3 Comparison principle

In this section, we discuss the comparison principle, which implies the uniqueness of viscosity solutions when their values on $\partial\Omega$ coincide (*i.e.* under the Dirichlet boundary condition). In the study of the viscosity solution theory, the comparison principle has been the main issue because the uniqueness of viscosity solutions is harder to prove than existence and stability of them.

First, we recall some "classical" comparison principles and then, show how to modify the proof to a modern "viscosity" version.

In this section, the comparison principle roughly means that

"Comparison principle"

$$\left.\begin{array}{c}\textbf{viscosity subsolution } u\\ \textbf{viscosity supersolution } v\\ u\le v \textbf{ on } \partial\Omega\end{array}\right\}\quad\Longrightarrow\quad u\le v \textbf{ in } \overline{\Omega}$$

Modifying our proofs of comparison theorems below, we obtain a slightly stronger assertion than the above one:

$$\left.\begin{array}{c}\text{viscosity subsolution } u\\ \text{viscosity supersolution } v\end{array}\right\}\quad\Longrightarrow\quad \max_{\overline{\Omega}}(u-v)=\max_{\partial\Omega}(u-v)$$

We remark that the comparison principle implies the uniqueness of (continuous) viscosity solutions under the Dirichlet boundary condition:

"Uniqueness for the Dirichlet problem"

$$\left.\begin{array}{c}\textbf{viscosity solutions } u \textbf{ and } v\\ u=v \textbf{ on } \partial\Omega\end{array}\right\}\quad\Longrightarrow\quad u=v \textbf{ in } \overline{\Omega}$$

Proof of "the comparison principle implies the uniqueness".

Since u (resp., v) and v (resp., u), respectively, are a viscosity subsolution and supersolution, by $u=v$ on $\partial\Omega$, the comparison principle yields $u\le v$ (resp., $v\le u$) in $\overline{\Omega}$. □

In this section, we mainly deal with the following PDE instead of (2.6).

$$\nu u+F(x,Du,D^2u)=0 \quad \text{in } \Omega, \tag{3.1}$$

where we suppose that

$$\nu\ge 0, \tag{3.2}$$

and

$$F:\Omega\times\mathbf{R}^n\times S^n\to\mathbf{R} \quad \text{is continuous.} \tag{3.3}$$

3.1 Classical comparison principle

In this subsection, we show that if one of viscosity sub- and supersolutions is a classical one, then the comparison principle holds true. We call this the "classical" comparison principle.

3.1.1 Degenerate elliptic PDEs

We first consider the case when F is (degenerate) elliptic and $\nu > 0$.

Proposition 3.1. *Assume that $\nu > 0$ and (3.3) hold. Assume also that F is elliptic. Let $u \in USC(\overline{\Omega})$ (resp., $v \in LSC(\overline{\Omega})$) be a viscosity subsolution (resp., supersolution) of (3.1) and $v \in LSC(\overline{\Omega}) \cap C^2(\Omega)$ (resp., $u \in USC(\overline{\Omega}) \cap C^2(\Omega)$) a classical supersolution (resp., subsolution) of (3.1).*

If $u \le v$ on $\partial\Omega$, then $u \le v$ in $\overline{\Omega}$.

Proof. We only prove the assertion when u is a viscosity subsolution of (3.1) since the other one can be shown similarly.

Set $\max_{\overline{\Omega}}(u - v) =: \theta$ and choose $\hat{x} \in \overline{\Omega}$ such that $(u - v)(\hat{x}) = \theta$.

Suppose that $\theta > 0$ and then, we will get a contradiction. We note that $\hat{x} \in \Omega$ because $u \le v$ on $\partial\Omega$.

Thus, the definition of u and v respectively yields

$$\nu u(\hat{x}) + F(\hat{x}, Dv(\hat{x}), D^2v(\hat{x})) \le 0 \le \nu v(\hat{x}) + F(\hat{x}, Dv(\hat{x}), D^2v(\hat{x})).$$

Hence, by these inequalities, we have

$$\nu\theta = \nu(u - v)(\hat{x}) \le 0,$$

which contradicts $\theta > 0$. □

3.1.2 Uniformly elliptic PDEs

Next, we present the comparison principle when $\nu = 0$ but F is uniformly elliptic in the following sense. Notice that if $\nu > 0$ and F is uniformly elliptic, then Proposition 3.1 yields Proposition 3.3 below because our uniform ellipticity implies (degenerate) ellipticity.

Throughout this book, we freeze the "uniform ellipticity" constants:

$$0 < \lambda \le \Lambda.$$

With these constants, we introduce the Pucci's operators: For $X \in S^n$,

$$\mathcal{P}^+(X) := \max\{-\text{trace}(AX) \mid \lambda I \le A \le \Lambda I \text{ for } A \in S^n\},$$

$$\mathcal{P}^-(X) := \min\{-\text{trace}(AX) \mid \lambda I \le A \le \Lambda I \text{ for } A \in S^n\}.$$

We give some properties of $\mathcal{P}^\pm$. We omit the proof since it is elementary.

Proposition 3.2. *For $X, Y \in S^n$, we have the following:*

(1) $\mathcal{P}^+(X) = -\mathcal{P}^-(-X)$,
(2) $\mathcal{P}^\pm(\theta X) = \theta \mathcal{P}^\pm(X)$ *for* $\theta \ge 0$,
(3) $\mathcal{P}^+$ *is convex,* $\mathcal{P}^-$ *is concave,*
(4) $\left\{ \begin{array}{lllll} \mathcal{P}^-(X) + \mathcal{P}^-(Y) & \le & \mathcal{P}^-(X+Y) & \le & \mathcal{P}^-(X) + \mathcal{P}^+(Y) \\ & \le & \mathcal{P}^+(X+Y) & \le & \mathcal{P}^+(X) + \mathcal{P}^+(Y). \end{array} \right.$

Definition. We say that $F : \Omega \times \mathbf{R}^n \times S^n \to \mathbf{R}$ is **uniformly elliptic** (with the uniform ellipticity constants $0 < \lambda \le \Lambda$) if

$$\mathcal{P}^-(X - Y) \le F(x, p, X) - F(x, p, Y) \le \mathcal{P}^+(X - Y)$$

for $x \in \Omega, p \in \mathbf{R}^n$, and $X, Y \in S^n$.

We also suppose the following continuity on F with respect to $p \in \mathbf{R}^n$: There is $\mu > 0$ such that

$$|F(x, p, X) - F(x, p', X)| \le \mu |p - p'| \tag{3.4}$$

for $x \in \Omega, p, p' \in \mathbf{R}^n$, and $X \in S^n$.

Proposition 3.3. *Assume that* (3.2), (3.3) *and* (3.4) *hold. Assume also that F is uniformly elliptic. Let $u \in USC(\overline{\Omega})$ (resp., $v \in LSC(\overline{\Omega})$) be a viscosity subsolution* (*resp., supersolution*) *of* (3.1) *and $v \in LSC(\overline{\Omega}) \cap C^2(\Omega)$ (resp., $u \in USC(\overline{\Omega}) \cap C^2(\Omega)$) a classical supersolution* (*resp., subsolution*) *of* (3.1).

If $u \le v$ on $\partial\Omega$, then $u \le v$ in $\overline{\Omega}$.

Proof. We give a proof only when u is a viscosity subsolution and v a classical supersolution of (3.1).

Suppose that $\max_{\overline{\Omega}}(u-v) =: \theta > 0$. Then, we will get a contradiction again.

For $\varepsilon > 0$, we set $\phi_\varepsilon(x) = \varepsilon e^{\delta x_1}$, where $\delta := \max\{(\mu+1)/\lambda, \nu+1\} > 0$. We next choose $\varepsilon > 0$ so small that

$$\varepsilon \max_{x\in\overline{\Omega}} e^{\delta x_1} \leq \frac{\theta}{2}$$

Let $\hat{x} \in \overline{\Omega}$ be the point such that $(u-v+\phi_\varepsilon)(\hat{x}) = \max_{\overline{\Omega}}(u-v+\phi_\varepsilon) \geq \theta$. By the choice of $\varepsilon > 0$, since $u \leq v$ on $\partial\Omega$, we see that $\hat{x} \in \Omega$.

From the definition of viscosity subsolutions, we have

$$\nu u(\hat{x}) + F(\hat{x}, D(v-\phi_\varepsilon)(\hat{x}), D^2(v-\phi_\varepsilon)(\hat{x})) \leq 0.$$

By the uniform ellipticity and (3.4), we have

$$\nu u(\hat{x}) + F(\hat{x}, Dv(\hat{x}), D^2v(\hat{x})) + \mathcal{P}^-(-D^2\phi_\varepsilon(\hat{x})) - \mu|D\phi_\varepsilon(\hat{x})| \leq 0.$$

Noting that $|D\phi_\varepsilon(\hat{x})| \leq \delta\varepsilon e^{\delta\hat{x}_1}$ and $\mathcal{P}^-(-D^2\phi_\varepsilon(\hat{x})) \geq \delta^2\varepsilon\lambda e^{\delta\hat{x}_1}$, we have

$$\nu u(\hat{x}) + F(\hat{x}, Dv(\hat{x}), D^2v(\hat{x})) + \delta\varepsilon(\lambda\delta-\mu)e^{\delta\hat{x}_1} \leq 0. \tag{3.5}$$

Since v is a classical supersolution of (3.1), by (3.5) and $\delta \geq (\mu+1)/\lambda$, we have

$$\nu(u-v)(\hat{x}) + \delta\varepsilon e^{\delta\hat{x}_1} \leq 0.$$

Hence, we have

$$\nu(\theta - \phi_\varepsilon(\hat{x})) \leq -\delta\varepsilon e^{\delta\hat{x}_1},$$

which gives a contradiction because $\delta \geq \nu + 1$. □

3.2 Comparison principle for first-order PDEs

In this subsection, without assuming that one of viscosity sub- and supersolutions is a classical one, we establish the comparison principle when F in (3.1) does not depend on D^2u; first-order PDEs. We will study the comparison principle for second-order ones in the next subsection.

In the viscosity solution theory, Theorem 3.4 below was the first surprising result.

Here, instead of (3.1), we shall consider the following PDE:

$$\nu u + H(x, Du) = 0 \quad \text{in } \Omega. \tag{3.6}$$

We shall suppose that

$$\nu > 0, \tag{3.7}$$

and that there is a continuous function $\omega_H : [0,\infty) \to [0,\infty)$ such that $\omega_H(0) = 0$ and

$$|H(x,p) - H(y,p)| \le \omega_H(|x-y|(1+|p|)) \quad \text{for } x,y \in \Omega \text{ and } p \in \mathbf{R}^n. \tag{3.8}$$

In what follows, we will call ω_H in (3.8) a modulus of continuity. For notational simplicity, we use the following notation:

$$\mathcal{M} := \{\omega : [0,\infty) \to [0,\infty) \mid \omega(\cdot) \text{ is continuous, } \omega(0) = 0\}.$$

Theorem 3.4. *Assume that* (3.7) *and* (3.8) *hold. Let* $u \in USC(\overline{\Omega})$ *and* $v \in LSC(\overline{\Omega})$ *be a viscosity sub- and supersolution of* (3.6), *respectively.*
If $u \le v$ *on* $\partial\Omega$, *then* $u \le v$ *in* $\overline{\Omega}$.

<u>*Proof.*</u> Suppose $\max_{\overline{\Omega}}(u - v) =: \theta > 0$ as usual. Then, we will get a contradiction.

Notice that since both u and v may not be differentiable, we **cannot** use the same argument as in Proposition 3.1.

Now, we present the most important idea in the theory of viscosity solutions to overcome this difficulty.

Setting $\Phi_\varepsilon(x,y) := u(x) - v(y) - (2\varepsilon)^{-1}|x-y|^2$ for $\varepsilon > 0$, we choose $(x_\varepsilon, y_\varepsilon) \in \overline{\Omega} \times \overline{\Omega}$ such that

$$\Phi_\varepsilon(x_\varepsilon, y_\varepsilon) = \max_{x,y\in\overline{\Omega}} \Phi_\varepsilon(x,y).$$

Noting that $\Phi_\varepsilon(x_\varepsilon, y_\varepsilon) \ge \max_{x\in\overline{\Omega}} \Phi_\varepsilon(x,x) = \theta$, we have

$$\frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon} \le u(x_\varepsilon) - v(y_\varepsilon) - \theta. \tag{3.9}$$

Since $\overline{\Omega}$ is compact, we can find $\hat{x}, \hat{y} \in \overline{\Omega}$, and $\varepsilon_k > 0$ such that $\lim_{k\to\infty} \varepsilon_k = 0$ and $\lim_{k\to\infty}(x_{\varepsilon_k}, y_{\varepsilon_k}) = (\hat{x}, \hat{y})$.

We shall simply write ε for ε_k (*i.e.* in what follows, "$\varepsilon \to 0$" means that $\varepsilon_k \to 0$ when $k \to \infty$).

Setting $M := \max_{\overline{\Omega}} u - \min_{\overline{\Omega}} v$, by (3.9), we have

$$|x_\varepsilon - y_\varepsilon|^2 \le 2\varepsilon M \to 0 \quad (\text{as } \varepsilon \to 0).$$

Thus, we have $\hat{x} = \hat{y}$.

Since (3.9) again implies

$$\begin{aligned} 0 \le \liminf_{\varepsilon\to 0} \frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon} &\le \limsup_{\varepsilon\to 0} \frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon} \\ &\le \limsup_{\varepsilon\to 0}(u(x_\varepsilon) - v(y_\varepsilon)) - \theta \\ &\le (u-v)(\hat{x}) - \theta \le 0, \end{aligned}$$

we have

$$\lim_{\varepsilon\to 0} \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} = 0. \tag{3.10}$$

Moreover, since $(u - v)(\hat{x}) = \theta > 0$, we have $\hat{x} \in \Omega$ from the assumption $u \le v$ on $\partial\Omega$. Thus, for small $\varepsilon > 0$, we may suppose that $(x_\varepsilon, y_\varepsilon) \in \Omega \times \Omega$.

Furthermore, ignoring the left hand side in (3.9), we have

$$\theta \le \liminf_{\varepsilon\to 0}(u(x_\varepsilon) - v(y_\varepsilon)). \tag{3.11}$$

Taking $\phi(x) := v(y_\varepsilon) + (2\varepsilon)^{-1}|x - y_\varepsilon|^2$, we see that $u - \phi$ attains its maximum at $x_\varepsilon \in \Omega$. Hence, from the definition of viscosity subsolutions, we have

$$\nu u(x_\varepsilon) + H\left(x_\varepsilon, \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}\right) \le 0.$$

On the other hand, taking $\psi(y) := u(x_\varepsilon) - (2\varepsilon)^{-1}|y - x_\varepsilon|^2$, we see that $v - \psi$ attains its minimum at $y_\varepsilon \in \Omega$. Thus, from the definition of viscosity supersolutions, we have

$$\nu v(y_\varepsilon) + H\left(y_\varepsilon, \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}\right) \ge 0.$$

The above two inequalities yield

$$\nu(u(x_\varepsilon) - v(y_\varepsilon)) \le \omega_H\left(|x_\varepsilon - y_\varepsilon| + \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon}\right).$$

Sending $\varepsilon \to 0$ in the above together with (3.10) and (3.11), we have $\nu\theta \le 0$, which is a contradiction. □

Remark. In the above proof, we could show that $\lim_{\varepsilon\to 0} u(x_\varepsilon) = u(\hat{x})$ and $\lim_{\varepsilon\to 0} v(y_\varepsilon) = v(\hat{x})$ although we do not need this fact. In fact, by (3.9), we have

$$v(y_\varepsilon) \le u(x_\varepsilon) - \theta,$$

which implies that

$$v(\hat{x}) \le \liminf_{\varepsilon\to 0} v(y_\varepsilon) \le \liminf_{\varepsilon\to 0} u(x_\varepsilon) - \theta \le \limsup_{\varepsilon\to 0} u(x_\varepsilon) - \theta \le u(\hat{x}) - \theta,$$

and

$$v(\hat{x}) \le \liminf_{\varepsilon\to 0} v(y_\varepsilon) \le \limsup_{\varepsilon\to 0} v(x_\varepsilon) \le \limsup_{\varepsilon\to 0} u(x_\varepsilon) - \theta \le u(\hat{x}) - \theta.$$

Hence, since all the inequalities become the equalities, we have

$$u(\hat{x}) = \liminf_{\varepsilon\to 0} u(x_\varepsilon) = \limsup_{\varepsilon\to 0} u(x_\varepsilon) \text{ and } v(\hat{x}) = \liminf_{\varepsilon\to 0} v(y_\varepsilon) = \limsup_{\varepsilon\to 0} v(y_\varepsilon).$$

We remark here that we cannot apply Theorem 3.4 to the eikonal equation (2.1) because we have to suppose $\nu > 0$ in the above proof.

We shall modify the above proof so that the comparison principle for viscosity solutions of (2.1) holds.

To simplify our hypotheses, we shall consider the following PDE:

$$H(x, Du) - f(x) = 0 \quad \text{in } \Omega. \tag{3.12}$$

Here, we suppose that H has homogeneous degree $\alpha > 0$ with respect to the second variable; there is $\alpha > 0$ such that

$$H(x, \mu p) = \mu^\alpha H(x, p) \quad \text{for } x \in \Omega, p \in \mathbf{R}^n \text{ and } \mu > 0. \tag{3.13}$$

To recover the lack of assumption $\nu > 0$, we suppose the positivity of $f \in C(\overline{\Omega})$; there is $\sigma > 0$ such that

$$\min_{x\in\overline{\Omega}} f(x) =: \sigma > 0. \tag{3.14}$$

<u>*Example.*</u> When $H(x, p) = |p|^2$ (*i.e.* $\alpha = 2$) and $f(x) \equiv 1$ (*i.e.* $\sigma = 1$), equation (3.12) becomes (2.1).

The second comparison principle for first-order PDEs is as follows:

Theorem 3.5. *Assume that* (3.8), (3.13) *and* (3.14) *hold. Let* $u \in USC(\overline{\Omega})$ *and* $v \in LSC(\overline{\Omega})$ *be a viscosity sub- and supersolution of* (3.12), *respectively.*

If $u \le v$ *on* $\partial\Omega$, *then* $u \le v$ *in* $\overline{\Omega}$.

Proof. Suppose that $\max_{\overline{\Omega}}(u - v) =: \theta > 0$ as usual. Then, we will get a contradiction.

If we choose $\mu \in (0, 1)$ so that

$$(1 - \mu) \max_{\overline{\Omega}} u \leq \frac{\theta}{2},$$

then we easily verify that

$$\max_{\overline{\Omega}}(\mu u - v) =: \tau \geq \frac{\theta}{2}.$$

We note that for any $z \in \overline{\Omega}$ such that $(\mu u - v)(z) = \tau$, we may suppose $z \in \Omega$. In fact, otherwise (*i.e.* $z \in \partial\Omega$), if we further suppose that $\mu < 1$ is close to 1 so that $-(1 - \mu) \min_{\partial\Omega} v \leq \theta/4$, then the assumption ($u \leq v$ on $\partial\Omega$) implies

$$\frac{\theta}{2} \leq \tau = \mu u(z) - v(z) \leq (\mu - 1)v(z) \leq \frac{\theta}{4},$$

which is a contradiction. For simplicity, we shall omit writing the dependence on μ for τ and $(x_\varepsilon, y_\varepsilon)$ below.

At this stage, we shall use the idea in the proof of Theorem 3.4: Consider the mapping $\Phi_\varepsilon : \overline{\Omega} \times \overline{\Omega} \to \mathbf{R}$ defined by

$$\Phi_\varepsilon(x, y) := \mu u(x) - v(y) - \frac{|x - y|^2}{2\varepsilon}.$$

Choose $(x_\varepsilon, y_\varepsilon) \in \overline{\Omega} \times \overline{\Omega}$ such that $\max_{x,y\in\overline{\Omega}} \Phi_\varepsilon(x, y) = \Phi_\varepsilon(x_\varepsilon, y_\varepsilon)$. Note that $\Phi_\varepsilon(x_\varepsilon, y_\varepsilon) \geq \tau \geq \theta/2$.

As in the proof of Theorem 3.4, we may suppose that $\lim_{\varepsilon\to 0}(x_\varepsilon, y_\varepsilon) = (\hat{x}, \hat{y})$ for some $(\hat{x}, \hat{y}) \in \overline{\Omega} \times \overline{\Omega}$ (by taking a subsequence if necessary). Also, we easily see that

$$\frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon} \leq \mu u(x_\varepsilon) - v(y_\varepsilon) - \tau \leq M_\mu := \mu \max_{\overline{\Omega}} u - \min_{\overline{\Omega}} v. \tag{3.15}$$

Thus, sending $\varepsilon \to 0$, we have $\hat{x} = \hat{y}$. Hence, (3.15) implies that $\mu u(\hat{x}) - v(\hat{x}) = \tau$, which yields $\hat{x} \in \Omega$ because of the choice of μ. Thus, we see that $(x_\varepsilon, y_\varepsilon) \in \Omega \times \Omega$ for small $\varepsilon > 0$.

Moreover, (3.15) again implies

$$\lim_{\varepsilon\to 0} \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} = 0. \tag{3.16}$$

Now, taking $\phi(x) := (v(y_\varepsilon) + (2\varepsilon)^{-1}|x - y_\varepsilon|^2)/\mu$, we see that $u - \phi$ attains its maximum at $x_\varepsilon \in \Omega$. Thus, we have

$$H\left(x_\varepsilon, \frac{x_\varepsilon - y_\varepsilon}{\mu\varepsilon}\right) \le f(x_\varepsilon).$$

Hence, by (3.13), we have

$$H\left(x_\varepsilon, \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}\right) \le \mu^\alpha f(x_\varepsilon). \tag{3.17}$$

On the other hand, taking $\psi(y) = \mu u(x_\varepsilon) - (2\varepsilon)^{-1}|y - x_\varepsilon|^2$, we see that $v - \psi$ attains its minimum at $y_\varepsilon \in \Omega$. Thus, we have

$$H\left(y_\varepsilon, \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}\right) \ge f(y_\varepsilon). \tag{3.18}$$

Combining (3.18) with (3.17), we have

$$\begin{aligned} f(y_\varepsilon) - \mu^\alpha f(x_\varepsilon) &\le H\left(y_\varepsilon, \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}\right) - H\left(x_\varepsilon, \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}\right) \\ &\le \omega_H\left(|x_\varepsilon - y_\varepsilon|\left(1 + \frac{|x_\varepsilon - y_\varepsilon|}{\varepsilon}\right)\right). \end{aligned}$$

Sending $\varepsilon \to 0$ in the above with (3.16), we have

$$(1 - \mu^\alpha) f(\hat{x}) \le 0,$$

which contradicts (3.14). $\square$

3.3 Extension to second-order PDEs

In this subsection, assuming a key lemma, we will present the comparison principle for fully nonlinear, second-order, (degenerate) elliptic PDEs (3.1).

We first remark that the argument of the proof of the comparison principle for first-order PDEs **cannot** be applied at least immediately.

Let us have a look at the difficulty. Consider the following simple PDE:

$$\nu u - \triangle u = 0, \tag{3.19}$$

where $\nu > 0$. As one can guess, if the argument does not work for this "easiest" PDE, then it must be hopeless for general PDEs.

However, we emphasize that the same argument as in the proof of Theorem 3.4 does not work. In fact, let $u \in USC(\overline{\Omega})$ and $v \in LSC(\overline{\Omega})$ be a viscosity sub- and supersolution of (3.19), respectively, such that $u \leq v$ on $\partial\Omega$. Setting $\Phi_\varepsilon(x, y) := u(x) - v(y) - (2\varepsilon)^{-1}|x - y|^2$ as usual, we choose $(x_\varepsilon, y_\varepsilon) \in \overline{\Omega} \times \overline{\Omega}$ so that $\max_{x,y\in\overline{\Omega}} \Phi_\varepsilon(x, y) = \Phi_\varepsilon(x_\varepsilon, y_\varepsilon) > 0$ as before.

We may suppose that $(x_\varepsilon, y_\varepsilon) \in \Omega \times \Omega$ converges to $(\hat{x}, \hat{x})$ (as $\varepsilon \to 0$) for some $\hat{x} \in \Omega$ such that $(u - v)(\hat{x}) > 0$. From the definitions of u and v, we have

$$\nu u(x_\varepsilon) - \frac{n}{\varepsilon} \leq 0 \leq \nu v(y_\varepsilon) + \frac{n}{\varepsilon}.$$

Hence, we only have

$$\nu(u(x_\varepsilon) - v(y_\varepsilon)) \leq \frac{2n}{\varepsilon},$$

which does not give any contradiction as $\varepsilon \to 0$.

How can we go beyond this difficulty ?

In 1983, P.-L. Lions first obtained the uniqueness of viscosity solutions for elliptic PDEs arising in stochastic optimal control problems (*i.e.* Bellman equations; F is convex in (Du, D^2u)). However, his argument heavily depends on stochastic representation of viscosity solutions as "value functions". Moreover, it seems hard to extend the result to Isaacs equations; F is fully nonlinear.

The breakthrough was done by Jensen in 1988 in case when the coefficients on the second derivatives of the PDE are constant. His argument relies purely on "real-analysis" and can work even for fully nonlinear PDEs.

Then, Ishii in 1989 extended Jensen's result to enable us to apply to elliptic PDEs with variable coefficients. We present here the so-called Ishii's lemma, which will be proved in Appendix.

Lemma 3.6. (Ishii's lemma) *Let u and w be in $USC(\overline{\Omega})$. For $\phi \in C^2(\overline{\Omega} \times \overline{\Omega})$, let $(\hat{x}, \hat{y}) \in \overline{\Omega} \times \overline{\Omega}$ be a point such that*

$$\max_{x,y\in\overline{\Omega}} (u(x) + w(y) - \phi(x, y)) = u(\hat{x}) + w(\hat{y}) - \phi(\hat{x}, \hat{y}).$$

Then, for each $\mu > 1$, there are $X = X(\mu), Y = Y(\mu) \in S^n$ such that

$$(D_x\phi(\hat{x}, \hat{y}), X) \in \overline{J}^{2,+}_{\overline{\Omega}} u(\hat{x}), \quad (D_y\phi(\hat{x}, \hat{y}), Y) \in \overline{J}^{2,+}_{\overline{\Omega}} w(\hat{y}),$$

and

$$-(\mu+\|A\|)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \le A + \frac{1}{\mu}A^2,$$

where $A = D^2\phi(\hat{x},\hat{y}) \in S^{2n}$.

Remark. We note that if we suppose that $u, w \in C^2(\overline{\Omega})$ and $(\hat{x},\hat{y}) \in \Omega\times\Omega$ in the hypothesis, then we easily have

$$X = D^2u(\hat{x}),\ Y = D^2w(\hat{y}),\ \text{and}\ \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \le A.$$

Thus, the last matrix inequality means that when u and w are only continuous, we get some error term $\mu^{-1}A^2$, where $\mu > 1$ will be large.

We also note that for $\phi(x,y) := |x-y|^2/(2\varepsilon)$, we have

$$A := D^2\phi(\hat{x},\hat{y}) = \frac{1}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \quad \text{and} \quad \|A\| = \frac{2}{\varepsilon}. \tag{3.20}$$

For the last identity, since

$$\|A\|^2 := \sup\left\{ \left\langle A\begin{pmatrix} x \\ y \end{pmatrix}, A\begin{pmatrix} x \\ y \end{pmatrix}\right\rangle \middle| \, |x|^2+|y|^2 = 1 \right\},$$

the triangle inequality yields $\|A\|^2 = 2\varepsilon^{-2}\sup\{|x-y|^2 \mid |x|^2+|y|^2 = 1\} \le 4/\varepsilon^2$. On the other hand, taking $x = -y$ (*i.e.* $|x|^2 = 1/2$) in the supremum of the definition of $\|A\|^2$ in the above, we have $\|A\|^2 \ge 4/\varepsilon^2$.

Remark. The other way to show the above identity, we may use the fact that for $B \in S^n$, in general,

$$\|B\| = \max\{|\lambda_k| \mid \lambda_k \text{ is the eigen-value of } B\}.$$

3.3.1 Degenerate elliptic PDEs

Now, we give our hypotheses on F, which is called the structure condition.

Structure condition
There is an $\omega_F \in \mathcal{M}$ such that if $X, Y \in S^n$ and $\mu > 1$ satisfy $-3\mu\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le 3\mu\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$, then $F(y,\mu(x-y),Y) - F(x,\mu(x-y),X) \le \omega_F(\lvert x-y\rvert(1+\mu\lvert x-y\rvert))$ for $x,y\in\Omega$.

(3.21)

In section 3.3.2, we will see that if F satisfies (3.21), then it is elliptic.

We first prove the comparison principle when (3.21) holds for F using this lemma. Afterward, we will explain why assumption (3.21) is reasonable.

Theorem 3.7. *Assume that* $\nu > 0$ *and* (3.21) *hold. Let* $u \in USC(\overline{\Omega})$ *and* $v \in LSC(\overline{\Omega})$ *be a viscosity sub- and supersolution of* (3.1), *respectively.*

If $u \leq v$ *on* $\partial\Omega$, *then* $u \leq v$ *in* $\overline{\Omega}$.

Proof. Suppose that $\max_{\overline{\Omega}}(u - v) =: \theta > 0$ as usual. Then, we will get a contradiction.

Again, for $\varepsilon > 0$, consider the mapping $\Phi_\varepsilon : \overline{\Omega} \times \overline{\Omega} \to \mathbf{R}$ defined by

$$\Phi_\varepsilon(x, y) = u(x) - v(y) - \frac{1}{2\varepsilon}|x - y|^2.$$

Let $(x_\varepsilon, y_\varepsilon) \in \overline{\Omega} \times \overline{\Omega}$ be a point such that $\max_{x,y\in\overline{\Omega}} \Phi_\varepsilon(x, y) = \Phi_\varepsilon(x_\varepsilon, y_\varepsilon) \geq \theta$. As in the proof of Theorem 3.4, we may suppose that

$$\lim_{\varepsilon\to 0}(x_\varepsilon, y_\varepsilon) = (\hat{x}, \hat{x}) \quad \text{for some } \hat{x} \in \Omega \quad (\textit{i.e. } x_\varepsilon, y_\varepsilon \in \Omega \text{ for small } \varepsilon > 0).$$

Moreover, since we have $(u - v)(\hat{x}) = \theta$,

$$\lim_{\varepsilon\to 0} \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} = 0, \tag{3.22}$$

and

$$\theta \leq \liminf_{\varepsilon\to 0}(u(x_\varepsilon) - v(y_\varepsilon)). \tag{3.23}$$

In view of Lemma 3.6 (taking $w := -v$, $\mu := 1/\varepsilon$, $\phi(x, y) = |x - y|^2/(2\varepsilon)$) and its Remark, we find $X, Y \in S^n$ such that

$$\left(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, X\right) \in \bar{J}^{2,+}u(x_\varepsilon), \quad \left(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, Y\right) \in \bar{J}^{2,-}v(y_\varepsilon),$$

and

$$-\frac{3}{\varepsilon}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \frac{3}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Thus, the equivalent definition in Proposition 2.6 implies that

$$\nu u(x_\varepsilon) + F\left(x_\varepsilon, \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, X\right) \leq 0 \leq \nu v(y_\varepsilon) + F\left(y_\varepsilon, \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, Y\right).$$

Hence, by virtue of our assumption (3.21), we have

$$\nu(u(x_\varepsilon) - v(y_\varepsilon)) \le \omega_F\left(|x_\varepsilon - y_\varepsilon| + \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon}\right). \tag{3.24}$$

Taking the limit infimum, as $\varepsilon \to 0$, together with (3.22) and (3.23) in the above, we have

$$\nu\theta \le 0,$$

which is a contradiction. □

3.3.2 Remarks on the structure condition

In order to ensure that assumption (3.21) is reasonable, we first present some examples. For this purpose, we consider the Isaacs equation as in section 1.2.2.

$$F(x,p,X) := \sup_{a\in A}\inf_{b\in B}\{L^{a,b}(x,p,X) - f(x,a,b)\},$$

where

$$L^{a,b}(x,p,X) := -\mathrm{trace}(A(x,a,b)X) + \langle g(x,a,b), p\rangle \quad \text{for } (a,b) \in A\times B.$$

If we suppose that A and B are compact sets in $\mathbf{R}^m$ (for some $m \ge 1$), and that the coefficients in the above and $f(\cdot,a,b)$ satisfy the hypotheses below, then F satisfies (3.21).

$$\begin{cases} (1) & \exists M_1 > 0 \text{ and } \exists \sigma_{ij}(\cdot,a,b) : \Omega \to \mathbf{R} \text{ such that } A_{ij}(x,a,b) = \\ & \displaystyle\sum_{k=1}^m \sigma_{ik}(x,a,b)\sigma_{jk}(x,a,b), \text{ and } |\sigma_{jk}(x,a,b) - \sigma_{jk}(y,a,b)| \le M_1|x-y| \\ & \text{for } x,y \in \Omega, i,j = 1,\ldots,n, k = 1,\ldots,m,\ a\in A, b\in B, \\ (2) & \exists M_2 > 0 \text{ such that } |g_i(x,a,b) - g_i(y,a,b)| \le M_2|x-y| \text{ for } x,y\in\Omega, \\ & i = 1,\ldots,n,\ a\in A, b\in B, \\ (3) & \exists \omega_f \in \mathcal{M} \text{ such that} \\ & |f(x,a,b) - f(y,a,b)| \le \omega_f(|x-y|) \text{ for } x,y\in\Omega, a\in A, b\in B. \end{cases}$$

We shall show (3.21) only when

$$F(x,p,X) := -\sum_{i,j=1}^n \sum_{k=1}^m \sigma_{ik}(x,a,b)\sigma_{jk}(x,a,b)X_{ij}$$

for a fixed $(a,b) \in A \times B$ because we can modify the proof below to general F.

Thus, we shall omit writing indices a and b.

To verify assumption (3.21), we choose $X, Y \in S^n$ such that

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3\mu \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Setting $\xi_k = {}^t(\sigma_{1k}(x), \ldots, \sigma_{nk}(x))$ and $\eta_k = {}^t(\sigma_{1k}(y), \ldots, \sigma_{nk}(y))$ for any fixed $k \in \{1, 2, \ldots, m\}$, we have

$$\begin{aligned} \left\langle \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \begin{pmatrix} \xi_k \\ \eta_k \end{pmatrix}, \begin{pmatrix} \xi_k \\ \eta_k \end{pmatrix} \right\rangle &\leq 3\mu \left\langle \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \begin{pmatrix} \xi_k \\ \eta_k \end{pmatrix}, \begin{pmatrix} \xi_k \\ \eta_k \end{pmatrix} \right\rangle \\ &= 3\mu |\xi_k - \eta_k|^2 \\ &\leq 3\mu n M_1^2 |x-y|^2. \end{aligned}$$

Therefore, taking the summation over $k \in \{1, \ldots, m\}$, we have

$$\begin{aligned} F(y, \mu(x-y), Y) - F(x, \mu(x-y), X) &\leq \sum_{i,j=1}^n (-A_{ij}(y) Y_{ij} + A_{ij}(x) X_{ij}) \\ &= \sum_{k=1}^m (-\langle Y\eta_k, \eta_k \rangle + \langle X\xi_k, \xi_k \rangle) \\ &\leq 3\mu m n M_1^2 |x-y|^2. \quad \square \end{aligned}$$

We next give other reasons why (3.21) is a suitable assumption. The reader can skip the proof of the following proposition if he/she feels that the above reason is enough to adapt (3.21).

Proposition 3.8. (1) (3.21) *implies ellipticity.*
(2) *Assume that F is uniformly elliptic. If $\bar\omega \in \mathcal{M}$ satisfies that* $\sup_{r\geq 0} \bar\omega(r)/(r+1) < \infty$, *and*

$$|F(x,p,X) - F(y,p,X)| \leq \bar\omega(|x-y|(\|X\| + |p| + 1)) \tag{3.25}$$

for $x, y \in \Omega, p \in \mathbf{R}^n, X \in S^n$, then (3.21) *holds for F.*

<u>*Proof.*</u> For a proof of (1), we refer to Remark 3.4 in [6].

For the reader's convenience, we give a proof of (2) which is essentially used in a paper by Ishii-Lions (1990). Let $X, Y \in S^n$ satisfy the matrix inequality in (3.21). Note that $X \leq Y$.

Multiplying $\begin{pmatrix} -I & -I \\ -I & I \end{pmatrix}$ to the last matrix inequality from both sides, we have

$$\begin{pmatrix} X-Y & X+Y \\ X+Y & X-Y \end{pmatrix} \leq 12\mu \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}.$$

Thus, multiplying $\begin{pmatrix} \xi \\ s\eta \end{pmatrix}$ for $s \in \mathbf{R}$ and $\xi, \eta \in \mathbf{R}^n$ with $|\eta| = |\xi| = 1$, we see that

$$0 \leq (12\mu - \langle (X-Y)\eta, \eta \rangle)s^2 - 2\langle (X+Y)\xi, \eta \rangle s - \langle (X-Y)\xi, \xi \rangle.$$

Hence, we have

$$|\langle (X+Y)\xi, \eta \rangle|^2 \leq |\langle (X-Y)\xi, \xi \rangle|(12\mu + |\langle (X-Y)\eta, \eta \rangle|),$$

which implies

$$\|X+Y\| \leq \|X-Y\|^{1/2}(12\mu + \|X-Y\|)^{1/2}.$$

Thus, we have

$$\|X\| \leq \frac{1}{2}(\|X-Y\| + \|X+Y\|) \leq \|X-Y\|^{1/2}(6\mu + \|X-Y\|)^{1/2}.$$

Since $X \leq Y$ (*i.e.* the eigen-values of $X-Y$ are non-positive), we see that

$$F(y,p,X) - F(y,p,Y) \geq \mathcal{P}^-(X-Y) \geq \lambda \|X-Y\|. \tag{3.26}$$

For the last inequality, we recall Remark after Lemma 3.6.

Since we may suppose $\bar{\omega}$ is concave, for any fixed $\varepsilon > 0$, there is $M_\varepsilon > 0$ such that $\bar{\omega}(r) \leq \varepsilon + M_\varepsilon r$ and $\bar{\omega}(r) = \inf_{\varepsilon>0}(\varepsilon + M_\varepsilon r)$ for $r \geq 0$. By (3.25) and (3.26), since $\|X\| \leq 3\mu$ and $\|Y\| \leq 3\mu$, we have

$$\begin{array}{rl} & F(y,p,Y) - F(x,p,X) \\ \leq & \varepsilon + M_\varepsilon |x-y|(|p|+1) + \sup\limits_{0\leq t\leq 6\mu} \left\{ M_\varepsilon |x-y| t^{1/2}(6\mu + t)^{1/2} - \lambda t \right\}. \end{array}$$

Noting that

$$M_\varepsilon |x-y| t^{1/2}(6\mu+t)^{1/2} - \lambda t \leq \frac{3}{\lambda} M_\varepsilon^2 \mu |x-y|^2,$$

we have

$$\begin{array}{rl} & F(y,\mu(x-y),Y) - F(x,\mu(x-y),X) \\ \leq & \varepsilon + M_\varepsilon |x-y|(\mu|x-y|+1) + 3\lambda^{-1} M_\varepsilon^2 \mu |x-y|^2, \end{array}$$

which implies the assertion by taking the infimum over $\varepsilon > 0$. □

3.3.3 Uniformly elliptic PDEs

We shall give a comparison result corresponding to Proposition 3.3; F is uniformly elliptic and $\nu \geq 0$.

Theorem 3.9. *Assume that* (3.2), (3.3), (3.4) *and* (3.21) *hold. Assume also that F is uniformly elliptic. Let $u \in USC(\overline{\Omega})$ and $v \in LSC(\overline{\Omega})$ be a viscosity sub- and supersolution of* (3.1), *respectively.*

If $u \leq v$ on $\partial\Omega$, then $u \leq v$ in $\overline{\Omega}$.

Remark. As in Proposition 3.3, we may suppose $\nu = 0$.

Proof. Suppose that $\max_{\overline{\Omega}}(u - v) =: \theta > 0$.

Setting $\sigma := (\mu + 1)/\lambda$, we choose $\delta > 0$ so that

$$\delta \max_{x \in \overline{\Omega}} e^{\sigma x_1} \leq \frac{\theta}{2}.$$

We then set $\tau := \max_{x\in\overline{\Omega}}(u(x) - v(x) + \delta e^{\sigma x_1}) \geq \theta > 0$.

Putting $\phi(x, y) := (2\varepsilon)^{-1}|x - y|^2 - \delta e^{\sigma x_1}$, we let $(x_\varepsilon, y_\varepsilon) \in \overline{\Omega} \times \overline{\Omega}$ be the maximum point of $u(x) - v(y) - \phi(x, y)$ over $\overline{\Omega} \times \overline{\Omega}$.

By the compactness of $\overline{\Omega}$, we may suppose that $(x_\varepsilon, y_\varepsilon) \to (\hat{x}, \hat{y}) \in \overline{\Omega} \times \overline{\Omega}$ as $\varepsilon \to 0$ (taking a subsequence if necessary). Since $u(x_\varepsilon) - v(y_\varepsilon) \geq \phi(x_\varepsilon, y_\varepsilon)$, we have $|x_\varepsilon - y_\varepsilon|^2 \leq 2\varepsilon(\max_{\overline{\Omega}} u - \min_{\overline{\Omega}} v + 2^{-1}\theta)$ and moreover, $\hat{x} = \hat{y}$. Hence, we have

$$u(\hat{x}) - v(\hat{x}) + \delta e^{\sigma \hat{x}_1} \geq \tau,$$

which implies $\hat{x} \in \Omega$ because of our choice of δ. Thus, we may suppose that $(x_\varepsilon, y_\varepsilon) \in \Omega \times \Omega$ for small $\varepsilon > 0$. Moreover, as before, we see that

$$\lim_{\varepsilon \to 0} \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} = 0. \tag{3.27}$$

Applying Lemma 3.6 to $\hat{u}(x) := u(x) + \delta e^{\sigma x_1}$ and $-v(y)$, we find $X, Y \in S^n$ such that $((x_\varepsilon - y_\varepsilon)/\varepsilon, X) \in \overline{J}^{2,+}\hat{u}(x_\varepsilon)$, $((x_\varepsilon - y_\varepsilon)/\varepsilon, Y) \in \overline{J}^{2,-}v(y_\varepsilon)$, and

$$-\frac{3}{\varepsilon}\begin{pmatrix} I & O \\ O & I \end{pmatrix} \leq \begin{pmatrix} X & O \\ O & -Y \end{pmatrix} \leq \frac{3}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

We shall simply write x and y for x_ε and y_ε, respectively.

Note that Proposition 2.7 implies

$$\left(\frac{x-y}{\varepsilon} - \delta\sigma e^{\sigma x_1} e_1, X - \delta\sigma^2 e^{\sigma x_1} I_1\right) \in \overline{J}^{2,+} u(x),$$

where $e_1 \in \mathbf{R}^n$ and $I_1 \in S^n$ are given by

$$e_1 := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{and} \quad I_1 := \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Setting $r := \delta\sigma e^{\sigma x_1}$, from the definition of u and v, we have

$$0 \le F\left(y, \frac{x-y}{\varepsilon}, Y\right) - F\left(x, \frac{x-y}{\varepsilon} - re_1, X - \sigma r I_1\right).$$

In view of the uniform ellipticity and (3.4), we have

$$0 \le r\mu + \sigma r \mathcal{P}^+(I_1) + F\left(y, \frac{x-y}{\varepsilon}, Y\right) - F\left(x, \frac{x-y}{\varepsilon}, X\right).$$

Hence, by (3.21) and the definition of $\mathcal{P}^+$, we have

$$0 \le r(\mu - \sigma\lambda) + \omega_F\left(|x-y| + \frac{|x-y|^2}{\varepsilon}\right),$$

which together with (3.27) yields $0 \le \delta\sigma e^{\sigma \hat{x}_1}(\mu - \sigma\lambda)$. This is a contradiction because of our choice of $\sigma > 0$. $\Box$