytic space $Y$ is called a *projective* analytic space if there is a closed immersion $Y \hookrightarrow \mathbb{P}^n$ for some $n$. We must be careful for the use of the word 'projective' when we discuss about a projective analytic space that is not a projective space $\mathbb{P}^n$. An analytic space is called *projective* if it is a projective analytic space. The name 'projective space' is used only for $\mathbb{P}^n$. If $A$ is a finitely generated graded $\mathbb{C}$-algebra, then $\operatorname{Projan} A$ is compact, since it is a closed analytic subset of the projective space $\mathbb{P}(A_d) = \operatorname{Projan} \operatorname{Sym} A_d$ for some $d$.

**1.2. Lemma** *Let $A = \bigoplus_{d=0}^{\infty} A_d$ be a graded $\mathbb{C}$-algebra and let $Y$ be a complex analytic space. Suppose that there exist*

- *a set $\{a_i\}_{i \in I}$ of non-zero homogeneous elements of $A$ with $A_{(a_i)}$ being finitely generated as $\mathbb{C}$-algebra,*
- *a graded $\mathcal{O}_Y$-algebra $\mathcal{R} = \bigoplus_{d=0}^{\infty} \mathcal{R}_d$,*
- *a graded $\mathbb{C}$-algebra homomorphism*

$$A = \bigoplus_{d=0}^{\infty} A_d \to \bigoplus_{d=0}^{\infty} \mathrm{H}^0(Y, \mathcal{R}_d),$$

- *an open covering $Y = \bigcup_{i \in I} Y_i$*

*satisfying the following conditions: let $d_i = \deg a_i$.*

(1) *the homomorphism $\mathcal{O}_Y \to \mathcal{R}_{d_i}$ induced by $a_i$ is isomorphic over $Y_i$;*
(2) *$\mathcal{R}_l \otimes \mathcal{R}_{d_i} \to \mathcal{R}_{l+d_i}$ is isomorphic over $Y_i$ for any $l \in \mathbb{Z}$.*

*Then there exist a morphism $f \colon Y \to X = \operatorname{Projan} A$ and natural homomorphisms $f^*\mathcal{O}_X(k) \to \mathcal{R}_k$ such that the composite $A_k \otimes \mathcal{O}_Y \to f^*\mathcal{O}_X(k) \to \mathcal{R}_k$ is the given homomorphism.*

Proof. Let us consider the homomorphism

$$A(k)_{(a_i)} \to \mathrm{H}^0(Y_i, \mathcal{R}_k)$$

for $k \geq 0$ that sends $b/a_i^l$ with $b \in A_{k+ld_i}$ to the image of $b$ under

$$A_{k+ld_i} \to \mathrm{H}^0(Y_i, \mathcal{R}_{k+ld_i}) \xleftarrow{\simeq} \mathrm{H}^0(Y_i, \mathcal{R}_k).$$

Since $\mathcal{R}_0 \simeq \mathcal{O}_Y$, we have a ring homomorphism $A_{(a_i)} \to \mathrm{H}^0(Y_i, \mathcal{O}_Y)$ and thus a morphism $f_i \colon Y_i \to \operatorname{Specan} A_{(a_i)}$. Here the homomorphism $f_i^*\mathcal{O}_X(k) \to \mathcal{R}_k|_{Y_i}$ is derived for $k \geq 0$. This is isomorphic if $d_i|k$. By patching $f_i$, we have $f\colon Y \to X = \operatorname{Projan} A$ and $f^*\mathcal{O}_X(k) \to \mathcal{R}_k$. □

**1.3. Corollary** *Let $A = \bigoplus_{d=0}^{\infty} A_d$ be a 1-generated graded $\mathbb{C}$-algebra and let $Y$ be a complex analytic space over* Specan $A_0$. *Then giving a morphism $Y \to \operatorname{Projan} A$ over* Specan $A_0$ *is equivalent to giving a surjective homomorphism $A_1 \otimes \mathcal{O}_Y \to \mathcal{L}$ into an invertible sheaf $\mathcal{L}$ such that $\operatorname{Sym}^k A_1 \otimes \mathcal{O}_Y \to \mathcal{L}^{\otimes k}$ factors through $A_k \otimes \mathcal{O}_Y \to \mathcal{L}^{\otimes k}$.*

Proof. The tautological line bundle $\mathcal{O}_X(1)$ on Projan $A$ is invertible, $\mathcal{O}_X(l) \simeq \mathcal{O}_X(1)^{\otimes l}$ for $l \in \mathbb{Z}$, and $A_1 \otimes \mathcal{O}_X \to \mathcal{O}_X(1)$ is surjective by **1.1**. For a morphism $f\colon Y \to \operatorname{Projan} A$, the pullback of $A_1 \otimes \mathcal{O}_X \to \mathcal{O}_X(1)$ satisfies the required condition. Conversely, let $A_1 \otimes \mathcal{O}_Y \to \mathcal{L}$ be the surjection satisfying the condition. It induces a surjective homomorphism

$$\psi\colon \left(\bigoplus_{d=0}^{\infty} A_d\right) \otimes \mathcal{O}_Y \to \operatorname{Sym} \mathcal{L}$$

of graded $\mathcal{O}_Y$-algebras. By **1.2**, we have a morphism $f\colon Y \to \operatorname{Projan} A$ where $A_1 \otimes \mathcal{O}_Y \to \mathcal{L}$ is induced from $A_1 \otimes \mathcal{O}_X \to \mathcal{O}_X(1)$. □

Let $X$ be a complex analytic space and let $\mathbb{C}[x] = \mathbb{C}[x_1, x_2, \ldots, x_l]$ be the polynomial ring of $l$-variables $x = (x_1, x_2, \ldots, x_l)$. An $\mathcal{O}_X$-algebra $\mathcal{A}$ is called *of finite presentation* if there is a surjective $\mathcal{O}_X$-algebra homomorphism

$$\mathcal{O}_X[x] = \mathcal{O}_X[x_1, x_2, \ldots, x_l] = \mathcal{O}_X \otimes_{\mathbb{C}} \mathbb{C}[x] \twoheadrightarrow \mathcal{A}$$

for some $l$ whose kernel is generated by a finite number of polynomials belonging to $\mathrm{H}^0(X, \mathcal{O}_X)[x]$. If $\mathcal{A}|_{X_\lambda}$ is of finite presentation for an open covering $X = \bigcup X_\lambda$, then $\mathcal{A}$ is called locally of finite presentation.

**1.4. Lemma** *Suppose that $\mathcal{A}$ is locally of finite presentation. Then there exist an analytic space $f\colon Y = \operatorname{Specan}_X \mathcal{A} \to X$ over $X$ and an $\mathcal{O}_X$-algebra homomorphism $\phi\colon \mathcal{A} \to f_*\mathcal{O}_Y$ satisfying the following universal property: If $g\colon Z \to X$ is an analytic space over $X$ and if $\varphi\colon \mathcal{A} \to g_*\mathcal{O}_Z$ is an $\mathcal{O}_X$-algebra homomorphism, then there is a unique morphism $h\colon Z \to Y$ such that $\varphi = h^* \circ \phi$.*

Proof. By the universal property, we may assume that $\mathcal{A}$ is of finite presentation. Then there is an exact sequence

$$\mathcal{O}_X[x]^{\oplus N} \to \mathcal{O}_X[x] \to \mathcal{A} \to 0$$

as $\mathcal{O}_X[x]$-modules, where the left homomorphism is given by $N$ polynomials contained in $\mathrm{H}^0(X, \mathcal{O}_X)[x]$. Let $B \subset \mathrm{H}^0(X, \mathcal{O}_X)$ be the subalgebra generated by the coefficients of the polynomials. This is finitely generated over $\mathbb{C}$ and there is a morphism $X \to \operatorname{Specan} B$. We can define an algebra $A$ by the similar exact sequence

$$B[x]^{\oplus N} \to B[x] \to A \to 0$$

of $B[x]$-modules. Then $X \times_{\operatorname{Specan} B} \operatorname{Specan} A \to X$ satisfies the universal property for $\operatorname{Specan}_X \mathcal{A}$. □

Next, we consider a graded $\mathcal{O}_X$-algebra $\mathcal{A} = \bigoplus_{n=0}^{\infty} \mathcal{A}_n$. For $d \in \mathbb{N}$, let $\mathcal{A}^{(d)}$ denote the graded algebra $\bigoplus_{n \geq 0} \mathcal{A}_{nd}$. If $\mathcal{A}_1^{\otimes n} \to \mathcal{A}_n$ is surjective for any $n$, then $\mathcal{A}$ is called 1-generated. If $\mathcal{A}$ is of finite presentation, then we have an exact sequence

$$\mathcal{O}_X[x]^N \to \mathcal{O}_X[x] \to \mathcal{A} \to 0$$

for $x = (x_1, x_2, \ldots, x_l)$, in which $x_i$ is mapped to a homogeneous element of $\mathrm{H}^0(X, \mathcal{A})$ and the left homomorphism is given by $N$-weighted homogeneous polynomials with respect to $x_i$. Let $B \subset \mathrm{H}^0(X, \mathcal{O}_X)$ be the subalgebra generated by all the coefficients of the polynomials. Then we have a graded algebra $A$ as the cokernel of the homomorphism $B[x]^N \to B[x]$ defined by the polynomials in which $A \otimes_B \mathcal{O}_X \simeq \mathcal{A}$ as $\mathcal{O}_X$-algebras. Here, $\operatorname{Proj} A$ is a scheme over $\operatorname{Spec} B$ and we have a morphism $X \to \operatorname{Specan} B$.

If $\mathcal{A} \simeq A' \otimes_{B'} \mathcal{O}_X$ for a finitely generated $\mathbb{C}$-algebra $B'$ contained in $\mathrm{H}^0(X, \mathcal{O}_X)$ and a finitely generated $B'$-graded algebra $A'$, then we can show $\operatorname{Projan} A \times_{\operatorname{Specan} B} X \simeq \operatorname{Projan} A' \times_{\operatorname{Specan} B'} X$ as follows: We can find an open covering $X = \bigcup X_\lambda$ and finitely generated $\mathbb{C}$-subalgebras $B_\lambda \subset \mathrm{H}^0(X_\lambda, \mathcal{O}_X)$ such that the images of $B$ and $B'$ in $\mathrm{H}^0(X_\lambda, \mathcal{O}_X)$ are contained in $B_\lambda$ and that there is an isomorphism $A \otimes_B B_\lambda \simeq A' \otimes_{B'} B_\lambda$ inducing the isomorphism $A \otimes_B \mathcal{O}_X \simeq A' \otimes_{B'} \mathcal{O}_X$ over $X_\lambda$. Let $Y$ be the fiber product $\operatorname{Projan} A \times_{\operatorname{Specan} B} X$ and let $a'_i \in A'$ be homogeneous elements generating $A'$ over $B'$. Under the isomorphism $A' \otimes_{B'} \mathcal{O}_X \simeq A \otimes_B \mathcal{O}_X$, $a'_i$ defines a homogeneous element $a_i \in \mathrm{H}^0(Y, p_1^* \mathcal{O}_A(d_i))$, where $d_i = \deg a'_i$. Let $Y_i \subset Y$ be the maximum open subset where $a_i \colon \mathcal{O}_Y \to p_1^* \mathcal{O}_A(d_i)$ is isomorphic. Then $Y = \bigcup Y_i$ and $p_1^* \mathcal{O}_A(l) \otimes p_1^* \mathcal{O}_A(d_i) \to p_1^* \mathcal{O}_A(l + d_i)$ is isomorphic for any $l \geq 0$, since $Y$ and $\operatorname{Projan} A' \times_{\operatorname{Specan} B'} X$ are isomorphic over $X_\lambda$. Thus we have a morphism $Y \to \operatorname{Projan} A'$ by **1.2**, which induces the isomorphism $Y \simeq \operatorname{Projan} A' \times_{\operatorname{Specan} B'} X$.

We define $\operatorname{Projan}_X \mathcal{A}$ to be the fiber product $\operatorname{Projan} A \times_{\operatorname{Specan} B} X$. We have $\operatorname{Projan}_X \mathcal{A}^{(d)} \simeq \operatorname{Projan}_X \mathcal{A}$ as $\operatorname{Proj} A^{(d)} \simeq \operatorname{Proj} A$ for $d \in \mathbb{N}$.

If $\mathcal{A}$ is locally of finite presentation, then the local $\operatorname{Projan}_X \mathcal{A}$ above can be patched and hence we can define an analytic space $\operatorname{Projan}_X \mathcal{A}$ proper over $X$. For a morphism $f \colon Y \to X$ from an analytic space, we have an isomorphism

$$\operatorname{Projan}_Y f^* \mathcal{A} \simeq \operatorname{Projan}_X \mathcal{A} \times_X Y$$

by the argument above. Let $\mathcal{M} = \bigoplus_{d \in \mathbb{Z}} \mathcal{M}_d$ be a graded $\mathcal{A}$-module which is locally of finite presentation, i.e., locally on $X$, there is an exact sequence

$$\bigoplus_{i=1}^{p} \mathcal{A}(m_i) \to \bigoplus_{j=1}^{q} \mathcal{A}(l_j) \to \mathcal{M} \to 0$$

of graded $\mathcal{A}$-modules for some $m_i$, $l_j \in \mathbb{Z}$, where $\mathcal{A}(l)$ stands for the twist of $\mathcal{A}$ by $l$. Then we can attach a coherent sheaf $\mathcal{M}^\sim$ on $\mathrm{Projan}_X \mathcal{A}$ as before. We also define $\mathcal{O}_{\mathcal{A}}(l)$ as $\mathcal{A}(l)^\sim$. If $\mathcal{A}$ is 1-generated, then $\mathcal{O}_{\mathcal{A}}(1)$ is invertible and is called the *tautological* invertible sheaf (line bundle) associated with $\mathcal{A}$. If $\mathcal{A}$ is specified, $\mathcal{O}_{\mathcal{A}}(l)$ is also denoted by $\mathcal{O}_P(l)$ for $P = \mathrm{Projan}_X \mathcal{A}$.

**1.5. Lemma** *Let $\mathcal{F}$ be an $\mathcal{O}_X[x] = \mathcal{O}_X[x_1, x_2, \ldots, x_l]$-module and let*

$$\phi\colon \mathcal{O}_X[x]^{\oplus r} \to \mathcal{F}$$

*be a surjective homomorphism of $\mathcal{O}_X[x]$-modules. Suppose either*

(1) *$\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, or*
(2) *$\mathcal{O}_X[x]$ is a graded $\mathcal{O}_X$-algebra for some weight of $x_i$, $\phi$ is regarded as a homomorphism*

$$\bigoplus \mathcal{O}_X[x](p_j) \to \mathcal{F} = \bigoplus\nolimits_{m\in\mathbb{Z}} \mathcal{F}_m$$

*of graded $\mathcal{O}_X[x]$-modules for some $p_j \in \mathbb{Z}$, and $\mathcal{F}_m$ are all coherent $\mathcal{O}_X$-modules.*

*Then the kernel* $\mathrm{Ker}\,\phi$ *is locally finitely generated as an $\mathcal{O}_X[x]$-module.*

Proof. We consider over open neighborhoods of a fixed point $P \in X$. First, we treat the case: $\mathcal{F}$ is coherent. Then there exist finitely many polynomials $\Phi_i(x) \in \mathrm{H}^0(U, \mathcal{O}_U)[x]$ over an open neighborhood $U$ such that $\Phi_i \cdot \mathcal{F}|_U = 0$ and the $\mathcal{O}_U$-algebra $\mathcal{A} = \mathcal{O}_U[x]/\mathcal{I}$ for the ideal $\mathcal{I}$ of $\mathcal{O}_U[x]$ generated by $\Phi_i$ is a coherent $\mathcal{O}_U$-module. Thus $\phi$ descends to $\phi_{\mathcal{A}}\colon \mathcal{A}^{\oplus r} \to \mathcal{F}|_U$. Since $\mathrm{Ker}\,\phi_{\mathcal{A}}$ is locally finitely generated as an $\mathcal{O}_U$-module, $\mathrm{Ker}\,\phi|_U$ is also locally finitely generated.

Next, we treat the homogeneous case. Let $U_0$ be a relatively compact Stein open neighborhood of $P$. Then there exists a Stein compact subset $K \supset U_0$ such that $\mathcal{O}_X(K) = \mathrm{H}^0(K, \mathcal{O}_X) = \varinjlim_{U\supset K} \mathrm{H}^0(U, \mathcal{O}_X)$ is Noetherian, by [**16**], [**129**]. Thus we have a Stein open subset $U \supset K$ and a homomorphism

$$\psi\colon \bigoplus \mathcal{O}_U[x](q_i) \to \bigoplus \mathcal{O}_U[x](p_j)$$

of graded $\mathcal{O}_U[x]$-modules such that the image of $\psi(K)$ is just $(\mathrm{Ker}\,\phi)(K)$. Let $(\mathrm{Coker}\,\psi)_m$ be the part of degree $m$ of the graded module $\mathrm{Coker}\,\psi$. Then we have the surjection $(\mathrm{Coker}\,\psi)_m \twoheadrightarrow \mathcal{F}_m|_U$ of coherent $\mathcal{O}_U$-modules which induces an isomorphism between the sections over $K$. In particular, $(\mathrm{Coker}\,\psi)_m \to \mathcal{F}_m$ is isomorphic over $U_0$. Therefore, the image of $\psi$ coincides with $\mathrm{Ker}\,\phi$ over $U_0$. □

**1.6. Corollary** *Let $\mathcal{A} = \bigoplus_{d=0}^{\infty} \mathcal{A}_d$ be a locally finitely generated graded $\mathcal{O}_X$-algebra such that $\mathcal{A}_d$ are all coherent $\mathcal{O}_X$-modules. Then $\mathcal{A}$ is locally of finite presentation. If $\mathcal{M} = \bigoplus_{d\in\mathbb{Z}} \mathcal{M}_d$ is a locally finitely generated graded $\mathcal{A}$-module for a graded $\mathcal{O}_X$-algebra $\mathcal{A}$ locally of finite presentation and if $\mathcal{M}_d$ are all coherent $\mathcal{O}_X$-modules, then $\mathcal{M}$ is an $\mathcal{A}$-module locally of finite presentation.*

**1.7. Example**

(1) Let $\mathcal{A}$ be an $\mathcal{O}_X$-algebra that is a coherent $\mathcal{O}_X$-module. Then $\mathcal{A}$ is locally of finite presentation by **1.5** and $f\colon Y = \operatorname{Specan}_X \mathcal{A} \to X$ is a finite morphism with an isomorphism $\mathcal{A} \simeq f_*\mathcal{O}_Y$. Conversely, if $f\colon Y \to X$ is a finite morphism, then $Y$ is isomorphic to $\operatorname{Specan}_X f_*\mathcal{O}_Y$.

(2) Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The symmetric algebra
$$\operatorname{Sym}\mathcal{F} = \bigoplus_{m\geq 0} \operatorname{Sym}^m \mathcal{F}$$
is an $\mathcal{O}_X$-algebra locally of finite presentation by **1.5**. The associated $\operatorname{Specan}_X$ is denoted by $\mathbb{L}(\mathcal{F})$. The morphism $f\colon Y = \mathbb{L}(\mathcal{F}) \to X$ is locally Stein and there is a natural homomorphism $\operatorname{Sym}\mathcal{F} \to f_*\mathcal{O}_Y$, which is not isomorphic if $\mathcal{F} \neq 0$.

(3) Let $\mathcal{E}$ be a vector bundle. Then $\mathbb{V}(\mathcal{E}) := \mathbb{L}(\mathcal{E}^\vee)$ for $\mathcal{E}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$ is the corresponding geometric vector bundle. The sheaf of germs of sections of the vector bundle is isomorphic to $\mathcal{E}$.

(4) Let $\mathcal{F}$ be a coherent sheaf on $X$. The $\operatorname{Projan}_X \operatorname{Sym}\mathcal{F}$ is denoted by $\mathbb{P}(\mathcal{F}) = \mathbb{P}_X(\mathcal{F})$ and its tautological line bundle by $\mathcal{O}_{\mathcal{F}}(1)$. We consider $P_0 = \mathbb{P}_X(\mathcal{O}_X \oplus \mathcal{F})$ and the closed embedding $P_1 = \mathbb{P}_X(\mathcal{F}) \subset P_0$ corresponding to $\mathcal{O}_X \oplus \mathcal{F} \to \mathcal{F}$. Then $\mathbb{L}(\mathcal{F})$ is isomorphic to the complement $P_0 \setminus P_1$.

**Remark** For a vector bundle $\mathcal{E}$ on $X$, $p\colon \mathbb{P}(\mathcal{E}) \to X$ is a $\mathbb{P}^{r-1}$-bundle for $r = \operatorname{rank}\mathcal{E}$. This is geometrically constructed as follows: let $\mathbb{V}(\mathcal{E}^\vee) \to X$ be the vector bundle associated with $\mathcal{E}^\vee$ defined as before and let $Z \subset \mathbb{V}(\mathcal{E}^\vee)$ be the zero section. Then $\mathbb{P}(\mathcal{E})$ is isomorphic to the quotient space of $\mathbb{V}(\mathcal{E}^\vee) \setminus Z$ by the scalar action of $\mathbb{C}^\star$ on fibers. For the tautological line bundle $\mathcal{O}_{\mathcal{E}}(1)$, we have $p_*\mathcal{O}_{\mathcal{E}}(l) \simeq \operatorname{Sym}^l \mathcal{E}$ for $l \geq 0$.

The following lemma is similar to **1.3**:

**1.8. Lemma** *Suppose that $\mathcal{A}$ is a 1-generated $\mathcal{O}_X$-graded algebra locally of finite presentation. Let $f\colon Y \to X$ be a morphism from an analytic space $Y$. Then giving a morphism $Y \to \operatorname{Projan}_X \mathcal{A}$ over $X$ is equivalent to giving a surjective homomorphism $f^*\mathcal{A}_1 \to \mathcal{L}$ into an invertible sheaf $\mathcal{L}$ on $Y$ that induces $f^*\mathcal{A}_d \to \mathcal{L}^{\otimes d}$ for $d > 0$.*

PROOF. The homomorphism to $\mathcal{L}$ is obtained as the pullback of $p^*\mathcal{A}_1 \to \mathcal{O}_{\mathcal{A}}(1)$ by $f$, where $p\colon \operatorname{Projan}_X \mathcal{A} \to X$ is the structure morphism. From a homomorphism to $\mathcal{L}$, we have a surjective homomorphism $f^*\mathcal{A} \to \operatorname{Sym}\mathcal{L}$ of graded $\mathcal{O}_Y$-algebras. Thus we have a closed immersion
$$Y \simeq \mathbb{P}_Y(\mathcal{L}) = \operatorname{Projan}_Y \operatorname{Sym}\mathcal{L} \hookrightarrow \operatorname{Projan}_Y(f^*\mathcal{A}) \simeq \operatorname{Projan}_X \mathcal{A} \times_X Y$$
and the morphism $Y \to \operatorname{Projan}_X \mathcal{A}$ over $X$. □

**§1.c. Ample line bundles.** Let $X$ be a compact complex analytic space. An invertible sheaf $\mathcal{L}$ of $X$ is called *very ample* if there is a closed immersion $i\colon X \hookrightarrow \mathbb{P}^N$ into an $N$-dimensional complex projective space such that $i^*\mathcal{O}_{\mathbb{P}^N}(1) \simeq \mathcal{L}$. An *ample* invertible sheaf is an invertible sheaf whose multiple by some positive integer is very ample. In particular, if $X$ admits an ample invertible sheaf, then $X$ is projective.

**Remark** (cf. [**68**]) Suppose that an invertible sheaf of a compact complex manifold $X$ admits a positive Hermitian metric. Then $X$ is a projective variety and the invertible sheaf is ample. The Kodaira vanishing theorem [**67**] is used for the proof.

**1.9. Definition**

(1) Let $\mathcal{F}$ be a coherent sheaf on a compact complex analytic variety $X$. It is called *generated by global sections* if the natural homomorphism
$$\mathrm{H}^0(X,\mathcal{F})\otimes\mathcal{O}_X\to\mathcal{F}$$
is surjective.

(2) Let $\mathcal{L}$ be an invertible sheaf on a compact complex analytic variety $X$. It is called *free* if it is generated by global sections. It is called *semi-ample* if $\mathcal{L}^{\otimes m}$ is free for some $m\in\mathbb{N}$.

(3) Let $f\colon Y\to X$ be a proper surjective morphism of complex analytic spaces. A coherent sheaf $\mathcal{F}$ of $Y$ is called *f-generated* or *relatively (globally) generated over* $X$ if the homomorphism
$$f^*f_*(\mathcal{F})\to\mathcal{F}$$
is surjective.

(4) Let $f\colon Y\to X$ be a proper surjective morphism of complex analytic spaces. An invertible sheaf $\mathcal{L}$ of $Y$ is called *f-free* if it is $f$-generated. If there exist an open covering $X=\bigcup U_\lambda$ and positive integers $m_\lambda$ such that $\mathcal{L}^{\otimes m_\lambda}|_{f^{-1}U_\lambda}$ is relatively generated over $U_\lambda$, then $\mathcal{L}$ is called *f-semi-ample* or *relatively semi-ample* over $X$.

**Remark** (1) Let $f\colon Y\to X$ be a proper morphism and let $\mathcal{L}$ be an $f$-generated line bundle of $Y$. Then there is a natural morphism $h\colon Y\to\mathbb{P}_X(f_*\mathcal{L})$ over $X$ such that $h^*\mathcal{O}_{f_*\mathcal{L}}(1)\simeq\mathcal{L}$.

(2) Let $X$ be a compact complex analytic variety and let $\mathcal{L}$ be an invertible sheaf. If there exist a morphism $g\colon X\to P$ into a projective analytic space $P$, an ample invertible sheaf $\mathcal{H}$ of $P$, and an integer $m\in\mathbb{N}$ with $\mathcal{L}^{\otimes m}\simeq g^*\mathcal{H}$, then $\mathcal{L}$ is semi-ample.

**1.10. Definition** Let $f\colon Y\to X$ be a proper morphism between complex analytic spaces. A line bundle $\mathcal{L}$ of $Y$ is called *f-very ample* or *relatively very ample over* $X$ if $\mathcal{L}$ is $f$-free and the morphism $Y\to\mathbb{P}_X(f_*\mathcal{L})$ is a closed immersion. A line bundle $\mathcal{L}$ is called *f-ample* or *relatively ample over* $X$ if, for any point $x\in X$, there exist an open neighborhood $U$ and an integer $n\in\mathbb{N}$ such that $\mathcal{L}^{\otimes n}|_{f^{-1}U}$ is relatively very ample over $U$. If there is an $f$-ample line bundle, then $f$ is called *projective*.

**Remark** Let $\mathcal{A}$ be a graded $\mathcal{O}_X$-algebra locally of finite presentation such that $\mathcal{O}_{\mathcal{A}}(l)$ is invertible for some $l>0$. Then $\mathcal{O}_{\mathcal{A}}(l)$ is relatively ample over $X$.

Let $f\colon Y \to X$ be a proper morphism of complex analytic spaces and let $\mathcal{L}$ be a line bundle of $Y$. The following conditions are known to be equivalent to each other:

(1) $\mathcal{L}$ is $f$-ample;
(2) There exist an open covering $X = \bigcup X_\lambda$ and closed immersions
$$\varphi_\lambda\colon f^{-1}X_\lambda \hookrightarrow \mathbb{P}^{n_\lambda} \times X_\lambda$$
over $X_\lambda$ for some $n_\lambda \in \mathbb{N}$ such that
$$\mathcal{L}^{\otimes m_\lambda} \simeq \varphi_\lambda^* p_1^* \mathcal{O}_{\mathbb{P}^{n_\lambda}}(1)$$
for some $m_\lambda \in \mathbb{N}$, where $p_1$ is the projection to $\mathbb{P}^{n_\lambda}$;
(3) (Theorem A [**29**], [**4**, Chapter IV]) For a compact subset $K \subset X$ and for a coherent sheaf $\mathcal{F}$ defined on a neighborhood of $f^{-1}K$, there is an integer $d \in \mathbb{N}$ such that
$$f^* f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes m}) \to \mathcal{F} \otimes \mathcal{L}^{\otimes m}$$
is surjective for $m \geq d$ along $f^{-1}K$;
(4) (Theorem B [**29**], [**4**, Chapter IV]) For a compact subset $K \subset X$ and for a coherent sheaf $\mathcal{F}$ defined on a neighborhood of $f^{-1}K$, there is an integer $d \in \mathbb{N}$ such that
$$\mathrm{R}^i f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes m}) = 0$$
for $m \geq d$ over $K$;
(5) ([**98**, 1.4]) Any fiber of $f$ is a projective analytic space and the restriction of $\mathcal{L}$ to any fiber is ample.

Theorem B above is called also the Serre vanishing theorem in the algebraic case.

**1.11. Lemma** *Let $f\colon Y \to X$ be a projective morphism. Then $Y \simeq \mathrm{Projan}_X \mathcal{A}$ for a graded $\mathcal{O}_X$-algebra $\mathcal{A}$ locally of finite presentation. If $\mathcal{F}$ is a coherent sheaf on $Y$, then it is isomorphic to $\mathcal{M}^\sim$ for a graded $\mathcal{A}$-module $\mathcal{M}$ locally of finite presentation.*

Proof. Let $\mathcal{L}$ be an $f$-ample invertible sheaf on $Y$. We shall show

- the graded $\mathcal{O}_X$-algebra
$$\mathcal{A} := \bigoplus\nolimits_{m \geq 0} \mathcal{A}_m := \bigoplus\nolimits_{m \geq 0} f_* \mathcal{L}^{\otimes m}$$
is locally of finite presentation,
- $Y \simeq \mathrm{Projan}_X \mathcal{A}$ over $X$,
- the graded $\mathcal{A}$-module
$$\mathcal{M} := \bigoplus\nolimits_{m \in \mathbb{Z}} \mathcal{M}_m := \bigoplus\nolimits_{m \in \mathbb{Z}} f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes m})$$
is locally of finite presentation, and
- $\mathcal{F} \simeq \mathcal{M}^\sim$.

We begin with the proof in the case $Y \simeq \mathbb{P}(V) \times X$ for a finite-dimensional $\mathbb{C}$-vector space $V$ and $\mathcal{L} = p_1^*\mathcal{O}_V(1)$ for the projection $p_1 \colon Y \to \mathbb{P}(V)$. Then $\mathcal{A} \simeq \operatorname{Sym} V \otimes \mathcal{O}_X$ and $Y \simeq \operatorname{Projan}_X \mathcal{A}$. Let $U \subset X$ be a relatively compact Stein open subset. Then $\mathcal{M}_m|_U = 0$ for $m \ll 0$ and $f^*\mathcal{M}_m \to \mathcal{F} \otimes \mathcal{L}^{\otimes m}$ is surjective on $f^{-1}U$ for $m \gg 0$ by Theorem A. We may assume that there is an exact sequence

$$\mathcal{O}_{f^{-1}U}^{\oplus s} \otimes \mathcal{L}^{\otimes(-b)} \to \mathcal{O}_{f^{-1}U}^{\oplus r} \otimes \mathcal{L}^{\otimes(-a)} \to \mathcal{F}|_{f^{-1}U} \to 0$$

for some positive integers $r$, $s$, $0 < a < b$. Then, for $m \gg 0$, the sequence

$$\operatorname{Sym}^{m-b} V \otimes \mathcal{O}_U^{\oplus s} \to \operatorname{Sym}^{m-a} V \otimes \mathcal{O}_U^{\oplus r} \to \mathcal{M}_m \to 0$$

is exact by Theorem B. The left homomorphism of the exact sequence is derived from

$$\mathcal{O}_U^{\oplus s} \to \mathcal{O}_U^{\oplus r} \otimes \operatorname{Sym}^{(b-a)} V.$$

Hence, for the cokernel $\mathcal{M}'$ of

$$\mathcal{A}(-b)^{\oplus s}|_U \to \mathcal{A}(-a)^{\oplus r}|_U,$$

we have $\mathcal{M}_m|_U \simeq \mathcal{M}'_m$ for $m \gg 0$. Therefore, $\mathcal{M}|_U$ is of finite presentation and $\mathcal{M}^\sim \simeq \mathcal{M}'^\sim \simeq \mathcal{F}$.

Next, we consider the general case. Let $U$ be the same as above. There exist a positive integer $m$, a finite-dimensional vector space $V$, and a closed immersion $i \colon f^{-1}U \hookrightarrow Z = \mathbb{P}(V) \times U$ such that $\mathcal{L}^{\otimes k}|_{f^{-1}U} \simeq \mathcal{O}_Z(1)|_{f^{-1}U}$, where $\mathcal{O}_Z(1)$ is the pullback of $\mathcal{O}_V(1)$ by the first projection. Then, for $0 \le j \le k-1$ and for the second projection $p_2 \colon Z \to U$,

$$\mathcal{A}^{(k,j)} = \bigoplus\nolimits_{m\in\mathbb{Z}} f_*\mathcal{L}^{\otimes(mk+j)}|_U \simeq \bigoplus\nolimits_{m\in\mathbb{Z}} p_{2*}(i_*\mathcal{L}^{\otimes j} \otimes \mathcal{O}_Z(m))$$

is a graded $\operatorname{Sym} V \otimes \mathcal{O}_U$-module of finite presentation by the previous argument, if we replace $U$ with a relatively compact open subset. Hence $\mathcal{A}$ is locally of finite presentation,

$$\operatorname{Projan}_U \mathcal{A}|_U \simeq \operatorname{Projan}_U \mathcal{A}^{(k)}|_U \simeq f^{-1}U \subset Z,$$

and $\mathcal{A}(1)^\sim \simeq \mathcal{L}$. For $\mathcal{F}$, we also have an exact sequence

$$\mathcal{O}_{f^{-1}U}^{\oplus s} \otimes \mathcal{L}^{\otimes(-b)} \to \mathcal{O}_{f^{-1}U}^{\oplus r} \otimes \mathcal{L}^{\otimes(-a)} \to \mathcal{F}|_{f^{-1}U} \to 0$$

some positive integers $r$, $s$, $0 < a < b$. Thus by the same argument as before, $\mathcal{M}$ is locally of finite presentation and $\mathcal{F} \simeq \mathcal{M}^\sim$. □

**Example** Let $\mathcal{I}$ be a coherent $\mathcal{O}_X$-ideal sheaf of $X$ and let $\mathcal{A} = \bigoplus_{d=0}^\infty \mathcal{I}^d$ be the graded $\mathcal{O}_X$-algebra naturally defined by the powers $\mathcal{I}^d \subset \mathcal{O}_X$. We set $V(\mathcal{I}) = \operatorname{Supp} \mathcal{O}_X/\mathcal{I}$. If $V(\mathcal{I}) = X$, then $\mathcal{I}^d = 0$ for $d \gg 0$ locally on $X$. Thus $\operatorname{Projan}_X \mathcal{A} = \emptyset$ in this case. Suppose that $V(\mathcal{I})$ is nowhere-dense in $X$. Then $f \colon Y = \operatorname{Projan}_X \mathcal{A} \to X$ is called the *blowing-up* (or the *blowup*) of $X$ along the ideal $\mathcal{I}$ or along $V(\mathcal{I})$. It is an isomorphism over $X \setminus V(\mathcal{I})$. The locus $V(\mathcal{I})$ is called the *center* of the blowing-up. The image $\mathcal{L}$ of $f^*\mathcal{I} \to f^*\mathcal{O}_X = \mathcal{O}_Y$ is invertible. In fact, $\mathcal{A}(1) \to \mathcal{A}$ given by $\mathcal{I}^{d+1} \subset \mathcal{I}^d$ is injective and $\mathcal{A}(1)^\sim \simeq \mathcal{L}$. If $X$ is a variety, then so is $Y$ and $f \colon Y \to X$ is a bimeromorphic morphism (cf. §**1.d**). Conversely, let $g\colon Z \to X$ be a morphism such that the image of $g^*\mathcal{I} \to g^*\mathcal{O}_X = \mathcal{O}_Y$ is an invertible sheaf $\mathcal{L}'$. Then there is a morphism $h\colon Z \to Y$ over $X$ such that $\mathcal{L}' \simeq h^*\mathcal{L}$. Let $E \subset Y$ be the analytic subspace defined by $0 \to \mathcal{L} \to \mathcal{O}_Y \to \mathcal{O}_E \to 0$. Then $E$ is an effective Cartier divisor of $Y$. This is isomorphic to $\operatorname{Projan}_V \mathcal{B}$, where $V = \operatorname{Specan}_X \mathcal{O}_X/\mathcal{I}$ and $\mathcal{B}$ is the graded $\mathcal{O}_V$-algebra $\bigoplus_{d\geq 0} \mathcal{I}^d/\mathcal{I}^{d+1}$.

**Remark** If $X$ is reduced and $\mathcal{J}$ is a torsion free sheaf of rank one of $X$, then we can define the power $\mathcal{J}^m$ as the quotient $\mathcal{J}^{\otimes m}/(\mathrm{tor})$ of $\mathcal{J}^{\otimes m}$ by the torsion part $(\mathcal{J}^{\otimes m})_{\mathrm{tor}}$ for $m \in \mathbb{N}$, and $\mathcal{J}^0$ as $\mathcal{O}_X$. Then the blowing-up $g\colon V(\mathcal{J}) = \operatorname{Projan}_X \bigoplus_{d=0}^{\infty} \mathcal{J}^d \to X$ along $\mathcal{J}$ is defined, where $g^*\mathcal{J}/(\mathrm{tor})$ is a $g$-ample invertible sheaf. Locally on $X$, the blowing-up $g$ is considered as a usual blowing-up along some ideal. In fact, we have an injection $i\colon \mathcal{J} \hookrightarrow \mathcal{O}_X$ locally on $X$, where $i(\mathcal{J})^m \simeq \mathcal{J}^m$ for any $m \geq 0$ and $V(\mathcal{J}) \simeq V(i(\mathcal{J}))$.

**§1.d. Bimeromorphic geometry.** A *meromorphic mapping* $f\colon Y \cdots\to X$ of complex analytic varieties is defined by the graph $\Gamma_f \subset Y \times X$ such that

(1) $\Gamma_f$ is a subvariety of $Y \times X$,
(2) the first projection $\Gamma_f \to Y$ is proper and is an isomorphism over a Zariski-open dense subset of $Y$.

The image $f(Y)$ is defined as the image of the graph $\Gamma_f$ under the second projection $Y \times X \to X$. If $f(Y)$ is dense in $X$, then we say that $f$ is *dominant* or that $Y$ dominates $X$. If the second projection $\Gamma_f \to X$ is proper, then $f$ is called *proper*. If $f$ is proper, $X$ and $Y$ are normal, and if a general fiber of the morphism $\widetilde{\Gamma}_f \to X$ induced from the normalization $\widetilde{\Gamma}_f$ of $\Gamma_f$ is connected, then $f$ is called a *meromorphic fiber space*. The composite of two meromorphic mappings $f\colon Y \cdots\to X$ and $g\colon X \cdots\to Z$ is well-defined when the first projection $\Gamma_g \to X$ is an isomorphism over some points of $f(Y)$. A meromorphic mapping $f\colon Y \cdots\to X$ is called *bimeromorphic* if the inverse $f^{-1}\colon X \cdots\to Y$ exists as a meromorphic mapping. This is the case the second projection $\Gamma_f \to X$ is proper and is an isomorphism over a Zariski-open dense subset of $X$. In particular, a bimeromorphic mapping is proper. A *bimeromorphic morphism* is a morphism that is a bimeromorphic mapping. Hence the first projection $\Gamma_f \to Y$ of the meromorphic mapping $f$ is a bimeromorphic morphism. Thus a meromorphic mapping $Y \cdots\to X$ is the composite of a morphism $Z \to X$ and the inverse of a bimeromorphic morphism $Z \to Y$. If $Z' \to X$ and $Z' \to Y$ are another morphism and another bimeromorphic morphism, respectively, and if the images of induced morphisms $Z \to Y \times X$ and $Z' \to Y \times X$ are the same, then we consider $Z$ and $Z'$ define the same meromorphic mapping $Y \cdots\to X$. By using $h\colon Z \to X$ and $\mu\colon Z \to Y$ above, we can define the fiber $f^{-1}(x)$ for $x \in X$ by $f^{-1}(x) := \mu(h^{-1}(x))$. Suppose that there are morphisms $Y \to S$ and $X \to S$ into another complex analytic space $S$. If there is a bimeromorphic mapping $Y \cdots\to X$ over $S$, then $Y$ is said to be *bimeromorphically equivalent* or *bimeromorphic* to $X$ over $S$.

**1.12. Lemma** (1) *Let $f\colon Y \cdots\to X$ be a meromorphic fiber space such that* $\dim Y = \dim X$. *Then $f$ is bimeromorphic.*

(2) *Let* $f\colon Y \dashrightarrow X$ *be a meromorphic fiber space and let* $h\colon Y \dashrightarrow Z$ *be a meromorphic map such that* $h(f^{-1}(x))$ *is a point for general* $x \in X$. *Then there exist a meromorphic map* $g\colon X \dashrightarrow Z$ *such that* $h = g \circ f$.

Proof. (1) We may assume that $f$ is holomorphic. Since the function $x \mapsto \dim f^{-1}(x)$ is upper semi-continuous, there is a normal dense Zariski-open subset $U \subset X$ such that $f^{-1}U \to U$ is a homeomorphism. Thus $f^{-1}U \simeq U$ by Zariski's Main Theorem and hence $f$ is bimeromorphic.

(2) Let $\phi = (f, h)\colon Y \dashrightarrow X \times Z$ be the induced meromorphic map and let $Y'$ be the normalization of the image $\phi(Y)$. Then $Y' \to X$ is proper and its general fiber consists of one point. Hence $Y' \to X$ is a bimeromorphic morphism. □

Let $f\colon Y \to X$ be a bimeromorphic morphism between normal varieties and let $U \subset X$ be the maximum open subset over which $f$ is an isomorphism. Then $Y \smallsetminus f^{-1}U$ is called the *exceptional locus* for $f$.

If $\mathcal{O}_{X,x}$ is not a regular local ring, then $x \in X$ is called a *singular* point. The set $\operatorname{Sing} X$ of singular points is called the *singular locus* and is a proper closed analytic subset if $X$ is reduced. If $\operatorname{Sing} X = \emptyset$, then $X$ is called *non-singular*. A non-singular complex analytic variety is called a *complex analytic manifold*. A non-singular complex analytic space is a disjoint union of countably many complex analytic manifolds. Hironaka's desingularization theorem [**40**] states that for a complex analytic variety $X$, there is a bimeromorphic morphism $\mu\colon Y \to X$ from a non-singular variety such that, over a relatively compact open subset of $X$, $\mu$ is the succession of blowups along non-singular centers contained in the singular loci.

Let $f\colon X \to Y$ be a morphism of complex analytic spaces and let $\mathcal{F}$ be a coherent sheaf of $X$. For a point $x \in X$, the sheaf $\mathcal{F}$ is called *$f$-flat* at $x$ or *flat* over $Y$ at $x$ if $\mathcal{F}_x$ is a flat $\mathcal{O}_{Y,f(x)}$-module. If $\mathcal{O}_X$ is flat over $Y$, then $f$ is called a flat morphism. A flat morphism is an open mapping and the dimensions of fibers are locally constant. The set of points $x \in X$ at which $\mathcal{F}$ is $f$-flat is Zariski-open by [**16**]. Suppose that $f$ is proper and $Y$ is a variety. Then, for a coherent sheaf $\mathcal{F}$, there is a dense Zariski-open subset $U \subset Y$ such that $\mathcal{F}|_{f^{-1}U}$ is flat over $U$. Moreover, Hironaka's flattening theorem [**41**] says that there is a proper morphism $\nu\colon Y' \to Y$ satisfying the following conditions:

(1) Over a relatively compact open subset of $Y$, $\nu$ is a succession of blowups along centers away from $U$;
(2) Let $\mu\colon X' = X \times_Y Y' \to X$ be the induced morphism and let $\mathcal{F}'$ be the quotient sheaf of $\mu^*\mathcal{F}$ by the maximum coherent subsheaf $\mathcal{G}$ such that $\operatorname{Supp} \mathcal{G}$ does not dominate $Y'$. Then $\mathcal{F}'$ is flat over $Y'$.

We say that the morphism $\nu$ flattens $\mathcal{F}$ or that $\nu$ is a *flattening* of $\mathcal{F}$. Combining with resolution of singularities, we may assume that $Y'$ above is non-singular. A flattening of $f$ means a flattening of $\mathcal{O}_X$.

Suppose that $\nu$ is a flattening of $f$ or $\mathcal{F} = \mathcal{O}_X$. Then $\mathcal{F}'$ above is the structure sheaf $\mathcal{O}_V$ of a closed subspace $V$ of $X \times_Y Y'$, where $V \to Y'$ is flat. We call $V$ the *main component* of $X \times_Y Y'$. If $\widetilde{X} \to V$ is a bimeromorphic morphism from a variety, then the induced morphism $\widetilde{X} \to Y'$ is called a *bimeromorphic transform* of $f$ by $\nu$.

Suppose that $f\colon X \to Y$ is a bimeromorphic morphism and let $\nu\colon Y' \to Y$ be a flattening of $f$. Then $Y' \to Y$ is a projective bimeromorphic morphism over a relatively compact open subset of $X$. This corresponds to a relative version of Chow's lemma: for a complete algebraic variety $X$, there exists a bimeromorphic morphism $X' \to X$ from a non-singular projective variety.

Let $f\colon X \to Y$ be a surjective morphism from a non-singular space. Then the fiber $f^{-1}(y)$ is also non-singular for a general point $y \in Y$. This is a theorem of Sard. Similarly, for a surjective morphism $f\colon X \to Y$ from a normal space, the general fiber $f^{-1}(y)$ is also normal.

## §2. Divisors

**§2.a. Weil and Cartier divisors.** Let $X$ be an $n$-dimensional normal complex analytic variety. A *prime divisor* is an irreducible and reduced subvariety of codimension one. Let $\mathrm{Div}'(X)$ be the free abelian group generated by prime divisors of $X$. By attaching an open subset $U \subset X$ the group $\mathrm{Div}'(U)$, we have a presheaf of abelian groups on $X$. Note that the restriction $\Gamma|_U$ might be reducible for a prime divisor $\Gamma$ of $X$. Let $\mathcal{D}iv_X$ be the sheafification. The *divisor group* $\mathrm{Div}(X)$ is defined to be $\mathrm{H}^0(X, \mathcal{D}iv_X)$ and an element of $\mathrm{Div}(X)$ is called a *divisor* or a *Weil divisor*. A divisor $D$ of $X$ is written as a formal sum

$$D = \sum a_\Gamma \Gamma, \tag{II-1}$$

where $\Gamma$ is a prime divisor of $X$, $a_\Gamma \in \mathbb{Z}$, and the *support*

$$\mathrm{Supp}\, D := \bigcup\nolimits_{a_\Gamma \neq 0} \Gamma$$

is an analytic subset of $X$. In other words, the sum $\sum a_\Gamma \Gamma$ is locally finite. The coefficient $a_\Gamma$ is denoted by $\mathrm{mult}_\Gamma D$ and is called the *multiplicity* along $\Gamma$. A prime divisor contained in the support of $D$ is called a *prime component* or an *irreducible component*. The presentation (II-1) is called the *prime decomposition* or the *irreducible decomposition* of $D$. We set

$$D_+ := \sum\nolimits_{a_\Gamma > 0} a_\Gamma \Gamma, \quad \text{and} \quad D_- := \sum\nolimits_{a_\Gamma < 0} (-a_\Gamma) \Gamma.$$

Then $D = D_+ - D_-$. The divisors $D_+$ and $D_-$ are called the positive and the negative parts of the prime decomposition of $D$, respectively. A divisor $D$ is called an *effective divisor* if $D_- = 0$. For two divisors $D_1$, $D_2$, if $D_1 - D_2$ is effective, then we write $D_1 \geq D_2$ or $D_2 \leq D_1$.

A holomorphic function $f$ on $X$ is called a *unit* function if $1/f$ is also holomorphic. This is also called a *nowhere-vanishing* function or an *invertible* holomorphic function. A meromorphic function $f$ is called an *invertible* meromorphic function if $1/f$ is also meromorphic. Since $X$ is a variety, a meromorphic $f$ is invertible unless $f$ is identically zero. The sheaf of germs of invertible holomorphic functions is denoted by $\mathcal{O}_X^\star$. The sheaf of germs of meromorphic functions (resp. invertible meromorphic functions) is denoted by $\mathfrak{M}_X$ (resp. $\mathfrak{M}_X^\star$). For a meromorphic function $\varphi \neq 0$ and a prime divisor $\Gamma$ of $X$, the order $\mathrm{ord}_\Gamma(\varphi)$ of $\varphi$ along $\Gamma$ is defined to be the order of zeros or the minus of the order of poles of $\varphi$ along $\Gamma$. The divisor

$$\mathrm{div}(\varphi) := \sum \mathrm{ord}_\Gamma(\varphi)\Gamma$$

is called a *principal divisor*. The div gives rise to a homomorphism $\mathfrak{M}_X^\star \to \mathcal{D}iv_X$ of sheaves. The image $\mathcal{CD}iv_X$ is called the sheaf of germs of Cartier divisors. The *Cartier divisor group* $\mathrm{CDiv}(X)$ is defined to be $\mathrm{H}^0(X, \mathcal{CD}iv_X)$. An element of $\mathrm{CDiv}(X)$ is called a *Cartier divisor*. The condition $\mathrm{div}(\varphi) = 0$ implies that $\varphi$ is a holomorphic unit function. Thus there is an exact sequence:

$$0 = \{1\} \to \mathcal{O}_X^\star \to \mathfrak{M}_X^\star \to \mathcal{CD}iv_X \to 0. \tag{II-2}$$

The principal divisor group $\mathrm{Princ}(X)$ is defined to be the image of $\mathrm{div}\colon \mathrm{H}^0(X, \mathfrak{M}_X^\star) \to \mathrm{CDiv}(X)$. For a point $x \in X$, $\mathcal{D}iv_{X,x} = \mathcal{CD}iv_{X,x}$ if and only if $\mathcal{O}_{X,x}$ is UFD. If $\mathcal{O}_{X,x}$ is UFD for any $x \in X$, then $X$ is called *locally factorial.* If $\mathrm{Div}(X) = \mathrm{CDiv}(X)$, then $X$ is called (globally) *factorial*.

Let $j\colon X_{\mathrm{reg}} = X \setminus \mathrm{Sing}\, X \hookrightarrow X$ be the open immersion from the non-singular part. Then the injection $\mathcal{CD}iv_X \hookrightarrow \mathcal{D}iv_X$ is an isomorphism over $X_{\mathrm{reg}}$. This induces an isomorphism $j_*\mathcal{CD}iv_{X_{\mathrm{reg}}} \simeq \mathcal{D}iv_X$ by the following:

**2.1. Lemma** *Let $Z \subset X$ be a Zariski-closed subset with* $\mathrm{codim}\, Z \geq 2$. *Then any prime divisor of $X \setminus Z$ extends to a prime divisor of $X$.*

Proof. The extension property is local on $X$. Thus, we may assume that there is a finite surjective morphism $p\colon X \to U$ into an open subset of $\mathbb{C}^n$. Then, for a prime divisor $\Gamma \subset X \setminus Z$, $p(\Gamma) \setminus p(Z)$ is a prime divisor of $U \setminus p(Z)$. Then $p(\Gamma) \setminus p(Z) = \Gamma' \setminus p(Z)$ for a prime divisor $\Gamma'$ of $U$ by a theorem of Thullen [**136**] (cf. [**118**]). Therefore, $p^{-1}\Gamma' \setminus Z$ contains $\Gamma$ as a prime component, and a prime component of $p^{-1}\Gamma'$ is the extension of $\Gamma$. □

Therefore, we have a long exact sequence:

$$0 \to \mathcal{O}_X^\star \to \mathfrak{M}_X^\star \to \mathcal{D}iv_X \to \mathrm{R}^1 j_*\mathcal{O}_{X_{\mathrm{reg}}}^\star \to \mathrm{R}^1 j_*\mathfrak{M}_{X_{\mathrm{reg}}}^\star \to \cdots . \tag{II-3}$$

**§2.b. Reflexive sheaves of rank one.** We define a subsheaf $\mathcal{O}_X(D) \subset \mathfrak{M}_X$ for a divisor $D$ as follows: For an open subset $U$,

$$\mathrm{H}^0(U, \mathcal{O}_X(D)) = \{\varphi \in \mathrm{H}^0(U, \mathfrak{M}_X^\star) \mid \mathrm{div}(\varphi) + D|_U \geq 0\} \cup \{0\}.$$

If $\Gamma$ is a prime divisor, then $\mathcal{O}_X(-\Gamma)$ is considered as the defining ideal sheaf of $\Gamma$. The sheaf $\mathcal{O}_X(D)$ is an invertible sheaf (locally free $\mathcal{O}_X$-module of rank one) if and only if $D$ is a Cartier divisor. If $D, D_1, D_2$ are Cartier divisors, then $\mathcal{O}_X(D_1 + D_2) \simeq \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)$ and $\mathcal{O}_X(-D) \simeq \mathcal{O}_X(D)^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X(D), \mathcal{O}_X)$. Hence, the homomorphism $D \mapsto \mathcal{O}_X(D)$ essentially coincides with the connecting homomorphism

$$\mathrm{CDiv}(X) = \mathrm{H}^0(X, \mathcal{CD}iv_X) \to \mathrm{H}^1(X, \mathcal{O}_X^\star) = \mathrm{Pic}(X)$$

of the exact sequence (II-2). The natural isomorphism $\mathcal{O}_X(D) \simeq j_*\mathcal{O}_{X_{\mathrm{reg}}}(D|_{X_{\mathrm{reg}}})$ exists by definition.

A *reflexive sheaf* is a coherent reflexive $\mathcal{O}_X$-module; a coherent sheaf $\mathcal{F}$ is reflexive if and only if its *double-dual* $\mathcal{F}^\wedge := (\mathcal{F}^\vee)^\vee$ is canonically isomorphic to $\mathcal{F}$. In particular, a reflexive sheaf is a torsion-free $\mathcal{O}_X$-module (a torsion-free sheaf).

**2.2. Lemma** *Let $\mathcal{F}$ be a coherent sheaf of a normal variety $X$.*

(1) *If $\mathcal{F} \simeq \mathcal{G}^\vee$ for a coherent sheaf $\mathcal{G}$, then $\mathcal{F}$ is reflexive.*
(2) *Assume that $\mathcal{F}$ is a subsheaf of a reflexive sheaf $\mathcal{G}$. Then $\mathcal{F}$ is reflexive if and only if* $\operatorname{codim}\operatorname{Supp}\mathcal{S} \le 1$ *for any non-zero coherent subsheaf $\mathcal{S} \subset \mathcal{G}/\mathcal{F}$.*
(3) *For an analytic subset $Z$ of* $\operatorname{codim} Z \ge 2$, *assume that $\mathcal{F}|_{X\setminus Z}$ is reflexive. Then, for the open immersion $j\colon X \setminus Z \hookrightarrow X$, the direct image sheaf $j_*(\mathcal{F}|_{X\setminus Z})$ is a reflexive sheaf.*
(4) *If $\mathcal{F}$ is reflexive, then $\mathcal{H}^p_Z(\mathcal{F}) = 0$ for an analytic subset $Z$ of* $\operatorname{codim} Z \ge 2$ *and for $p = 0$, 1.*

In particular, the sheaf $\mathcal{O}_X(D)$ for a divisor $D$ is a reflexive sheaf of rank one.

Proof. (1) There are natural homomorphisms $\mathcal{G} \to \mathcal{G}^\wedge$ and $\mathcal{G}^\vee \to (\mathcal{G}^\vee)^\wedge$. The dual of the first one is the inverse to the second.

(2) For a non-zero coherent subsheaf $\mathcal{S}$ of $\mathcal{G}/\mathcal{F}$, there is an intermediate coherent sheaf $\mathcal{F} \subset \mathcal{F}' \subset \mathcal{G}$ with $\mathcal{F}'/\mathcal{F} \simeq \mathcal{S}$. We have an exact sequence

$$0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{S}, \mathcal{O}_X) \to \mathcal{F}'^\vee \to \mathcal{F}^\vee \to \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{S}, \mathcal{O}_X).$$

If $\operatorname{codim}\operatorname{Supp}\mathcal{S} \ge 2$, then $\mathcal{E}xt^i(\mathcal{S}, \mathcal{O}_X) = 0$ for $i \le 1$, by a property of depth. Hence, if $\mathcal{F}$ is reflexive, then $\operatorname{codim}\operatorname{Supp}\mathcal{S} \le 1$. If $\mathcal{F}$ is not reflexive, then $\mathcal{F} \subset \mathcal{F}^\wedge \subset \mathcal{G}$ and $\mathcal{S} = \mathcal{F}^\wedge/\mathcal{F}$ is a non-zero subsheaf of $\mathcal{G}/\mathcal{F}$ with $\operatorname{codim}\operatorname{Supp}\mathcal{S} \ge 2$.

(3) We may consider locally on $X$. Thus we may assume that $\mathcal{F}$ is torsion-free and that there is a surjective homomorphism $\mathcal{O}_X^{\oplus k} \twoheadrightarrow \mathcal{F}^\vee$. Then $\mathcal{F} \hookrightarrow \mathcal{O}_X^{\oplus k}$. Let $\mathcal{S} \subset \mathcal{O}_X^{\oplus k}/\mathcal{F}$ be the subsheaf defined as the union

$$\bigcup \mathcal{H}^0_W(\mathcal{O}_X^{\oplus k}/\mathcal{F})$$

for all the analytic subsets $W \subset X$ of $\operatorname{codim} W \ge 2$. Then $\mathcal{S}$ is a coherent sheaf. Let $\mathcal{F} \subset \mathcal{F}' \subset \mathcal{O}_X^{\oplus k}$ be the intermediate sheaf satisfying $\mathcal{F}'/\mathcal{F} \simeq \mathcal{S}$. Then

$$j_*(\mathcal{F}|_{X\setminus Z}) \subset \mathcal{F}' \subset \mathcal{O}_X^{\oplus k} = j_*\mathcal{O}_{X\setminus Z}^{\oplus k}$$

and $\mathcal{F}'|_{X\setminus Z} \simeq \mathcal{F}|_{X\setminus Z}$. Thus $\mathcal{F}' = j_*(\mathcal{F}|_{X\setminus Z})$.

(4) follows from the isomorphism $\mathcal{F} \simeq j_*(\mathcal{F}|_{X\setminus Z})$ given in (3). □

**Remark** The second condition of (2) is equivalent to that, for any $x \in X$, the height of an associated prime of the $\mathcal{O}_{X,x}$-module $(\mathcal{G}/\mathcal{F})_x$ is 0 or 1.

An effective divisor $D$ is considered as an analytic subspace defined by the ideal sheaf $\mathcal{O}_X(-D) \subset \mathcal{O}_X$. For an effective divisor $D$, its reduced part $D_{\mathrm{red}}$ is defined to be the reduced structure of the analytic subspace $D$. In other words, it is defined by

$$D_{\text{red}} := \sum_{\Gamma\text{: prime component of } D} \Gamma,$$

which is identified with $\operatorname{Supp} D$. An effective divisor $D$ is called *reduced* if $D = D_{\text{red}}$.

Let $\zeta \neq 0$ be a meromorphic section of a reflexive sheaf $\mathcal{L}$ of rank one on $X$. Then, as in the case of meromorphic functions, the order $\operatorname{ord}_\Gamma(\zeta)$ for any prime divisor $\Gamma$ is defined. The divisor $\operatorname{div}(\zeta) := \sum \operatorname{ord}_\Gamma(\zeta)\Gamma$ is also defined and there is an isomorphism $\mathcal{O}_X(\operatorname{div}(\zeta)) \simeq \mathcal{L}$. Therefore, a reflexive sheaf of rank one admitting non-zero meromorphic sections is derived from a divisor. Every reflexive sheaf of rank one is derived from a divisor if $X$ is a projective variety or a Stein space. The set $\operatorname{Ref}_1(X)$ of isomorphism classes of all the reflexive sheaves of rank one of $X$ has a structure of abelian group; the product is given by the double-dual of the tensor product and the inverse is given by the dual. The Picard group $\operatorname{Pic}(X) = \mathrm{H}^1(X, \mathcal{O}_X^\star)$ is a subgroup. We have a group homomorphism $\operatorname{Div}(X) \ni D \mapsto \mathcal{O}_X(D) \in \operatorname{Ref}_1(X)$.

**Example** For an analytic subset $Z \subset X$ of $\operatorname{codim} Z \geq 2$, the restriction homomorphism $\operatorname{Div}(X) \to \operatorname{Div}(X \setminus Z)$ is bijective by **2.1**. On the other hand, $\operatorname{Ref}_1(X) \to \operatorname{Ref}_1(X \setminus Z)$ is not necessarily surjective as in the following example: Let $X$ be the two-dimensional unit polydisc

$$\{(x, y) \in \mathbb{C}^2 \mid |x| < 1, |y| < 1\}$$

and let $Z = \{(0,0)\}$. Then $\operatorname{Ref}_1(X \setminus Z) = \operatorname{Pic}(X \setminus Z) \simeq \mathrm{H}^1(X \setminus Z, \mathcal{O}_X)$, which is an infinite-dimensional vector space. In fact, the isomorphism is derived from the exponential sequence and the vanishing $\mathrm{H}^p(X \setminus Z, \mathbb{Z}) = 0$ for $p \leq 2$. On the other hand, $\operatorname{Ref}_1(X) = \operatorname{Pic}(X) = \{1\}$.

**2.3. Lemma** *Let* $\mathcal{R}ef_X^\bullet$ *be the complex*:

$$[\cdots \to 0 \to \mathfrak{M}_X^\star \to \mathcal{D}iv_X \to 0 \to \cdots]$$

*of sheaves of abelian groups on* $X$*, where* $\mathfrak{M}_X^\star$ *lies in the degree* 0 *and the homomorphism appears in* (II-3)*. Then the hyper-cohomology group* $\mathbb{H}^1(X, \mathcal{R}ef_X^\bullet)$ *is isomorphic to* $\operatorname{Ref}_1(X)$*.*

Proof. Let $\mathcal{I}$ be the image of $\mathrm{R}^1 j_* \mathcal{O}_{X_{\text{reg}}}^\star \to \mathrm{R}^1 j_* \mathfrak{M}_{X_{\text{reg}}}^\star$. Then there is a distinguished triangle

$$\cdots \to \mathcal{I}[-2] \to \mathcal{R}ef_X^\bullet \to \tau_{\leq 1}(Rj_* \mathcal{O}_{X_{\text{reg}}}^\star) \xrightarrow{+1} \mathcal{I}[-1] \to \cdots$$

in the derived category of sheaves of abelian groups of $X$. Thus $\mathbb{H}^1(X, \mathcal{R}ef_X^\bullet)$ is isomorphic to the kernel of

$$\mathrm{H}^1(X_{\text{reg}}, \mathcal{O}_{X_{\text{reg}}}^\star) \to \mathrm{H}^0(X, \mathcal{I}).$$

Since every reflexive sheaf of rank one is locally derived from divisors, we have the isomorphism. □

Two divisors $D_1$ and $D_2$ are said to be *linearly equivalent* if $D_1 - D_2$ is a principal divisor, equivalently, $\mathcal{O}_X(D_1) \simeq \mathcal{O}_X(D_2)$. We write $D_1 \sim D_2$ for the linear equivalence relation. Let $\mathrm{C}\ell(X)$ and $\mathrm{CC}\ell(X)$ be the *divisor class group* $\mathrm{Div}(X)/\mathrm{Princ}(X)$ and the *Cartier divisor class group* $\mathrm{CDiv}(X)/\mathrm{Princ}(X)$, respectively. We have canonical injections $\mathrm{C}\ell(X) \hookrightarrow \mathrm{Ref}_1(X)$ and $\mathrm{CC}\ell(X) \hookrightarrow \mathrm{Pic}(X)$ by $D \mapsto \mathcal{O}_X(D)$. The sheafification $\mathcal{C}\ell_X$ of the presheaf $U \mapsto \mathrm{C}\ell(U)$ is canonically isomorphic to the cohomology sheaf

$$\mathcal{H}^1(\mathcal{R}ef^\bullet_X) \simeq \mathcal{D}iv_X/\mathcal{CD}iv_X \simeq \mathcal{H}^1_{\mathrm{Sing}\,X}(\mathcal{CD}iv_X).$$

The injections $\mathrm{C}\ell(X) \hookrightarrow \mathrm{Ref}_1(X)$ and $\mathrm{CC}\ell(X) \hookrightarrow \mathrm{Pic}(X)$ are not necessarily isomorphic, in general.

**§2.c. Intersection numbers.** Let $X$ be a complex analytic space. For the connecting homomorphism $\mathrm{Pic}(X) \to \mathrm{H}^2(X, \mathbb{Z})$ derived from the exponential sequence of $X$, we denote by $c_1(\mathcal{L})$ the image of an invertible sheaf $\mathcal{L}$ and call it the *first Chern class* of $\mathcal{L}$. Let $Z$ be a compact analytic subvariety of $X$ of dimension $d$. Then it is also regarded as a generator of $H_{2d}(Z, \mathbb{Z})$ and defines a homology class $\mathrm{cl}(Z) \in H_{2d}(X, \mathbb{Z})$. Let $\mathcal{L}_1$, $\mathcal{L}_2$, ..., $\mathcal{L}_d$ be invertible sheaves on $X$. By the natural pairing $\langle\ ,\ \rangle\colon H^{2d}(X, \mathbb{Z}) \times H_{2d}(X, \mathbb{Z}) \to \mathbb{Z}$, we can define the intersection number

$$\begin{aligned}(\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_d; Z) &:= \mathcal{L}_1\mathcal{L}_2 \cdots \mathcal{L}_d \cdot Z \\ &:= \langle c_1(\mathcal{L}_1) \cup c_1(\mathcal{L}_2) \cup \cdots \cup c_1(\mathcal{L}_d), \mathrm{cl}(Z)\rangle,\end{aligned}$$

where $\cup$ stands for the cup-product. If $X$ itself is a compact variety of dimension $n$, then we denote $\mathcal{L}_1\mathcal{L}_2 \cdots \mathcal{L}_n = (\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_n; X)$.

Let $Z$ be a reduced complex analytic space purely of dimension $d$, $U$ a dense Zariski-open subset contained in the non-singular locus $Z_{\mathrm{reg}}$, and let $W := Z \setminus U$. Note that $U$ is a disjoint union of complex analytic manifolds of dimension $d$. Let $\boldsymbol{\omega}_Z^{\mathrm{top}\bullet}$ be the *topological dualizing complex* defined by Verdier [**144**] as the twisted inverse image $\varepsilon^!\mathbb{Z}$ for the structure morphism $\varepsilon\colon Z \to \mathrm{Specan}\,\mathbb{C}$ in the derived category of sheaves of abelian groups of $Z$. Then, for the open immersion $j\colon U \subset Z$, we have a distinguished triangle

$$\cdots \xrightarrow{+1} \boldsymbol{\omega}_W^{\mathrm{top}\bullet} \to \boldsymbol{\omega}_Z^{\mathrm{top}\bullet} \to Rj_*\mathbb{Z}_U[2d] \xrightarrow{+1} \boldsymbol{\omega}_W^{\mathrm{top}\bullet}[1] \to \cdots.$$

Then the cohomology sheaf $\mathcal{H}^{-2d}(\boldsymbol{\omega}_Z^{\mathrm{top}\bullet})$ is isomorphic to $j_*\mathbb{Z}_U$ and $\mathcal{H}^i(\boldsymbol{\omega}_Z^{\mathrm{top}\bullet}) = 0$ for $i < -2d$.

**2.4. Lemma** *Let $X$ be a normal complex analytic variety of dimension $n$ and let $Z$ be a reduced divisor. Then the group $\mathrm{Div}_Z(X) = \mathrm{H}^0_Z(X, \mathcal{D}iv_X)$ of divisors of $X$ supported in $Z$ is isomorphic to $\mathrm{H}^{2-2n}_Z(X, \boldsymbol{\omega}_X^{\mathrm{top}\bullet})$. Moreover, the isomorphism extends to*

$$\mathrm{Div}(X) \simeq \varinjlim_{Z \subset X} \mathrm{H}^{2-2n}_Z(X, \boldsymbol{\omega}_X^{\mathrm{top}\bullet}).$$

Proof. Since $\mathrm{H}^i_Z(X, \boldsymbol{\omega}_X^{\mathrm{top}\bullet}) \simeq \mathrm{H}^i(Z, \boldsymbol{\omega}_Z^{\mathrm{top}\bullet})$ for any $i$, we have

$$\mathrm{H}^{2-2n}_Z(X, \boldsymbol{\omega}_X^{\mathrm{top}\bullet}) \simeq H^0(U, \mathbb{Z})$$

for any Zariski-open dense subset $U$ of $Z$ contained in $Z_{\mathrm{reg}}$. Thus we have an isomorphism $\mathrm{Div}_Z(X) \simeq \mathrm{H}^0(Z_{\mathrm{reg}}, \mathbb{Z}) \simeq \mathrm{H}^0(U, \mathbb{Z})$. For another reduced divisor $Z' \supset Z$, the homomorphism $\mathrm{H}^{2-2n}_Z(X, \boldsymbol{\omega}_X^{\mathrm{top}\bullet}) \to \mathrm{H}^{2-2n}_{Z'}(X, \boldsymbol{\omega}_X^{\mathrm{top}\bullet})$ is described as follows: Let $U$ be the intersection $Z'_{\mathrm{reg}} \cap Z_{\mathrm{reg}}$ and $U'$ be the complement $Z'_{\mathrm{reg}} \setminus Z$. Then

$$\mathrm{H}^0(Z'_{\mathrm{reg}}, \mathbb{Z}) \simeq \mathrm{H}^0(U, \mathbb{Z}) \oplus \mathrm{H}^0(U', \mathbb{Z}) \simeq \mathrm{H}^0(Z_{\mathrm{reg}}, \mathbb{Z}) \oplus \mathrm{H}^0(U', \mathbb{Z}).$$

The natural homomorphism $\mathrm{H}^0(Z_{\mathrm{reg}}, \mathbb{Z}) \to \mathrm{H}^0(Z'_{\mathrm{reg}}, \mathbb{Z})$ as the extension by zero is isomorphic to the homomorphism above. Therefore, it corresponds to the natural inclusion $\mathrm{Div}_Z(X) \subset \mathrm{Div}_{Z'}(X)$. Since $\mathrm{Div}(X) = \bigcup \mathrm{Div}_Z(X)$, we are done. □

Let $X$ be a normal complex analytic variety of dimension $n$ and let $\mathrm{Div}_c(X) = \mathrm{H}^0_c(X, \mathcal{D}iv_X)$ be the group of divisors with compact support. Then there is a natural homomorphism $\mathrm{Div}_c(X) \to \mathrm{H}^{2-2n}_c(X, \boldsymbol{\omega}_X^{\mathrm{top}\bullet}) = \mathrm{H}_{2n-2}(X, \mathbb{Z})$. This is just the homomorphism giving the homology class $\mathrm{cl}(D)$ for a divisor $D$ with compact support. Let $\mathrm{CDiv}_c(X)$ be the intersection $\mathrm{CDiv}(X) \cap \mathrm{Div}_c(X) = \mathrm{H}^0_c(X, \mathcal{CD}iv_X)$. Then $\mathrm{CDiv}_c(X) \to \mathrm{H}^2_c(X, \mathbb{Z})$ is induced from (II-2) and the exponential sequence of $X$. We have another homomorphism

$$\mathrm{H}^2_c(X, \mathbb{Z}) \to \mathrm{H}^{2-2n}_c(X, \boldsymbol{\omega}_X^{\mathrm{top}\bullet}) = \mathrm{H}_{2n-2}(X, \mathbb{Z})$$

since $\mathbb{Z} \simeq \mathcal{H}^{-2n}(\boldsymbol{\omega}_X^{\mathrm{top}\bullet})$. This is nothing but the Poincaré isomorphism.

**2.5. Lemma** *The diagram*

$$\begin{array}{ccc} \mathrm{CDiv}_c(X) & \longrightarrow & \mathrm{Div}_c(X) \\ \big\downarrow & & \big\downarrow \\ \mathrm{H}^2_c(X, \mathbb{Z}) & \longrightarrow & H_{2n-2}(X, \mathbb{Z}) \end{array}$$

*is commutative.*

Proof. Let $Z \subset X$ be a reduced divisor. Then there is a Zariski-open dense subset $V \subset X$ such that $V$ and $V \cap Z$ are non-singular and $V \cap Z$ is dense in $Z$. Let

$$\psi \colon \mathcal{H}^0_Z(\mathcal{CD}iv_X) \to \mathcal{H}^1_Z(\mathcal{O}_X^\star) \to \mathcal{H}^2_Z(\mathbb{Z}_X)$$

be a homomorphism induced from (II-2) and the exponential sequence of $X$. For a point $x \in V \cap Z$, there is an open neighborhood $\mathcal{U} \subset V$ isomorphic to an $n$-dimensional unit polydisc with a coordinate system $z_1, z_2, \ldots z_n$ such that $\mathcal{U} \cap Z = \mathrm{div}(z_1)$. Then $\mathcal{H}^0_Z(\mathcal{CD}iv_X)_x \simeq \mathbb{Z}\,\mathrm{div}(z_1)$ and $\mathcal{H}^2_Z(\mathbb{Z}_X)_x \simeq (R^1 j_* \mathbb{Z}_{X \setminus Z})_x \simeq \mathbb{Z}$ for the open immersion $j \colon X \setminus Z \hookrightarrow X$. Here $\psi(\mathrm{div}(z_1))$ corresponds to giving the integral

$$\frac{1}{2\pi\sqrt{-1}} \int_\gamma \frac{\mathrm{d}\, z_1}{z_1}$$

for $\gamma \in H_1(\mathcal{U} \setminus Z, \mathbb{Z})$. Thus $\psi$ is an isomorphism. Hence the homomorphism $\mathrm{CDiv}_{V \cap Z}(V) \to \mathrm{H}^2_{V \cap Z}(V, \mathbb{Z})$, which is induced from (II-2) and the exponential sequence of $V$, is an isomorphism and is isomorphic to $\mathrm{Div}_{V \cap Z}(V) \to \mathrm{H}^{2-2n}_{V \cap Z}(V, \boldsymbol{\omega}_V^{\mathrm{top}\bullet})$

described in the proof of **2.4**. Here the restrictions $\mathrm{Div}_Z(X) \to \mathrm{Div}_{V\cap Z}(V)$ and $\mathrm{H}^{2-2n}_Z(X, \boldsymbol{\omega}^{\mathrm{top}\bullet}_X) \to \mathrm{H}^{2-2n}_{V\cap Z}(V, \boldsymbol{\omega}^{\mathrm{top}\bullet}_V)$ are both isomorphic. Hence

$$\begin{array}{ccccc} \mathrm{CDiv}_Z(X) & \longrightarrow & \mathrm{Div}_{V\cap Z}(V) & = & \mathrm{Div}_Z(X) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{H}^2_Z(X, \mathbb{Z}) & \longrightarrow & \mathrm{H}^2_{V\cap Z}(V, \mathbb{Z}) & = & \mathrm{H}^2_Z(X, \boldsymbol{\omega}^{\mathrm{top}\bullet}_X) \end{array}$$

is commutative. By considering the inductive limit for compact reduced divisors $Z$, we have the commutativity of the diagram in question. □

**2.6. Corollary** *If $D$ is a principal divisor with compact support, then* $\mathrm{cl}(D) = 0$*. In particular, the intersection number*

$$\mathcal{L}_1\mathcal{L}_2 \cdots \mathcal{L}_{n-1} \cdot D = 0$$

*for any $\mathcal{L}_i \in \mathrm{Pic}(X)$.*

**2.7. Corollary** *Suppose that $X$ is a compact normal variety of dimension $n$ and let $D$ be a Cartier divisor. Then*

$$(\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_{n-1}; D) = (\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_{n-1}, \mathcal{O}_X(D); X)$$

*for invertible sheaves $\mathcal{L}_1$, $\mathcal{L}_2$, ..., $\mathcal{L}_{n-1}$ of $X$.*

Proof. The homology class $\mathrm{cl}(D)$ comes from the first Chern class $c_1(\mathcal{O}_X(D))$ in $\mathrm{H}^2(X, \mathbb{Z}) = \mathrm{H}^2_c(X, \mathbb{Z})$ and the intersection is induced from

$$\mathrm{H}^{2n-2}(X, \mathbb{Z}) \times \mathrm{H}^2_c(X, \mathbb{Z}) \to \mathrm{H}^{2n-2}(X, \mathbb{Z}) \times \mathrm{H}_{2n-2}(X, \mathbb{Z}) \to \mathbb{Z}. \qquad □$$

**§2.d. $\mathbb{Q}$-divisors and $\mathbb{R}$-divisors.** Let $\mathbb{Q}$ be the field of rational numbers and $\mathbb{R}$ be that of real numbers. Let $\mathfrak{K}$ be $\mathbb{Q}$ or $\mathbb{R}$. A $\mathfrak{K}$-*divisor* and a $\mathfrak{K}$-*Cartier divisor* are defined to be elements of

$$\mathrm{Div}(X, \mathfrak{K}) := \mathrm{H}^0(X, \mathcal{D}iv_X \otimes \mathfrak{K}) \quad \text{and} \quad \mathrm{CDiv}(X, \mathfrak{K}) := \mathrm{H}^0(X, \mathcal{CD}iv_X \otimes \mathfrak{K}),$$

respectively. Note that $\mathrm{Div}(X, \mathfrak{K})$ is not necessarily isomorphic to $\mathrm{Div}(X) \otimes \mathfrak{K}$. But

$$\mathrm{H}^0_c(X, \mathcal{D}iv_X) \otimes \mathfrak{K} \simeq \mathrm{H}^0_c(X, \mathcal{D}iv_X \otimes \mathfrak{K})$$

holds. Hence, if $X$ is compact, then $\mathrm{Div}(X) \otimes \mathfrak{K} \simeq \mathrm{Div}(X, \mathfrak{K})$. Under the natural inclusions $\mathrm{Div}(X) \subset \mathrm{Div}(X, \mathbb{Q}) \subset \mathrm{Div}(X, \mathbb{R})$, a divisor is considered as a $\mathfrak{K}$-divisor. A divisor $D$ is $\mathbb{Q}$-Cartier if and only if some multiple of $D$ is a Cartier divisor, locally on $X$. A divisor $D$ is $\mathbb{R}$-Cartier if and only if it is $\mathbb{Q}$-Cartier by **2.9** below. In order to distinguish "$\mathfrak{K}$-Cartier Weil divisor" from "$\mathfrak{K}$-Cartier $\mathfrak{K}$-divisor", we sometimes call a usual (Weil) divisor by a $\mathbb{Z}$-*divisor*. If $X$ is locally factorial, then $\mathcal{CD}iv_X \otimes \mathbb{Q} = \mathcal{D}iv_X \otimes \mathbb{Q}$ and hence $\mathrm{CDiv}(X, \mathbb{Q}) = \mathrm{Div}(X, \mathbb{Q})$. If $\mathcal{CD}iv_X \otimes \mathbb{Q} = \mathcal{D}iv_X \otimes \mathbb{Q}$, then $X$ is called *locally $\mathbb{Q}$-factorial.* If $\mathrm{CDiv}(X, \mathbb{Q}) = \mathrm{Div}(X, \mathbb{Q})$, then $X$ is called (globally) $\mathbb{Q}$-*factorial*. A *principal $\mathfrak{K}$-divisor* is an element of the image $\mathrm{Princ}(X, \mathfrak{K})$ of

$$\mathrm{div}\colon \mathrm{H}^0(X, \mathfrak{M}^\star_X \otimes \mathfrak{K}) \to \mathrm{Div}(X, \mathfrak{K}).$$

Let $\Gamma$ be a prime divisor of $X$. For an $\mathbb{R}$-divisor $D$, the *multiplicity* $\operatorname{mult}_\Gamma D \in \mathbb{R}$ is similarly defined. An $\mathbb{R}$-divisor is called *effective* if $\operatorname{mult}_\Gamma D \geq 0$ for any prime divisor of $X$. For two $\mathbb{R}$-divisors $D_1$ and $D_2$, we also write $D_1 \geq D_2$ or $D_2 \leq D_1$, if $D_1 - D_2$ is effective. The prime decomposition and the support of $D$ are defined similarly. For the prime decomposition $D = \sum a_i \Gamma_i$ of an $\mathbb{R}$-divisor, the *round-down* (or the *integral part*) is defined to be the divisor

$$\llcorner D \lrcorner = \sum \llcorner a_i \lrcorner \Gamma_i,$$

where $\llcorner a_i \lrcorner$ is the maximal integer not greater than $a_i$. The *round-up* and the *fractional part* are defined to be $\ulcorner D \urcorner := -\llcorner -D \lrcorner$ and $\langle D \rangle := D - \llcorner D \lrcorner$, respectively. For an effective $\mathfrak{K}$-divisor $D$, its reduced part $D_{\mathrm{red}}$ is defined by

$$D_{\mathrm{red}} = \sum_{\Gamma\text{: prime component of } D} \Gamma,$$

which is identified with $\operatorname{Supp} D$.

Two $\mathbb{R}$-divisors $D_1$ and $D_2$ are said to be *linearly equivalent* if $D_1 - D_2 \in \operatorname{Princ}(X) \subset \operatorname{Div}(X, \mathbb{R})$. We denote the linear equivalence also by $D_1 \sim D_2$. If $D_1 \sim D_2$, then

$$\llcorner D_1 \lrcorner \sim \llcorner D_2 \lrcorner, \quad \ulcorner D_1 \urcorner \sim \ulcorner D_2 \urcorner, \quad \text{and} \quad \langle D_1 \rangle = \langle D_2 \rangle.$$

Two $\mathbb{R}$-divisors $D_1$ and $D_2$ are said to be $\mathfrak{K}$-*linearly equivalent* if $D_1 - D_2 \in \operatorname{Princ}(X, \mathfrak{K})$. We denote the $\mathfrak{K}$-linear equivalence by $D_1 \sim_{\mathfrak{K}} D_2$.

**2.8. Lemma** *Let $D$ be a $\mathbb{Q}$-divisor and let $\Delta$ be an effective $\mathbb{R}$-divisor on a normal variety. Suppose that*

$$\Delta = D + \sum r_i \operatorname{div}(f_i)$$

*for some finitely many meromorphic functions $f_i$ and for some real numbers $r_i$. Then, for any $\varepsilon > 0$, there is an effective $\mathbb{Q}$-divisor $\Delta'$ such that*

(1) $\operatorname{Supp} \Delta = \operatorname{Supp} \Delta'$,
(2) $|\operatorname{mult}_\Gamma \Delta - \operatorname{mult}_\Gamma \Delta'| < \varepsilon$ *for any prime divisor* $\Gamma$,
(3) $D \sim_{\mathbb{Q}} \Delta'$.

Proof. Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ be all the prime components of the reduced divisor

$$S := \operatorname{Supp} D \cup \operatorname{Supp} \Delta \cup \bigcup \operatorname{Supp} \operatorname{div}(f_i).$$

Let $V_{\mathbb{Q}}$ be the $k$-dimensional $\mathbb{Q}$-vector space generated by $\Gamma_1, \ldots, \Gamma_k$: $V_{\mathbb{Q}} = \bigoplus \mathbb{Q}\Gamma_i$. A vector in $V_{\mathbb{Q}}$ corresponds to a $\mathbb{Q}$-divisor supported on $S$. We set $V = V_{\mathbb{Q}} \otimes \mathbb{R}$. The first quadrant cone $V^{\geq 0}$ of $V$ with respect to the base $(\Gamma_1, \ldots, \Gamma_k)$ is identified with the set of effective $\mathbb{R}$-divisors supported on $S$. Let $W_{\mathbb{Q}} \subset V_{\mathbb{Q}}$ be the $\mathbb{Q}$-vector subspace generated by prime components of $\Delta$ and set $W = W_{\mathbb{Q}} \otimes \mathbb{R}$. We have

$$W \cap V^{\geq 0} \ni \Delta = D + \sum r_i \operatorname{div}(f_i).$$

Let $L_{\mathbb{Q}} \subset V_{\mathbb{Q}}$ be the $\mathbb{Q}$-vector subspace generated by $\operatorname{div}(f_i)$ and set $L = L_{\mathbb{Q}} \otimes \mathbb{R}$. If $L \cap W = 0$, then $\{\Delta\} = (\{D\} + L) \cap W$ and hence $\Delta \in (\{D\} + L_{\mathbb{Q}}) \cap W_{\mathbb{Q}}$. In other words, $\Delta \sim_{\mathbb{Q}} D$. Suppose that $L \cap W \neq 0$. Let $C$ be the interior of the cone $W \cap V^{\geq 0}$. Then

$$\Delta \in (\{D\} + L) \cap C = (\{\Delta\} + (L \cap W)) \cap C.$$

There is an open neighbourhood $\mathcal{U}$ of 0 in $L \cap W$ with $\{\Delta\} + \mathcal{U} \subset C$, since $C$ is open in $W$. Hence,

$$\emptyset \neq (\{\Delta\} + \mathcal{U}) \cap W_{\mathbb{Q}} \subset (\{D\} + L_{\mathbb{Q}}) \cap C.$$

Thus the expected $\mathbb{Q}$-divisor $\Delta'$ exists for any $\varepsilon > 0$. □

**2.9. Corollary** *Let $D_1$ and $D_2$ be two $\mathbb{Q}$-divisors with $D_1 \sim_{\mathbb{R}} D_2$. Then $D_1 \sim_{\mathbb{Q}} D_2$ holds on any relatively compact open subset.*

Proof. Apply **2.8** to $D = D_1 - D_2$ and $\Delta = 0$. □

The first Chern classes $c_1(D) \in \mathrm{H}^2(X, \mathfrak{K})$ for $\mathfrak{K}$-Cartier divisors $D$ and the homology classes $\mathrm{cl}(E) \in \mathrm{H}_{2n-2}(X, \mathfrak{K})$ for $\mathfrak{K}$-divisors $E$ with compact support are naturally defined, where $n = \dim X$. In particular, we can consider intersection numbers for $\mathbb{R}$-divisors with compact support with a cohomology class in $\mathrm{H}^{2n-2}(X, \mathbb{R})$.

We define the following $\mathfrak{K}$-versions of the divisor class group, the Cartier divisor class group, the Picard group, and the group of reflexive sheaves of rank one:

$$\mathrm{C}\ell(X, \mathfrak{K}) := \mathrm{Div}(X, \mathfrak{K}) / \mathrm{Princ}(X, \mathfrak{K}), \qquad \mathrm{CC}\ell(X, \mathfrak{K}) := \mathrm{CDiv}(X, \mathfrak{K}) / \mathrm{Princ}(X, \mathfrak{K}),$$

$$\mathrm{Pic}(X, \mathfrak{K}) := \mathrm{H}^1(X, \mathcal{O}_X^{\star} \otimes \mathfrak{K}), \qquad \mathrm{Ref}_1(X, \mathfrak{K}) := \mathbb{H}^1(X, \mathcal{R}ef_X^{\bullet} \overset{\mathrm{L}}{\otimes} \mathfrak{K}).$$

Then the following commutative diagram exists:

$$\begin{array}{ccc} \mathrm{CC}\ell(X, \mathfrak{K}) & \longrightarrow & \mathrm{C}\ell(X, \mathfrak{K}) \\ \downarrow & & \downarrow \\ \mathrm{Pic}(X, \mathfrak{K}) & \longrightarrow & \mathrm{Ref}_1(X, \mathfrak{K}). \end{array}$$

Let $\mathcal{R}ef^{\bullet}_{X,\mathfrak{K}}$ be the following complex of sheaves of abelian groups on $X$:

$$[\cdots \to 0 \to \mathfrak{M}_X^{\star} \to \mathcal{D}iv_X \otimes \mathfrak{K} \to 0 \to \cdots],$$

where $\mathfrak{M}_X^{\star}$ lies in the degree 0. An element of the hyper-cohomology group

$$\mathrm{Ref}_1(X)_{\mathfrak{K}} := \mathbb{H}^1(X, \mathcal{R}ef^{\bullet}_{X,\mathfrak{K}})$$

is called a *reflexive $\mathfrak{K}$-sheaf of rank one*. Similarly for the complex

$$\mathcal{P}ic^{\bullet}_{X,\mathfrak{K}} := [\cdots \to 0 \to \mathfrak{M}_X^{\star} \to \mathcal{CD}iv_X \otimes \mathfrak{K} \to 0 \to \cdots]$$

of sheaves of abelian groups on $X$, an element of the hyper-cohomology group

$$\mathrm{Pic}(X)_{\mathfrak{K}} := \mathbb{H}^1(X, \mathcal{P}ic^{\bullet}_{X,\mathfrak{K}})$$

is called an *invertible* $\mathfrak{K}$*-sheaf*. There is a canonical injection $\mathrm{Pic}(X)_{\mathfrak{K}} \hookrightarrow \mathrm{Ref}_1(X)_{\mathfrak{K}}$. For a reflexive $\mathfrak{K}$-sheaf $\boldsymbol{\mathcal{L}}$ of rank one, the round-down $\llcorner \boldsymbol{\mathcal{L}} \lrcorner \in \mathrm{Ref}_1(X)$, the round-up $\ulcorner \boldsymbol{\mathcal{L}} \urcorner \in \mathrm{Ref}_1(X)$, and the fractional part $\langle \boldsymbol{\mathcal{L}} \rangle \in \mathrm{Div}(X, \mathfrak{K})$ are naturally defined by the homomorphism

$$\mathbb{H}^1(X, \mathcal{R}ef^{\bullet}_{X,\mathfrak{K}}) \to \mathrm{H}^0(X, \mathcal{D}iv_X \otimes \mathfrak{K}/\mathbb{Z}).$$

**2.10. Lemma** *There is a short exact sequence*

$$0 \to \mathrm{Div}(X) \to \mathrm{Ref}_1(X) \oplus \mathrm{Div}(X, \mathfrak{K}) \to \mathrm{Ref}_1(X)_{\mathfrak{K}} \to 0.$$

Proof. By definition, there exists a distinguished triangle

$$\cdots \to \mathcal{D}iv_X[-1] \to \mathcal{R}ef^{\bullet}_X \oplus (\mathcal{D}iv_X \otimes \mathfrak{K}[-1]) \to \mathcal{R}ef^{\bullet}_{X,\mathfrak{K}} \xrightarrow{+1} \mathcal{D}iv_X \to \cdots.$$

It is enough to show that

$$\mathrm{H}^1(X, \mathcal{D}iv_X) \to \mathrm{H}^1(X, \mathcal{D}iv_X \otimes \mathfrak{K})$$

is injective. Let $\{U_i\}_{i \in I}$ be an open covering of $X$ and let $\Delta_i$ be a $\mathfrak{K}$-divisor of $U_i$ such that $D_{i,j} := (\Delta_i)|_{U_i \cap U_j} - (\Delta_j)|_{U_i \cap U_j}$ are $\mathbb{Z}$-divisors for $i, j \in I$. Then $D_{i,j} = (\llcorner \Delta_i \lrcorner)|_{U_i \cap U_j} - (\llcorner \Delta_j \lrcorner)|_{U_i \cap U_j}$. Thus we have the injectivity. □

For a reflexive sheaf $\mathcal{L}$ of rank one and for a $\mathfrak{K}$-divisor $D$, we write by $\mathcal{L}(D)$ the reflexive $\mathfrak{K}$-sheaf of rank one corresponding to the image of $(\mathcal{L}, D)$ under $\mathrm{Ref}_1(X) \oplus \mathrm{Div}(X, \mathfrak{K}) \to \mathrm{Ref}_1(X)_{\mathfrak{K}}$. Here $\llcorner \mathcal{L}(D) \lrcorner = \mathcal{L}(\llcorner D \lrcorner)$, $\ulcorner \mathcal{L}(D) \urcorner = \mathcal{L}(\ulcorner D \urcorner)$, and $\langle \mathcal{L}(D) \rangle = \langle D \rangle$. Note that $\mathcal{L}(D)$ is not a usual sheaf in general, but if $D$ is a $\mathbb{Z}$-divisor, then it is regarded as a reflexive sheaf of rank one isomorphic to the double-dual of $\mathcal{L} \otimes \mathcal{O}_X(D)$. If $\mathcal{L}(D) = \mathcal{L}'(D')$ in $\mathrm{Ref}_1(X)_{\mathfrak{K}}$ for another reflexive sheaf $\mathcal{L}'$ of rank one and another $\mathfrak{K}$-divisor $D'$, then $D - D' \in \mathrm{Div}(X)$ and $\mathcal{O}_X(D - D') \simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}')$. Thus $\mathcal{L}(D)$ and $\mathcal{L}'(D')$ are considered to be linearly equivalent: $\mathcal{L}(D) \sim \mathcal{L}'(D')$. Note that

$$0 \to \mathrm{Pic}(X)_{\mathfrak{K}} \to \mathrm{Ref}_1(X)_{\mathfrak{K}} \to \mathrm{H}^0(X, \mathcal{C}\ell_X \otimes \mathfrak{K})$$

is exact. Thus a reflexive $\mathfrak{K}$-sheaf $\boldsymbol{\mathcal{L}}$ of rank one is contained in $\mathrm{Pic}(X)_{\mathfrak{K}}$ if and only if it is linearly equivalent to a $\mathfrak{K}$-Cartier divisor locally on $X$.

**§2.e. Pullback and push-forward.** Let $f\colon Y \to X$ be a morphism of normal complex analytic varieties. The pullback $f^*\mathcal{L}$ of an invertible sheaf $\mathcal{L} \in \mathrm{Pic}(X)$ is defined by the natural pullback homomorphism $f^*\colon \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ induced from the sheaf homomorphism $\mathcal{O}_X^{\star} \to f_*\mathcal{O}_Y^{\star}$. It also induces $f^*\colon \mathrm{Pic}(X, \mathfrak{K}) \to \mathrm{Pic}(Y, \mathfrak{K})$ for $\mathfrak{K} = \mathbb{Q}$ or $\mathbb{R}$. If $C \subset X$ is a compact irreducible curve and if $f\colon Y \to C \subset X$ is the normalization of $C$, then we have $L \cdot C = \deg f^*L \in \mathbb{R}$ for $L \in \mathrm{Pic}(X, \mathbb{R})$.

Suppose that the image $f(Y)$ has the following property: if $U \subset X$ is an open subset with $f(Y) \cap U \neq \emptyset$, then $f(Y) \cap U$ is not contained in any proper analytic subset of $U$. This condition is satisfied, for example, if $f$ is surjective. Then there is a natural homomorphism $\mathfrak{M}_X^{\star} \to f_*\mathfrak{M}_Y^{\star}$ which induces $\mathcal{CD}iv_X \to f_*\mathcal{CD}iv_Y$. Hence, the pullback homomorphisms

$$f^*\colon \mathrm{CDiv}(X) \to \mathrm{CDiv}(Y), \quad f^*\colon \mathrm{CDiv}(X, \mathfrak{K}) \to \mathrm{CDiv}(Y, \mathfrak{K})$$

are derived. These are compatible with $f^*$ for $\mathrm{Pic}(X)$ and $\mathrm{Pic}(X, \mathfrak{K})$.

Suppose that $f\colon Y \to X$ is proper surjective. A prime divisor $\Gamma$ of $Y$ is called $f$-*horizontal* if $f(\Gamma) = X$, and is called $f$-*vertical*, otherwise. An $f$-vertical prime divisor $\Gamma$ is called $f$-*exceptional* if $\mathrm{codim}_X f(\Gamma) \geq 2$. Let $I$ be the set of $f$-vertical but not $f$-exceptional prime divisors. Let $D$ be a $\mathfrak{K}$-divisor of $X$. Then it is $\mathfrak{K}$-Cartier on the non-singular locus $X_{\mathrm{reg}}$. Thus we can consider the multiplicity of $f^*(D|_{X_{\mathrm{reg}}})$ along $\Gamma \in I$. We define

$$f^{[*]}D = f^{[*]}(D) = \sum\nolimits_{\Gamma \in I} \mathrm{mult}_\Gamma(f^*(D|_{X_{\mathrm{reg}}}))\Gamma,$$

which is called the *proper inverse image* or the *proper pullback* of $D$ by $f$ (cf. [**45**]).

**2.11. Lemma** *Let $D$ be an $\mathbb{R}$-Cartier divisor of $X$. If $f$ is surjective, then there is a canonical homomorphism*

$$\mathcal{O}_X(\llcorner D \lrcorner) \to f_*\mathcal{O}_Y\left(\llcorner f^*D \lrcorner\right).$$

*If $f$ is a fiber space, then the homomorphism is isomorphic.*

PROOF. Let $\varphi$ be a meromorphic function defined on an open subset $U$ of $X$ such that $\mathrm{div}(\varphi) + D|_U \geq 0$. Then $\mathrm{div}(f^*\varphi) + f^*D|_{f^{-1}U} \geq 0$. This defines the canonical homomorphism. Next, suppose that $f$ is a fiber space. Let $\psi$ be a meromorphic function defined on $f^{-1}U$ such that $\Delta := \mathrm{div}(\psi) + f^*D|_{f^{-1}U} \geq 0$. Then, for a general fiber $F = f^{-1}(x)$, $\Delta|_F$ is an effective $\mathbb{R}$-divisor $\mathbb{R}$-linearly equivalent to zero. Hence any component of $\Delta$ is $f$-vertical. In particular, $\psi|_F$ is a constant function. Since $F$ is connected, $\psi$ descends to a meromorphic function on $U$. Thus the homomorphism is isomorphic. □

Let $f\colon Y \to X$ be a morphism of normal complex analytic varieties. Suppose that $f$ is a proper surjective and generically finite morphism. The norm mapping $\mathrm{Nm}_{Y/X}\colon f_*\mathcal{O}_Y \to \mathcal{O}_X$ is defined as follows: a holomorphic function $\alpha \in \mathrm{H}^0(Y, \mathcal{O}_Y)$ induces an $\mathcal{O}_X$-linear homomorphism

$$f_*\mathcal{O}_Y \ni \varphi \longmapsto \alpha\varphi \in f_*\mathcal{O}_Y$$

and an $\mathcal{O}_X$-linear homomorphism between the double-duals of determinants:

$$\det(f_*\mathcal{O}_Y)^\wedge \to \det(f_*\mathcal{O}_Y)^\wedge.$$

The latter corresponds to the multiplication by $\mathrm{Nm}_{Y/X}(a) \in \mathrm{H}^0(X, \mathcal{O}_X)$. The norm mapping extends to $\mathrm{Nm}_{Y/X}\colon f_*\mathfrak{M}_Y \to \mathfrak{M}_X$. Let $\Delta$ be an effective divisor of $Y$ and let $\mathcal{O}_Y \to \mathcal{O}_Y(\Delta)$ be a natural injective homomorphism. Then $f_*\mathcal{O}_Y \to f_*\mathcal{O}_Y(\Delta)$ induces an effective divisor $f_*\Delta$ such that

$$\mathcal{O}_X(f_*\Delta) \simeq \left(\det(f_*\mathcal{O}_Y(\Delta)) \otimes \det(f_*\mathcal{O}_Y)^\vee\right)^\wedge.$$

Let $\Gamma$ be a prime divisor of $Y$. If $\Gamma$ is $f$-exceptional, then $f_*\Gamma = 0$. If $f(\Gamma)$ is a divisor, then $f_*\Gamma = af(\Gamma)$ for the mapping degree $a$ of $\Gamma \to f(\Gamma)$. For two effective divisors $\Delta_1$, $\Delta_2$, we have $f_*(\Delta_1 + \Delta_2) = f_*\Delta_1 + f_*\Delta_2$. Thus $f_*$ gives rise to a homomorphism $f_*\colon f_*\mathcal{D}iv_Y \to \mathcal{D}iv_X$. In particular, $f_*\operatorname{div}(\alpha) = \operatorname{div}(\mathrm{Nm}_{Y/X}(\alpha))$ for $0 \neq \alpha \in \mathrm{H}^0(Y, \mathcal{O}_Y)$. Hence we have a commutative diagram of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & f_*\mathcal{O}_Y^\star & \longrightarrow & f_*\mathfrak{M}_Y^\star & \longrightarrow & f_*\mathcal{D}iv_Y \\ & & \big\downarrow{\scriptstyle \mathrm{Nm}_{Y/X}} & & \big\downarrow{\scriptstyle \mathrm{Nm}_{Y/X}} & & \big\downarrow{\scriptstyle f_*} \\ 0 & \longrightarrow & \mathcal{O}_X^\star & \longrightarrow & \mathfrak{M}_X^\star & \longrightarrow & \mathcal{D}iv_X. \end{array}$$

Note that if $D_1 \sim D_2$, then $f_*D_1 \sim f_*D_2$. But $f_*D$ for a Cartier divisor $D$ is not necessarily a Cartier divisor. If $E$ is a Cartier divisor of $X$, then $f_*f^*E = (\deg f)E$. Let $Z$ be a reduced divisor of $X$. Then $f_*$ induces $\mathrm{Div}_{f^{-1}Z}(Y) \to \mathrm{Div}_Z(X)$. By **2.4**, it is isomorphic to the natural homomorphism

$$\mathrm{H}^{2-2n}_{f^{-1}Z}(Y, \boldsymbol{\omega}_Y^{\mathrm{top}\bullet}) \to \mathrm{H}^{2-2n}_Z(X, \boldsymbol{\omega}_X^{\mathrm{top}\bullet})$$

induced from the property of $\boldsymbol{\omega}^{\mathrm{top}\bullet}$, where $n = \dim X$. In particular,

$$\mathcal{L}_1\mathcal{L}_2\cdots\mathcal{L}_{n-1}\cdot f_*D = f^*\mathcal{L}_1 f^*\mathcal{L}_2\cdots f^*\mathcal{L}_{n-1}\cdot D$$

for a divisor $D$ of $Y$ with compact support and invertible sheaves $\mathcal{L}_i$ of $X$. The push-forward $f_*$ is also defined for $\mathfrak{K}$-divisors by the linearity of $f_*$. Here the linear equivalence $\sim$ and the $\mathfrak{K}$-linear equivalence $\sim_{\mathfrak{K}}$ are also preserved. The push-forward extends to reflexive sheaves of rank one by

$$\mathrm{Ref}_1(Y) \ni \mathcal{L} \mapsto (\det(f_*\mathcal{L}) \otimes \det(f_*\mathcal{O}_Y)^\vee)^\wedge \in \mathrm{Ref}_1(X).$$

Note that the push-forward is different from the direct image $f_*$ as a sheaf.

Next, we consider the pullback and push-forward in the case of meromorphic mappings. Let $f\colon Y \dashrightarrow X$ be a meromorphic mapping of normal varieties. Let $\mu\colon Z \to Y$ be a bimeromorphic morphism from a normal variety such that $f \circ \mu$ is a morphism $g\colon Z \to X$. If $g$ is surjective, then, for a $\mathfrak{K}$-Cartier divisor $D$, we have the pullback $g^*D$ and its push-forward $\mu_*(g^*D)$. We define the pullback $f^*D$ as the $\mathfrak{K}$-divisor $\mu_*(g^*D)$, which does not depend on the choice of $\mu\colon Z \to Y$. If $f$ is bimeromorphic, then $f^*D$ is called the *total transform* of $D$. Similarly, if $g$ is proper and surjective, then we define the proper pullback $f^{[*]}D$ for a $\mathfrak{K}$-divisor $D$ of $X$ as $\mu_*(g^{[*]}D)$. If $f$ is bimeromorphic, then $f^{[*]}D$ is called the *proper transform* or the *strict transform* of $D$. Suppose that $g$ is a generically finite proper surjective morphism. Let $E$ be a $\mathfrak{K}$-divisor of $Y$. We define the push-forward $f_*E$ as $g_*(\mu^{[*]}E)$. Note that $f_*E \neq g_*\mu^*E$ for some $\mathfrak{K}$-Cartier divisor $E$. If $f$ is bimeromorphic, then $f^{[*]}D = (f^{-1})_*D$ for any $\mathfrak{K}$-divisor $D$ of $X$.

As a final remark on the pullback of divisors, we consider some divisors which are not Cartier but admit reasonable pullbacks. Let $f\colon Y \to X$ be a bimeromorphic morphism from a non-singular variety onto a normal variety. Then $f_*\mathcal{D}iv_Y \to \mathcal{D}iv_X$ is surjective and $\mathcal{CD}iv_X \to f_*\mathcal{D}iv_Y$ is injective. Let us consider the composition

$$\text{(II-4)} \qquad f_*\mathcal{D}iv_Y \to \mathrm{R}^1 f_*\mathcal{O}_Y^\star \to (\mathrm{R}^2 f_*\mathbb{Z}_Y) \otimes \mathbb{Q} \simeq \mathrm{R}^2 f_*\mathbb{Q}_Y.$$

Let $\mathcal{K}$ be the kernel of (II-4) and let $\mathcal{ND}iv_X \subset \mathcal{D}iv_X$ be the image of $\mathcal{K}$ under $f_*\mathcal{D}iv_Y \to \mathcal{D}iv_X$. Then the following properties hold:

(1) $\mathcal{ND}iv_X$ does not depend on the choice of $Y$;
(2) $\mathcal{K}$ is isomorphic to $\mathcal{ND}iv_X$;
(3) $\mathcal{CD}iv_X \subset \mathcal{ND}iv_X \subset \mathcal{D}iv_X$.

(2) is proved by an argument in Chapter **III**, §**5.a**. A divisor $D$ contained in the group $\mathrm{NDiv}(X) := \mathrm{H}^0(X, \mathcal{ND}iv_X)$ is called a *numerically* $\mathbb{Q}$*-Cartier* $\mathbb{Z}$-divisor. In this case, there is a $\mathbb{Q}$-divisor $E$ of $Y$ such that $f_*E = D$ and $E \cdot \gamma = 0$ for any irreducible curve $\gamma$ contained in fibers of $f$.

**2.12. Lemma** (cf. [**99**]) *Let $f\colon Y \to X$ be a bimeromorphic morphism from a non-singular variety.*

(1) *For a point $x \in X$, $(\mathrm{R}^1 f_*\mathcal{O}_Y)_x = 0$ if and only if the stalk $\mathcal{C}\ell_{X,x}$ is a finitely generated abelian group.*
(2) *If $\mathrm{R}^1 f_*\mathcal{O}_Y = 0$, then $\mathcal{ND}iv_X \subset \mathcal{CD}iv_X \otimes \mathbb{Q}$.*

In particular, every numerically $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor is $\mathbb{Q}$-Cartier if $\mathrm{R}^1 f_*\mathcal{O}_Y = 0$.

Proof. (1) We have a surjection

$$\mathrm{Im}(f_*\mathcal{D}iv_Y \to \mathrm{R}^1 f_*\mathcal{O}_Y^\star) \twoheadrightarrow \mathcal{D}iv_X/\mathcal{CD}iv_X \simeq \mathcal{C}\ell_X.$$

If $(\mathrm{R}^1 f_*\mathcal{O}_Y)_x = 0$, then $(\mathrm{R}^1 f_*\mathcal{O}_Y^\star)_x \subset (\mathrm{R}^2 f_*\mathbb{Z}_Y)_x \simeq \mathrm{H}^2(f^{-1}(x), \mathbb{Z})$ is a finitely generated abelian group and so is $\mathcal{C}\ell_{X,x}$. The kernel of $(f_*\mathcal{D}iv_Y)_x \to (\mathcal{D}iv_X)_x$ is generated by the $f$-exceptional prime divisors over an open neighborhood of $x$. Hence if $\mathcal{C}\ell_{X,x}$ is finitely generated, then so is $\mathrm{Im}(f_*\mathcal{D}iv_Y \to \mathrm{R}^1 f_*\mathcal{O}_Y^\star)_x$. However the image contains $(\mathrm{R}^1 f_*\mathcal{O}_Y)_x/(\mathrm{R}^1 f_*\mathbb{Z}_Y)_x$. Therefore $(\mathrm{R}^1 f_*\mathcal{O}_Y)_x = 0$.

(2) For the kernel $\mathcal{K}$ of (II-4), the stalk at a point $x$ of the image of $\mathcal{K} \to \mathrm{R}^1 f_*\mathcal{O}_Y^\star$ is a torsion group. Since $\mathcal{ND}iv_X$ is the image of $\mathcal{K}$, $\mathcal{ND}iv_{X,x}/\mathcal{CD}iv_{X,x}$ is a finite group. Therefore $\mathcal{ND}iv_X \subset \mathcal{CD}iv_X \otimes \mathbb{Q}$. □

Let $\mathrm{NDiv}(X, \mathfrak{K})$ denote the group $\mathrm{H}^0(X, \mathcal{ND}iv_X \otimes \mathfrak{K})$. For a surjective morphism $g\colon Z \to X$ of normal varieties, the pullback $g^*\colon \mathrm{CDiv}(X, \mathfrak{K}) \to \mathrm{CDiv}(Z, \mathfrak{K})$ extends to $g^*\colon \mathrm{NDiv}(X, \mathfrak{K}) \to \mathrm{NDiv}(Z, \mathfrak{K})$. The pullback $g^*D$ is called the *numerical pullback* for $D \in \mathrm{NDiv}(X, \mathfrak{K})$ (cf. Chapter **III**, §**5.b**).

Suppose that $n = \dim X = 2$. Then, for a numerically $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor $D$ and for an irreducible compact curve $\gamma$, the intersection number is well-defined by

$$D \cdot \gamma := f^*D \cdot \gamma',$$

where $f\colon Y \to X$ is a desingularization and $\gamma'$ is the proper transform of $\gamma$. However if $n > 2$, then the intersection number is not well-defined in general:

**Example** Let $E$ be an elliptic curve and $\mathcal{L}$ be a very ample invertible sheaf of $E$. We consider the surface $S = E \times E$ and the $\mathbb{P}^1$-bundle $\pi\colon \mathbb{P} = \mathbb{P}_S(\mathcal{O}_S \oplus p_1^*\mathcal{L}) \to S$, where $p_1\colon S \to E$ is the first projection. Let $\Sigma \subset \mathbb{P}$ be the section of $\pi$ corresponding to the projection $\mathcal{O}_S \oplus p_1^*\mathcal{L} \to \mathcal{O}_S$. For the tautological invertible sheaf $\mathcal{O}_{\mathbb{P}}(1)$, we have an isomorphism $\mathcal{O}_{\mathbb{P}}(1)|_\Sigma \simeq \mathcal{O}_\Sigma$. Further $\mathcal{O}_\Sigma(-\Sigma)$ is isomorphic to $p_1^*\mathcal{L}$ by $\pi$. Let $H$ be a divisor of $\mathbb{P}$ such that $\mathcal{O}_{\mathbb{P}}(H)$ is isomorphic to $\mathcal{O}_{\mathbb{P}}(1) \otimes \pi^*p_2^*\mathcal{L}$, where $p_2$ is the second projection. Then the linear system $|H|$ is base-point free (cf. §**3.a**) and defines a birational morphism $f\colon \mathbb{P} \to X$ onto a three-dimensional normal projective variety such that

(1) $C = f(\Sigma)$ is isomorphic to $E$,
(2) $f^{-1}C = \Sigma$,
(3) $f|_\Sigma$ is isomorphic to $p_2\colon S \to E$,
(4) $f$ induces an isomorphism $\mathbb{P} \setminus \Sigma \simeq X \setminus C$.

Let $\Delta \subset S = E \times E$ be the diagonal and let $F \subset S$ be a fiber of $p_1$. Let $D$ be a divisor of $\mathbb{P}$ such that $\mathcal{O}_{\mathbb{P}}(D)$ is isomorphic to $\mathcal{O}_{\mathbb{P}}(1) \otimes \pi^*\mathcal{O}_S(\Delta - F)$. Then, for a fiber $\gamma$ of $f$, we have $D \cdot \gamma = 0$. Thus $f_*D$ is numerically $\mathbb{Q}$-Cartier. Let $C_0$ and $C_1$, respectively, be the inverse images of $\Delta$ and $F$ under the isomorphism $\pi\colon \Sigma \xrightarrow{\simeq} S$. Then $f(\Delta) = f(F) = C$, $D \cdot C_0 = (\Delta - F) \cdot \Delta = -1$, and $D \cdot C_1 = (\Delta - F) \cdot F = 1$. Therefore, it is not possible to define $f_*D \cdot C$ in a natural way.

## §3. $D$-dimension

**§3.a. Linear systems of $\mathbb{R}$-divisors.** Let $X$ be a normal complex analytic variety and let $\boldsymbol{\mathcal{L}}$ be a reflexive $\mathbb{R}$-sheaf of rank one (cf. §**2.d**). We denote by $|\boldsymbol{\mathcal{L}}|$ the set of effective $\mathbb{R}$-divisors linearly equivalent to $\boldsymbol{\mathcal{L}}$. Note that if an $\mathbb{R}$-divisor $\Delta$ is linearly equivalent to $\boldsymbol{\mathcal{L}}$, then $\llcorner \Delta \lrcorner \sim \llcorner \boldsymbol{\mathcal{L}} \lrcorner$ and $\langle \Delta \rangle = \langle \boldsymbol{\mathcal{L}} \rangle$. Hence we have the identification $|\boldsymbol{\mathcal{L}}| = |\llcorner \boldsymbol{\mathcal{L}} \lrcorner| + \langle \boldsymbol{\mathcal{L}} \rangle$. An effective $\mathbb{Z}$-divisor $\Delta$ defines an ideal $\mathcal{O}_X(-\Delta) \subset \mathcal{O}_X$ and equivalently an injective homomorphism $\mathcal{O}_X \hookrightarrow \mathcal{O}_X(\Delta)$ up to unit holomorphic functions. Thus any member of $|\boldsymbol{\mathcal{L}}|$ is derived from a non-zero global section of $\llcorner \boldsymbol{\mathcal{L}} \lrcorner$. Hence $|\boldsymbol{\mathcal{L}}|$ is set-theoretically identified with the quotient space

$$\mathrm{H}^0(X, \llcorner \boldsymbol{\mathcal{L}} \lrcorner) \setminus \{0\} / \mathrm{H}^0(X, \mathcal{O}_X^\star)$$

by the scalar action.

A *linear system* $\Lambda = \Lambda(L, \boldsymbol{\mathcal{L}})$ is defined to be the projective space

$$\Lambda = \mathbb{P}(L^\vee) = L \setminus \{0\}/\mathbb{C}^\star$$

associated with a finite-dimensional vector subspace $L \subset \mathrm{H}^0(X, \llcorner \boldsymbol{\mathcal{L}} \lrcorner)$. Usually, we assume that $\Lambda \neq \emptyset$ and hence $L \neq 0$. A point $\lambda \in \Lambda$ defines an effective $\mathbb{R}$-divisor $\Delta_\lambda$ linearly equivalent to $\boldsymbol{\mathcal{L}}$. If $\mathrm{H}^0(X, \mathcal{O}_X) \simeq \mathbb{C}$ and if $L = \mathrm{H}^0(X, \llcorner \boldsymbol{\mathcal{L}} \lrcorner)$, then $\Lambda$ is set-theoretically identified with $|\boldsymbol{\mathcal{L}}|$ and is called a *complete* linear system. We denote the linear system $\Lambda(L, \llcorner \boldsymbol{\mathcal{L}} \lrcorner)$ by $\llcorner \Lambda \lrcorner$ and the $\mathbb{R}$-divisor $\langle \boldsymbol{\mathcal{L}} \rangle$ by $\langle \Lambda \rangle$. Then we can write $\Lambda = \llcorner \Lambda \lrcorner + \langle \Lambda \rangle$. The *base locus* $\mathrm{Bs}\,\Lambda$ of the linear system $\Lambda$ is defined set-theoretically as the intersection of $\mathrm{Supp}\,\Delta_\lambda$ for all $\lambda \in \Lambda$. Thus $\mathrm{Bs}\,\Lambda = \mathrm{Bs}\,\llcorner \Lambda \lrcorner \cup \mathrm{Supp}\langle \Lambda \rangle$. If $\mathrm{Bs}\,\Lambda = \emptyset$, then $\Lambda$ is called *base-point free*. In this case, $\boldsymbol{\mathcal{L}}$ is regarded as an invertible sheaf which is generated by finitely many global sections.

The evaluation mapping

$$L \otimes_{\mathbb{C}} \mathcal{O}_X \to \mathrm{H}^0(X, \llcorner \boldsymbol{\mathcal{L}} \lrcorner) \otimes_{\mathbb{C}} \mathcal{O}_X \to \llcorner \boldsymbol{\mathcal{L}} \lrcorner \tag{II-5}$$

is not zero. Let $\mathcal{G}$ be the image. Then $\mathcal{G} \hookrightarrow \llcorner \boldsymbol{\mathcal{L}} \lrcorner$ is isomorphic over a dense Zariski-open subset of $X$. Thus $\mathbb{P}_X(\mathcal{G}) \to X$ admits a meromorphic section $X \dashrightarrow \mathbb{P}_X(\mathcal{G})$.

Therefore, we have a meromorphic mapping

$$\Phi_\Lambda : X \dashrightarrow \mathbb{P}_X(\mathcal{G}) \dashrightarrow \mathbb{P}(L) = \Lambda^\vee$$

into the dual projective space of $\Lambda$. By taking the dual of (II-5), we have an injection

$$(\llcorner \mathcal{L} \lrcorner)^\vee \to L^\vee \otimes_{\mathbb{C}} \mathcal{O}_X.$$

It defines an effective $\mathbb{R}$-divisor $\mathcal{D} = \mathcal{D}_\Lambda$ of $X \times \Lambda$ such that $\langle \mathcal{D} \rangle = p_1^* \langle \Lambda \rangle$,

$$\mathcal{O}_{X \times \Lambda}(\llcorner \mathcal{D} \lrcorner) \simeq p_1^*(\llcorner \mathcal{L} \lrcorner) \otimes p_2^* \mathcal{O}_\Lambda(1), \quad \text{and} \quad \mathcal{D}|_{X \times \{\lambda\}} = \Delta_\lambda \subset X$$

for the projections $p_1$, $p_2$ and for $\lambda \in \Lambda$. The base locus $\operatorname{Bs} \Lambda$ is the set of points $x \in X$ with $p_1^{-1}(x) = \mathcal{D}_x = \Lambda$. If $x \notin \operatorname{Bs} \Lambda$, then the fiber $p_1^{-1}(x) = \mathcal{D}_x \subset \Lambda$ is a hyperplane and it specifies a point of the dual space $\Lambda^\vee$. The point coincides with the image $\Phi_\Lambda(x)$. Let

$$L \otimes_{\mathbb{C}} (\llcorner \mathcal{L} \lrcorner)^\vee \to \mathrm{H}^0(X, \llcorner \mathcal{L} \lrcorner) \otimes_{\mathbb{C}} (\llcorner \mathcal{L} \lrcorner)^\vee \to \mathcal{O}_X$$

be the homomorphism induced from (II-5) and let $\mathcal{I}$ be the image. Then the base locus $\operatorname{Bs} \llcorner \Lambda \lrcorner$ is regarded as a closed subspace defined by the ideal $\mathcal{I}$. The *fixed part* or the *fixed* ($\mathbb{R}$-)*divisor* is defined to be the maximum effective $\mathbb{R}$-divisor $\Lambda_{\mathrm{fix}}$ satisfying $\Delta_\lambda \geq \Lambda_{\mathrm{fix}}$ for any $\lambda \in \Lambda$. In other words, $\langle \Lambda_{\mathrm{fix}} \rangle = \langle \Lambda \rangle$ and

$$\mathcal{O}_X(-\llcorner \Lambda_{\mathrm{fix}} \lrcorner) = \mathcal{I}^\wedge.$$

In particular, $\mathcal{L}(-\Lambda_{\mathrm{fix}})$ is a usual reflexive sheaf of rank one. The linear system $\Lambda(L, \mathcal{L}(-\Lambda_{\mathrm{fix}}))$ is denoted by $\Lambda_{\mathrm{red}}$. Then $\Lambda = \Lambda_{\mathrm{red}} + \Lambda_{\mathrm{fix}}$. We can identify $\Phi_{\Lambda_{\mathrm{red}}}$ with $\Phi_\Lambda$. Here, $\operatorname{Bs} \Lambda_{\mathrm{red}}$ is the locus of indeterminacy of $\Phi_\Lambda$. If $\Lambda_{\mathrm{fix}} = 0$, then $\Lambda$ is called *reduced* or *fixed-part free*. If $\Lambda(L, \mathcal{L})_{\mathrm{fix}} = 0$ for some linear subspace $L$, then $\mathcal{L}$ is called *fixed-part free*.

Let $f : Y \to X$ be a proper surjective morphism from a normal variety. Let $\Lambda = \Lambda(L, \mathcal{L})$ be a linear system in which $\mathcal{L}$ is $\mathbb{R}$-Cartier: $\mathcal{L} \in \operatorname{Pic}(X)_{\mathbb{R}}$. Then we can define its pullback $f^*\Lambda$ as follows: For the pullback $f^*\mathcal{L}$, there is a canonical injective homomorphism $\llcorner \mathcal{L} \lrcorner \to f_*\left(\llcorner f^*\mathcal{L} \lrcorner\right)$ by **2.11**. For the image $L' \subset \mathrm{H}^0\left(Y, \llcorner f^*\mathcal{L} \lrcorner\right)$ of $L$ under the homomorphism above, we define $f^*\Lambda = \Lambda(L', f^*\mathcal{L})$. Here, $\Lambda$ and $f^*\Lambda$ are isomorphic to each other as a projective space and $\Phi_{f^*\Lambda} = \Phi_\Lambda \circ f$.

We can generalize the notion of linear systems to the following relative situation: Let $\pi : X \to S$ be a proper surjective morphism into another complex analytic variety. Let $\mathcal{F} \subset \pi_*(\llcorner \mathcal{L} \lrcorner)$ be a non-zero coherent subsheaf. Then $\mathbb{P}_S(\mathcal{F}^\vee)$ should be the relative linear system $\Lambda = \Lambda(\mathcal{F}, \mathcal{L}/S)$. The evaluation homomorphism

$$\pi^*\mathcal{F} \to \pi^*\pi_*(\llcorner \mathcal{L} \lrcorner) \to \llcorner \mathcal{L} \lrcorner$$

corresponds to (II-5). We have the associated meromorphic map $\Phi_\Lambda : X \dashrightarrow \mathbb{P}_S(\mathcal{F})$ over $S$ and we can define the relative base locus $\operatorname{Bs} \Lambda/S$ and the relative fixed part $\Lambda_{\mathrm{fix}/S}$ in a natural way.

**3.1. Proposition** *Let $\pi : X \to S$ be a proper surjective morphism from a normal variety and let $\mathcal{L}$ be a reflexive $\mathbb{R}$-sheaf of rank one of $X$. For $k \in \mathbb{Z}$, let*

$\mathcal{L}^{[k]}$ *denote the* $k$*-th power as a reflexive* $\mathbb{R}$*-sheaf of rank one. Suppose that there exist coherent subsheaves* $\mathcal{F}_k \subset \pi_*(\llcorner \mathcal{L}^{[k]} \lrcorner)$ *for* $k \geq 0$ *such that*

$$R(\mathcal{F}_\bullet, \mathcal{L}/S) := \bigoplus\nolimits_{k=0}^{\infty} \mathcal{F}_k \subset R(\mathcal{L}/S) := \bigoplus\nolimits_{k=0}^{\infty} \pi_*(\llcorner \mathcal{L}^{[k]} \lrcorner)$$

*is an* $\mathcal{O}_S$*-subalgebra. Then the following three conditions are mutually equivalent:*

(1) $R(\mathcal{F}_\bullet, \mathcal{L}/S)$ *is a locally finitely presented graded* $\mathcal{O}_S$*-algebra;*

(2) *Locally on* $S$*, there is a positive integer* $k$ *such that* $R(\mathcal{F}_\bullet, \mathcal{L}/S)^{(k)} = R(\mathcal{F}_{k\bullet}, \mathcal{L}^{[k]}/S)$ *is* 1*-generated;*

(3) *Locally on* $S$*, there exist a positive integer* $k$*, a bimeromorphic morphism* $\mu\colon Y \to X$ *from a normal variety, and an effective* $\mathbb{R}$*-divisor* $E$ *of* $Y$ *such that* $\mathcal{M} = \mu^*\mathcal{L}^{[k]}(-E)$ *is an invertible sheaf and the image of*

$$\mu^*\pi^*\mathcal{F}_{mk} \to \llcorner \mu^*(\mathcal{L}^{[mk]}) \lrcorner$$

*is* $\mathcal{M}^m$ *for any* $m \in \mathbb{N}$*.*

Proof. (1) $\Rightarrow$ (2) is shown by **1.1**-(2).

(2) $\Rightarrow$ (3): Assume that $R(\mathcal{F}_\bullet, \mathcal{L}/S)^{(k)}$ is 1-generated. Let $\mathcal{G}$ be the image of $\mu^*\pi^*\mathcal{F}_k \to \llcorner \mathcal{L}^{[k]} \lrcorner$ and let $V$ be the blowing-up $\mathrm{Projan}_X \bigoplus_i \mathcal{G}^i$, where $\mathcal{G}^i = \mathcal{G}^{\otimes i}/(\mathrm{tor})$. Let $Y \to V$ be the normalization and let $\mu\colon Y \to X$ be the composite. Then $\mu^*\mathcal{G}/(\mathrm{tor}) = \mathcal{M}$ is an invertible sheaf which is the image of $\mu^*\pi^*\mathcal{F}_k \to \llcorner \mu^*\mathcal{L}^{[k]} \lrcorner$. Let $E$ be the effective $\mathbb{R}$-divisor of $Y$ with $\mathcal{M}(E) = \mu^*\mathcal{L}^{[k]}$. Then the image of $\mu^*\pi^*\mathcal{F}_{mk} \to \llcorner \mu^*\mathcal{L}^{[mk]} \lrcorner$ is $\mathcal{M}^{\otimes m}$ for any $m \in \mathbb{N}$, since $\mathrm{Sym}^m \mathcal{F}_k \to \mathcal{F}_{km}$ is surjective.

(3) $\Rightarrow$ (1): We consider locally on $Y$. If $s$ is a section of $\mathcal{F}_i$ for some $i \in \mathbb{N}$, then $s^k$ is a section of $\mathcal{F}_{ki}$. Thus the corresponding effective $\mathbb{R}$-divisor $\mathrm{div}(s) \sim \llcorner \mu^*\mathcal{L}^{[i]} \lrcorner$ satisfies

$$k\,\mathrm{div}(s) + k\langle \mu^*\mathcal{L}^{[i]} \rangle \geq iE.$$

Therefore, for any $0 \leq i < k$ and $m \geq 0$,

$$\mathcal{F}_{mk+i} \subset \pi_*\mu_*(\llcorner \mu^*\mathcal{L}^{[mk+i]}(-mE) \lrcorner) = \pi_*\mu_*(\mathcal{M}^m \otimes \llcorner \mu^*\mathcal{L}^{[i]} \lrcorner).$$

Let $f\colon Y \to \mathbb{P}_S(\mathcal{F}_k)$ be the morphism over $S$ associated with the relative linear system $\Lambda(\mathcal{F}_k, \mathcal{M}/S)$ which is relatively base point free. Then $\mathcal{M} \simeq f^*\mathcal{O}(1)$ for the tautological invertible sheaf $\mathcal{O}(1)$ associated with $\mathcal{F}_k$. Let $p\colon \mathbb{P}_S(\mathcal{F}_k) \to S$ be the structure morphism. Since $\mathcal{O}(1)$ is $p$-ample,

$$\bigoplus\nolimits_{m=0}^{\infty} p_*\left(\mathcal{O}(m) \otimes f_*\llcorner \mu^*\mathcal{L}^{[i]} \lrcorner\right)$$

is a locally finitely presented graded $\mathrm{Sym}^\bullet(\mathcal{F}_k)$-module. Thus the submodule

$$R^{(k,i)} := \bigoplus\nolimits_{m=0}^{\infty} \pi_*\mathcal{F}_{mk+i}$$

is also locally of finitely presented by the argument in **1.5**. Hence $R(\mathcal{F}_\bullet, \mathcal{L}/S)$ is locally finitely presented. $\square$

The criterion above is well-known in algebraic case which is related to the Iitaka fibration and the sectional decomposition.

Suppose that $X$ is compact. For a divisor $D$, we have the associated reflexive sheaf $\mathcal{O}_X(D)$ of rank one. We denote by $|D|$ the complete linear system $|\mathcal{O}_X(D)|$, which is not empty when $D$ is linearly equivalent to an effective divisor. A Cartier divisor $A$ of $X$ is called *very ample* if so is $\mathcal{O}_X(A)$. This is the case $\mathrm{Bs}\,|A| = \emptyset$ and $\Phi_{|A|}\colon X \to |A|^\vee$ is a closed immersion. An *ample* divisor is a divisor whose multiple by some positive integer is very ample. A *base-point free* (or *free*) divisor $D$ is a Cartier divisor with $|D|$ is base-point free. A *fixed-part free* divisor $D$ is a divisor with $|D|_{\mathrm{fix}} = 0$. A divisor $D$ is called *semi-ample* if $\mathcal{O}_X(D)$ is so.

**Notation** We denote the cohomology group $\mathrm{H}^i(X, \mathcal{O}_X(D))$ for a divisor $D$ of $X$ simply by $\mathrm{H}^i(X, D)$ and the dimension $\dim \mathrm{H}^i(X, D)$ by $\mathrm{h}^i(X, D)$.

**§3.b. $D$-dimensions of $\mathbb{R}$-divisors.** We shall generalize Iitaka's theory of $D$-dimension to $\mathbb{R}$-divisors on normal varieties in §§**3.b** and **3.c** by using a property proved in Chapter **III**. We follow Iitaka's argument in the book [**44**].

Let $D$ be an $\mathbb{R}$-divisor of a compact complex normal variety $X$ of dimension $n$. Assume that $|D| \neq \emptyset$. Then we have a meromorphic mapping

$$\Phi_D := \Phi_{|D|}\colon X \dashrightarrow |D|^\vee = \mathbb{P}(\mathrm{H}^0(X, \llcorner D \lrcorner))$$

associated with the complete linear system $|D|$. We set $W_D$ to be the image of $\Phi_D$.

**3.2. Definition** We set

$$\mathbb{N}(D) := \{m \in \mathbb{N}\,;\, |mD| \neq \emptyset\}.$$

The *$D$-dimension* $\kappa(D) = \kappa(D, X)$ of $X$ is defined as follows:

$$\kappa(D) = \begin{cases} -\infty, & \text{if } \mathbb{N}(D) = \emptyset; \\ \max\{\dim W_{mD} \mid m \in \mathbb{N}(D)\}, & \text{if } \mathbb{N}(D) \neq \emptyset. \end{cases}$$

In case $\kappa(D) \geq 0$, we set $m_0(D) = \gcd \mathbb{N}(D)$.

Here, $\mathbb{N}(D)$ is a semi-group and $km_0(D) \in \mathbb{N}(D)$ for $k \gg 0$. We infer that $\kappa(D) \in \{-\infty, 0, 1, \ldots, a(X)\}$ for the algebraic dimension $a(X)$ of $X$ and that the equality $\kappa(D) = \max\{\kappa(\llcorner mD \lrcorner) \mid m \in \mathbb{N}\}$ holds. If $D' \sim_{\mathbb{Q}} D$, then $\kappa(D) = \kappa(D')$.

**3.3. Lemma** *Let $D$ be an $\mathbb{R}$-divisor with $\kappa(D) = 0$. Then $m_0(D) \in \mathbb{N}(D)$. In particular,*

$$\mathrm{h}^0(X, \llcorner mD \lrcorner) = \begin{cases} 1, & \text{if } m_0(D) | m; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. By definition, $\mathrm{h}^0(X, \llcorner mD \lrcorner) \leq 1$. If $\Delta_1$ and $\Delta_2$ are effective $\mathbb{R}$-divisors with $\Delta_1 \sim m_1 D$ and $\Delta_2 \sim m_2 D$ for some $m_1$, $m_2 \in \mathbb{N}$, then

$$\frac{m_2}{\gcd(m_1, m_2)} \Delta_1 = \frac{m_1}{\gcd(m_1, m_2)} \Delta_2.$$

Hence there is an effective $\mathbb{R}$-divisor $\Delta_0$ with

$$\Delta_1 = \frac{m_1}{\gcd(m_1, m_2)}\Delta_0, \quad \Delta_2 = \frac{m_2}{\gcd(m_1, m_2)}\Delta_0.$$

Therefore, $\Delta_0 \sim \gcd(m_1, m_2)D$. Thus $m_0(D) \in \mathbb{N}(D)$. □

Let $\Delta$ be an effective $\mathbb{R}$-divisor of $X$. For the open immersion $j\colon X \setminus \Delta \hookrightarrow X$,

$$\mathcal{O}_X(*\Delta) := j_*\mathcal{O}_{X\setminus\Delta} \cap \mathfrak{M}_X \subset j_*\mathfrak{M}_{X\setminus\Delta}$$

is the sheaf of germs of meromorphic functions of $X$ holomorphic outside $\Delta$. We consider the integral domain

$$\mathcal{O}(X, *\Delta) := \mathrm{H}^0(X, \mathcal{O}_X(*\Delta)) \subset \mathfrak{M}(X) := \mathrm{H}^0(X, \mathfrak{M}_X)$$

and its quotient field $\mathfrak{M}(X, *\Delta) := Q\mathcal{O}(X, *\Delta)$. Note that $a(X) = \operatorname{tr.deg}_{\mathbb{C}} \mathfrak{M}(X)$. We can show that the extension $\mathfrak{M}(X, *\Delta) \subset \mathfrak{M}(X)$ is algebraically closed as follows: Let $\varphi$ be a meromorphic function integral over $\mathcal{O}(X, *\Delta)$. Then

$$\varphi^N + a_1\varphi^{N-1} + \cdots + a_{N-1}\varphi + a_N = 0$$

for some $a_1, a_2, \ldots, a_N \in \mathcal{O}(X, *\Delta)$. Hence $\varphi$ has no poles outside $\Delta$, which means $\varphi \in \mathcal{O}(X, *\Delta)$. If a meromorphic function $\varphi$ is integral over $\mathfrak{M}(X, *\Delta)$, then $\varphi\psi$ is integral over $\mathcal{O}(X, *\Delta)$ for some $0 \neq \psi \in \mathcal{O}(X, *\Delta)$, and hence $\varphi \in \mathfrak{M}(X, *\Delta)$.

Suppose that $\Delta' \sim \Delta$ for another effective $\mathbb{R}$-divisor $\Delta'$. Then $\Delta' - \Delta = \operatorname{div}(\varphi)$ for a meromorphic function $\varphi \in \mathcal{O}(X, *\Delta)$. If $\varphi' \in \mathcal{O}(X, *\Delta')$, then $\varphi'\varphi^k \in \mathcal{O}(X, *\Delta)$ for some $k > 0$. Hence $\mathfrak{M}(X, *\Delta') = \mathfrak{M}(X, *\Delta) \subset \mathfrak{M}(X)$.

**3.4. Definition** For an $\mathbb{R}$-divisor $D$ with $\kappa(D) \geq 0$, we define the subfield $\mathfrak{M}(X, *D) \subset \mathfrak{M}(X)$ as $\mathfrak{M}(X, *\Delta)$ for $\Delta \in |mD|$ for some $m \in \mathbb{N}(D)$.

**3.5. Lemma** (cf. [**44**, Proposition 10.1]) *Let $D$ be an $\mathbb{R}$-divisor with $\kappa(D) \geq 0$. Then $\kappa(D) = \operatorname{tr.deg}_{\mathbb{C}} \mathfrak{M}(X, *D)$ and the set*

$$\mathbb{I}(D) := \{m \in \mathbb{N}(D) \mid \mathfrak{M}(X, *D) = \mathfrak{M}(W_{mD})\}$$

*is a semi-group with* $\gcd \mathbb{I}(D) = m_0(D)$.

Proof. Let $\Delta$ be an effective $\mathbb{R}$-divisor with $\Delta \sim kD$ for some $k \in \mathbb{N}(D)$. Then we have a natural injection $\mathcal{O}_X(\llcorner\Delta\lrcorner) \subset \mathfrak{M}_X$ and equalities

$$\mathcal{O}_X(*\Delta) = \bigcup\nolimits_{m\in\mathbb{N}} \mathcal{O}_X(\llcorner m\Delta\lrcorner), \quad \text{and} \quad \mathcal{O}(X, *\Delta) = \bigcup\nolimits_{m\in\mathbb{N}} \mathrm{H}^0(X, \llcorner m\Delta\lrcorner) \subset \mathfrak{M}(X).$$

The subfield of $\mathfrak{M}(X)$ generated by $\mathrm{H}^0(X, \llcorner\Delta\lrcorner)$ is identified with $\mathfrak{M}(W_{kD})$ by $\Phi_{kD}^*\colon \mathfrak{M}(W_{kD}) \hookrightarrow \mathfrak{M}(X)$. Therefore,

$$\mathfrak{M}(X, *D) = \bigcup\nolimits_{m\in\mathbb{N}} \mathfrak{M}(W_{mkD}) = \bigcup\nolimits_{m\in\mathbb{N}(D)} \mathfrak{M}(W_{mD}) \subset \mathfrak{M}(X).$$

This implies $\kappa(D) = \operatorname{tr.deg}_{\mathbb{C}} \mathfrak{M}(X, *D)$. Furthermore, $mk \in \mathbb{I}(D)$ for $m \gg 0$. Since $k^l \in \mathbb{I}(D)$ for $l \gg 0$, any element $k$ of $\mathbb{N}(D)$ is divisible by $\gcd \mathbb{I}(D)$. Hence $\gcd \mathbb{I}(D) = \gcd \mathbb{N}(D) = m_0(D)$. □

In particular, if $D_1$ and $D_2$ are effective $\mathbb{R}$-divisors with $\operatorname{Supp} D_1 \subset \operatorname{Supp} D_2$, then $\kappa(D_1) \le \kappa(D_2)$.

**3.6. Remark** Suppose that $D$ is $\mathbb{R}$-Cartier. Then, for a bimeromorphic morphism $\mu\colon Y \to X$ from a normal variety, we have an isomorphism $\mathcal{O}_X(\llcorner mD \lrcorner) \simeq \mu_*\mathcal{O}_Y(\llcorner m\mu^*D \lrcorner)$ for any $m$ by **2.11**. Hence $\mu^*|mD| = |m\mu^*D|$ and $\kappa(D, X) = \kappa(\mu^*D, Y)$. Even if $D$ is not $\mathbb{R}$-Cartier, for a bimeromorphic morphism $\mu\colon Y \to X$ from a non-singular variety, there is a $\mu$-exceptional effective divisor $E$ such that

$$\mathcal{O}_X(\llcorner mD \lrcorner) \simeq \mu_*\mathcal{O}_Y(\llcorner m(\mu^{[*]}D + E) \lrcorner)$$

for any $m > 0$. This follows from **III.5.10**-(3), or **III.5.11**. In particular,

$$\kappa(D) = \max\{\kappa(\mu^{[*]}D + E) \mid E \text{ is a } \mu\text{-exceptional effective divisor}\}.$$

**3.7. Theorem** (Estimate) *Let $D$ be an $\mathbb{R}$-divisor with $\kappa(D) \ge 0$. Then there exist positive rational numbers $\alpha < \beta$ such that*

$$\alpha m^{\kappa(D)} \le \mathrm{h}^0(X, \llcorner m m_0(D)\, D \lrcorner) \le \beta m^{\kappa(D)}$$

*for $m \gg 0$.*

Proof. We may assume that $X$ is non-singular by **3.6** and that $\kappa(D) > 0$ by **3.3**. For $m \in \mathbb{I}(D)$, the meromorphic mapping $\Phi_{mD}\colon X \dashrightarrow W_{mD}$ induces an algebraically closed extension $\mathfrak{M}(W_{mD}) = \mathfrak{M}(X, *D) \subset \mathfrak{M}(X)$. Thus, the meromorphic mappings $\Phi_{mD}$ are mutually bimeromorphically equivalent for all $m \in \mathbb{I}(D)$, in the sense that there is a bimeromorphic mapping $i\colon W_{mD} \dashrightarrow W_{m'D}$ such that $\Phi_{m'D} = i \circ \Phi_{mD}$ for $m$, $m' \in \mathbb{I}(D)$. Let $W$ be a non-singular projective variety birational to $W_{mD}$ above and let $\mu\colon Y \to X$ be a bimeromorphic morphism from a non-singular variety such that $\Phi_{mD} \circ \mu$ induces a holomorphic mapping $f\colon Y \to W$. Then $f$ has only connected fibers. Suppose first that the birational mapping $\nu\colon W \dashrightarrow W_{kD}$ is holomorphic for a fixed $k \in \mathbb{I}(D)$. Then $|k\mu^*D|_{\mathrm{red}}$ is base-point free and

$$k\mu^*D - |k\mu^*D|_{\mathrm{fix}} \sim f^*\nu^*H_k \tag{II-6}$$

for an ample and free divisor $H_k$ of $W_{kD}$. In particular, for $m \in \mathbb{N}$, we have

$$\mathrm{h}^0(W_{kD}, mH_k) \le \mathrm{h}^0(Y, \llcorner mk\mu^*D \lrcorner) = \mathrm{h}^0(X, \llcorner mkD \lrcorner).$$

Since the left hand side is a polynomial of degree $\dim W$ for $m \gg 0$, there is a positive rational number $a_k$ such that

$$a_k m^{\dim W} \le \mathrm{h}^0(X, \llcorner mkD \lrcorner) \tag{II-7}$$

for $m \gg 0$. For a member $\Delta \in |k\mu^*D|$, let $\Delta = \Delta^h + \Delta^v$ be the decomposition into the *$f$-vertical part* $\Delta^v$ and the *$f$-horizontal part* $\Delta^h$; components of $\Delta^h$ are $f$-horizontal and components of $\Delta^v$ are $f$-vertical. Then we infer that $\Delta^h \le |k\mu^*D|_{\mathrm{fix}}$ by the linear equivalence (II-6). Thus $\Delta^h$ coincides with the $f$-horizontal part of $|k\mu^*D|_{\mathrm{fix}}$. There is an ample effective divisor $A$ of $W$ such that $\Delta^v \le f^*A$. Hence

$$\mathrm{h}^0(X, \llcorner mkD \lrcorner) = \mathrm{h}^0(Y, \llcorner mk\mu^*D \lrcorner) \le \mathrm{h}^0(W, mkA),$$

for $m \in \mathbb{N}$. Since the right hand side is a polynomial of degree $\dim W$ for $m \gg 0$, there is a positive rational number $b_k$ such that

$$\mathrm{h}^0(X, \llcorner mkD \lrcorner) \le b_k m^{\dim W} \tag{II-8}$$

for $m \gg 0$.

Let $\ell(m)$ be $\mathrm{h}^0(X, \llcorner mm_0(D)D \lrcorner)$. Then $\ell(m) > 0$ for $m \gg 0$ and $\ell(m+r) \ge \ell(m)$ for $m \in \mathbb{N}$ and for $r \in \mathbb{N}$ with $\ell(r) > 0$. Let $r_1, r_2, \ldots, r_{k-1}$ be natural numbers such that $r_i \equiv i \mod k$ and $\ell(r_i) > 0$. We set $r_0 = 0$ and $r_+ := \max\{r_i\}$. If $m \ge k + r_+$, then $m = q_1 k + r_i$ for some $i$ and $q_1 \in \mathbb{N}$. Hence $\ell(m) \ge \ell(q_1 k)$ and $q_1 \ge (m - r_+)/k$. Any $m \in \mathbb{N}$ is written as $m = q_2 k - r_j$ for some $j$ and $q_2 \in \mathbb{N}$. Hence $\ell(m) \le \ell(q_2 k)$ and $q_2 \le (m + r_+)/k$. Therefore, from (II-7), (II-8), we have

$$a_k \left(\frac{m - r_+}{k}\right)^{\dim W} \le \ell(m) \le b_k \left(\frac{m + r_+}{k}\right)^{\dim W}$$

for $m \ge k + r_+$. Thus we can find the required numbers $\alpha$ and $\beta$. □

**3.8. Corollary** *If $\kappa(D) \ge 0$, then*

$$\begin{aligned}\kappa(D) &= \max\{k \in \mathbb{Z}_{\ge 0} \mid \varlimsup_{m\to\infty} m^{-k}\,\mathrm{h}^0(X, \llcorner mD \lrcorner) > 0\} \\ &= \min\{k \in \mathbb{Z}_{\ge 0} \mid \varlimsup_{m\to\infty} m^{-k}\,\mathrm{h}^0(X, \llcorner mD \lrcorner) < +\infty\} \\ &= \lim_{m\to\infty} \frac{\log \mathrm{h}^0(X, \llcorner mm_0(D)D \lrcorner)}{\log m}.\end{aligned}$$

*In particular, the equality*

$$\kappa(D) = \varlimsup_{m\to\infty} \frac{\log \mathrm{h}^0(X, \llcorner mD \lrcorner)}{\log m}$$

*holds for any case including $\kappa(D) = -\infty$, under the notation*: $\log 0 = -\infty$.

**3.9. Corollary** *If $\kappa(D) \ge 0$, then there exist positive numbers $\alpha$ and $\beta$ such that*

$$\alpha m^{\kappa(D)} \le \mathrm{h}^0(X, \ulcorner mm_0(D)D \urcorner) \le \beta m^{\kappa(D)}$$

*for $m \gg 0$.*

Proof. It is enough to show the existence of $\beta$. Let $k$ be a positive integer such that $km_0(D)D \sim \Delta$ for an effective $\mathbb{R}$-divisor $\Delta$. Then $\langle km_0(D)D \rangle = \langle \Delta \rangle$. There is a positive integer $b$ such that $\ulcorner \langle \Delta \rangle \urcorner \le b\Delta$. Then $\ulcorner \langle \Delta \rangle \urcorner \le \llcorner b\Delta \lrcorner$. Therefore,

$$\ulcorner mkm_0(D)D \urcorner - \llcorner mkm_0(D)D \lrcorner = \ulcorner \langle mkm_0(D)D \rangle \urcorner \le \ulcorner \langle \Delta \rangle \urcorner \le \llcorner b\Delta \lrcorner$$

for any $m > 0$. Hence

$$\begin{aligned}\mathrm{h}^0(X, \ulcorner mkm_0(D)D \urcorner) &\le \mathrm{h}^0(X, \llcorner mkm_0(D)D \lrcorner + \llcorner bkm_0(D)D \lrcorner) \\ &\le \mathrm{h}^0(X, \llcorner (m+b)km_0(D)D \lrcorner).\end{aligned}$$

Thus $\beta$ exists by the same argument as in the proof of **3.7**. □

**Example** There is an example of $\mathbb{R}$-divisor $D$ such that $\kappa(D) = -\infty$ and $\mathrm{h}^0(X, \ulcorner mD \urcorner) > 0$ for $m \gg 0$. Let $X = \mathbb{P}^1$ and let $D = rP_1 - rP_2$ for $0 < r \in \mathbb{R} \setminus \mathbb{Q}$ and for two points $P_1$ and $P_2$. Then $\llcorner mD \lrcorner = \llcorner mr \lrcorner P_1 - \ulcorner mr \urcorner P_2 \sim -P_2$. Hence $\kappa(D) = -\infty$. But $\ulcorner mD \urcorner \sim P_1$. Thus $\mathrm{h}^0(X, \ulcorner mD \urcorner) > 0$ for $m > 0$.

**3.10. Lemma** (Fibration) *Let $D$ be an $\mathbb{R}$-Cartier divisor with $\kappa(D) > 0$. Suppose that there exist a morphism $f\colon X \to W$ into a normal variety and a bimeromorphic mapping $i\colon W \dashrightarrow W_{mD}$ with $i \circ f = \Phi_{mD}$ for $m \in \mathbb{I}(D)$. Then every fiber $f^{-1}(w)$ is connected and, for any $m \in \mathbb{N}(D)$, there exists a dense Zariski-open subset $U_m \subset W$ such that $f$ is flat over $U_m$ and*

$$f_*\mathcal{O}_X(\llcorner mD \lrcorner) \otimes \mathbb{C}(w) \simeq \mathrm{H}^0(f^{-1}(w), \mathcal{O}_X(\llcorner mD \lrcorner) \otimes \mathcal{O}_{f^{-1}(w)}) \simeq \mathbb{C}(w)$$

*for $w \in U_m$. In particular,*

$$\kappa(D|_{f^{-1}(w)}) = 0, \quad \textit{for} \quad w \in \bigcap\nolimits_{m \in \mathbb{N}(D)} U_m.$$

Proof. The connectedness of $f^{-1}(w)$ follows from that $\mathfrak{M}(W) = \mathfrak{M}(X, *D) \subset \mathfrak{M}(X)$ is algebraically closed. We have only to show that $\operatorname{rank} f_*\mathcal{O}_X(\llcorner mD \lrcorner) = 1$ for $0 \ll m \in \mathbb{N}(D)$. Let $\Delta$ be a member of $|mD|$ and set

$$M = \mathrm{H}^0(W, f_*\mathcal{O}_X(*\Delta) \otimes \mathfrak{M}_W).$$

Since $\mathfrak{M}_W \otimes f_*\mathfrak{M}_X \simeq f_*\mathfrak{M}_X$, we have an inclusion $\mathfrak{M}(W) \subset M \subset \mathfrak{M}(X)$. A meromorphic function $0 \neq \varphi \in \mathfrak{M}(X)$ belongs to $M$ if and only if the $f$-horizontal part $\operatorname{div}(\varphi)^h_-$ of the negative part $\operatorname{div}(\varphi)_-$ of the prime decomposition of $\operatorname{div}(\varphi)$ is supported in $\operatorname{Supp}\Delta$. Hence $M$ is generated by $\mathfrak{M}(W)$ and $\mathfrak{M}(X, *\Delta)$. Since $m \in \mathbb{I}(D)$, we have $\mathfrak{M}(W) = \mathfrak{M}(X, *\Delta) = M$. It implies that $\operatorname{rank} f_*\mathcal{O}_X(t\Delta) = 1$ for any $t \in \mathbb{N}$. Thus we are done. □

**3.11. Lemma** (Covering lemma) *Let $f\colon Y \to X$ be a proper surjective morphism of normal varieties and let $D$ be an $\mathbb{R}$-Cartier divisor of $X$. Then*

$$\kappa(f^*D + E) = \kappa(D)$$

*for an $f$-exceptional effective $\mathbb{R}$-divisor $E$.*

Proof. Let $Y \to V \to X$ be the Stein factorization of $f$ and set $g\colon Y \to V$ and $\tau\colon V \to X$. If $\operatorname{div}(\varphi) + m(f^*D + E) \geq 0$ for a non-zero meromorphic function $\varphi$ of $Y$ and for a positive integer $m$, then $\operatorname{div}(\varphi)|_{g^{-1}(v)} = 0$ for a general point $v \in V$, thus $\varphi$ is constant along $g^{-1}(v)$. Hence $\varphi \in \mathfrak{M}(V)$ by **1.12**-(2). Therefore, $\operatorname{div}(\varphi) + m\tau^*D \geq 0$, since $E$ is $g$-exceptional. This observation implies that $\mathfrak{M}(Y, *(f^*D + E)) = \mathfrak{M}(V, *\tau^*D)$ and hence $\kappa(f^*D + E) = \kappa(\tau^*D)$. Let $\mathrm{Nm}\colon \tau_*\mathfrak{M}_V \to \mathfrak{M}_X$ be the norm map. Then $\operatorname{div}(\varphi) + m\tau^*D \geq 0$ implies $\operatorname{div}(\mathrm{Nm}(\varphi)) + m(\deg\tau)D \geq 0$. In particular, $\kappa(\tau^*D) = -\infty$ if and only if $\kappa(D) = -\infty$. Hence we may assume that $D$ is effective. The multiplication by $\varphi$ defines an endomorphism of $\tau_*\mathcal{O}_V(*\tau^*D)$

and that of $\tau_*\mathfrak{M}_V \simeq \mathfrak{M}_X \otimes \tau_*\mathcal{O}_V$. Let $P(x) \in \mathfrak{M}(X)[x]$ be the polynomial defined by

$$\det(x \cdot \mathrm{id} - \varphi) \in \mathrm{End}(\det(\mathfrak{M}_X \otimes \tau_*\mathcal{O}_V))[x] \simeq \mathfrak{M}(X)[x].$$

Then $P(\varphi) = 0$. For the non-singular locus $U = X_{\mathrm{reg}}$, we have an isomorphism

$$\begin{aligned}\tau_*\mathcal{O}_V(*\tau^*D)|_U &= \bigcup\nolimits_{m>0} \tau_*\mathcal{O}_{\tau^{-1}U}(\tau^*(\llcorner mD|_U \lrcorner))\\ &\simeq \bigcup\nolimits_{m>0} \mathcal{O}_U(\llcorner mD \lrcorner) \otimes \tau_*\mathcal{O}_V = \mathcal{O}_X(*D)|_U \otimes \tau_*\mathcal{O}_V|_U.\end{aligned}$$

Since $\mathcal{O}(U, *D|_U) = \mathcal{O}(X, *D)$, the polynomial $P(x)$ belongs to $\mathcal{O}(X, *D)[x]$. Hence $\varphi$ is integral over $\mathcal{O}(X, *D)$. Therefore, $\mathcal{O}(V, *\tau^*D)$ is integral over $\mathcal{O}(X, *D)$ and $\mathrm{tr.deg}\,\mathfrak{M}(V, *\tau^*D) = \mathrm{tr.deg}\,\mathfrak{M}(X, *D)$. □

**3.12. Corollary** *Let $f\colon Y \to X$ be a proper surjective morphism of normal varieties and let $D$ be an $\mathbb{R}$-divisor of $X$. Then*

$$\kappa(D) = \max\{\kappa(f^{[*]}D + E) \mid E \textit{ is an } f\textit{-exceptional effective divisor}\}.$$

Proof. We may assume that $X$ and $Y$ are non-singular by **3.6**. Then it follows from **3.11**. □

**§3.c. Relative $D$-dimension.** Let $f\colon X \to Y$ be a proper surjective morphism from a non-singular variety and let $D$ be an $\mathbb{R}$-Cartier divisor of $X$. For a general point $y \in Y$, the fiber $f^{-1}(y)$ is non-singular and the restriction $D|_{f^{-1}(y)}$ is well-defined as the pullback of the $\mathbb{R}$-divisor $D$ by $f^{-1}(y) \hookrightarrow X$. Since $y$ is general, we have an isomorphism

$$\mathcal{O}_X(\llcorner D \lrcorner) \otimes \mathcal{O}_{f^{-1}(y)} \simeq \mathcal{O}_{f^{-1}(y)}(\llcorner D|_{f^{-1}(y)} \lrcorner).$$

For a positive integer $m$, by the upper-semicontinuity theorem and the flattening theorem, we can find a Zariski-open dense subset $U_m \subset Y$ such that

(1) $f$ is flat over $U_m$,
(2) $f^{-1}(y)$ is non-singular for any $y \in U_m$,
(3)
$$y \longmapsto \mathrm{h}^0(f^{-1}(y), \mathcal{O}_X(\llcorner mD \lrcorner) \otimes \mathcal{O}_{f^{-1}(y)})$$
is constant on $U_m$.

Let $X \to V \to Y$ be the Stein factorization of $f$. Then a connected component of a general fiber $f^{-1}(y)$ is a general fiber of $g\colon X \to V$. Therefore,

$$\mathrm{rank}\, f_*\mathcal{O}_X(\llcorner mD \lrcorner) = (\deg \tau)\, \mathrm{rank}\, g_*\mathcal{O}_X(\llcorner mD \lrcorner)$$

for the finite morphism $\tau\colon V \to Y$. Therefore, by **3.8**, we have

$$\kappa(D|_\Gamma) = \varlimsup_{m\to\infty} \frac{\log \mathrm{rank}\, f_*\mathcal{O}_X(\llcorner mD \lrcorner)}{\log m}$$

for a connected component $\Gamma$ of a 'general' fiber $f^{-1}(y)$. The *relative $D$-dimension* $\kappa(D; X/Y)$ is defined as $\kappa(D|_\Gamma)$.

Next, we consider a dominant proper meromorphic mapping $f\colon X \dashrightarrow Y$ from a normal variety and an $\mathbb{R}$-divisor $D$ of $X$. Let $\mu\colon Z \to X$ be a bimeromorphic morphism from a non-singular variety such that $g = f \circ \mu$ is holomorphic. If $Y$ is a point, then $X$ is compact and

$$\kappa(D, X) = \max\{\kappa(\mu^{[*]}D + E) \mid E \text{ is a } \mu\text{-exceptional effective divisor}\}$$

by **3.6**. We define the *relative $D$-dimension* by

$$\kappa(D; X/Y) := \max\{\kappa(\mu^{[*]}D + E; Z/Y) \mid E \text{ is a } \mu\text{-exceptional effective divisor}\}.$$

If $f$ is holomorphic and $D$ is $\mathbb{R}$-Cartier, then $\kappa(D; X/Y) = \kappa(\mu^* D; Z/Y)$.

**3.13. Theorem** (Easy addition) *Let $f\colon X \to Y$ be a proper surjective morphism of compact normal varieties and let $D$ be an $\mathbb{R}$-divisor of $X$. Then the easy addition formula*:

$$\kappa(D, X) \leq \kappa(D; X/Y) + \dim Y$$

*holds. If $\kappa(D - \varepsilon f^* H) \geq 0$ for some ample divisor $H$ and for some $\varepsilon > 0$, then*

$$\kappa(D) = \kappa(D; X/Y) + \dim Y.$$

Proof. We may assume that $X$ is non-singular and $f$ is a fiber space by taking a desingularization of $X$ and the Stein factorization of $f$. There is a countable intersection $\mathcal{Y}$ of dense Zariski-open subsets of $Y$ such that

(1) $f^{-1}(y)$ is non-singular,
(2) $\mathcal{O}_X(\llcorner mD \lrcorner) \otimes \mathcal{O}_{f^{-1}(y)} \simeq \mathcal{O}_{f^{-1}(y)}\left(\llcorner mD|_{f^{-1}(y)} \lrcorner\right)$,
(3) $\mathrm{h}^0\left(f^{-1}(y), \llcorner mD|_{f^{-1}(y)} \lrcorner\right) = \operatorname{rank} f_* \mathcal{O}_X(\llcorner mD \lrcorner)$,

for any $y \in \mathcal{Y}$, $m \in \mathbb{N}$. The evaluation mapping $f^* f_* \mathcal{O}_X(\llcorner mD \lrcorner) \to \mathcal{O}_X(\llcorner mD \lrcorner)$ defines a meromorphic mapping

$$\Phi_m\colon X \dashrightarrow \mathbb{P}_Y(f_* \mathcal{O}_X(\llcorner mD \lrcorner))$$

over $Y$. The restriction of $\Phi_m$ to the fiber $f^{-1}(y)$ over a point $y \in \mathcal{Y}$ is the meromorphic mapping associated with $|mD|_{f^{-1}(y)}|$. Let $Z_m$ be the image $\Phi_m(X)$. Then $\kappa(D; X/Z_m) = 0$ for some $m \in \mathbb{N}$. Replacing $X$ by a blowing-up, we may assume that there exist a fiber space $\phi\colon X \to Z$ into a non-singular variety $Z$ over $Y$ and a bimeromorphic mapping $\rho\colon Z \dashrightarrow Z_m$ with $\Phi_m = \rho \circ \phi$. Then, for a member $\Delta \in |mD|$, the $\phi$-horizontal part $\Delta^h$ is contained in the fixed part $|mD|_{\mathrm{fix}}$ since $\kappa(D; X/Z) = 0$. There are an effective $\mathbb{R}$-divisor $G$ of $Z$ and an effective $\phi$-exceptional $\mathbb{R}$-divisor $E$ of $X$ such that $\phi^* G + E \geq \Delta^v$. Hence $\kappa(D, X) \leq \kappa(\phi^* G + E) = \kappa(G)$ by **3.11**. Since $\kappa(G) \leq \dim Z$ and the dimension of the fiber of $Z \to Y$ over $y \in Y$ is $\kappa(D; X/Y)$, we have $\kappa(G) \leq \kappa(D; X/Y) + \dim Y$.

We may assume that $\rho\colon Z \dashrightarrow Z_m$ is also holomorphic. Let $B$ be a divisor such that $\mathcal{O}_Z(B)$ is isomorphic to the pullback of the tautological line bundle $\mathcal{O}(1)$ of $\mathbb{P}_Y(f_* \mathcal{O}_X(\llcorner mD \lrcorner))$. Then $mD - \phi^* B$ is linearly equivalent to an effective $\mathbb{R}$-divisor. Let $p\colon Z \to Y$ be the induced morphism. Then $B + bp^* H$ is free for some $b > 0$. Since $\kappa(D - \varepsilon f^* H) \geq 0$, we have

$$\kappa(D) \geq \kappa(mD + bf^* H) \geq \dim Z = \kappa(D; X/Y) + \dim Y. \qquad \square$$

**3.14. Theorem-Definition** *Let $X$ be a compact normal complex analytic variety, $D$ an $\mathbb{R}$-divisor of $X$ with $\kappa(D) > 0$ and $f\colon X \dashrightarrow Y$ a meromorphic fiber space. If $\kappa(D; X/Y) = 0$, then there exists a meromorphic mapping $\rho\colon Y \dashrightarrow W_{mD}$ with $\Phi_{mD} = \rho \circ f$ for $m \in \mathbb{I}(D)$. In particular, the following conditions are mutually equivalent:*

1. $\kappa(D; X/Y) = 0$ *and* $\dim Y = \kappa(D, X)$;
2. *There is a bimeromorphic mapping $\rho\colon Y \dashrightarrow W_{mD}$ for $m \in \mathbb{I}(D)$ such that $\Phi_{mD} = \rho \circ f$.*

*If $f$ satisfies the conditions above, then it is called the $D$-canonical fibration or the* Iitaka fibration *for $D$.*

Proof. We may assume that $f$ is holomorphic and $X$ is non-singular. By considering the restriction homomorphism

$$\mathrm{H}^0(X, \llcorner mD \lrcorner) \to \mathrm{H}^0\left(f^{-1}(y), \llcorner mD|_{f^{-1}(y)} \lrcorner\right) \simeq \mathbb{C},$$

we infer that the image of $f^{-1}(y)$ under $\Phi_{mD}$ is a point for $m \in \mathbb{I}(D)$. Hence the existence of the meromorphic mapping $\rho$ follows from **1.12**-(2). The implication (1) $\Rightarrow$ (2) follows from **1.12**-(1). The inverse implication is shown in **3.10**. □

**§3.d. Big divisors.** Let $X$ be a compact normal variety and let $D$ be an $\mathbb{R}$-divisor.

**3.15. Definition** $D$ is called *big* if $\kappa(D, X) = \dim X$.

If $X$ admits a big $\mathbb{R}$-divisor, then the algebraic dimension $a(X)$ is equal to $\dim X$. Hence $X$ is a Moishezon variety, which is a compact complex variety bimeromorphic to a projective variety, by definition.

**3.16. Lemma** (Kodaira's lemma) *Let $D$ be a big $\mathbb{R}$-divisor and let $H$ be an $\mathbb{R}$-divisor. Then there exist a positive integer $m$ and an effective $\mathbb{R}$-divisor $\Delta$ such that $mD \sim H + \Delta$.*

Proof. Let $\mu\colon X' \to X$ be a bimeromorphic morphism from a non-singular projective variety. There is a big $\mathbb{R}$-divisor $D'$ of $X'$ with $D = \mu_* D'$ by **3.6**. Suppose that there exist a positive integer $m$ and an effective $\mathbb{R}$-divisor $\Delta'$ of $X'$ such that $mD' \sim \mu^{[*]}H + \Delta'$. Then $mD \sim H + \mu_*\Delta'$. Thus we may assume that $X$ is non-singular projective. Let $A$ be an ample divisor. Then $|kA - H| \neq \emptyset$ for $k \gg 0$ by Theorem A. Hence we may assume that $H$ is a very ample non-singular divisor that does not contain any intersection $\Gamma \cap \Gamma'$ of two mutually distinct prime components $\Gamma$ and $\Gamma'$ of $D$. We consider the exact sequence

$$0 \to \mathcal{O}_X(\llcorner mD \lrcorner - H) \to \mathcal{O}_X(\llcorner mD \lrcorner) \to \mathcal{O}_X(\llcorner mD \lrcorner) \otimes \mathcal{O}_H \to 0$$

for $m \in \mathbb{N}$. Here we have an isomorphism $\mathcal{O}_X(\llcorner mD \lrcorner) \otimes \mathcal{O}_H \simeq \mathcal{O}_H(\llcorner mD|_H \lrcorner)$. By applying **3.7** to $D$ and $D|_H$, we infer that $\mathrm{h}^0(X, \llcorner mD \lrcorner - H) \neq 0$ for some $m$. □

**3.17. Corollary** *If $D$ is a big $\mathbb{R}$-divisor, then there is a positive integer $c$ such that $|tD| \neq \emptyset$ for any real number $t \geq c$. In particular, $m_0(D) = \gcd \mathbb{N}(D) = 1$.*

PROOF. We may assume that $X$ is non-singular projective. Then $aD \sim H + \Delta$ for a very ample divisor $H$, an effective $\mathbb{R}$-divisor, and a positive integer $a$. There is a positive integer $b$ such that $\mathrm{h}^0(X, \llcorner rD \lrcorner + bH) \neq 0$ for any real number $0 \leq r \leq a$, since we have only finitely many divisors $\llcorner rD \lrcorner$. If $t \geq ba$, then $t = ka + r$ for some integer $k \geq b$ and $0 \leq r < a$. Hence

$$tD = (k-b)aD + (ba + r)D \sim (k-b)(H + \Delta) + b\Delta + (bH + rD).$$

Thus $|tD| \neq \emptyset$. □

The following theorem is proved by Fujita [**26**] in the case where $D$ is $\mathbb{Q}$-Cartier and $t \in \mathbb{N}$:

**3.18. Theorem** *Let $D$ be a big $\mathbb{R}$-divisor of a compact normal variety $X$ of dimension $n$. Then the limit*

$$v(D) = n! \lim_{t \to \infty} \frac{1}{t^n} \mathrm{h}^0(X, \llcorner tD \lrcorner)$$

*exists. Here $v(rD) = r^n v(D)$ holds for any positive real number $r$. If in addition $D$ is $\mathbb{R}$-Cartier and $\alpha < v(D)$, then there exist a bimeromorphic morphism $\mu \colon Y \to X$ from a non-singular projective variety and an effective $\mathbb{R}$-divisor $E$ of $Y$ such that $k(\mu^* D - E)$ is a free $\mathbb{Z}$-divisor for some $k \in \mathbb{N}$ and $(\mu^* D - E)^n > \alpha$.*

PROOF. We follow the proof by Fujita. We may assume that $X$ is a non-singular projective variety as before. Thus $D$ is $\mathbb{R}$-Cartier. We consider

$$v(D) := n! \varlimsup_{t \to \infty} \frac{1}{t^n} \mathrm{h}^0(X, \llcorner tD \lrcorner) \quad \text{and} \quad w(D) := n! \varlimsup_{\mathbb{N} \ni m \to \infty} \frac{1}{m^n} \mathrm{h}^0(X, \llcorner mD \lrcorner).$$

Then $v(D) \geq w(D)$. For $a \in \mathbb{N}$, we have $w(aD) \leq a^n w(D)$. Let $c$ be a positive integer such that $|tD| \neq \emptyset$ for any real number $t \geq c$. For $t \geq ca$, we write $t = ka - r$ for a real number $0 \leq r < a$ and an integer $k \geq c$. Thus we have

$$\frac{1}{t^n} \mathrm{h}^0(X, \llcorner tD \lrcorner) \leq \frac{1}{t^n} \mathrm{h}^0(X, \llcorner (k+c)aD \lrcorner) \leq \frac{1}{(k-1)^n a^n} \mathrm{h}^0(X, \llcorner (k+c)aD \lrcorner),$$

since $tD = (k+c)aD - c(a-1)D - (c+r)D$. Thus $a^n v(D) \leq w(aD)$. Therefore, $v(D) = w(D)$ and $v(aD) = a^n v(D)$. Consequently, $v(qD) = q^n v(D)$ for $0 < q \in \mathbb{Q}$. If $q_1 < r < q_2$ for $q_1, q_2 \in \mathbb{Q}_{>0}$, then $v(q_1 D) \leq v(rD) \leq v(q_2 D)$. Hence $v(rD) = r^n v(D)$ holds for any $r > 0$.

For $m \in \mathbb{I}(D)$, let $\mu_m \colon Y_m \to X$ be a birational morphism from a non-singular projective variety such that $|\mu_m^*(mD)|_{\mathrm{red}}$ is base-point free. We set $E_m := (1/m)|\mu_m^*(mD)|_{\mathrm{fix}}$ and $L_m := \mu_m^* D - E_m$. Then we have

$$v(D) \geq n! \varliminf_{\mathbb{N} \ni k \to \infty} \frac{1}{k^n} \mathrm{h}^0(X, \llcorner kD \lrcorner) \geq n! \varliminf_{\mathbb{N} \ni k \to \infty} \frac{1}{k^n} \mathrm{h}^0(X, \llcorner kL_m \lrcorner) = v(L_m) = L_m^n.$$

Suppose that $v(D) > v := \sup\{v(L_m) \mid m \in \mathbb{I}(D)\}$. Then, for any $\varepsilon > 0$ with $v + n(n!)\varepsilon < v(D)$, there is an $m$ with $L_m^n > v - \varepsilon$. For $s \in \mathbb{N}$, we have

$$\begin{aligned}\mathrm{h}^0(X, \llcorner smD \lrcorner) = \mathrm{h}^0(Y_{sm}, smL_{sm}) &\leq \mathrm{h}^0(Y_m, smL_m) + n(sm)^n(L_{sm}^n - L_m^n)\\ &\leq \mathrm{h}^0(Y_m, smL_m) + \varepsilon n(sm)^n\end{aligned}$$

by the key lemma [**26**, Lemma 2]. Thus $v(D) = m^{-n}v(mD) \leq v(L_m) + n(n!)\varepsilon < v(D)$. This is a contradiction. Thus $v(D) = v$. □

## §4. Canonical divisor

The *canonical sheaf* (or the *dualizing sheaf*) $\omega_X$ of an $n$-dimensional normal complex analytic variety $X$ is the unique reflexive sheaf whose restriction to $X_{\mathrm{reg}}$ is isomorphic to the sheaf $\Omega^n_{X_{\mathrm{reg}}}$ of germs of holomorphic differential $n$-forms. For the trivial morphism $f_X : X \to \operatorname{Specan}\mathbb{C} = (\text{point})$, we have a *dualizing complex* $\boldsymbol{\omega}^\bullet_X \simeq_{\mathrm{qis}} f_X^!\mathbb{C}$ (cf. [**37**], [**116**]). Then $\omega_X \simeq \mathcal{H}^{-n}(\boldsymbol{\omega}^\bullet_X)$ (cf. [**113**]). A non-zero meromorphic $n$-form $\eta$ on $X_{\mathrm{reg}}$ is regarded as a meromorphic section of $\omega_X$. The associated divisor $\operatorname{div}(\eta)$ is called the *canonical divisor* and is denoted by $K_X$ even though it depends on the choice of $\eta$. In order to make the definition reasonable, we must define $K_X$ as a divisor class. Since $\mathcal{O}_X(K_X) \simeq \omega_X$, the role of $K_X$ is almost identical to that of $\omega_X$. Some complex analytic variety $X$ does not admit any non-zero meromorphic section of $\omega_X$. However, we use the symbol $K_X$ as a formal divisor class with an isomorphism $\mathcal{O}_X(K_X) \simeq \omega_X$ and call it the *canonical divisor* of $X$. If $K_X$ is Cartier, in other words $\omega_X$ is invertible, then $X$ is called 1-*Gorenstein*. Note that $X$ is Gorenstein if and only if $X$ is 1-Gorenstein and Cohen-Macaulay. If $K_X$ is $\mathbb{Q}$-Cartier, then $X$ is called $\mathbb{Q}$-*Gorenstein*. In this case, there exists a positive integer $m$ locally on $X$ such that $mK_X$ is Cartier.

**§4.a. Kodaira dimension.** Let $f : Y \to X$ be a generically finite morphism of $n$-dimensional non-singular varieties. The sheaf $\Omega^1_X$ of germs of holomorphic 1-forms is locally free and the natural pullback homomorphism

$$f^*\Omega^1_X \to \Omega^1_Y$$

is injective. By taking determinant, we have a natural injection $f^*\omega_X \hookrightarrow \omega_Y$ and the *ramification formula*:

$$K_Y \sim f^*K_X + R_f.$$

The divisor $R_f$ is effective and is called the *ramification divisor* of $f$. The support $\operatorname{Supp} R_f$ coincides with the ramification locus that is the set of points of $Y$ at which $f$ is not étale. If $f$ is proper, then the induced homomorphism $\omega_X \to f_*\omega_Y$ is an isomorphism into a direct summand since the composite

$$\omega_X \to f_*\omega_Y \to \omega_X$$

with the trace map of $f$ is the multiplication map by $\deg f$. In what follows, we shall write the ramification formula as $K_Y = f^*K_X + R_f$ by replacing $\sim$ with $=$, because we can compare $K_Y$ and $f^*K_X$ in such a way that the difference remains only over the ramification locus. Suppose that $X$ and $Y$ are compact and $f$ is bimeromorphic. Then $R_f$ is $f$-exceptional. Therefore, by the covering lemma **3.11**, we have $\kappa(K_Y, Y) = \kappa(K_X, X)$. In particular, if $X_1$ and $X_2$ are mutually bimeromorphically equivalent compact complex manifolds, then $\kappa(K_{X_1}) = \kappa(K_{X_2})$. Iitaka has defined the *Kodaira dimension* $\kappa(X)$ for a compact complex analytic variety $X$ as $\kappa(K_Y, Y)$ for a compact complex manifold $Y$ bimeromorphically equivalent to $X$. Similarly, the *m-genus* $P_m(X)$ for $m \geq 1$ is defined as $\mathrm{h}^0(Y, mK_Y)$. Here, $P_1(X)$ is just the *geometric genus* $p_g(X)$.

**Remark** (1) If $X$ is singular, then $\kappa(X) \leq \kappa(K_X, X)$ and the equality does not hold in general.
(2) We write the $D$-dimension of $X$ explicitly by $\kappa(D, X)$ when we must distinguish it from the Kodaira dimension $\kappa(D)$ of a prime divisor $D$.
(3) If $\kappa(X) = \dim X$, then $X$ is called of *general type*.

For a meromorphic fiber space $f : X \dashrightarrow Y$, we define $\kappa(X/Y)$ as $\kappa(K_{X'}; X'/Y')$ for bimeromorphic morphisms $X' \to X$ and $Y' \to Y$ from compact complex manifolds such that the induced $X' \dashrightarrow Y'$ is holomorphic. Then we have the easy addition formula: $\kappa(X) \leq \kappa(X/Y) + \dim Y$. If $f$ is holomorphic, then $\kappa(X/Y) = \kappa(f^{-1}(y))$ for a 'general' $y \in Y$. If $\kappa(X/Y) = 0$ and $\kappa(X) = \dim Y$, then $f$ is called an *Iitaka fibration* of $X$. An Iitaka fibration is bimeromorphic to $\Phi_{mK_{X'}}$ for a compact complex manifold $X'$ bimeromorphic to $X$ and for $m \in \mathbb{I}(K_{X'})$.

If $f : X \dashrightarrow Y$ is a dominant proper generically finite meromorphic map, then $\kappa(X) \geq \kappa(Y)$ by the ramification formula. If $f$ is holomorphic and étale in addition, then $\kappa(X) = \kappa(Y)$ by the covering lemma **3.11**.

By the Iitaka fibration, the study of compact complex manifolds $X$ with $0 < \kappa(X) < \dim X$ is reduced to that of fiber spaces whose 'general' fiber is a compact complex manifold with $\kappa = 0$. The Kodaira dimension is one of the most important bimeromorphic invariant for the classification of compact complex manifolds. Here, the following conjecture posed by Iitaka was considered as a central problem for the bimeromorphic classification:

**Conjecture** ($C_n$ or $C_{n,m}$) Let $f : X \to Y$ be a fiber space of compact complex manifolds with $\dim X = n$, $\dim Y = m$. Then $\kappa(X) \geq \kappa(X/Y) + \kappa(Y)$.

This is sometimes called Iitaka's addition conjecture. Iitaka proved $C_2$ by applying the classification theory of compact complex surfaces. Conversely, the classification theory of surfaces is simplified if we can assume the conjecture to be true. In fact, Ueno gave a proof of $C_2$ without using the classification theory. There are counterexamples to $C_n$ for $n \geq 3$ found by Nakamura. But $C_n$ still is expected to be true in case $X$ belongs to the class $\mathcal{C}$ in the sense of Fujiki [**18**]. During ten years from the middle of 1970's, remarkable progress was made in the case of projective varieties by Ueno, Viehweg, Fujita, Kawamata, and Kollár. We discuss the details of the addition conjecture $C_n$ in Chapter **V**.

### §4.b. Logarithmic ramification formula.

**4.1. Definition** A reduced divisor $D$ of an $n$-dimensional non-singular variety $X$ is called a *normal crossing divisor* if $D$ is locally expressed as $\operatorname{div}(z_1 z_2 \cdots z_l)$ for a local coordinate $(z_1, z_2, \cdots, z_n)$ and for some $1 \leq l \leq n$. This is called *simple normal crossing* if furthermore every irreducible component of $D$ is non-singular. A meromorphic 1-form $\eta$ is said to have at most logarithmic poles along $D$ if locally $\eta$ is expressed as

$$\eta = \sum\nolimits_{i=1}^{l} a_i(z) \frac{\mathrm{d}\, z_i}{z_i} + \sum\nolimits_{j=l+1}^{n} a_j(z)\, \mathrm{d}\, z_j$$

for holomorphic functions $a_i(z) = a_i(z_1, z_2, \ldots, z_n)$, in which $D = \operatorname{div}(z_1 z_2 \cdots z_l)$. Such a form $\eta$ is called a *logarithmic* 1*-form* along $D$. The sheaf of germs of logarithmic 1-forms along $D$ is denoted by $\Omega^1_X(\log D)$. The $p$-th wedge product $\bigwedge^p \Omega^1_X(\log D)$ is denoted by $\Omega^p_X(\log D)$ and is called the sheaf of germs of logarithmic $p$-forms along $D$ for $p \geq 1$.

Let $D$ be a normal crossing divisor. For a generically finite morphism $f \colon Y \to X$ from an $n$-dimensional non-singular variety, suppose that $E = (f^*D)_{\mathrm{red}}$ is also a normal crossing divisor. Then the injection $f^*\Omega^1_X \hookrightarrow \Omega^1_Y$ extends to the injection

$$f^*\Omega^1_X(\log D) \hookrightarrow \Omega^1_Y(\log E).$$

The isomorphism $\det \Omega^1_X(\log D) \simeq \omega_X(D) = \mathcal{O}_X(K_X + D)$ induces the *logarithmic ramification formula*:

$$K_Y + E = f^*(K_X + D) + R,$$

where the effective divisor $R$ is called the *logarithmic ramification divisor*. Note that $R = E - f^*D + R_f \leq R_f$. In particular, if $f$ is bimeromorphic, then any prime component of $R$ is $f$-exceptional.

Iitaka has introduced the logarithmic Kodaira dimension for open varieties. An open variety is a complex analytic variety $X$ together with its compactification $\overline{X}$ as a complex analytic space in which $X$ is a Zariski-open subset. Note that there is an example of complex manifold $X$ admitting two such compactifications with different algebraic dimensions. If we consider only algebraic varieties (an integral scheme of finite type over $\mathbb{C}$), then we can take $\overline{X}$ as a complete algebraic variety which is unique up to the bimeromorphic equivalence. The logarithmic Kodaira dimension $\overline{\kappa}(X)$ of the open variety $X = (X, \bar{X})$ is defined as follows: Let $\mu \colon \overline{Y} \to \overline{X}$ be a bimeromorphic morphism from a compact complex manifold such that $D := \overline{Y} \setminus \mu^{-1}(X)$ is a normal crossing divisor. The existence of $\mu$ follows from Hironaka's desingularization theorem. Then $\overline{\kappa}(X) := \kappa(K_{\overline{Y}} + D, \overline{Y})$. It is well-defined by the logarithmic ramification formula. Iitaka proceeded the study of birational classification of open algebraic varieties and posed a logarithmic version $\overline{C}_{n,m}$ of the conjecture $C_{n,m}$.

**Remark** In the definition $\overline{\kappa}$, we consider $K_X + D$ for a normal crossing divisor $D = \sum D_i$ of a compact complex manifold $X$. Before Iitaka introduced $\overline{\kappa}$, Sakai [**123**], [**124**] found a similar invariant related to the $\mathbb{Q}$-divisor $K_X + \sum(1 - e_i^{-1})D_i$

for $e_i \geq 2$ in the study of a higher-dimensional version of the Second Main Theorem in the Nevanlinna theory.

The following generalization of logarithmic ramification formula is due to Iitaka [**45**, Proposition 1, Part 2] which improves the proof by Suzuki [**133**]:

**4.2. Theorem** *Let $\rho\colon W \to V$ be a generically finite morphism of non-singular varieties of the same dimension, $X \subset V$ a non-singular divisor, and $Y \subset W$ a reduced divisor such that $\rho^{[*]}X \leq Y$. Let $B \subset V$ and $D \subset W$ be effective divisors such that*

(1) *$X + B$ is a reduced normal crossing divisor,*
(2) *$Y + D$ is reduced,*
(3) *$\rho^{-1}(\operatorname{Supp} B) \subset \operatorname{Supp} D$.*

*Then*

$$K_W + Y + D = \rho^*(K_V + X + B) + R^{\&}$$

*for an effective divisor $R^{\&}$.*

Proof. We may assume that $X + B$ and $\rho^{-1}X \cup D$ are simple normal crossing divisors and $Y = \rho^{[*]}X$. If $Y = (\rho^*X)_{\mathrm{red}}$, then $R^{\&}$ is effective by the usual logarithmic ramification formula. Thus it is enough to show $\operatorname{mult}_\Gamma R^{\&} \geq 0$ for any $\rho$-exceptional prime component $\Gamma$ of $\rho^*X$. Let $P$ be a general point of $\Gamma$ such that $\rho(P)$ is a non-singular point of $\rho(\Gamma)$. Let $(w_1, w_2, \ldots, w_n)$ be a local coordinate system of $W$ at $P$ and let $(v_1, v_2, \ldots, v_n)$ be that of $V$ at $\rho(P)$. We may assume that $w_1$ is a defining equation of $\Gamma$ at $P$, $v_1$ is a defining equation of $X$ at $\rho(P)$, and that $v_2 = 0$ on $\rho(\Gamma)$. Then we can replace coordinates so that $\rho^*v_1 = w_1^k$ and $\rho^*v_2 = w_1^l\varepsilon$ for some $k$, $l \in \mathbb{N}$ and for a holomorphic function $\varepsilon = \varepsilon(w)$ with $\varepsilon(0, w_2, \ldots, w_n) \neq 0$. Then

$$\begin{aligned}\rho^*(\frac{\mathrm{d}\,v_1}{v_1} \wedge \mathrm{d}\,v_2 \wedge \cdots \wedge \mathrm{d}\,v_n) &= k\frac{\mathrm{d}\,w_1}{w_1} \wedge (lw_1^{l-1}\varepsilon\,\mathrm{d}\,w_1 + w_1^l\,\mathrm{d}\,\varepsilon) \wedge \rho^*(\mathrm{d}\,v_3 \wedge \cdots \wedge \mathrm{d}\,v_n)\\ &= kw_1^{l-1}\,\mathrm{d}\,w_1 \wedge \mathrm{d}\,\varepsilon \wedge \rho^*(\mathrm{d}\,v_3 \wedge \cdots \wedge \mathrm{d}\,v_n)\\ &= \psi(w)\,\mathrm{d}\,w_1 \wedge \mathrm{d}\,w_2 \wedge \cdots \wedge \mathrm{d}\,w_n\end{aligned}$$

for a holomorphic function $\psi(w)$. Thus $R^{\&} = \operatorname{div}(\psi) \geq 0$. □

We generalize the logarithmic ramification formula to the case of $\mathbb{R}$-divisors:

**4.3. Lemma** *Let $f\colon Y \to X$ be a generically finite morphism between non-singular varieties of the same dimension. Let $R_f$ be the ramification divisor of $f$.*

(1) *Let $\Delta$ be an effective $\mathbb{R}$-divisor of $X$ such that $\Delta_{\mathrm{red}}$ is a normal crossing divisor and $\llcorner\Delta\lrcorner = 0$. Then the $\mathbb{R}$-divisor*

$$R_\Delta := K_Y - f^*(K_X + \Delta)$$

*satisfies the following properties*:

(a) $0 \leq \ulcorner R_\Delta \urcorner \leq R_f$;

(b) $\mathrm{mult}_E f^*(\Delta_{\mathrm{red}}) = \mathrm{mult}_E R_f + 1$ *for any prime component* $E$ *of* $f^*\Delta$ *not contained in* $\ulcorner R_\Delta \urcorner$.

(2) *Let* $L$ *be an* $\mathbb{R}$*-divisor of* $X$ *such that* $\mathrm{Supp}\langle L\rangle$ *is a normal crossing divisor. Then*

$$K_Y + \ulcorner f^*L \urcorner = f^*(K_X + \ulcorner L \urcorner) + \ulcorner R_{\langle -L\rangle} \urcorner$$

*for the* $\mathbb{R}$*-divisor* $R_{\langle -L\rangle}$ *defined in* (1).

Proof. (1) We may assume that $\Delta_{\mathrm{red}}$ and $(f^*\Delta)_{\mathrm{red}}$ are simple normal crossing divisors. By the logarithmic ramification formula,

$$\begin{aligned} R &:= K_Y + (f^*\Delta)_{\mathrm{red}} - f^*(K_X + \Delta_{\mathrm{red}}) = R_f + (f^*\Delta)_{\mathrm{red}} - f^*(\Delta_{\mathrm{red}}) \\ &= R_\Delta + (f^*\Delta)_{\mathrm{red}} - f^*(\Delta_{\mathrm{red}} - \Delta) \end{aligned}$$

is effective. Hence $R_\Delta + (f^*\Delta)_{\mathrm{red}}$ is effective. If $E$ is a prime component of $f^*\Delta$, then

$$\mathrm{mult}_E(R_\Delta + (f^*\Delta)_{\mathrm{red}}) = \mathrm{mult}_E R + \mathrm{mult}_E f^*(\Delta_{\mathrm{red}} - \Delta) > 0.$$

Hence $\mathrm{mult}_E R_\Delta > -1$. If further $\mathrm{mult}_E R_\Delta \leq 0$, then $\mathrm{mult}_E R = 0$. Combining with $R_f = R_\Delta + f^*\Delta \geq R_\Delta$, we infer that $R_\Delta$ satisfies the expected properties.

(2) Let $\Delta$ be the $\mathbb{R}$-divisor $\langle -L\rangle$. Since $\ulcorner L \urcorner = L + \Delta$, we have

$$K_Y + f^*L = f^*(K_X + \ulcorner L \urcorner) + R_\Delta.$$

Hence $\langle -f^*L\rangle = \langle -R_\Delta\rangle$ and $\ulcorner f^*L \urcorner = f^*L + \langle -R_\Delta\rangle$. Thus

$$K_Y + \ulcorner f^*L \urcorner - f^*(K_X + \ulcorner L \urcorner) = \ulcorner R_\Delta \urcorner. \qquad \square$$

We have the following variant:

**4.4. Lemma** *Let* $\rho\colon W \to V$ *be a generically finite morphism of non-singular varieties of the same dimension and let* $B$ *be an effective* $\mathbb{R}$*-divisor of* $V$ *such that* $\ulcorner B \urcorner$ *is reduced and is a non-singular divisor.*

(1) *Let* $\Delta$ *be an effective* $\mathbb{R}$*-divisor of* $V$ *such that* $\llcorner \Delta \lrcorner = 0$ *and* $\Delta_{\mathrm{red}} + B_{\mathrm{red}}$ *is a normal crossing divisor. Then, for the* $\mathbb{R}$*-divisor*

$$R^{\&}_\Delta := K_W + \rho^{[*]}B - \rho^*(K_V + B + \Delta),$$

*its round-up* $\ulcorner R^{\&}_\Delta \urcorner$ *is effective.*

(2) *Let* $L$ *be an* $\mathbb{R}$*-divisor of* $V$ *such that* $\langle L\rangle_{\mathrm{red}} + B_{\mathrm{red}}$ *is a normal crossing divisor. Then*

$$K_W + \rho^{[*]}B + \ulcorner \rho^*L \urcorner = \rho^*(K_V + B + \ulcorner L \urcorner) + \ulcorner R^{\&}_{\langle -L\rangle} \urcorner$$

*for the* $\mathbb{R}$*-divisor* $R^{\&}_{\langle -L\rangle}$ *defined in* (1).

Proof. (1) We may assume that $\Delta_{\mathrm{red}} + B_{\mathrm{red}}$ and $(\rho^*(\Delta + B))_{\mathrm{red}}$ are simple normal crossing divisors. By **4.2**,

$$\begin{aligned} &K_W + \rho^{[*]}B_{\mathrm{red}} + (\rho^*\Delta)_{\mathrm{red}} - \rho^*(K_V + B_{\mathrm{red}} + \Delta_{\mathrm{red}}) \\ &\quad = R^{\&}_\Delta + (\rho^*\Delta)_{\mathrm{red}} - \rho^*(\Delta_{\mathrm{red}} - \Delta) - (\rho^*(B_{\mathrm{red}} - B) - \rho^{[*]}(B_{\mathrm{red}} - B)) \end{aligned}$$

is an effective divisor. Hence $R^{\&}_{\Delta} + (\rho^*\Delta)_{\mathrm{red}}$ is effective. For any prime component $E$ of $\rho^*\Delta$, we have $\mathrm{mult}_E\,\rho^*(\Delta_{\mathrm{red}} - \Delta) > 0$. Thus $\ulcorner R^{\&}_{\Delta} \urcorner \geq 0$.

(2) We set $\Delta = \langle -L \rangle$. Then $\ulcorner L \urcorner = L + \Delta$ and

$$K_W + \rho^{[*]}B + \rho^* L = \rho^*(K_V + B + \ulcorner L \urcorner) + R^{\&}_{\Delta}.$$

Hence $\langle -\rho^* L \rangle = \langle -R^{\&}_{\Delta} \rangle$ and $\ulcorner \rho^* L \urcorner = \rho^* L + \langle -R^{\&}_{\Delta} \rangle$. Thus

$$K_W + \rho^{[*]}B + \ulcorner \rho^* L \urcorner - \rho^*(K_V + B + \ulcorner L \urcorner) = \ulcorner R^{\&}_{\Delta} \urcorner. \qquad \square$$

**§4.c. Terminal, canonical, and log-terminal singularities.** Let $f\colon Y \to X$ be a bimeromorphic morphism from a non-singular variety into a normal $\mathbb{Q}$-Gorenstein variety. Then we can write

$$K_Y = f^* K_X + \sum a_i E_i$$

for $f$-exceptional prime divisors $E_i$ and for $a_i \in \mathbb{Q}$.

**4.5. Definition** (Reid [**113**], [**114**]) A germ $(X, x)$ of a normal $\mathbb{Q}$-Gorenstein variety is called a *canonical* singularity if there is a bimeromorphic morphism $f\colon Y \to X$ as above over a neighborhood of $x$ such that $a_i \geq 0$ for all $i$. The germ $(X, x)$ is called a *terminal* singularity if $a_i > 0$ for all $i$.

Note that a non-singular germ is a terminal and canonical singularity.

**Remark** If a normal variety $X$ has only terminal (resp. canonical) singularities, then, for any bimeromorphic morphism $f\colon Y \to X$ from a non-singular variety, $a_i > 0$ (resp. $a_i \geq 0$ ) in the formula: $K_Y = f^* K_X + \sum a_i E_i$. This follows from the relative Chow lemma [**41**] and the ramification formula.

**4.6. Definition** Let $(X, \Delta)$ be a pair of a normal variety and an effective $\mathbb{R}$-divisor. It is called *log-canonical* if the following conditions are satisfied:

(1) $K_X + \Delta$ is $\mathbb{R}$-Cartier;
(2) There exist a bimeromorphic morphism $f\colon Y \to X$ from a non-singular variety and a normal crossing divisor $E = \sum E_i$ on $Y$ such that
  (a) $E$ contains the $f$-exceptional locus and $f^{-1}(\mathrm{Supp}\,\Delta)$,
  (b) $a_i \geq -1$ for any $i$ in the formula:

$$K_Y = f^*(K_X + \Delta) + \sum a_i E_i.$$

The pair $(X, \Delta)$ is called *log-terminal* if further $a_i > -1$ for any $i$ in the condition above.

**Remark** If $(X, \Delta)$ is log-terminal (resp. log-canonical), then $a_i > -1$ (resp. $a_i \geq -1$ ) hold for all $i$ for any bimeromorphic morphism $f\colon Y \to X$ from a non-singular variety such that the union of the $f$-exceptional locus and $f^{-1}(\mathrm{Supp}\,\Delta)$ is a normal crossing divisor $E = \sum E_i$. This follows from the logarithmic ramification formula (cf. **4.3**).

The germ $(X, \Delta, x)$ for the pair $(X, \Delta)$ and for a point $x \in X$ is called a log-terminal (resp. log-canonical) singularity if $(U, \Delta|_U)$ is log-terminal (resp. log-canonical) for an open neighborhood $U$ of $x$. If $(X, \Delta)$ has only log-terminal (resp. log-canonical) singularities, then $(X, \Delta)$ is log-terminal (resp. log-canonical). If $(X, \Delta)$ is log-terminal, then $X$ has only rational singularities: $\mathrm{R}^i f_* \mathcal{O}_Y = 0$ for $i > 0$ for a bimeromorphic morphism $f: Y \to X$ from a non-singular variety (cf. [**61**, 1-3-6], **VII.1.1**).

**§4.d. Bimeromorphic pairs.** As an analogy of *birational pair* defined by Iitaka [**45**], we shall introduce the notion of *bimeromorphic pair*. A bimeromorphic pair consists of a normal complex analytic variety $V$ and an effective $\mathbb{R}$-divisor $B$ of $V$ such that $\ulcorner B \urcorner$ is reduced. We denote the bimeromorphic pair by the symbol $V \& B$. A morphism $\rho: W \& C \to V \& B$ of bimeromorphic pairs is defined to be a morphism $\rho: W \to V$ such that $C \geq \rho^{[*]} B$. If $\rho: W \to V$ is a bimeromorphic morphism and if $V$, $W$, $B$, and $C$ are all non-singular, then

$$\mathrm{H}^0(V, m(K_V + B)) \to \mathrm{H}^0(W, m(K_W + C))$$

is an isomorphism for $m \geq 0$ by **4.2**. Hence $\mathrm{H}^0(V, m(K_V + B))$ is a bimeromorphic invariant for the bimeromorphic pair $V \& B$. If $V$ is compact, then its dimension $P_m(V \& B) = \mathrm{h}^0(V, m(K_V + B))$ is called the *m-genus* of $V \& B$.

**4.7. Definition** A bimeromorphic pair $V \& B$ is called *canonical* if the following two conditions are satisfied:

(1) $K_V + B$ is $\mathbb{R}$-Cartier;
(2) For any bimeromorphic morphism $\rho: W \to V$ from a non-singular variety,
$$K_W + \rho^{[*]} B = \rho^*(K_V + B) + R^{\&}$$
for an effective $\mathbb{R}$-divisor $R^{\&}$.

A canonical bimeromorphic pair $V \& B$ is called *terminal* if, in the second condition above, $\operatorname{mult}_E R^{\&} > 0$ for any $\rho$-exceptional prime divisor $E$.

For a point $x \in V$, the germ $(V \& B, x)$ is called a canonical (resp. terminal) singularity if $U \& (B|_U)$ is canonical (resp. terminal) for an open neighborhood $U$ of $x$. By definition, if $V \& B$ is canonical, then $(V, B)$ is log-terminal. If $V \& B$ is canonical, then $\mathrm{H}^0(V, \llcorner m(K_V + B) \lrcorner)$ is a bimeromorphic invariant.

**4.8. Definition** Let $V \& B$ be a bimeromorphic pair and let $\Delta$ be an effective $\mathbb{R}$-divisor of $V$ having no common prime component with $B$. The symbol $(V \& B, \Delta)$ is called log-terminal if the following conditions are satisfied:

(1) $K_V + B + \Delta$ is $\mathbb{R}$-Cartier;
(2) For any bimeromorphic morphism $\rho: W \to V$ from a non-singular variety and for the $\mathbb{R}$-divisor
$$R^{\&}_{\Delta} = K_W + \rho^{[*]} B - \rho^*(K_V + B + \Delta),$$
its round-up $\ulcorner R^{\&}_{\Delta} \urcorner$ is effective.

By definition, $(V\&B, \Delta)$ is log-terminal if and only if $(V\& \llcorner B \lrcorner, \langle B\rangle + \Delta)$ is log-terminal. If $K_V + B + \Delta$ is $\mathbb{R}$-Cartier and if there exists a bimeromorphic morphism $\rho\colon W \to V$ such that $\rho^{[*]}B_{\mathrm{red}}$ is non-singular and $\ulcorner R^{\&}_{\Delta} \urcorner$ above is effective, then $(V\&B, \Delta)$ is log-terminal, by **4.4**.

**Remark** The condition: $(V\&B, \Delta)$ is log-terminal is equivalent to the condition: $(V, B + \Delta)$ is *purely log terminal* (plt) in the sense of [**132**] and [**74**].

The following properties are proved in [**132**] and [**74**]:

**4.9. Lemma** *Suppose that* $(V\&B, \Delta)$ *is log-terminal. Then*:

(1) $\llcorner B \lrcorner$ *is normal*;
(2) *For any component* $X$ *of* $\llcorner B \lrcorner$, *there is a natural effective* $\mathbb{R}$*-divisor* $\Delta_X$ *such that* $(K_V + B + \Delta)|_X \sim_{\mathbb{R}} K_X + \Delta_X$ *and* $(X, \Delta_X)$ *is log-terminal.*

In the following proof, we use some notation and results discussed in later sections.

Proof. There is a bimeromorphic morphism $\rho\colon W \to V$ from a non-singular variety such that $\rho^{-1}(B \cup \Delta)$ is a normal crossing divisor and $\rho^{[*]}B_{\mathrm{red}}$ is a non-singular divisor. Let us consider the $\mathbb{R}$-divisor

$$R^{\&}_{\Delta} = K_W + \rho^{[*]}B - \rho^*(K_V + X + \Delta)$$

and $Y = \rho^{[*]}(\llcorner B \lrcorner) = \llcorner \rho^{[*]}B \lrcorner$. Then $f(Y) = \llcorner B \lrcorner$. Let $R$ be the $\rho$-exceptional effective divisor $\ulcorner R^{\&}_{\Delta} \urcorner$ (cf. **4.4**) and set

$$\Delta_W := \langle -R^{\&}_{\Delta}\rangle + \rho^{[*]}\langle B\rangle.$$

Then

$$R - Y - (K_W + \Delta_W) = -\rho^*(K_V + X + \Delta)$$

is $\rho$-numerically trivial. Thus $R^1\rho_*\mathcal{O}_W(R - Y) = 0$ by **5.12** below. Furthermore, we have the surjection

$$\mathcal{O}_V \simeq \rho_*\mathcal{O}_W \simeq \rho_*\mathcal{O}_W(R) \twoheadrightarrow \rho_*\mathcal{O}_Y(R).$$

In particular, $\llcorner B \lrcorner$ is normal by

$$\mathcal{O}_{\llcorner B \lrcorner} \simeq \rho_*\mathcal{O}_Y \simeq \rho_*\mathcal{O}_Y(R).$$

For the proof of (2), we may assume $X = \llcorner B \lrcorner$ is irreducible. We set $\Delta_Y := \Delta_W|_Y$. Then $\llcorner \Delta_Y \lrcorner = 0$, $(\Delta_Y)_{\mathrm{red}}$ is a normal crossing divisor of $Y$,

$$R|_Y - (K_Y + \Delta_Y) = -(\rho^*(K_V + B + \Delta))|_Y,$$

and $R|_Y$ is $\rho|_Y$-exceptional. Hence for the push-forward $\Delta_X := \rho_*\Delta_Y$, we infer that $K_X + \Delta_X$ is an $\mathbb{R}$-Cartier divisor $\mathbb{R}$-linearly equivalent to $(K_V + B + \Delta)|_X$ and that

$$K_Y = (\rho|_Y)^*(K_X + \Delta_X) + R|_Y - \Delta_Y,$$

in which $\ulcorner R|_Y - \Delta_Y \urcorner = R|_Y$ is effective. Thus $(X, \Delta_X)$ is log-terminal. □

**Remark** In **VI.5.1**, we shall prove a kind of inverse to **4.9**.

## §5. Numerical properties of divisors

**§5.a. Ample and nef cones.** Let $X$ be an $n$-dimensional normal projective variety. Let $\mathrm{NS}(X)$ be the Néron-Severi group and let $\mathrm{N}^1(X)$ be the real vector space $\mathrm{NS}(X)\otimes\mathbb{R}$. If $X$ is non-singular, then $\mathrm{N}^1(X)$ is isomorphic to the vector subspace in $\mathrm{H}^2(X,\mathbb{R})$ generated by the first Chern classes of all the invertible sheaves. The dimension $\dim_{\mathbb{R}}\mathrm{N}^1(X)$ is called the *Picard number* of $X$ and denoted by $\rho(X)$. Let $c_1(D)$ denote the image of an $\mathbb{R}$-Cartier divisor $D$ under $\mathrm{CDiv}(X,\mathbb{R})\to\mathrm{N}^1(X)$. Note that $c_1(D)=0$ if and only if $D\cdot C=0$ for any irreducible curve $C$. If $D_1-D_2$ is an $\mathbb{R}$-Cartier divisor with $c_1(D_1-D_2)=0$, then two $\mathbb{R}$-divisors $D_1$ and $D_2$ are called *numerically equivalent*. The numerical equivalence relation is denoted by $D_1\approx D_2$. An $\mathbb{R}$-Cartier divisor $D$ of $X$ is called *nef* if $D\cdot C\geq 0$ for any irreducible curve $C\subset X$. The *nef cone* $\mathrm{Nef}(X)\subset\mathrm{N}^1(X)$ is the set of first Chern classes $c_1(D)$ of nef $\mathbb{R}$-Cartier divisors $D$ of $X$. This is a strictly convex closed cone. The dual space $\mathrm{N}_1(X)$ of $\mathrm{N}^1(X)$ is considered as the real vector space generated by the numerical equivalence classes of all the algebraic 1-cycles of $X$. Let $\mathrm{NE}(X)$ be the cone of the numerical equivalence classes of effective 1-cycles and let $\overline{\mathrm{NE}}(X)$ be the closure in $\mathrm{N}_1(X)$ (cf. [**86**]). Kleiman's criterion [**64**] asserts that $\overline{\mathrm{NE}}(X)$ and $\mathrm{Nef}(X)$ are dual to each other and that a Cartier divisor $A$ is ample if and only if $c_1(A)$ is contained in the interior of $\mathrm{Nef}(X)$. The interior $\mathrm{Amp}(X)$ is an open convex cone and is called the *ample cone*. Its closure is $\mathrm{Nef}(X)$. An $\mathbb{R}$-Cartier divisor $D$ is called *ample* if $c_1(D)\in\mathrm{Amp}(X)$.

**5.1. Lemma** *Let $C$ be a convex cone of a finite-dimensional real vector space $V$ such that $C$ generates $V$ as an $\mathbb{R}$-module. Let $\overline{C}$ be the closure of $C$ in $V$. Then the interior $\mathrm{Int}\,\overline{C}$ is contained in $C$. If $V=L\otimes\mathbb{R}$ for a finitely generated abelian group $L\subset V$, then*

$$\mathrm{Int}\,\overline{C}=\sum\nolimits_{w\in L\cap\mathrm{Int}\,C}\mathbb{R}_{>0}w.$$

Proof. Let $\mathcal{U}$ be an open neighborhood of 0 in $V$ and let $v$ be a vector contained in $\mathrm{Int}\,\overline{C}$. We can find vectors $u_1, u_2, \dots, u_n\in\mathcal{U}$ such that $v+u_i\in C$ for all $i$ and $\{u_1,u_2,\dots,u_n\}$ is a basis of $V$. There is also a vector $u=\sum r_iu_i$ such that $r_i>0$ for all $i$ and $v-u\in C$. The vector $v+\lambda u$ is contained in $C$ if $\lambda\sum r_i=1$. Hence $(\lambda+1)v=\lambda(v-u)+(v+\lambda u)\in C$. Thus $\mathrm{Int}\,C=\mathrm{Int}\,\overline{C}$. Let $\{e_1,e_2,\dots,e_n\}$ be a basis of $L$. Then any $v\in V$ is written uniquely by $v=\sum a_ie_i$ for $a_i\in\mathbb{R}$. We define $\llcorner v\lrcorner$ by $\sum\llcorner a_i\lrcorner e_i$. For $v\in\mathrm{Int}\,C$, there is a positive integer $m$ such that $v_m:=\llcorner mv\lrcorner\in\mathrm{Int}\,C$ and $v_m+ne_i\in\mathrm{Int}\,C$ for all $i$. Then

$$mv=\frac{1}{n}\sum\nolimits_{i=1}^{n}\langle ma_i\rangle(v_m+ne_i)+\Big(1-\frac{1}{n}\sum\nolimits_{i=1}^{n}\langle ma_i\rangle\Big)v_m.\qquad\square$$

**5.2. Corollary** *Let $A$ be an ample $\mathbb{R}$-divisor of a normal projective variety. Then $A=\sum s_jH_j$, for some ample Cartier divisors $H_j$ and $s_j\in\mathbb{R}_{>0}$. In other words, $\mathrm{Amp}(X)$ is generated by $c_1(A)$ for ample Cartier divisors $A$ of $X$.*

**5.3. Lemma** *Let $f\colon X\to Y$ be a generically finite surjective morphism between $n$-dimensional normal projective varieties and let $D_1$ and $D_2$ be $\mathbb{R}$-divisors of $X$.*

*If $D_1 \approx D_2$, then*

$$L_1 \cdot L_2 \cdots L_{n-1} \cdot f_* D_1 = L_1 \cdot L_2 \cdots L_{n-1} \cdot f_* D_2$$

*for any Cartier divisors $L_1, L_2, \dots L_{n-1}$ of $Y$. If $Y$ is non-singular, then $f_* D_1 \approx f_* D_2$.*

Proof. Let $D$ be a divisor of $X$ and let $H$ be a general very ample divisor of $Y$. Then $H$ and $T = f^* H$ are also normal and $f_* \mathcal{O}_X(D) \otimes \mathcal{O}_H \simeq (f|_T)_* \mathcal{O}_T(D|_T)$. In particular, $(f_* D)|_H = (f|_T)_*(D|_T)$. Therefore, for general very ample divisors $A_1, A_2, \dots A_{n-1}$ of $Y$, we have the equality

$$A_1 \cdot A_2 \cdots A_{n-1} \cdot f_* D = f^* A_1 \cdot f^* A_2 \cdots f^* A_{n-1} \cdot D.$$

In fact, this is shown in the case $f$ is bimeromorphic and in the case $X$ and $Y$ are non-singular. The equality in general case is reduced to these cases by a standard argument. Since any Cartier divisor is expressed as a linear combination of ample divisors, the first assertion is proved. Next, suppose that $Y$ is non-singular. Then, for the divisor $C = f_* D_1 - f_* D_2$, we have

$$C \cdot A^{n-1} = C^2 \cdot A^{n-2} = 0$$

for any ample divisor $A$. Then $C \approx 0$ by the Hodge index theorem. □

**§5.b. Big and pseudo-effective cones.** Suppose that $X$ is a non-singular projective variety. Let $\mathrm{Eff}(X) \subset \mathrm{N}^1(X)$ be the subset consisting of $c_1(D)$ of all effective $\mathbb{R}$-divisors $D$. It is called the *effective cone*. The closure of $\mathrm{Eff}(X)$ is denoted by $\mathrm{PE}(X)$ and is called the *pseudo-effective cone*. The interior of $\mathrm{PE}(X)$ is denoted by $\mathrm{Big}(X)$ and is called the *big cone*. Note that $\mathrm{Amp}(X) \subset \mathrm{Big}(X) \subset \mathrm{Eff}(X)$ by **5.1** and $\mathrm{Nef}(X) \subset \mathrm{PE}(X)$.

**5.4. Lemma** *Let $X$ be an $n$-dimensional non-singular projective variety and let $B$ be an $\mathbb{R}$-divisor on $X$. Then the following conditions are mutually equivalent*:

(1) $c_1(B) \in \mathrm{Big}(X)$;
(2) *For any ample divisor $A$, there exist a positive rational number $\delta$ and an effective $\mathbb{R}$-divisor $\Delta$ such that $B \sim_{\mathbb{Q}} \delta A + \Delta$;*
(3) *There exists an effective $\mathbb{R}$-divisor $\Delta$ such that $B - \Delta$ is ample*;
(4) *$B$ is big.*

Proof. (2) ⇒ (3) is trivial. (4) ⇒ (2) is done in **3.16**.

(1) ⇒ (3): By applying **5.1** to $C = \mathrm{Eff}(X)$, we infer that $c_1(B) \in \mathrm{Int}\,\mathrm{Eff}(X)$. Thus for an ample divisor $A$, there exist a positive number $\delta$ and an effective $\mathbb{R}$-divisor $\Delta$ such that $c_1(B - \delta A) = c_1(\Delta)$. Hence $B - \Delta$ is ample.

(3) ⇒ (1): For the ample $\mathbb{R}$-divisor $A := B - \Delta$, let $\mathcal{U}$ be an open neighborhood of 0 in $\mathrm{N}^1(X)$ such that $D + A$ is ample for any $\mathbb{R}$-divisor $D$ with $c_1(D) \in \mathcal{U}$. Then $B + \mathcal{U} \subset \mathrm{PE}(X)$.

(3) ⇒ (4): There is a positive integer $m_0$ such that $A := \llcorner m_0 B - m_0 \Delta \lrcorner$ is an ample divisor. There is a positive integer $k$ such that $|iB + kA| \neq \emptyset$ for $0 \leq i \leq m_0$.

For $m \geq m_0$, we have

$$mB + kA = \llcorner m/m_0 \lrcorner (A + m_0\Delta) + m_0\langle m/m_0\rangle B + \llcorner m/m_0 \lrcorner \langle m_0 B - m_0\Delta\rangle + kA.$$

Hence, there is an injection

$$\mathcal{O}_X(\llcorner m/m_0 \lrcorner A) \hookrightarrow \mathcal{O}_X(\llcorner mB \lrcorner + kA),$$

which induces the inequality

$$\mathrm{h}^0(X, \llcorner m/m_0 \lrcorner A) \leq \mathrm{h}^0(X, \llcorner mB \lrcorner + kA) \leq \mathrm{h}^0(X, \llcorner (m + km_0)B \lrcorner).$$

Hence $\kappa(B) = n$, since $\mathrm{h}^0(X, mA)$ is a polynomial of degree $n$ for $m \gg 0$. □

**5.5. Definition** An $\mathbb{R}$-divisor $D$ of a normal projective variety $X$ is called *pseudo-effective* if there exist a birational morphism $\mu\colon Y \to X$ from a non-singular projective variety and an $\mathbb{R}$-divisor $D'$ of $Y$ such that $c_1(D') \in \mathrm{PE}(Y)$ and $\mu_* D' = D$.

An $\mathbb{R}$-divisor is pseudo-effective if and only if $D + A$ is big for any ample $\mathbb{R}$-divisor $A$.

**5.6. Lemma** *Let $f\colon Y \to X$ be a surjective morphism of non-singular projective varieties and let $D$ be an $\mathbb{R}$-divisor of $X$.*

(1) *Suppose that $f$ is a generically finite morphism. Then $f^*D$ is big if and only if so is $D$.*
(2) *The pullback $f^*D$ is pseudo-effective if and only if so is $D$.*

Proof. It is enough to show the 'only if' parts.

(1) If $f^*D$ is big, then there exist an ample divisor $A$ of $X$, an effective $\mathbb{R}$-divisor $\Delta$ on $Y$, and a positive number $k$ such that $kf^*D \sim_{\mathbb{Q}} f^*A + \Delta$. Then, by taking $f_*$, we have $k(\deg f)D \sim_{\mathbb{Q}} (\deg f)A + f_*\Delta$. Thus $D$ is big.

(2) If $f$ is a generically finite morphism, then this is derived from (1) above. Thus we may assume that $\dim Y > \dim X$. Let $H \subset Y$ be a 'general' hyperplane section. Then the restriction $f^*(D)|_H$ is also pseudo-effective. Thus we can replace the situation to $f|_H\colon H \to X$. Therefore, by induction on $\dim Y$, we can conclude that $D$ is pseudo-effective. □

If $D$ is pseudo-effective, then the intersection number $D \cdot A_1 \cdot A_2 \cdots A_{n-1}$ is non-negative for any ample divisors $A_1, A_2, \ldots, A_{n-1}$. If $n = 2$, then an $\mathbb{R}$-divisor $D$ is pseudo-effective if $D \cdot A \geq 0$ for any ample divisor $A$ of $X$. This is a consequence of Kleiman's criterion [**64**]. However, $D$ is not necessarily pseudo-effective even if $D \cdot A_1 \cdot A_2 \cdots A_{n-1} \geq 0$ in the case $n \geq 3$.

**5.7. Example** Let $X \to \mathbb{P}^2$ be the blowing-up at a point, $E$ the exceptional divisor, and $F$ a fiber of the induced $\mathbb{P}^1$-bundle structure $X \to \mathbb{P}^1$. Let $p\colon \mathbb{P} \to X$ be the $\mathbb{P}^1$-bundle associated with the vector bundle $\mathcal{O}(F) \oplus \mathcal{O}(E)$ and let $H$ be the tautological divisor. For an $\mathbb{R}$-divisor $D$ of $X$, we have the following properties by **IV.2.6**:

(1) $p^*D + H$ is nef if and only if $D - F$ is nef (Note that $D + E$ and $D + F$ are both nef if and only if $D - F$ is nef);
(2) $p^*D + H$ is pseudo-effective if and only if there is a real number $0 \leq s \leq 1$ such that $D + (1 - s)F + sE$ is pseudo-effective.

Consequently, the divisor $p^*(-2F) + H$ is not pseudo-effective. But $(p^*(-2F) + H)A_1A_2 \geq 0$ for any ample divisors $A_1$, $A_2$ of $\mathbb{P}$.

**5.8. Remark** Let $W$ be a compact complex analytic variety. An $\mathbb{R}$-Cartier divisor of $W$ is defined as an $\mathbb{R}$-linear combination of Cartier divisors of $W$. An $\mathbb{R}$-divisor $D$ is called pseudo-effective, nef, big, or ample, according as $\nu^*D$ is so, for the normalization $\nu\colon V \to W$. Let $X$ be a non-singular projective variety, $D$ an $\mathbb{R}$-divisor, and $W$ a closed subvariety of $X$. Suppose that $W \not\subset \operatorname{Supp} D$. Then we can define the restriction $D|_W$ as an $\mathbb{R}$-Cartier divisor. If $D$ is effective, then $D|_W$ is effective. Next, suppose that $W \subset \operatorname{Supp} D$. Then $D|_W$ is defined only as an $\mathbb{R}$-Cartier divisor class of $W$. Even though, we can say $D|_W$ is pseudo-effective, nef, big, or ample if $\nu^*(D|_W)$ is so as an element of $\mathrm{CC}\ell(V, \mathbb{R})$.

**§5.c. Vanishing theorems.** Let $X$ be a compact Kähler manifold of dimension $n$. An invertible sheaf $\mathcal{H}$ of $X$ is called *positive* if it admits a Hermitian metric with positive Ricci curvature form. Then we have the following results:

(1) (Kodaira vanishing theorem [**67**]) $\mathrm{H}^p(X, \omega_X \otimes \mathcal{H}) = 0$ for $p > 0$.
(2) (Kodaira's embedding theorem [**68**]) $X$ is projective and $\mathcal{H}$ is ample.
(3) (Akizuki–Nakano vanishing theorem [**1**]) $\mathrm{H}^q(X, \Omega^p_X \otimes \mathcal{H}) = 0$ for $p+q > n$.

The Kodaira vanishing theorem is generalized to the following form by Kawamata [**51**] and Viehweg [**146**] independently.

**5.9. Theorem** *Let $X$ be a non-singular projective variety and let $D$ be a nef and big $\mathbb{R}$-divisor of $X$. Suppose that* $\operatorname{Supp}\langle D\rangle$ *is a normal crossing divisor. Then* $\mathrm{H}^p(X, K_X + \ulcorner D \urcorner) = 0$ *for any $p > 0$.*

Their proofs need some covering tricks. Viehweg has prepared the following lemma on cyclic coverings (cf. [**147**]).

**5.10. Lemma** *Let $D$ be a $\mathbb{Q}$-divisor of a non-singular variety $X$ such that* $\operatorname{Supp}\langle D\rangle$ *is a normal crossing divisor and $rD \sim 0$ for an integer $r > 1$. Let $i\colon \mathcal{O}_X(-rD) \to \mathcal{O}_X$ be an isomorphism. Then $Y = \operatorname{Specan} \mathcal{A}$ is normal with only quotient singularities over* $\operatorname{Sing}\operatorname{Supp}\langle D\rangle$ *for the $\mathcal{O}_X$-algebra*

$$\mathcal{A} = \bigoplus\nolimits_{m=0}^{r-1} \mathcal{O}_X(\llcorner -mD \lrcorner)$$

*defined by $i$. Here, $\tau^*D$ is a Cartier divisor linearly equivalent to zero and there are isomorphisms*

$$\tau_*\omega_Y \simeq \bigoplus\nolimits_{m=0}^{r-1} \mathcal{O}_X(K_X + \ulcorner mD \urcorner), \quad \tau_*\mathcal{O}_Y(a\tau^*D) \simeq \bigoplus\nolimits_{m=0}^{r-1} \mathcal{O}_X(\llcorner (a-m)D \lrcorner)$$

*for $a \in \mathbb{Z}$ and for the structure morphism $\tau\colon Y \to X$.*

By composing cyclic coverings, Kawamata [**50**] has obtained a Kummer covering from a non-singular variety which changes a $\mathbb{Q}$-divisor to a $\mathbb{Z}$-divisor. His argument is also effective also for non-algebraic cases:

**5.11. Lemma** ([**50**, Theorem 17] (cf. [**98**])) *Let $D_1, D_2, \ldots, D_k$ be non-singular prime divisors of a non-singular variety $X$ and let $m_1, m_2, \ldots, m_k$ be integers greater than* 1*. Suppose that $\sum_{i=1}^{k} D_i$ is a simple normal crossing divisor and $X$ is a weakly* 1*-complete manifold with a positive line bundle. Then, for a relatively compact open subset $U \subset X$, there is a finite Galois morphism $\tau\colon Y \to U$ from a non-singular variety such that $\tau^*(D_i|_U) = m_i D_i'$ for divisors $D_i'$ of $Y$.*

Proof of **5.9**. There is an effective divisor $\Delta$ such that $D - \varepsilon\Delta$ is ample for $0 < \varepsilon \ll 1$. Let $\mu\colon X' \to X$ be a birational morphism from a non-singular projective variety such that the union of the $\mu$-exceptional locus and $\mu^{-1}\Delta$ is a simple normal crossing divisor. We may assume that there is a $\mu$-exceptional divisor $E$ with $-E$ being $\mu$-ample. Hence $D' = \mu^*(D - \varepsilon\Delta) - \varepsilon' E$ is ample and $\ulcorner D' \urcorner = \ulcorner \mu^* D \urcorner$ for $0 < \varepsilon' \ll \varepsilon$. We have

$$\mu_* \mathcal{O}_{X'}(K_{X'} + \ulcorner \mu^* D \urcorner) \simeq \mathcal{O}_X(K_X + \ulcorner D \urcorner)$$

by **4.3**-(2). Hence we may assume that $D$ is ample from the beginning. Further, we can assume that $D$ is a $\mathbb{Q}$-divisor since $(1/m) \llcorner mD \lrcorner$ is ample and $\ulcorner (1/m) \llcorner mD \lrcorner \urcorner = \ulcorner D \urcorner$ for $m \gg 0$. Replacing $X$ by a blowing-up of $X$ and applying **4.3**-(2), we may also assume that $\mathrm{Supp}\langle D \rangle$ is a simple normal crossing divisor. Let $\tau\colon Y \to X$ be a finite Galois morphism from a non-singular projective variety obtained by **5.11** such that $\tau^* D$ is a Cartier divisor. Then $\tau_* \mathcal{O}_Y(-\tau^* D)$ contains $\mathcal{O}_X(\llcorner -D \lrcorner)$ as the direct summand corresponding to the invariant part of the Galois action. Thus $\tau_* \omega_Y(\tau^* D)$ contains $\omega_X(\ulcorner D \urcorner)$ as a direct summand. Hence the vanishing for $\omega_X(\ulcorner D \urcorner)$ follows from the Kodaira vanishing for $\omega_Y(\tau^* D)$. □

The following variant is proved in [**98**] by Nakano's vanishing theorem [**95**] for weakly 1-complete manifolds and by **5.11**:

**5.12. Corollary** *Let $f\colon X \to S$ be a projective morphism from a non-singular complex analytic variety and let $D$ be an $\mathbb{R}$-divisor of $X$. Suppose that $D$ is $f$-nef and $f$-big and that* $\mathrm{Supp}\langle D \rangle$ *is a normal crossing divisor. Then, for $p > 0$,*

$$\mathrm{R}^p f_* \mathcal{O}_X(K_X + \ulcorner D \urcorner) = 0.$$

It induces the Grauert–Riemenschneider vanishing theorem [**30**].

We insert the following application of Kodaira's vanishing theorem:

**5.13. Lemma** *Let $P$ be a nef and big $\mathbb{R}$-divisor of a non-singular projective variety $X$ of dimension $n$ such that* $\mathrm{Supp}\langle P \rangle$ *is a normal crossing divisor. Then*

$$\lim_{m \to \infty} \frac{1}{m^{n-1}} \mathrm{h}^1(X, \llcorner mP \lrcorner) = 0.$$

Proof. By **2.11**, we can replace $X$ by a blowing-up and $P$ by the total transform. Thus, we may assume that there exist an effective divisor $\Delta$ and a positive integer $m_0$ such that

(1) $\operatorname{Supp}\Delta \cup \operatorname{Supp}\langle P\rangle$ is a simple normal crossing divisor,
(2) $\llcorner mP \lrcorner - \Delta - K_X$ is ample for any $m \geq m_0$.

Hence $\mathrm{H}^1(X, \llcorner mP \lrcorner - \Delta) = 0$ by Kodaira's vanishing theorem. In particular, we have $\mathrm{h}^1(X, \llcorner mP \lrcorner) \leq \mathrm{h}^1(\Delta, \mathcal{O}_\Delta(\llcorner mP \lrcorner))$. It is enough to show that

$$\lim_{m\to\infty} \frac{1}{m^{n-1}} \mathrm{h}^1(E, \mathcal{L} \otimes \mathcal{O}_E(\llcorner mP \lrcorner)) = 0$$

for any prime component $E$ of $\Delta$ and for any line bundle $\mathcal{L}$ of $E$. There is an ample effective divisor $H$ of $X$ such that

$$\mathcal{L} + \llcorner mP \lrcorner|_E + H|_E - K_E$$

is ample for any $m > 0$, since $P$ is nef. Thus $\mathrm{H}^1(E, \mathcal{L} \otimes \mathcal{O}_E(\llcorner mP \lrcorner + H)) = 0$ by Kodaira's vanishing theorem. Hence

$$\mathrm{h}^1(E, \mathcal{L} \otimes \mathcal{O}_E(\llcorner mP \lrcorner)) \leq \mathrm{h}^0(E \cap H, \mathcal{L} \otimes \mathcal{O}_{E\cap H}(\llcorner mP \lrcorner + H)),$$

which is bounded by a polynomial of $m$ of order at most $n - 2$. □

## §5.d. Relative numerical properties.

**5.14. Definition** Let $\pi\colon X \to S$ be a projective surjective morphism from a normal complex analytic variety and let $W \subset S$ be a subset. An $\mathbb{R}$-Cartier divisor $D$ of $X$ is called *$\pi$-ample*, *$\pi$-nef*, and *$\pi$-numerically trivial* over $W$ if $D|_{X_s}$ is ample, nef, and numerically trivial for any $s \in W$, respectively, where $X_s = \pi^{-1}(s)$. It is also called *relatively ample*, *relatively nef*, or *relatively numerically trivial* over $W$. If $W = S$, we drop the phrase 'over $S$.'

**5.15. Lemma** *Let $\pi\colon X \to S$ be a proper surjective morphism from a normal complex analytic space and let $D$ be an $\mathbb{R}$-Cartier divisor on $X$.*

(1) *Suppose that $\pi$ is projective. If $D$ is $\pi$-ample over a point $s \in S$, then there is a Zariski-open neighborhood $U \subset S$ over which $D$ is $\pi$-ample.*
(2) *Suppose that $\pi$ is projective. If $D$ is $\pi$-nef over a point $s \in S$, then there is a countable union $W$ of proper Zariski-closed subsets of $S$ such that $s \notin W$ and $D$ is $\pi$-nef over $S \setminus W$.*
(3) *Suppose that $S$ is connected and $\pi$ is a smooth morphism whose fibers are bimeromorphically equivalent to projective analytic spaces. If $D$ is $\pi$-numerically trivial over a point $s \in S$, then $D$ is $\pi$-numerically trivial.*
(4) *Suppose that $\pi$ is a projective morphism. For a point $s \in S$, there is a Zariski-open subset $U \subset S$ containing $s$ having the following property: If an $\mathbb{R}$-Cartier divisor of $X$ is $\pi$-numerically trivial over the point $s \in S$, then it is $\pi$-numerically trivial over $U$.*

Proof. (1) Let $D_i$ $(1 \leq i \leq l)$ be a finite number of $\mathbb{Q}$-divisors of $X$ such that $D_i$ is $\pi$-ample over $s$ for any $i$ and $D = \sum s_i D_i$ for some positive real numbers $s_i$ (cf. **5.2**). Since the ampleness is an open condition, we can find a Zariski-open neighborhood $U_i \subset S$ such that $D_i$ is $\pi$-ample over $U_i$. Thus $D$ is $\pi$-ample over $\bigcap U_i$.

(2) Let $\mathcal{A}$ be a $\pi$-ample invertible sheaf. By (1), for any positive integer $m$, there is a Zariski-open neighborhood $U_m \subset S$ of $s$ such that $mD|_{X_s} + \mathcal{A}|_{X_s}$ is ample for any $s \in U_m$. We can take $W$ to be the complement of $\bigcap U_m$.

(3) The real first Chern class $c_1(D)$ is an element of $\mathrm{H}^2(X, \mathbb{R})$. Let $c$ be the image under $\mathrm{H}^2(X, \mathbb{R}) \to \mathrm{H}^0(S, \mathrm{R}^2\, \pi_* \mathbb{R}_X)$. Now $\mathrm{R}^2\, \pi_* \mathbb{R}_X$ is a locally constant sheaf whose stalk at $s$ is canonically isomorphic to $\mathrm{H}^2(X_s, \mathbb{R})$. Thus $c_s = 0$ implies $c = 0$. This means that $D$ is $\pi$-numerically trivial.

(4) Let $X_0 \to X$ be a bimeromorphic morphism from a non-singular space obtained by Hironaka's desingularization [**40**] and let $\pi_0$ be the composite $X_0 \to X \to S$. Let $S_1 \subset S$ be an analytic subset such that $\dim S_1 < \dim S$ and $\pi_0$ is smooth over $S \setminus S_1$. Let $X_1 \to \pi_0^{-1}(S_1)$ be a proper surjective morphism from a non-singular analytic space obtained by Hironaka's desingularizations of irreducible components of $\pi_0^{-1}(S_1)$. We can define inductively a sequence of analytic subsets

$$S = S_0 \supset S_1 \supset \cdots \supset S_l \supset S_{l+1},$$

proper surjective morphisms $\pi_i \colon X_i \to S_i$, and proper surjective morphisms $X_i \to \pi_{i-1}^{-1}(S_i)$ for $1 \leq i \leq l$ satisfying the following conditions:

- $\dim_t S_i < \dim_t S_{i-1}$ for any $t \in S_i$;
- $s \in S_l \setminus S_{l+1}$;
- $\pi_i$ is smooth over $S_i \setminus S_{i+1}$;
- $\pi_i$ is isomorphic to the composite $X_i \to \pi_{i-1}^{-1}(S_i) \to S_i$;
- $\pi_i$ is, locally on $S_i$, bimeromorphic to a projective morphism.

Let $C$ be a connected component of $S_i \setminus S_{i+1}$ for $i \leq l$ such that $s \notin \overline{C}$. Note that $\overline{C}$ is an analytic subset of $S$. Let $U \subset S$ be the Zariski-open subset whose complement is the union of all such $\overline{C}$ for all $i$ above and of $S_{l+1}$.

Let $\mathcal{A}$ be a $\pi$-ample invertible sheaf of $X$ and let $D$ be an $\mathbb{R}$-divisor of $X$ which is $\pi$-numerically trivial over $s$. For any integer $m \in \mathbb{Z}$, there is a Zariski-open neighborhood $U_m$ of $s$ such that $mD + \mathcal{A}$ is $\pi$-ample over $U_m$. Hence the set $\Sigma(D)$ of points over which $D$ is $\pi$-numerically trivial is a countable intersection of Zariski-open subsets. Since $\Sigma(D)$ is dense, $(S_i \setminus S_{i+1}) \cap \Sigma(D) \neq \emptyset$ for any $i$. Therefore, $U \subset \Sigma(D)$ by (3). □

**5.16. Definition** Let $\pi \colon X \to S$ be a locally projective morphism. An $\mathbb{R}$-divisor $D$ of $X$ is called *$\pi$-big* or *relatively big over $S$* if there exist an open covering $S = \bigcup S_\lambda$, Cartier divisors $A_\lambda$ of $X_\lambda := \pi^{-1}(S_\lambda)$, and positive integers $m_\lambda$ such that $A_\lambda$ is $\pi$-ample over $S_\lambda$ and

$$\pi_{\lambda *} \mathcal{O}_{X_\lambda}(\llcorner m_\lambda D \lrcorner|_{X_\lambda} - A_\lambda) \neq 0,$$

for the restriction $\pi_\lambda\colon X_\lambda \to S_\lambda$ of $\pi$. An $\mathbb{R}$-divisor $D$ is called *$\pi$-pseudo-effective* or *relatively pseudo-effective over $S$* if there exist an open covering $S = \bigcup S_\lambda$ and $\pi_\lambda$-ample Cartier divisors $A_\lambda$ of $X_\lambda$ such that $D|_{X_\lambda} + \varepsilon A_\lambda$ is $\pi_\lambda$-big for any $\varepsilon > 0$.

Let $B$ be an $\mathbb{R}$-divisor of $X$ and set $d := \dim X - \dim S$. Then the following conditions are mutually equivalent by the same argument as **5.4**:

(1) $B$ is $\pi$-big;

(2) There exist integer $m_1$ and a positive number $C$ such that

$$\operatorname{rank} \pi_*\mathcal{O}_X(\llcorner mB \lrcorner) \geq Cm^d$$

for $m \geq m_1$;

(3) $\overline{\lim}_{m\to\infty} m^{-d} \operatorname{rank} \pi_*\mathcal{O}_X(\llcorner mB \lrcorner) > 0$.

If there is a $\pi$-ample invertible sheaf $\mathcal{A}$, then the following condition also is equivalent to the conditions above:

(4) $\pi_*(\mathcal{O}_X(\llcorner mB \lrcorner) \otimes \mathcal{A}^{-1}) \neq 0$ for a positive integer $m$.

We can define the notion of $\pi$-bigness also for the case $\pi$ is not locally projective by the properties above. If a $\pi$-big $\mathbb{R}$-divisor exists, then, locally over $S$, $\pi$ is bimeromorphic to a projective morphism.

**5.17. Corollary** *An $\mathbb{R}$-divisor $D$ of $X$ is $\pi$-big (resp. $\pi$-pseudo-effective) if and only if, for any component $F$ of a 'general' fiber, $D|_F$ is big (resp. pseudo-effective).*

**Remark** (1) If $D$ is $\pi$-nef over a point, then it is $\pi$-pseudo-effective, by **5.15**,

(2) If $\pi$ is generically finite, then every $\mathbb{R}$-divisor is $\pi$-big.

(3) If a projective morphism $\pi$ is the composite of two surjective morphisms $f\colon X \to Y$ and $g\colon Y \to S$ of complex analytic varieties, then every $\pi$-big divisor is $f$-big and every $\pi$-pseudo-effective divisor is $f$-pseudo-effective. Moreover, if $g$ is generically finite, then $D$ is $\pi$-big (resp. $\pi$-pseudo-effective) if and only if $D$ is $f$-big (resp. $f$-pseudo-effective).

**5.18. Example** On deformation of divisors, pseudo-effectivity and bigness are not open conditions: Over the projective line $\mathbb{P}^1$, let us consider the group $\operatorname{Ext}^1_{\mathbb{P}^1}(\mathcal{O}, \mathcal{O}(-2)) \simeq \mathbb{C}$ of extensions:

$$0 \to \mathcal{O}(-2) \to \mathcal{E} \to \mathcal{O} \to 0.$$

If the extension is non-trivial, then $\mathcal{E} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Therefore, we can construct a family of ruled surfaces $\pi\colon X \to \mathbb{P}^1 \times \mathbb{C} \to \mathbb{C}$ and a Cartier divisor $H$ of $X$ such that

(1) $X_t := \pi^{-1}(t)$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ for $t \neq 0$,

(2) $X_0 \simeq \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-2))$,

(3) for $t \neq 0$, the restriction $H_t := H|_{X_t}$ is linearly equivalent to $\ell_1 - \ell_2$, where $\ell_i$ is a fiber of the $i$-th projection $X_t \to \mathbb{P}^1$,

(4) the restriction $H_0 := H|_{X_0}$ is linearly equivalent to the negative section of the ruled surface $X_0 \to \mathbb{P}^1$.

Thus $H_0$ is pseudo-effective and $H_t$ is not pseudo-effective for $t \neq 0$. Let $F$ be a fiber of $X \to \mathbb{P}^1 \times \mathbb{C} \to \mathbb{P}^1$. Then $(H + xF)|_{X_0}$ is big for $x > 0$ and $(H + xF)|_{X_t}$ is not big for any $x \leq 1$.

Let $\pi\colon X \to S$ be a projective surjective morphism of complex analytic spaces and let $W$ be a compact subset of $S$. Let $Z_1(W)$ be the free abelian group generated by the irreducible curves $\gamma \subset X$ with $\pi(\gamma)$ being a point of $W$. For an open neighborhood $U$ of $W$, we have the intersection pairing

$$\mathrm{Pic}(\pi^{-1}(U)) \times Z_1(W) \ni (\mathcal{L}, \gamma) \mapsto \mathcal{L} \cdot \gamma \in \mathbb{Z}.$$

If $\mathcal{L} \cdot \gamma = 0$ for any $\gamma \in Z_1(W)$, then $\mathcal{L}$ is $\pi$-numerically trivial over $W$. Let $\widetilde{A}(U, W)$ be the quotient group of $\mathrm{Pic}(\pi^{-1}U)$ by the $\pi$-numerical trivial relation over $W$. We define

$$A^1(X/S; W) := \varinjlim_{W \subset U} \widetilde{A}(U, W),$$

where $U$ runs through all the open neighborhoods of $W$. This definition coincides with that in [**98**, 4.1]. We also define $\mathrm{N}^1(X/S; W) := A^1(X/S; W) \otimes \mathbb{R}$. We correct the statements [**98**, 4.3, 4.4] as follows:

**5.19. Lemma** *Suppose that $W \cap Y$ has only finitely many connected components for any analytic subset $Y$ defined over an open neighborhood of $W$. Then $A^1(X/S; W)$ is a finitely generated abelian group.*

Proof. Let $S = S_0 \supset S_1 \supset \cdots$ and $\pi_i\colon X_i \to S_i$ be the objects constructed in the proof of **5.15**-(4). Let $W_{i,j}$ for $1 \leq j \leq k_i$ be the connected components of $W \cap S_i$. We choose a point $w_{i,j} \in W_{i,j} \setminus S_{i+1}$ for $(i, j)$ with $W_{i,j} \not\subset S_{i+1}$. It is enough to show that

$$A^1(X/S; W) \to \bigoplus \mathrm{NS}(\pi_i^{-1}(w_{i,j}))/(\mathrm{tor})$$

is injective. For a line bundle $\mathcal{L}$ on $\pi^{-1}U$ for an open neighborhood $U$ of $W$, assume that $\mathcal{L}$ is $\pi$-numerically trivial over all $w_{i,j}$. Then $\mathcal{L}$ is $\pi$-numerically trivial over $U_{i,j} \setminus S_{i+1}$ for the connected component $U_{i,j}$ of $S_i \cap U$ containing $w_{i,j}$. Note that $W \cap S_i \subset \bigcup_j U_{i,j}$. Therefore, $\mathcal{L}$ is $\pi$-numerically trivial over $W = \bigcup_i W \cap S_i$. □

Assume that the compact subset $W \subset S$ satisfies the condition of **5.19**. Then we can define the relative Picard number $\rho(X/S; W)$ to be the rank of $A^1(X/S; W)$. We can consider similarly several cones such as: the $\pi$-ample cone $\mathrm{Amp}(X/S; W)$, the $\pi$-nef cone $\mathrm{Nef}(X/S; W)$, the $\pi$-big cone $\mathrm{Big}(X/S; W)$, and the $\pi$-pseudo-effective cone $\mathrm{PE}(X/S; W)$, over $W$. Let $A_1(X/S; W)$ be the image of

$$Z_1(W) \to \mathrm{Hom}(A^1(X/S; W), \mathbb{Z})$$

given by the intersection pairing. We set $\mathrm{N}_1(X/S; W) := A_1(X/S; W) \otimes \mathbb{R}$ and let $\mathrm{NE}(X/S; W)$ be the set of the numerical equivalence classes of effective 1-cycles contracted to points of $W$. Then the following Kleiman's criterion holds: $\mathrm{Nef}(X/S; W)$ and the closure $\overline{\mathrm{NE}}(X/S; W)$ of $\mathrm{NE}(X/S; W)$ are dual to each other (cf. [**98**, 4.7]).

Even if the compact set $W$ does not satisfy the condition of **5.19**, we can consider another abelian group $\widehat{A}^1(X/S;W)$ similar to $A^1(X/S;W)$ above as follows: For a while, let $W$ be a subset of $S$. A coherent sheaf $\mathcal{F}$ of $X$ is called invertible over $W$ if the restriction to $\pi^{-1}U$ is an invertible sheaf for some open neighborhood $U$ of $W$. A homomorphism $\mathcal{F}_1 \to \mathcal{F}_2$ of coherent sheaves of $X$ is called an isomorphism over $W$ if the restriction to $\pi^{-1}U$ is an isomorphism for an open neighborhood $U$ of $W$. We define $\widehat{\mathrm{Pic}}(X;W)$ to be the set of coherent sheaves of $X$ which are invertible over $W$, modulo the isomorphisms over $W$. Then $\widehat{\mathrm{Pic}}(X;W)$ has an abelian group structure by the tensor-product and the restriction map $\widehat{\mathrm{Pic}}(X;W) \to \mathrm{Pic}(\pi^{-1}(w))$ is a homomorphism for $w \in W$. Let $\pi' \colon X' \to S$ be another projective surjective morphism from a normal variety. A meromorphic map $\varphi \colon X' \dashrightarrow X$ over $S$ is called a morphism over $W$ if $\varphi \colon \pi'^{-1}U \to \pi^{-1}U$ is a morphism over $U$ for an open neighborhood $U$ of $W$. In this situation, we have the pullback homomorphism $\varphi^* \colon \widehat{\mathrm{Pic}}(X;W) \to \widehat{\mathrm{Pic}}(X';W')$. If $\varphi$ is dominant, then $\varphi^*$ is injective, and if $\varphi^*$ is isomorphic in addition, then $\varphi$ is an isomorphism over an open neighborhood of $W$. We have the natural intersection pairing $\widehat{\mathrm{Pic}}(X;W) \times Z_1(W) \to \mathbb{Z}$, where $Z_1(W)$ is the free abelian group generated by the curves of $X$ contracted to points of $W$. Let $\widehat{A}^1(X/S;W)$ be the quotient of $\widehat{\mathrm{Pic}}(X;W)$ defined as the image of

$$\widehat{\mathrm{Pic}}(X;W) \to \mathrm{Hom}(Z_1(W), \mathbb{Z}).$$

**5.20. Lemma** *If $W$ is a compact subset, then $\widehat{A}^1(X/S;W)$ is a finitely generated abelian group.*

Proof. Let $S = S_0 \supset S_1 \supset \cdots$ and $\pi_i \colon X_i \to S_i$ be the objects constructed in the proof of **5.15**-(4). We have an injection

$$\widehat{A}^1(X/S;W) \hookrightarrow \bigoplus\nolimits_{i\geq 0} \widehat{A}^1(X_i \setminus \pi_i^{-1}S_{i+1}/(S_i \setminus S_{i+1}); W \cap S_i \setminus S_{i+1}).$$

Since $W \cap S_i$ is compact, we may assume that $S_i$ has only finitely many connected components. Therefore, the target of the injection above is a finitely generated abelian group by **5.15**-(3). □

We can define another candidate $\hat{\rho}(X/S;W)$ for the relative Picard number over the compact subset $W$ as the rank of $\widehat{A}^1(X/S;W)$. We can consider similarly several cones in the vector space $\widehat{\mathrm{N}}^1(X/S;W) := \widehat{A}^1(X/S;W) \otimes \mathbb{R}$ such as: the $\pi$-ample cone $\widehat{\mathrm{Amp}}(X/S;W)$, the $\pi$-nef cone $\widehat{\mathrm{Nef}}(X/S;W)$, the $\pi$-big cone $\widehat{\mathrm{Big}}(X/S;W)$, and the $\pi$-pseudo-effective cone $\widehat{\mathrm{PE}}(X/S;W)$, over $W$. Let $\widehat{A}_1(X/S;W)$ be the image of

$$Z_1(W) \to \mathrm{Hom}(\widehat{A}^1(X/S;W), \mathbb{Z})$$

given by the intersection pairing. We set $\widehat{\mathrm{N}}_1(X/S;W) := \widehat{A}_1(X/S;W) \otimes \mathbb{R}$ and let $\widehat{\mathrm{NE}}(X/S;W)$ be the set of the numerical equivalence classes of effective 1-cycles contracted to points of $W$. Then Kleiman's criterion also holds: $\widehat{\mathrm{Nef}}(X/S;W)$ and the closure $\overline{\widehat{\mathrm{NE}}}(X/S;W)$ of $\widehat{\mathrm{NE}}(X/S;W)$ are dual to each other.

We can consider the relative minimal model program by applying $\widehat{\mathrm{N}}^1(X/S;W)$. For example, if $\mathcal{F}$ is a coherent sheaf of $X$ invertible over $W$ and if $\mathcal{F}$ is $\pi$-semi-ample over $W$, then some positive multiple of $\mathcal{F}$ is the pullback of a relatively ample element of $\widehat{\mathrm{Pic}}(X'';W)$ by a meromorphic map $X \dashrightarrow X''$ over $S$ which is a morphism over $W$. In fact, there exist an open neighborhood $U$ of $W$ and a positive integer $k$ such that $\mathcal{F}$ is invertible over $U$ and $\pi^*\pi_*\mathcal{F}^{\otimes k} \to \mathcal{F}^{\otimes k}$ is surjective over $\pi^{-1}U$. It induces a meromorphic map $X \dashrightarrow \mathbb{P}_S(\pi_*\mathcal{F}^{\otimes k})$ over $S$, which is holomorphic over $U$, and $\mathcal{F}^{\otimes k}$ is considered as the pullback of the tautological line bundle.

## §6. Algebraic cycles

**§6.a. Chow groups.** Let $X$ be an $n$-dimensional non-singular projective variety. Let $\mathrm{CH}^i(X)$ denote the Chow group of algebraic cycles of codimension $i \geq 0$. There is a homomorphism $\mathrm{cl}\colon \mathrm{CH}^i(X) \to \mathrm{H}^{2i}(X,\mathbb{Z})$ called the *cycle map*. Here $\mathrm{CH}^1(X) \simeq \mathrm{Pic}(X)$ and the cycle map $\mathrm{CH}^1(X) \to \mathrm{H}^2(X,\mathbb{Z})$ is induced from the connecting homomorphism $\mathrm{H}^1(X,\mathcal{O}_X^\star) \to \mathrm{H}^2(X,\mathbb{Z})$ of the exponential exact sequence of $X$. A cycle is called *homologically equivalent* to zero if it goes to zero by the composite $\mathrm{CH}^i(X) \to \mathrm{H}^{2i}(X,\mathbb{Z}) \to \mathrm{H}^{2i}(X,\mathbb{Q})$. Let $\mathrm{N}^i(X) \subset \mathrm{H}^{2i}(X,\mathbb{R})$ be the real vector subspace generated by the image $\mathrm{cl}(\mathrm{CH}^i(X))$. By the Poincaré duality, the vector subspace $\mathrm{N}_i(X) \subset \mathrm{H}_{2i}(X,\mathbb{R})$ generated by algebraic cycles of dimension $i$ is isomorphic to $\mathrm{N}^{n-i}(X)$. The cup product of $\mathrm{H}^\bullet(X,\mathbb{R})$ induces the intersection homomorphism $\mathrm{N}^i(X) \times \mathrm{N}^j(X) \to \mathrm{N}^{i+j}(X)$, which is compatible with the ring structure of the Chow ring $\mathrm{CH}^\bullet(X) = \bigoplus \mathrm{CH}^i(X)$. A cycle $\zeta$ of codimension $i$ is called *numerically trivial* or numerically equivalent to zero if $\zeta \cdot \eta = 0$ for any $\eta \in \mathrm{CH}^{n-i}(X)$. By the trace map $\mathrm{H}^{2n}(X,\mathbb{R}) \simeq \mathbb{R}$, two vector spaces $\mathrm{H}^{2i}(X,\mathbb{R})$ and $\mathrm{H}^{2n-2i}(X,\mathbb{R})$ are dual to each other by the intersection pairing. However, it is still conjectural that $\mathrm{N}^i(X)$ and $\mathrm{N}^{n-i}(X)$ are dual to each other. This is equivalent to saying that the numerical equivalence and the homological equivalence on $\mathrm{CH}^i(X)$ coincide. For $i = 1$, it is true by $\mathrm{H}^2(X,\mathbb{Q}) \cap \mathrm{H}^{1,1}(X) = \mathrm{NS}(X) \otimes \mathbb{Q}$.

**6.1. Definition** An *algebraic* $\mathbb{R}$-*cycle* of codimension $k$ is a finite $\mathbb{R}$-linear combination $\zeta = \sum c_i W_i$ of subvarieties $W_i$ of codimension $k$. The $\mathbb{R}$-cycle $\zeta$ is called *effective* if all the coefficients $c_i$ are non-negative. We call $c_i$ the multiplicity of $\zeta$ along $W_i$ and denote $c_i = \mathrm{mult}_{W_i} \zeta$. For cycles $\zeta_1$, $\zeta_2$ of codimension $i$, both of the relations $\zeta_1 \geq \zeta_2$ and $\zeta_2 \leq \zeta_1$ indicate that $\zeta_1 - \zeta_2$ is effective.

**6.2. Definition** Let $\mathrm{Eff}^k(X) \subset \mathrm{N}^k(X)$ be the cone of the cohomology classes of effective algebraic $\mathbb{R}$-cycles of codimension $k$. The closure $\mathrm{PE}^k(X)$ is called the pseudo-effective cone of algebraic cycles of codimension $k$. Note that $\mathrm{PE}^{n-1}(X) = \overline{\mathrm{NE}}(X)$. An algebraic $\mathbb{R}$-cycle $\zeta$ is called pseudo-effective if $\mathrm{cl}(\zeta) \in \mathrm{PE}^k(X)$.

**6.3. Proposition** *Let $\zeta$ be a pseudo-effective $\mathbb{R}$-cycle of codimension $k$. Then, for any nef $\mathbb{R}$-divisor $D$, the intersection number $\zeta \cdot D^{n-k}$ is non-negative. If $\zeta \cdot A^{n-k} = 0$ for an ample $\mathbb{R}$-divisor, then $\zeta$ is homologically equivalent to zero.*

PROOF. It is enough to show the second statement. Let $P^k \subset \mathrm{H}^{k,k}(X,\mathbb{R})$ be the set of cohomology classes $[\omega]$ of global $C^\infty$-real $d$-closed $(k,k)$-forms

$$\omega = (\sqrt{-1})^k \sum\nolimits_{I,J\subset\{1,2,\ldots,n\}} \omega_{I,J} dz_I \wedge d\bar{z}_J,$$

where the matrix $(\omega_{I,J})$ is positive definite everywhere. Then $P^k$ is an open convex cone in the space $\mathrm{H}^{k,k}(X,\mathbb{R})$ and $c_1(A)^{n-k}$ belongs to $P^{n-k}$. Since $\zeta$ is pseudo-effective, $\mathrm{cl}(\zeta) \cup [\omega] = 0$ for any $[\omega] \in P^{n-k}$. Thus $\mathrm{cl}(\zeta) = 0$, since $\mathrm{H}^{k,k}(X,\mathbb{R})$ and $\mathrm{H}^{n-k,n-k}(X,\mathbb{R})$ are dual to each other by the intersection pairing. □

**Remark** The proposition above proves the conjecture [**98**, 2.12] affirmatively.

For a morphism $f\colon Y \to X$ from a non-singular projective variety $Y$ of dimension $m$, we have the natural homomorphisms

$$f^*\colon \mathrm{N}^i(X) \to \mathrm{N}^i(Y), \quad f_*\colon \mathrm{N}_i(Y) \simeq \mathrm{N}^{m-i}(Y) \to \mathrm{N}_i(X) \simeq \mathrm{N}^{n-i}(X),$$

where the projection formula

$$f_*(f^*u \cdot v) = u \cdot f_*v \quad \in \mathrm{N}^{n-m+(i+j)}(X)$$

holds for $u \in \mathrm{N}^i(X)$ and $v \in \mathrm{N}^j(Y)$.

**Remark** Let $f\colon Y \to X$ be a morphism from a non-singular projective variety $Y$ of dimension $m$. Then $f_*(\mathrm{PE}^{m-i}(Y)) \subset \mathrm{PE}^{n-i}(X)$.

**§6.b. Chern classes of vector bundles.** Let $X$ be a non-singular projective variety of dimension $n$. For a vector bundle $\mathcal{E}$ of $X$ of rank $r$, its $i$-th *Chern classes* $\hat{c}_i(\mathcal{E})$ is defined as an element of the Chow group $\mathrm{CH}^i(X)$ as follows: Let $p\colon \mathbb{P} = \mathbb{P}_X(\mathcal{E}) \to X$ be the projective bundle and let $H = H_\mathcal{E}$ be a *tautological divisor* associated with $\mathcal{E}$: $\mathcal{O}_\mathbb{P}(H)$ is the tautological line bundle. There is an isomorphism

$$\mathrm{CH}^i(\mathbb{P}) \simeq \mathrm{CH}^i(X) \oplus \mathrm{CH}^{i-1}(X) \cdot H \oplus \cdots \oplus \mathrm{CH}^0(X) \cdot H^i$$

for any $i < r$, where $H$ is regarded as an element of $\mathrm{CH}^1(\mathbb{P}) = \mathrm{Pic}(\mathbb{P})$. Under the natural isomorphism $\mathrm{CH}^0(X) \simeq \mathrm{H}^0(X,\mathbb{Z})$, $\hat{c}_0(\mathcal{E})$ is defined to be 1. The other Chern classes $\hat{c}_i(\mathcal{E})$ are defined as elements of $\mathrm{CH}^i(X)$ satisfying the equality:

$$\sum\nolimits_{i=0}^r (-1)^{r-i} p^*\hat{c}_i(\mathcal{E}) \cdot H^{r-i} = 0.$$

The usual $i$-th Chern class $c_i(\mathcal{E})$ is defined as the image of $\hat{c}_i(\mathcal{E})$ under $\mathrm{cl}\colon \mathrm{CH}^i(X) \to \mathrm{H}^{2i}(X,\mathbb{Z})$. Let us introduce polynomials

$$\begin{aligned} C_\mathcal{E}(t) &:= \sum\nolimits_{i=1}^r \hat{c}_i(\mathcal{E})t^i, \\ P_\mathcal{E}(t) &:= \sum\nolimits_{i=1}^r (-1)^{r-i}\hat{c}_i(\mathcal{E})t^{r-i} = (-1)^r t^r C_{\mathcal{E}^\vee}(1/t), \\ Q_\mathcal{E}(t) &:= P_\mathcal{E}\left(t + \frac{1}{r}\hat{c}_1(\mathcal{E})\right) = \sum\nolimits_{i=0}^r (-1)^i \widehat{\Delta}_i(\mathcal{E}) t^{r-i}. \end{aligned}$$

Here, $C_{\mathcal{E}}(t)$ is called the Chern polynomial which belongs to $\mathrm{CH}^\bullet(X)[t]$. For other polynomials, we have $P_{\mathcal{E}}(t) \in \mathrm{CH}^\bullet(X)[t]$ and $Q_{\mathcal{E}}(t) \in \mathrm{CH}^\bullet(X)[t] \otimes \mathbb{Q}$. The coefficients $\widehat{\Delta}_k(\mathcal{E}) \in \mathrm{CH}^k(X)_{\mathbb{Q}}$ are written in terms of Chern classes $\hat{c}_i(\mathcal{E})$ by

$$\widehat{\Delta}_k(\mathcal{E}) = \sum\nolimits_{j=0}^{k} \frac{(-1)^j}{r^j}\binom{r-k+j}{j}\hat{c}_1(\mathcal{E})^j \cdot \hat{c}_{k-j}(\mathcal{E}).$$

If $k \leq 2$, we have

$$\widehat{\Delta}_0(\mathcal{E}) = \hat{c}_0(\mathcal{E}) = 1, \quad \widehat{\Delta}_1(\mathcal{E}) = 0, \quad \widehat{\Delta}_2(\mathcal{E}) = \hat{c}_2(\mathcal{E}) - \frac{r-1}{2r}\hat{c}_1(\mathcal{E})^2.$$

**6.4. Definition** A *normalized tautological divisor* $\Lambda = \Lambda_{\mathcal{E}}$ of $\mathcal{E}$ is a $\mathbb{Q}$-divisor of $\mathbb{P} = \mathbb{P}(\mathcal{E})$ such that $r\Lambda$ is a $\mathbb{Z}$-divisor and

$$\mathcal{O}_{\mathbb{P}}(r\Lambda) \simeq \mathcal{O}_{\mathcal{E}}(r) \otimes p^*(\det \mathcal{E})^{-1}.$$

In particular, $r\Lambda \sim -K_{\mathbb{P}/X}$.

From the vanishing $P_{\mathcal{E}}(H) = 0$, we have

$$Q_{\mathcal{E}}(\Lambda) = \sum\nolimits_{i=0}^{r} (-1)^i p^* \widehat{\Delta}_i(\mathcal{E}) \Lambda^{r-i} = 0.$$

Note that $p_*\Lambda^j = 0$ for $j < r-1$ and $p_*\Lambda^{r-1} = 1 \in \mathrm{CH}^0(X)$. Thus

$$p_*\Lambda^r = \widehat{\Delta}_1(\mathcal{E}) = 0, \quad p_*\Lambda^{r+1} = -\widehat{\Delta}_2(\mathcal{E}), \quad p_*\Lambda^{r+2} = \widehat{\Delta}_3(\mathcal{E}),$$
$$p_*\Lambda^{r+3} = \widehat{\Delta}_2(\mathcal{E})^2 - \widehat{\Delta}_4(\mathcal{E}), \text{ etc.}$$

For an exact sequence $0 \to \mathcal{E}_0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to 0$ of vector bundles $\mathcal{E}_i$, we have

$$C_{\mathcal{E}_1}(t) = C_{\mathcal{E}_0}(t) \cdot C_{\mathcal{E}_2}(t).$$

Let $K_0(X)$ be the Grothendieck $K$-group of vector bundles of $X$. Then $\mathcal{E} \mapsto C_{\mathcal{E}}(t)$ gives rise to a homomorphism $K_0(X) \to \mathrm{CH}^\bullet(X)[t]/(t^{n+1})$ from the additive group structure $(K_0(X), +)$ into the semi-group structure $(\mathrm{CH}^\bullet(X)[t]/(t^{n+1}), \times)$. Since $X$ has an ample divisor, every coherent sheaf $\mathcal{F}$ has an exact sequence

$$0 \to \mathcal{E}_n \to \mathcal{E}_{n-1} \to \cdots \to \mathcal{E}_0 \to \mathcal{F} \to 0$$

such that $\mathcal{E}_i$ are all vector bundles. Thus the $K$-group of coherent sheaves coincides with $K_0(X)$ and hence the Chern classes $\hat{c}_i(\mathcal{F})$, $c_i(\mathcal{F})$, and also $\widehat{\Delta}_i(\mathcal{F})$ of a coherent sheaf $\mathcal{F}$ are well-defined. The Chern character $\mathrm{ch}(\mathcal{E})$ and the Todd character $\mathrm{Todd}(\mathcal{E})$ of a vector bundle $\mathcal{E}$ of rank $r$ are defined as elements of $\mathrm{CH}^\bullet(X) \otimes \mathbb{Q}$ as follows: For formal elements $\xi_1, \xi_2, \ldots, \xi_r$ satisfying $C_{\mathcal{E}}(t) = \prod(1 + \xi_i t)$,

$$\mathrm{ch}(\mathcal{E}) := \sum\nolimits_{i=1}^{r} \exp(\xi_i) \quad \text{and} \quad \mathrm{Todd}(\mathcal{E}) = \prod\nolimits_{i=1}^{r} \frac{\xi_i}{1 - \exp(-\xi_i)}.$$

The Chern character extends to a ring homomorphism $\mathrm{ch}\colon K_0(X) \to \mathrm{CH}^\bullet(X) \otimes \mathbb{Q}$. We denote $\mathrm{Todd}(T_X)$ for the tangent bundle $T_X$ by $\mathrm{Todd}(X)$.

Let $\mathcal{F}$ be a coherent sheaf of $X$ with $\operatorname{codim} \operatorname{Supp} \mathcal{F} = k \geq 0$ and let $Z \subset \operatorname{Supp} \mathcal{F}$ be an irreducible component of codimension $k$. We define the length $l_Z(\mathcal{F})$ of $\mathcal{F}$ along $Z$ as follows: There is a filtration

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_l = \mathcal{F}$$

of coherent sheaves such that $\mathcal{F}_i/\mathcal{F}_{i-1}$ is a non-zero torsion-free $\mathcal{O}_Z$-module for $i > 0$ and $Z \not\subset \operatorname{Supp} \mathcal{F}_0$. Here, we set

$$l_Z(\mathcal{F}) := \sum\nolimits_{i>0} \operatorname{rank} \mathcal{F}_i/\mathcal{F}_{i-1},$$

which does not depend on the choice of such filtrations. We define

$$\operatorname{cl}(\mathcal{F}) := \sum\nolimits_{Z \subset \operatorname{Supp} \mathcal{F},\, \operatorname{codim} Z = k} l_Z(\mathcal{F}) \operatorname{cl}(Z) \in H^{2k}(X, \mathbb{Z}).$$

**6.5. Lemma** *Under the situation above,* $c_i(\mathcal{F}) = 0$ *for* $0 < i < k$ *and*

$$c_k(\mathcal{F}) = (-1)^{k-1}(k-1)! \operatorname{cl}(\mathcal{F}).$$

PROOF. We shall prove by induction on $\dim \operatorname{Supp} \mathcal{F}$.

Let $Z_1, Z_2, \ldots, Z_l$ be the irreducible component of codimension $k$ of $\operatorname{Supp} \mathcal{F}$. Then there exist coherent sheaves $\mathcal{F}_{(j)}$ with $\operatorname{Supp} \mathcal{F}_{(j)} = Z_j$ and a homomorphism $\mathcal{F} \to \bigoplus_{j=1}^{l} \mathcal{F}_{(j)}$ whose kernel and cokernel are sheaves supported on analytic subset of codimension greater than $k$. Hence we are reduced to the case: $Z = \operatorname{Supp} \mathcal{F}$ is irreducible.

Let $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_l = \mathcal{F}$ be the filtration above calculating $l = l_Z(\mathcal{F})$. Then $\operatorname{ch}(\mathcal{F}) = \operatorname{ch}(\mathcal{F}_0) + \sum_{j=1}^{l} \operatorname{ch}(\mathcal{F}_j/\mathcal{F}_{j-1})$. Thus we are reduced to the case: $Z = \operatorname{Supp} \mathcal{F}$ is irreducible and $\mathcal{F}$ is a torsion-free $\mathcal{O}_Z$-module.

Let $f \colon Y \to Z$ be a resolution of singularities of $Z$ and set $\mathcal{G} := f^*\mathcal{F}/(\mathrm{tor})$. Then the kernel and the cokernel of $\mathcal{F} \to f_*\mathcal{G}$ are torsion sheaves on $Z$. There are homomorphisms $f_! \colon K_0(Y) \to K_0(X)$, $f_* \colon \mathrm{CH}^i(Y) \to \mathrm{CH}^{i+k}(X)$ given by $f_!\mathcal{G} = \sum (-1)^i \mathrm{R}^i f_*\mathcal{G}$ and

$$f_* \colon \mathrm{CH}^i(Y) \simeq \mathrm{CH}_{n-k-i}(Y) \to \mathrm{CH}_{n-k-i}(X) \simeq \mathrm{CH}^{i+k}(X).$$

By the Grothendieck–Riemann–Roch formula [**5**], we have

$$\operatorname{ch}(f_!\mathcal{G}) \cdot \operatorname{Todd}(X) = f_*(\operatorname{ch}(\mathcal{G}) \cdot \operatorname{Todd}(Y)).$$

Let $\operatorname{ch}(\mathcal{F})^{(i)} \in \mathrm{CH}^i(X) \otimes \mathbb{Q}$ denote the $i$-th component of $\operatorname{ch}(\mathcal{F})$ in $\mathrm{CH}^\bullet(X) \otimes \mathbb{Q}$: $\operatorname{ch}(\mathcal{F}) = \sum \operatorname{ch}(\mathcal{F})^{(i)}$. By induction, we infer that $\operatorname{ch}(\mathcal{F})^{(i)} = 0$ for $i < k$ and $\operatorname{ch}(\mathcal{F})^{(k)} = (\operatorname{rank} \mathcal{G}) Z \in \mathrm{CH}^i(X) \otimes \mathbb{Q}$. Since $\operatorname{ch}(\mathcal{F})^{(i)} = 0$ for $i < k$, we have

$$\hat{c}_k(\mathcal{F}) = (-1)^{k-1}(k-1)! \operatorname{ch}(\mathcal{F})^{(k)}.$$

Thus we are done. □

For example, if $\mathcal{F}$ is a skyscraper sheaf, then

$$\dim \mathrm{H}^0(X, \mathcal{F}) = \chi(X, \mathcal{F}) = (-1)^{n-1} \frac{1}{(n-1)!} \deg c_n(\mathcal{F}).$$

If $\operatorname{codim} \operatorname{Supp} \mathcal{F} \geq k$, then

$$(-1)^{k-1} c_k(\mathcal{F}) \cdot A^{n-k} \geq 0$$

for an ample divisor $A$ and the equality holds only when $\operatorname{codim} \operatorname{Supp} \mathcal{F} \geq k+1$.

An entire holomorphic function $\Psi(x) = \Psi(x_1, x_2, \ldots, x_d)$ of $d$-variables is written by the following form:

$$\Psi(x) = \sum\nolimits_{i_1, i_2, \ldots, i_d \geq 0} c_{i_1, i_2, \ldots, i_d} \frac{x_1^{[i_1]} x_2^{[i_2]} \cdots x_d^{[i_d]}}{i_1!\, i_2! \cdots i_d!},$$

where $c_{i_1, i_2, \ldots, i_d} = c_{i_1, i_2, \ldots, i_d}(\Psi)$ are constants and

$$x^{[k]} := \begin{cases} \prod_{j=0}^{k-1}(x+j), & k \geq 1; \\ 1, & k = 0. \end{cases}$$

Let $\Delta_j$ be the $j$-th difference operator defined by

$$(\Delta_j \Psi)(x) = \Psi(x_1, \ldots, x_j, \ldots, x_d) - \Psi(x_1, \ldots, x_j - 1, \ldots, x_d)$$

for $1 \leq j \leq d$. Then $\Delta_j x_j^{[k]} = k x_j^{[k-1]}$ for $k \geq 1$. Thus we have

$$c_{1,1,\ldots,1} = c_{1,1,\ldots,1}(\Psi) = (\Delta_1 \Delta_2 \cdots \Delta_d \Psi)(0).$$

For example, for the function $\psi(x) = \exp(\sum_{i=1}^{d} \lambda_i x_i)$ for $\lambda_i \in \mathbb{C}$, we have

$$(\Delta_j \psi)(x) = (1 - \exp(-\lambda_j))\psi(x), \quad \text{and} \quad c_{1,\ldots,1}(\psi) = \prod\nolimits_{j=1}^{d} (1 - \exp(-\lambda_j)).$$

**6.6. Lemma** *Let $\mathcal{F}$ be a coherent sheaf with* $\operatorname{codim} \operatorname{Supp} \mathcal{F} = k = n - d$ *and let $\mathcal{L}_1$, $\mathcal{L}_2$, ..., $\mathcal{L}_d$ be invertible sheaves on $X$. Let $F(x) = F(x_1, x_2, \ldots, x_d)$ be the polynomial satisfying*

$$F(m_1, m_2, \ldots, m_d) = \chi(X, \mathcal{F} \otimes \mathcal{L}_1^{\otimes m_1} \otimes \cdots \otimes \mathcal{L}_d^{\otimes m_d})$$

*for $m_i \in \mathbb{Z}$. Then*

$$c_{1,1,\ldots,1}(F) = \mathcal{L}_1 \cdot \mathcal{L}_2 \cdots \mathcal{L}_d \cdot \operatorname{cl}(\mathcal{F}).$$

Proof. Since ch is a ring homomorphism, we have

$$\operatorname{ch}(\mathcal{F} \otimes \mathcal{L}_1^{\otimes m_1} \otimes \cdots \otimes \mathcal{L}_d^{\otimes m_d}) = \operatorname{ch}(\mathcal{F}) \operatorname{ch}(\mathcal{L}_1)^{m_1} \cdots \operatorname{ch}(\mathcal{L}_d)^{m_d},$$

where

$$\operatorname{ch}(\mathcal{L}_i)^{m_i} = \exp(m_i \mathcal{L}_i) = \sum\nolimits_{p=0}^{n} \frac{1}{p!} m_i^p \mathcal{L}_i^p \in \operatorname{CH}^\bullet(X) \otimes \mathbb{Q}$$

for $1 \leq i \leq d$. By the Riemann–Roch formula, $F(x)$ is regarded as the $n$-th component of

$$\operatorname{ch}(\mathcal{F}) \cdot f(x) \cdot \operatorname{Todd}(X) \in \operatorname{CH}^\bullet(X) \otimes \mathbb{Q},$$

where

$$f(x) = \exp\left(\sum\nolimits_{i=1}^{d} \mathcal{L}_i x_i\right).$$

Thus $c_{1,\dots,1}(F)$ is the $n$-th component of

$$\mathrm{ch}(\mathcal{F})\cdot\left(\prod\nolimits_{j=1}^{d}(1-\exp(-\mathcal{L}_j))\right)\cdot\mathrm{Todd}(X).$$

By **6.5**, we have

$$c_{1,\dots,1}(F)=\mathrm{ch}(\mathcal{F})^{(k)}\cdot\mathcal{L}_1\cdots\mathcal{L}_d=\mathcal{L}_1\cdots\mathcal{L}_d\cdot\mathrm{cl}(\mathcal{F}).\qquad\square$$

**§6.c. Semistable vector bundles.** Let $X$ be a non-singular projective variety of dimension $d$. Let $\mathcal{F}$ be a non-zero torsion-free coherent sheaf of $X$. The *averaged first Chern class* $\mu(\mathcal{F})$ is defined by

$$\mu(\mathcal{F})=\frac{1}{\operatorname{rank}\mathcal{F}}c_1(\mathcal{F}).$$

For an ample divisor $A$, we set $\mu_A(\mathcal{F})=\mu(\mathcal{F})\cdot A^{d-1}$. A torsion-free sheaf $\mathcal{F}$ is called *$A$-$\mu$-stable* and *$A$-$\mu$-semi-stable* if the inequalities $\mu_A(\mathcal{G})<\mu_A(\mathcal{F})$ and $\mu_A(\mathcal{G})\le\mu_A(\mathcal{F})$ hold for any coherent subsheaf $0\subsetneq\mathcal{G}\subsetneq\mathcal{F}$, respectively. There is a notion of $A$-stable sheaf which is different from the notion of $A$-$\mu$-stable sheaf. The first notion is important when we consider some moduli space of vector bundles. However, in our article, we call an $A$-$\mu$-stable sheaf by an $A$-stable sheaf and an $A$-$\mu$-semi-stable sheaf by an $A$-semi-stable sheaf, for short.

Let $\mathcal{F}$ be a non-zero torsion-free sheaf of $X$. The *Harder–Narasimhan filtration* [**35**] of $\mathcal{F}$ with respect to $A$ is a filtration

$$0=\mathcal{F}_0\subset\mathcal{F}_1\subset\cdots\subset\mathcal{F}_l=\mathcal{F}$$

of coherent subsheaves satisfying the following conditions:

(1) $\mathcal{F}_i/\mathcal{F}_{i-1}$ are non-zero $A$-semi-stable sheaves;
(2) $\mu_A(\mathcal{F}_i/\mathcal{F}_{i-1})>\mu_A(\mathcal{F}_{i+1}/\mathcal{F}_i)$ for $1\le i\le l-1$.

This exists uniquely up to isomorphisms. The existence essentially follows from the lower-boundedness of $c_1(\mathcal{G})\cdot A^{d-1}$ for all quotient sheaves $\mathcal{G}$ of $\mathcal{F}$. The number $l$ is called the length of the filtration.

Assume that $\dim X=1$. Then the notion of stability is independent of the choice of ample divisors. A vector bundle $\mathcal{E}$ on $X$ is semi-stable if and only if the normalized tautological divisor $\Lambda$ is nef (cf. [**107**], [**82**, 3.1]). Moreover if $\Lambda$ is nef, then every effective divisor of $\mathbb{P}$ is nef. If $\mathcal{E}$ is not semi-stable, then $\Lambda$ is big. Therefore, if $\dim X=1$, then $\Lambda$ is always pseudo-effective.

**Example** In higher dimension, the normalized tautological divisor is not necessarily pseudo-effective. Let $T_X$ be the tangent bundle of the projective plane $X=\mathbb{P}^2$. Then $\mathbb{P}=\mathbb{P}_X(T_X)$ is a hypersurface of $\mathbb{P}^2\times\mathbb{P}^2$ and there are two $\mathbb{P}^1$-bundle structures $p_1,p_2\colon\mathbb{P}\to\mathbb{P}^2$. We consider $p_1$ as the associated $\mathbb{P}^1$-bundle structure of $T_X$. The tautological divisor $H$ associated with $T_X$ is linearly equivalent to $p_1^*\ell+p_2^*\ell$ for a line $\ell\subset\mathbb{P}^2$. Thus the normalized tautological divisor is written by $\Lambda=p_2^*\ell-(1/2)p_1^*\ell$. Then we infer that $\Lambda$ is not pseudo-effective by $\Lambda\cdot(p_2^*\ell)^2=-1/2<0$.

Suppose that $\dim X \geq 2$. Then, for an $A$-semi-stable reflexive sheaf $\mathcal{F}$ on $X$, we have the Bogomolov inequality

$$\widehat{\Delta}_2(\mathcal{F}) \cdot A^{n-2} = \left(c_2(\mathcal{F}) - \frac{r-1}{2r} c_1(\mathcal{F})^2\right) \cdot A^{n-2} \geq 0.$$

For a short exact sequence $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ of non-zero torsion-free coherent sheaves, we have the following formula:

$$\text{(II-9)} \qquad \widehat{\Delta}_2(\mathcal{F}) = \widehat{\Delta}_2(\mathcal{E}) + \widehat{\Delta}_2(\mathcal{G}) - \frac{(\operatorname{rank}\mathcal{E})(\operatorname{rank}\mathcal{G})}{2(\operatorname{rank}\mathcal{F})}(\mu(\mathcal{E}) - \mu(\mathcal{G}))^2.$$

Thus, if $\mu_A(\mathcal{E}) = \mu_A(\mathcal{F}) = \mu_A(\mathcal{G})$ for an ample divisor $A$, then

$$\widehat{\Delta}_2(\mathcal{F}) \cdot A^{d-2} \geq \widehat{\Delta}_2(\mathcal{E}) \cdot A^{d-2} + \widehat{\Delta}_2(\mathcal{G}) \cdot A^{d-2}$$

by the Hodge index theorem. Here, the equality holds if and only if $\mu(\mathcal{F}) = \mu(\mathcal{E}) = \mu(\mathcal{G})$.