7 Appendix

In this appendix, we present proofs of the propositions, which appeared in the previous sections. However, to prove them, we often need more fundamental results, for which we only give references. One of such results is the following "Area formula", which will be employed in sections 7.1 and 7.2. We refer to [9] for a proof of a more general Area formula.

Area formula
$\left.\begin{array}{c} \xi \in C^1(\mathbf{R}^n, \mathbf{R}^n), \\ g \in L^1(\mathbf{R}^n), \\ A \subset \mathbf{R}^n \text{ measurable} \end{array}\right\} \Longrightarrow \int_{\xi(A)} \lvert g(y) \rvert dy \le \int_A \lvert g(\xi(x)) \rvert \lvert \det(D\xi(x)) \rvert dx$

We note that the Area formula is a change of variable formula when $|\det(D\xi)|$ may vanish. In fact, the equality holds if $|\det(D\xi)| > 0$ and ξ is injective.

7.1 Proof of Ishii's lemma

First of all, we recall an important result by Aleksandrov. We refer to the Appendix of [6] and [10] for a "functional analytic" proof, and to [9] for a "measure theoretic" proof.

Lemma 7.1. (Theorem A.2 in [6]) *If $f : \mathbf{R}^n \to \mathbf{R}$ is convex, then for a.a. $x \in \mathbf{R}^n$, there is $(p, X) \in \mathbf{R}^n \times S^n$ such that*

$$f(x+h) = f(x) + \langle p, h \rangle + \frac{1}{2}\langle Xh, h \rangle + o(|h|^2) \quad \text{as } |h| \to 0.$$

(i.e., f is twice differentiable at a.a. $x \in \mathbf{R}^n$.)

We next recall Jensen's lemma, which is a version of the ABP maximum principle in 7.2 below.

Lemma 7.2. (Lemma A.3 in [6]) *Let $f : \mathbf{R}^n \to \mathbf{R}$ be semi-convex (i.e. $x \to f(x) + C_0|x|^2$ is convex for some $C_0 \in \mathbf{R}$). Let $\hat{x} \in \mathbf{R}^n$ be a strict maximum point of f. Set $f_p(x) := f(x) - \langle p, x \rangle$ for $x \in \mathbf{R}^n$ and $p \in \mathbf{R}^n$.*

Then, for $r > 0$, there are $C_1, \delta_0 > 0$ such that

$$|\Gamma_{r,\delta}| \ge C_1 \delta^n \quad \text{for } \delta \in (0, \delta_0],$$

where

$$\Gamma_{r,\delta} := \left\{ x \in B_r(\hat{x}) \,\middle|\, \exists p \in \overline{B}_\delta \textit{ such that } f_p(y) \le f_p(x) \textit{ for } y \in B_r(\hat{x}) \right\}.$$

<u>*Proof.*</u> By translation, we may suppose $\hat{x} = 0$.

For integers m, we set $f^m(x) = f * \rho_{1/m}(x)$, where $\rho_{1/m}$ is the mollifier. Note that $x \to f^m(x) + C_0|x|^2$ is convex.

Setting

$$\Gamma^m_{r,\delta} = \left\{ x \in B_r \,\middle|\, \exists p \in \overline{B}_\delta \text{ such that } f^m_p(y) \le f^m_p(x) \text{ for } y \in \overline{B}_r \right\},$$

where $f^m_p(x) = f^m(x) - \langle p, x\rangle$, we claim that there are $C_1, \delta_0 > 0$, independent of large integers m, such that

$$|\Gamma^m_{r,\delta}| \ge C_1 \delta^n \quad \text{for } \delta \in (0, \delta_0].$$

We remark that this concludes the assertion. In fact, setting $A_m := \cup_{k=m}^\infty \Gamma^k_{r,\delta}$, we have $\cap_{m=1}^\infty A_m \subset \Gamma_{r,\delta}$. Because, for $x \in \cap_{m=1}^\infty A_m$, we can select $p_k \in \overline{B}_\delta$ and m_k such that $\lim_{k\to\infty} m_k = \infty$, and

$$\max_{\overline{B}_r} f^{m_k}_{p_k} = f^{m_k}_{p_k}(x).$$

Hence, sending $k \to \infty$ (along a subsequence if necessary), we find $\hat{p} \in \overline{B}_\delta$ such that $\max_{\overline{B}_r} f_{\hat{p}} = f_{\hat{p}}(x)$, which yields $x \in \Gamma_{r,\delta}$.

Therefore, we have

$$C_1\delta^n \le \lim_{m\to\infty} |A_m| = |\cap_{m=1}^\infty A| \le |\Gamma_{r,\delta}|.$$

Now we shall prove our claim. First of all, we notice that $x \to f^m(x) + C_0|x|^2$ is convex.

Since 0 is the strict maximum of f, we find $\varepsilon_0 > 0$ such that

$$\varepsilon_0 = f(0) - \max_{\overline{B}_{4r/3} \setminus B_{r/3}} f.$$

Fix $p \in \overline{B}_{\delta_0}$, where $\delta_0 = \varepsilon_0/(3r)$. For $m \ge 3/r$, we note that

$$f^m(x) - \langle p, x\rangle \le f(0) - \varepsilon_0 + \delta_0 r \le f(0) - \frac{2\varepsilon_0}{3} \quad \text{in } \overline{B}_r \setminus B_{2r/3}.$$

On the other hand, for large m, we verify that

$$f^m(0) \geq f(0) - \omega_f(m^{-1}) > f(0) - \frac{\varepsilon_0}{3},$$

where ω_f denotes the modulus of continuity of f. Hence, in view of these observations, for any $p \in \overline{B}_{\delta_0}$, if $\max_{\overline{B}_r} f_p^m = f_p^m(x)$ for $x \in \overline{B}_r$, then $x \in B_r$. In other words, we see that

$$\overline{B}_\delta = Df^m(\Gamma_{r,\delta}^m) \quad \text{for } \delta \in (0, \delta_0].$$

Thanks to the Area formula, we have

$$|B_\delta| = \int_{Df^m(\Gamma_{r,\delta}^m)} dy \leq \int_{\Gamma_{r,\delta}^m} |\mathrm{det} D^2 f^m| dx \leq (2C_0)^n |\Gamma_{r,\delta}^m|.$$

Here, we have employed that $-2C_0 I \leq D^2 f^m \leq O$ in $\Gamma_{r,\delta}^m$. □

Although we can find a proof of the next proposition in [6], we recall the proof with a minor change for the reader's convenience.

Proposition 7.3. (Lemma A.4 in [6]) *If $f \in C(\mathbf{R}^m)$, $B \in S^m$, $\xi \to f(\xi) + (\lambda/2)|\xi|^2$ is convex and $\max_{\xi \in \mathbf{R}^m}\{f(\xi) - 2^{-1}\langle B\xi, \xi\rangle\} = f(0)$, then there is an $X \in S^m$ such that*

$$(0, X) \in \overline{J}^{2,+} f(0) \cap \overline{J}^{2,-} f(0) \quad \text{and} \quad -\lambda I \leq X \leq B.$$

Proof. For any $\delta > 0$, setting $f_\delta(\xi) := f(\xi) - 2^{-1}\langle B\xi, \xi\rangle - \delta|\xi|^2$, we notice that the semi-convex f_δ attains its strict maximum at $\xi = 0$.

In view of Lemmas 7.1 and 7.2, there are $\xi_\delta, q_\delta \in B_\delta$ such that $\xi \to f_\delta(\xi) + \langle q_\delta, \xi\rangle$ has a maximum at ξ_δ, at which f is twice differentiable.

It is easy to see that $Df(\xi_\delta) \to 0$ (as $\delta \to 0$) and, moreover, from the convexity of $\xi \to f(\xi) + (\lambda/2)|\xi|^2$,

$$-\lambda I \leq D^2 f(\xi_\delta) \leq B + 2\delta I.$$

Noting $(Df(\xi_\delta), D^2 f(\xi_\delta)) \in J^{2,+} f(\xi_\delta) \cap J^{2,-} f(\xi_\delta)$, we conclude the assertion by taking the limit as $\delta \to 0$. □

We next give a "magic" property of sup-convolutions. For the reader's convenience, we put the proof of [6].

Lemma 7.4. (Lemma A.5 in [6]) *For $v \in USC(\mathbf{R}^n)$ with $\sup_{\mathbf{R}^n} v < \infty$ and $\lambda > 0$, we set*

$$\hat{v}(\xi) := \sup_{x \in \mathbf{R}^n} \left(v(x) - \frac{\lambda}{2}|x - \xi|^2 \right).$$

For $\eta, q \in \mathbf{R}^n$, $Y \in S^n$, and $(q, Y) \in J^{2,+}\hat{v}(\eta)$, we have

$$(q, Y) \in J^{2,+}v(\eta + \lambda^{-1}q) \quad \text{and} \quad \hat{v}(\eta) + \frac{|q|^2}{2\lambda} = v(\eta + \lambda^{-1}q).$$

In particular, if $(0, Y) \in \overline{J}^{2,+}\hat{v}(0)$, then $(0, Y) \in \overline{J}^{2,+}v(0)$.

<u>*Proof.*</u> For $(q, Y) \in J^{2,+}\hat{v}(\eta)$, we choose $y \in \mathbf{R}^n$ such that

$$\hat{v}(\eta) = v(y) - \frac{\lambda}{2}|y - \eta|^2.$$

Thus, from the definition, we see that for any $x, \xi \in \mathbf{R}^n$,

$$\begin{array}{rcl} v(x) - \frac{\lambda}{2}|\xi - x|^2 \leq \hat{v}(\xi) & \leq & \hat{v}(\eta) + \langle q, \xi - \eta \rangle \\ & & + \frac{1}{2}\langle Y(\xi - \eta), \xi - \eta \rangle + o(|\xi - \eta|^2) \\ & = & v(y) - \frac{\lambda}{2}|y - \eta|^2 + \langle q, \xi - \eta \rangle \\ & & + \frac{1}{2}\langle Y(\xi - \eta), \xi - \eta \rangle + o(|\xi - \eta|^2). \end{array}$$

Taking $\xi = x - y + \eta$ in the above, we have $(q, Y) \in J^{2,+}v(y)$.

To verify that $y = \eta + \lambda^{-1}q$, putting $x = y$ and $\xi = \eta - \varepsilon(\lambda(\eta - y) + q)$ for $\varepsilon > 0$ in the above again, we have

$$\varepsilon|\lambda(\eta - y) + q|^2 \leq o(\varepsilon),$$

which yield $y = \eta + \frac{1}{\lambda}q$.

When $(0, Y) \in \overline{J}^{2,+}\hat{v}(0)$, we can choose (η_k, q_k, Y_k) such that $\lim_{k\to\infty}(\eta_k, \hat{v}(\eta_k), q_k, Y_k) = (0, \hat{v}(0), 0, O)$, and $(q_k, Y_k) \in J^{2,+}\hat{v}(\eta_k)$. Since $(q_k, Y_k) \in J^{2,+}v(\eta_k + \lambda^{-1}q_k)$ and $\hat{v}(\eta_k) + (2\lambda)^{-1}|q_k|^2 = v(\eta_k + \lambda^{-1}q_k)$, sending $k \to \infty$, we have $(0, Y) \in \overline{J}^{2,+}v(0)$. $\square$

Proof of Lemma 3.6. First of all, extending upper semi-continuous functions u, w in $\overline{\Omega}$ into $\mathbf{R}^n$ by $-\infty$ in $\mathbf{R}^n \setminus \overline{\Omega}$, we shall work in $\mathbf{R}^n \times \mathbf{R}^n$ instead of $\overline{\Omega} \times \overline{\Omega}$.

By translation, we may suppose that $\hat{x} = \hat{y} = 0$, at which $u(x) + w(y) - \phi(x, y)$ attains its maximum.

Furthermore, replacing $u(x)$, $w(y)$ and $\phi(x, y)$, respectively, by

$$u(x) - u(0) - \langle D_x\phi(0,0), x\rangle, \quad w(y) - w(0) - \langle D_y\phi(0,0), y\rangle$$

and

$$\phi(x, y) - \phi(0,0) - \langle D_x\phi(0,0), x\rangle - \langle D_y\phi(0,0), y\rangle,$$

we may also suppose that $\phi(0,0) = u(0) = w(0) = 0$ and $D\phi(0,0) = (0,0) \in \mathbf{R}^n \times \mathbf{R}^n$.

Since $\phi(x, y) = \left\langle \frac{A}{2}\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right\rangle + o(|x|^2+|y|^2)$, where $A := D^2\phi(0,0) \in S^{2n}$, for each $\eta > 0$, we see that the mapping $(x, y) \to u(x) + w(y) - \frac{1}{2}\left\langle (A + \eta I)\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right\rangle$ attains its (strict) maximum at $0 \in \mathbf{R}^{2n}$.

We will show the assertion for $A + \eta I$ in place of A. Then, sending $\eta \to 0$, we can conclude the proof. Therefore, we need to prove the following:

Simplified version of Ishii's lemma.
For upper semi-continuous functions u and w in $\mathbf{R}^n$, we suppose that $u(x) + w(y) - \left\langle \frac{A}{2}\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right\rangle \leq u(0) + w(0) = 0$ in $\mathbf{R}^n \times \mathbf{R}^n$. Then, for each $\mu > 1$, there are $X, Y \in S^n$ such that $(0, X) \in \overline{J}^{2,+}u(0)$, $(0, Y) \in \overline{J}^{2,+}w(0)$ and $-(\mu + \|A\|)\begin{pmatrix} I & O \\ O & I \end{pmatrix} \leq \begin{pmatrix} X & O \\ O & Y \end{pmatrix} \leq A + \frac{1}{\mu}A^2$.

Proof of the simplified version of Lemma 3.6. Since Hölder's inequality implies

$$\left\langle A\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right\rangle \leq \left\langle \left(A + \frac{1}{\mu}A^2\right)\begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix}\right\rangle + (\mu + \|A\|)(|x-\xi|^2 + |y-\eta|^2)$$

for $x, y, \xi, \eta \in \mathbf{R}^n$ and $\mu > 0$, setting $\lambda = \mu + \|A\|$, we have

$$u(x) - \frac{\lambda}{2}|x-\xi|^2 + w(y) - \frac{\lambda}{2}|y-\eta|^2 \leq \frac{1}{2}\left\langle \left(A + \frac{1}{\mu}A^2\right)\begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix}\right\rangle.$$

Using the notation in Lemma 7.4, we denote by $\hat{u}$ and $\hat{w}$ the sup-convolution of u and w, respectively, with the above $\lambda > 0$. Thus, we have

$$\hat{u}(\xi) + \hat{w}(\eta) \le \frac{1}{2}\left\langle \left(A + \frac{1}{\mu}A^2\right)\begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix}\right\rangle \quad \text{for all } \xi, \eta \in \mathbf{R}^n.$$

Since $\hat{u}(0) \ge u(0) = 0$ and $\hat{w}(0) \ge w(0) = 0$, the above inequality implies $\hat{u}(0) = \hat{w}(0) = 0$.

In view of Proposition 7.3 with $m = 2n$, $f(\xi, \eta) = \hat{u}(\xi) + \hat{w}(\eta)$ and $B = A + \mu^{-1}A^2$, there is $Z \in S^{2n}$ such that $(0, Z) \in \overline{J}^{2,+} f(0,0) \cap \overline{J}^{2,-} f(0,0)$ and $-\lambda I \le Z \le B$.

Hence, from the definition of $\overline{J}^{2,\pm}$, it is easy to verify that there are $X, Y \in S^n$ such that $(0, X) \in \overline{J}^{2,+}\hat{u}(0) \cap \overline{J}^{2,-}\hat{u}(0)$, $(0, Y) \in \overline{J}^{2,+}\hat{w}(0) \cap \overline{J}^{2,-}\hat{w}(0)$, and

$$Z = \begin{pmatrix} X & O \\ O & Y \end{pmatrix}.$$

Applying the last property in Lemma 7.4 to $\hat{u}$ and $\hat{w}$, we see that

$$(0, X) \in \overline{J}^{2,+} u(0) \quad \text{and} \quad (0, Y) \in \overline{J}^{2,+} w(0). \quad \Box$$

7.2 Proof of the ABP maximum principle

First of all, we remind the readers of our strategy in this and the next subsections.

We first show that the ABP maximum principle holds under $f \in L^n(\Omega) \cap C(\Omega)$ in Steps 1 and 2 of this subsection. Next, using this fact, we establish the existence of L^p-strong solutions of "Pucci" equations in the next subsection when $f \in L^p(\Omega)$.

Employing this existence result, in Step 3, we finally prove Proposition 6.2; the ABP maximum principle when $f \in L^n(\Omega)$.

ABP maximum principle for $f \in L^n(\Omega) \cap C(\Omega)$ (Section 7.2)
$\Downarrow$
Existence of L^p-strong solutions of Pucci equations (Section 7.3)
$\Downarrow$
ABP maximum principle for $f \in L^n(\Omega)$ (Section 7.2)

Proof of Proposition 6.2. We give a proof in [5] for the subsolution assertion of Proposition 6.2.

By scaling, we may suppose that $\mathrm{diam}(\Omega) \le 1$.

Setting

$$r_0 := \max_{\overline{\Omega}} u - \max_{\partial\Omega} u^+,$$

we may also suppose that $r_0 > 0$ since otherwise, the conclusion is obvious.

We first introduce the following notation: For $u : \Omega \to \mathbf{R}$ and $r \ge 0$,

$$\Gamma_r := \left\{ x \in \Omega \,\middle|\, \exists p \in \overline{B}_r \text{ such that } u(y) \le u(x) + \langle p, y - x\rangle \text{ for } y \in \Omega \right\} .$$

Recalling the upper contact set in section 6.2, we note that

$$\Gamma[u, \Omega] = \bigcup_{r>0} \Gamma_r.$$

Step 1: $u \in C^2(\Omega) \cap C(\overline{\Omega})$. We first claim that for $r \in (0, r_0)$,

$$\begin{cases} (i) & \overline{B}_r = Du(\Gamma_r), \\ (ii) & D^2 u \le O \quad \text{in } \Gamma_r. \end{cases} \tag{7.1}$$

To show (i), for $p \in \overline{B}_r$, we take $\hat{x} \in \overline{\Omega}$ such that $u(\hat{x}) - \langle p, \hat{x}\rangle = \max_{x\in\overline{\Omega}}(u(x) - \langle p, x\rangle)$. Since $u(x) - u(\hat{x}) \le r < r_0$ for $x \in \overline{\Omega}$, taking the maximum over $\overline{\Omega}$, we have $\hat{x} \in \Omega$. Hence, we see $p = Du(\hat{x})$, which concludes (i).

For $x \in \Gamma_r$, Taylor's formula yields

$$u(y) = u(x) + \langle Du(x), y - x\rangle + \frac{1}{2}\langle D^2u(x)(y - x), y - x\rangle + o(|y - x|^2).$$

Hence, we have $0 \ge \langle D^2u(x)(y - x), y - x\rangle + o(|y - x|^2)$, which shows (ii).

Now, we introduce functions $g_\kappa(p) := \left(|p|^{n/(n-1)} + \kappa^{n/(n-1)}\right)^{1-n}$ for $\kappa > 0$. We shall simply write g for g_κ.

Thus, for $r \in (0, r_0)$, we see that

$$\begin{aligned} \int_{Du(\Gamma_r)} g(p)dp \le & \int_{\Gamma_r} g(Du(x))|\det(D^2u(x))|dx \\ = & \int_{\Gamma_r} \left(|Du|^{n/(n-1)} + \kappa^{n/(n-1)}\right)^{1-n} |\det D^2u(x)|dx. \end{aligned}$$

Recalling (7.1), we utilize $|\mathrm{det}D^2u| \le (-\mathrm{trace}(D^2u)/n)^n$ in Γ_r to find $C>0$ such that

$$\int_{B_r} g(p)dp \le C\int_{\Gamma_r}\left(|Du|^{n/(n-1)}+\kappa^{n/(n-1)}\right)^{1-n}(-\mathrm{trace}(D^2u))^n dx. \quad (7.2)$$

Thus, since $(\mu|Du|+f^+)^n \le g(Du)^{-1}(\mu^n+\kappa^{-n}(f^+)^n)$ by Hölder's inequality, we have

$$\int_{B_r} g(p)dp \le C\int_{\Gamma_r}\left(\mu^n+\left(\frac{f^+}{\kappa}\right)^n\right)dx. \quad (7.3)$$

On the other hand, since $(|p|^n+\kappa^n)^{-1}\le g(p)$, we have

$$\log\left(\left(\frac{r}{\kappa}\right)^n+1\right)\le C\int_{B_r}\frac{1}{|p|^n+\kappa^n}dp \le C\int_{B_r} g(p)dp.$$

Hence, noting $\Gamma_r\subset\Omega^+[u]$ for $r\in(0,r_0)$, by (7.3), we have

$$r\le\kappa\left[\exp\left\{C\int_{\Gamma[u,\Omega]\cap\Omega^+[u]}\left(\mu^n+\left(\frac{f^+}{\kappa}\right)^n\right)dx\right\}-1\right]^{1/n}. \quad (7.4)$$

When $\|f^+\|_{L^n(\Gamma[,\Omega]\cap\Omega^+[u])}=0$, then sending $\kappa\to 0$, we get a contradiction. Thus, we may suppose that $\|f^+\|_{L^n(\Gamma[,\Omega]\cap\Omega^+[u])}>0$.

Setting $\kappa:=\|f^+\|_{L^n(\Gamma[u,\Omega]\cap\Omega^+[u])}$ and $r:=r_0/2$, we can find $C>0$, independent of u, such that $r_0\le C\|f^+\|_{L^n(\Gamma[u,\Omega]\cap\Omega^+[u])}$.

Remark. We note that we do not need to suppose f to be continuous in Step 1 while we need it in the next step.

Step 2: $u\in C(\overline{\Omega})$ and $f\in L^n(\Omega)\cap C(\Omega)$. First of all, because of $f\in C(\Omega)$, we remark that u is a "standard" viscosity subsolution of

$$\mathcal{P}^-(D^2u)-\mu|Du|\le f \quad \text{in } \Omega^+[u].$$

(See Proposition 2.9 in [5].)

Let u^ε be the sup-convolution of u for $\varepsilon>0$;

$$u^\varepsilon(x):=\sup_{y\in\Omega}\left\{u(y)-\frac{|x-y|^2}{2\varepsilon}\right\}.$$

Note that u^ε is semi-convex and thus, twice differentiable *a.e.* in $\mathbf{R}^n$.

We claim that for small $\varepsilon > 0$, u^ε is a viscosity subsolution of

$$\mathcal{P}^-(D^2u^\varepsilon) - \mu|Du^\varepsilon| \leq f^\varepsilon \quad \text{in } \Omega_\varepsilon, \tag{7.5}$$

where $f^\varepsilon(x) := \sup\{f^+(y) \mid |x-y| \leq 2(\|u\|_{L^\infty(\Omega)}\varepsilon)^{1/2}\}$ and $\Omega_\varepsilon := \{x \in \Omega^+[u] \mid \mathrm{dist}(x, \partial\Omega^+[u]) > 2(\|u\|_{L^\infty(\Omega)}\varepsilon)^{1/2}\}$. Indeed, for $x \in \Omega_\varepsilon$ and $(q, X) \in J^{2,+}u^\varepsilon(x)$, choosing $\hat{x} \in \overline{\Omega}$ such that $u^\varepsilon(x) = u(\hat{x}) - (2\varepsilon)^{-1}|x-\hat{x}|^2$, we easily verify that $|q| = \varepsilon^{-1}|\hat{x} - x| \leq 2\sqrt{\|u\|_{L^\infty(\Omega)}/\varepsilon}$. Thus, by Lemma 7.4, we see that $(q, X) \in J^{2,+}u(x+\varepsilon q)$. Hence, we have

$$\mathcal{P}^-(X) - \mu|q| \leq f^+(x+\varepsilon q) \leq f^\varepsilon(x).$$

We note that for small $\varepsilon > 0$, we may suppose that

$$r^\varepsilon := \max_{\overline{\Omega}_\varepsilon} u^\varepsilon - \max_{\partial\Omega_\varepsilon}(u^\varepsilon)^+ > 0. \tag{7.6}$$

Here, we list some properties on upper contact sets: For small $\delta > 0$, we set

$$\Omega^\delta := \{x \in \Omega \mid \mathrm{dist}(x, \partial\Omega) > \delta\}.$$

Lemma 7.5. *Let $v_\delta \in C(\overline{\Omega}^\delta)$ and $v \in C(\overline{\Omega})$ satisfy that $v_\delta \to v$ uniformly on any compact sets in Ω as $\delta \to 0$. Assume that $\hat{r} := \max_{\overline{\Omega}} v - \max_{\partial\Omega} v^+ > 0$. Then, for $r \in (0, \hat{r})$, we have the following properties:*

$$\begin{cases} (1) & \Gamma_r[v,\Omega] \text{ is a compact set in } \Omega^+[v], \\ (2) & \limsup\limits_{\delta\to 0} \Gamma_r[v_\delta, \Omega^\delta] \subset \Gamma_r[v,\Omega], \\ (3) & \text{for small } \alpha > 0, \text{ there is } \delta_\alpha \text{ such that } \cup_{0\leq\delta<\delta_\alpha} \Gamma_r[v_\delta, \Omega^\delta] \subset \hat{\Gamma}_r^\alpha, \\ & \text{where } \hat{\Gamma}_r^\alpha := \{x \in \Omega \mid \mathrm{dist}(x, \Gamma_r[v,\Omega]) < \alpha\}, \\ (4) & x_k \in \Omega^{\delta_k} \to x \in \overline{\Omega} \text{ as } k \to \infty, \text{ then}, \liminf\limits_{k\to\infty} v_{\delta_k}(x_k) \leq v(x). \end{cases}$$

Proof of Lemma 7.5. To show (1), we first need to observe that for $r \in (0, \hat{r})$, $\mathrm{dist}(\Gamma_r[v,\Omega], \partial\Omega) > 0$. Suppose the contrary; if there is $x_k \in \Gamma_r[v,\Omega]$ such that $x_k \in \Omega \to \hat{x} \in \partial\Omega$, then there is $p_k \in \overline{B}_r$ such that $v(y) \leq v(x_k) + \langle p_k, y - x_k\rangle$ for $y \in \Omega$. Hence, sending $k \to \infty$, we have

$$\max_{\overline{\Omega}} v - \max_{\partial\Omega} v^+ \leq r < \hat{r},$$

which is a contradiction. Thus, we can find a compact set $K \subset \Omega$ such that $\Gamma_r[v,\Omega] \subset K$.

Moreover, if $v(x) \le 0$ for $x \in \Gamma_r[v,\Omega]$, then we get a contradiction:

$$\hat{r} \le \max_{\overline{\Omega}} v \le r < \hat{r}.$$

Next, choose $x \in \limsup_{\delta\to 0} \Gamma_r[v_\delta, \Omega^\delta]$. Then, for any $k \ge 1$, there are $\delta_k \in (0,1/k)$ and $p_k \in \overline{B}_r$ such that

$$v_{\delta_k}(y) \le v_{\delta_k}(x) + \langle p_k, y-x\rangle \quad \text{for } y \in \Omega^{\delta_k}.$$

We may suppose $p_k \to p$ for some $p \in \overline{B}_r$ taking a subsequence if necessary. Sending $k \to \infty$ in the above, we see that $x \in \Gamma_r[v,\Omega]$.

If (3) does not hold, then there are $\alpha_0 > 0$, $\delta_k \in (0,1/k)$ and $x_k \in \Gamma_r[v_{\delta_k}, \Omega^{\delta_k}] \setminus \hat{\Gamma}_r^{\alpha_0}$. We may suppose again that $\lim_{k\to\infty} x_k = \hat{x}$ for some $\hat{x} \in \overline{\Omega}$. When $\hat{x} \in \partial\Omega$, since there is $p_k \in \overline{B}_r$ such that $v_{\delta_k}(y) \le v_{\delta_k}(x_k) + \langle p_k, y - x_k\rangle$ for $y \in \Omega$, we have $\hat{r} < \hat{r}$, which is a contradiction. Thus, we may suppose that $\hat{x} \in \Omega$ and, then $\hat{x} \in \Gamma_r[v,\Omega]$. Thus, there is $k_0 \ge 1$ such that $x_k \in \hat{\Gamma}_r^{\alpha_0}$ for $k \ge k_0$, which is a contradiction. $\square$

For $\delta > 0$, we set $u_\delta^\varepsilon := u^\varepsilon * \rho_\delta$, where ρ_δ is the standard mollifier. We set $\tilde{\Gamma}_r^{\varepsilon,\delta} := \Gamma_r[u_\delta^\varepsilon, \Omega_\varepsilon]$ for $r \in (0, r_\delta^\varepsilon)$, where $r_\delta^\varepsilon := \max_{\overline{\Omega}_\varepsilon} u_\delta^\varepsilon - \max_{\partial\Omega_\varepsilon} (u_\delta^\varepsilon)^+$. Notice that for small $\delta > 0$, $r_\delta^\varepsilon > 0$.

In view of the argument to derive (7.2) in Step 1, we have

$$\int_{B_r} g(p)dp \le C \int_{\tilde{\Gamma}_r^{\varepsilon,\delta}} \left(|Du_\delta^\varepsilon|^{n/(n-1)} + \kappa^{n/(n-1)}\right)^{1-n} (-\mathrm{trace}(D^2 u_\delta^\varepsilon))^n dx$$

for small $r > 0$.

Also, by the same argument for (ii) in (7.1), we can show that $D^2 u_\delta^\varepsilon(x) \le O$ in $\tilde{\Gamma}_r^{\varepsilon,\delta}$. Furthermore, from the definition of u^ε, we verify that $-\varepsilon^{-1} I \le D^2 u_\delta^\varepsilon(x)$ in Ω_ε.

Hence, sending $\delta \to 0$ with Lemma 7.5 (3), we have

$$\begin{aligned}\int_{B_r} g(p)dp &\le C \int_{\Gamma_r[u^\varepsilon,\Omega_\varepsilon]} \left(|Du^\varepsilon|^{n/(n-1)} + \kappa^{n/(n-1)}\right)^{1-n} (-\mathrm{trace}(D^2 u^\varepsilon))^n dx \\ &\le C \int_{\Gamma_r[u^\varepsilon,\Omega_\varepsilon]} \left(\mu^n + \left(\frac{f^\varepsilon}{\kappa}\right)^n\right) dx.\end{aligned}$$

Therefore, sending $\varepsilon \to 0$ (again with Lemma 7.5 (3)), we obtain (7.4), which implies the conclusion.

Remark. Using the ABP maximum principle in Step 2 (*i.e.* $f \in C(\Omega)$), we can give a proof of Proposition 6.3, which will be seen in section 7.3. Thus, in Step 3 below, we will use Proposition 6.3.

Step 3: $u \in C(\overline{\Omega})$ and $f \in L^n(\Omega)$. Let $f_k \in C(\overline{\Omega})$ be nonnegative functions such that $\|f_k - f^+\|_{L^n(\Omega)} \to 0$ as $k \to \infty$.

In view of Proposition 6.3, we choose $\phi_k \in C(\overline{\Omega}) \cap W^{2,n}_{loc}(\Omega)$ such that

$$\begin{cases} \mathcal{P}^+(D^2\phi_k) + \mu|D\phi_k| = f_k - f^+ & a.e. \text{ in } \Omega, \\ \phi_k = 0 & \text{on } \partial\Omega, \\ \|\phi_k\|_{L^\infty(\Omega)} \leq C\|f_k - f^+\|_{L^n(\Omega)}. \end{cases}$$

Setting $w_k := u + \phi_k - \|\phi_k\|_{L^\infty(\Omega)}$, we easily verify that w_k is an L^n-viscosity subsolution of

$$\mathcal{P}^-(D^2w_k) - \mu|Dw_k| \leq f_k \quad \text{in } \Omega.$$

Note that $\Omega^+[w_k] \subset \Omega^+[u]$.

Thus, by Step 2, we have

$$\max_{\overline{\Omega}} w_k \leq \max_{\partial\Omega} w_k + C\|(f_k)^+\|_{L^n(\Gamma_r[w_k,\Omega]\cap\Omega^+[u])}.$$

Therefore, sending $k \to \infty$ with Lemma 7.5 (2), we finish the proof. □

7.3 Proof of existence results for Pucci equations

We shall solve Pucci equations under the Dirichlet condition in Ω. For simplicity of statemants, we shall treat the case when Ω is a ball though we will need the existence result in smooth domains later. To extend the result for general Ω with smooth boundary, we only need to modify the function v^z in the argument below.

For $\mu \geq 0$ and $f \in L^p(B_1)$ with $p \geq n$,

$$\begin{cases} \mathcal{P}^-(D^2u) - \mu|Du| \geq f & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases} \tag{7.7}$$

and

$$\begin{cases} \mathcal{P}^+(D^2u) + \mu|Du| \leq f & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1. \end{cases} \tag{7.8}$$

Note that the first estimate of (7.10) is valid by Proposition 6.2 when the inhomogeneous term is continuous.

Sketch of proof. We only show the assertion for (7.8).

Step 1: $f \in C^\infty(\overline{B}_1)$. We shall consider the case when $f \in C^\infty(\overline{B}_1)$.

Set $\mathcal{S}_{\lambda,\Lambda} := \{A := (A_{ij}) \in S^n \mid \lambda I \le A \le \Lambda I\}$. We can choose a countable set $\mathcal{S}_0 := \{A^k := (A^k_{ij}) \in \mathcal{S}_{\lambda,\Lambda}\}_{k=1}^\infty$ such that $\overline{\mathcal{S}_0} = \mathcal{S}_{\lambda,\Lambda}$.

Noting that $\mu|q| = \max\{\langle b, q\rangle \mid b \in \partial B_\mu\}$ for $q \in \mathbf{R}^n$, we choose $\mathcal{B}_0 := \{b^k \in \partial B_\mu\}_{k=1}^\infty$ such that $\overline{\mathcal{B}}_0 = \partial B_\mu$.

According to Evans' result in 1983, we can find classical solutions $u^N \in C(\overline{\Omega}) \cap C^2(\Omega)$ of

$$\begin{cases} \max\limits_{k=1,\ldots,N} \left\{-\mathrm{trace}(A^k D^2 u) + \langle b^k, Du\rangle\right\} = f & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1. \end{cases} \tag{7.9}$$

Moreover, we find $\sigma = \sigma(\varepsilon) \in (0,1)$, $C_\varepsilon > 0$ (for each $\varepsilon \in (0,1)$) and $C_1 > 0$, which are independent of $N \ge 1$, such that

$$\|u^N\|_{L^\infty(B_1)} \le C_1 \|f\|_{L^n(B_1)} \quad \text{and} \quad \|u^N\|_{C^{2,\sigma}(B_{1-\varepsilon})} \le C_\varepsilon. \tag{7.10}$$

Note that the first estimate of (7.10) is valid by Proposition 6.2 **when the inhomogeneous term is continuous**.

More precisely, by the classical comparison principle, Proposition 3.3, we have

$$u^N \le u^1 \quad \text{in } \overline{B}_1. \tag{7.11}$$

Furthermore, we can construct a subsoluion of (7.9) for any $N \ge 1$ in the following manner: Fix $z \in \partial B_1$. Set $v^z(x) := \alpha(e^{-\beta|x-2z|^2} - e^{-\beta})$, where $\alpha, \beta > 0$ (independent of $z \in \partial B_1$) will be chosen later. We first note that $v^z(z) = 0$ and $v^z(x) \le 0$ for $x \in \overline{B}_1$.

Setting $L^k w(x) := -\mathrm{trace}(A^k D^2 w(x)) + \langle b^k, Dw(x)\rangle$, we verify that

$$\begin{aligned} L^k v^z(x) &\le 2\alpha\beta e^{-\beta|x-2z|^2}(\Lambda n - 2\beta\lambda|x-2z|^2 + \mu|x-2z|) \\ &\le 2\alpha\beta e^{-9\beta}(\Lambda n - 2\beta\lambda + 3\mu). \end{aligned}$$

Thus, fixing $\beta := (\Lambda n + 3\mu + 1)/(2\lambda)$, we have $L^k v^z(x) \le -2\alpha\beta e^{-9\beta}$. Hence, taking $\alpha > 0$ large enough so that $2\alpha\beta e^{-9\beta} \ge \|f\|_{L^\infty(B_1)}$, we have

$$\max_{k=1,2,\ldots,N} L^k v^z(x) \le f(x) \quad \text{in } B_1.$$

Now, putting $V(x) := \sup_{z \in \partial B_1} v^z(x)$, in view of Theorem 4.2, we see that V is a viscosity subsolution of

$$\max_{k=1,2,\ldots,N} L^k u(x) - f(x) \le 0 \quad \text{in } B_1.$$

Moreover, it is easy to check that $V^*(x) = 0$ for $x \in \partial B_1$. Thus, by Proposition 3.3 again, we obtain that

$$V \le u^N \quad \text{in } \overline{B}_1. \tag{7.12}$$

Therefore, in view of (7.10)-(7.12), we can choose a sequence N_k and $u \in C^2(B_1)$ such that $\lim_{k\to\infty} N_k = \infty$,

$$(u^{N_k}, Du^{N_k}, D^2u^{N_k}) \to (u, Du, D^2u) \quad \text{uniformly in } B_{1-\varepsilon}$$

for each $\varepsilon \in (0,1)$, and

$$V \le u \le u^1 \quad \text{in } \overline{B}_1. \tag{7.13}$$

We note that (7.13) implies that $u^* = u_*$ on ∂B_1.

By virtue of the stability result (Proposition 4.8), we see that u is a viscosity solution of

$$\mathcal{P}^+(D^2u) + \mu|Du| - f = 0 \quad \text{in } B_1$$

since $\sup_{k\ge1}\{-\mathrm{trace}(A^kX) + \langle b^k, p\rangle\} = \mathcal{P}^+(X) + \mu|p|$. Hence, Theorem 3.9 yields $u \in C(\overline{B}_1)$.

Therefore, by Proposition 2.3, we see that $u \in C(\overline{B}_1) \cap C^2(B_1)$ is a classical solution of (7.8).

Step 2: $f \in L^p(B_1)$. (Lemma 3.1 in [5]) Choose $f_k \in C^\infty(\overline{B}_1)$ such that $\|f_k - f\|_{L^p(\Omega)} \to 0$ as $k \to \infty$.

Let $u_k \in C(\overline{B}_1) \cap C^2(B_1)$ be a classical solution of

$$\mathcal{P}^+(D^2u) + \mu|Du| - f_k = 0 \quad \text{in } B_1$$

such that $u_k = 0$ on ∂B_1. Proposition 6.2 implies that $-C\|f_k^-\|_{L^n(B_1)} \le u_k \le C\|f_k^+\|_{L^p(B_1)}$ in B_1.

We first claim that $\{u_k\}_{k=1}^\infty$ is a Cauchy sequence in $L^\infty(B_1)$. Indeed, since (1) and (4) of Proposition 3.2 imply that

$$\begin{aligned}
&\mathcal{P}^-(D^2(u_j - u_k)) - \mu|D(u_j - u_k)| \\
\le\ &\mathcal{P}^+(D^2u_j) + \mathcal{P}^-(-D^2u_k) + \mu|Du_j| - \mu|Du_k| \\
=\ &f_j - f_k \\
\le\ &\mathcal{P}^+(D^2u_j) - \mathcal{P}^+(D^2u_k) + \mu|D(u_j - u_k)| \\
\le\ &\mathcal{P}^+(D^2(u_j - u_k)) + \mu|D(u_j - u_k)|,
\end{aligned}$$

using Proposition 6.2 **when the inhomogeneous term is continuous**, we have

$$\max_{\overline{B}_1} |u_j - u_k| \le C\|f_j - f_k\|_{L^n(B_1)}.$$

Recalling $p \ge n$, we thus have

$$\|u_j - u_k\|_{L^\infty(B_1)} \le C\|f_j - f_k\|_{L^p(B_1)}.$$

Hence, we find $u \in C(\overline{B}_1)$ such that u_k converges to u uniformly in $\overline{B}_1$ as $k \to \infty$. Moreover, we see that $-C\|f^-\|_{L^p(B_1)} \le u \le C\|f^+\|_{L^p(B_1)}$ in B_1.

Therefore, by the standard covering and limiting arguments with weakly convergence in $W^{2,p}$ locally, it suffices to find $C > 0$, independent of $k \ge 1$, such that

$$\|u_k\|_{W^{2,p}(B_{1/2})} \le C.$$

Moreover, we see that $-C\|f^-\|_{L^p(B_1)} \le u \le C\|f^+\|_{L^p(B_1)}$ in B_1.

For $\varepsilon \in (0, 1/2)$, we select $\eta := \eta_\varepsilon \in C^2(B_1)$ such that

$$\begin{cases} (i) & 0 \le \eta \le 1 \quad \text{in } B_1, \\ (ii) & \eta = 0 \quad \text{in } B_1 \setminus B_{1-\varepsilon}, \\ (iii) & \eta = 1 \quad \text{in } B_{1-2\varepsilon}, \\ (iv) & |D\eta| \le C_0\varepsilon^{-1}, \ |D^2\eta| \le C_0\varepsilon^{-2} \text{ in } B_1, \end{cases}$$

where $C_0 > 0$ is independent of $\varepsilon \in (0, 1/2)$.

Now, we recall Caffarelli's result (1989) (see also [4]): There is a universal constant $\hat{C} > 0$ such that

$$\|D^2(\eta u_k)\|_{L^p(B_{1-\varepsilon})} \le \hat{C}\|\mathcal{P}^+(D^2(\eta u_k))\|_{L^p(B_{1-\varepsilon})}.$$

Hence, we find $C_1 > 0$ such that for $0 < \varepsilon < 1/4$,

$$\begin{aligned} \|D^2 u_k\|_{L^p(B_{1-2\varepsilon})} &\le \|D^2(\eta u_k)\|_{L^p(B_{1-\varepsilon})} \le \hat{C}\|\mathcal{P}^+(D^2(\eta u_k))\|_{L^p(B_{3/4})} \\ &\le C_1\left(\|f_k\|_{L^p(B_{1-\varepsilon})} + \varepsilon^{-1}\|Du_k\|_{L^p(B_{1-\varepsilon})} + \varepsilon^{-2}\|u_k\|_{L^p(B_{1-\varepsilon})}\right) \end{aligned}$$

Multiplying $\varepsilon^2 > 0$ in the above, we get

$$\varepsilon^2\|D^2 u_k\|_{L^p(B_{1-2\varepsilon})} \le C_1(\|f_k\|_{L^p(B_1)} + \phi_1(u_k) + \phi_0(u_k)),$$

where $\phi_j(u_k) := \sup_{0<\varepsilon<1/2} \varepsilon^j \|D^j u_k\|_{L^p(B_{1-\varepsilon})}$ for $j = 0, 1, 2$.

Therefore, in view of the "interpolation" inequality (see [13] for example), *i.e.* for any $\delta > 0$, there is $C_\delta > 0$ such that

$$\phi_1(u_k) \leq \delta\phi_2(u_k) + C_\delta\phi_0(u_k),$$

we find $C_3 > 0$ such that

$$\phi_2(u_k) \leq C_3 \left(\|f_k\|_{L^p(B_1)} + \phi_0(u_k) \right).$$

On the other hand, since we have L^∞-estimates for u_k, we conclude the proof. $\square$

Remark. It is possible to show that the uniform limit u in Step 2 is an L^p-viscosity solution of (7.8) by Proposition 6.13. Moreover, since it is known that if L^p-viscosity supersolution of (7.8) belongs to $W^{2,p}_{loc}(B_1)$, then it is an L^p-strong supersolution (see [5]), u satisfies $\mathcal{P}^+(D^2u) + \mu|Du| = f(x)$ *a.e.* in B_1.

7.4 Proof of the weak Harnack inequality

We need a modification of Lemma 4.1 in [4] since our PDE (7.14) below has the first derivative term.

Lemma 7.6. (cf. Lemma 4.1 in [4]) *There are* $\phi \in C^2(\overline{B}_{2\sqrt{n}})$ *and* $\xi \in C(B_{2\sqrt{n}})$ *such that*

$$\begin{cases} (1) & \mathcal{P}^-(D^2\phi) - \mu|D\phi| \geq -\xi \text{ in } B_{2\sqrt{n}}, \\ (2) & \phi(x) \leq -2 \text{ for } x \in Q_3, \\ (3) & \phi(x) = 0 \text{ for } x \in \partial B_{2\sqrt{n}}, \\ (4) & \xi(x) = 0 \text{ for } x \in B_{2\sqrt{n}} \setminus B_{1/2}. \end{cases}$$

<u>*Proof.*</u> Set $\phi_0(r) := A\{1 - (2\sqrt{n}/r)^\alpha\}$ for $A, \alpha > 0$ so that $\phi_0(2\sqrt{n}) = 0$.
Since

$$\begin{cases} D\phi_0(|x|) = A(2\sqrt{n})^\alpha \alpha |x|^{-\alpha-2}x, \\ D^2\phi_0(|x|) = A(2\sqrt{n})^\alpha \alpha |x|^{-\alpha-4}\{|x|^2 I - (\alpha+2)x \otimes x\}, \end{cases}$$

we caluculate in the following way: At $x \neq 0$, we have

$$\mathcal{P}^-(D^2\phi_0(|x|)) - \mu|D\phi_0(|x|)| \geq A(2\sqrt{n})^\alpha \alpha |x|^{-\alpha-2}\{(\alpha+2)\lambda - n\Lambda - \mu|x|\}.$$

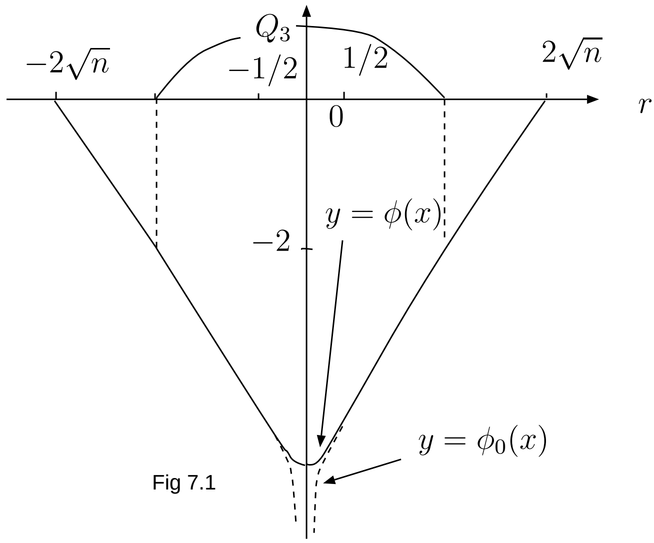

Fig 7.1

Setting $\alpha := \lambda^{-1}(n\Lambda + 2\mu\sqrt{n}) - 2$ so that $\alpha > 0$ for $n \geq 2$, we see that the right hand side of the above is nonnegative for $x \in B_{2\sqrt{n}} \setminus \{0\}$. Thus, taking $\phi \in C^2(B_{2\sqrt{n}})$ such that $\phi(x) = \phi_0(|x|)$ for $x \in B_{2\sqrt{n}} \setminus B_{1/2}$ and $\phi(x) \leq \phi_0(3\sqrt{n}/2)$ for $x \in B_{3\sqrt{n}/2}$, we can choose a continuous function ξ satisfying (1) and (4). See Fig 7.1.

Moreover, taking $A := 2/\{(4/3)^\alpha - 1\}$ so that $\phi_0(3\sqrt{n}/2) = -2$, we see that (2) holds. $\square$

We now present an important "cube decomposition lemma".

We shall explain a terminology for the lemma: For a cube $\tilde{Q} := Q_r(x)$ with $r > 0$ and $x \in \mathbf{R}^n$, we call Q a **dyadic cube** of $\tilde{Q}$ if it is one of cubes $\{Q_k\}_{k=1}^{2^n}$ so that $Q_k := Q_{r/2}(x_k)$ for some $x_k \in \tilde{Q}$, and $\cup_{k=1}^{2^n} Q_k \subset \tilde{Q} \subset \cup_{k=1}^{2^n} \overline{Q}_k$.

Lemma 7.7. (Lemma 4.2 in [4]) *Let $A \subset B \subset Q_1$ be measurable sets and $0 < \delta < 1$ such that*

(a) $|A| \leq \delta$,

(b) *Assume that if a dyadic cube Q of $\tilde{Q} \subset Q_1$ satisfies $|A \cap Q| > \delta|Q|$, then $\tilde{Q} \subset B$.*

Then, $|A| \leq \delta|B|$.

<u>*Proof of Proposition 6.4.*</u> Assuming that $u \in C(\overline{B}_{2\sqrt{n}})$ is a nonnegative

viscosity supersolution of

$$\mathcal{P}^+(D^2u) + \mu|Du| \geq 0 \quad \text{in } B_{2\sqrt{n}}, \tag{7.14}$$

we shall show that for some constants $p_0 > 0$ and $C_1 > 0$,

$$\|u\|_{L^{p_0}(Q_1)} \leq C_1 \inf_{Q_{1/2}} u.$$

To this end, it is sufficient to show that if $u \in C(\overline{B}_{2\sqrt{n}})$ satisfies that $\inf_{Q_{1/2}} u \leq 1$, then we have $\|u\|_{L^{p_0}(Q_1)} \leq C_1$ for some constants $p_0, C_1 > 0$. Indeed, by taking $v(x) := u(x)\left(\inf_{Q_{1/2}} u + \delta\right)^{-1}$ for any $\delta > 0$ in place of u, we have $\|v\|_{L^{p_0}(Q_1)} \leq C_1$, which implies the assertion by sending $\delta \to 0$.

Lemma 7.8. *There are $\theta > 0$ and $M > 1$ such that if $u \in C(\overline{B}_{2\sqrt{n}})$ is a nonnegative L^p-viscosity supersolution of* (7.14) *such that*

$$\inf_{Q_3} u \leq 1, \tag{7.15}$$

then we have

$$|\{x \in Q_1 \mid u(x) \leq M\}| \geq \theta.$$

Remark. In our setting of proof of Proposition 7.4, assumption (7.15) is automatically satisfied.

Proof of Lemma 7.8. Choose $\phi \in C^2(B_{2\sqrt{n}})$ and $\xi \in C(B_{2\sqrt{n}})$ from Lemma 7.6. Using (4) of Proposition 3.2, we easily see that $w := u + \phi$ is an L^n-viscosity supersolution of

$$\mathcal{P}^+(D^2w) + \mu|Dw| \geq -\xi \quad \text{in } B_{2\sqrt{n}}.$$

Since $\inf_{Q_3} w \leq -1$ and $w \geq 0$ on $\partial B_{2\sqrt{n}}$ by (2) and (3) in Lemma 7.6, respectively, by Proposition 6.2, we find $\hat{C} > 0$ such that

$$1 \leq \sup_{Q_3}(-w) \leq \sup_{B_{2\sqrt{n}}}(-w) \leq \hat{C}\|\xi\|_{L^n(\Gamma[-w, B_{2\sqrt{n}}] \cap B^+_{2\sqrt{n}}[-w])}. \tag{7.16}$$

In view of (4) of Lemma 7.6, (7.16) implies that

$$1 \leq \hat{C} \max_{B_{1/2}} |\xi| |\{x \in Q_1 \mid (u + \phi)(x) < 0\}|.$$

Since

$$u(x) \leq -\phi(x) \leq \max_{\overline{B}_{2\sqrt{n}}}(-\phi) =: M \quad \text{for } x \in B_{2\sqrt{n}}.$$

Therefore, setting $\theta = (\hat{C} \sup_{Q_1} |\xi|)^{-1} > 0$ and $M = \sup_{B_{2\sqrt{n}}}(-\phi) \geq 2$, we have

$$\theta \leq |\{x \in Q_1 \mid u(x) \leq M\}|. \quad \square$$

We next show the following:

Lemma 7.9. *Under the same assumptions as in Lemma 7.8, we have*

$$|\{x \in Q_1 \mid u(x) > M^k\}| \leq (1-\theta)^k \quad \textit{for all } k = 1, 2, \ldots$$

Proof. Lemma 7.8 yields the assertion for $k = 1$.

Suppose that it holds for $k-1$. Setting $A := \{x \in Q_1 \mid u(x) > M^k\}$ and $B := \{x \in Q_1 \mid u(x) > M^{k-1}\}$, we shall show $|A| \leq (1-\theta)|B|$.

Since $A \subset B \subset Q_1$ and $|A| \leq |\{x \in Q_1 \mid u(x) > M\}| \leq \delta := 1 - \theta$, in view of Lemma 7.8, it is enough to check that property (b) in Lemma 7.7 holds.

To this end, let $Q := Q_{1/2^j}(z)$ be a dyadic cube of $\tilde{Q} := Q_{1/2^{j-1}}(\hat{z})$ (for some $z, \hat{z} \in Q_1$ and $j \geq 1$) such that

$$|A \cap Q| > \delta |Q| = \frac{1-\theta}{2^{jn}}. \tag{7.17}$$

It remains to show $\tilde{Q} \subset B$.

Assuming that there is $\tilde{x} \in \tilde{Q}$ such that $\tilde{x} \notin B$; *i.e.* $u(\tilde{x}) \leq M^{k-1}$.

Set $v(x) := u(z + 2^{-j}x)/M^{k-1}$ for $x \in B_{2\sqrt{n}}$. Since $|\tilde{x}_i - z_i| \leq 3/2^{j+1}$, we see that $\inf_{Q_3} v \leq u(\tilde{x})/M^{k-1} \leq 1$. Furthermore, since $z \in Q_1$, $z + 2^{-j}x \in B_{2\sqrt{n}}$ for $x \in B_{2\sqrt{n}}$.

Thus, since v is an L^p-viscosity supersolution of

$$\mathcal{P}^+(D^2 v) + \mu |Dv| \geq 0,$$

Lemma 7.8 yields $|\{x \in Q_1 \mid v(x) \leq M\}| \geq \theta$. Therefore, we have

$$|\{x \in Q \mid u(x) \leq M^k\}| \geq \frac{\theta}{2^{jn}} = \theta |Q|.$$

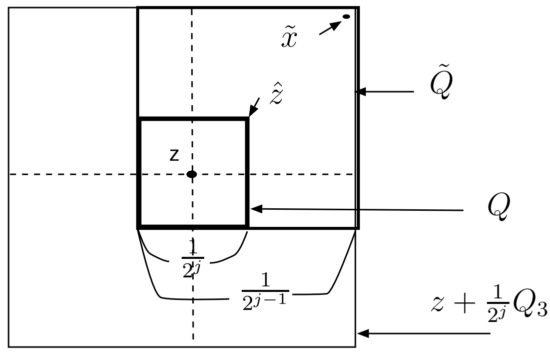

Fig 7.2

Thus, we have $|Q \setminus A| \geq \theta|Q|$. Hence, in view of (7.17), we have

$$|Q| = |A \cap Q| + |Q \setminus A| > \delta|Q| + \theta|Q| = |Q|,$$

which is a contradiction. $\Box$

<u>*Back to the proof of Proposition 6.4.*</u> A direct consequence of Lemma 7.9 is that there are $\tilde{C}, \varepsilon > 0$ such that

$$|\{x \in Q_1 \mid u(x) \geq t\}| \leq \tilde{C}t^{-\varepsilon} \quad \text{for } t > 0. \tag{7.18}$$

Indeed, for $t > M$, we choose an integer $k \geq 1$ so that $M^{k+1} \geq t > M^k$. Thus, we have

$$|\{x \in Q_1 \mid u(x) \geq t\}| \leq |\{x \in Q_1 \mid u(x) > M^k\}| \leq (1-\theta)^k \leq \tilde{C}_0 t^{-\varepsilon},$$

where $\tilde{C}_0 := (1-\theta)^{-1}$ and $\varepsilon := -\log(1-\theta)/\log M > 0$.

Since $1 \leq M^\varepsilon t^{-\varepsilon}$ for $0 < t \leq M$, taking $\tilde{C} := \max\{\tilde{C}_0, M^\varepsilon\}$, we obtain (7.18).

Now, recalling Fubini's theorem,

$$\begin{aligned}\int_{Q_1} u^{p_0}(x)dx &\leq \int_{\{x \in Q_1 \mid u(x) \geq 1\}} u^{p_0}(x)dx + 1 \\ &= p_0 \int_1^\infty t^{p_0-1}|\{x \in Q_1 \mid u(x) \geq t\}|dt + 1,\end{aligned}$$

(see Lemma 9.7 in [13] for instance), in view of (7.18), for any $p_0 \in (0, \varepsilon)$, we can find $C(p_0) > 0$ such that $\|u\|_{L^{p_0}(Q_1)} \leq C(p_0)$. $\Box$

7.5 Proof of the local maximum principle

Although our proof is a bit technical, we give a modification of Trudinger's proof in [13] (Theorem 9.20), in which he observed a precise estimate for "strong" subsolutions on the upper contact set. Recently, Fok in [11] (1996) gave a similar proof to ours.

We note that we can find a different proof of the local maximum principle in [4] (Theorem 4.8 (2)).

Proof of Proposition 6.5. We give a proof only when $q \in (0,1]$ because it is immediate to show the assertion for $q > 1$ by Hölder's inequality.

Let $x_0 \in \overline{Q}_1$ be such that $\max_{\overline{Q}_1} u = u(x_0)$. It is sufficient to show that

$$\max_{\overline{B}_{1/4}(x_0)} u \le C_2 \|u^+\|_{L^q(B_{1/2}(x_0))}$$

since $B_{1/2}(x_0) \subset Q_2$. Thus, by considering $u((x - x_0)/2)$ instead of $u(x)$, it is enough to find $C_2 > 0$ such that

$$\max_{\overline{B}_{1/2}} u \le C_2 \|u^+\|_{L^q(B_1)}.$$

We may suppose that

$$\max_{\overline{B}_1} u > 0 \tag{7.19}$$

since otherwise, the conclusion is trivial.

Furthermore, by the continuity of u, we can choose $\tau \in (0, 1/4)$ such that $1 - 2\tau \ge 1/2$ and

$$\max_{\overline{B}_{1-2\tau}} u > 0.$$

We shall consider the sup-convolution of u again: For $\varepsilon \in (0, \tau)$,

$$u^\varepsilon(x) := \sup_{y \in B_1} \left\{ u(y) - \frac{|x-y|^2}{2\varepsilon} \right\}.$$

By the uniform convergence of u^ε to u, (7.19) yields

$$\max_{\overline{B}_{1-\tau}} u^\varepsilon > 0 \quad \text{for small } \varepsilon > 0. \tag{7.20}$$

For small $\varepsilon > 0$, we can choose $\delta := \delta(\varepsilon) \in (0, \tau)$ such that $\lim_{\varepsilon \to 0} \delta = 0$, and

$$\mathcal{P}^-(D^2 u^\varepsilon) - \mu |Du^\varepsilon| \le 0 \quad a.e. \text{ in } B_{1-\delta}.$$

Putting $\eta^\varepsilon(x) := \{(1-\delta)^2 - |x|^2\}^\beta$ for $\beta := 2n/q \geq 2$, we define $v^\varepsilon(x) := \eta^\varepsilon(x)u^\varepsilon(x)$. We note that

$$r^\varepsilon := \max_{\overline{B}_{1-\delta}} v^\varepsilon > 0.$$

Fix $r \in (0, r^\varepsilon)$ and set $\Gamma_r^\varepsilon := \Gamma_r[v^\varepsilon, B_{1-\delta}]$. By (1) in Lemma 7.5, we see $\Gamma_r^\varepsilon \subset B_{1-\delta}^+[v^\varepsilon]$.

For later convenience, we observe that

$$Dv^\varepsilon(x) = -2\beta x\eta(x)^{(\beta-1)/\beta}u^\varepsilon(x) + \eta(x)Du^\varepsilon(x), \tag{7.21}$$

$$\begin{array}{rl} D^2v^\varepsilon(x) = & -2\beta\eta(x)^{(\beta-1)/\beta}\{u^\varepsilon(x)I + x\otimes Du^\varepsilon(x) + Du^\varepsilon(x)\otimes x\} \\ & +4\beta(\beta-1)\eta(x)^{(\beta-2)/\beta}u^\varepsilon(x)x\otimes x + \eta(x)D^2u^\varepsilon(x). \end{array} \tag{7.22}$$

Since u^ε is twice differentiable almost everywhere, we can choose a measurable set $N_\varepsilon \subset B_{1-\delta}$ such that $|N_\varepsilon| = 0$ and u^ε is twice differentiable at $x \in B_{1-\delta} \setminus N_\varepsilon$. Of course, v^ε is also twice differentiable at $x \in B_{1-\delta} \setminus N_\varepsilon$.

By (7.22), we have

$$\mathcal{P}^-(D^2v^\varepsilon) \leq \eta\mathcal{P}^-(D^2u^\varepsilon) + 2\beta\eta^{(\beta-1)/\beta}\{\Lambda nu^\varepsilon - \mathcal{P}^-(x\otimes Du^\varepsilon + Du^\varepsilon\otimes x)\}$$

in $B_{1-\delta}^+[v^\varepsilon]$. By using (7.21), the last term in the above can be estimated from above by

$$C\{\eta^{-2/\beta}(v^\varepsilon)^+ + \eta^{-1/\beta}|Dv^\varepsilon|\}.$$

Moreover, using (7.21) again, we have

$$\mathcal{P}^-(D^2u^\varepsilon) \leq \mu|Du^\varepsilon| \leq \mu\eta^{-1}|Dv^\varepsilon| + C\eta^{-1/\beta}(u^\varepsilon)^+.$$

Hence, we find $C > 0$ such that

$$\mathcal{P}^-(D^2v^\varepsilon) \leq C\eta^{-1/\beta}|Dv^\varepsilon| + C\eta^{-2/\beta}(v^\varepsilon)^+ =: g^\varepsilon \quad \text{in } B_{1-\delta}\setminus N_\varepsilon. \tag{7.23}$$

We next claim that there is $C > 0$ such that

$$|Dv^\varepsilon(x)| \leq C\eta^{-1/\beta}(x)v^\varepsilon(x) \quad \text{for } x \in \Gamma_r^\varepsilon \setminus N_\varepsilon. \tag{7.24}$$

First, we note that at $x \in \Gamma_r^\varepsilon \setminus N_\varepsilon$, $v^\varepsilon(y) \leq v^\varepsilon(x) + \langle Dv^\varepsilon(x), y - x\rangle$ for $y \in B_{1-\delta}$.

To show this claim, since we may suppose $|Dv^\varepsilon(x)|>0$ to get the estimate, setting $y:=x-tDv^\varepsilon(x)|Dv^\varepsilon(x)|^{-1}\in\partial B_{1-\delta}$ for $t\in[1-\delta-|x|,1-\delta+|x|]$, we see that

$$0=v^\varepsilon(y)\le v^\varepsilon(x)-t|Dv^\varepsilon(x)|,$$

which implies

$$|Dv^\varepsilon(x)|\le Cv^\varepsilon(x)\eta^{-1/\beta}(x)\quad\text{in }\Gamma_r^\varepsilon\setminus N_\varepsilon. \tag{7.25}$$

Here, we use Lemma 2.8 in [5], which will be proved in the end of this subsection for the reader's convenience:

Lemma 7.10. *Let $w\in C(\Omega)$ be twice differentiable a.e. in Ω, and satisfy*

$$\mathcal{P}^-(D^2w)\le g\quad\text{a.e. in }\Omega,$$

where $g\in L^p(\Omega)$ with $p\ge n$. If $-C_1I\le D^2w(x)\le O$ a.e. in Ω for some $C_1>0$, then w is an L^p-viscosity subsolution of

$$\mathcal{P}^-(D^2w)\le g\quad\text{in }\Omega. \tag{7.26}$$

Since u^ε is Lipschitz continuous in $B_{1-\delta}$, by (7.22), we see that v^ε is an L^n-viscosity subsolution of

$$\mathcal{P}^-(D^2v^\varepsilon)\le g^\varepsilon\quad\text{in }B_{1-\delta}.$$

Noting (7.25), in view of Proposition 6.2, we have

$$\begin{aligned}\max_{\overline{B}_{1-\delta}}v^\varepsilon &\le C\|\eta^{-2/\beta}(v^\varepsilon)^+\|_{L^n(\Gamma_r^\varepsilon)}\\ &\le C\left(\max_{\overline{B}_{1-\delta}}(v^\varepsilon)^+\right)^{\frac{\beta-2}{\beta}}\|((u^\varepsilon)^+)^{2/\beta}\|_{L^n(B_{1-\delta})},\end{aligned}$$

which together with our choice of β yields

$$\max_{\overline{B}_{1-\delta}}v^\varepsilon\le C\|(u^\varepsilon)^+\|_{L^q(B_{1-\delta})}.$$

Therefore, by (7.20), we have

$$\max_{\overline{B}_{1/2}}u^\varepsilon\le C\max_{\overline{B}_{1-\delta}}v^\varepsilon\le C\|(u^\varepsilon)^+\|_{L^q(B_{1-\delta})},$$

Therefore, sending $\varepsilon \to 0$ in the above, we finish the proof. $\Box$

<u>*Proof of Lemma 7.10.*</u> In order to show that $w \in C(\Omega)$ is an L^p-viscosity subsolution of (7.26), we suppose the contrary; there are $\varepsilon, r > 0$, $\hat{x} \in \Omega$ and $\phi \in W^{2,p}_{loc}(\Omega)$ such that $0 = (w - \phi)(\hat{x}) = \max_{\overline{\Omega}}(w - \phi)$, $B_{2r}(\hat{x}) \subset \Omega$, and

$$\mathcal{P}^-(D^2\phi) - g \geq 2\varepsilon \quad a.e. \text{ in } B_r(\hat{x}).$$

We may suppose that $\hat{x} = 0 \in \Omega$. Setting $\psi(x) := \phi(x) + \tau|x|^4$ for small $\tau > 0$, we observe that

$$h := \mathcal{P}^-(D^2\psi) - g \geq \varepsilon \quad a.e. \text{ in } B_r.$$

Notice that $0 = (w - \psi)(0) > (w - \psi)(x)$ for $x \in B_r \setminus \{0\}$.

Moreover, we observe

$$\mathcal{P}^-(D^2(w - \psi)) \leq -\varepsilon \quad a.e. \text{ in } B_r. \tag{7.27}$$

Consider $w_\delta := w * \rho_\delta$, where ρ_δ is the standard mollifier for $\delta > 0$. From our assumption, we see that, as $\delta \to 0$,

$$\begin{cases} (1) & w_\delta \to w \quad \text{uniformly in } B_r, \\ (2) & D^2 w_\delta \to D^2 w \quad a.e. \text{ in } B_r. \end{cases}$$

By Lusin's Theorem, for any $\alpha > 0$, we find $E_\alpha \subset B_r$ such that $|B_r \setminus E_\alpha| < \alpha$,

$$\int_{B_r \setminus E_\alpha} (1 + |\mathcal{P}^-(-D^2\psi)|)^p dx < \alpha,$$

and

$$D^2 w_\delta \to D^2 w \quad \text{uniformly in } E_\alpha \quad (\text{as } \delta \to 0).$$

Setting $h_\delta := \mathcal{P}^-(D^2(w_\delta - \psi))$, we find $C > 0$ such that

$$h_\delta \leq C + \mathcal{P}^-(-D^2\psi)$$

because of our hypothesis. Hence, we have

$$\|(h_\delta)^+\|^p_{L^p(B_r)} \leq C \int_{B_r \setminus E_\alpha} (1 + |\mathcal{P}^-(-D^2\psi)|)^p dx + \int_{E_\alpha} |(h_\delta)^+|^p dx.$$

Sending $\delta \to 0$ in the above, by (7.27), we have

$$\limsup_{\delta \to 0} \|(h_\delta)^+\|_{L^p(B_r)} \le C\|(1 + |\mathcal{P}^-(-D^2\psi)|)\|_{L^p(B_r \setminus E_\alpha)} \le C\alpha. \tag{7.28}$$

On the other hand, in view of Proposition 6.2, we see that

$$\max_{\overline{B}_r}(w_\delta - \psi) \le \max_{\partial B_r}(w_\delta - \psi) + C\|(h_\delta)^+\|_{L^p(B_r)}.$$

Hence, by sending $\delta \to 0$, this inequality together with (7.28) implies that

$$0 = \max_{\overline{B}_r}(w - \psi) \le \max_{\partial B_r}(w - \psi) + C\alpha \quad \text{for any } \alpha > 0.$$

This is a contradiction since $\max_{\partial B_r}(w - \psi) < 0$. $\square$