CHAPTER II

Introduction to the theory of p-adic period mappings.

Contents

1. A survey of moduli and period mappings over $\mathbb{C}$.

ABSTRACT: Review of moduli, Shimura varieties, Hodge structures, period mappings, period domains, Gauss-Manin connections.

1.1. Moduli spaces.

1.1.1. The term "moduli" was coined by Riemann to describe the continuous essential parameters of smooth complex compact curves of a given genus g. The use of this term has been extended to many classification problems in analytic or algebraic geometry: typically, a classification problem consists of a discrete part (the dimension, and other numerical invariants...) and a continuous part (the moduli).

One can distinguish between local and global moduli spaces:

(i) local moduli are the parameters of the most general small (or infinitesimal) deformation of a given analytic manifold or algebraic variety. Let for instance X be a compact analytic manifold. Then a famous theorem of Kuranishi asserts that there is a local deformation $\pi : \underline{X} \to S$ with base point s ($X = \underline{X}_s$) such that any other deformation comes, locally around s, from π by base change; moreover the tangent space of S at s is isomorphic to $\mathrm{H}^1(X, (\Omega^1_X)^\vee)$. For curves, the dimension of this space is $3g-3$ if $g \geq 2$ (the number of Riemann's complex moduli).

(ii) Global moduli spaces, when they exist, provide the solution of the continuous part of moduli problems. For instance, Douady has constructed the global moduli space for closed analytic subspaces of a given compact complex manifold. Although global moduli spaces were already studied in the nineteenth century (in the case of curves), the concept was first put on firm foundations by Grothendieck in terms of representable functors.

The existence of a global moduli space is often obstructed by the presence of automorphisms of the objects under classification. One possibility is to rule out these automorphisms by imposing some extra structure (rigidification). When all automorphism groups are finite, another possibility is to look for moduli orbifolds (or moduli stacks) rather than moduli spaces.

1.1.2. The most classical moduli problems concern complex elliptic curves. These are classified by their j-invariant. However the affine j-line $\mathbb{A}^1$ is not a fine moduli space: indeed, in any family of elliptic curves parametrized by the whole affine line, all the fibers are isomorphic.

One can rigidify the problem by introducing a "level N structure" on elliptic curves E, *i.e.* fixing an identification $\mathrm{Ker}[N]_A \cong (\mathbb{Z}/N\mathbb{Z})^2$ such that the Weil pairing is given by $(\xi, \eta) \mapsto \exp(2\pi i(\xi \wedge \eta))$ (here $[N]_A$ denotes the multiplication by N on the elliptic curve A).

For $N \geq 3$, this rules out automorphisms of A, and the corresponding moduli functor is representable. For $N = 3$, the moduli space is $\mathbb{A}^1 \setminus \{\mu^3 =$

1}; the universal elliptic curve with level 3 structure is the Dixon elliptic curve

$$x^3 + y^3 + 3\mu xy = 1 \text{ (in affine coordinates } x, y)$$

with $\mathrm{Ker}[3]_A$ = the set of flexes.

For $N = 2$, the automorphism $[-1]_A$ remains; there is an elliptic curve with level 2 structure over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, namely the Legendre elliptic curve

$$y^2 = x(x-1)(x-\lambda) \text{ (in affine coordinates } x, y)$$

with $\mathrm{Ker}[2]_A = \{(0,0), (1,0), (\lambda, 0), (\infty, \infty)\}$, but it is not universal: another one is given by $\lambda y^2 = x(x-1)(x-\lambda)$.

1.1.3. These moduli problems admit two natural generalizations: moduli problems for curves of higher genus, and moduli problems for polarized abelian varieties (polarizations ensure that the automorphism groups are finite).

There is a moduli orbifold for smooth compact complex curves of genus $g > 0$, denoted by $\mathcal{M}_g$, which can be described as follows. The set of complex structures on compact connected oriented surface S_g of genus g is in a natural way a contractible complex manifold of dimension $3g - 3$ (*resp.* the Poincaré upper half plane $\mathfrak{h}$ if $g = 1$), the Teichmüller space $\mathcal{T}_g$. The mapping class group Γ_g, *i.e.* the component of connected components of the group of orientation-preserving self-homeomorphisms of S_g, acts on right on $\mathcal{T}_g$, and $\mathcal{M}_g$ is the quotient $\mathcal{T}_g/\Gamma_g$. For $g = 1$, one recovers $\mathcal{M}_1 = \mathfrak{h}/SL(2, \mathbb{Z})$.

There is a moduli orbifold for principally polarized abelian varieties of dimension $g > 0$, denoted by $\mathcal{A}_g$, which can be described as follows. For any abelian variety A, one has an exact sequence

$$0 \to L \to \mathrm{Lie}\, A \to A \to 0$$

and a principal polarization corresponds to a positive definite hermitian form H on $\mathrm{Lie}\, A$ whose imaginary part induces a perfect alternate form on L. Choosing a $\mathbb{C}$-basis $\omega_1, \ldots, \omega_g$ of $\Omega^1(A) \cong (\mathrm{Lie}\, A)^\vee$ and a symplectic $\mathbb{Z}$-basis $\gamma_1^+, \gamma_1^-, \ldots, \gamma_g^+, \gamma_g^-$ of L gives rise to a period matrices $\Omega^+ = (\int_{\gamma_j^+} \omega_i)$, $\Omega^- = (\int_{\gamma_j^-} \omega_i)$, and $\tau = (\Omega^-)^{-1}\Omega^+$ does not depend on the ω_i's. By the properties of H, τ belongs to the Siegel upper half space $\mathfrak{h}_g$ of symmetric matrices with positive definite imaginary part (Riemann relations); it is well-defined up to right multiplication by a matrix in $Sp(2g, \mathbb{Z})$ (which is the same as the standard left action on $\mathfrak{h}_g$ of the transposed matrix). This construction identifies $\mathcal{A}_g$ with $\mathfrak{h}_g/Sp(2g, \mathbb{Z})$.

The construction which associates to a curve its jacobian variety with the principal polarization given by the theta divisor gives rise to an immersion $\mathcal{M}_g \hookrightarrow \mathcal{A}_g$, (at the stack level) the Torelli map.

1.1.4. *Shimura varieties of P.E.L. type.* Following G. Shimura, one also studies refined moduli problems for "decorated" (principally polarized) abelian varieties, by prescribing in addition that the endomorphism algebra contains a given simple $\mathbb{Q}$-algebra, and imposing a level N structure. This gives rise to moduli orbifolds (*resp.* moduli spaces if $N \geq 3$) of "PEL type".

More precisely, let B be a simple finite-dimensional $\mathbb{Q}$-algebra, with a positive involution $*$. Let V be a B-module of finite type, endowed with an alternate $\mathbb{Q}$-bilinear form such that $\langle bv, w\rangle = \langle v, b^* w\rangle$. One denotes by G the algebraic $\mathbb{Q}$-group of B-linear symplectic similitudes of V.

Let $\mathcal{B}$ be a maximal order in B stable under $*$, and let L be a lattice in V, stable under $\mathcal{B}$ and autodual for $\langle\ ,\ \rangle$ (this will represent the structure of $\mathrm{H}^1(A, \mathbb{Z})$ for the abelian varieties A under consideration). We choose in addition a lagrangian subspace $F_0^1 \subset V_{\mathbb{C}}$ stable under B (which will be $\Omega^1(A_0) \cong (\mathrm{Lie}\, A_0)^\vee$ for one particular abelian variety A_0 under consideration).

There is a moduli orbifold (*resp.* moduli space if $N \geq 3$) for principally $*$-polarized[(1)] abelian varieties A, together with a given action $\iota : \mathcal{B} \to \mathrm{End}\, A$ such that $\det(\iota(b)|\Omega^1(A)) = \det(b|F_0^1)$ for all $b \in \mathcal{B}$ (Shimura type condition), and a given $\mathcal{B}/N\mathcal{B}$-linear symplectic similitude $A[N] = \mathrm{Ker}[N]_A \to \mathrm{Hom}(L, \mathbb{Z}/N\mathbb{Z})$.

It amounts to the same to consider abelian varieties A up to isogeny, with B-action (with Shimura type condition), together with a $\mathbb{Q}$-homogeneous principal $*$-polarization and a class of B-linear symplectic similitudes $\mathrm{H}^1(A, \hat{\mathbb{Z}} \otimes \mathbb{Q}) \to \hat{\mathbb{Z}} \otimes V$ modulo the group $\{g \in G(\hat{\mathbb{Z}} \otimes \mathbb{Q}) \mid (g-1)(\hat{\mathbb{Z}} \otimes L) \subset \hat{\mathbb{Z}} \otimes L\}$.

These moduli orbifolds are called Shimura orbifolds of "PEL type". They are algebraic, and defined over the number field $E = \mathbb{Q}[\mathrm{tr}(b|F_0^1)]$ (the so-called reflex field). Their connected components are defined over finite abelian extensions of E, *cf.* [**Del71**] for all of this. Further comments and examples will be supplied in 1.2.5, 7.1.1, 7.4.2.

1.2. Period mappings.

1.2.1. In the case of curves, the construction of period mappings belongs to the nineteenth century. The idea is to study the deformations of a curve by looking at the variation of its periods. Let $f : \underline{X} \to S$ be an algebraic family of projective smooth (connected) curves of genus g over $\mathbb{C}$. Let $\tilde{S}$ be the universal covering of S. Then the homology groups $\mathrm{H}_1(-, \mathbb{Z})$ of the fibers form a constant local system on $\tilde{S}$. Let us choose a symplectic basis $\gamma_1^+, \gamma_1^-, \ldots, \gamma_g^+, \gamma_g^-$. Choosing, locally, an auxiliary $\mathbb{C}$-basis $\omega_1, \ldots, \omega_g$ of $f_* \Omega^1_{\underline{X}/S}$ gives rise as above to a period matrices $\Omega^+ = (\int_{\gamma_j^+} \omega_i)$, $\Omega^- = (\int_{\gamma_j^-} \omega_i)$, and $\mathcal{P} = (\Omega^-)^{-1}\Omega^+$ does not depend on the ω_i's and is well-defined on $\tilde{S}$.

We get a commutative square

(1) a principal $*$-polarization is a symmetric $\mathcal{B}$-linear isomorphism between the abelian variety and its dual, endowed with the transposed action of $\mathcal{B}$ twisted by $*$

$$\begin{array}{ccc} \widetilde{S} & \xrightarrow{\mathcal{P}} & \mathfrak{h}_g \\ \downarrow & & \downarrow \\ S & \xrightarrow{\mathcal{P}} & \mathfrak{h}_g/Sp(2g,\mathbb{Z}) \end{array}$$

Applying this construction to the universal case $S = \mathcal{M}_g$ ($\mathcal{T}_g = \widetilde{\mathcal{M}_g}$ in the orbifold sense), the bottom line of this commutative square is nothing but an analytic interpretation of the Torelli map.

1.2.2. An important feature of the period mapping $\mathcal{P}$ is that it is a *quotient of solutions of a fuchsian differential system*, the Picard-Fuchs differential system (of Gauss-Manin connection). This comes out as follows. The vector bundle $f_*\Omega^1_{\underline{X}/S}$ is a subbundle (of rank g) of a vector bundle $\mathcal{H}$ (of rank $2g$) whose fibers are the De Rham cohomology spaces $\mathrm{H}^1_{\mathrm{DR}}(\underline{X}_s)$. This bundle $\mathcal{H}$ carries an integrable connection, the Gauss-Manin connection

$$\nabla_{\mathrm{GM}} : \mathcal{H} \to \Omega^1_S \otimes \mathcal{H}.$$

The local system of germs of horizontal analytic sections is $R^1 f_*\mathbb{C}$ (with fibers $\mathrm{H}^1(\underline{X}_s, \mathbb{C})$). Completing, locally, the basis $\omega_1, \ldots, \omega_g$ of sections of $f_*\Omega^1_{\underline{X}/S}$ into a basis $\omega_1, \ldots, \omega_g, \eta_1, \ldots, \eta_g$ of sections $\mathcal{H}$ allows to write the connection in form of a differential system, and a full *solution matrix* is given by the "full $2g \times 2g$-period matrix" $\begin{pmatrix} \Omega^+ & \Omega^- \\ N^+ & N^- \end{pmatrix}$, while $\mathcal{P} = (\Omega^-)^{-1}\Omega^+$ (we have set $N^+ = \left(\int_{\gamma_j^+} \eta_i\right)$, $N^- = \left(\int_{\gamma_j^-} \eta_i\right)$).

In the case of the Legendre elliptic curve over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ (*cf.* I.3), one finds that $\mathcal{P}$ is given by the quotient τ of two solutions of the hypergeometric differential equation with parameters $\left(\frac{1}{2}, \frac{1}{2}, 1\right)$.

1.2.3. The theory of period mappings has been generalized to any projective smooth morphism $f : \underline{X} \to S$ by P. Griffiths [**Gri71**]. We assume that S and the fibers of f are connected for simplicity. The idea is the following: by abstracting the structure of the periods of the fibers of f, one arrives at the notion of polarized Hodge structure; one then construct a map from the universal covering $\tilde{S}$ to a classifying space $\mathcal{D}$ of polarized Hodge structures.

For a fixed n, the real cohomology spaces $V_s^{\mathbb{R}} := \mathrm{H}^n(\underline{X}_s, \mathbb{R}))$ form a local system $V^{\mathbb{R}}$ on S. We denote by $V^{\mathbb{C}}$ its complexification. The vector bundle $\mathcal{O}_S \otimes V^{\mathbb{C}}$ is the analytification of an algebraic vector bundle $\mathcal{H}$ on S, and $V^{\mathbb{C}}$ is the local system of germs of analytic solutions of an algebraic fuchsian integrable connection ∇_{GM} on $\mathcal{H}$, the Gauss-Manin connection.

On the other hand, the spaces $V_s^{\mathbb{R}}$ carry a natural algebraic representation of $\mathbb{C}^\times$ (Hodge structure), which amounts to a decomposition $V_s^{\mathbb{C}} = \bigoplus_{p+q=n,\, p\geq 0} V^{p,q}$, $\overline{V^{p,q}} = V^{q,p}$. When s varies, the filtrations

$$F_s^p := \bigoplus_{p'+q=n,\, p'\geq p} V^{p',q}$$

are the fibers of a decreasing filtration F_s^p of $\mathcal{H}$ by vector subbundles (the Hodge filtration). The Gauss-Manin connection fails to preserve the Hodge filtration by just one notch: $\nabla_{\mathrm{GM}}(F^p) \subset \Omega^1_S \otimes F^{p-1}$.

The last piece of data is the polarization: a bilinear form Q on $V_s^{\mathbb{R}}$, $Q(v,v') = (-1)^n Q(v',v)$, such that $(-i)^n Q(v,\overline{v})$ is hermitian on $V_s^{\mathbb{C}}$, definite (of sign $(-1)^p$) on each $V^{p,q} = F_s^p \cap \overline{F}_s^q$, and such that F_s^p is the orthogonal of F_s^{n-p-1} in $V_s^{\mathbb{C}}$.

Let $G_{\mathbb{R}}$ denote the real group of similitudes of Q. The space of isotropic flags of type $0 \subset \ldots \subset F_s^p \subset F_s^{p-1} \subset \ldots \subset V_s^{\mathbb{C}}$ is a homogeneous space

$$\mathcal{D}^\vee = P\backslash G_{\mathbb{C}}$$

for a suitable parabolic subgroup P.

The flags satisfying the above positivity condition (on each $F_s^p \cap \overline{F}_s^q$) are classified by an analytic open submanifold, the *period domain*,

$$\mathcal{D} \subset \mathcal{D}^\vee; \quad \mathcal{D} \cong K\backslash G_{\mathbb{R}}^{\mathrm{ad}}(\mathbb{R})^0$$

for a suitable compact subgroup K ($\mathcal{D}^\vee$ is called the compact dual of $\mathcal{D}$).

Over $\tilde{S}$, the local system $V^{\mathbb{C}}$ becomes constant, and the construction which attaches to any $s \in \tilde{S}$ the classifying point of the corresponding flag $0 \subset \ldots \subset F_s^p \subset F_s^{p-1} \subset \ldots \subset V_s^{\mathbb{C}}$ gives rise to a holomorphic mapping, the *period mapping*

$$\tilde{S} \xrightarrow{\mathcal{P}} \mathcal{D}.$$

Via Plücker coordinates, $\mathcal{P}$ is again given by quotients of solutions of the Gauss-Manin connection ∇_{GM}. The projective monodromy group of ∇_{GM} is a subgroup $\Gamma \subset G_{\mathbb{R}}^{\mathrm{ad}}$ (well-defined up to conjugation), and we get a commutative square

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\mathcal{P}} & \mathcal{D} \subset \mathcal{D}^\vee \\ \downarrow & & \downarrow \\ S & \xrightarrow{\mathcal{P}} & \mathcal{D}/\Gamma. \end{array}$$

1.2.4. In the simple case of a family of elliptic curves and $n = 1$, $G_{\mathbb{R}} = GL_{2,\mathbb{R}}$, the flag space is just $\mathcal{D}^\vee = \mathbb{P}^1_{\mathbb{C}}$, $K = PSO(2)$ (the isotropy group of $i \in \mathbb{P}^1_{\mathbb{C}}$ in $PSL_2(\mathbb{R}) = G_{\mathbb{R}}^{\mathrm{ad}}(\mathbb{R})^0$), and $\mathcal{D} \cong K\backslash G_{\mathbb{R}}^{\mathrm{ad}}(\mathbb{R})^0$ is the Poincaré upper half plane $\mathfrak{h}$.

More generally, for a polarized abelian scheme, $G_{\mathbb{R}}$ is the group of symplectic similitudes, and $\mathcal{D}^\vee$ is the grassmannian of lagrangian subspaces of $\mathbb{C}^{2g}$, and $\mathcal{D}$ is the Siegel upper half space.

1.2.5. *Shimura varieties and period mappings.* When one deals with refined moduli problems for abelian schemes with PEL decoration as in 1.1.4, one should work with the group $G_{\mathbb{R}}$ of *B-linear* symplectic similitudes of $V_{\mathbb{R}}$ (notation as in 1.1.4). In that case, the corresponding period domain $\mathcal{D}$ is a symmetric domain.

More precisely, the datum of our B-stable lagrangian subspace $F_0^1 \subset V_{\mathbb{C}}$ (attached to our particular A_0 with $\mathrm{H}^1(A_0, \mathbb{C}) = V_{\mathbb{C}}$) amounts to the datum of an algebraic $\mathbb{R}$-homomorphism $h_0 : \mathbb{C}^\times \to G_{\mathbb{R}} \subset GL(V_{\mathbb{R}})$ such that h_0 acts by the characters z and $\overline{z}$ on $V_{\mathbb{C}}$ (F^1 is the subspace where h_0 acts through z), and such that the symmetric form $\langle v, h(i)w \rangle$ is positive definite on $V_{\mathbb{R}}$. For any abelian variety A with PEL structure of the correct type, together with a fixed isomorphism $\mathrm{H}^1(A, \mathbb{C}) \cong V_{\mathbb{C}}$, the corresponding subspace $F^1\mathrm{H}^1(A, \mathbb{C}) = \Omega^1(A) \subset V_{\mathbb{C}}$ is an eigenspace for some conjugate of h_0. The set of conjugates of h_0 under $G(\mathbb{R})$ is a finite union of copies of a symmetric domain $\mathcal{D}$.

For any component S of the *Shimura orbifold*, we thus get a commutative square

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\mathcal{P}} & \mathcal{D} \subset \mathcal{D}^\vee \\ {\scriptstyle \mathcal{Q}}\downarrow & & \downarrow{\scriptstyle \mathcal{Q}} \\ S & \xrightarrow{\mathcal{P}} & \mathcal{D}/\Gamma \end{array}$$

where the horizontal maps are *isomorphisms*, $\mathcal{Q}$ denotes the quotient maps, and Γ is a congruence subgroup of level N in the semi-simple group G^{ad} (a conjugate of the standard one).

The complex manifold $\tilde{S}$ admits the following modular description. Let U be the oriented real Lie group $\mathrm{Hom}(L, (\mathbb{R}/\mathbb{Z})^\times)$ (with $\mathcal{B}$-action and polarization), and let $\hat{U}$ be the associated infinitesimal Lie group. The automorphism group J of the decorated $\hat{U}$ coincides with $G(\mathbb{R})^0$. Then $\tilde{S}$ is a moduli space for *marked* decorated abelian varieties, where the marking is an isomorphism $\rho : \hat{A} \to \hat{U}$ of associated infinitesimal decorated oriented real Lie groups (note that, by taking duals of Lie algebras, ρ amounts to an B-linear symplectic isomorphism $\mathrm{H}^1(A, \mathbb{R}) \cong V^{\mathbb{R}}$). Note that the standard right action of J on $\tilde{S}$ is obtained by functoriality.

2. Preliminaries on p-divisible groups.

ABSTRACT: Definitions and basic theorems about p-divisible groups, quasi-isogenies, liftings and deformations.

2.1. p-divisible groups and quasi-isogenies.

2.1.1. We begin with four definitions. Let S be a scheme. Let Λ and Λ' be commutative group schemes over S.

A homomorphism $f : \Lambda \to \Lambda'$ is called an *isogeny* if it is an (f.p.p.f.) epimorphism with finite locally free kernel.

Let p be a prime number. Λ is a *p-divisible group* (or Barsotti-Tate group) if $\Lambda = \varinjlim_n \mathrm{Ker}[p^n]$ and the multiplication $[p]$ by p on Λ is an isogeny.

If so, $[p^n] = [p]^n$ is an isogeny for every $n \geq 0$, and if S is connected, the rank of $\mathrm{Ker}[p^n]$ is then of the form p^{hn}, where h is an integer called the height of Λ.

From the fact that $[p]$ is an isogeny, it follows that for p-divisible groups, $\mathrm{Hom}_S(\Lambda, \Lambda')$ is a torsion-free $\mathbb{Z}_p$-module.

A *quasi-isogeny* of p-divisible groups Λ, Λ' is a global section ρ of the Zariski sheaf $\underline{\mathrm{Hom}}_S(\Lambda, \Lambda') \otimes_{\mathbb{Z}} \mathbb{Q}$ such that there exists locally an integer n for which $p^n\rho$ is an isogeny. By abuse, one writes $\rho : \Lambda \to \Lambda'$ as for homomorphisms. We denote by $\mathrm{qisog}_S(\Lambda, \Lambda')$ the $\mathbb{Q}_p$-space of quasi-isogenies.

The objects of the category of *p-divisible groups on S up to isogeny* are the p-divisible groups over S; morphisms between Λ and Λ' are global sections of $\underline{\mathrm{Hom}}_S(\Lambda, \Lambda') \otimes_{\mathbb{Z}} \mathbb{Q}$. This is a $\mathbb{Q}_p$-linear category.

2.1.2. An important example of a p-divisible group is $\Lambda = A[p^\infty]$, the (inductive system of) p-primary torsion of an abelian scheme A over S ([**Ta66**]); in this case, the height h is twice the relative dimension of A.

2.1.3. When p is locally nilpotent on S, any p-divisible group Λ over S is formally smooth; the completion $\hat{\Lambda}$ of a p-divisible group Λ along the zero section is a formal Lie group [**Gro74**, VI,3.1]. However $\hat{\Lambda}$ is not necessarily itself a p-divisible group; it is if and only if the separable rank of the fibers of $\mathrm{Ker}[p]$ is a locally constant function on S (in which case Λ is an extension

$$1 \to \hat{\Lambda} \to \Lambda \to \Lambda^{\mathrm{et}} \to 1$$

of an ind-etale p-divisible group Λ^{et} by the infinitesimal p-divisible group $\hat{\Lambda}$, *cf.* [**Gro74**, III,7.4]). An example of an ind-etale (*resp.* infinitesimal) p-divisible group of height $h = 1$ is $\mathbb{Q}_p/\mathbb{Z}_p = \varinjlim_n \mathbb{Z}/p^n\mathbb{Z}$ (*resp.* $\hat{\mathbb{G}}_m$).

2.2. Two theorems on p-divisible groups.

2.2.1. A considerable amount of work has been devoted to p-divisible groups and their applications to the p-adic study of abelian varieties and their local moduli. Among the pioneers, let us mention: Barsotti, Tate, Serre, Grothendieck, Lubin, Messing...

In the sequel, we shall outline the main concepts and results of the Rapoport-Zink theory of period mappings for p-divisible groups, emphasizing the analogy with the complex case. We refer to their book [**RZ96**] for details and proofs. We shall also emphasize the relation to differential equations, which does not appear in [**RZ96**].

The whole theory relies upon three basic theorems on p-divisible groups and their Dieudonné modules, which we shall recall below:

(i) the rigidity theorem for p-divisible groups up to isogeny,
(ii) the Serre-Tate theorem, and
(iii) the Grothendieck-Messing theorem,

which deal with infinitesimal deformations of p-divisible groups.

2.2.2. Let $\mathfrak{v}$ be a complete discrete valued ring $\mathfrak{v}$ of mixed characteristic $(0,p)$. Let $\mathrm{Nil}_{\mathfrak{v}}$ denote the category of locally noetherian $\mathfrak{v}$-schemes S on which p is locally nilpotent. For any S in $\mathrm{Nil}_{\mathfrak{v}}$, the ideal of definition $\mathcal{J} = \mathcal{J}_{S_{\mathrm{red}}}$ of the closed subscheme S_{red} (of characteristic p) is locally nilpotent.

Theorem 2.2.3 (Rigidity theorem for p-divisible groups up to isogeny). *Every homomorphism $\overline{\rho} : \Lambda \times_S S_{\mathrm{red}} \to \Lambda' \times_S S_{\mathrm{red}}$ of p-divisible groups up to isogeny admits a unique lifting $\rho : \Lambda \to \Lambda'$. Moreover, ρ is a quasi-isogeny if $\overline{\rho}$ is.*

(It is crucial to work with p-divisible groups *up to quasi-isogeny*: if $\bar{\rho}$ were a "true" homomorphism, one could only expect to have a lifting if one allows denominators).

PROOF. (following an argument of V. Drinfeld). We may assume that S is affine, and that $\mathcal{J}$ is nilpotent. Then the connected part $\hat{\Lambda}'$ is a formal Lie group, hence there is a power p^n of p such that for any affine S-scheme S', $\hat{\Lambda}'(S')$ is killed by $[p^n]$ (in fact if $\mathcal{J}^{r+1} = 0$ on S, one can take $n = r^2$).

We observe that $\mathrm{Ker}(\Lambda'(S') \to \Lambda'(S'_{\mathrm{red}})) = \hat{\Lambda}'(S')$. We have

$$\begin{aligned}&\mathrm{Ker}(\mathrm{Hom}(\Lambda(S'), \Lambda'(S')) \to \mathrm{Hom}(\Lambda(S'_{\mathrm{red}}), \Lambda'(S'_{\mathrm{red}}))\\ &= \mathrm{Hom}(\Lambda(S'), \mathrm{Ker}(\Lambda'(S') \to \Lambda'(S'_{\mathrm{red}})))\end{aligned}$$

which is zero because $\mathrm{Ker}(\Lambda'(S') \to \Lambda'(S'_{\mathrm{red}})$ is killed by $[p^n]$ while Λ is p-divisible.

This implies the injectivity of $\mathrm{qisog}_S(\Lambda, \Lambda') \to \mathrm{qisog}_{S_{\mathrm{red}}}(\Lambda \times_S S_{\mathrm{red}}, \Lambda' \times_S S_{\mathrm{red}})$ (taking into account the fact that $\mathrm{Hom}(\Lambda(S'_{\mathrm{red}}), \Lambda'(S'_{\mathrm{red}}))$ is torsion-free).

For the surjectivity, let us first show that for any $\overline{f} \in \mathrm{Hom}(\Lambda(S'_{\mathrm{red}}), \Lambda'(S'_{\mathrm{red}}))$, there is a lifting $g \in \mathrm{Hom}(\Lambda(S'), \Lambda'(S'))$ of $p^n \overline{f}$. For any $x \in \Lambda(S')$, let $\overline{x}$ denote its image in $\Lambda(S'_{\mathrm{red}})$. Since Λ' is formally smooth, $\overline{f}(\overline{x})$ admits a lifting $y \in \Lambda'(S')$. Since $\mathrm{Ker}(\Lambda'(S') \to \Lambda'(S'_{\mathrm{red}})))$ is killed by p^n, $g(x) := [p^n]y$ is well-defined. This construction provides the desired lifting g.

It remains to show that g is an isogeny if $\overline{f}$ is. Let $\overline{f}'$ be a quasi-inverse of $\overline{f}$ ($\overline{f}'\overline{f} = [p^m]$), and g' be a lifting of $p^n\overline{f}'$. Then $g'g = [p^{2n+m}]$ by unicity of liftings. Thus g is an epimorphism, and the subscheme $\mathrm{Ker}\, g$ of $\mathrm{Ker}[p^{2n+m}]$

is finite over S. On the other hand, Λ is flat over S and the fibers of g' are flat (being isogenies). The criterium of flatness fiber by fiber [**EGA IV**, 11.3.10] implies that g' is flat, and we conclude that it is an isogeny. □

Let us observe that there is variant of both the statement and the proof of 2.2.3 for abelian schemes, instead of p-divisible groups.

Variant 2.2.4 (of theorem 2.2.3). *Let A, A' be abelian schemes over S, and let $\bar{f}$ be a homomorphism $A \times_S S_{red} \to A' \times_S S_{\rm red}$. Then there exists $n \geq 0$ and a unique lifting $g : A \to A'$ of $p^n \bar{f}$.*

The next theorem asserts that deforming an abelian scheme of characteristic p is equivalent to deforming its p-divisible group. For any S in $\mathrm{Nil}_{\mathfrak{o}}$, let $\mathrm{Def}_{p\text{-div}}(S)$ be the category of triples $(\overline{A}, \Lambda, \epsilon)$ consisting of an abelian scheme over $S_{\rm red}$, a p-divisible group Λ over S and an isomorphism $\epsilon : \overline{A}[p^\infty] \cong \Lambda \times_S S_{\rm red}$.

Theorem 2.2.5 (The Serre-Tate theorem). *The functor*

$$A \mapsto (A_{\rm red},\ \Lambda := A[p^\infty],\ \epsilon : A_{\rm red}[p^\infty] \cong \Lambda \times_S S_{\rm red})$$

induces an equivalence of categories between the category of abelian schemes over S and the category $\mathrm{Def}_{p\text{-}div}(S)$.

We again follow Drinfeld's argument [**Dri76**], *cf.* also [**Ka81**]. We may assume that S is affine and that $\mathcal{J}^{n+1} = 0$. We begin with the full faithfulness: given a homomorphism $f^\infty : A[p^\infty] \to B[p^\infty]$ such that f^∞ (mod $\mathcal{J}$) comes from a homomorphism $\overline{f} : A_{\rm red} \to B_{\rm red}$ of abelian schemes over $S_{\rm red}$, we have to show that there exists a unique homomorphism $f : A \to B$ compatible with f^∞ and $\overline{f}$. The unicity follows from the injectivity of $\mathrm{Hom}(A, B) \to \mathrm{Hom}(A_{\rm red}, B_{\rm red})$ (2.2.3). Furthermore, 2.2.4 shows the existence of a homomorphism g compatible with $p^n f^\infty$ and $p^n \overline{f}$. To show the existence of an f such that $g = p^n f = f \circ [p^n]$, the point is to show that g kills $A[p^n]$, which can be seen on g^∞.

For the essential surjectivity, we have to construct an abelian scheme A over S compatible with a given datum $(\overline{A}, \Lambda, \epsilon)$ in $\mathrm{Def}_{p\text{-div}}(S)$. By unicity of liftings, it suffices to do so locally on S. Hence we may assume that S is affine, and choose an abelian scheme A' over S and an isogeny $A'_{\rm red} \to \overline{A}$. By the above argument, p^n times the given isogeny $A'_{\rm red} \to \overline{A}$ lifts to a (unique) isogeny $e : A'[p^\infty] \to \Lambda$ of p-divisible groups over S. Its kernel is a finite locally free subgroup scheme of A', and we can form the abelian scheme $A'' := A'/\operatorname{Ker} e$. The isogeny $A''[p^\infty] \to \Lambda$ induced by e is an isomorphism (mod $\mathcal{J}$), hence is an isomorphism.

2.2.6. In the sequel, we shall have to deal occasionally with p-divisible groups over formal schemes rather than over schemes. Our formal schemes $\mathfrak{X}$ will be adic, locally noetherian; hence there is a largest ideal of definition $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$, and $\mathfrak{X} = \varinjlim \mathfrak{X}_n$, $\mathfrak{X}_n = \mathrm{Spec}(\mathcal{O}_{\mathfrak{X}}/\mathcal{J}^{n+1})$ $(\mathfrak{X}_{\rm red} = \mathfrak{X}_0)$.

A p-divisible group Λ over $\mathfrak{X}$ will be an inductive system of p-divisible groups Λ_n over $\mathfrak{X}_n$ such that $\Lambda_{n+1} \times_{\mathfrak{X}_{n+1}} \mathfrak{X}_n \cong \Lambda_n$.

If $\mathfrak{X} = \operatorname{Spf} \mathcal{A}$, this is the same as a p-divisible group over $\operatorname{Spec} \mathcal{A}$ (see [**dJ95a**, lemma 2.4.4]).

3. A stroll in the crystalline world.

ABSTRACT: Review of Dieudonné modules, crystals, Grothendieck-Messing theory, convergent isocrystals.

3.1. From Dieudonné modules to crystals.

3.1.1. At first, Dieudonné theory presents itself as an analogue of Lie theory for formal Lie groups over a perfect field k of characteristic $p > 0$. In practice, its scope is limited to the case of *commutative* formal Lie groups (in which case the classical counterpart is not very substantial). The theory extends to the case of p-divisible groups over k.

Let W be the Witt ring of k, with its Frobenius automorphism σ. The theory associates to any p-divisible group Λ over k a *Dieudonné module, i.e.* a finitely generated free module over W endowed with a σ-linear Frobenius F and a σ^{-1}-linear Verschiebung V satisfying $FV = VF = p.\mathrm{id}$ (we work with the contravariant theory: $\mathbf{D}(\Lambda) = \mathrm{Hom}(\Lambda, \mathcal{W}_k)$, where $\mathcal{W}_k$ is the Witt scheme of k, as in [**Dem72**, pp.63–71], and in [**LiO98**, 5]). This provides an *anti-equivalence of categories between p-divisible groups over k and Dieudonné modules.* The rank of $\mathbf{D}(\Lambda)$ is the height h of Λ.

Examples 3.1.2. • $\mathbf{D}(\mathbb{Q}_p/\mathbb{Z}_p) = W$, with $F = \sigma$, $V = p\sigma^{-1}$.
• $\mathbf{D}(\hat{\mathbb{G}}_m) = W(-1)$, *i.e.* W as an underlying module, with $F = p\sigma$, $V = \sigma^{-1}$.
• When $\Lambda = A[p^\infty]$ for an abelian variety A over k, $\mathbf{D}(\Lambda)$ may be identified with $\mathrm{H}^1_{\mathrm{cris}}(A/W)$.

However, the structure of Dieudonné modules is rather complicated in general unless one inverts p.

3.1.3. When k is algebraically closed, Dieudonné modules $\otimes\mathbb{Q}$ were completely classified by Dieudonné (this provides a classification of p-divisible groups over k up to isogeny). They are classified by their *slopes* $\lambda \in [0,1]\cap\mathbb{Q}$ and multiplicities $m_\lambda \in \mathbb{Z}_{>0}$ (these data are better recorded in the form of a Newton polygon). This comes as follows: each Dieudonné module $\mathbf{D}$ has a canonical increasing finite filtration by Dieudonné submodules $\mathbf{D}_\lambda$, such that the associated graded W-module is free, and with the following property: if $\lambda = a/b$ in irreducible form, $\mathrm{Gr}_\lambda \otimes\mathbb{Q}$ admits a basis of m_λ vectors x satisfying $F^b x = p^a x$.

For instance, the slope of $W(-1)$ is 1; the slope of the Dieudonné module attached to (the p-divisible group of) a supersingular elliptic curve over k is 1/2, with multiplicity 2.

3.1.4. The problem of generalizing Dieudonné theory to p-divisible groups over more general bases S (over which p is nilpotent) has been tackled and advertised by Grothendieck [**Gro74**], and further studied by many geometers.

What should be the right substitutes for the W-modules $\mathbf{D}(\Lambda)$ (*resp.* for the $W[\frac{1}{p}]$-spaces $\mathbf{D}(\Lambda)_{\mathbb{Q}}$)?

Grothendieck's proposal was to define $\mathbf{D}(\Lambda)$ as a *F-crystal* on the crystalline site of S.

The right substitute for $W[\frac{1}{p}]$-spaces was proposed later by P. Berthelot and A. Ogus under the name of *convergent F-isocrystal.* These notions, which we are about to describe, are of a higher technical level, and more abstract, than the rest of the text. Our goal being to define and study the Hodge filtration and the Gauss-Manin connection on convergent Dieudonné isocrystals, the reader who is ready to accept 3.3.3 and 3.6.7 may safely get out of this crystalline stroll and resume with §4.

3.2. The Dieudonné crystal of a p-divisible group.

3.2.1. Let $\mathfrak{v}$ be a finite extension of W of degree e. We assume that $e < p$, and that ϖ is a uniformizing parameter of $\mathfrak{v}$ such that σ extends to $\mathfrak{v}$ by setting $\sigma(\varpi) = \varpi$.

Let S be in $\mathrm{Nil}_{\mathfrak{v}}$. An S-divided power thickening $T_0 \hookrightarrow T$ is given by

(i) an S-scheme T_0,
(ii) a $\mathfrak{v}$-scheme T on which p is locally nilpotent,
(iii) a closed immersion $T_0 \hookrightarrow T$ over $\mathfrak{v}$,
(iv) a collection of maps "$\frac{x^n}{n!}$" from the ideal of definition of T_0 in T to $\mathcal{O}_T$, which satisfy the formal properties of the divided powers.

S-divided power thickenings form a category $(S/\mathfrak{v})_{\mathrm{cris}}$ (which we endow with its fppf topology). A sheaf on $(S/\mathfrak{v})_{\mathrm{cris}}$ is the data, for every $T_0 \hookrightarrow T$, of a fppf sheaf $\mathcal{E}_{T_0,T}$ on T, and for every morphism $f : (T_0 \hookrightarrow T) \to (T_0' \hookrightarrow T')$ in $(S/\mathfrak{v})_{\mathrm{cris}}$, of a homomorphism $f^*\mathcal{E}_{T_0,T} \to \mathcal{E}_{T_0',T'}$ satisfying the obvious transitivity condition. Example: $\mathcal{O}_{S_0/\mathfrak{v}}$, defined by $(\mathcal{O}_{S_0/\mathfrak{v}})_{T_0,T} = \mathcal{O}_T$. Another example: any fppf sheaf $\mathcal{F}$ on S_0 gives rise to a sheaf $\mathcal{F}_{\mathrm{cris}}$ on $(S/\mathfrak{v})_{\mathrm{cris}}$ by $(\mathcal{F}_{\mathrm{cris}})_{T_0,T} = \mathcal{F}_{T_0}$.

A *crystal* on $(S/\mathfrak{v})_{\mathrm{cris}}$ (or, abusively, on S) is a sheaf $\mathcal{E}$ of $\mathcal{O}_{S/\mathfrak{v}}$-modules on $(S/\mathfrak{v})_{\mathrm{cris}}$ such that the homomorphisms $f^*\mathcal{F}_{T_0,T} \to \mathcal{F}_{T_0',T'}$ are isomorphisms (in Grothendieck's terms: "crystals grow and are rigid"). One denotes by $\mathcal{E}_T$ the $\mathcal{O}_T$-module obtained by evaluating the crystal on $(\mathrm{id} : T \hookrightarrow T) \in (S/\mathfrak{v})_{\mathrm{cris}}$.

Because of this rigidity, and using the canonical divided powers on the ideal $\varpi\mathcal{O}_S$ (since $e < p$, $\frac{\varpi^n}{n!} \in \mathfrak{v}$ for any n), one gets an equivalence of categories between crystals on S and crystals on the scheme $S_0 = S/\varpi$ of characteristic p.

3.2.2. The Dieudonné crystal attached to a p-divisible group Λ over S is the crystalline $\mathcal{E}xt$-sheaf $\mathcal{E}xt^1(\Lambda_{\mathrm{cris}}, \mathcal{O}_{S/\mathfrak{v}})$. It can be shown that $\mathbf{D}(\Lambda)$ is a finite locally free crystal on S, of rank the height of Λ. It depends functorially in Λ (in a contravariant way).

An important, though formal, consequence of the crystalline local character of this definition is that *the formation of* $\mathbf{D}(\Lambda)$ *commutes with base change in* $(S/\mathfrak{v})_{\mathrm{cris}}$, and for any S-divided power thickening $(T_0 \hookrightarrow T)$, there is a canonical isomorphism of $\mathcal{O}_T$-modules $\mathbf{D}(\Lambda)_{T_0,T} \cong \mathbf{D}(\Lambda_{T_0})_{T_0,T}$.

It follows that the datum of $\mathbf{D}(\Lambda)$ amounts to the datum of $\mathbf{D}(\Lambda \times_S S_0)$: the Dieudonné crystal depends only on the p-divisible group modulo ϖ.

3.2.3. The pull-back of a crystal $\mathcal{E}$ on S_0 via the absolute Frobenius F_{S_0} is denoted by $\mathcal{E}^{(p)}$. An F-crystal structure on $\mathcal{E}$ is the datum of a morphism of crystals $F : \mathcal{E}^{(p)} \to \mathcal{E}$ which admits an inverse up to a power of p.

If Λ is a p-divisible group over S_0, $\mathbf{D}(\Lambda)$ is endowed by functoriality with a structure of F-crystal. The classical Dieudonné module is obtained for $S_0 = T_0 = \mathrm{Spec}(k)$, $T = \mathrm{Spf}(W) = \varinjlim \mathrm{Spec}(W/p^n)$.

Although we shall not use this fact, let us mention that the "Dieudonné functor" $\mathbf{D}$ from p-divisible groups on S_0 to F-crystals is fully faithful under further mild assumptions on S_0 (and even an equivalence with values in the category of "Dieudonné crystals", *i.e.* finite locally free F-crystals for which F admits an inverse up to p — the Verschiebung); without any further assumption on S_0, it induces an equivalence of categories between ind-etale p-divisible groups on S_0 and unit-root F-crystals (*i.e.* finite locally free F-crystals for which F is an isomorphism), *cf.* [**BM79**], [**dJ95a**].

3.3. The Hodge filtration.

3.3.1. Let $\Lambda^\vee$ be the (Serre) dual of Λ (the $\varinjlim$ of the Cartier duals of the $\Lambda[p^n]$'s). There is a canonical, locally split, exact sequence of $\mathcal{O}_S$-modules

$$0 \to F^1 \to \mathbf{D}(\Lambda)_S \to \mathrm{Lie}\,\Lambda^\vee \to 0$$

where $F^1 = \mathrm{CoLie}\,\Lambda = \omega_\Lambda$ may be identified with the module of invariant differentials on the formal Lie group $\hat{\Lambda}$. In case Λ is the p-divisible group attached to an abelian S-scheme $\underline{A}$, this exact sequence reduces to the Hodge exact sequence

$$0 \to F^1 \to \mathrm{H}^1_{\mathrm{DR}}(\underline{A}/S) \to \mathrm{Lie}\,\underline{A}^\vee \to 0.$$

3.3.2. Now let $S_0 \hookrightarrow S$ be a *nilpotent* divided power thickening (this means that the products "$\frac{x_1^{n_1}}{n_1!}$"$\ldots$"$\frac{x_1^{n_m}}{n_m!}$" vanish for $n_1 + \ldots + n_m \gg 0$); this is for instance the case if $S_0 = S/\varpi$ as before and $e < p - 1$.

The next theorem asserts that deformations of p-divisible groups are controlled by the variation of the Hodge filtration in the Dieudonné module [**Mes72**]. For any p-divisible group $\overline{\Lambda}$ on S_0, and any lifting Λ over S, we have seen that $\mathbf{D}(\Lambda)_S$ identifies with $\mathbf{D}(\overline{\Lambda})_S$, which is a lifting of $\mathbf{D}(\overline{\Lambda})_{S_0}$ to S.

Theorem 3.3.3 (The Grothendieck-Messing theorem). *The functor*

$$\Lambda \mapsto (\Lambda \times_S S_0,\ \omega_\Lambda)$$

induces an equivalence of categories between the category of p-divisible groups on S, and the category of pairs consisting of a p-divisible group $\overline{\Lambda}$ over S_0 (Zariski-locally liftable to S) and a locally direct factor vector subbundle of $\mathbf{D}(\overline{\Lambda})_S$ which lifts $\omega_{\overline{\Lambda}} \subset \mathbf{D}(\overline{\Lambda})_{S_0}$.

We have already seen the faithfulness in 2.2.3. We just indicate here the principle of proof of essential surjectivity and fullness. Working with nilpotent divided power thickenings allows one to construct, by the exponential method, a formally smooth group scheme $\mathbb{E}(\overline{\Lambda})$ over S together with a canonical isomorphism $\mathbf{D}(\overline{\Lambda})_{S_0} = \mathrm{Lie}(\mathbb{E}(\overline{\Lambda}) \times_S S_0)$, in such a way that the canonical exact sequence

$$0 \to \omega_{\overline{\Lambda}} \to \mathbf{D}(\overline{\Lambda})_{S_0} \to \mathrm{Lie}\,\overline{\Lambda}^{\vee} \to 0$$

lifts to an exact sequence of formally smooth group schemes

$$0 \to \omega_{\overline{\Lambda}} \to \mathbb{E}(\overline{\Lambda}) \times_S S_0 \to \overline{\Lambda}^{\vee} \to 0$$

Via the exponential, it is equivalent to give a local summand F^1 of $\mathbf{D}(\overline{\Lambda})_S$ which lifts $\omega_{\overline{\Lambda}}$, or to give a vector sub-group scheme V of $\mathbb{E}(\overline{\Lambda})$ which lifts $\omega_{\overline{\Lambda}}$. It then turns out that $\mathbb{E}(\overline{\Lambda})/V$ is a p-divisible group $\Lambda^{\vee}$, whose Serre dual is the looked for p-divisible group Λ over S: $\Lambda \times_S S_0 = \overline{\Lambda}$ by duality, and $\omega_\Lambda = F^1$.

The formation of $\mathbb{E}(\overline{\Lambda})$ is functorial (contravariant in $\overline{\Lambda}$). If we have two p-divisible groups $\overline{\Lambda}, \overline{\Lambda}'$ and a morphism $\overline{f} : \overline{\Lambda} = \Lambda \times_S S_0 \to \overline{\Lambda}' = \Lambda' \times_S S_0$ such that $\mathbf{D}(f)$ respects the F^1's, then the homomorphism $\mathbb{E}(\overline{f}) : \mathbb{E}(\overline{\Lambda}) \to \mathbb{E}(\overline{\Lambda}')$ preserves the vector subgroup schemes, hence induces a homomorphism $f : \mathbb{E}(\overline{\Lambda})/V \cong \Lambda \to \mathbb{E}(\overline{\Lambda}')/V' \cong \Lambda'$ which lifts $\overline{f}$.

The exponential construction becomes simpler when the ideal of S_0 in S is of square zero (and endowed with the trivial divided powers). This is the case which will be used in the sequel.

3.4. Crystals and connections.

3.4.1. We have already encountered the terms "F-crystal", "unit-root F-crystal" in I.2.4.2, 2.4.3 in the more down-to-earth context of connections. This comes as follows. Assume for simplicity that S_0 is affine of characteristic p and lifts to a formally smooth p-adic affine formal scheme $\mathcal{S}$ over $\mathfrak{o} = W$. Then there is an equivalence of categories between finite locally free crystals on $(S_0/W)_{\mathrm{cris}}$ and finite locally free $\mathcal{O}(\mathcal{S})$-modules with integrable topologically nilpotent connection (this interpretation of crystals extends to much more general situations, *cf.* [**dJ95a**, 2.2.2]).

Using this equivalence of categories, the Katz (covariant) functor

$$\begin{pmatrix}\text{continuous } \mathbb{Z}_p\text{-representations} \\ \text{of } \pi_1^{\mathrm{alg}}(S_0, s_0)\end{pmatrix} \longrightarrow (\text{unit-root } F\text{-crystals})$$

is the composite of the following (anti)equivalences of categories

$$\begin{pmatrix}\text{continuous } \mathbb{Z}_p\text{-representations} \\ \text{of } \pi_1^{\mathrm{alg}}(S_0, s_0)\end{pmatrix} \longrightarrow (\text{ind-etale } p\text{-divisible groups over } S_0)$$

$$\xrightarrow{\mathbf{D}} (\text{unit-root } F\text{-crystals over } S_0).$$

3.4.2. Let us briefly recall the construction of the connection attached to a finite locally free crystal $\mathcal{E}$ on $(S_0/W)_{\text{cris}}$. It depends on the technique of formally adding divided powers to an ideal (the pd-hull). Let Δ_0 be the diagonal in $S_0 \times S_0$, and let $\hat{\Delta}$ be the p-completion of the pd-hull of the diagonal in $\mathcal{S}\hat{\times}\mathcal{S}$, endowed with the two projections p_1, p_2 to $\mathcal{S}$. Then by rigidity of crystals, we have for any n an isomorphism

$$p_2^*(\mathcal{E}_{S_0,\mathcal{S}/p^n}) \cong \mathcal{E}_{\Delta_0,\hat{\Delta}/p^n} \cong p_1^*(\mathcal{E}_{S_0,\mathcal{S}/p^n})$$

and at the limit $n \to \infty$, an isomorphism of finite locally free $\mathcal{O}(\hat{\Delta})$-modules

$$\epsilon : \; p_2^*(\mathcal{E}_{S_0,\mathcal{S}}) \cong p_1^*(\mathcal{E}_{S_0,\mathcal{S}})$$

satisfying a cocycle condition. Such an isomorphism is the "Taylor series"

$$\epsilon(1 \otimes e) = \sum_{n_1,\ldots,n_d \geq 0} \left(\nabla\Big(\frac{\partial}{\partial t_1}\Big)^{n_1} \cdots \nabla\Big(\frac{\partial}{\partial t_d}\Big)^{n_d} \right)(e) \otimes \prod_i \frac{\text{"}(1 \otimes t_i - t_i \otimes 1)^{n_i}\text{"}}{n_i!}$$

of an integrable connection ∇ on the finite locally free $\mathcal{O}(\mathcal{S})$-module $\mathcal{E}_{S_0,\mathcal{S}}$ (here $t_1, \ldots t_d$ denote local coordinates on $\mathcal{S}$).

3.4.3. In general, the Taylor series converges only in polydiscs of radius $|p|^{1/p-1}$, due to the presence of the factorials. One says that ∇ is *convergent* if its Taylor series converges in (open) unit polydiscs. A well-known argument due to Dwork shows that F-crystals give rise to convergent connections ∇ ([**Ka73**]).

In order to put the discussion of convergence on proper foundations, we need the notion of tube in analytic geometry.

3.5. Interlude : analytic spaces associated with formal schemes and tubes.

3.5.1. Let us first review the Raynaud-Berthelot construction of the analytic space attached to a $\mathfrak{o}$-formal scheme $\mathfrak{X}$. It will be convenient for later purpose to formulate the construction in the frame of Berkovich spaces rather than rigid geometry (we refer to [**Ber93**, 1.6] for the translation).

As in 2.2.6, we assume that $\mathfrak{X}$ is adic, locally noetherian; moreover, we assume that $\mathfrak{X}_{\text{red}} = \text{Spec}(\mathcal{O}_{\mathfrak{X}}/\mathcal{J})$ is a separated k-scheme locally of finite type (in particular, $p \in \mathcal{J}$). However, we *do not* assume that $\mathfrak{X}$ is p-adic, *i.e.* that the topology of $\mathcal{O}_{\mathfrak{X}}$ is p-adic.

We describe the construction in the affine case $\mathfrak{X} = \text{Spf}(\mathcal{A})$; the general case follows by gluing. One chooses generators $f_1, \ldots, f_r$ of $J = \Gamma(\mathfrak{X}, \mathcal{J})$, and defines, for any $n > 0$,

$$\mathcal{B}_n = \mathcal{A}\{T_1, \ldots, T_r\}/(f_1^n - \varpi T_1, \ldots, f_r^n - \varpi T_r)$$

where $\mathcal{A}\{T_1, \ldots, T_r\}$ stands for the ϖ-adic completion of $\mathcal{A}[T_1, \ldots, T_r]$. One has $\mathcal{B}_n/\varpi\mathcal{B}_n \cong (\mathcal{A}/(\varpi, f_1^n, \ldots, f_r^n))[T_1, \ldots, T_r]$, a k-algebra of finite type, and $\mathcal{B}_n[\frac{1}{p}]$ is an affinoid algebra. Let $\mathcal{M}(\mathcal{B}_n[\frac{1}{p}])$ be its Berkovich spectrum.

The homomorphism $\mathcal{B}_{n+1} \to \mathcal{B}_n$ sending T_i to $f_i T_i$ identifies $\mathsf{M}(\mathcal{B}_n[\frac{1}{p}])$ with an affinoid subdomain of $\mathsf{M}(\mathcal{B}_{n+1}[\frac{1}{p}])$ [**Be96**, 0.2.6].

The analytic space $\mathfrak{X}^{\mathrm{an}}$ is defined as the union $\bigcup_n \mathsf{M}(\mathcal{B}_n[\frac{1}{p}])$. It does not depend on the choice of $f_1, \ldots, f_r$. This is a *paracompact (strictly) analytic space* over the fraction field K of $\mathfrak{v}$ (one can replace the nested "ball-like" $\mathsf{M}(\mathcal{B}_n[\frac{1}{p}])$ by nested "annulus-like" spaces in order to obtain locally finite coverings).

Examples 3.5.2. (i) if $\mathfrak{X} = \mathrm{Spf}(\mathfrak{v}\{t\})$, $\mathfrak{X}^{\mathrm{an}}$ is the closed unit disc.

(ii) if $\mathfrak{X} = \mathrm{Spf}(\mathfrak{v}[[t]])$, $\mathcal{J} = (\varpi, t)$, then $\mathsf{M}(\mathcal{B}_n[\frac{1}{p}])$ is the closed disc $\mathsf{D}_K(0, |\varpi|^{1/n})$, and $\mathfrak{X}^{\mathrm{an}}$ is the open unit disc.

3.5.3. Let S_0 be a closed subscheme of $\mathfrak{X}_{\mathrm{red}}$, and let $\mathfrak{X}_{S_0}$ be the formal completion of $\mathfrak{X}$ along S_0. The *tube of* S_0 *in* $\mathfrak{X}$ is $\mathfrak{X}^{\mathrm{an}}_{S_0}$. It is denoted by $]S_0[_{\mathfrak{X}}$ (and considered as a subspace of $\mathfrak{X}^{\mathrm{an}}$). It depends only on $S_{0,\mathrm{red}}$.

Example: if $\mathfrak{X}$ is as in example (i) above (so that $\mathfrak{X}_{\mathrm{red}} = \mathbb{A}^1_k$), and S_0 is the point 0, then $\mathfrak{X}_{S_0}$ is as in example (ii) above: $]0[_{\mathfrak{X}} = \mathsf{D}(0, 1^-) \subset \mathsf{D}(0, 1^+)$. More generally, the polydiscs mentioned in 3.4.3 are the tubes of closed points.

3.6. Convergent isocrystals.

3.6.1. Let S_0 be a separated k-scheme of finite type, and $S_0 \hookrightarrow \mathfrak{Y}$ a closed immersion into a flat p-adic formal $\mathfrak{v}$-scheme that is formally smooth in a neighborhood of S_0. Let E be a vector bundle on the tube $]S_0[_{\mathfrak{Y}}$. An integrable connection ∇ on E is *convergent* if its Taylor series

$$\epsilon(1 \otimes e) = \sum_{n_1, \ldots, n_d \geq 0} \left(\nabla\left(\frac{\partial}{\partial t_1}\right)^{n_1} \cdots \nabla\left(\frac{\partial}{\partial t_d}\right)^{n_d} \right)(e) \otimes \prod_i \frac{\text{“}(1 \otimes t_i - t_i \otimes 1)^{n_i}\text{”}}{n_i!}$$

is induced by an isomorphism on the tube of the diagonal $]S_0[_{\mathfrak{Y} \times \mathfrak{Y}}$

$$\epsilon : p_2^*(E) \cong p_1^*(E)$$

It amounts to the same to say that for any affine U in $\mathfrak{Y}$ with local coordinates $t_1, \ldots, t_d$, any section $e \in \Gamma(U^{\mathrm{an}}, E)$ and any $\eta < 1$, one has

$$\left\| \frac{1}{\prod n_i!} \nabla\left(\frac{\partial}{\partial t_1}\right)^{n_1} \cdots \nabla\left(\frac{\partial}{\partial t_d}\right)^{n_d} (e) \right\| . \eta^{n_1 + \ldots + n_d} \to 0$$

where $\| \;\|$ denotes a Banach norm on $\Gamma(U^{\mathrm{an}}, E)$.

Example: (Dwork's convergent isocrystal) One fixes π such that $\pi^{p-1} = -p$ and take $\mathfrak{v} = \mathbb{Z}_p[\pi]$, $S_0 = \mathbb{A}^1_{\mathbb{F}_p}$, $\mathfrak{Y} = \widehat{\mathbb{A}}^1_{\mathfrak{v}}$, t = a coordinate on $\mathbb{A}^1$. One has $]S_0[_{\mathfrak{Y}} = \mathsf{D}(0, 1^+)$. Then $E = \mathcal{O}_{\mathsf{D}(0,1^+)}$ endowed with the connection $\nabla(1) = -\pi\, dt$ is a convergent isocrystal: the convergence is ensured since $\pi^n/n! \in \mathfrak{v}$ (see also I.2.4.5).

According to Berthelot, *the category of vector bundles with convergent connection depends only on* S_0 — even only on $S_{0,\mathrm{red}}$ — (up to canonical equivalence), and is functorial in S_0/K; this is the category of *convergent isocrystals on* S_0/K (or, abusively, on S_0).

Since any separated k-scheme locally of finite type can be locally embedded into a $\mathfrak{X}$ as above, one can define the category of convergent isocrystals on such a scheme by gluing [**Be96**, 2.3.2].

3.6.2. There is a natural functor from the category of F-isocrystals on $S_0/\mathfrak{v}$ to the category of convergent isocrystals on S_0/K [**Be96**, 2.4]. The composite of this functor with the Dieudonné functor is denoted by $\mathbf{D}(-)_{\mathbb{Q}}$. It factors through a $\mathbb{Q}_p$-linear functor, still denoted by $\mathbf{D}(-)_{\mathbb{Q}}$:

$$\left\{\begin{matrix} p\text{-divisible groups over } S_0 \\ \text{up to isogeny} \end{matrix}\right\} \longrightarrow \{\text{convergent isocrystals on } S_0\}$$

3.6.3. The advantage of convergent isocrystals over "crystals up to isogeny" lies in their *strong "continuity property", which allows to remove the restriction* $e < p$ on ramification (as we shall do henceforth), and work conveniently with singular schemes S_0 (while divided powers present certain "pathologies" in the singular case). In our applications, S_0 will be a countable (non-disjoint) union of irreducible projective varieties, and $\mathfrak{v}$ may be highly ramified.

3.6.4. If S_0 is the reduced underlying scheme $\mathfrak{X}_{\mathrm{red}}$ of a formal scheme $\mathfrak{X}$ as in 3.5.1, there is a notion of *evaluation of* E *on* $\mathfrak{X}^{\mathrm{an}}$ [**Be96**, 2.3.2]: this is a vector bundle on $\mathfrak{X}^{\mathrm{an}}$ with an integrable connection.

By functoriality of the construction, it will be enough to deal with the affine case $\mathfrak{X} = \mathrm{Spf}\,\mathcal{A}$ and to assume that there is a closed immersion $S_0 = (\mathrm{Spf}\,\mathcal{A})_{\mathrm{red}} \hookrightarrow \mathfrak{Y}$ into a formally smooth p-adic formal $\mathfrak{v}$-scheme. One can construct inductively a compatible family of $\mathfrak{v}$-morphisms $\mathrm{Spec}(\mathcal{O}_{\mathfrak{X}}/\mathcal{J}^n) \to \mathfrak{Y}$, whence a $\mathfrak{v}$-morphism $\mathfrak{X} \to \mathfrak{Y}$. Let $v :]S_0[_{\mathfrak{X}} = \mathfrak{X}^{\mathrm{an}} \to]S_0[_{\mathfrak{Y}}$ be the associated analytic map. The looked for evaluation of E on $\mathfrak{X}^{\mathrm{an}}$ is the pull-back by v of the vector bundle with integrable connection on $]S_0[_{\mathfrak{Y}}$ defined by E.

3.6.5. Let now S be a separated scheme in $\mathrm{Nil}_{\mathfrak{v}}$ (it may also be viewed in Nil_W). Let $\mathcal{E}$ be a finite locally free F-crystal on S/W. This provides a vector bundle $\mathcal{E}_S$ on S, which only depends on the inverse image $\mathcal{E} \times_W k$ of $\mathcal{E}$ on the k-scheme S/p (*cf.* 3.2.1). On the other hand, the convergent isocrystal E on S/p attached to F-crystal $\mathcal{E} \times_W k$ (*cf.* 3.6.2) depends only on S_{red}.

3.6.6. Let us apply this to Dieudonné crystals. Let $\mathfrak{X}$ be a formal scheme as in 3.5.1, and let $\Lambda = \varinjlim \Lambda_n$ be a p-divisible on $\mathfrak{X} = \varinjlim \mathfrak{X}_n$. This gives rise to a vector bundle $\mathbf{D}(\Lambda)_{\mathfrak{X}} = \varinjlim \mathbf{D}(\Lambda_n)_{\mathfrak{X}_n}$ on $\mathfrak{X}$. On the other hand, we have the vector bundle $\mathbf{D}(\Lambda_0)_{\mathfrak{X}} =$ on $\mathfrak{X}$ and a natural homomorphism $\mathbf{D}(\Lambda_0)_{\mathfrak{X}} \to \mathbf{D}(\Lambda)_{\mathfrak{X}}$ obtained by pull-back $\mathfrak{X}_{\mathrm{red}} = \mathfrak{X}_0 \to \mathfrak{X}$.

When $e \geq p$, this is not an isomorphism in general. However, *the associated morphism of analytic vector bundles* $(\mathbf{D}(\Lambda_0)_{\mathfrak{X}})^{\mathrm{an}} \to (\mathbf{D}(\Lambda)_{\mathfrak{X}})^{\mathrm{an}}$ *is an* *isomorphism. Moreover,* $(\mathbf{D}(\Lambda_0)_{\mathfrak{X}})^{\mathrm{an}}$ *is the analytic vector bundle underlying the evaluation of* $\mathbf{D}(\Lambda_0)_{\mathbb{Q}}$ *on* $\mathfrak{X}^{\mathrm{an}}$.

Indeed, working locally, and taking in account the very construction of $\mathfrak{X}^{\mathrm{an}}$, we reduce to the case when $\mathfrak{X}$ is p-adic. Then $\mathfrak{X}/p$ is a k-scheme. Using the canonical divided powers on (p) and the rigidity of crystals, $\mathbf{D}(\Lambda \times_{\mathfrak{X}} \mathfrak{X}/p)_{\mathfrak{X}} \cong \mathbf{D}(\Lambda)_{\mathfrak{X}}$. It is thus enough to see that $\mathbf{D}(\Lambda_0)^{\mathrm{an}} \cong \mathbf{D}(\Lambda \times_{\mathfrak{X}} \mathfrak{X}/p)^{\mathrm{an}}_{\mathfrak{X}}$. This follows from the fact that the convergent isocrystals $\mathbf{D}(\Lambda_0)_{\mathbb{Q}}$ and $\mathbf{D}(\Lambda \times_{\mathfrak{X}} \mathfrak{X}/p)_{\mathbb{Q}}$ are "the same", since $\mathfrak{X}_0 = (\mathfrak{X}/p)_{\mathrm{red}}$: indeed, it is clear that $(\mathbf{D}(\Lambda_0)_{\mathfrak{X}})^{\mathrm{an}}$ (*resp.* $\mathbf{D}(\Lambda \times_{\mathfrak{X}} \mathfrak{X}/p)^{\mathrm{an}}_{\mathfrak{X}}$) coincides with the evaluation of the convergent isocrystal $\mathbf{D}(\Lambda_0)_{\mathbb{Q}}$ (*resp.* $\mathbf{D}(\Lambda \times_{\mathfrak{X}} \mathfrak{X}/p)_{\mathbb{Q}}$) on $\mathfrak{X}^{\mathrm{an}}$ (*cf.* also [**dJ95b**, 6.4], in a slightly less general context).

3.6.7. Let us draw from this two consequences:

(i) taking into account the fact that the functor $\mathbf{D}(-)_{\mathbb{Q}}$ factors through the category of p-divisible groups up to isogeny, we see that if we are given a quasi-isogeny $\Lambda_0 \to \Lambda'_0$, then there is a canonical isomorphism of vector bundles $\left(\mathbf{D}(\Lambda'_0)_{\mathfrak{X}}\right)^{\mathrm{an}} \to \left(\mathbf{D}(\Lambda)_{\mathfrak{X}}\right)^{\mathrm{an}}$.

(ii) Through the isomorphism $\left(\mathbf{D}(\Lambda_0)_{\mathfrak{X}}\right)^{\mathrm{an}} \to \left(\mathbf{D}(\Lambda)_{\mathfrak{X}}\right)^{\mathrm{an}}$ (or equivalently, through the isomorphism $\mathbf{D}\left(\Lambda \times_{\mathfrak{X}} \mathfrak{X}/p\right)^{\mathrm{an}}_{\mathfrak{X}} \to \left(\mathbf{D}(\Lambda)_{\mathfrak{X}}\right)^{\mathrm{an}}$), the analytic vector bundle $\left(\mathbf{D}(\Lambda)_{\mathfrak{X}}\right)^{\mathrm{an}}$ is naturally endowed with an integrable connection, which deserves to be called the *Gauss-Manin* connection attached to Λ.

Indeed, when $\Lambda = \underline{A}[p^\infty]$ for an abelian scheme $\underline{A}$ over $\mathfrak{X}$, $(\mathbf{D}(\Lambda)_{\mathfrak{X}})^{\mathrm{an}}$ may then be identified with the De Rham cohomology bundle $\mathrm{H}^1_{\mathrm{DR}}(\underline{A}^{\mathrm{an}}/\mathfrak{X}^{\mathrm{an}})$ together with its Gauss-Manin connection. Here, one can use the comparison theorem between crystalline and De Rham cohomology, or the theory of universal vectorial extensions and Grothendieck's $\natural$-structures, *cf.* [**MM74**], [**BBM82**].

In the first alternative, one uses the divided powers on (p) and construct a canonical isomorphism of vector bundles with connection $\mathbf{D}(\Lambda \times_{\mathfrak{X}} \mathfrak{X}/p)_{\mathfrak{X}} \cong \mathrm{H}^1_{\mathrm{DR}}(\underline{A}/\mathfrak{X})$.

In the second alternative, one uses Grothendieck's construction of a canonical connection on the universal vectorial extension $\mathbb{E}(\Lambda)$ of $\Lambda^\vee$ — *cf.* 3.3.2; the induced connection on the Lie algebra $\mathbf{D}(\Lambda)_{\mathfrak{X}}$ turns out to be the Gauss-Manin connection on $\mathrm{H}^1_{\mathrm{DR}}(\underline{A}/\mathfrak{X})$.

The Gauss-Manin connection is especially significant when $\mathfrak{X}^{\mathrm{an}}$ is smooth (even though $\mathfrak{X}$ itself may not be formally smooth).

4. Moduli problems for p-divisible groups.

ABSTRACT: Review of moduli spaces of p-divisible groups quasi-isogenous to a fixed one modulo p, with examples.

4.1. Moduli problem for p-divisible groups quasi-isogenous to a fixed one modulo p.

4.1.1. Here, we work over the discrete valuation ring $\mathfrak{o} = W = \widehat{\mathbb{Z}_p^{\mathrm{ur}}}$ (the completion of the maximal unramified extension of $\mathbb{Z}_p$), with uniformizing parameter $\varpi = p$. Recall that $\mathrm{Nil}_{\mathfrak{o}}$ denote the category of locally noetherian $\mathfrak{o}$-schemes S on which p is locally nilpotent.

Let $\overline{\Lambda}$ be a fixed p-divisible group over $k = \overline{\mathbb{F}}_p$. The following theorem is due to M. Rapoport and T. Zink [**RZ96**, 2.16, 2.32].

Theorem 4.1.2. *The functor*

$$\mathrm{Nil}_{\mathfrak{o}} \to \mathrm{Sets}$$

$$S \mapsto \left\{ (\Lambda, \rho) \;\middle|\; \begin{array}{l} \Lambda\text{: } p\text{-divisible group on } S, \\ \rho \in \mathrm{qisog}(\Lambda \times_S S_{\mathrm{red}}, \overline{\Lambda} \times_{\overline{\mathbb{F}}_p} S_{\mathrm{red}}) \end{array} \right\} \Big/ \cong$$

is representable by a formal scheme $\mathfrak{M}$ over $\mathfrak{o}$; $\mathfrak{M}_{\mathrm{red}}$ is locally of finite type over $\overline{\mathbb{F}}_p$ and its irreducible components are projective $\overline{\mathbb{F}}_p$-varieties.

Of course, the rigidity of p-divisible groups up to isogeny (2.2.3) is crucial here. The main difficulty in proving 4.1.2 is to control the powers of p which are needed in the process of lifting isogenies.

The separated formal scheme $\mathfrak{M}$ is not formally smooth in general, but has been conjectured to be flat over $\mathfrak{o}$. The group J of self-quasi-isogenies of $\overline{\Lambda}$ acts (on the right) on $\mathfrak{M}$ by $(\Lambda, \rho).j = (\Lambda, \rho \circ j)$. This group J is actually the group of $\mathbb{Q}_p$-points of an algebraic group over $\mathbb{Q}_p$.

We denote by $\mathcal{M}$ the Berkovich analytic space over $\mathbb{C}_p$ attached to $\mathfrak{M}$: $\mathcal{M} = \mathfrak{M}^{\mathrm{an}} \hat{\otimes}_K \mathbb{C}_p$ (*cf.* 3.5). This is a paracompact (strictly) analytic space. We shall see later that it is smooth, and a p-adic manifold in the sense of I.1.3.7.

4.2. Examples.

We take $\overline{\Lambda} = \overline{A}[p^\infty]$, the p-primary torsion of an *elliptic curve* $\overline{A}$ *over* $\overline{\mathbb{F}}_p$ $(h = 2)$.

4.2.1. *The ordinary case, i.e.* $\overline{A}[p](\overline{\mathbb{F}}_p) \neq 0$.

Then $\overline{\Lambda} \cong \hat{\mathbb{G}}_m \oplus \mathbb{Q}_p/\mathbb{Z}_p$, and $J = (\mathbb{G}_{m\mathbb{Q}_p})^2$. For any (Λ, ρ) in $\mathfrak{M}(S)$, the separable rank of $\mathrm{Ker}[p]$ is constant (due to the quasi-isogeny ρ). Therefore Λ is an extension

$$\hat{\Lambda} \to \Lambda \to \Lambda^{\mathrm{et}}.$$

The quasi-isogeny ρ respects this extension, hence splits into two parts $(\hat{\rho}, \rho^{\mathrm{et}})$. We have $\hat{\Lambda} \cong \hat{\mathbb{G}}_m$, $\Lambda^{\mathrm{et}} \cong \mathbb{Q}_p/\mathbb{Z}_p$. Up to isomorphism, the pair

$(\hat{\rho}, \rho^{\mathrm{et}})$ thus amounts to an element of $(\mathbb{Q}_p^\times/\mathbb{Z}_p^\times)^2 \cong \mathbb{Z}^2$. Therefore

$$\mathfrak{M} = \coprod_{\mathbb{Z}^2} \mathfrak{M}^0$$

is a disjoint sum of copies of a formal (group) scheme $\mathfrak{M}^0$ which parametrizes extensions of $\mathbb{Q}_p/\mathbb{Z}_p$ by $\hat{\mathbb{G}}_m$. It is known that

$$\mathfrak{M}^0 = \hat{\mathbb{G}}_m$$

This identification is given by the following recipe. Recall that if $S = \mathrm{Spec}(R)$, R local artinian with residue field $\overline{\mathbb{F}}_p$ and radical $\mathfrak{m}$ ($\mathfrak{m}^{n+1} = 0$), then $\hat{\mathbb{G}}_m(S) = 1 + \mathfrak{m}$; in particular $\hat{\mathbb{G}}_m(S)$ is killed by $[p^n]$. Let us choose any lift q_n of $1/p^n \in \mathbb{Q}_p/\mathbb{Z}_p$ in Λ. Then $[p^n]q_n$ is a well-defined element q in $\hat{\Lambda}(S) \cong 1 + \mathfrak{m}$, which is unchanged if n is replaced by a bigger integer (but another choice of the isomorphism $\hat{\Lambda} \cong \hat{\mathbb{G}}_m$ would change q into q^a, $a \in \mathbb{Z}_p^\times$). As a formal W-scheme, $\mathfrak{M}^0 = \mathrm{Spf}(\mathfrak{o}[[q-1]])$, with the $(p, q-1)$-adic topology. The associated analytic space $\mathcal{M}^0$ over $\mathbb{C}_p$ is the open unit disc $|q-1| < 1$.

The parameter q is thus the local modulus for deformations of $\overline{A}[p^\infty]$. On the other hand, if $p \neq 2$, we have an algebraic local modulus for deformations of $\overline{A}$ (say, with level two structure): the Legendre parameter $z = \lambda$ around the Legendre parameter ζ_{can} of the canonical lifting A_{can} of $\overline{A}$ (*cf.* I.3.4.2). The p-divisible group $A_{\mathrm{can}}[p^\infty]$ splits: its parameter is $q = 1$. Therefore, in this special case, the Serre-Tate theorem 2.2.5 asserts that $q \in 1 + (z - \zeta_{\mathrm{can}})\widehat{\mathbb{Z}}^{\mathrm{ur}}[[z - \zeta_{\mathrm{can}}]]$. It has been proved by W. Messing and N. Katz that q is the Dwork-Serre-Tate parameter discussed in I.3.4, ([**Mes76**], [**Ka81**]) — which explains the terminology.

4.2.2. *The supersingular case, i.e.* $\overline{A}[p](\overline{\mathbb{F}}_p) = 0$.

In this case $\overline{\Lambda} = \hat{\overline{\Lambda}}$ is a so-called Lubin-Tate formal group, End $\overline{\Lambda} = \mathcal{B}_p$, "the" maximal order in the (non-split) quaternion algebra B_p over $\mathbb{Q}_p$, and $J = B_p^\times$.

Here again, there is a discrete invariant: the "height" of the quasi-isogeny ρ, which amounts to the p-adic valuation of the norm of its image in $J = B_p^\times$. Therefore

$$\mathfrak{M} = \coprod_{\mathbb{Z}} \mathfrak{M}^0$$

is a disjoint sum of copies of a formal scheme $\mathfrak{M}^0$ which parametrizes deformations of the Lubin-Tate group $\hat{\overline{\Lambda}}$. According to Lubin-Tate, $\mathfrak{M}^0 \cong \mathrm{Spf}(\mathfrak{o}[[t]])$, with the (p,t)-adic topology. The associated analytic space $\mathcal{M}^0$ over $\mathbb{C}_p$ is the open unit disc $|t| < 1$.

Assume for simplicity that $\overline{A}$ is defined over $\mathbb{F}_p$. Then one can describe the universal formal deformation Λ over $\mathrm{Spf}\,\mathbb{Z}_p[[t]]$ through its logarithm $\sum b_n(t) X^{p^{2n}}$, which is given recursively by

$$b_0(t) = 1,\ b_n(t) = \frac{1}{p} \sum_{0 \leq m < n} b_m(t) t_{n-m}^{p^{2m}}$$

The so-called canonical lifting, corresponding to $t = 0$, has formal complex multiplication by $\mathbb{Z}_{p^2}$, *cf.* [**GH94a**], [**GH94b**].

4.3. Decorated variants.

4.3.1. Just as in the complex case 1.1.4, it is useful to consider variants or the moduli problem 4.1 for "decorated" p-divisible groups, with prescribed endomorphisms and/or polarization.

Instead of the $\mathbb{Q}$-algebra B of 1.1.4, we shall consider a finite-dimensional semi-simple $\mathbb{Q}_p$-algebra B_p, since the category of p-divisible groups up to isogeny is $\mathbb{Q}_p$-linear. Let V_p be a B_p-module of finite type, and let $G_{\mathbb{Q}_p}$ be the group of B_p-linear endomorphisms of V_p.
Let $\mathcal{B}_p$ be a maximal order in B_p, and let L_p be a lattice in V_p, stable under $\mathcal{B}_p$.

4.3.2. We assume that $\mathcal{B}_p$ acts on our p-divisible group $\overline{\Lambda}$. One then redefines J to be the group of self-quasi-isogenies of $\overline{\Lambda}$ which respect the $\mathcal{B}_p$-action.

For technical reasons, one picks up a lifting $\tilde{\overline{\Lambda}}$ of $\overline{\Lambda}$ with $\mathcal{B}_p$-action to some finite extension of W, and looks at the $\mathcal{B}_p$-module $F_0^1 := \omega_{\tilde{\overline{\Lambda}}}$. We denote by $\mathfrak{v}$ the finite extension $W[\mathrm{tr}(b|F_0^1)]$ of W.

One strengthens the moduli problem 4.1 by imposing to our pairs (Λ, ρ) that Λ is endowed with a $\mathcal{B}_p$-action $\iota : \mathcal{B}_p \longrightarrow \mathrm{End}\,\Lambda$, ρ respects the $\mathcal{B}_p$-action, and moreover a "Shimura type condition" $\det(\iota(b)|\omega(\Lambda)) = \det(b|F_0^1)$.

It follows easily from 4.1.2 [**RZ96**, 3.25] that this moduli problem is *representable by a formal scheme, still denoted by $\mathfrak{M}$, over $\mathfrak{v}$. It is acted on by J. $\mathfrak{M}_{\mathrm{red}}$ is locally of finite type over $\overline{\mathbb{F}}_p$ and its irreducible components are projective $\overline{\mathbb{F}}_p$-varieties.*

4.3.3. Let us say a word about the polarized variant (here, it is safer to assume $p \neq 2$). One assumes that $\mathcal{B}_p$ is endowed with an involution $*$, and that L_p is autodual for an alternate $\mathbb{Z}_p$-bilinear form such that $\langle bv, w\rangle = \langle v, b^*w\rangle$. Of course, here, $G_{\mathbb{Q}_p}$ denotes the algebraic $\mathbb{Q}_p$-group of B_p-linear symplectic similitudes of V_p. One assumes that $\overline{\Lambda}$ is endowed with a $*$-polarization [(2)], and modify J to respect the polarization up to a multiple in $\mathbb{Q}_p$.

The corresponding moduli problem is again representable by a formal scheme with the same properties. This formal scheme has been conjectured to be flat by Rapoport-Zink; although this has been verified in many cases (*cf. e.g.* [**Gor01**]), there are some counter-examples (*cf.* [**P00**]).

Example 4.3.4 (fake elliptic curves at a critical prime p). Let us fix a indefinite quaternion algebra B over $\mathbb{Q}$ and a maximal order $\mathcal{B}$ in B (after Eichler, they are all conjugate). A prime p is called *critical* if it divides the discriminant of B, *i.e.* if $B_p = B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is the (non-split) quaternion algebra over

[(2)] a $*$-polarization is a symmetric $\mathcal{B}_p$-linear quasi-isogeny between the p-divisible and its Serre dual, endowed with the transposed action of $\mathcal{B}_p$ twisted by $*$

$\mathbb{Q}_p$. Note that B_p then contains copies of the unramified quadratic extension on $\mathbb{Q}_p$; we denote by $\mathbb{Q}_{p^2}$ one of them. The maximal order (unique up to conjugation) can be written $\mathcal{B}_p = \mathcal{B} \otimes \mathbb{Z}_p = \mathbb{Z}_{p^2}[\Pi]$, $\Pi^2 = p$, $\Pi.a = a^\sigma \Pi$ for $a \in \mathbb{Z}_{p^2}$. We fix a critical prime p.

We consider abelian surfaces with quaternionic multiplication by $\mathcal{B}$. They are sometimes called "*fake elliptic curves*" because they share many features with elliptic curves, *cf.* [**BC91**, III, 1].

We fix a fake elliptic curve $\overline{A}$ over $\overline{\mathbb{F}}_p$, and set $\Lambda = \overline{A}[p^\infty]$. This is a special formal $\mathcal{B}_p$-modules of height 4 in the following sense.

Let S be in $\mathrm{Nil}_{\mathfrak{v}}$ (or else an inductive limit of such schemes — as a formal scheme), and let Λ be a p-divisible group over S. After Drinfeld, one says that Λ is a *special formal $\mathcal{B}_p$-modules of height* 4 if

(i) Λ is infinitesimal ($\Lambda = \hat{\Lambda}$) of height $h = 4$,
(ii) $\mathcal{B}_p$ acts on Λ,
(iii) via this action, ω_Λ is a locally free $(\mathbb{Z}_{p^2} \otimes_{\mathbb{Z}_p} \mathcal{O}_S)$-module of rank 1.

We take $V_p = B_p$ where B_p acts by left multiplication. In this situation, $G_{\mathbb{Q}_p} \cong B_p^\times$. When $S = \mathrm{Spec}(\overline{\mathbb{F}}_p)$, there is a unique class of $\mathcal{B}_p$-isogeny of special formal $\mathcal{B}_p$-modules of height 4. The group J of self-quasi-isogenies of a special formal $\mathcal{B}_p$-module of height 4 is $J = GL_2(\mathbb{Q}_p)$. This can be easily read off the Dieudonné module $\otimes\mathbb{Q}$.

In this example, there is again a discrete invariant: the "height" of the quasi-isogeny ρ, which amounts to the p-adic valuation of the determinant of its image in J. Therefore

$$\mathfrak{M} = \coprod_{\mathbb{Z}} \mathfrak{M}^0$$

is a disjoint sum of copies of a formal scheme $\mathfrak{M}^0$ which parametrizes deformations of the special formal module $\hat{\overline{\Lambda}}$. This formal scheme has been described by Drinfeld, *cf.* 6.3.4.

4.3.5. This example and examples 4.2 were the sources of the general theory. Let us at once suggest a vague analogy with the archimedean situation — especially 1.1.4, 1.2.5 — which will become more and more precise in the next sections: one can see the p-divisible group $\overline{\Lambda}$ in characteristic p (*resp.* J) as an analogue of the infinitesimal oriented real Lie group $\hat{U}$ (*resp.* $G(\mathbb{R})^0$).

5. p-adic period domains.

ABSTRACT: Review of p-adic symmetric domains and period domains via stability theory.

5.1. p-adic flag spaces.

5.1.1. We begin with the p-adic analogue of the flag space $\mathcal{D}^\vee$ of 1.2.3, 1.2.5. In 1.2.5, this flag space parametrized the lagrangian subspaces F^1 in the $\mathrm{H}^1_{\mathrm{DR}}$ of abelian varieties of a certain type, marked by a fixed isomorphism $\mathrm{H}^1_{\mathrm{DR}} \cong V^{\mathbb{C}}$.

In the p-adic case, $\mathcal{D}^\vee$ will parametrize the Hodge filtration F^1 of Dieudonné modules of p-divisible groups of a certain type (3.3.1).

5.1.2. More precisely, let us consider a moduli problem for p-divisible groups as in 4.1, possibly decorated as in 4.3 (from which we follow the notation). The decoration reflects on the Dieudonné module $\mathbf{D}(\overline{\Lambda})_{\mathbb{Q}} = \mathbf{D}(\overline{\Lambda}) \otimes \mathbb{Q}$: one has an isomorphism of B_p-module (with symplectic structure, in the polarized case):

$$\mathbf{D}(\overline{\Lambda})_{\mathbb{Q}} \cong V_p \otimes_{\mathbb{Q}_p} \widehat{\mathbb{Q}_p^{\mathrm{ur}}}.$$

We fix such an isomorphism, and consider that $G_{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} \widehat{\mathbb{Q}_p^{\mathrm{ur}}}$ acts on $\mathbf{D}(\overline{\Lambda})_{\mathbb{Q}}$. But $\mathbf{D}(\overline{\Lambda})_{\mathbb{Q}}$ has an extra structure: the σ-linear Frobenius automorphism. The group J is nothing but the ***subgroup*** of $G_{\mathbb{Q}_p}(\widehat{\mathbb{Q}_p^{\mathrm{ur}}})$ of elements commuting with Frobenius.

5.1.3. Let $\mathfrak{o}$ be the finite extension of $W = \mathbb{Z}_p^{\mathrm{ur}}$ considered in 4.3.2 and let K be its fraction field. We denote here by $F_0^1 \subset V_p^{\mathbb{C}_p} = V_p \otimes \mathbb{C}_p$ the $\mathbb{C}_p$-span of the space $\omega_{\overline{\Lambda}} \subset V_p \otimes_{\mathbb{Q}_p} K$ introduced in 4.3.2. We set $G_{\mathbb{C}_p} = G_{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} \mathbb{C}_p$. The flag variety $\mathcal{D}^\vee$ of all $G_{\mathbb{C}_p}$-conjugates of F_0^1 in $V_p^{\mathbb{C}_p} = \mathbf{D}(\overline{\Lambda}) \otimes_W \mathbb{C}_p$ is a homogeneous space

$$\mathcal{D}^\vee = P \backslash G_{\mathbb{C}_p}$$

for a suitable parabolic subgroup P. There is an ample line bundle $\mathcal{L}$ on $\mathcal{D}^\vee$ which is homogeneous under the derived group $G_{\mathbb{C}_p}^{\mathrm{der}}$; we denote by $\mathcal{D}^\vee \subset \mathbb{P}(W)$ an associated $G_{\mathbb{C}_p}^{\mathrm{der}}$-equivariant projective embedding (if $G_{\mathbb{C}_p} = GL_{2g}$, $\mathcal{D}^\vee = \mathrm{Grass}(g, 2g)$, one can just take the SL_{2g}-equivariant Plücker embedding).

Note that, there is an obvious right action of J on $\mathcal{D}^\vee$, which factors through the adjoint group $J^{\mathrm{ad}} := \mathrm{Im}(J \to G_{\mathbb{C}_p}^{\mathrm{ad}})$.

5.1.4. In our three basic examples (4.2.1, 4.2.2, 4.3.4), we have $G_{\mathbb{C}_p} = GL_2$, hence $\mathcal{D}^\vee = \mathbb{P}^1_{\mathbb{C}_p}$.

5.2. Symmetric domains via stability theory.

5.2.1. In the p-adic case, one cannot define symmetric spaces by a positivity condition as in 1.2.3, 1.2.5. One uses instead the Hilbert-Mumford notion of semi-stability, following an idea of M. van der Put and H. Voskuil [**vdPV92**].

5.2.2. We set $J' = J \cap G^{\mathrm{der}}_{\mathbb{C}_p}$. This is again the group of $\mathbb{Q}_p$-points of an algebraic group over $\mathbb{Q}_p$. Let $T \cong (\mathbb{G}_{m\mathbb{Q}_p})^j$ be a maximal $\mathbb{Q}_p$-split torus in J'. The *semi-stable locus* of $\mathcal{D}^\vee$ with respect to T-action is the open subset of points $x \in \mathcal{D}^\vee$ such that there is a T-invariant function of $\mathbb{P}(W)$ which does not vanish on x.

One can check whether x is semi-stable with respect to T using the Hilbert criterion: let λ be any cocharacter (= one-parameter subgroup) of T. By properness of $\mathcal{D}^\vee$, the map $z \in \mathbb{G}_m \mapsto x.\lambda(z) \in \mathcal{D}^\vee$ extends to $\mathbb{A}^1$, *i.e.* the limit $x_0 = \lim_{z \to 0} x.\lambda(z)$ exists. It is clearly fixed by $\mathbb{G}_m$, thus the fiber $\mathcal{L}_{x_0}$ corresponds to a character $z \mapsto z^{-\mu(x,\lambda)}$ of $\mathbb{G}_m$. Then x is semi-stable if and only if the Mumford invariant $\mu(x, \lambda)$ is ≥ 0 for every λ.

5.2.3. One defines the *symmetric space* $\mathcal{D}$ *to be the intersection of the semi-stable loci of* $\mathcal{D}^\vee$ *with respect to all maximal* $\mathbb{Q}_p$*-split tori* T *in* J'. This is a p-adic manifold in the sense of I.1.3.6. It is also called the *p-adic period domain* (associated with the moduli problem).

Since J acts on the set of tori T by conjugation, it is clear that $\mathcal{D}$ is stable under the J-action on $\mathcal{D}^\vee$.

5.3. Examples.

5.3.1. In our three basic examples, we have $G_{\mathbb{C}_p} = GL_2$, $\mathcal{D}^\vee = \mathbb{P}^1$.

In example 4.2.1, $J' \subset J = (\mathbb{G}_m)^2$ is the one-dimensional torus T with elements (z^{-1}, z). Its action on a point x is by z^2-scaling; the limit point x_0 is always 0, except if $x = \infty$. The only non-semi-stable point is ∞, hence $\mathcal{D} = \mathbb{A}^1$.

In example 4.2.2, $J' = SL_1(B_p) \subset J = B_p^\times$. There is no non-trivial $\mathbb{Q}_p$-split torus in J', hence $\mathcal{D} = \mathcal{D}^\vee = \mathbb{P}^1$.

5.3.2. We now turn to example 4.3.4. In this example, $J = GL_2(\mathbb{Q}_p)$ and $J' = SL_2(\mathbb{Q}_p)$. All maximal $\mathbb{Q}_p$-split tori are conjugate $g^{-1}Tg$ of the above T. Therefore, $\mathcal{D} = \mathbb{P}^1 \setminus \bigcup \infty.g$ is the *Drinfeld space* over $\mathbb{C}_p$

$$\mathcal{D} = \Omega_{\mathbb{C}_p} := \mathbb{P}^1_{\mathbb{C}_p} \setminus \mathbb{P}^1(\mathbb{Q}_p)$$

To be consistent, we consider the right action of PGL_2-action on $\mathbb{P}^1$, which is deduced from the customary left action by the rule $x.g = g^{-1}.x$.

The Drinfeld space is actually defined over $\mathbb{Q}_p$, *i.e.* is a $\mathbb{Q}_p$-analytic manifold Ω. It would follow from the *Bruhat-Tits tree* $\mathcal{T}$ of $J^{\mathrm{ad}} = PGL_2(\mathbb{Q}_p)$. The vertices of this tree correspond to closed discs in $\mathbb{Q}_p$; two such vertices are connected by an edge if for the corresponding discs $\mathrm{D}', \mathrm{D}''$, one has $\mathrm{D}' \subset \mathrm{D}''$ and the radius of D'' is p times the radius of D' (each vertex has $p + 1$ neighbors). The group $PGL_2(\mathbb{Q}_p)$ acts on $\mathcal{T}$ (on the right, say).

For any disc D centered in $\mathbb{Q}_p$, let η_{D} denote its Berkovich "generic point" (which is a point in Berkovich's affine line $\mathbb{A}^1$ over $\mathbb{Q}_p$). Then the

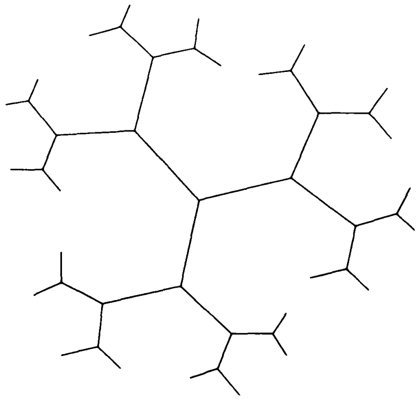

FIGURE 1. $\mathcal{T}$ for $p = 2$

geometric realization of $\mathcal{T}$ may be identified with the closed subset

$$|\mathcal{T}| = \{\eta_{\mathrm{D}(a,r^+)} \mid a \in \mathbb{Q}_p,\ 0 < r < \infty\} \subset \Omega$$

(a $PGL_2(\mathbb{Q}_p)$-equivariant embedding), and the set of vertices with the subset

$$\operatorname{ver} \mathcal{T} = \{\eta_{\mathrm{D}(a,r^+)} \mid a \in \mathbb{Q}_p,\ r \in p^{\mathbb{Z}}\} \subset |\mathcal{T}|$$

According to Berkovich, $|\mathcal{T}|$ is a retract of Ω [**Ber90**]; in particular, Ω is simply connected.

On the other hand, Ω is the generic fiber of a p-adic formal scheme $\hat{\Omega}$ whose reduction modulo p is an infinite tree of projective lines, with dual graph $\mathcal{T}$ (*cf.* [**BC91**, I]).

The Drinfeld space Ω is a p-adic analogue of the Poincaré double half-plane $\mathbb{P}^1_{\mathbb{C}} \setminus \mathbb{P}^1(\mathbb{R})$. Note that the latter space could also be defined using stability, in the same way (with $J' = SL_2(\mathbb{R})$). Another similarity: according to [**Ber90**], there is a $PGL_2(\mathbb{Q}_p)$-invariant metric on Ω (which extends the standard metric on $|\mathcal{T}|$).

6. The p-adic period mapping and the Gauss-Manin connection.

ABSTRACT: Rapoport-Zink's period mapping as classifying map for the Hodge filtration in the Dieudonné module. Basic properties: equivariance, étaleness. Its relation to the Gauss-Manin connection. Examples and formulas.

6.1. Construction.

6.1.1. The construction of the p-adic period mapping is parallel to the complex case: one attaches to a point $s \in \mathcal{M}$ representing a pair (Λ, ρ) the point $\mathcal{P}(s)$ of $\mathcal{D}^\vee$ which parametrizes the notch F^1 of the Hodge filtration of the Dieudonné module of Λ. The idea of the construction is already present in Grothendieck's talk [**Gro70**].

6.1.2. More precisely, notation being as in 4.1.1–4.3.1, let us consider the universal object $(\underline{\Lambda}, \underline{\rho})$ on the moduli formal scheme $\mathfrak{M}$, and the Dieudonné crystal $\mathbf{D}(\underline{\Lambda})$. Its evaluation $\mathbf{D}(\underline{\Lambda})_{\mathfrak{M}}$ on $\mathfrak{M}$ is a vector bundle on $\mathfrak{M}$.

Lemma 6.1.3. *The quasi-isogeny $\underline{\rho}$ induces a trivialization of the associated analytic vector bundle on $\mathcal{M} = \mathfrak{M}^{\mathrm{an}}_{\mathbb{C}_p}$:*

$$(\mathbf{D}(\underline{\Lambda})_{\mathfrak{M}})^{\mathrm{an}} \cong V_p^{\mathbb{C}_p} \otimes_{\mathbb{C}_p} \mathcal{O}_{\mathcal{M}}$$

PROOF. ρ is a quasi-isogeny between $\underline{\Lambda} \times_{\mathfrak{M}} \mathfrak{M}_{\mathrm{red}}$ and $\overline{\Lambda} \times_{\overline{\mathbb{F}}_p} \mathfrak{M}_{\mathrm{red}}$. According to 3.6.7 (i), by the rigidity of convergent isocrystals, it induces a canonical isomorphism $(\mathbf{D}(\underline{\Lambda})_{\mathfrak{M}})^{\mathrm{an}} \cong \left(\mathbf{D}(\overline{\Lambda} \times_{\overline{\mathbb{F}}_p} \mathfrak{M}_{\mathrm{red}})_{\mathfrak{M}}\right)^{\mathrm{an}}$. On the other hand, since the formation of $\mathbf{D}(\Lambda)$ commutes with base change (3.2.2), we have

$$\left(\mathbf{D}(\overline{\Lambda} \times_{\overline{\mathbb{F}}_p} \mathfrak{M}_{\mathrm{red}})_{\mathfrak{M}}\right)^{\mathrm{an}} = \mathbf{D}(\overline{\Lambda}) \otimes_W \mathcal{O}_{\mathcal{M}} = V_p \otimes_{\mathbb{Q}_p} \mathcal{O}_{\mathcal{M}}.$$

□

6.1.4. On the other hand, we have a locally direct summand $F^1 = (\omega_{\underline{\Lambda}})^{\mathrm{an}}$ in $(\mathbf{D}(\underline{\Lambda})_{\mathfrak{M}})^{\mathrm{an}} \cong V_p \otimes_{\mathbb{Q}_p} \mathcal{O}_{\mathcal{M}}$ (3.3.1). This determines an $\mathcal{M}$-point of the flag space $\mathcal{D}^\vee$, *i.e.* an analytic mapping

$$\mathcal{P} : \mathcal{M} \longrightarrow \mathcal{D}^\vee$$

This is called the *period mapping* for the moduli problem (decorated or not). It is clear from the construction that $\mathcal{P}$ is *J-equivariant.*

If $\mathcal{M}^0$ is any connected component of $\mathcal{M}$, it is stable under $J' \subset J$, hence the induced period mapping $\mathcal{P} : \mathcal{M}^0 \longrightarrow \mathcal{D}^\vee$ is J'-equivariant (the J'-action factors through J'^{ad}).

6.2. Properties.

Theorem 6.2.1. *The period mapping $\mathcal{P}$ is etale. Its image lies in the period domain $\mathcal{D}$.*

The first assertion reduces, via an infinitesimal criterion of etaleness, to the "essential surjectivity part" of the Grothendieck-Messing theorem (3.3.3), in the case of an ideal of square zero, *cf.* [**RZ96**, 5.17]. It corresponds to what is called the "local Torelli property" in the complex case.

The second assertion is a consequence of a theorem of B. Totaro on the equivalence of semi-stability and "weak admissibility" [**Tot96**]. The second assertion may be viewed as a p-adic analogue of the "Riemann relations" (1.1.3).

Corollary 6.2.2. *$\mathcal{M}$ is smooth, and any component $\mathcal{M}^0$ is a connected p-adic manifold in the sense of I.1.3.7.*

6.2.3. It is very likely that $\mathcal{P}$, viewed as a mapping $\mathcal{M} \to \mathcal{D}$, is surjective. This is closely related to Fontaine's conjecture on the realizability of filtered Dieudonné modules by p-divisible groups. Recent work by C. Breuil [**Bre99**], [**Bre00**] settles the Fontaine conjecture at least for finite residue fields of characteristic $\neq 2$, but it seems that the surjectivity conjecture is not yet fully established.

On the other hand, the *fibers* of the period mapping are relatively well-understood, *cf.* [**RZ96**, 5.37].

6.2.4. The argument in the proof of 6.1.3 shows a little more. Recall from 3.6.7 (ii) that the analytic vector bundle $(\mathbf{D}(\underline{\Lambda})_{\mathfrak{M}})^{\mathrm{an}}$ on $\mathcal{M}$ comes together with an integrable connection ∇_{GM} (the Gauss-Manin connection).

Lemma 6.2.5. *The trivialization*

$$(\mathbf{D}(\underline{\Lambda})_{\mathfrak{M}})^{\mathrm{an}} \cong V_p^{\mathbb{C}_p} \otimes_{\mathbb{C}_p} \mathcal{O}_{\mathcal{M}}$$

of 6.1.3 induces a trivialization of the Gauss-Manin connection:

$$\left((\mathbf{D}(\underline{\Lambda})_{\mathfrak{M}})^{\mathrm{an}}\right)^{\nabla_{\mathrm{GM}}} \cong V_p^{\mathbb{C}_p}.$$

The point is that the convergent isocrystal $\mathbf{D}(\Lambda \times_{\mathfrak{M}} \mathfrak{M}_{\mathrm{red}})$ is constant, since it comes by base change, from $\mathbf{D}(\overline{\Lambda})_{\mathbb{Q}}$.

This useful lemma will allow us to give explicit expression of the period mapping in terms of quotients of solutions of differential equations, as in the complex case.

6.3. Examples and formulas.

6.3.1. We start with example 4.1.1 (p-divisible groups of elliptic curves with ordinary reduction). The extension

$$1 \to \hat{\mathbb{G}}_m \to \Lambda \to \mathbb{Q}_p/\mathbb{Z}_p \to 1$$

parametrized by $q \in 1 + pW$ gives rise, by contravariance, to an extension of Dieudonné modules (*cf.* 3.1.1)

$$0 \to W \to \mathbf{D}(\Lambda) \to W(-1) \to 0$$

which splits canonically due to the rigidity of Dieudonné crystals: $\mathbf{D}(\Lambda) = \mathbf{D}(\overline{\Lambda}) = W \oplus W(-1)$. We denote by e_0, e_{-1} the corresponding canonical basis of $\mathbf{D}(\Lambda)$. The image $\tau = \mathcal{P}(q) \in W[\frac{1}{p}]$ of q under the period mapping is, by definition, the unique element such that $\tau e_0 + e_{-1}$ generates F^1. It turns out that

$$\tau = \log q,$$

thus

$$\mathcal{M}^0 = \mathsf{D}(1, 1^-) \xrightarrow{\mathcal{P}=\log} \mathbb{A}^1 = \mathcal{D}.$$

Let us indicate a sketch of proof of this remarkable formula (the formula goes back to Messing-Katz [**Mes76**], [**Ka81**]). The argument is more transparent if we think in terms of one-motives instead of elliptic curves: that is, rather than viewing Λ as the p-divisible attached to some elliptic curve over W, one remarks that it is also the p-divisible group attached to the one-motive $M = [\mathbb{Z} \xrightarrow{1\mapsto q} \mathbb{G}_m]$. Then $\mathbf{D}(\Lambda)$ can be identified with $H_{\mathrm{DR}}(M)$, the dual of the Lie algebra of $G^\natural$, where $[\mathbb{Z} \to G^\natural]$ denotes the universal vectorial extension of M (*cf.* [**Del74**]); this extension splits canonically ($G^\natural = \mathbb{G}_a \times \mathbb{G}_m$), and this splitting agrees with the above splitting of $\mathbf{D}(\Lambda)$. We are thus reduced to prove the following purely algebraic fact: let us view $M = [\mathbb{Z} \xrightarrow{1\mapsto q} \mathbb{G}_m]$ as a one-motive over $K((q-1))$; then in terms of the canonical basis e_0, e_{-1} of $H_{\mathrm{DR}}(M)$, $F^1 \subset H_{\mathrm{DR}}(M)$ is generated by the vector $(\log q)e_0 + e_{-1}$. This well-known fact is easily proved by transcendental means, replacing K by $\mathbb{C}$.

6.3.2. Since the basis e_0, e_{-1} is horizontal under the Gauss-Manin connection ∇ on $\mathsf{D}(1, 1^-)$ (6.2.5)[(3)], τ is a quotient of solutions of the Gauss-Manin connection. In terms of the Legendre parameter $z = \lambda$ (around the parameter ζ_{can} of the canonical lifting A_{can} of the fixed ordinary elliptic curve $\overline{A}$), the equality $\tau = \log q$ means that $q \in 1 + (z - \zeta_{\mathrm{can}})W[[z - \zeta_{\mathrm{can}}]]$ is the Dwork-Serre-Tate parameter discussed in I.3.4 (for $p \neq 2$).

This allows to give explicit formulas for $\mathcal{P}$ as a function of the algebraic parameter z rather than q. For instance, let us assume that $p \equiv 1$ (4). The disc $|z - \frac{1}{2}| < 1$ is an ordinary disc. The canonical lifting is $\zeta_{\mathrm{can}} = \frac{1}{2}$ (complex multiplication by $\mathbb{Z}[\sqrt{-1}]$). The Gauss-Manin connection is represented by the hypergeometric differential operator with parameter $(\frac{1}{2}, \frac{1}{2}, 1)$. Let us recall the setting, which we have already met in I.5.3.11.

We consider the symplectic basis $\omega_1 = \left[\frac{dx}{2y}\right]$, $\omega_2 = \left[\frac{(2x-1)dx}{4y}\right]$ of relative De Rham cohomology. It has the virtue to be, at $z = \frac{1}{2}$, a basis of eigenvectors for the complex multiplication. The Gauss-Manin connection satisfies

$$\omega_2 = 2z(z-1)\nabla\Big(\frac{d}{dz}\Big)\omega_1 + \frac{4z-5}{6}\,\omega_1.$$

The first row of the solution matrix Y normalized by $Y(\frac{1}{2}) = \mathrm{id}$ is given by

$$y_{11} = F\Big(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}; (1-2z)^2\Big) + \frac{1}{2}(1-2z)F\Big(\frac{3}{4}, \frac{3}{4}, \frac{3}{2}; (1-2z)^2\Big),$$

$$y_{12} = (1-2z)F\Big(\frac{3}{4}, \frac{3}{4}, \frac{3}{2}; (1-2z)^2\Big).$$

We have to relate $\omega_1|_{1/2}$, $\omega_2|_{1/2}$ to the symplectic basis e_0, e_{-1}. Because $z = 1/2$ gives the canonical lifting, the canonical splitting of the Dieudonné

(3) which is induced by the usual Gauss-Manin connection attached to the Legendre elliptic pencil, *cf.* 3.6.6 (ii).

module is given by the complex multiplication: e_0, e_{-1} is a basis of eigenvectors. In fact, taking into account the fact that we deal with symplectic bases, we find

$$\omega_1|_{1/2} = \frac{\Theta}{2} e_{-1}, \quad \omega_2|_{1/2} = \frac{2}{\Theta} e_0,$$

where $\Theta \in W^\times$ denotes the Tate constant (for $z = \frac{1}{2}$) discussed in I.3.4.1, 4.5. (the ambiguity — by a factor in $\mathbb{Z}_p^\times$ — is removed since the basis e_0, e_{-1} has been fixed). The period mapping $\tau = \tau(z)$ is given by the quotient of the entries in the first row of the matrix $Y.\begin{pmatrix} \Theta/2 & 0 \\ 0 & 2/\Theta \end{pmatrix}$, that is to say:

$$\tau(z) = \left(\frac{\Theta^2}{4}\right) \frac{(1-2z)F(\frac{3}{4},\frac{3}{4},\frac{3}{2};(1-2z)^2)}{F(\frac{1}{4},\frac{1}{4},\frac{1}{2};(1-2z)^2) + \frac{1}{2}(1-2z)F(\frac{3}{4},\frac{3}{4},\frac{3}{2};(1-2z)^2)}$$

Note that since the target of the period mapping is $\mathbb{A}^1$, the denominator does not vanish on $\mathsf{D}(\frac{1}{2}, 1^-)$. Note also that, in analytic terms, the etaleness of $\mathcal{P}$ (6.2.1) reflects the fact that $\frac{d\tau}{dz}$ does not vanish, being the quotient of linearly independent solutions of a linear differential equation of rank two.

6.3.3. We now turn to example 4.1.2 (p-divisible groups of elliptic curves with supersingular reduction). In this case, the period mapping

$$\mathcal{M}^0 \cong \mathsf{D}(0,1^-) \xrightarrow{\mathcal{P}} \mathbb{P}^1 = \mathcal{D}$$

was thoroughly investigated by Gross-Hopkins [**GH94a**], [**GH94b**]. In the special case considered in 4.1.2. ii) (the fixed supersingular elliptic curve $\overline{A}$ being defined over $\mathbb{F}_p$), they obtained the following formula

$$\mathcal{P}(t) = \lim_{n\to\infty} p^n b_{2n}(t)/p^{n+1} b_{2n+1}(t) \in \mathbb{Q}_p[[t]]$$

and a closed formula was proposed by J. K. Yu in [**Yu95**, 11].
In terms of the algebraic Legendre parameter $z = \lambda$ in a supersingular disc, it is still true, via 6.2.5, that $\tau = \mathcal{P}(z)$ is a quotient of solutions of the Gauss-Manin connection. This allows to give explicit formulas for $\mathcal{P}$ as a function of the algebraic parameter z. For instance, let us assume that $p \equiv 3$ (4). The disc $|z - \frac{1}{2}| < 1$ is a supersingular disc. In terms of a symplectic basis of eigenvectors for the action of $[\sqrt{-1}]$ on $V_p^{\mathbb{C}_p}$, we get the following formula for the Gross-Hopkins period mapping

$$\tau(z) = \kappa \frac{(1-2z)F(\frac{3}{4},\frac{3}{4},\frac{3}{2};(1-2z)^2)}{F(\frac{1}{4},\frac{1}{4},\frac{1}{2};(1-2z)^2) + \frac{1}{2}(1-2z)F(\frac{3}{4},\frac{3}{4},\frac{3}{2};(1-2z)^2)}$$

where κ is a constant depending on the choice of the basis (a natural choice would be to take this basis inside the p-adic Betti lattice discussed in I.5.3; the corresponding constant κ is then the one occuring in the proof of I.Theorem 5.3.9.

Note that since the target of the period mapping is $\mathbb{P}^1$, the denominator vanishes somewhere on $\mathsf{D}(\frac{1}{2}, 1^-)$.

Note also that the period mapping in the complex situation[4] is given, up to a constant factor, by *the same hypergeometric formula, viewed as a complex analytic function.*

6.3.4. At last, let us turn to example 4.3.4 (p-divisible groups of fake elliptic curves at a critical prime p). In this case, the period mapping

$$\mathcal{M}^0 \xrightarrow{\mathcal{P}} \Omega_{\mathbb{C}_p} = \mathcal{D}$$

is an *isomorphism.* This is a reinterpretation of Drinfeld's work, *cf.* [**RZ96**] (and [**BC91**, II8]). Actually, Drinfeld establishes an isomorphism at the level of formal schemes

$$\mathfrak{M}^0 \cong \hat{\Omega}\, \hat{\otimes}_{\mathbb{Z}_p} \widehat{\mathbb{Z}_p^{\mathrm{ur}}}$$

which is J^1-equivariant (where J^1 denotes the image in $J^{\mathrm{ad}} = PGL_2(\mathbb{Q}_p)$ of the elements of J whose determinant is a p-adic unit).

6.3.5. Recall that the p-adic manifold $\Omega_{\mathbb{C}_p}$ is simply connected (in the usual topological sense). However, the above isomorphism and the modular property of $\mathfrak{M}$, allowed Drinfeld to construct a tower of finite etale connected Galois coverings of $\Omega_{\mathbb{C}_p}$ (or even of $\Omega_{\widehat{\mathbb{Q}_p^{\mathrm{ur}}}}$), as follows.

One considers the universal special formal module $\underline{\Lambda}$ of height 4 over $\mathfrak{M}^0 \cong \hat{\Omega}\, \hat{\otimes} \widehat{\mathbb{Z}_p^{\mathrm{ur}}}$. For any $n \geq 1$, $\mathrm{Ker}[p^n]$ is a finite locally free formal group scheme of rank p^{4n} over $\hat{\Omega}\, \hat{\otimes} \widehat{\mathbb{Z}_p^{\mathrm{ur}}}$. This gives rise to a finite etale covering of the analytic space $\Omega_{\widehat{\mathbb{Q}_p^{\mathrm{ur}}}}$ (fibered in rank-one $\mathcal{B}_p/p^n\mathcal{B}_p$-modules).

To get a connected Galois covering, one looks at the Π-action (recall from 4.3.4 that $\mathcal{B}_p = \mathbb{Z}_{p^2}[\Pi]$, $\Pi^2 = p$): one has $\mathrm{Ker}[p^n] = \mathrm{Ker}[\Pi^{2n}]$, and the complement of $\mathrm{Ker}[\Pi^{2n-1}]$ in $\mathrm{Ker}[\Pi^{2n}]$ provides a finite etale Galois covering Σ^n of $\Omega_{\widehat{\mathbb{Q}_p^{\mathrm{ur}}}}$ with group $(\mathcal{B}_p/p^n\mathcal{B}_p)^\times$. When n grows, these form a projective system of Galois coverings (with Galois group the profinite completion of $\mathcal{B}_p^\times$). This tower is equivariant with respect to the $GL_2(\mathbb{Q}_p)$-action.

Over $\mathbb{C}_p$, Σ^n decomposes into finitely many copies of a connected p-adic manifold Υ^n which is a finite etale Galois covering of $\Omega_{\mathbb{C}_p}$ with group $SL_1(\mathcal{B}_p/p^n\mathcal{B}_p)$. This can be seen either by using various maximal commutative $\mathbb{Q}_p$-subalgebras of B_p and reasoning as in [**Var98a**, 1.4.6], or by global means, using the Čerednik-Drinfeld theorem (7.4.7), or else by the properties of the so-called determinant map [**BZ95**, II].

Up to now, there is no other method for constructing non-abelian finite etale coverings of $\Omega_{\mathbb{C}_p}$. The coverings Υ^n remain rather mysterious: are they simply-connected? Is any connected finite etale covering of $\Omega_{\mathbb{C}_p}$ a quotient of some Υ^n? Does there exist an infinite etale Galois covering of analytic manifolds $\Upsilon \to \Omega_{\mathbb{C}_p}$ with Galois group $SL_1(\mathcal{B}_p)$?

Drinfeld's construction works for any GL_n over any local field L, replacing Ω by the complement ${}_L\Omega^{(n-1)}$ of L-rational hyperplanes in $\mathbb{P}^{n-1}$.

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[4] for the Legendre elliptic pencil, and computed with help of a basis of cycles — with coefficients in the Gauss integers — which are eigenvectors for complex multiplication at z=1/2.

6.4. De Jong's viewpoint on p-adic period mappings.

J. de Jong [**dJ95b**] has proposed an interpretation of the period mapping which clarifies a lot the nature of its fibers. We briefly present this interpretation under the following mild simplifying assumption:

$*$ that there is a family of affinoid domains (X_i) of $\mathcal{M}^0$ such that the $\mathcal{P}(X_i)$ (which are finite unions of affinoid domains in $\mathcal{D}$ since $\mathcal{P}$ is etale) form an admissible covering of $\mathcal{D}$.

This assumption is fulfilled in our three basic examples (but not always, *cf.* [**RZ96**, 5.53]).

Under $(*)$, De Jong shows (*loc. cit.* [5], intr. and 7.2) that there is an etale local system of $\mathbb{Q}_p$-spaces $\mathcal{V}$ on $\mathcal{D}$ such that $\mathcal{M}^0$ is a component of the space of $\mathbb{Z}_p$-lattices in $\mathcal{V}$. Here, an "etale local system of $\mathbb{Q}_p$-spaces" is given by the data

$$\mathcal{V} = (\{U_i \hookrightarrow \mathcal{D}\}, \mathcal{V}_i, \phi_{ij})$$

where $U_i \hookrightarrow \mathcal{D}$ are open immersions, $\mathcal{V}_i = \varprojlim_n \mathcal{V}_i/p^n\mathcal{V}_i$ are etale local systems of $\mathbb{Z}_p$-lattices (each $\mathcal{V}_i/p^n\mathcal{V}_i$ being a finite locally free sheaf of $\mathbb{Z}/p^n\mathbb{Z}$-modules on the etale site of U_i), and $\phi_{ij} : (\mathcal{V}_i \otimes \mathbb{Q}_p)_{|U_i \times_{\mathcal{D}} U_j} \to (\mathcal{V}_j \otimes \mathbb{Q}_p)_{|U_i \times_{\mathcal{D}} U_j}$ are isomorphisms satisfying the usual cocycle condition (*cf.* [**dJ95b**, 4]).

Of course, in the "decorated case", $\mathcal{V}$ inherits the relevant decoration.

It follows from this interpretation that in example 4.1.2, the fibers of the period mapping are in natural bijection with the set of vertices of the Bruhat-Tits tree $\mathcal{T}$.

Similarly, in example 4.3.4, one can see that the fibers of $\mathcal{P}$ are in bijection with the one-point set $SL_1(B_p)/SL_1(\mathcal{B}_p)$, *cf.* 7.4.4.

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[5] of course $p = \ell$ throughout §§6 and 7 of *loc. cit.*

7. p-adic uniformization of Shimura varieties.

ABSTRACT: Application of the theory of p-adic period mappings: Rapoport-Zink's p-adic uniformization theorem. The case of global uniformization. Čerednik-Drinfeld uniformization of Shimura curves and other examples.

7.1. p-integral models of Shimura varieties.

7.1.1. Let $\mathcal{S}h$ be a Shimura variety of PEL type as in 1.1.4. We recall the setting of 1.1.4: B is a simple finite-dimensional $\mathbb{Q}$-algebra, with a positive involution $*$, V is a B-module of finite type, endowed with an alternate $\mathbb{Q}$-bilinear form such that $\langle bv, w\rangle = \langle v, b^*w\rangle$; G denotes the $\mathbb{Q}$-group of B-linear symplectic similitudes of V; $\mathcal{B}$ is a maximal order in B stable under $*$, and L is a lattice in V, stable under $\mathcal{B}$ and autodual for $\langle\ ,\ \rangle$.

The (non-connected) Shimura variety $\mathcal{S}h$ is defined over the reflex field E (which is a number field in $\mathbb{C}$). There are variants of $\mathcal{S}h$ in which the principal congruence subgroup of level N of $G(\hat{\mathbb{Z}})$ is replaced by any open compact subgroup C of $G(\mathbb{A}_f)$, where $\mathbb{A}_f = \hat{\mathbb{Z}}\otimes\mathbb{Q}$ is the ring of finite adeles of $\mathbb{Q}$. The complex points of $\mathcal{S}h$ admit the following adelic description

$$\mathcal{S}h(\mathbb{C}) = C\backslash\big((G(\mathbb{R})^{\mathrm{ad}}.h_0) \times G(\mathbb{A}_f)\big)/G(\mathbb{Q})$$

(*cf.* 1.2.5 about the one-parameter group h_0). We refer to [**Moo98**] for a recent survey of Shimura varieties in adelic setting (and their integral models).

7.1.2. We set $B_p = B\otimes_{\mathbb{Q}}\mathbb{Q}_p$, $\mathcal{B}_p = \mathcal{B}\otimes_{\mathbb{Z}}\mathbb{Z}_p$, and so on. We denote by C_p the stabilizer of L_p in $G(\mathbb{Q}_p)$

We assume that C is of the form C^pC_p for some open compact subgroup C^p of $G(\mathbb{A}_f^p)$, where $\mathbb{A}_f^p = (\prod_{\ell\neq p}\mathbb{Z}_\ell)\otimes\mathbb{Q}$.

In the case of a congruence subgroup of level N, this condition means that *the level is prime to p.*

We fix an embedding $v : E \hookrightarrow \mathbb{C}_p$, denote by E_v the v-completion of E, by $\mathfrak{v}$ the discrete valuation ring $\widehat{\mathbb{Z}^{\mathrm{ur}}}.\mathcal{O}_{E_v} = \widehat{\mathbb{Z}^{\mathrm{ur}}}.\mathcal{O}_E$ (compositum in $\mathbb{C}_p$), and by K the fraction field of $\mathfrak{v}$.

7.1.3. Let S be a scheme. The objects of the the category of S-abelian schemes *up to prime to p isogeny* are the abelian schemes over S; morphisms between A and A' are global sections of $\underline{\mathrm{Hom}}_S(A, A')\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)}$, where $\mathbb{Z}_{(p)}$ is the ring of rational numbers with denominator prime to p. This is a $\mathbb{Z}_{(p)}$-linear category. Its relation to p-divisible groups is given by the following elementary lemma:

Lemma 7.1.4. *The faithful functor $A \mapsto A[p^\infty]$*

$\{S$-abelian schemes up to prime to p isogeny$\} \to \{p$-divisible groups over $S\}$

is conservative, i.e. reflects isomorphisms.

The following result (which we state in a rather vague manner) is due to R. Kottwitz [**Kot92**] (Compare with 1.2.5.).

Proposition 7.1.5. *For sufficiently small C^p, there is a moduli scheme $\mathcal{S}h$ over $\mathcal{O}_{E_v}$ for isomorphism classes of abelian varieties A up to prime to p isogeny, with $\mathcal{B}$-action (with Shimura type condition), together with a $\mathbb{Q}$-homogeneous principal $*$-polarization and a class of B-linear symplectic similitudes $\mathrm{H}^1_{\mathrm{ct}}(A, \mathbb{A}^p_f) \to \mathbb{A}^p_f \otimes V$ modulo the group C^p. The generic fiber $\mathcal{S}h \otimes_{\mathcal{O}_{E_v}} E_v$ is a finite sum of copies of $\mathcal{S}h \otimes_E E_v$.*

In the case of a congruence subgroup of level N, this condition that C^p is sufficiently small means that N is sufficiently large (and kept prime to p).

7.2. The uniformization theorem.

7.2.1. One fixes a point of $\mathcal{S}h(\overline{\mathbb{F}}_p)$, hence a $\overline{\mathbb{F}}_p$-abelian variety $\overline{A}$ up to prime to p isogeny with $\mathcal{B}$-action, principal $*$-polarization, and level structure. We consider the p-divisible group $\overline{\Lambda} = \overline{A}[p^\infty]$ (with $\mathcal{B}_p$-action, principal $*$-polarization, level structure). We denote by $\overline{G}$ the $\mathbb{Q}$-group of self-quasi-isogenies of $\overline{A}$ respecting the additional structure. It is clear that $\overline{G}(\mathbb{Q}_p) \subset J$; on the other hand $\overline{G}(\mathbb{A}^p_f) \subset G(\mathbb{A}^p_f)$.

We also consider an auxiliary lifting $\tilde{\overline{A}}$ of $\overline{A}$ (with $\mathcal{B}$-action) over some finite extension $\mathcal{O}_{E'}$ of $\mathcal{O}_E$ in $\mathfrak{v}$, and set $\tilde{\overline{\Lambda}} = \tilde{\overline{A}}_{\mathfrak{v}}[p^\infty]$. We can then identify $\mathbf{D}(\overline{\Lambda}) \otimes_W K$ with $\mathrm{H}^1_{\mathrm{DR}}(\tilde{\overline{A}}) \otimes_{\mathcal{O}_{E'}} K$, and the submodule $\omega_{\tilde{\overline{\Lambda}}}$ with $F^1\mathrm{H}^1_{\mathrm{DR}}(\tilde{\overline{A}}) \otimes_{\mathcal{O}_{E'}} K$. Thus if the complex lagrangian space F^1_0 of 1.1.4 is taken to be $F^1\mathrm{H}^1_{\mathrm{DR}}(\tilde{\overline{A}}) \otimes_{\mathcal{O}_{E'}} \mathbb{C}$ (for some complex embedding of E' extending the natural complex embedding of E), the complex and p-adic Shimura type conditions "agree" (1.1.4, 4.3.2). We can consider the corresponding decorated moduli problem for p-divisible groups, and the formal moduli scheme $\mathfrak{M}$ over $\mathfrak{v}$ as in 4.3. Moreover, the flags spaces $\mathcal{D}^\vee$ considered in the complex and p-adic situations agree, *i.e.* come from the same flag space defined over the reflex field E.

7.2.2. One defines a morphism of functors on $\mathrm{Nil}_{\mathfrak{v}}$:

$$\mathfrak{M} \to \mathcal{S}h$$

as follows [**RZ96**, 6.14]. Let S be in $\mathrm{Nil}_{\mathfrak{v}}$ and let us consider a pair $(\Lambda, \rho) \in \mathfrak{M}(S)$. By rigidity of p-divisible groups up to isogeny (2.2.3), the $\mathcal{B}_p$-quasi-isogeny $\rho \in \mathrm{qisog}(\Lambda \times_S S_{\mathrm{red}}, \overline{\Lambda} \times_{\overline{\mathbb{F}}_p} S_{\mathrm{red}})$ lifts to a unique $\mathcal{B}_p$-quasi-isogeny $\tilde{\rho} \in \mathrm{qisog}(\Lambda, \tilde{\overline{\Lambda}} \times_{\mathfrak{v}} S)$. This can be algebraized in a weak sense: there is an S-abelian scheme $A_{(\Lambda,\rho)}$ up to prime to p isogeny, with $\mathcal{B}$-action and with p-divisible group Λ, and a quasi-isogeny $A_{(\Lambda,\rho)} \to \tilde{\overline{A}} \times_{\mathcal{O}_{E'}} S$ which induces $\tilde{\rho}$ at the level of p-divisible groups. Moreover $A_{(\Lambda,\rho)}$ is unique in the category of S-abelian schemes up to prime to p isogeny with $\mathcal{B}$-action (7.1.4), and its formation is functorial in (Λ, ρ). One then defines the required morphism of functors on $\mathfrak{M}(S)$ by setting $(\Lambda, \rho) \mapsto A_{(\Lambda,\rho)}$ (endowed with the polarization and level structure inherited from $\overline{A}$).

This defines a morphism of formal schemes, hence a morphism of associated analytic spaces over $\mathbb{C}_p$:

$$\mathcal{Q} : \mathcal{M} \to \mathcal{S}h^{\mathrm{an}}_{\mathbb{C}_p}.$$

7.2.3. It turns out that the decorated abelian varieties (up to prime to p isogeny) which are *isogenous* to $\overline{A}$ form in a natural way a *countable union* $Z = \bigcup Z_n$ *of projective irreducible subvarieties* Z_n *of* $\mathcal{S}h_{\overline{\mathbb{F}}_p}$, each of them meeting only finitely many members of the family ([**RZ96**, 6.23, 6.34]). One can slightly generalize the notion of tubes (3.5) and define the tube $]Z[\subset \mathcal{S}h^{\mathrm{an}}_{\mathbb{C}_p}$. We denote by S the connected component of $]Z[$ which contains the modular point s of our chosen lifting $\tilde{\overline{A}}$.

Theorem 7.2.4. *For sufficiently small C^p, there is a connected component $\mathcal{M}^0$ of $\mathcal{M}$ such that the restriction $\mathcal{Q} : \mathcal{M}^0 \to S$ is a topological covering of S. More precisely, S is the (right) quotient of $\mathcal{M}^0$ by some torsion-free discrete subgroup of J.*

cf. [**RZ96**, 6.23, 6.31]. Roughly speaking, the proof of 7.2.4 involves two steps: using the Serre-Tate theorem 2.2.5, one shows that the morphism of $\mathfrak{v}$-formal schemes $\mathfrak{M} \to \widehat{\mathcal{S}h}$ defined above is formally etale. One concludes by a careful study of this morphism on geometric points.

In our examples 4.2, S is an open unit disc, and the map $\mathcal{Q}$ is an isomorphism (in 4.2.1, it is induced by $q \mapsto z = \lambda$).

7.2.5. Let us consider the universal decorated abelian scheme $\underline{A}$ over $\mathcal{S}h$, and the vector bundle $\mathcal{H} = \mathrm{H}^1_{\mathrm{DR}}(\underline{A}/\mathcal{S}h)$ with its Gauss-Manin connection $\nabla_{\mathrm{GM}} : \mathcal{H} \to \Omega^1_{\mathcal{S}h} \otimes \mathcal{H}$.

Corollary 7.2.6. *Viewed as a p-adic connection and restricted to S, the Gauss-Manin connection $\nabla_{\mathrm{GM}|S}$ comes from a representation of $\pi_1^{\mathrm{top}}(S, s)$ on $V^{\mathbb{C}_p}$ (hence it satisfies Cauchy's theorem, cf. I.1.5).*

PROOF. Let $\tilde{S}$ be the universal covering of S. According to I.1.5), we have to show that the pull-back of ∇_{GM} on $\tilde{S}$ is trivial ($\cong V^{\mathbb{C}_p} \otimes \mathcal{O}_{\tilde{S}}$). By 7.2.4 and 6.2.5, this already holds on the quotient $\mathcal{M}^0$ of $\tilde{S}$. □

Remark. Using the Weil descent data of [**RZ96**, 6.21], one can show that the representation is defined over a finite unramified extension E'_v of E_v, *i.e.* comes from a representation of $\pi_1^{\mathrm{top}}(S, s)$ on $V \otimes E'_v$.

7.2.7. The assumption $C = C^p C_p$ is necessary to define the morphism $\mathcal{Q} : \mathcal{M}^0 \to S$. The assumption "$C^p$ small enough" is much less important. Theorem 7.2.4. holds without it, except that the discrete subgroup of J is then no longer torsion-free [the point is that there is always a normal subgroup C'^p of finite index which is small enough, one applies the theorem to C'^p, and one passes to the quotient by C'^p/C^p]. However, to give a modular interpretation of the (ramified) quotient of $\mathcal{M}^0$ in this case, one needs the notion of p-adic orbifold (to be discussed in III); moreover the Gauss-Manin connection may acquire logarithmic singularities along the branched locus.

7.3. Global uniformization

We say that there is *global uniformization* in case S is (the analytification of) a whole connected component of the Shimura variety $\mathcal{S}h$. This occurs when the reduction mod p of the decorated abelian varieties A parametrized by $\mathcal{S}h$ form a *single isogeny class.*

In the situation of global uniformization, the group $\overline{G}$ of self-quasi-isogenies of $\overline{A}$ (with decoration) is an *inner form of* G.

In view of our analogy 4.3.5, this situation can be expected to be closest to the complex case (in the notation of 1.2.5, the $\hat{A}$'s are all isomorphic to $\hat{U}$). Indeed, we then get a commutative diagram of p-adic manifolds (similar to 1.2.5):

$$\begin{array}{ccc} \mathcal{M}^0 & \xrightarrow{\ \mathcal{P}\ } & \mathcal{D} \subset \mathcal{D}^\vee \\ {\scriptstyle \mathcal{Q}}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{Q}} \\ S & \xrightarrow{\ \mathcal{P}\ } & \mathcal{D}/\Gamma. \end{array}$$

where the horizontal maps are etale, $\mathcal{Q}$ denotes the quotient maps, and Γ is an arithmetic subgroup in the semi-simple group $\overline{G}^{\mathrm{ad}}$. In all known examples, $\mathcal{P}$ is actually an isomorphism and $\mathcal{M}^0 = \tilde{S}$ [**RZ96**, 6], [**BZ95**], [**Var98a**], [**Var98b**].

Here, the archimedean—non archimedean correspondence goes beyond a mere analogy: indeed, here and in 1.2.5, what is denoted by S (*resp.* $\mathcal{D}^\vee$) is the — complex or p-adic — analytification of the same algebraic variety.

Remark. To avoid any ambiguity, let us point out that we consider here only phenomena of global uniformization which can be interpreted in terms of moduli spaces of p-divisible groups. This does not account for all cases of global p-adic uniformization of Shimura varieties. For instance, it is known that the elliptic modular curve $X_0(p)$ is a Mumford curve over $\mathbb{Q}_p$, whereas the p-adic uniformization of 7.2.4 (or variants with level p structure) is local.

7.4. Example: the Čerednik-Drinfeld uniformization of Shimura curves at a critical prime.

7.4.1. The first historical example of global uniformization in the above sense is the Čerednik uniformization of "Shimura curves" (modular curves for fake elliptic curves) at a critical prime p, *cf.* [**Čer76**]. Čerednik's method was of group-theoretic nature and did not involve formal groups. A modular proof was subsequently proposed by Drinfeld [**Dri76**] (*cf.* also [**BC91**, III]). For examples, and related topics, we refer to [**vdP89**], [**vdP92a**], [**vdP92b**].

7.4.2. We fix an indefinite quaternion algebra B over $\mathbb{Q}$, a maximal order $\mathcal{B}$ in B, and a positive involution $*$ of $\mathcal{B}$. We consider the standard representation $V = B$ of B, so that $G = GL_B(V) \cong B^\times$.

We refer to the foundational paper [**Shi67**] for the properties of the (connected) Shimura curve $\mathcal{X}^+(N)$ of level N attached to B. It parametrizes principally $*$-polarized abelian surfaces with multiplication by $\mathcal{B}$ (fake elliptic curves, *cf.* 4.3.4), with level N structure. We recall the following properties of $\mathcal{X}^+(N)$, which actually characterize it (*loc. cit.* 3.2):

(a) $\mathcal{X}^+(N)$ is a projective geometrically connected smooth curve defined over the cyclotomic field $\mathbb{Q}(\zeta_N)$,

(b) $\mathcal{X}^+(N)(\mathbb{C}) = \mathfrak{h}/\Gamma^+(N)$, that is to say: $\mathcal{X}^+(N)$ is the quotient of the Poincaré upper half plane by the image $\Gamma^+(N) = PSL_1(\mathcal{B})(N)$ of the group $\{g \in (1+N\mathcal{B})^\times, \mathrm{Nr}(g) > 0\}$ in $PSL_2(\mathbb{R})$, where Nr stands for the reduced norm,

(c) for any imaginary quadratic field K in B such that $\mathcal{O}_K \subset \mathcal{B}$, let z be its fixed point in $\mathfrak{h}$. Then the extension of $\mathbb{Q}(\zeta_N)$ generated by the image of z in $\mathcal{X}^+(N)$ is the class field over K with conductor $N\mathcal{B} \cap \mathcal{O}_K$.

The commutative diagram of 1.2.5, involving the complex period mapping, specializes to the following

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\mathcal{P}} & \mathcal{D} = \mathfrak{h} \subset \mathcal{D}^\vee = \mathbb{P}^1 \\ {\scriptstyle\mathcal{Q}}\big\downarrow & & \big\downarrow{\scriptstyle\mathcal{Q}} \\ S = \mathcal{X}^+(N)^{\mathrm{an}}_{\mathbb{C}} & \xrightarrow{\mathcal{P}} & \mathcal{D}/\Gamma^+(N) \end{array}$$

where the horizontal maps are isomorphisms (and the vertical maps are topological coverings if $\Gamma^+_p(N)$ is torsion-free[(6)]).

There is a useful variant $\mathcal{X}^*$ of $\mathcal{X}^+ = \mathcal{X}^+(1)$, already considered by Shimura ([**Shi67**, 3.13], in which Γ^+ is replaced by the image Γ^* in $PSL_2(\mathbb{R})$ of the group $\{g \in B^\times \mid g\mathcal{B} = \mathcal{B}g,\ \mathrm{Nr}(g) > 0\}$:

$$\mathcal{X}^*(\mathbb{C}) = \mathfrak{h}/\Gamma^*.$$

It is defined over $\mathbb{Q}$. More generally, there is a Shimura curve $\mathcal{X}_\Gamma$ attached to any congruence subgroup Γ in $B^{\times^+}/\mathbb{Q}^\times$, where $B^{\times^+}$ denotes the group of elements of positive reduced norm.

7.4.3. It is known that $\mathcal{X}^+(N)$ has good reduction at any prime p which does not divide the discriminant $d(B)$ of B nor the level N (Y. Morita). If instead p is critical (but still *does not divide* N), as we shall assume henceforth, Čerednik's theorem shows that $\mathcal{X}^+(N)_{\mathbb{C}_p}$ is a *Mumford curve* uniformized by the Drinfeld space $\Omega_{\mathbb{C}_p} = \mathbb{P}^1_{\mathbb{C}_p} \setminus \mathbb{P}^1(\mathbb{Q}_p)$ (at least if N is large enough); in fact, Sh becomes a Mumford curve over some finite unramified extension of $\mathbb{Q}_p$ (note that $\mathbb{Q}(\zeta_N) \subset \mathbb{Q}_p^{\mathrm{ur}}$).

To be more explicit, let $\overline{B}$ be the definite quaternion $\mathbb{Q}$-algebra which is ramified at the same primes as B except p: $\overline{B}_p \cong M_2(\mathbb{Q}_p)$. Let $\overline{\mathcal{B}}$ be a maximal order in $\overline{B}$. Let $\Gamma^+_p(N)$ be the image in $PGL_2(\mathbb{Q}_p)$ of the group $\{g \in (1+N\overline{\mathcal{B}}[\frac{1}{p}])^\times \mid \mathrm{ord}_p(\mathrm{Nr}(g))\ \text{even}\}$. Let Γ^*_p be the image in $PGL_2(\mathbb{Q}_p)$

(6) if this group is not torsion-free $\tilde{S}$ is the universal covering in the sense of orbifolds.

of the normalizer $\{g \in \overline{B}^\times \mid g\overline{\mathcal{B}}[\frac{1}{p}] = \overline{\mathcal{B}}[\frac{1}{p}]g\}$. These are discrete subgroups, and one has the *Čerednik uniformization*:

$$\mathcal{X}^+(N)^{\mathrm{an}}_{\mathbb{C}_p} \cong \Omega_{\mathbb{C}_p}/\Gamma_p^+(N),$$
$$\mathcal{X}^{*\mathrm{an}}_{\mathbb{C}_p} \cong \Omega_{\mathbb{C}_p}/\Gamma_p^*.$$

For recent applications to p-adic L-functions, Heegner points, etc... *cf.* [**BD98**].

7.4.4. The case of $\mathcal{X}^*$ is the one originally considered in [**Čer76**], except that Čerednik uses the adelic language. In order to make the translation, it is useful to keep in mind the following few facts (*cf.* [**Vig80**, pp. 40, 99 and passim]).

For critical p, the unique maximal order $\mathcal{B}_p$ of B_p is $\{b \in B_p \mid \mathrm{Nr}(b) \in \mathbb{Z}_p\}$. Hence, we have an equality $SL_1(\mathcal{B}_p) = SL_1(B_p)$ and a chain of compact normal subgroups

$$PSL_1(\mathcal{B}_p) = PSL_1(B_p) \subset PGL_1(\mathcal{B}_p) \subset PGL_1(B_p)$$

with $PGL_1(B_p)/PSL_1(\mathcal{B}_p) \cong \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$ (which is of type $(2,2)$ if $p \neq 2$, of type $(2,2,2)$ if $p = 2$).

On the other hand, the normalizer of $\mathcal{B}_\ell$ in $B_\ell^\times$ is $B_\ell^\times$ if ℓ is critical, and $\mathcal{B}_\ell^\times.\mathbb{Q}_\ell^\times$ otherwise. It follows that

$$\Gamma^* = \{g \in B^{\times^+} \mid \forall \ell \nmid d(B),\ g \in \mathcal{B}_\ell^\times.\mathbb{Q}_\ell^\times\}/\mathbb{Q}^\times$$
$$\Gamma_p^* = \{g \in \overline{B}^\times \mid \forall \ell \nmid d(B),\ g \in \overline{\mathcal{B}}_\ell^\times.\mathbb{Q}_\ell^\times\}/\mathbb{Q}^\times$$
$$\Gamma^+(N) = \{g \in B^{\times^+} \mid \forall \ell,\ g \in (1 + N\mathcal{B}_\ell)^\times.\mathbb{Q}_\ell^\times\}/\mathbb{Q}^\times$$
$$\Gamma_p^+(N) = \left\{g \in \overline{B}^\times \;\middle|\; \begin{array}{l}\mathrm{ord}_p(\mathrm{Nr}(g)) \text{ even;}\\ \forall \ell \neq p,\ g \in (1 + N\overline{\mathcal{B}}_\ell)^\times.\mathbb{Q}_\ell^\times\end{array}\right\}\Big/\mathbb{Q}^\times$$

If N is prime to $d(B)$, this can also be rewritten

$$\Gamma^+(N) = \left\{g \in B^{\times^+} \;\middle|\; \begin{array}{l}\forall \ell \mid d(B),\ \mathrm{ord}_\ell(\mathrm{Nr}(g)) \text{ even;}\\ \forall \ell \nmid d(B),\ g \in (1 + N\mathcal{B}_\ell)^\times.\mathbb{Q}_\ell^\times\end{array}\right\}\Big/\mathbb{Q}^\times$$
$$\Gamma_p^+(N) = \left\{g \in \overline{B}^\times \;\middle|\; \begin{array}{l}\forall \ell \mid d(B),\ \mathrm{ord}_\ell(\mathrm{Nr}(g)) \text{ even;}\\ \forall \ell \nmid d(B),\ g \in (1 + N\overline{\mathcal{B}}_\ell)^\times.\mathbb{Q}_\ell^\times\end{array}\right\}\Big/\mathbb{Q}^\times$$

In particular, for $N = 1$, we see that there are canonical bijections

$$\Gamma^*/\Gamma^+ = \Gamma_p^*/\Gamma_p^+ = (\mathbb{Z}/2\mathbb{Z})^{\sharp\{\ell \mid d(B)\}}$$

7.4.5. We now give some indications on Drinfeld's modular approach to Čerednik's uniformization.

Lemma 7.4.6. (i) *Fake elliptic curves have potentially good reduction at every prime; the reduction is supersingular reduction at the critical prime p. In particular, over $\overline{\mathbb{F}}_p$, they form a unique isogeny class.*

(ii) *Fake elliptic curves have a unique principal $*$-polarization.*

PROOF. (i): due to the presence of many endomorphisms, the potential good reduction follows easily from the semi-stable reduction theorem for abelian schemes; also, the non-split quaternion algebra B_p acts on the p-divisible group up to isogeny, and this forces the slopes to be $1/2$; for details and (ii), *cf.* [**BC91**, III]. □

It follows from (i) that any fake elliptic curve over $\overline{\mathbb{F}}_p$ is isogenous to the square of a supersingular elliptic curve $\overline{E}$. However, while there are only finitely many supersingular elliptic curves up to isomorphism, fake elliptic curves over $\overline{\mathbb{F}}_p$ have continuous moduli! The moduli spaces are chains of $\mathbb{P}^1$'s.(7)

Let us introduce the quaternion $\mathbb{Q}$-algebra D ramified only at p and ∞: $D \cong \mathrm{End}^{\mathbb{Q}}(\overline{E})$. Let $\overline{A}$ be a fake elliptic curve over $\overline{\mathbb{F}}_p$. It follows from 7.4.6 (i) that $\mathrm{End}^{\mathbb{Q}}(\overline{A}) \cong M_2(D)$, and $\mathrm{End}^{\mathbb{Q}}_{\mathcal{B}}(\overline{A}) \cong \overline{B}$.

In particular, the inner form $\overline{G}$ of G considered in 7.3 is $\overline{B}^{\times}$, and $\overline{G}(\mathbb{Q}_p) = GL_2(\mathbb{Q}_p) = J$.

It follows from 7.4.6 (ii) that we may disregard polarizations. The p-divisible groups attached to fake elliptic curves with $\mathcal{B}$-multiplication are special formal $\mathcal{B}_p$-modules of height 4; recall from 4.3.4 the corresponding formal moduli space $\mathfrak{M} = \coprod_{\mathbb{Z}} \mathfrak{M}^0$. The period mapping is an isomorphism (6.3.4, 6.4), and the commutative diagram of 7.3 specializes to

$$\begin{array}{ccc} \tilde{S} = \mathcal{M}^0 & \xrightarrow{\ \mathcal{P}\ } & \mathcal{D} = \Omega_{\mathbb{C}_p} \subset \mathcal{D}^{\vee} = \mathbb{P}^1 \\ \mathcal{Q}\big\downarrow & & \big\downarrow\mathcal{Q} \\ S = \mathcal{X}^+(N)^{\mathrm{an}}_{\mathbb{C}_p} & \xrightarrow{\ \mathcal{P}\ } & \mathcal{D}/\Gamma^+_p(N) \end{array}$$

where the horizontal maps are isomorphisms (and the vertical maps are topological coverings if $\Gamma^+_p(N)$ is torsion-free), in complete analogy with 7.2.

7.4.7. Let us end this section by considering the case where p *divides the level* N: $N = p^n.M$, $p \nmid M$.

In this case, $\Gamma^+_p(N) = \Gamma^+_p(M)$. The finite etale covering $\mathcal{X}^+(N)^{\mathrm{an}}_{\mathbb{C}_p} \to \mathcal{X}^+(M)^{\mathrm{an}}_{\mathbb{C}_p}$ is not a topological covering anymore (even if $\Gamma^+_p(M)$ is torsion-free). As was shown by Drinfeld, it is hidden in the finite etale covering Υ^n of $\Omega_{\mathbb{C}_p}$ (6.3.5).

Let us assume for simplicity that $M > 2$, so that $-1 \notin \Gamma^+(M)$. This implies that $\Gamma^+(N)/\Gamma^+(M) \cong SL_1(\mathcal{B}/p^n\mathcal{B})$. Then Υ^n is a topological covering of $\mathcal{X}^+(N)^{\mathrm{an}}_{\mathbb{C}_p}$, and more precisely, one has the commutative diagram

(7) Another curious feature of the higher dimensional case is that the product of $n > 1$ supersingular elliptic curves over $\overline{\mathbb{F}}_p$ form a single isomorphism class (Deligne, Ogus)

$$\begin{array}{ccc} \mathcal{X}^+(N)^{\mathrm{an}}_{\mathbb{C}_p} & \xrightarrow{\sim} & \Upsilon^n/\Gamma_p^+(M) \\ \downarrow & & \downarrow \\ \mathcal{X}^+(M)^{\mathrm{an}}_{\mathbb{C}_p} & \xrightarrow{\mathcal{P}} & \Omega_{\mathbb{C}_p}/\Gamma_p^+(M) \end{array}$$

where the vertical maps are finite etale Galois covering maps with group $SL_1(\mathcal{B}/p^n\mathcal{B})$, *cf.* [**BC91**, III.5.5], [**Var98a**]. Similarly, there is such a p-adic global uniformization for any Shimura curve $\mathcal{X}_\Gamma$ (*cf.* 7.4.1).

7.5. Other examples of global p-adic uniformization.

7.5.1. Shimura's paper [**Shi67**] deals not only with quaternion algebra over $\mathbb{Q}$, but also with quaternion algebras B/E over a totally real number field E. When B *splits at a single real place,* he constructs curves $\mathcal{X}^+(N)$ and $\mathcal{X}^*$ satisfying properties similar to (a),(b),(c) of 7.4.2.

In this context, N may be an arbitrary ideal of $\mathcal{O}_E$, $\mathbb{Q}(\zeta_N)$ has to be replaced by the ray class field with conductor $N.\infty$, and the condition $N(g) > 0$ means that the reduced norm of g is totally positive, or, what amounts to the same, is positive at the split real place ∞_0.

The novelty, however, is that these Shimura curves have *no direct interpretation* as moduli spaces for decorated abelian varieties, since B is not in the Albert list of endomorphism algebras of abelian varieties when $[E:\mathbb{Q}] > 1$.

To overcome this problem, Shimura introduces the following remarkable trick. Let K be a totally imaginary quadratic extension of E. Then $B^\bullet = B \otimes_E K$ is in Albert's list (we endow it with the involution which is the tensor product of $*$ on B and complex conjugation on K). It turns out that the Shimura varieties of PEL type for abelian varieties of dimension $g = 4[E:\mathbb{Q}]$ ($*$-polarized, with level structure) with complex multiplication by $B^\bullet$ are twisted forms of the same curve. The Shimura curve attached to B is obtained by Weil descent from the Shimura curves attached to $B \otimes_E K$, for sufficiently many quadratic extensions K.

7.5.2. Čerednik's original p-adic uniformization theorem already applies to Shimura curves attached to quaternion algebras over a totally real number field E. His method has been further developed in [**Var98a**], [**Var98b**]. On the other hand, Drinfeld's modular approach has been extended by Boutot-Zink [**BZ95**], using Shimura's trick. Let us sketch the principle of the construction.

Let G be the multiplicative group of $B^\times$, viewed as an algebraic group over $\mathbb{Q}$. Let Γ be a congruence subgroup of $G^{\mathrm{ad}}(\mathbb{Q})$ (alternatively, one can consider an open compact subgroup C of $G(\mathbb{A}_f)$). Let us first assume that Γ is *maximal at* p (alternatively C is of the form C^pC_p for some open compact subgroup C^p of $G(\mathbb{A}_f^p)$, *cf.* 7.1.2). The (non-connected) Shimura curve $\mathcal{S}h$ given by

$$\mathcal{S}h^{\mathrm{an}}_{\mathbb{C}} = C\backslash\big((\mathbb{P}^1_{\mathbb{C}} \setminus \mathbb{P}^1(\mathbb{R})) \times G(\mathbb{A}_f)\big)/G(\mathbb{Q})$$

is defined over the reflex field, which is E (embedded into $\mathbb{R}$ via ∞_0). Over some finite abelian extension E'/E, it decomposes as a disjoint union of geometrically connected Shimura curves $\mathcal{X}_\Gamma$ ($\mathfrak{h}/\Gamma$ over $\mathbb{C}$; one possible Γ is the image of $G(\mathbb{Q}) \cap C$ in $\mathbb{G}^{\mathrm{ad}}(\mathbb{Q})$).

Let $v = v_0$ be a *critical* finite place of E, *i.e.* such that B_v is a non-split quaternion algebra. Let p be the residue characteristic of v_0 and let $v_1, \ldots, v_m$ be the other places of E above p.

Let K be a totally imaginary quadratic extension of E, such that every place v_i splits in K. We denote by $w_i, \overline{w_i}$ the two places of K above v_i. The quaternion algebra $B^\bullet = B \otimes_E K$ is ramified at $w = w_0$: $B^\bullet_w \cong B_v$.

We fix an extension $\infty_0 : K \hookrightarrow \mathbb{C}$ of the real embedding ∞_0 of E, and a double embedding $\mathbb{C} \hookleftarrow \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ whose restriction to K is compatible with (∞_0, w). The embeddings $K \hookrightarrow \mathbb{C}_p$ which factor through some w_i but not through some $\overline{w_i}$ form a CM type Ψ: $\Psi \coprod \overline{\Psi} = \mathrm{Hom}(K, \mathbb{C}_p)$ (identified with $\mathrm{Hom}(K, \mathbb{C})$ through our chosen double embedding).

There is a natural $\mathbb{Q}$-group $G^\bullet$ such that $G^\bullet(\mathbb{Q}) = \{b \in B^\bullet | bb^* \in E\}$, acting on $V^\bullet = V \otimes_E K$. One can attach to C an open compact subgroup $C^\bullet$ of $G^\bullet(\mathbb{A}_f)$, maximal at p, and a Shimura curve $\mathcal{S}h^\bullet$ such that $\mathcal{S}h$ is open and closed in $\mathcal{S}h^\bullet$ ([**BZ95**, 3.11]). Alternatively, one can attach to Γ a congruence subgroup $\Gamma^\bullet$ in $G^{\bullet ad}(\mathbb{Q})$ and a corresponding connected Shimura curve $\mathcal{X}_{\Gamma^\bullet}$ which is a twisted form of $\mathcal{X}_\Gamma$ (defined over some abelian extension of K). The point is that this new Shimura curve $\mathcal{S}h^\bullet$ is of PEL type. More precisely, for C small enough, it is a moduli space for $*$-polarized abelian varieties of dimension $g = 4[E : \mathbb{Q}]$ with multiplication by $\mathcal{B}^\bullet = \mathcal{B} \otimes_{\mathcal{O}_E} \mathcal{O}_K$, Shimura type Ψ, and level structure. As in 7.1.5, it admits a model $\mathcal{S}h^\bullet$ over $\mathcal{O}_{E_v}$.

Let $\underline{A} \to \mathcal{S}h^\bullet$ be the universal abelian scheme up to prime to p isogeny. Due to the $\mathcal{O}_K$-action, the p-divisible group of $\underline{A}$ splits as $\underline{A}[p^\infty] \cong \prod_i \Lambda_{w_i} \times \Lambda_{\overline{w_i}}$, where Λ_{w_i} and $\Lambda_{\overline{w_i}}$ are p-divisible groups of height $4[E_{v_i} : \mathbb{Q}_p]$ with formal multiplication by E_{v_i}. Moreover, $\Lambda_w = \Lambda_{w_0}$ is a special formal $\mathcal{B}_v$-module of height $4.[E_v : \mathbb{Q}_p]$ (defined as in 4.3.4, replacing $\mathbb{Q}_p$ by E_v), and $\Lambda_{\overline{w}}$ is the Serre dual of Λ_w; the other Λ_{w_i}, $i > 0$, are etale, and the $\Lambda_{\overline{w_i}}$ are their respective Serre duals. Hence the Newton polygon of the Dieudonné module of $\underline{A}[p^\infty]$ has slopes $0, \frac{1}{2}, 1$ (only $\frac{1}{2}$ if $m = 0$).

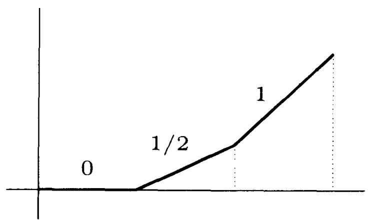

FIGURE 2

Let $\mathfrak{o}$ be the discrete valuation ring $\widehat{\mathbb{Z}^{\mathrm{ur}}}.\mathcal{O}_{E_v}$ as in 7.1.2. The moduli problem for $\mathcal{B}_v$-module of height $4.[E_v : \mathbb{Q}_p]$ is representable by the p-adic formal scheme $\coprod_{\mathbb{Z}} ({}_{E_v}\hat{\Omega})_{\mathfrak{v}}$, where ${}_{E_v}\hat{\Omega}$ is the analogue for (and over) $\mathcal{O}_{E_v}$ of the formal scheme $\hat{\Omega}$, with associated analytic space $({}_{E_v}\Omega)_{\mathbb{C}_p} = \mathbb{P}^1_{\mathbb{C}_p} \backslash \mathbb{P}^1(E_v)$, *cf.* 6.3.5.

7.5.3. This is the modular way taken by Boutot and Zink to reprove (and strengthen) Čerednik's theorem. As was remarked by Varshavsky [**Var98b**, 3.13], one obtains a simpler adelic formulation if one works with a twisted version ${}^{\mathrm{ad}}\mathcal{S}h$ of $\mathcal{S}h$ (corresponding to a different normalization of h_0 in 1.2.5); this does not change the adelic description of the associated complex space

$$ {}^{\mathrm{ad}}\mathcal{S}h^{\mathrm{an}}_{\mathbb{C}} = C\backslash\big((\mathbb{P}^1_{\mathbb{C}} \setminus \mathbb{P}^1(\mathbb{R})) \times G(\mathbb{A}_f)\big)/G(\mathbb{Q}), $$

but makes the analogy with the following description of the associated p-adic space more striking.

Let $\overline{B}$ be the quaternion algebra over E obtained from B by interchanging the invariants at ∞_0, v, and let $\overline{G}$ be the $\mathbb{Q}$-algebraic group attached to $\overline{B}^{\times}$. Let us identify $\overline{G}(\mathbb{A}^p_f)$ with $G(\mathbb{A}^p_f)$ via an anti-isomorphism $\overline{B} \otimes_E \mathbb{A}^p_f \to B \otimes_E \mathbb{A}^p_f$, and set $\overline{C} = GL_2(E_v)C^p$ viewed as a subgroup of $\overline{G}(\mathbb{A}_f)$. Then

$$ \begin{aligned} {}^{\mathrm{ad}}\mathcal{S}h^{\mathrm{an}}_{\mathbb{C}_p} &= C^p\backslash\big(({}_{E_v}\Omega)_{\mathbb{C}_p} \times G(\mathbb{A}^p_f)\big)/\overline{G}(\mathbb{Q}) \\ &= \overline{C}\backslash\big((\mathbb{P}^1_{\mathbb{C}_p} \setminus \mathbb{P}^1(E_v)) \times \overline{G}(\mathbb{A}_f)\big)/\overline{G}(\mathbb{Q}) \end{aligned} $$

where $\overline{G}(\mathbb{Q})$ acts on $({}_{E_v}\Omega)_{\mathbb{C}_p}$ through $\overline{G}^{\mathrm{ad}}(\mathbb{Q}_p) \cong PGL_2(E_v)$.

Moreover, one can drop the assumption that C is maximal at p, on replacing $({}_{E_v}\Omega)_{\mathbb{C}_p}$ by some finite etale covering as in 7.4.7. [**BZ95**, 3], [**Var98b**, 5].

7.5.4. Rapoport, Zink and Boutot on one hand (by modular methods à la Drinfeld), and Varshavsky on the other hand (by group-theoretic methods à la Čerednik), have pointed out similar global p-adic uniformizations in much more general cases.

Let us only mention a remarkable example: Mumford's *fake projective plane* [**Mu79**]. This is a smooth projective complex surface S with the same Betti numbers as $\mathbb{P}^2$, but not isomorphic to $\mathbb{P}^2$. Actually, it is a surface of general type. Mumford's construction is of diadic nature. It is a quotient of the two-dimensional generalization $\Omega^{(2)}$ of Ω by an explicit arithmetic group. This $\Omega^{(2)}$ is the complement of the lines defined over $\mathbb{Q}_2$ in the projective plane.

Using the last theorem of [**RZ96**] or [**Var98b**], one can see that S is actually a Shimura surface of PEL type, parametrizing polarized abelian varieties of dimension 9, with multiplication by an order in a simple $\mathbb{Q}$-algebra B of dimension 18 with center $\mathbb{Q}(\sqrt{-7})$. The invariants of B at the two primes of $\mathbb{Q}(\sqrt{-7})$ above 2 are 1/3 and 2/3 (*cf.* a recent preprint by F. Kato for more detail). The associated group G is a group of unitary similitudes of signature $(1, 2)$.

The "diadic ball" $\Omega^{(2)}$ (or rather its formal avatar) has an modular interpretation for certain p-divisible groups (with multiplication by B_2). These p-divisible groups split as a product of three p-divisible groups of slope $1/3$ and height 3, and three p-divisible groups of slope $2/3$ and height 3.

The associated Gauss-Manin connection splits into six factors of rank 3 (in two variables). It resembles the Appell-Lauricella connection studied in [**Ter73**], [**DM93**], [**Yo87**], but is different.

7.6. Application of the theory of p-adic Betti lattices.

7.6.1. We consider more closely the Gauss-Manin connection $(\mathcal{H}, \nabla_{\mathrm{GM}})$ in the case of Shimura curves.

We begin with the case $E = \mathbb{Q}$ as in 7.4. Then $\mathcal{H}$ is a vector bundle of rank 4, and the B-action on $\mathcal{H}$ obtained by functoriality of De Rham cohomology commutes with ∇_{GM}.

Let us choose an imaginary quadratic field $F = \mathbb{Q}(\sqrt{-d})$ inside B. Then $(\mathcal{H}, \nabla_{\mathrm{GM}}) \otimes F$ splits into two parts: $(\mathcal{H}, \nabla_{\mathrm{GM}})_+ \oplus (\mathcal{H}, \nabla_{\mathrm{GM}})_-$ (on which $F \subset B$ acts through identity and complex conjugation respectively).

On the other hand, let us fix embeddings $F \subset \overline{\mathbb{Q}} \subset \mathbb{C}$, and let us fix a base point $s \in \mathcal{X}_\Gamma(\overline{\mathbb{Q}})$ of our Shimura curve. This is the moduli point of a decorated fake elliptic curve $A/\overline{\mathbb{Q}}$. We have $V^{\mathbb{C}} = \mathrm{H}^1_B(A_{\mathbb{C}}, \mathbb{C}) \cong (\mathcal{H}^{\mathrm{an}}_{\mathbb{C}})^{\nabla_{\mathrm{GM}}}$, and this again splits into $\pm$ parts. The Betti cohomology F-space $\mathrm{H}^1_B(A_{\mathbb{C}}, F)$ already splits into two parts: $\mathrm{H}^1_B(A_{\mathbb{C}}, F)_+ \oplus \mathrm{H}^1_B(A_{\mathbb{C}}, F)_-$, both of rank two and stable under the action of $B \otimes F \cong M_2(F)$.

The flag space $\mathcal{D}^\vee$ is actually defined over the reflex field $\mathbb{Q}$. Over F, it becomes isomorphic to $\mathbb{P}^1$, or more precisely, to $\mathbb{P}(\mathrm{H}^1_B(A_{\mathbb{C}}, F)_+)$. The period mapping describes the "slope" of the Hodge line $F^1 \cap V^{\mathbb{C}}_+$ with respect to a basis of $\mathrm{H}^1_B(A_{\mathbb{C}}, F)_+ \subset ((\mathcal{H}_+)^{\mathrm{an}}_{\mathbb{C}})^{\nabla_{\mathrm{GM}}}$, and is thus given by a quotient of solutions of the (partial) Gauss-Manin connection $\nabla_{\mathrm{GM}+}$.

Assume for simplicity that $\Gamma \subset B^\times/\mathbb{Q}^\times$ is torsion-free. Then Γ may be identified with the projective monodromy group, *i.e.* with the quotient by the homotheties of the image of the monodromy representation

$$\pi_1((\mathcal{X}_\Gamma)_{\mathbb{C}}, s) \to GL(\mathrm{H}^1_B(A_{\mathbb{C}}, F)_+).$$

7.6.2. Let us now turn to the p-adic side. We fix an embedding $\overline{\mathbb{Q}} \subset \mathbb{C}_p$. We shall limit ourselves to the case of a prime p of *supersingular reduction* for A. Note that this includes the case of critical primes (7.4.6).

Let D be the definite quaternion algebra over $\mathbb{Q}$ which is ramified only at p, and let $\overline{B}$ be the definite quaternion algebra over $\mathbb{Q}$ which is ramified at the same primes as B except p. Then, whether p is critical or not, the group J (of self-quasi-isogenies of $\overline{A}[p^\infty]$ commuting with B) is $\overline{B}_p^\times$.

By the local criterion for splitting fields [**Vig80**, III.3.5], F is also a splitting field for the quaternion algebra D (and $\overline{B}$ as well). We can therefore apply the theory of I.5.3, and get an F-structure $\mathrm{H}^1_B(A_{\mathbb{C}_p}, F)$ inside

$\mathrm{H}^1_{\mathrm{cris}}(\overline{A}, /\mathbb{C}_p) \cong V^{\mathbb{C}_p} = \mathbf{D}(\overline{\Lambda}) \otimes \mathbb{C}_p$. By construction, it is stable under the action of $M_2(D) \otimes F \cong M_4(F) \cong B \otimes \overline{B} \otimes F$.

Using the embedding $F \otimes 1 \otimes F \subset B \otimes \overline{B} \otimes F$, one splits this F-structure $\mathrm{H}^1_B(A_{\mathbb{C}_p}, F)$ into two parts $\mathrm{H}^1_B(A_{\mathbb{C}_p}, F)_+ \oplus \mathrm{H}^1_B(A_{\mathbb{C}_p}, F)_-$ (on which $F \otimes 1 \otimes 1 \subset B \otimes \overline{B} \otimes F$ acts through identity and complex conjugation respectively), both of rank two and stable under the action of $\overline{B} \otimes F \subset M_4(F)$, hence under J.

The p-adic flag space $\mathcal{D}^\vee$ is again defined over the reflex field $\mathbb{Q}$. Over F, it becomes isomorphic to $\mathbb{P}(\mathrm{H}^1_B(A_{\mathbb{C}_p}, F)_+)$ (note that the scaling ambiguity of $\mathrm{H}^1_B(A_{\mathbb{C}_p}, F)_+$ by a factor in $\sqrt{F^\times}$ disappears here). The J-equivariant period mapping $\mathcal{P}$ describes the "slope" of the Hodge line $F^1 \cap V_+^{\mathbb{C}_p}$ with respect to a fixed basis (e_0, e_1) of $\mathrm{H}^1_B(A_{\mathbb{C}_p}, F)_+ \subset ((\mathcal{H}_+)^{\mathrm{an}}_{\mathbb{C}_p})^{\nabla_{\mathrm{GM}}}$, and is thus given by a quotient of solutions of $\nabla_{\mathrm{GM}+}$.

Note that if B is split at p, then $\mathcal{P}$ is of Gross-Hopkins type, and one has only a local p-adic uniformization of $\mathcal{X}_\Gamma$ around s (in a supersingular disk).

If instead p is critical, then $\mathcal{P}$ is of Drinfeld type, and the group $\Gamma_p \subset J^1$ appearing in the global p-adic uniformization of $\mathcal{X}_\Gamma$ may be identified with the quotient by the homotheties of the image of the monodromy representation (in the sense of 7.2.6)

$$\pi_1^{\mathrm{top}}((\mathcal{X}_\Gamma)_{\mathbb{C}_p}, s) \to GL(\mathrm{H}^1_B(A_{\mathbb{C}_p}, F)_+).$$

A slight refinement provides a rank two $\mathcal{O}_F[\frac{1}{p}]$-lattice $\mathrm{H}^1_B(A_{\mathbb{C}_p}, \mathcal{O}_F[\frac{1}{p}])_+$ in $((\mathcal{H}_+)^{\mathrm{an}}_{\mathbb{C}_p})^{\nabla_{\mathrm{GM}}}$ stable under p-adic monodromy.

7.6.3. One can go one step further and see that the phenomenon encountered in 6.3.3 shows up here again: "the" quotient τ of solutions of solutions of ∇_{GM} which expresses locally the period mapping $\mathcal{P}$ is essentially given by the same formula in the complex and the p-adic cases.

More precisely, assume that A has complex multiplication by an order in $M_2(F)$ (so that A is isogenous to the square of an elliptic curve with complex multiplication by $\mathcal{O}_{\mathbb{Q}(\sqrt{-d})}$ with [fundamental] discriminant $-d$), and identify $\mathrm{End}_B(A)$ with F. We denote by ϵ the Dirichlet character, w the number of roots of unity, h the class number attached to $\mathcal{O}_{\mathbb{Q}(\sqrt{-d})}$.
We may choose our symplectic basis e_0, e_1 of $\mathrm{H}^1_B(A_{\mathbb{C}}, F)_+$ (*resp.* $\mathrm{H}^1_B(A_{\mathbb{C}_p}, F)_+$) in such a way that $[\sqrt{-d}]^* e_0 = -\sqrt{-d}\, e_0$, $[\sqrt{-d}]^* e_1 = \sqrt{-d}\, e_1$ (here $[\sqrt{-d}]$ is viewed as an element of $\mathrm{End}_B(A) = F$).
Let ω be a section of $\mathcal{H}_+$ in a Zariski neighborhood U of s, which corresponds to a relative differential of the first kind (no pole) on the universal abelian scheme. Then ω is a cyclic vector for $\mathcal{H}_+$ over U with respect to ∇_{GM}, so that ∇_{GM} amounts to a concrete linear differential equation of order two over U.

We calculate τ with respect the basis e_0, e_1 of $\mathrm{H}^1_B(A_{\mathbb{C}}, F)_+$ (*resp.* $\mathrm{H}^1_B(A_{\mathbb{C}_p}, F)_+$) in a small neighborhood of the base point s. Denoting by z an algebraic local parameter at s in U, and extending e_0, e_1 by horizontality, we then have the proportionality

$$\omega \sim \tau(z)e_0 + e_1$$

Theorem 7.6.4. *There is a quotient $y = y_1/y_2$ of solutions of ∇_{GM} in $\overline{\mathbb{Q}}[[z]]$ such that τ takes the form $\tau(z) = \kappa^{-1}\, y(z)$ for a suitable constant κ. In the complex case, $\kappa \sim \prod_{u\in(\mathbb{Z}/d)^\times} \left(\Gamma\left\langle\frac{u}{d}\right\rangle\right)^{\epsilon(u)w/2h} \in \mathbb{C}^\times/\overline{\mathbb{Q}}^\times$.*
In the p-adic case, assuming moreover that $p \nmid d$,

$$\kappa \sim \prod_{u\in(\mathbb{Z}/d)^\times} \left(\Gamma_p\left\langle\frac{pu}{d}\right\rangle\right)^{-\epsilon(u)w/4h} \in \mathbb{C}_p^\times/\overline{\mathbb{Q}}^\times.$$

PROOF. Let us write $\omega_{|z=0} = \omega_{11}e_1$. Then $\tau(z) = \kappa^{-1}y(z)$, with $\kappa = \omega_{11}^2/2\pi i$ in the complex case, $\kappa = \omega_{11}^2$ in the p-adic case (the factor $2\pi i$ which arises in the complex case is the factor of proportionality of the symplectic forms in De Rham and in Betti cohomology respectively). The evaluation of ω_{11} in the complex case is given by the Lerch-Chowla-Selberg formula (*cf.* I.4.6.3). In the p-adic case, it is given by I.5.3.9. □

7.6.5. This discussion generalizes without much complication to the case of Shimura curves attached to a quaternion algebra D over a totally real number field E of degree > 1. We concentrate on the case of a critical place v and assume, for simplicity, that E is Galois over $\mathbb{Q}$ and that v is the unique place of E with the same residue characteristic p (*i.e.* $m = 0$ in the notation of 7.5.2).

Let F be a totally imaginary quadratic extension of E contained in B. Then F is a splitting field for $B, \overline{B}$, and D. Let us fix a double embedding $\mathbb{C} \hookleftarrow \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ as in 7.5.2, and an extension $FK \hookrightarrow \overline{\mathbb{Q}}$ of ∞_0. For C small enough, $\mathcal{S}h^\bullet$ carries a universal abelian scheme of relative dimension $g = [FK : \mathbb{Q}]$. After tensoring $\otimes_K\overline{\mathbb{Q}}$, its Gauss-Manin connection splits into pieces of rank two, indexed by the embeddings of FK into $\overline{\mathbb{Q}}$. We select the piece $(\mathcal{H}, \nabla_{\mathrm{GM}})_+$ corresponding to ∞_0.

Let us fix a base point $s \in \mathcal{S}h(\overline{\mathbb{Q}}) \subset \mathcal{S}h^\bullet(\overline{\mathbb{Q}})$ of our Shimura curve. This is the moduli point of a decorated abelian variety $A^\bullet/\overline{\mathbb{Q}}$ (of dimension $g = [FK : \mathbb{Q}]$ and multiplication by $\mathcal{B}^\bullet$). We have $V^\bullet = V \otimes_E K \cong \mathrm{H}^1_B(A^\bullet_{\mathbb{C}}, \mathbb{Q})$. Note that $\mathrm{H}^1_B(A_{\mathbb{C}}, F) \cong V \otimes_{\mathbb{Q}} FK$ splits into pieces of FK-rank two, indexed by the embeddings of F into $\overline{\mathbb{Q}}$. We select the piece $\mathrm{H}^1_B(A^\bullet_{\mathbb{C}}, F)_+$ corresponding to ∞_0. We then have a natural embedding $\mathrm{H}^1_B(A^\bullet_{\mathbb{C}}, F)_+ \subset ((\mathcal{H}_+)^{\mathrm{an}}_{\mathbb{C}})^{\nabla_{\mathrm{GM}}}$, and $\mathrm{H}^1_B(A^\bullet_{\mathbb{C}}, F)_+$ is stable under monodromy.

On the other hand, it follows from 7.4.6, that $A^\bullet$ has supersingular reduction $\overline{A}^\bullet$ at v. We can construct the F-structure $\mathrm{H}^1_B(A^\bullet_{\mathbb{C}_p}, F)$ inside $\mathrm{H}^1_{\mathrm{cris}}(\overline{A}^\bullet, /\mathbb{C}_p) \cong V^{\mathbb{C}_p} = \mathbf{D}(\overline{\Lambda}^\bullet) \otimes \mathbb{C}_p$. By construction, it is stable under the action of $M_g(D) \otimes F \cong M_{2g}(F)$, which contains naturally $B \otimes_{\mathbb{Q}} \overline{B} \otimes_{\mathbb{Q}} F$.

Using the embedding $F \otimes 1 \otimes F \subset B \otimes \overline{B} \otimes F$, one splits this F-structure $\mathrm{H}^1_B(A_{\mathbb{C}_p}, F)$ into parts of FK-rank two, indexed by the embeddings of F into $\overline{\mathbb{Q}}$. We select the piece $\mathrm{H}^1_B(A^\bullet_{\mathbb{C}_p}, F)_+$ corresponding to ∞_0. We then have a natural embedding $\mathrm{H}^1_B(A^\bullet_{\mathbb{C}_p}, F)_+ \subset ((\mathcal{H}_+)^{\mathrm{an}}_{\mathbb{C}_p})^{\nabla_{\mathrm{GM}}}$, and $\mathrm{H}^1_B(A^\bullet_{\mathbb{C}_p}, F)_+$ is stable under p-adic monodromy.

7.7. Conclusion.

Roughly speaking, the period mapping $\mathcal{P}$ attaches to each member of an algebraic family of algebraic complex varieties (with additional structure) a point in some flag space, which encodes the Hodge filtration in the cohomology of this variety. This mapping can be expressed in terms of quotients of solutions of the fuchsian differential equation (Picard-Fuchs/Gauss-Manin) which controls the variation of the cohomology.

We have described the theory of period mappings for p-divisible groups in somewhat similar terms.

Sometimes, it is possible to go through the looking-glass of sheer analogies. This occurs when we restrict our attention to p-divisible groups attached to polarized abelian varieties with prescribed endomorphisms and fixed "Shimura type". In this situation, we have three algebraic objects defined over a number field (the reflex field):

(a) the Shimura variety $\mathcal{S}h$ parametrizing certain "decorated" abelian varieties,
(b) the Gauss-Manin connection ∇_{GM} describing the variation of these abelian varieties over $\mathcal{S}h$, and
(c) the flag variety $\mathcal{D}^\vee$.

These objects are transcendentally related, both in the complex and p-adic sense.

The period mapping may be viewed as multivalued and defined on p-adic open domains in $\mathcal{S}h$; it is given as in the complex case by quotients of solutions of the Gauss-Manin connection (as a p-adic connection).

In most situations, these open domains are small and $\mathcal{P}$ is single-valued.

However, it happens in some remarkable cases that $\mathcal{P}$ is a global multivalued function. This occurs when the abelian varieties under consideration form a *single isogeny class modulo p*. In this situation, the complex and p-adic transcendental relations between $\mathcal{S}h, \nabla_{\mathrm{GM}}$ are completely similar: $\mathcal{S}h$ is uniformized by an analytic space which is etale over some open subset $\mathcal{D}$ of $\mathcal{D}^\vee$, and the group of deck transformations Γ is isomorphic to the projective global monodromy group of ∇_{GM}. In III, we shall analyze in detail the group-theoretic aspects of this situation.