CHAPTER I

Analytic aspects of p-adic periods.

Contents

1. Analytic continuation; topological point of view.

ABSTRACT: Thanks to Berkovich's presentation of p-adic analytic geometry, it is possible to make sense of the familiar monodromy principle in the exotic world of p-adic manifolds. Its application is however more limited than in the classical case, because (1) sheaves of solutions of linear differential equations are usually not locally constant, and (2) many spaces (for instance, annuli) are simply connected.

1.1. The monodromy principle.

Let S be a topological space, and $\mathcal{F}$ an abelian sheaf on S. For an open set $U \subseteq S$ and a section $f \in \Gamma(U, \mathcal{F})$, the support of f is the subset $\mathrm{Supp}(f) = \{u \in U \mid f_u \neq 0\}$, which is easily seen to be closed in U.

Definition 1.1.1. We say that the sheaf $\mathcal{F}$ satisfies the *principle of unique continuation* if for any open set $U \subseteq S$ and any section $f \in \Gamma(U, \mathcal{F})$ the support $\mathrm{Supp}(f)$ is open in U.

Note that the principle of unique continuation is a local property.

Lemma 1.1.2. *If $\mathcal{F}$ satisfies the principle of unique continuation, then any two sections $f, g \in \Gamma(U, \mathcal{F})$ on a connected open subset $U \subseteq S$ coincide if (and only if) their germs f_s, g_s at some point $s \in U$ coincide. The converse also holds if S is locally connected.*

PROOF. The first assertion is clear. For the converse, let $f \in \Gamma(U, \mathcal{F})$ and assume that for any $u \in U$, any connected open neighborhood V of u (which exists since S is locally connected), and any $s \in V$, $f_s = 0$ implies $f|_V = 0$. We choose u to be a point adherent to the complement of $\mathrm{Supp}(f)$, so that $V \cap (U \setminus \mathrm{Supp}(f)) \neq \emptyset$ and we can pick s in that set. We conclude that $f|_V = 0$. Hence $u \notin \mathrm{Supp}(f)$. This shows that $\mathrm{Supp}(f)$ is open. □

Lemma 1.1.3. *Let $p : F \to S$ be the local homeomorphism canonically attached to $\mathcal{F}$: $\mathcal{F}(U)$ is the set of (continuous) sections of p over U. If F is separated (i.e. Hausdorff), then $\mathcal{F}$ satisfies the principle of unique continuation. The converse also holds if S is separated.*

PROOF. If F is separated, then for any two sections f, g of p over an open subset $U \subset S$, the set of $s \in U$ such that $f(s) = g(s)$ is closed. Applying this to the zero-section $g = 0$, we see that the support of f is open.

Conversely, let x, y be two points of F. If $p(x) \neq p(y)$, there are disjoint open neighborhoods V_x, V_y of $p(x)$ and $p(y)$ respectively in the separated space S; then $p^{-1}V_x$ and $p^{-1}V_y$ are disjoint open neighborhoods of x and y respectively. If $p(x) = p(y)$, there are open neighborhoods U_x, U_y of x and y respectively, an open subset $V \subset S$, and sections f, g of p over V, such that $f(V) = U_x$, $g(V) = U_y$. By the principle of unique continuation, $f \neq g$ defines an open subset $W \subset V$ containing $p(x) = p(y)$; then $f(W)$ and $g(W)$ are disjoint open neighborhoods of x and y respectively. □

Definition 1.1.4. We say that the sheaf $\mathcal{F}$ satisfies the *monodromy principle* if it has the following property:

let $\Gamma : [a,b] \times [0,1] \to S$ be any continuous map with $\Gamma(\{a\} \times [0,1]) = \{x_a\}, \Gamma(\{b\} \times [0,1]) = \{x_b\}$. Let f_{x_a} be an element of the stalk $\mathcal{F}_{x_a}$. Assume that for any $t \in [0,1]$, f_{x_a} extends to a global section f^{Γ_t} of $\Gamma_t^*\mathcal{F}$ on $[a,b]$. Then this extension is unique and $f^{\Gamma_t}(b) \in \mathcal{F}_{x_b}$ is independent of t.

In this situation, the section f^{Γ_t} of $\Gamma_t^*\mathcal{F}$ is called the *continuation of f_{x_a} along the path Γ_t*, and $f^{\Gamma_t}(b) \in \mathcal{F}_{x_b}$ its *value at x_b*.

Proposition 1.1.5. *If $\mathcal{F}$ satisfies the principle of unique continuation, then it satisfies the principle of monodromy. The converse also holds if S is locally arcwise connected.*

PROOF. Assume that $\mathcal{F}$ satisfies the principle of unique continuation. This guarantees the uniqueness of the continuation f^{Γ_t} of f_{x_a} along the path Γ_t for any t. Moreover, by a special case of the proper base change theorem (applied to the first projection $[a,b] \times [0,1] \to [a,b]$, *cf.* [**Iv86**, IV, 1.4]) f^{Γ_t} extends to a section of $\Gamma^*\mathcal{F}$ on a suitable subset of the form $[a,b] \times (t-\epsilon, t+\epsilon)$. By unicity, these sections glue together to a global section of $\Gamma^*\mathcal{F}$. The restriction of this sheaf to $\{b\} \times [0,1]$ is the constant sheaf with stalk $\mathcal{F}_{x_b}$. Therefore the value $f^{\Gamma_t}(b) \in \mathcal{F}_{x_b}$ is independent of t.

Conversely, let U be an arcwise connected open subset of S, let x_a, x_b be two points of U, and let f be a section of $\mathcal{F}$ over U. Let $\gamma : [a,b] \to S$ be any path from x_a to x_b. It follows from the unicity of continuation of f_{x_a} along γ (requested in the monodromy principle) that if f_{x_a} is 0, so is f_{x_b}. □

Example 1.1.6. (1) Any locally constant abelian sheaf $\mathcal{F}$ on a topological space S satisfies the principle of unique continuation, hence the principle of monodromy.

In fact, for any $\Gamma : [a,b] \times [0,1] \to S$ as in Lemma 1.1.3, the inverse image $\Gamma^*\mathcal{F}$ is locally constant, hence constant and canonically isomorphic to the constant sheaf attached to $\mathcal{F}_{x_a}$. Therefore the extension f^γ of any $f_{x_a} \in \mathcal{F}_{x_a}$ along any path γ exists, and the value at the other extremity $x_b = \gamma(b)$ depends only on the *homotopy class* of γ.

(2) When S is a complex manifold, the structure sheaf $\mathcal{O}_S$ satisfies the principle of unique continuation, hence the principle of monodromy.

1.1.7. Let S be a topological space, connected and locally arcwise (or simply) connected, and $\mathfrak{F}$ a locally constant abelian sheaf on S. We fix a point $s \in S$, and denote by $\pi_1(S,s)$ the fundamental group based at s.[1] To any loop $\gamma\colon [0,1] \to S$ based at s, let us associate the so-called *monodromy* along γ, defined by the composite

$$\mathfrak{F}_s \xrightarrow{\sim} (\gamma^*\mathfrak{F})_0 \xrightarrow{\sim} \Gamma([0,1], \gamma^*\mathfrak{F}) \xrightarrow{\sim} (\gamma^*\mathfrak{F})_1 \xrightarrow{\sim} \mathfrak{F}_s.$$

[1] to remove any ambiguity, let us say that we adopt Deligne's convention: the composition in $\pi_1(S,s)$ is induced by the juxtaposition of loops in the reverse order. As such, $\pi_1(S,s)$ acts *on the right* on the pointed universal covering.

By 1.1.6 (1), the monodromy $\mathcal{F}_s \xrightarrow{\sim} \mathcal{F}_s$ depends only on the class of γ in $\pi_1(S,s)$, hence gives rise to a left $\mathbb{Z}[\pi_1(S,s)]$-module structure on $\mathcal{F}_s$. This construction yields an equivalence of the categories

$$\{\text{locally constant sheaves on } S\} \xrightarrow{\sim} \{\text{left } \mathbb{Z}[\pi_1(S,s)]\text{-modules}\} \ :$$

giving a locally constant sheaf amounts to giving its value at s together with the monodromy action.

1.2. Rigid geometry and the problem of unique continuation.

1.2.1. For p a prime number, let $\mathbb{Q}_p$ denote as usual the completion of $\mathbb{Q}$ for the p-adic absolute value $|\ |_p$: $|p^n \frac{a}{b}|_p = p^{-n}$ if the rational integers a, b are prime to p. This ultrametric absolute value extends in a unique way to each finite extension of $\mathbb{Q}_p$. These finite extensions are locally compact and totally disconnected. They are all complete, but "the" algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ itself is not complete. Its completion, denoted by $\mathbb{C}_p$, turns out to be algebraically closed, and plays the role of $\mathbb{C}$ in p-adic analysis.
In the sequel, $\mathrm{D}(a, r^+)$ (*resp.* $\mathrm{D}(a, r^-)$) stands for the disk — archimedean or not — of radius r centered at a with (*resp.* without) circumference.

It might be surprising at first that geometries can be built upon p-adic numbers, whose "fractal" nature makes them hardly amenable to intuition as a continuum. Nevertheless, Bourbaki's presentation of analytic geometry [**Bou83**] treats the real, complex and p-adic cases on equal footing. This approach is based on a local definition of analytic functions as sums of convergent power series. Its major drawback is that these analytic functions fail to satisfy the principle of unique continuation, essentially because two ultrametric disks are either concentric or disjoint (like drops of mercury)

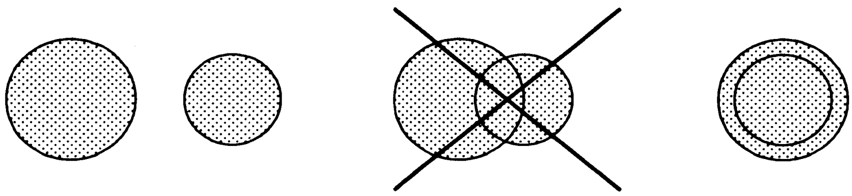

FIGURE 1

1.2.2. M. Krasner had the idea to remedy this by using a definition of analytic functions *à la Runge*. He properly founded ultrametric analysis by introducing his *analytic elements* defined as uniform limits of rational functions, a global notion which overcomes, to some extent, problems stemming from the disconnectedness of the p-adics.

The next step was taken by J. Tate. In order to deal with more general spaces than just subsets of the line, he introduced and developed the so-called *rigid analytic geometry* (as opposed to Bourbaki's "wobbly" analytic geometry), based on affinoid algebras (topological K-algebras isomorphic to quotients of rings of restricted formal power series, *i.e.* whose coefficients tends to 0) and a suitable Grothendieck topology [**Ta71**].

In a connected rigid analytic variety S, one has the following avatar of analytic continuation [**Be96**, 0.1.13]: assume that there is no admissible covering formed by two disjoint non-empty open subsets, and let f be an analytic function on S. If there is a connected open subset U over which f vanishes, then $f = 0$.

1.2.3. The first achievement of rigid geometry was Tate's representation of an elliptic curve with "bad reduction at p" as a rigid analytic quotient $\mathbb{C}_p^\times/q^{\mathbb{Z}}$ of the multiplicative group by the discrete group generated by q. Here q is given by the usual series $\frac{1}{j} + \dots$ in the j-invariant of the elliptic curve ($|j| > 1$), interpreted p-adically. The quotient is obtained by gluing two affinoid annuli of width $|q|^{-1/2}$ along their boundary; the inverse image in $\mathbb{C}_p^\times$ of each of these annuli consists of countably many disjoint copies of it. This is analogous to Jacobi's partial uniformization $\mathbb{C}^\times/q^{\mathbb{Z}}$ of a complex elliptic curve.

1.2.4. This result was extended by D. Mumford to curves of higher genus [**Mu72b**]. Here, the partial uniformization which serves as a complex model is the Schottky uniformization of complex curves of genus $g > 1$, which we briefly recall (*cf.* [**Heh75**]). Let D be a connected open subset of $\mathbb{P}^1(\mathbb{C})$ whose boundary consists of $2g$ disjoint circles $C_1, C_1', \dots, C_g, C_g'$. One assumes the existence, for any $i = 1\dots, g$, of an element $\gamma_i \in PGL_2(\mathbb{C})$, with two distinct fixed points, which sends C_i to C_i' and D outside itself. Then the subgroup Γ generated by the γ_i's is discrete and freely generated by them, the complement $\Omega \subset \mathbb{P}^1(\mathbb{C})$ of the topological closure of the set of fixed points of Γ is an open dense subset ($= \bigcup_{\gamma\in\Gamma}\gamma(D)$), and Ω/Γ is a projective smooth complex curve of genus g.

When $\mathbb{C}$ is replaced by $\mathbb{C}_p$, the same construction applies. The rigid analytic quotient Ω/Γ exists and is called a Mumford curve. Among projective smooth curves of genus g over $\mathbb{C}_p$, Mumford curves are characterized by the existence of a reduction over the residue field $\overline{\mathbb{F}}_p$ such that every irreducible component is isomorphic to $\mathbb{P}^1$ and intersect the others at double points, *cf.* [**GvdP80**].

Let us mention that T. Ichikawa has proposed a unified theory of the archimedean and non-archimedean Schottky-Mumford uniformizations, *cf.* [**Ic97**].

1.3. Berkovich geometry and the principle of monodromy

1.3.1. Rigid analytic spaces are endowed with a Grothendieck generalized topology, and their structure sheaf is a sheaf with respect to this topology. Hence it cannot be said to satisfy the principle of unique continuation in the strict sense of Definition 1.1.1. Moreover, there is no non-trivial path in such spaces. Therefore, rigid geometry is not a suitable setting for discussing p-adic analytic continuation in an intuitive way.

In contrast, Berkovich's viewpoint on p-adic geometry does not suffer from these drawbacks: Berkovich's analytic spaces are genuine locally ringed topological spaces, which are even locally arcwise connected. We refer to [**Ber98**] for a compact technical presentation.

1.3.2. The buildings blocks are the same: affinoid algebras A (called strictly affinoid algebras by Berkovich), *i.e.* topological K-algebras isomorphic to quotients of rings of restricted power series. However, instead of attaching to A its maximal spectrum $\mathrm{Spm}(\mathsf{A})$ as in rigid geometry, Berkovich analytic geometry deals with the *affinoid space* $\mathsf{M}(\mathsf{A})$ of all bounded multiplicative seminorms on A.

This "spectrum" contains "more points" than the maximal spectrum $\mathrm{Spm}(\mathsf{A})$, namely something like "generic points", which "complete" $\mathrm{Spm}(\mathsf{A})$. There is a natural inclusion $\mathrm{Spm}(\mathsf{A}) \subseteq \mathsf{M}(\mathsf{A})$ which identifies $\mathrm{Spm}(\mathsf{A})$ with the subspace of all seminorms $|\cdot|$ with $\mathsf{A}/\operatorname{Ker}|\cdot| = \mathbb{C}_p$. This "completion", in fact, simplifies the topology; e.g. Berkovich analytic spaces are locally arcwise connected.

1.3.3. *Berkovich's affinoids.* Let us be a little more precise about the definition of *affinoid spaces* in Berkovich geometry. Let A be an affinoid algebra over a complete subfield K of $\mathbb{C}_p$.

(i) A point of $\mathsf{M}(\mathsf{A})$ is a bounded multiplicative seminorm on A.
(ii) The topology of $\mathsf{M}(\mathsf{A})$ is the weakest one so that the mapping $\mathsf{M}(\mathsf{A}) \ni \chi \mapsto \chi(f) \in \mathbb{R}_{\geq 0}$ is continuous for any $f \in \mathsf{A}$.
(iii) The sheaf of rings $\mathcal{O}_{\mathsf{M}(\mathsf{A})}$ is defined by $\Gamma(U, \mathcal{O}_{\mathsf{M}(\mathsf{A})}) = \varprojlim \mathsf{A}_V$, where V runs over finite unions $\bigcup V_i$ of affinoid domains contained in U, and $\mathsf{A}_V = \operatorname{Ker}(\prod_i \mathsf{A}_{V_i} \rightrightarrows \prod_{ij} \mathsf{A}_{V_i \cap V_j})$, A_{V_i} being the affinoid algebra attached to the affinoid domain V_i.

The value of a "function" $f \in \mathsf{A}$ at a point $\chi \in \mathsf{M}(\mathsf{A})$ is its image in the field of fractions of $\mathsf{A}/\operatorname{Ker}\chi$. This field inherits the absolute value induced by the seminorm χ, and its completion is denoted by $\mathcal{H}(\chi)$.

Example 1.3.4. Let $K\{t\}$ denote the ring of restricted power series in one variable. Let us assume, for simplicity, that K is algebraically closed. In rigid geometry, the maximal spectrum $\mathrm{Spm}(K\{t\})$ is just the closed disk $\mathsf{D}(0, 1^+)$ in K of radius 1 in the usual sense. In $\mathsf{M}(K\{t\})$, the following four kinds of points occur:

(1) classical points (*i.e.* those coming from $\mathrm{Spm}(K\{t\})$): $x \in \mathsf{D}(0, 1^+)$, $\chi_x(f) = |f(x)|_K$ for $f \in K\{t\}$,
(2) generic points of disks: $\chi = \eta_{x,r}$ for $0 < r \leq 1$ with $r \in |K^\times|$, $\chi(f) = |f|_{\mathsf{D}(x,r^+)}$ (the sup-norm),
(3) the same, for $r \notin |K^\times|$,
(4) generic points of infinite decreasing families $\{\mathsf{D}_\alpha\}$ of closed disks with radius ≤ 1: $\chi(f) = \inf |f|_{\mathsf{D}_\alpha}$.

Therefore, two arbitrary distinct points in $\mathsf{M}(K\{t\})$ can be connected by a unique path. For example, two points of type (1), x and y, are connected by a path consisting of points of type (2) and (3) associated to the disks

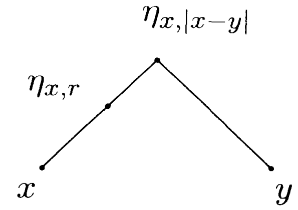

FIGURE 2. Generic points and paths on a Berkovich space.

$\mathsf{D}(x, r^+)$ and $\mathsf{D}(y, r^+)$ for $0 < r \leq |x - y|$. The complement of a point of type (1) or (4) is connected. The affine line $\mathbb{A}^1$ is the union of the affinoid spaces associated to the algebras of power series convergent in $\mathsf{D}(0, r^+)$, and the projective line $\mathbb{P}^1$ is the Alexandroff compactification of $\mathbb{A}^1$.

Remark 1.3.5. Adding "generic points" to rigid spaces to come up with Berkovich spaces simplifies the topology in contrast to what happens in algebraic geometry, when generic points are added to varieties to produce schemes. Actually Berkovich's "generic points" are *closed*, unlike Grothendieck's ones.

1.3.6. The construction of (strictly) analytic K-spaces by gluing affinoid spaces together is a little delicate (one is not gluing open subspaces); we refer to [**Ber93**]. These spaces are *locally compact, locally countable at infinity, and locally arcwise connected.*

There is a fully faithful functor between Berkovich's Hausdorff (strictly) analytic K-spaces and rigid analytic varieties: at the level of underlying sets, this functor sends a space S to the subset of "classical points", *i.e.* points s for which $[\mathcal{H}(s) : K] < \infty$.

This functor establishes *an equivalence of categories between the category of paracompact (strictly) analytic K-spaces and quasi-separated rigid spaces over K having an admissible affinoid covering of finite type.*

This equivalence respects the notion of dimension (topological dimension in the case of Berkovich analytic spaces), as well as the standard properties of local rings such as: reduced, normal, smooth..., and the properties: finite, étale... of morphisms. Also, a paracompact analytic space is connected (in the usual sense) if and only if the corresponding rigid space does not admit an admissible covering by two disjoint non-empty open subsets.

Furthermore, there is a canonical functor "analytification"

$$\left\{\begin{matrix}\text{separated schemes}\\ \text{locally of finite type}\\ \text{over } K\end{matrix}\right\} \to \left\{\begin{matrix}\text{paracompact (strictly)}\\ \text{analytic } K\text{-spaces}\end{matrix}\right\}$$

and a canonical functor "generic fiber"

$$\left\{\begin{matrix}\text{separated formal schemes locally}\\ \text{finitely presented over } \mathcal{O}_K\end{matrix}\right\} \to \left\{\begin{matrix}\text{paracompact (strictly)}\\ \text{analytic } K\text{-spaces}\end{matrix}\right\}.$$

1.3.7. *p-adic manifolds.* We shall be mainly concerned with paracompact (strictly) K-analytic spaces S which satisfy the following assumption:

any $s \in S$ has a neighborhood $U(s)$ which is isomorphic to an affinoid subdomain of some space V_s which admits locally an étale morphism to the affine space $\mathbb{A}^{\dim S}$.

For convenience, we shall call such a space S a *K-manifold*, and simply a *p-adic manifold* if $K = \mathbb{C}_p$.
It is an important result of V. Berkovich [**Ber99**] that *p-adic manifolds are locally contractible.* Therefore they have universal coverings.

Proposition 1.3.8. *The structure sheaf of any p-adic manifold S satisfies the principle of unique continuation, and (equivalently) the principle of monodromy.*

PROOF. Since the principle of unique continuation is local, we may replace S by the neighborhood $U(s)$ of any point s given in advance. Thus we may assume that S itself is affinoid: $S = \mathsf{M}(\mathsf{A})$, and connected, and we have to show that the homomorphism $\iota_s : \ \mathsf{A} \to \mathsf{A}_s \simeq \varinjlim_{s\in V} \mathsf{A}_V$ is injective (V runs over the affinoid neighborhoods of s). This follows from the fact that the homomorphism of completion at $s : \mathsf{A} \to \widehat{\mathsf{A}}_s = \varprojlim_n \mathsf{A}/\mathcal{I}_s^n$ is injective and factors through ι_s. $\square$

In the case of curves (*i.e.* p-adic manifolds of dimension 1), analytic continuation is particularly intuitive, because there is a basis of open subsets $\mathcal{U}$ with finite boundary such that two arbitrary points in $\mathcal{U}$ are connected by a unique geometric path lying in $\mathcal{U}$.

1.4. Topological coverings and étale coverings.

1.4.1. Complex manifolds are locally contractible, and have universal coverings. There is no need to distinguish between topological coverings and étale coverings (finite or infinite).

In p-adic rigid geometry, the situation is more complicated. It is natural to call topological covering any morphism $f : Y \to X$ such that there is an admissible cover (X_i) of X and an admissible cover (Y_{ij}) of $f^{-1}(X_i)$ with disjoint Y_{ij} isomorphic to (X_i) via f. Indeed, such topological coverings correspond to locally constant sheaves of sets on X. It is still true that *topological coverings are étale, but the converse is wrong*, even if one restricts to finite surjective morphisms. Indeed, the Kummer covering $z \mapsto z^n$ of the punctured unit disk is an étale covering, but not a topological covering, if $n > 1$.

1.4.2. It is again more convenient to tackle these questions in the framework of Berkovich's geometry. For instance, one sees immediately that a Kummer covering $z \mapsto z^n$ as above is not a topological covering because a classical point has n preimages, while the "generic point" $\eta_{0,1}$ (corresponding to the sup-norm on the disk) is its own single preimage. Topological coverings of a p-adic manifold X are defined in the usual way; they correspond to locally constant sheaves of sets on X. They coincide [**Ber90**, 3.3.4] with topological coverings of the rigid analytic variety associated to X.

1.4.3. There are only three one-dimensional simply connected complex manifolds up to isomorphism: the projective line $\mathbb{P}^1(\mathbb{C})$, the affine line $\mathbb{C}$, and the disk D ($\simeq \mathfrak{h}$, the complex upper half plane). In contrast, there are plenty of *simply-connected* one-dimensional p-adic manifolds, for instance: any annulus, the line minus a finite number of points, more generally any connected p-adic manifold homeomorphic to a *subset of* $\mathbb{P}^1(\mathbb{C}_p)$, any smooth projective curve with "good reduction"... (all these Berkovich spaces look like "bushy trees").

1.4.4. One defines the (discrete) topological fundamental group $\pi_1^{\mathrm{top}}(S,s)$ of a pointed p-adic manifold (S,s) in the usual way. The general topological theory of coverings applies: (S,s) is naturally isomorphic to the quotient of the pointed universal covering $(\widetilde{S},\tilde{s})$ by $\pi_1^{\mathrm{top}}(S,s)$, and $\pi_1^{\mathrm{top}}(S,s)$ classify the topological coverings of (S,s). In the one-dimensional case, the topological fundamental group is a discrete *free* group (more precisely, it is isomorphic to the fundamental group of the dual graph of the so-called semistable reduction of S ([**dJ95b**, 5.3])).

For instance, if $S = \Omega/\Gamma$ is the uniformization of a Mumford curve (*cf.* 1.2), Ω is the universal covering of S and the Schottky group Γ is isomorphic to the topological fundamental group.

On the other hand, one can define the (profinite) algebraic fundamental group $\pi_1^{\mathrm{alg}}(S,s)$ à la Grothendieck, classifying the finite étale coverings of (S,s). In contrast to the complex situation, the natural map $\pi_1^{\mathrm{alg}}(S,s) \to \pi_1^{\mathrm{top}}(S,s)^\wedge$ to the profinite completion of $\pi_1^{\mathrm{top}}(S,s)$ is *generally not injective* (*e.g.* for annuli).

1.5. Connections with locally constant sheaves of solutions.

1.5.1. Let us briefly recall the complex situation. Let S be a complex connected manifold, $(\mathcal{M},\nabla)$ a vector bundle of rank r with integrable connection on S. The classical Cauchy theorem shows that for any $s \in S$, the solution space $(\mathcal{M}\otimes\mathcal{O}_{S,s})^\nabla$ at s has dimension r. Analytic continuation along paths gives rise to a homomorphism $\pi_1^{\mathrm{top}}(S,s) \to \mathrm{Aut}_{\mathbb{C}}((\mathcal{M}\otimes\mathcal{O}_{S,s})^\nabla)$ (the *monodromy*). The sheaf of germs of solutions $\mathcal{M}^\nabla$ is locally constant on S: its pull-back over the universal covering $\widetilde{S}$ of S is constant. Conversely, any complex representation V of $\pi_1^{\mathrm{top}}(S,s)$ of dimension r gives rise naturally to a vector bundle $\mathcal{M}$ of rank r with integrable connection ∇: $\mathcal{M} = (V\times\widetilde{S})/\pi_1^{\mathrm{top}}(S,s)$, $\nabla(V)=0$. This sets up an equivalence of categories:

$$\left\{\begin{matrix}\text{finite dimensional}\\ \text{representations of } \pi_1^{\mathrm{top}}(S,s)\end{matrix}\right\} \simeq \left\{\begin{matrix}S\text{-vector bundles with}\\ \text{integrable connection}\end{matrix}\right\}.$$

1.5.2. Let (S,s) be now a pointed connected p-adic manifold. It is still true that any $\mathbb{C}_p$-linear representation V of $\pi_1^{\mathrm{top}}(S,s)$ of dimension r gives rise to a vector bundle $\mathcal{M} = \mathcal{M}_V$ a vector bundle of rank r with integrable connection $\nabla = \nabla_V$ (same formula). The functor

$$\left\{\begin{matrix}\text{finite dimensional} \\ \text{representations of } \pi_1^{\mathrm{top}}(S,s)\end{matrix}\right\} \to \left\{\begin{matrix}S\text{-vector bundles with} \\ \text{integrable connection}\end{matrix}\right\}$$

is still fully faithful, but ***no longer surjective***; its essential image consists of those connections whose sheaf of solutions is locally constant, *i.e.* becomes constant over $\widetilde{S}$. In fact, the classical "***Cauchy theorem***" according to which the solution space $(\mathcal{M} \otimes \mathcal{O}_{S,s})^\nabla$ at s has dimension r holds for every classical point s of S — which corresponds to a point of the associated rigid variety —, but ***does not hold*** for non-classical points s of the Berkovich space S in general.

1.5.3. Any non-trivial connection over the projective line minus a few points gives an example when the p-adic analogue of the "Cauchy theorem" does not hold: indeed, in this case, the topological fundamental group is trivial.

However, a p-adic variant of Cauchy's theorem in the neighborhood of a non-classical point s may be restored as follows (Dwork's technique of generic points): it suffices to extend the scalars from $\mathbb{C}_p$ to a complete algebraically closed extension of $\mathcal{H}(s)$. This transforms s into a classical point, and neighborhoods of s after scalar extensions are "smaller" than before.

1.5.4. When "Cauchy's theorem" holds at every point of S, one can continue the local solutions along paths and get the monodromy representation just as in the complex situation.

This nice category of p-adic connections has not yet attracted much attention.

Example 1.5.5. Let us consider the case when S is a Tate elliptic curve: $S = \mathbb{C}_p^\times/q^{\mathbb{Z}}$, with $s =$ its origin. Then $\pi_1^{\mathrm{top}}(S,s) = q^{\mathbb{Z}}$, and the connections on S which arise from representations of $q^{\mathbb{Z}}$ are those which become trivial over $\widetilde{S} = \mathbb{C}_p^\times$. In this correspondence, the multivalued solutions $\vec{y}$ (*i.e.* horizontal sections) of such a connection are the solutions of the associated linear q-difference equation $\vec{y}(qt) = M(q)\vec{y}(t)$, where t is the standard coordinate on $\mathbb{C}_p^\times$ and the matrix $M(q)$ is the image of q in the monodromy representation.

The simplest example is given by the obvious one-dimensional representation $M(q) = q$. It corresponds to $\mathcal{M} = \mathcal{O}_S$, $\nabla(1) = \omega_{\mathrm{can}}$ (the canonical differential on S induced by dt/t); here the q-difference equation is $t.dy = y.dt$ with obvious solution $y = t$.

Let us now look at the representation $M(q) = \sqrt{q}$. The associated q-difference equation is $t.dy = \frac{1}{2}y.dt$. There is the obvious solution $\sqrt{t}$, which leads to $\mathcal{M} = \mathcal{O}_S$, $\nabla(1) = \frac{1}{2}\omega_{\mathrm{can}}$. Here we encounter an interesting paradox: $\sqrt{t}$ is not a multivalued analytic function on S (*i.e.* it is not an analytic function on the universal covering $\mathbb{C}_p^\times$. In the complex situation, a similar paradox arose in the work of G. Birkhoff in his theory of q-difference equations, which was pointed out and analyzed by M. van der Put and M. Singer in the last chapter of their book [**SvdP97**]. The solution of the paradox is that the vector bundle $\mathcal{M}$ associated to the representation $q \to \sqrt{q}$ of $q^{\mathbb{Z}}$ (or to the q-difference equation $y(qt) = \sqrt{q}.t$) is in fact a non-trivial vector bundle of rank one, and the basic solution is not $\sqrt{t}$, but $\frac{\theta(t/\sqrt{q})}{\theta(t)}$, where $\theta(t) = \prod_{n>0}(1 - q^n t)\prod_{n\leq 0}(1 - q^n/t)$.

1.5.6. In the previous example, it is easily seen that rank-one vector bundles with connection on S which arise from a representation of $\pi_1^{\mathrm{top}}(S,s)$ form a space of dimension one, while the space of all rank-one vector bundles with connection on S has dimension two.

We next consider the case of a p-adic manifold S which "is" a geometrically irreducible algebraic $\mathbb{C}_p$-curve. Let $\overline{S}$ be its projective completion. It follows from the Van Kampen theorem, together with the fact that punctured disks are simply-connected, that $\pi_1^{\mathrm{top}}(S,s) \to \pi_1^{\mathrm{top}}(\overline{S},s)$ is an isomorphism. It follows that the vector bundles with connection attached to representations of $\pi_1^{\mathrm{top}}(S,s)$ automatically extend to $\overline{S}$. Hence we may assume without loss of generality that S is compact.

Vector bundles with connection on S are algebrizable, and one can use C. Simpson's construction ([**Si94**]) to define the *moduli space of connections* of rank r over S, denoted by $M_{\mathrm{dR}}(S,r)$. On the other hand, we have seen that the topological fundamental group $\pi_1^{\mathrm{top}}(S,s)$ is free on $b_1(\Delta)$ generators, being isomorphic to the fundamental group of the dual graph Δ of the semistable reduction of S. Simpson has also studied the *moduli space of representations of dimension* r of such a group. We denote it by $M_{\mathrm{B}}(S,r)$; in fact, it depends only on the couple of integers $(b_1(\Delta), r)$.

Let us assume that S is of genus $g \geq 2$. Simpson shows that $M_{\mathrm{dR}}(S,r)$ is algebraic irreducible of dimension $2(r^2(g-1)+1)$. On the other hand, $M_{\mathrm{B}}(S,r)$ is an algebraic irreducible affine variety of dimension $(r^2(b_1(\Delta) - 1) + 1)$. We note that this dimension is maximal when the Betti number $b_1(\Delta)$ takes its maximal value, namely g. This corresponds to the case where S is a Mumford curve (a curve with totally degenerate reduction).

It turns out that the functor which associates a vector bundle with connection to any finite-dimensional representation of the topological fundamental group induces an injective analytic map of moduli spaces ι : $M_{\mathrm{B}}(S,r) \to M_{\mathrm{dR}}(S,r)$.

[The map is induced by the functor $V \to (\mathcal{M}_V, \nabla_V)$, and its injectivity follows from the faithfulness of this functor. The difficulty in showing that ι is analytic is that $M_{\mathrm{dR}}(S,r)$ is a priori a moduli space for algebraic connections, not for all analytic connections; we shall not pursue here in this direction].

In the complex situation, the corresponding map ι would always be an analytic isomorphism (Riemann-Hilbert-Simpson). In the p-adic cases, we see that the connections which satisfy Cauchy's theorem at every point (classical or not) form a stratum of maximal dimension half of the dimension of the total moduli space.

We shall leave the closer analysis of this kind of p-adic differential equations with global monodromy until later chapters, where we show how they arise in the context of period mappings. In the next section, we shall deal with a very different kind of differential equations, which play a distinguished role in p-adic analysis under the name of *unit-root F-crystals.*

2. Analytic continuation; algebraic approach.

ABSTRACT: An equivalent, but more algebraic, approach to analytic continuation consists in interpreting complex analytic multivalued functions as limits of algebraic functions. Since the concept of topological coverings and that of étale coverings do not coincide in the p-adic setting, the algebraic approach leads in that case to a theory quite different from that of the previous section. It turns out to be well-suited to the function-theoretic study of so-called unit-root F-isocrystals, and induces us to revisit two fundamental notions due to Dwork: Frobenius structure and overconvergence.

2.1. Limits of rational functions.

2.1.1. *Over* $\mathbb{C}$. Let us briefly survey the complex analytic situation. Let S be a connected complex analytic curve, and U an open set of S (not necessarily connected) which is holomorphically convex; *i.e.* $S \setminus U$ has no compact connected component. Then $\mathcal{O}(S)$ is dense in $\mathcal{O}(U)$ for the topology of uniform convergence on every compact set. We denote this situation by $\mathcal{O}(U) = \widehat{\mathcal{O}(S)}^U$ (*cf. e.g.* [**Re89**, 13]).

Moreover, if S is the Riemann surface coming from an affine algebraic curve S^{alg}, then we also have $\mathcal{O}(U) = \widehat{\mathcal{O}(S^{\mathrm{alg}})}^U$, which generalizes the theorem of Runge on approximation by rational functions. Indeed, considering an embedding $S^{\mathrm{alg}} \hookrightarrow (\mathbb{A}^N)^{\mathrm{alg}}$, we may extend analytic functions on S to analytic functions on $(\mathbb{A}^N)^{\mathrm{alg}}$ (due to the vanishing of the first cohomology group of the coherent ideal sheaf defining S). Any analytic function on $(\mathbb{A}^N)^{\mathrm{alg}}$ can be approximated by polynomials, which we restrict to S.

2.1.2. *Over the p-adics.* We have an analogous situation. Let K be a complete subfield of $\mathbb{C}_p$, and let S be an analytic curve coming from a smooth affine algebraic curve over K. We consider a closed immersion $S \hookrightarrow \mathbb{A}^N$. For any $r > 0$, $S_r := S \cap \mathrm{D}_{\mathbb{A}^N}(0, r^+)$ is an affinoid domain in S. The same argument of polynomial approximation used in 2.1.1 shows that $\mathcal{O}(S_r)$ is the completion of $\mathcal{O}(S^{\mathrm{alg}})$.

For any compact $Z \subset S$ (*e.g.* an affinoid domain), we define the topological ring $\mathcal{H}(Z)$ of *analytic elements* on Z to be the completion of $\Gamma(Z, \mathcal{O}_S) = \varinjlim_{Z \subset U:\ \mathrm{open}} \Gamma(U, \mathcal{O}_S)$ under the sup-norm. Whenever Z is contained in a S_r, $\mathcal{H}(Z)$ is also the completion of $\mathcal{O}(S_r)_Z$, where the subscript Z denotes the localization with respect to the set of elements which do not vanish on Z (*cf.* [**Ber90**] and [**Ray94**] for a more precise version of the Runge theorem).

The holomorphic convexity condition on Z is that for some r, Z is the intersection of affinoid neighborhoods defined by inequalities of the form $|f_i| \leq 1$ in S_r. If this condition is satisfied, $\mathcal{H}(Z)$ is in fact the completion of $\mathcal{O}(S_r)$ itself.

2.1.3. *The Krasner-Dwork viewpoint.* For $S = \mathbb{A}^1$, and $K = \mathbb{C}_p$, we deduce $\mathcal{H}(Z) = (\widehat{K[z]_Z})$, which is nothing but the M. Krasner's original definition of analytic elements on Z.

Let $\widehat{K(z)}$ be the completion of $K(z)$ by means of Gauss norm. One can interpret its elements as the analytic elements on a generic disk, and then specialize in the complement $Z \subseteq \mathbb{P}^1$ of a finite union of disks $\mathrm{D}(a_i, 1^-)$, $(i = 1, 2, \ldots)$. This viewpoint leads us to understand analytic continuation as a *specialization.*

2.2. Limits of algebraic functions; complex case.

Let S be a Riemann surface coming from a complex affine algebraic curve S^{alg}, smooth and connected. Let $\widetilde{S}$ be the universal covering of S (here we tacitly fix a base point $s \in S$). We endow $\mathcal{O}(\widetilde{S})$ with the topology of uniform convergence on every compact; note that the group $\pi_1(S)$ acts (continuously) on $\mathcal{O}(\widetilde{S})$.

On the other hand, let $\mathcal{O}(S^{\mathrm{alg}})^{\mathrm{ct}}$ be the maximal unramified integral extension of $\mathcal{O}(S^{\mathrm{alg}})$. Elements in $\mathcal{O}(S^{\mathrm{alg}})^{\mathrm{ct}}$ can be regarded as unramified algebraic functions on S^{alg}; thus we have $\mathcal{O}(S^{\mathrm{alg}})^{\mathrm{ct}} \subset \mathcal{O}(\widetilde{S})$.

Proposition 2.2.1. *$\mathcal{O}(S^{\mathrm{alg}})^{\mathrm{et}}$ is dense in $\mathcal{O}(\widetilde{S})$.*

For example, when $S^{\mathrm{alg}} = \mathbb{P}^1 \setminus \{0, \infty\}$, the function $\log z \in \mathcal{O}(\widetilde{S})$ can be written as $\lim\limits_{n \to \infty} n(z^{1/n} - 1)$, uniformly on every compact subset of $\widetilde{S} \simeq \mathbb{A}^1$. We can use this formula to compute $\log_\gamma 1 = \lim n(\zeta_n - 1) = 2i\pi$, where γ is the counter-clockwise loop around 0 with the base point 1.

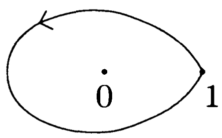

FIGURE 3

PROOF OF 2.2.1. (after O. Gabber). Consider a countable covering of $\widetilde{S}$ $(\simeq \mathbb{P}^1(\mathbb{C}), \mathbb{C}$ or $\mathrm{D}(0, 1^-))$ by relatively compact contractible open subsets which are oriented manifolds U_n (*e.g.* disks), such that $\overline{U}_n \subset U_{n+1}$. For any n, let us consider the set

$$\mathcal{F}_n = \{\gamma \in \pi_1(S, s) \mid \gamma \neq 1 \text{ and } \gamma \overline{U}_n \cap \overline{U}_n \neq \emptyset\}.$$

If this set were infinite, we could find a sequence of pairwise distinct elements $\gamma_m \in \pi_1(S, s)$ and a sequence of points $s_m \in \overline{U}_n$ such that $\gamma_m s_m \in \overline{U}_n$. By compacity of $\overline{U}_n$, we might assume that the sequences s_m and $\gamma_m s_m$ converge to points s' and s'' respectively. Then the sequence $\gamma_m s'$ converges to s'', which contradicts the discreteness of the $\pi_1(S, s)$-orbits in $\widetilde{S}$. So $\mathfrak{F}_n$ is finite.

On the other hand, S is topologically a surface of genus g with $t \geq 1$ punctures, so $\pi_1(S)$ is generated by $2g + t$ generators γ_i subject to the relation

$$[\gamma_1, \gamma_{g+1}] \cdots [\gamma_g, \gamma_{2g}]\gamma_{2g+1} \cdots \gamma_{2g+t} = 1.$$

Thus $\pi_1(S)$ is free with $2g+t-1$ generators. Therefore it is residually finite (*cf.* [**Bou64**, p. 150, ex. 34]); so we can find a subgroup $\Gamma_n \subseteq \pi_1(S)$ of finite index which avoids $\mathcal{F}_n$.

We immediately see that $\big((\Gamma_n \setminus \{1\})\cdot \overline{U}_n\big) \cap \overline{U}_n = \emptyset$. The restriction to $\overline{U}_n$ of the canonical projection $\widetilde{S} \to S_n := \widetilde{S}/\Gamma_n$ is thus an embedding. We identify $\overline{U}_n$ with its image.

The part of the exact sequence of cohomologies with compact support

$$\cdots \longrightarrow \mathrm{H}^1(\overline{U}_n, \mathbb{Z}) \longrightarrow \mathrm{H}^2_{\mathrm{c}}(S_n \setminus \overline{U}_n, \mathbb{Z}) \longrightarrow \mathrm{H}^2_{\mathrm{c}}(S_n, \mathbb{Z}) \longrightarrow \cdots$$

which can also be written as

$$\ldots \to \mathrm{H}_1(\overline{U}_n, \mathbb{Z}) = 0 \to \mathrm{H}_0(S_n \setminus \overline{U}_n, \mathbb{Z}) \to \mathrm{H}_0(S_n, \mathbb{Z}) = \mathbb{Z} \to \ldots$$

and from which we deduce that $S_n \setminus \overline{U}_n$ is connected. Because $\partial U_n \subset U_{n+1} \setminus \overline{U}_n$, we have $\overline{S_n \setminus \overline{U}_n} = S_n \setminus U_n$, hence $S_n \setminus U_n$ is connected. Since $S_n \setminus U_n$ is not compact, U_n is holomorphically convex in S_n. Riemann's existence theorem assures that S_n is the analytification of an affine algebraic curve S_n^{alg}. Here we can apply 2.1.1 to deduce $\mathcal{O}(U_n) = \widehat{\mathcal{O}(S_n^{\mathrm{alg}})}^{U_n}$. Note that $\mathcal{O}(S_n^{\mathrm{alg}}) \subset \mathcal{O}(S^{\mathrm{alg}})^{\mathrm{et}}$. Since every compact set of $\widetilde{S}$ is contained in some U_n, we conclude the desired equality $\mathcal{O}(\widetilde{S}) = \widehat{\mathcal{O}(S^{\mathrm{alg}})^{\mathrm{et}}}$. □

2.2.2. It follows from Proposition 2.2.1 that if Ω/Γ is a Schottky partial uniformization of S, then the intersection $\mathcal{O}(S^{\mathrm{alg}})^{\mathrm{et}} \cap \mathcal{O}(\Omega)$ is dense in $\mathcal{O}(\Omega)$. A variant of this statement (with a similar proof) holds, in the p-adic case, for a Mumford curve. We leave it to the reader.

2.3. Limits of algebraic functions; p-adic case.

2.3.1. In the p-adic situation, we have seen that there are "much more" étale coverings than topological coverings. For instance, the unit disk $\mathrm{D} = \mathrm{D}(0, 1^+)$ is simply-connected, but admits many non-trivial finite étale coverings, *e.g.* the Artin-Schreier covering $\mathrm{D} \to \mathrm{D}$, $y \mapsto z = y^p - y$.

Proposition 2.2.1 suggests to replace, in the p-adic case, the ring $\mathcal{O}(\widetilde{S})$, which is often too small, by some kind of completion of $\mathcal{O}(S)^{\mathrm{et}} = \varinjlim \mathcal{O}(S')$, where S' runs over the finite étale connected coverings of S.

Let us now assume that S is an affinoid curve with good reduction (hence simply connected). Then there is a Gauss p-adic norm on $\mathcal{O}(S)$, which extends uniquely to $\mathcal{O}(S)^{\mathrm{et}}$. The completion $\widehat{\mathcal{O}(S)^{\mathrm{et}}}$ is however pathological in several senses: for instance, it is difficult to give a function-theoretic meaning to its elements, and the continuous derivations of $\mathcal{O}(S)$ extends to $\mathcal{O}(S)^{\mathrm{et}}$ but not to $\widehat{\mathcal{O}(S)^{\mathrm{et}}}$ in a natural way.

2.3.2. To bypass such problems, we are going to look for some convenient *subring* of $\widehat{\mathcal{O}(S)^{\mathrm{ct}}}$. For simplicity, let us limit ourselves to the following situation:

- $K = \widehat{\mathbb{Q}}_p^{\mathrm{ur}} \subset \mathbb{C}_p$ is the completion of the maximal unramified algebraic extension of $\mathbb{Q}_p$,
- $\mathfrak{v} = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$ is its ring of integers (the Witt ring of $\overline{\mathbb{F}}_p$),
- S_0 is the affine line over $\overline{\mathbb{F}}_p$, deprived from finitely many points $\overline{\zeta}_1 \dots, \overline{\zeta}_\nu$; we choose a point s_0 on S_0,
- $\mathcal{R}$ is the p-adic completion of $\mathfrak{v}[z, \frac{1}{(z-\zeta_1)\dots(z-\zeta_\nu)}]$, where $\zeta_1 \dots \zeta_\nu$ are liftings of $\overline{\zeta}_1 \dots, \overline{\zeta}_\nu$ in $\mathfrak{v}$; its residue ring is $\mathcal{O}(S_0)$,
- $S = \mathsf{M}(\mathcal{R}[\frac{1}{p}]) = \mathsf{D}(0, 1^+)\backslash\bigcup \mathsf{D}(\zeta_i, 1^-)$, the associated affinoid domain over K. There is a natural specialization map $\mathrm{sp} : S \longrightarrow S_0$ from characteristic 0 to characteristic p.

Let us consider the integral closure $\mathcal{R}^{\mathrm{ct}}$ of $\mathcal{R}$ in $\mathcal{O}(S)^{\mathrm{ct}}$ and its p-adic completion $\widehat{\mathcal{R}^{\mathrm{ct}}} \subset \widehat{\mathcal{O}(S)^{\mathrm{ct}}}$. It is thus endowed with (a natural extension of) the p-adic valuation, and its residue ring is $\mathcal{O}(S_0)^{\mathrm{ct}}$. Moreover, there is a natural structure of "differential ring" (better: a connection) on $\widehat{\mathcal{R}^{\mathrm{ct}}}$ with $\mathfrak{v}$ as ring of constants. The "remarkable equivalence of categories" of A. Grothendieck [**EGA IV**, 18.1.2] allows to identify $\mathrm{Aut}_{\mathrm{cont}}(\widehat{\mathcal{R}^{\mathrm{ct}}}/\mathcal{R})$ with the algebraic fundamental group $\pi_1^{\mathrm{alg}}(S_0, s_0)$.

2.3.3. *"Analytic continuation" as specialization.* Let $\mathsf{D}(s_0, 1^-) = \mathrm{sp}^{-1}\{s_0\}$ be the residue class of s_0 in S. The morphism $\mathcal{R} \longrightarrow \mathcal{O}(\mathsf{D}(s_0, 1^-))$ extends non-canonically to a continuous morphism $\widehat{\mathcal{R}^{\mathrm{ct}}}[\frac{1}{p}] \longrightarrow \mathcal{O}(\mathsf{D}(s_0, 1^-))$, determined *only up to the action of* $\pi_1^{\mathrm{alg}}(S_0, s_0)$. This allows to interpret elements of $\widehat{\mathcal{R}^{\mathrm{ct}}}[\frac{1}{p}]$ as certain multivalued locally analytic functions (in the "wobbly" sense).

The main difference here with Krasner's analytic continuation (*cf.* §§2.1) is the ambiguity coming from $\pi_1^{\mathrm{alg}}(S_0, s_0)$. This provides a kind of multivalued analytic continuation, which may be interpreted as analytic continuation in characteristic 0 along an "étale path" in characteristic p (figure 4).

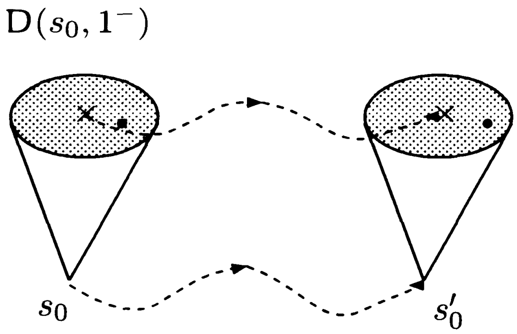

FIGURE 4. analytic continuation along an underground path

This is very close to the Robba-Christol theory of "algebraic elements" [**C86**], in which the main player is the *complete* subalgebra of $H^\infty(\mathrm{D}_{\mathbb{C}_p}(0,1^-))$ (the algebra of bounded analytic functions) generated the algebraic functions analytic on $\mathrm{D}_{\mathbb{C}_p}(0,1^-)$. Underlying this "algebraic" analytic continuation, there is a combinatorics of *automata* (whereas a combinatorics of graphs underlies the "topological" analytic continuation, as we have seen).(2)

Remark 2.3.4. The complex formula $\log z = \lim_n n\big(z^{\frac{1}{n}} - 1\big)$ has the following p-adic analogue: $\log z = \lim_n p^{-n}(z^{p^n} - 1)$ for $z \in \mathrm{D}(1,1^-)$. However, the convergence is not uniform on $\mathrm{D}(1,1^-)$, and the analytic function $\log z$ is not bounded on that disk. Nevertheless $\log : \mathrm{D}(1,1^-) \to \mathbb{A}^1$ defines an infinite Galois étale covering of the p-adic affine line (not at all a topological covering), with Galois group μ_{p^∞}, the p-primary torsion in $\mathbb{C}_p$.

2.4. $\widehat{\mathcal{R}^{\mathrm{et}}}$ and unit-root F-crystals.

2.4.1. Let us assume for technical simplicity that $p \neq 2$. Let $\mathcal{M}$ be a free $\mathcal{R}$-module of finite rank μ, endowed with a connection $\nabla : \mathcal{M} \to \Omega_{\mathcal{R}} \otimes_{\mathcal{R}} \mathcal{M}$, where $\Omega_{\mathcal{R}}$ is the rank-one module of continuous differentials (relative to $\mathfrak{o}$).

In general, solutions make sense only very locally: typically, analytic solutions exist in disks of radius $p^{\frac{-1}{p-1}}$ (the radius of p-adic convergence of the exponential function, which is the basic example). A very important criterion, due to B. Dwork, for the convergence of analytic solutions in any open disk of radius 1, is the existence of a so-called Frobenius structure.

2.4.2. *F-crystals.* Let σ be the Frobenius automorphism of $\mathfrak{o}$ lifting the pth-power map in $\overline{\mathbb{F}}_p$. There are many σ-linear endomorphisms ϕ of $\mathcal{R}$ which reduce to the pth-power map of $\mathcal{O}(S_0)$ in characteristic p. For instance, we can take $\phi(z) = z^p$, so that $\phi\big((z-\zeta_1)\dots(z-\zeta_\nu)\big) = (z^p - \sigma(\zeta_1))\dots(z^p - \sigma(\zeta_\nu)) \equiv \big((z-\zeta_1)\dots(z-\zeta_\nu)\big)^p \pmod p$. One can show that any lifting ϕ extends uniquely to a $\pi_1^{\mathrm{alg}}(S_0, s_0)$-equivariant endomorphism of $\widehat{\mathcal{R}^{\mathrm{et}}}$.

A Frobenius structure on $(\mathcal{M}, \nabla)$ is a rule F which associates to every lifting ϕ a homomorphism $F(\phi) : \phi^*\mathcal{M} \to \mathcal{M}$ (that is to say, a ϕ-linear endomorphism of $\mathcal{M}$) such that:

- (i) $F(\phi) \otimes \mathbb{Q}$ is an *isomorphism*,
- (ii) $F(\phi)$ is *horizontal*, *i.e.* compatible with the connections $\phi^*\nabla$ and ∇ respectively,
- (iii) for any two liftings ϕ and ϕ', the homomorphisms $F(\phi)$ and $F(\phi')$ are related by $F(\phi') = F(\phi) \circ \chi(\phi', \phi)$, where $\chi(\phi', \phi) : \phi'^*\mathcal{M} \to \phi^*\mathcal{M}$

(2) for these and further aspects of the Krasner, resp. Christol-Robba, analytic continuation, we refer to [**MoRo69**] and to the mimeographed notes of the numerous conferences of the Groupe d'étude d'analyse ultramétrique, Paris, devoted to this subject (1973-1980).

is the "Taylor isomorphism" given by the formulas

$$\chi(\phi',\phi)(\phi'^*m) = \sum_{n\geq 0} \phi^*\Big(\nabla\Big(\frac{d}{dz}\Big)^n(m)\Big)(\phi'(z)-\phi(z))^n/n!$$
$$= \sum_{n\geq 0} \phi^*\Big(\nabla\Big(z\frac{d}{dz}\Big)^n(m)\Big)\Big(\log\frac{\phi'(z)}{\phi(z)}\Big)^n/n!.$$

In virtue of (iii), a Frobenius structure amounts to the data of the semi-linear horizontal endomorphism $F(\phi)$, for the standard $\phi(z)=z^p$. The triple $(\mathcal{M},\nabla,F)$ is called an *F-crystal*, *cf.* [**Ka73**][(3)]. The very existence of a horizontal isomorphism $\phi^*\mathcal{M}\otimes\mathbb{Q}\simeq\mathcal{M}\otimes\mathbb{Q}$ implies that the radius of convergence ρ of any local solution of $(\mathcal{M},\nabla)$ satisfies $\min(1,\rho)=\min(1,\rho^p)$, that is to say: $\rho\geq 1$ (since $\rho\neq 0$).

2.4.3. *Unit-root F-crystals.* This is the case where $F(\phi)$ (not only $F(\phi)\otimes\mathbb{Q}$) is an isomorphism (for one, or equivalently, for all ϕ). The name comes from their first appearance in Dwork's computation of the p-adic units among the reciprocal zeroes of the zeta-function of hypersurfaces in characteristic p.

N. Katz has constructed a functor:

$$\begin{pmatrix}\text{continuous } \mathbb{Z}_p\text{-representations}\\ \text{of } \pi_1^{\mathrm{alg}}(S_0,s_0)\end{pmatrix} \longrightarrow (\text{unit-root } F\text{-crystals})$$

which associates to any free $\mathbb{Z}_p$-module V of rank r with a continuous action of $\pi_1^{\mathrm{alg}}(S_0,s_0)$ a unit-root F-crystal $(\mathcal{U}_V,\nabla_V,F_V)$ over $\mathcal{R}$ of rank r. Let us recall its definition at the level of objects. For any $m\in\mathbb{N}$, we set $S_m=\operatorname{Spec}\mathcal{R}/p^m\mathcal{R}$. Starting from a representation ρ of $\pi_1^{\mathrm{alg}}(S_0,s_0)$, let G_n denote the image of ρ in $GL(V/p^nV)$. The homomorphism $\pi_1^{\mathrm{alg}}(S_0,s_0)\to G_n$ corresponds to an étale covering $S_{n,0}\to S_0$, which has a unique étale lifting $\pi_{n,m}: S_{n,m}\to S_m$. The action of G_n on $S_{n,0}$ extends uniquely to $S_{n,m}$; on the other hand, the action of ϕ on S_m extends uniquely to $S_{n,m}$, and the ϕ- and G_n-actions commute. The opposite action makes $\mathcal{O}_{S_{n,m}}$ into a right G_n-module. If we set $\mathcal{U}_n=\pi_{n,n*}\mathcal{O}_{S_{n,n}}\otimes_{(\mathfrak{o}/p^n\mathfrak{o})[G_n]}V$, we then have a compatible system of isomorphisms

$$\Phi_n=\phi\otimes\mathrm{id}:\ \phi^*\mathcal{U}_n\to\mathcal{U}_n$$

and a compatible system of connections

$$\nabla_n=d\otimes\mathrm{id}:\ \mathcal{U}_n\to\pi_{n,n*}\Omega_{S_{n,n}}\otimes_{\mathcal{R}/p^m\mathcal{R}}\mathcal{U}_n\simeq\Omega_{\mathcal{R}}\otimes_{\mathcal{R}}\mathcal{U}_n$$

The unit-root crystal attached to ρ is given by $\mathcal{U}=\varprojlim_n\mathcal{U}_n$, $\nabla=\varprojlim_n\nabla_n$, $\Phi=\varprojlim_n\Phi_n$.

The next statement summarizes results of Katz and R. Crew [**Cr85**].

Proposition 2.4.4. (i) *The Katz functor is an equivalence of categories;*

---

(3) the endomorphism F_{S_0} of S_0 given by the pth-power map, as well as σ, ϕ and F are all called "Frobenius" in the usual jargon, without causing too much confusion, it seems...

(ii) *any unit-root F-crystal $(\mathcal{U}, \nabla, F)$ is solvable in $\widehat{\mathcal{R}^{\mathrm{et}}}$, i.e. :* $\mathcal{U} \otimes_{\mathcal{R}} \widehat{\mathcal{R}^{\mathrm{et}}} \simeq (\mathcal{U} \otimes_{\mathcal{R}} \widehat{\mathcal{R}^{\mathrm{et}}})^{\nabla} \otimes_{\mathfrak{o}} \widehat{\mathcal{R}^{\mathrm{et}}}$;

(iii) *the rule* $(\mathcal{U}, \nabla, F) \mapsto (\mathcal{U} \otimes_{\mathcal{R}} \widehat{\mathcal{R}^{\mathrm{et}}})^{\nabla=0, F(\phi)=\mathrm{id}}$ *(with Galois action coming from that on $\widehat{\mathcal{R}^{\mathrm{et}}}$) provides an inverse of the Katz functor.*

More precisely, we have the action of ϕ and d/dz on $\widehat{\mathcal{R}^{\mathrm{et}}}$ commute with the action of $\pi_1^{\mathrm{alg}}(S_0, s_0)$, and we have a canonical isomorphism

$$\mathcal{U} \otimes_{\mathcal{R}} \widehat{\mathcal{R}^{\mathrm{et}}} \simeq V \otimes_{\mathbb{Z}_p} \widehat{\mathcal{R}^{\mathrm{et}}}$$

compatible with $\phi, d/dz$ and $\pi_1^{\mathrm{alg}}(S_0, s_0)$ (diagonal action of $\phi, d/dz$ on the left hand side, diagonal action of $\pi_1^{\mathrm{alg}}(S_0, s_0)$ on the right hand side), which allows to reconstruct the representation V from the unit-root F-crystal and conversely. In fact, the connection as well as V can be reconstructed from $(\mathcal{U}, \Phi)$ alone.

This proposition may be compared with 1.5.4, though it applies to a quite different type of p-adic connections. In 1.5.4 (for dimension 1), the analytic curve S had typically bad reduction and the main player was the *discrete fundamental group* $\pi_1(\Delta)$ of the dual graph of the semistable reduction, together with the $\pi_1(\Delta)$-module $\mathcal{O}(\widetilde{S})$. Here the curve S has good reduction S_0 and the main player is the *compact fundamental group* $\pi_1^{\mathrm{alg}}(S_0)$, together with the $\pi_1^{\mathrm{alg}}(S_0)$-module $\widehat{\mathcal{R}^{\mathrm{et}}}$.

Before presenting one of Dwork's classical unit-root F-crystals, let us just mention that the above theory extends with little change to the case when the base ring $\mathfrak{o}$ is a finite ramified extension of $\widehat{\mathbb{Z}}_p^{\mathrm{ur}}$: one has to fix an extension of σ to $\mathfrak{o}$, to replace $\mathbb{Z}_p$ by $\mathfrak{o}^{\sigma}$, and to impose some mild nilpotence constraint on ∇ if the ramification index is $\geq p-1$ (also, it is customary to extend the definition of Frobenius structure on replacing ϕ by some power).

Example 2.4.5. *Dwork's exponential.* We denote by π a fixed $(p-1)$th root of $-p$, and set $\mathfrak{o} = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}[\pi]$, with $\sigma(\pi) = \pi$. Let us consider the differential equation

$$(*) \qquad f'(z) = -\pi f(z)$$

over $S = \mathrm{D}(0, 1^+)$, which has the analytic solution $f_a = e^{-\pi(z-a)}$ in any residue class $\mathrm{D}(a, 1^-) \subset \mathrm{D}(0, 1^+)$. The change of variable $z \mapsto z^p$ leads to the differential equation

$$(*_p) \qquad g'(z) = -p\pi z^{p-1} g(z) = \pi^p z^{p-1} g(z)$$

Dwork's exponential function is

$$E_{\pi}(z) = e^{\pi(z - z^p)} \in \mathbb{Z}_p[\pi][[z]].$$

This is an invertible element of $\mathcal{R}$, the π-adic completion of $\mathfrak{o}[z]$. This function provides the unit-root Frobenius structure which relates $(*)$ and $(*_p)$.

Let us reformulate this in the setting of 2.4.3: the relevant F-crystal is $(\mathcal{R}, \nabla, F)$, with $\nabla(1) = \pi dz$, $F(\phi)(1) = E_\pi(z)$ for the standard ϕ. This is the unit-root F-crystal attached to the finite character of $\pi_1^{\mathrm{alg}}(\mathbb{A}^1_{\mathbb{F}_p}, 0)$ corresponding to the Artin-Schreier covering $z = y^p - y$. The solution f_0 of $(*)$ belongs to $\widehat{\mathcal{R}^{\mathrm{ct}}}$: indeed, $e^{-\pi z} = E_\pi(y) \in \widehat{\mathfrak{o}[y]} \subset \widehat{\mathcal{R}^{\mathrm{ct}}}$; explicitly, $e^{-\pi z} = \lim_n (1 - \pi p^n z)^{p^{-n}}$.

Dwork has computed the value of his exponential for any $(p-1)$th root of unity ζ_{p-1} (*e.g.* $\zeta_{p-1} = 1$): this is

$$E_\pi(\zeta_{p-1}) = \zeta_p^{\zeta_{p-1}}$$

where ζ_p is the unique pth root of unity $\equiv 1 + \pi \pmod{\pi}$ (*cf.* [**La90**, chap. 14]). In the spirit of 2.3.3, this may be understood as follows:

$\zeta_p^{\zeta_{p-1}}$ = *the value at* $(z = 0)$ *of the analytic continuation of* f_0 *along the "wild loop"* corresponding to the path from $(y = 0)$ to $(y = \zeta_{p-1})$ on the Artin-Schreier covering $z = y^p - y$ in characteristic p (figure 5). If one changes ζ_{p-1}, this value $\zeta_p^{\zeta_{p-1}}$ is multiplied by some pth root of unity.

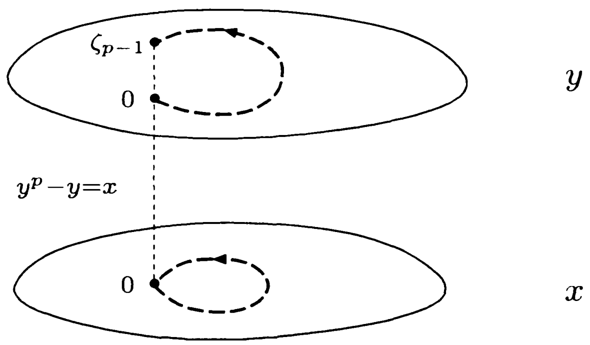

FIGURE 5. a wild underground loop

In our special case, the discussion of 2.3.3 tells us that $e^{-\pi z}$ *admits an extension to any disk* $\mathrm{D}(a, 1^-) \subset \mathrm{D}(0, 1^+)$, *analytic in that disk, and well-defined up to multiplication by some pth root of unity.* It is easy to find a formula for this extension: it must be proportional to $e^{\pi(a-z)}$, and by evaluation at a, we find that it is $e^{\pi(a-z)} E_\pi(b)$, where b is any solution of the equation $b^p - b = a$. Its pth power is, as expected, $e^{-p\pi z}$ itself.

Therefore, this multivalued exponential $e^{-\pi z}$ on $\mathrm{D}(0, 1^+)$ appears as a multivalued section of the logarithmic étale covering $\frac{-1}{\pi} \log : \mathrm{D}\big(1, p^{-\frac{1}{p-1}+}\big) \to \mathrm{D}(0, 1^+)$. This is just the *opposite* way from the complex situation, where the logarithm is a multivalued section of the étale covering of $\mathbb{C}^\times$ given by the exponential.

2.5. p-adic chiaroscuro: overconvergence.

2.5.1. In fact, Dwork's exponential E_π is more than just an element of the π-adic completion of $\mathfrak{v}[z]$: it is *overconvergent, i.e.* extends to an analytic function on a disk of radius $\rho > 1$ ($\rho = p^{(p-1)/p^2}$).

Overconvergence is a fundamental notion in p-adic analysis: dimming the contours of an affinoid turns out to be the key to the finiteness properties of p-adic cohomology, as was recognized by Dwork, and subsequently developed by P. Monsky, G. Washnitzer, P. Berthelot (rigid cohomology, $\mathcal{D}^\dagger$-modules)...

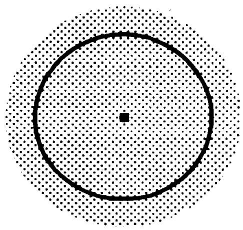

FIGURE 6

2.5.2. Let again S be $\mathrm{D}(0,1^+) \setminus \bigcup \mathrm{D}(\zeta_i, 1^-)$. For any $\rho > 1$, let us consider the bigger affinoids $S_\rho = \mathrm{D}(0,\rho^+) \setminus \bigcup \mathrm{D}(\zeta_i, \frac{1}{\rho}^-)$. The ring of *overconvergent analytic functions* on S is

$$\mathcal{H}^\dagger(S) = \bigcup_{\rho>1} \mathcal{O}(S_\rho).$$

Its relevance to the algebraic viewpoint on analytic continuation comes from the following result [**BDR80**]:

Proposition 2.5.3. *Let $f \in \mathcal{O}(S)$ satisfy a* monic *polynomial equation with coefficients in $\mathcal{H}^\dagger(S)$. Then $f \in \mathcal{H}^\dagger(S)$.*

2.5.4. A F-crystal $(\mathcal{M}, \nabla, F)$ is *overconvergent* if $(\mathcal{M}, \nabla)$ as well as $F(\phi)$ extends over some S_ρ. This is the case in example 2.4.5. For unit-root F-crystals, Crew has given the following characterization:

Proposition 2.5.5. *The unit-root F-crystal $(\mathcal{U}_V, \nabla_V, F_V)$ attached to a p-adic representation V is overconvergent if and only if the images in $GL(V)$ of the inertia groups at the missing points $\overline{\zeta}_1 \ldots, \overline{\zeta}_\nu, \infty$ are finite.*

2.6. (Overconvergence and Frobenius) Dwork's derivation of the p-adic Gamma function and exponential sums.

2.6.1. We come back to the situation of 2.4.5. For any $\alpha \in \mathbb{Z}_p$, let us consider $M_\alpha^\dagger := z^\alpha e^{\pi z}\mathcal{H}^\dagger(S)$ endowed with the derivation $z\frac{d}{dz}$. A simple computation shows that

$$z^\alpha e^{\pi z} z^{k+1} \equiv -\frac{\alpha + k}{\pi} z^\alpha e^{\pi z} z^k$$

in $M_\alpha^\dagger/z\frac{d}{dz}M_\alpha^\dagger$, from which one deduces that this cokernel has dimension 1 over K and is generated by $[z^\alpha e^{\pi z}]$.

Following Dwork, let us introduce the operator ψ defined by

$$\psi\Big(\sum a_n z^n\Big) = \sum a_{pn} z^n.$$

This is a left inverse of the Frobenius operator induced by $\phi : z \mapsto z^p$. It acts on $\mathcal{H}^\dagger(S)$ and commutes with $z\frac{d}{dz}$ up to multiplication by p: $\psi \circ z\frac{d}{dz} = pz\frac{d}{dz} \circ \psi$. The operator ψ can also be applied to any element of $M_\alpha^\dagger$ for $\alpha \in \mathbb{N}$: one finds that

$$\psi(z^\alpha e^{\pi z} f) = z^\beta e^{\pi z}\psi\big(z^{\alpha-p\beta}E_\pi(z)f\big)$$

where β is the ***successor*** of $\alpha \in \mathbb{Z}_p$, *i.e.* the unique p-adic integer β such that $p\beta-\alpha \in \mathbb{Z}\cap[0,p[$ (note that since $\alpha-p\beta > -p$, the terms containing a negative power of z disappear when applying ψ). This formula makes sense for any $\alpha \in \mathbb{Z}_p$, and we can see that ψ applies $M_\alpha^\dagger$ into $M_\beta^\dagger$, and commutes with $z\frac{d}{dz}$. Let us write the induced map of one-dimensional cokernels $[\psi]$: $M_\alpha^\dagger/z\frac{d}{dz}M_\alpha^\dagger \to M_\beta^\dagger/z\frac{d}{dz}M_\beta^\dagger$ in the form

$$[\psi]([z^\alpha e^{\pi z}]) = \pi^{p\beta-\alpha}\Gamma_p(\alpha)[z^\beta e^{\pi z}].$$

2.6.2. This function Γ_p is then nothing but Morita's p-adic Gamma function, characterized by its continuity and the functional equation

$$\Gamma_p(0) = 1,\quad \Gamma_p(\alpha+1)/\Gamma_p(\alpha) = \begin{cases} -\alpha & \text{if } \alpha \text{ is a unit,} \\ -1 & \text{if } |\alpha|_p < 1. \end{cases}$$

Let us check this functional equation:

- for $\alpha = 0$, one has $[z^k e^{\pi z}] = 0$ if $k > 0$; hence $\Gamma_p(0)[e^{\pi z}] = [e^{\pi z}][\psi(E_\pi(z))]$ $= [e^{\pi z}]$;
- if $|\alpha|_p = 1$, β is also the successor of $\alpha+1$, and $\Gamma_p(\alpha+1)[z^\beta e^{\pi z}] = \pi^{\alpha-p\beta+1}[\psi]([z^{\alpha+1}e^{\pi z}]) = \pi^{\alpha-p\beta+1}[\psi](-\frac{\alpha}{\pi}[z^\alpha e^{\pi z}]) = -a\Gamma_p(\alpha)[z^\beta e^{\pi z}]$;
- finally, if $|\alpha|_p < 1$, one has $\alpha = p\beta$, and $\beta+1$ is the successor of $\alpha+1$; one gets $\Gamma_p(\alpha+1)[z^{\beta+1}e^{\pi z}] = \pi^{-p+1}[\psi]([z^{\alpha+1}e^{\pi z}]) = -\frac{1}{p}[\psi](-\frac{\alpha}{\pi}[z^\alpha e^{\pi z}])$ $= \frac{\beta}{\pi}\Gamma_p(\alpha)[z^\beta e^{\pi z}] = -\Gamma_p(\alpha)[z^{\beta+1}e^{\pi z}]$.

2.6.3. Let us now check the continuity of Γ_p on $\mathbb{Z}_p$, or better, its analyticity on any disk $\mathrm{D}(-k,|p|^+)$, $k = 0,1,\ldots,p-1$. Let us write $E_\pi(z) = e^{\pi(z-z^p)} = \sum_0^\infty e_n z^n$.

For any $\alpha \in \mathbb{Z}_p \cap \mathrm{D}(-k,1^-)$, one has

$$\psi(z^\alpha e^{\pi z}) \equiv \pi^k\Gamma_p(\alpha)z^\beta e^{\pi z} = z^\beta e^{\pi z}\psi(z^{-k}E_\pi(z))$$

$$= z^\beta e^{\pi z}\sum_{n=0}^\infty e_{pn+k}z^n \equiv z^\beta e^{\pi z}\sum_{n=0}^\infty e_{pn+k}(-\pi)^{-n}(\beta)_n,$$

where $(\beta)_n = (\frac{\alpha+k}{p})_n$ is the Pochhammer symbol. Easy estimates now show that $\Gamma_p(\alpha) = \sum_0^\infty e_{pn+k}(-\pi)^{-n-k}(\frac{\alpha+k}{p})_n$ is analytic on $\mathrm{D}(-k,|p|^+)$.

2.6.4. *Gross-Koblitz formula:* for $k = 0, 1, \ldots, p-2$, one has

$$\Gamma_p\left(\frac{k}{p-1}\right) = -\pi^{-k} \sum_{\zeta_{p-1}} \zeta_{p-1}^{-k} \zeta_p^{\zeta_{p-1}} \quad \in \mathbb{Q}_p[\pi] \cap \overline{\mathbb{Q}}$$

where ζ_{p-1} runs over the $(p-1)$th roots of unity, and $\zeta_p^{\zeta_{p-1}}$ is as before the unique pth root of unity $\equiv 1 + \zeta_{p-1}\pi \pmod{\pi}$.

Let us sketch the proof. We choose $\alpha = \frac{k}{p-1}$, so that $\alpha = \beta$ and ψ acts on the ind-Banach-space $M_\alpha^\dagger$ via the formula: $\psi(z^{-k}E_\pi(z)f) = (z^\alpha e^{\pi z})^{-1}\psi(z^\alpha e^{\pi z} f)$. Coming back to the definition of ψ, one observes that for any $g \in \mathcal{H}^\dagger(S)$,

$$\psi(z^{-k}g)(z) = \frac{1}{p} \sum_{t \in \phi^{-1}(z)} t^{-k} g(t).$$

Using the fact that the domain of analyticity of this function is bigger than the domain of analyticity of g itself, one shows that ψ is a nuclear operator of $M_\alpha^\dagger$. In particular, it has a trace, which is the trace of the "composition operator" $\Psi : g \in \mathcal{H}^\dagger(S) \mapsto \frac{1}{p}\sum_{t\in\phi^{-1}(z)} t^{-k}E_\pi(t)g(t)$. The computation of this trace is done by approximating E_π by polynomials and studying the resulting action on the subspace of polynomials. One finds $\operatorname{Tr}\Psi = \frac{1}{p-1}\sum_{\zeta_{p-1}} \zeta_{p-1}^{-k} E_\pi(\zeta_{p-1})$.

At last, because $\psi \circ z\frac{d}{dz} = pz\frac{d}{dz} \circ \psi$, one has

$$\pi^k \Gamma_p\left(\frac{k}{p-1}\right) = \operatorname{Tr}\left([\psi]\Big| M_\alpha^\dagger / z\frac{d}{dz} M_\alpha^\dagger\right) = (1-p)\operatorname{Tr}(\psi | M_\alpha^\dagger)$$
$$= -\sum_{\zeta_{p-1}} \zeta_{p-1}^{-k} E_\pi(\zeta_{p-1}) = -\sum_{\zeta_{p-1}} \zeta_{p-1}^{-k}\zeta_p^{\zeta_{p-1}}.$$

We refer to [**CR94**] for a detailed account. A more general form of the Gross-Koblitz, proved along the same lines, shows that for any $1 \le k < p^r$, the product

$$\prod_0^{r-1} \Gamma_p\left(\frac{p^i k}{p^r - 1}\right)$$

belongs to the cyclotomic field $\mathbb{Q}(\zeta_{p^r})$.

2.6.5. A more straightforward and elementary proof has been discovered by A. Robert. It goes as follows.

As we have just seen, the right hand side of the Gross-Koblitz formula can be written

$$-\pi^{-k} \sum_{\zeta_{p-1}} \zeta_{p-1}^{-k} E_\pi(\zeta_{p-1}) = -\pi^{-k} \sum_{n=0}^{\infty} \left(\sum_{\zeta_{p-1}} \zeta_{p-1}^{n-k} \right) e_n$$
$$= (1-p)\pi^{-k} \sum_{m=0}^{\infty} e_{(p-1)m+k}.$$

Using the expansion of $\Gamma(\alpha)$ given in 2.6.3, in the case of $\alpha = \frac{k}{p-1}$, we thus have to show that

$$\sum_{n=0}^{\infty} e_{pn+k}(-\pi)^{-n}\left(\frac{k}{p-1}\right)_n = (1-p)\sum_{m=0}^{\infty} e_{(p-1)m+k}.$$

Denote the left hand side by G_k (for any $k \in \mathbb{N}$). Due to the overconvergence of E_π, it is not difficult to see that $\lim_{k\to\infty} G_k = 0$. One has Robert's formula:

$$G_k - G_{p-1+k} = (1-p)e_k.$$

Summing up consecutive expressions, one gets a telescoping sum which yields the desired equality $G_k = G_k - \lim_{m\to\infty} G_{(p-1)m+k} = (1-p)\sum_{m=0}^{\infty} e_{(p-1)m+k}$.

It remains to prove Robert's formula. One first observes that $z\frac{d}{dz}E_\pi = (\pi z - p\pi z^p)E_\pi(z)$, which yields the relation $ne_n = \pi(e_{n-1} - pe_{n-p})$ for $n \geq p$, hence $\pi e_{p-1+m} = p\pi e_m + (m+p)e_{m+p}$ for $m \geq 0$. Then

$$\begin{aligned}
&G_k - G_{p-1+k} \\
&= e_k + \sum_0^\infty e_{p(n+1)+k}(-\pi)^{-n-1}\left(\frac{k}{p-1}\right)_{n+1} \\
&\qquad\qquad - \sum_0^\infty e_{p-1+pn+k}(-\pi)^{-n}\left(\frac{k}{p-1}+1\right)_n \\
&= e_k + \sum_0^\infty \left[\frac{k}{p-1}e_{p(n+1)+k} + \pi e_{p-1+pn+k}\right](-\pi)^{-n-1}\left(\frac{k}{p-1}+1\right)_n \\
&= e_k + \sum_0^\infty \left[p\frac{(k+(n+1)(p-1))}{p-1}e_{p(n+1)+k} + p\pi e_{pn+k}\right](-\pi)^{-n-1}\left(\frac{k}{p-1}+1\right)_n \\
&= (1-p)e_k + p\sum_0^\infty \frac{(k+(n+1)(p-1))}{p-1}e_{p(n+1)+k}(-\pi)^{-n-1}\left(\frac{k}{p-1}+1\right)_n \\
&\qquad\qquad - p\sum_1^\infty \frac{k+n(p-1)}{p-1}e_{pn+k}(-\pi)^{-n}\left(\frac{k}{p-1}+1\right)_{n-1} \\
&= (1-p)e_k.
\end{aligned}$$

2.6.6. The right hand side of the Gross-Koblitz formula is a special case of an *exponential sum*, *i.e.* an expression of the form

$$S_r(\overline{f},\overline{g},\overline{h}) = \sum_{x_1,\dots,x_d\in\mathbb{F}_{p^r},\ \overline{h}(\underline{x})\neq 0} \chi\big(N_{\mathbb{F}_{p^r}/\mathbb{F}_p}\overline{g}(\underline{x})\big)\exp\left(\frac{2\pi i}{p}\operatorname{Tr}_{\mathbb{F}_{p^r}/\mathbb{F}_p}\overline{f}(\underline{x})\right)$$

where $\overline{h}$ is a polynomial with coefficients in $\mathbb{F}_{p^r}$, $\overline{f}$ and $\overline{g}$ are rational functions with coefficients in $\mathbb{F}_{p^r}$ with no pole where $\overline{h}$ vanishes, and where χ is a character of $\mathbb{F}_p^\times$. This includes Gauss, Jacobi and Kloosterman sums as special cases. In fact, it is known, after the classical works of Gauss, Artin, Weil, that counting solutions of systems of polynomial equations in finite fields amounts to the computation of exponential sums $S_r(\overline{f},\overline{g},\overline{h})$. They are studied via their generating series, the so-called L-series:

$$L(\overline{f},\overline{g},\overline{h};t) = \exp\Big(\sum_{r\geq 1} S_r(\overline{f},\overline{g},\overline{h})\frac{t^r}{r}\Big).$$

Dwork's methods, refined by P. Robba and others, allow to tackle the questions of the rationality of L, its degree, and of the functional equation relating $L(\overline{f},\overline{g},\overline{h};t)$ to $L(\overline{f},\overline{g},\overline{h};\frac{1}{p^r t})$. In the one-dimensional case, the solution is elementary and follows the pattern sketched in 2.6.4. Namely **[CR94]**:

(i) One considers liftings f,g,h of $\overline{f},\overline{g},\overline{h}$ in characteristic zero, and one introduces the affinoid set S defined by $|h| = 1$. One sets $F = g(z)^{1/p-1}\exp(\pi f(z))$. Then F'/F is a rational function, and one shows that the differential operator $\dfrac{d}{dz}+\dfrac{F'}{F}$ has an index in $\mathcal{H}^\dagger(S)$ (in a slightly generalized sense, and which can be computed thanks to the work of Robba). This is the crucial point. Hence the cohomology spaces of the de Rham complex: $\Omega^0 = F\mathcal{H}^\dagger(S) \to \Omega^1 = F\mathcal{H}^\dagger(S)dz$ are finite-dimensional over $\mathbb{C}_p$.

(ii) One observes that $E := F^{\phi-\mathrm{id}}$ (*i.e.* $E(z) = F(z^p)/F(z)$) is an element of $\mathcal{H}^\dagger(S)$. This allows to define two endomorphisms of the de Rham complex by setting:

$$\phi^0(F.g) = FEg^\phi,\ \ \phi^1(Fgdz) = FE\phi' g^\phi dz \ \ (\text{with } \phi'(z) = pz^{p-1});$$

$$\psi^0(F.g)(z) = F(z)\sum_{\phi(t)=z}\frac{g(t)}{E(t)},\ \ \psi^1(Fgdz) = \sum_{\phi(t)=z}\frac{g(t)}{E(t)\phi'(t)}.$$

One has $\psi^*\circ\phi^* = p\cdot\mathrm{id}$. In particular, the operators $\mathrm{H}^i(\psi^*)$, $i = 0,1$, are invertible on the cohomology spaces. The same argument as in 2.6.4 shows that ψ^i is nuclear on Ω^i. In particular, it has a trace.

(iii) Polynomial approximation allows to prove the trace formula:

$$\mathrm{tr}(\psi^1) - \mathrm{tr}(\psi^0) = \mathrm{tr}(\mathrm{H}^1(\psi^*)) - \mathrm{tr}(\mathrm{H}^0(\psi^*)) = S_r(\overline{f},\overline{g},\overline{h}).$$

It follows that

$$L(\overline{f},\overline{g},\overline{h};t) = \frac{\det(\mathrm{id}-t\mathrm{H}^1(\psi^*))}{\det(\mathrm{id}-t\mathrm{H}^0(\psi^*))}.$$

This is a rational function since the cohomology is finite-dimensional. Moreover, since the $\mathrm{H}^i(\psi^*)$ are invertible, the computation of its degree amounts to the computation of the index of $\dfrac{d}{dz}+\dfrac{F'}{F}$ (the computation of the dimension of H^0, 0 or 1, being essentially trivial).

(iv) The functional equation follows from a topological "dual theory" in which the transpose of ϕ and ψ play the roles of ψ and ϕ respectively.

Let us remark at last that this method has an *archimedean analogue* in the so-called thermodynamic formalism —in dimension one. The analogy is especially striking in the presentation of D. Mayer [**May91**]: exponential sums correspond to "partition functions", L to the Ruelle zeta-function, Frobenius to the "shift", Dwork's operator ϕ^* to the "transfer operator" (or Ising-Perron-Frobenius-Ruelle operator) and is given by a "composition operator" (loc. cit. 7.2.2), its nuclearity is established by the same argument, and the trace formula has the same form (loc. cit. 7.17).

In view of this close analogy, one could dream of an archimedean proof of the rationality of Weil zeta functions, parallel to Dwork's p-adic proof...

3. The tale of $F(\frac{1}{2},\frac{1}{2},1;z)$.

ABSTRACT: This is a detailed concrete illustration of the somewhat abstract non-archimedean notions discussed in the previous section. We refer to [**Yo97**] (and [**Hu87**, 9]) for an excellent account of the archimedean tale of $F(\frac{1}{2},\frac{1}{2},1;z)$.

3.1. Dwork's hypergeometric function.

Let us consider the Legendre pencil of elliptic curves with parameter $z \neq 0,1,\infty$, given in inhomogeneous coordinates by

$$y^2 = x(x-1)(x-z).$$

As a scheme over $\mathbb{Z}[z,\frac{1}{2z(1-z)}]$, we denote it by X. The first de Rham cohomology module $\mathrm{H}^1_{\mathrm{dR}}(X)$ is free of rank 2 over $\mathbb{Z}[z,\frac{1}{2z(1-z)}]$, and endowed with the Gauss-Manin connection ∇ (derivation with respect to the parameter z); it is generated by the class ω of $\frac{dx}{y}$, and $\nabla(\frac{d}{dz})(\omega)$. The canonical symplectic form (cup-product) satisfies $\langle \omega, \nabla(\frac{d}{dz})(\omega)\rangle = \frac{2}{z(z-1)}$. The Gauss-Manin connection is given by the hypergeometric differential equation with parameters $(\frac{1}{2},\frac{1}{2},1)$

$$\nabla(L_{\frac{1}{2},\frac{1}{2},1})(\omega) = 0, \text{ with } L_{\frac{1}{2},\frac{1}{2},1} = z(1-z)\frac{d^2}{dz^2} + (1-2z)\frac{d}{dz} - \frac{1}{4}.$$

Let p be an odd prime. The *Hasse invariant* is the polynomial $h_p(z) \in \mathbb{Z}[\frac{1}{2}][z]$ obtained by truncating the hypergeometric series $(-1)^{\frac{p-1}{2}}F(\frac{1}{2},\frac{1}{2},1;z)$ at order $\frac{p-1}{2}$. It enjoys the following well-known properties:

- functional equations: $h_p(z) \equiv (-1)^{\frac{p-1}{2}}h_p(1-z) \equiv z^{\frac{p-1}{2}}h_p(\frac{1}{z}) \pmod p$,
- for any $\overline{\zeta} \in \overline{\mathbb{F}}_p \setminus \{0,1\}$, one has $h_p(\overline{\zeta}) = 0$ if and only if the elliptic curve $X_{\overline{\zeta}}$ over $\overline{\mathbb{F}}_p$ is *supersingular*, *i.e.* has no geometric point of order p, *cf.* [**Hu87**, 13.3].

The roots of $h_p \pmod p$ are distinct and lie in $\mathbb{F}_{p^2}$: we denote them by $\overline{\zeta}_1,\ldots,\overline{\zeta}_{(p-1)/2}$, and choose liftings $\zeta_1,\ldots,\zeta_{(p-1)/2}$ in $\widehat{\mathbb{Z}_p^{\mathrm{ur}}}$.

Non-supersingular elliptic curves are called *ordinary*; they have exactly p geometric points of order p.

Dwork's hypergeometric function is

$$f_p(z) = (-1)^{\frac{p-1}{2}}\frac{F(\frac{1}{2},\frac{1}{2},1;z)}{F(\frac{1}{2},\frac{1}{2},1;z^p)} \in \mathbb{Z}\left[\frac{1}{2}\right][[z]].$$

One has $f_p(z) \equiv h_p(z) \pmod p$. Dwork discovered that although the p-adic radius of convergence of this series is exactly 1, f_p *does extend to a p-adic analytic function on $\mathbb{A}^1$ deprived from the supersingular disks* $\mathrm{D}(\zeta_j,1^-)$; we denote this extension by the same symbol f_p.

In terms of this function, he obtained in 1958 his famous p-adic formula for the number of rational points of an ordinary elliptic curve defined over $\mathbb{F}_{p^n}$:

for any $s_0 \in \mathbb{F}_{p^n}$, $s_0 \neq 0, 1, \overline{\zeta}_j$, *the number of* $\mathbb{F}_{p^n}$*-points of* X_{s_0} *is*

$$1 - \prod_{k=0}^{n-1} f_p(\omega^{p^k}) + p^n \left(1 - \prod_{k=0}^{n-1} f_p(\omega^{p^k})^{-1}\right),$$

where ω denotes the unique (p^n-1)th root of unity $\equiv s_0 \pmod p$.

3.2. The ordinary unit-root F-crystal.

On completing $\widehat{\mathbb{Z}_p^{\mathrm{ur}}}[z, \frac{1}{2z(1-z)}]$ p-adically, $(\mathrm{H}^1_{\mathrm{dR}}(X), \nabla)$ gives rise to an overconvergent F-crystal $(\mathcal{H}, \nabla, F)$. We do not discuss here the construction of the Frobenius structure, which is a general p-adic feature of Gauss-Manin connections. In fact, various analytic or geometric constructions are available, but in our present case, the Frobenius structure can be made quite explicit, *cf.* [**Dw69**].

Let us introduce a few notations:

- $\mathcal{R}$ = the p-adic completion of $\widehat{\mathbb{Z}_p^{\mathrm{ur}}}[z, \frac{1}{h_p(z)}]$
- $\mathcal{R}_{\mathrm{ord}}$ = the p-adic completion of $\widehat{\mathbb{Z}_p^{\mathrm{ur}}}[z, \frac{1}{z(1-z)h_p(z)}]$
- $S = \mathsf{M}(\mathcal{R}[\frac{1}{p}]) = \mathsf{D}(0, 1^+) \setminus \left(\bigcup \mathsf{D}(\zeta_j, 1^-)\right)$
- $S_{\mathrm{ord}} = \mathsf{M}(\mathcal{R}_{\mathrm{ord}}[\frac{1}{p}]) = S \setminus \left(\mathsf{D}(0, 1^-) \cup \mathsf{D}(1, 1^-)\right)$ is the *ordinary locus*
- $S_{\mathrm{nss}} = \mathbb{A}^1 \setminus \left(\bigcup \mathsf{D}(\zeta_j, 1^-)\right)$ is the *non-supersingular locus.*

The restriction of $(\mathcal{H}, \nabla, F)$ to the ordinary locus possesses a unique nonzero unit-root sub-F-crystal $(\mathcal{U}, \nabla, F)$. This unit-root F-crystal extends over S (and even over S_{nss} as an F-isocrystal with logarithmic singularity at ∞ in the sense of [**Scho85**]) *cf.* also [**O00**]; we use the same symbol for the extension. It can be described along the following lines:

- $\mathcal{U}$ is the unique rank-one horizontal submodule of $\mathcal{H}$.
- $\mathcal{U} \otimes_{\mathcal{R}_{\mathrm{ord}}} \widehat{\mathcal{R}^{\mathrm{et}}_{\mathrm{ord}}} = (\mathcal{H} \otimes_{\mathcal{R}_{\mathrm{ord}}} \widehat{\mathcal{R}^{\mathrm{et}}_{\mathrm{ord}}})^{\nabla} \otimes_{\widehat{\mathbb{Z}_p^{\mathrm{ur}}}} \widehat{\mathcal{R}^{\mathrm{et}}_{\mathrm{ord}}}$. (Here $\widehat{\mathcal{R}^{\mathrm{et}}_{\mathrm{ord}}}$ is defined as in 2.3.2.)
- for any ordinary $s_0 \in \overline{\mathbb{F}}_p$, the associated p-adic representation of $\pi_1^{\mathrm{alg}}(S_0, s_0)$ is $(\mathcal{H} \otimes_{\mathcal{R}_{\mathrm{ord}}} \widehat{\mathcal{R}^{\mathrm{et}}_{\mathrm{ord}}})^{\nabla=0, F(\phi)=\mathrm{id}} \simeq \mathrm{H}^1_{\mathrm{et}}(X_{s_0}, \mathbb{Z}_p)$.
- For any $u \in (\mathcal{U} \otimes_{\mathcal{R}} \widehat{\mathcal{R}^{\mathrm{et}}})^{\nabla}$, "the" image of u in $\mathcal{O}(\mathsf{D}(s_0, 1^-))$ is a bounded solution of the differential operator $L_{\frac{1}{2}, \frac{1}{2}, 1}$ on $\mathsf{D}_{\mathbb{C}_p}(s_0, 1^-)$; this also characterizes $\mathcal{U}$.
- $\mathcal{U}|^{\nabla}_{\mathsf{D}(0,1^-)}$ has a canonical $\mathbb{Z}$-submodule, which can be identified with $\mathrm{H}^1((X_z)^{\mathrm{an}}, \mathbb{Z})$ for any $z \in \mathsf{D}(0, 1^-) \setminus \{0\}$; for a generator u, one has

  $$\langle \omega, u \rangle = \sqrt{-1} F\Big(\frac{1}{2}, \frac{1}{2}, 1; z\Big) \text{ in } \mathcal{O}(\mathsf{D}(0, 1^-))$$

  ($\sqrt{-1}$ appears as residue of $\frac{dx}{y}|_{z=0}$ at $x = 0$), *cf.* [**And90**]. Similarly, $\mathcal{U}|^{\nabla}_{\mathsf{D}(1,1^-)}$ has a canonical $\mathbb{Z}$-submodule; for a generator u, one has

  $$\langle \omega, u \rangle = \sqrt{-1} F\Big(\frac{1}{2}, \frac{1}{2}, 1; 1-z\Big) \text{ in } \mathcal{O}(\mathsf{D}(1, 1^-)).$$

The existence of the unit-root F-crystal $(\mathcal{U},\nabla,F)$ over $\mathcal{R}$ then amounts to the two function-theoretic facts:

$$\operatorname{dlog} F\Big(\frac{1}{2},\frac{1}{2},1;z\Big) \quad\text{and}\quad f_p(z)=\Big(\sqrt{-1}F\Big(\frac{1}{2},\frac{1}{2},1;z\Big)\Big)^{1-\sigma}$$

both extend to units in $\mathcal{R}$.

The inertia at any supersingular point maps *onto* $\operatorname{Aut}\mathrm{H}^1_{\mathrm{et}}(X_{s_0},\mathbb{Z}_p)\simeq\mathbb{Z}_p^\times$ (J.-I. Igusa, *cf. e.g.* [**vdP87**]). The inertia at ∞ acts as ±1. According to Crew's criterion, the F-crystal $(\mathcal{U},\nabla,F)$ is *not* overconvergent.

We refer to [**SPM91**] for a study in the same spirit of more general hypergeometric equations and other differential equations "coming from geometry".

3.3. Analytic continuation: a précis.

3.3.1. We have seen that Dwork's hypergeometric function extends to an analytic function on the whole of S_{nss}. Some of its special values have been computed. Let us mention [**Ko79**], [**You92**]

(i) $f_p(1)=1$ (Koblitz),
(ii) if $p\equiv1 \pmod 4$,

$$f_p(-1)=(-1)^{\frac{p-1}{4}}\frac{\Gamma_p(1/4)^2}{\Gamma_p(1/2)}=\frac{\Gamma_p(1/4)}{\Gamma_p(1/2)\Gamma_p(3/4)} \qquad \text{(Young)}.$$

(the condition $p\equiv1\pmod 4$ ensures that -1 is an ordinary modulus); $f_p(-1)$ is a Gauss integer (Van Hamme).

3.3.2. We have seen that the logarithmic derivative $\operatorname{dlog}F(\frac{1}{2},\frac{1}{2},1;z)$ also extends to an analytic function on S_{nss}. It satisfies the functional equations

$$\begin{aligned}\operatorname{dlog}F\Big(\frac{1}{2},\frac{1}{2},1;z\Big)&=-\operatorname{dlog}F\Big(\frac{1}{2},\frac{1}{2},1;1-z\Big)\\&=-\frac{1}{2z}-\frac{1}{z^2}\operatorname{dlog}F\Big(\frac{1}{2},\frac{1}{2},1;\frac{1}{z}\Big).\end{aligned}$$

3.3.3. Let us turn to the more subtle case of $F(\frac{1}{2},\frac{1}{2},1;z)$ itself. The general discussion of 2.3.3/2.4.3 applies to the ordinary unit-root F-crystal and tells us that $F(\frac{1}{2},\frac{1}{2},1;z)$ *admits an extension to any unit disk* $\mathrm{D}(s,1^-)\subset S$, *analytic in that disk, and well-defined up to multiplication by a unit in* $\mathbb{Z}_p$. In other words, $F(\frac{1}{2},\frac{1}{2},1;z)$ is the specialization of an element (in fact, a unit) of $\widehat{\mathcal{R}^{\mathrm{et}}_{\mathrm{nss}}}$. It may be suggestive to denote such an element by $F(\frac{1}{2},\frac{1}{2},1;\eta)$, where η stands for the Berkovich generic point of S corresponding to the sup-norm).

In $\mathrm{D}(1,1^-)$, this extension is $F(\frac{1}{2},\frac{1}{2},1;1-z)$ (up to $\mathbb{Z}_p^\times$). The refined structure of F-isocrystal with logarithmic singularity at ∞ allows to construct an extension of $F(\frac{1}{2},\frac{1}{2},1;z)^2$ to $\mathrm{D}(\infty,1^-)$, analytic in that disk: it is $\frac{1}{z}F(\frac{1}{2},\frac{1}{2},1;\frac{1}{z})^2$ (up to $\mathbb{Z}_p^\times$); notice the square, which comes from the exponent $\pm1/2$ at ∞.

3.3.4. The Krasner representation of $f_p(z)$ and $\operatorname{dlog} F(\frac{1}{2}, \frac{1}{2}, 1; z)$ as uniform limits of rational functions without pole (nor zero) on S can be made explicit: for $n > 0$, let g_n be the polynomial obtained by truncating $F(\frac{1}{2}, \frac{1}{2}, 1; z)$ at order $p^n - 1$; then $f_p(z) = \lim (-1)^{\frac{p-1}{2}} \frac{g_{n+1}(z)}{g_n(z^p)}$ and $\operatorname{dlog} F(\frac{1}{2}, \frac{1}{2}, 1; z) = \lim \operatorname{dlog} g_n(z)$ [**Dw69**, 3.4]. On the other hand, $F(\frac{1}{2}, \frac{1}{2}, 1; z)$ cannot be approximated by rational functions on S, but $L_{\frac{1}{2}, \frac{1}{2}, 1}(g_n)$ tends uniformly to 0 on $\mathrm{D}(0, 1^+)$ [**Ro75**]. One has

$$\sqrt{-1} F\Big(\frac{1}{2}, \frac{1}{2}, 1; z\Big) \equiv \big(h_p(z)\big)^{-\frac{1}{p-1}} \pmod{p}$$

but we do not know any explicit representation of $F(\frac{1}{2}, \frac{1}{2}, 1; z)$ as a p-adic limit of algebraic functions on S.

3.3.5. Any solution of $L_{\frac{1}{2}, \frac{1}{2}, 1}$ on an ordinary disk $\mathrm{D}(s, 1^-)$, which is not proportional to (the extended) $F(\frac{1}{2}, \frac{1}{2}, 1; z)$, is not bounded in $\mathrm{D}_{\mathbb{C}_p}(s, 1^-)$, hence does not extend to neighboring disks in any reasonable *analytic* sense. In spite of this obstruction to continuation, one can nevertheless "jump from disk to disk" (see figure 7) and extend them as locally analytic functions, in

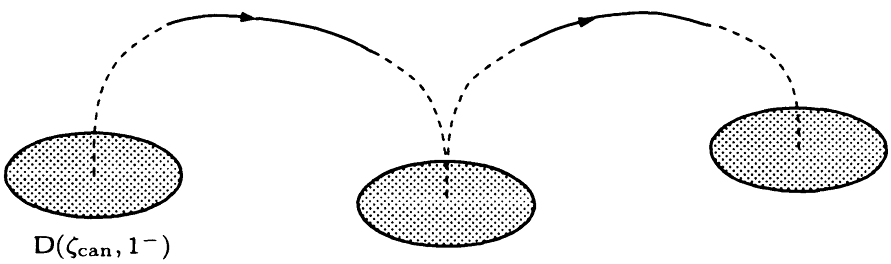

FIGURE 7

the following way.

For each $\overline{\zeta} \in S_0 \setminus \{0, 1\}$, there is a *canonical lifting* $\zeta_{\mathrm{can}} \in \mathbb{Z}_p^{\mathrm{ur}}$: namely, the modulus ζ_{can} for which $X_{\zeta_{\mathrm{can}}}$ has complex multiplication by the quadratic order $\mathrm{End}(X_{\overline{\zeta}})$. Let us fix a branch of $F(\frac{1}{2}, \frac{1}{2}, 1; z)$ in $\mathrm{D}(\zeta_{\mathrm{can}}, 1^-)$, *i.e.* an analytic solution $F_{\overline{\zeta}} \in \widehat{\mathbb{Z}_p^{\mathrm{ur}}}[[z - \zeta_{\mathrm{can}}]]$ of $L_{\frac{1}{2}, \frac{1}{2}, 1}$ in $\mathrm{D}(\zeta_{\mathrm{can}}, 1^-)$ which is a specialization of $F(\frac{1}{2}, \frac{1}{2}, 1; \eta) \in (\widehat{\mathcal{R}^{\mathrm{et}}})^\times$ (any other branch is of the form $c.F_{\overline{\zeta}}$ with $c \in \mathbb{Z}_p^\times$. Let $u_{\overline{\zeta}} \in \mathcal{U}|^{\nabla}_{\mathrm{D}(\zeta_{\mathrm{can}}, 1^-)}$ be defined by

$$\langle \omega, u_{\overline{\zeta}} \rangle = \sqrt{-1} F_{\overline{\zeta}} \text{ in } \mathcal{O}(\mathrm{D}(\zeta_{\mathrm{can}}, 1^-)).$$

On the other hand, let $v_{\overline{\zeta}} \in \mathcal{M}|^{\nabla}_{\mathrm{D}(\zeta_{\mathrm{can}}, 1^-)}$ be defined by

$$\sqrt{-1} F_{\overline{\zeta}}(\zeta_{\mathrm{can}}).v_{\overline{\zeta}} = \omega(\zeta_{\mathrm{can}}), \quad \text{so that } \langle v_{\overline{\zeta}}, u_{\overline{\zeta}} \rangle = 1.$$

Then $\langle \omega, v_{\overline{\zeta}} \rangle$ defines an unbounded solution of $L_{\frac{1}{2}, \frac{1}{2}, 1}$ in $\mathrm{D}_{\mathbb{C}_p}(\zeta_{\mathrm{can}}, 1^-)$.

3.3.6. Finally, in any supersingular disk $\mathrm{D}_{\mathbb{C}_p}(\zeta_i, 1^-)$, there is no non-zero bounded solution of $L_{\frac{1}{2},\frac{1}{2},1}$.

3.4. The Tate and Dwork-Serre-Tate parameters.

3.4.1. It turns out that the formal group $\widehat{X}_{(\widehat{\mathcal{R}^{\mathrm{et}}_{\mathrm{ord}}})}$ of X over $\widehat{\mathcal{R}^{\mathrm{et}}_{\mathrm{ord}}}$ is isomorphic to $\widehat{\mathbb{G}}_m$ (*cf.* [**Ka81**], [**vdM89**, I]). Any such isomorphism transforms the canonical differential ω_{can} on $\widehat{\mathbb{G}}_m$ into $\Theta\frac{dx}{y}$, for a suitable element $\Theta \in \widehat{\mathcal{R}^{\mathrm{et}}_{\mathrm{ord}}}$ well-defined up to multiplication by an element of $\mathbb{Z}_p^\times$. It is possible to arrange normalizations so that the following relation hold (loc. cit.):

$$\Theta = \left(\sqrt{-1}F\Big(\frac{1}{2}, \frac{1}{2}, 1; \eta\Big)\right)^{-1}.$$

Let $\zeta \in \widehat{\mathbb{Z}_p^{\mathrm{ur}}}, \zeta \neq 0, 1$, be a point of S. By specialization, the element $\Theta(\zeta) \in \widehat{\mathbb{Z}_p^{\mathrm{ur}}}^\times$ links up ω_{can} and $\frac{dx}{y}$ on $\widehat{X}_\zeta \simeq \widehat{\mathbb{G}}_m$; it is called the *Tate parameter* or *Tate constant* of X_ζ. It is well-defined up to multiplication by an element of $\mathbb{Z}_p^\times$.

This is the p-adic analogue of the following familiar situation: let us consider the two-step uniformization of a complex elliptic curve

$$\mathbb{C} \xrightarrow{\exp\left(\frac{2i\pi}{\omega_1}\cdot\ \right)} \mathbb{C}^\times \longrightarrow \mathbb{C}^\times/q^{\mathbb{Z}} \simeq \mathbb{C}/(\omega_1\mathbb{Z} + \omega_2\mathbb{Z})$$

with $q = \exp(2i\pi\frac{\omega_2}{\omega_1})$. If the elliptic curve is X_ζ with $\zeta \in \mathrm{D}(0, 1^-) \setminus \{0\}$, and if ω_1 and ω_2 are fundamental periods of $\frac{dx}{y}$ (ω_1 being the period attached to the vanishing cycle), then $\frac{2i\pi}{\omega_1}$ appears as the analogue of $\Theta(\zeta)$, and it is well-known that $\omega_1/2i\pi = iF(\frac{1}{2}, \frac{1}{2}, 1; \zeta)$ (up to sign).

3.4.2. We come back to the symplectic basis $v_{\overline{\zeta}}, u_{\overline{\zeta}}$ and write $\frac{1}{\sqrt{-1}F_{\overline{\zeta}}}\omega$ in the form $v_{\overline{\zeta}} + \tau u_{\overline{\zeta}}$, where

$$\tau = -\frac{\langle \omega, v_{\overline{\zeta}}\rangle}{\langle \omega, u_{\overline{\zeta}}\rangle} \in (z - \zeta_{\mathrm{can}})\widehat{\mathbb{Q}_p^{\mathrm{ur}}}[[z - \zeta_{\mathrm{can}}]].$$

This defines an unbounded element of $\mathcal{O}(\mathrm{D}_{\mathbb{C}_p}(\zeta_{\mathrm{can}}, 1^-))$ (notice that another choice of $u_{\overline{\zeta}}$ multiplies τ by an element of $\mathbb{Z}_p^\times$).
Applying $\nabla(d/dz)$ to $\frac{1}{\sqrt{-1}F_{\overline{\zeta}}}\omega = v_{\overline{\zeta}} + \tau u_{\overline{\zeta}}$ and using $\langle \omega, \nabla(\frac{d}{dz})(\omega)\rangle = \frac{2}{z(z-1)}$, one derives:

$$\frac{d\tau}{dz} = \frac{2}{z(1-z)F_{\overline{\zeta}}^2}.$$

The exponential of τ is called the *Dwork-Serre-Tate parameter*. We shall recall later its meaning as a parameter for p-divisible groups. It satisfies the following remarkable integrality property [**Dw69**, Th.4], [**Ka73**]

$$q = e^\tau \in 1 + (z - \zeta_{\mathrm{can}})\widehat{\mathbb{Z}_p^{\mathrm{ur}}}[[z - \zeta_{\mathrm{can}}]].$$

This suggests the following question: does q arise from an element of $\widehat{\mathcal{R}^{\mathrm{et}}_{\mathrm{ord}}}$? The answer is *no*. This may be seen by considering $q \pmod p$: the formula for $d\tau/dz = \mathrm{dlog}(q)$ shows that $q \pmod p$ is non-constant ($\mathrm{dlog}(q) \equiv \frac{2(h_p)^{2/p-1}}{z(z-1)} \pmod p$). If q comes from an element of $\widehat{\mathcal{R}^{\mathrm{et}}_{\mathrm{ord}}}$ which specializes to 1 at every canonical modulus ζ_{can}, one has $q \equiv 1 \pmod p$, a contradiction.

3.5. Complex counterpart: the supersingular locus.

Let us now revert things and try to understand the complex situation from the p-adic viewpoint!

Let D^+ and D^- be the (complex-conjugate) connected components of $\mathsf{D}(\frac{1}{2}, \frac{3}{2}^-) \setminus (\mathsf{D}(\frac{-1}{4}, \frac{3}{4}^+) \cup \mathsf{D}(\frac{5}{4}, \frac{3}{4}^+))$. Let S be the complement of $\mathsf{D}^+ \cup \mathsf{D}^-$ in the complex plane. Note that S is closed and arcwise connected, and that $\pi_1^{\mathrm{top}}(S)$ is a free group with two generators γ^+, γ^- (see figure 8). Its interior S° has three connected (simply-connected) components $\mathsf{D}_0 = \mathsf{D}(\frac{-1}{4}, \frac{3}{4}^-)$, $\mathsf{D}_1 = \mathsf{D}(\frac{5}{4}, \frac{3}{4}^-)$ and D_∞.

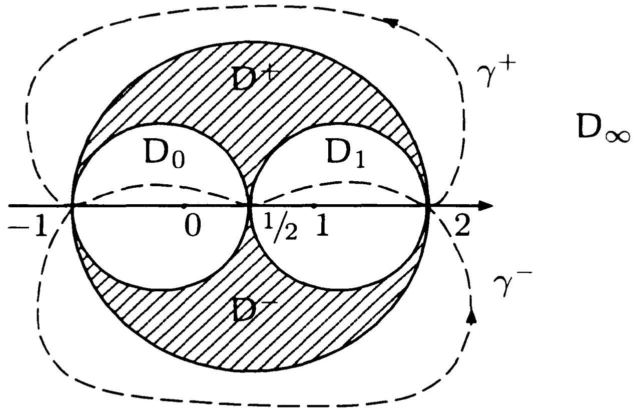

FIGURE 8

Let $\mathcal{O}(S)$ denote the ring of continuous functions on S analytic in S°. According to Mergelyan's theorem [**Mer54**], they are uniform limits of rational functions on every compact $K \subset S$ such that $\pi_0(\mathbb{C} \setminus K, 0)$ is finite. This ring is however not stable under differentiation, and we consider its differential closure $\mathcal{R}$ in $\mathcal{O}(S^\circ)$. We also consider the integral closure $\mathcal{R}^{\mathrm{et}}$ of $\mathcal{R}$ in $\mathcal{O}(S^\circ)$. Every element of $\mathcal{R}^{\mathrm{et}}$ defines a multivalued locally analytic function on S, *i.e.* an analytic germ which may be analytically continued along any path of S not ending at $-1, \frac{1}{2}, 2$, in such a way that the germ at the other extremity is analytic.

For instance, $F(\frac{1}{2}, \frac{1}{2}, 1; z)$ may be viewed as an element of $\mathcal{R}^{\mathrm{et}}$: in fact, $F(\frac{1}{2}, \frac{1}{2}, 1; z)^4 \in \mathcal{O}(S)$. An explicit continuous extension of $F(\frac{1}{2}, \frac{1}{2}, 1; z)^4$ from D_0 to S is given by $F(\frac{1}{2}, \frac{1}{2}, 1; 1-z)^4$ in D_1, $\frac{1}{z^2}F(\frac{1}{2}, \frac{1}{2}, 1; \frac{1}{z})^4$ in D_∞, and the values $\frac{\Gamma(1/4)^8}{64\pi^6}$, $\frac{\Gamma(1/4)^8}{16\pi^6}$, $\frac{\Gamma(1/4)^8}{64\pi^6}$ at -1, $\frac{1}{2}$, 2 respectively [**Car61**, p.189]. On the other hand, note that $\operatorname{dlog} F(\frac{1}{2}, \frac{1}{2}, 1; z)$ is in $\mathcal{R}$ but not in $\mathcal{O}(S)$.

The de Rham cohomology of the Legendre elliptic pencil gives rise to an $\mathcal{R}[1/z(1-z)]$-module with connection $(\mathcal{H}, \nabla)$. It admits a ***unique*** non-zero horizontal submodule $\mathcal{U}$ which extends to S; we use the same symbol for the extension. It can be described along the following lines:

- For any $u \in (\mathcal{U} \otimes_{\mathcal{R}} \mathcal{R}^{\mathrm{et}})^{\nabla}$, "the" image of u in $\mathcal{O}(\mathrm{D}_j)(j = 0, 1,$ or $\infty)$, is a bounded solution of the differential operator $L_{\frac{1}{2}, \frac{1}{2}, 1}$.
- $\mathcal{U} \otimes_{\mathcal{R}} \mathcal{R}^{\mathrm{et}}[\frac{1}{z(1-z)}] = \big(\mathcal{H} \otimes_{\mathcal{R}[\frac{1}{z(1-z)}]} \mathcal{R}^{\mathrm{et}}[\frac{1}{z(1-z)}]\big)^{\nabla} \otimes_{\mathbb{C}} \mathcal{R}^{\mathrm{et}}[\frac{1}{z(1-z)}]$.
- $\big(\mathcal{H} \otimes_{\mathcal{R}[\frac{1}{z(1-z)}]} \mathcal{R}^{\mathrm{et}}[\frac{1}{z(1-z)}]\big)^{\nabla}$ has a canonical $\mathbb{Z}[i]$-submodule ($i = \sqrt{-1}$) which can be locally identified with the part $\mathrm{H}^1(X_z^{\mathrm{an}}, \mathbb{Z}[i])_{\mathrm{isotriv}}$ of $\mathrm{H}^1(X_z^{\mathrm{an}}, \mathbb{Z}[i])$ where $\pi_1^{\mathrm{top}}(S, z)$ acts through a finite group.
- $\mathcal{U}|_{\mathrm{D}_0}^{\nabla}$ has a canonical $\mathbb{Z}$-submodule, which can be identified with the part of $\mathrm{H}^1(X_z^{\mathrm{an}}, \mathbb{Z}[i])_{\mathrm{isotriv}}$ invariant under complex conjugation. One of the two generators u satisfies
$$\Big\langle \omega, \frac{u}{2i\pi} \Big\rangle = iF\Big(\frac{1}{2}, \frac{1}{2}, 1; z\Big) \quad \text{in} \quad \mathcal{O}(\mathrm{D}_0).$$
Similarly, $\mathcal{U}|_{\mathrm{D}_1}^{\nabla}$ has a canonical $\mathbb{Z}$-submodule, and for one of the two generators u, one has
$$\Big\langle \omega, \frac{u}{2i\pi} \Big\rangle = iF\Big(\frac{1}{2}, \frac{1}{2}, 1; 1-z\Big) \quad \text{in} \quad \mathcal{O}(\mathrm{D}_1).$$
Note the occurrence of $2i\pi$ in these formulas.
- The local monodromy γ^+ around D^+ maps *onto*
$$\operatorname{Aut} \mathrm{H}^1(X_s, \mathbb{Z}[i])_{\mathrm{isotriv}} \simeq \mathbb{Z}/4\mathbb{Z}.$$
Same for γ^-. The local monodromy at ∞ acts as ± 1.
- In D^+ and in D^-, there is no non-zero bounded analytic solution of $L_{\frac{1}{2}, \frac{1}{2}, 1}$.

This is quite similar to §§3.2, §§3.3, $\mathrm{D}^+ \cup \mathrm{D}^-$ playing the role of the *supersingular locus.* This picture is assuredly very different from the traditional view of monodromy for $F(\frac{1}{2}, \frac{1}{2}, 1; z)$, and may shed some light upon the divergences between the topological and algebraic approaches to analytic continuation in the p-adic case (here of course, the coexistence of the two pictures is explained by the fact that the principle of unique continuation (Definition 1.1.1) fails for the sheaf of germs of continuous functions on S, analytic on S°).

The tale of $F(\frac{1}{2}, \frac{1}{2}, 1; z)$ is not finished: we have not yet explored the islands of supersingularity. We shall reach them in §§5.2.

4. Abelian periods as algebraic integrals.

ABSTRACT: We discuss periods of abelian varieties. Their p-adic counterparts live naturally in Fontaine's ring $\mathrm{B_{dR}}$. We present Colmez' construction of abelian p-adic periods, which relies on p-adic integration and reflects as closely as possible the complex picture. We also deal with the concrete evaluation of elliptic p-adic periods.

4.1. Over $\mathbb{C}$.

Let A be an abelian variety over $\mathbb{C}$ of dimension g, and $\omega_1, \ldots, \omega_g$ a basis of invariant differential forms. Let Λ be the image of the map

$$\iota : \mathrm{H}_1(A(\mathbb{C}), \mathbb{Z}) \to \mathbb{C}^g \; ; \;\; \gamma \mapsto \left(\int_\gamma \omega_i \right)_i .$$

One has a canonical isomorphism

$$\mathbb{C}^g/\Lambda \xrightarrow{\sim} A(\mathbb{C}).$$

Let us denote the projection $\mathbb{C}^g \to A(\mathbb{C})$ by pr. For any differential one-form ω of the second kind, one can consider a primitive function f_ω of $\mathrm{pr}^*\omega$; it is meromorphic (univalued) on $\mathbb{C}^g$, because all residues of ω are 0 by definition, and it is unique up to addition of an arbitrary constant. Then $f_\omega(z_1+z_2+z_3) - f_\omega(z_1+z_2) - f_\omega(z_1+z_3) + f_\omega(z_1)$ defines a periodic function on $(\mathbb{C}^g)^3$, hence induces a meromorphic function on $A(\mathbb{C})^3$, which we denote by F^3_ω. Note that the function F^3_ω can also be defined purely algebraically by the following conditions:

- $F^3_\omega(z_1, 0, z_3) = F^3_\omega(z_1, z_2, 0) = 0$.
- $dF^3_\omega = m^*_{123}\omega - m^*_{12}\omega - m^*_{13}\omega + m^*_1\omega$,

where m_{123} is the addition $A^3 \to A$ sending (z_1, z_2, z_3) to $z_1 + z_2 + z_3$, etc. For $\gamma \in \mathrm{H}_1(A(\mathbb{C}), \mathbb{Z})$ the value $f_\omega(i(\gamma) + a) - f_\omega(a)$ does not depend on $a \in \mathbb{C}^g$ (a being chosen so that $i(\gamma) + a$ and a are not poles of f_ω), and one has the equality

$$f_\omega(i(\gamma) + a) - f_\omega(a) = \int_\gamma \omega,$$

which describes the period pairing

$$\mathrm{H}^1_{\mathrm{dR}}(A) \times \mathrm{H}_1(A(\mathbb{C}), \mathbb{Z}) \longrightarrow \mathbb{C}$$

(or equivalently, the isomorphism $\mathrm{H}^1_{\mathrm{dR}}(A) \otimes \mathbb{C} \xrightarrow{\sim} \mathrm{H}^1_{\mathrm{B}}(A(\mathbb{C}), \mathbb{Z}) \otimes \mathbb{C}$.) Here $\mathrm{H}^1_{\mathrm{dR}}(A)$ denotes the first *algebraic* de Rham cohomology group of A, which coincides with the group of differential forms of the second kind modulo exact forms.

4.2. Over the p-adics; prolegomena

If we try to translate this into the p-adic setting, one has to face at once the problem: what is *integration over a loop*?

4.2.1. One tentative way is via Berkovich's theory, where loops do exist. Let us for instance consider the Legendre elliptic curve X_z. Viewed as a p-adic space, for $z \in \mathrm{D}(0,1^-) \setminus \{0\}$ $(p \neq 2)$, this is a Tate curve: $X_z^{\mathrm{an}} \simeq \mathbb{C}_p^\times / q^{\mathbb{Z}}$. The canonical differential ω_{can} inherited from $\mathbb{C}_p^\times$ and $\frac{dx}{y}$ are proportional:

$$\omega_{\mathrm{can}} = \Theta(z).\frac{dx}{y} \quad \text{with} \quad 1/\Theta(z) = \sqrt{-1}F\Big(\frac{1}{2},\frac{1}{2},1;z\Big).$$

The basic element γ of $\mathrm{H}_1(X_z^{\mathrm{an}},\mathbb{Z})$ can be identified with the generator q of $q^{\mathbb{Z}}$, and it is natural to set $\int_\gamma \omega_{\mathrm{can}} = \int_0^q \frac{dt}{t} = \log(q)$ (choosing a branch of the p-adic logarithm). One then has

$$\omega_2^{(p)} := \int_\gamma^{(p)} \frac{dx}{y} = \sqrt{-1}F\Big(\frac{1}{2},\frac{1}{2},1;z\Big)\log(q).$$

This formula, as well as

$$\sqrt{q} = \frac{z}{16} e^{\frac{F^*(\frac{1}{2},\frac{1}{2},1;z)}{F(\frac{1}{2},\frac{1}{2},1;z)}}$$

$$\left(\text{with } F^*\Big(\frac{1}{2},\frac{1}{2},1;z\Big) = 4\sum_{n>0}\binom{2n}{n}^2\Big(\sum_1^{2n}\frac{(-1)^{m-1}}{m}\Big)\Big(\frac{z}{16}\Big)^n\right)$$

are the same as those encountered in the complex situation. This leaves the open problem: how can one construct "the other period" $\omega_1^{(p)}$, *i.e.* the p-adic analogue of $\omega_1 = 2\pi\sqrt{-1}F(\frac{1}{2},\frac{1}{2},1;z)$, since the corresponding loop is missing?

4.2.2. One simple tentative answer would be to replace the topological covering $\mathbb{C}_p^\times \to X_z^{\mathrm{an}}$ by other étale coverings. Natural candidates for this purpose are étale coverings of order p^n, especially those corresponding to torsion points of order p^n which are close to the origin. Let ζ_{p^n} be a p^nth root of unity in $\mathbb{C}_p^\times$ and let x_n be its image in X_z^{an}. In the corresponding complex case, taking $\zeta_{p^n} = e^{2i\pi/p^n}$ would give the right answer $p^n \int_0^{x_n} \frac{dx}{y} = \omega_1$. In the p-adic case, we get instead $p^n \int_0^{x_n} \frac{dx}{y} = 0$. Indeed, already in the case of the multiplicative group, we have $p^n \int_1^{\zeta_{p^n}} \frac{dt}{t} = p^n \log(1 + (\zeta_{p^n} - 1)) = 0$ p-adically: in other words, $2i\pi$ "***is missing***".

4.2.3 (*Riemann-Shnirelman sums.*). It is appropriate to evoke here Shnirelman's approach to integration over loops, indeed one of the earliest works in p-adic analysis. This is an adaptation to the p-adic case of the computation of integrals by Riemann sums

$$\frac{1}{2i\pi}\int_{C(a,r)} f(z)dz = \int_0^1 f(a+re^{2i\pi\theta})re^{2i\pi\theta}d\theta \;=\; \lim_{m\to\infty}\frac{1}{m}\sum_{\zeta^m=1} f(a+r\zeta)r\zeta.$$

Now, let $a, r \in \mathbb{C}_p$ and let f be a $\mathbb{C}_p$-valued function on the circumference $C(a, |r|)$. Let us denote the limit

$$\lim_{m \to \infty} \frac{1}{m} \sum_{\zeta^m = 1} f(a + r\zeta).r\zeta$$

symbolically by $\int_{C(a,r)} f(z)dz$. For better convergence, Shnirelman actually restricted the limit to those integers m prime to p. In the case of an analytic function f on $C(a, |r|)$, this restriction is unnecessary (one can even take $m = p^k$) and one has the following analogue of Cauchy's theorem of residues (*cf.* [**Ko80**, app.])

Lemma 4.2.4. *Assume that f is a meromorphic function on $\mathrm{D}(a, |r|^+)$, and that its poles $z_1, \ldots, z_\nu$ all lie in $\mathrm{D}(a, |r|^-)$. Then $\int_{C(a,r)} f(z)dz$ exists and equals $\sum \mathrm{Res}_{z_i} f$.*

In the (trivial) special case $f = 1/z$, we get $\int_{C(0,1)} dz/z = 1$ (not 0 as in 4.2.2!), but $2i\pi$ is still missing.

4.2.5. Actually, there is no way to remedy this if one remains in $\mathbb{C}_p$. A deeper reason for that, due to Tate, is that while $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ acts on the inverse system of p^nth roots of unity ($n \geq 0$) through the cyclotomic character χ, there is no element $(2i\pi)_p$ in $\mathbb{C}_p$ such that $g((2i\pi)_p) = \chi(g)(2i\pi)_p$ for every $g \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$.
It turns out that $(2i\pi)_p$ is in a sense the only "missing piece": there is a good theory of p-adic periods (due to J.M. Fontaine, W. Messing [**FM87**], G. Faltings) which lives in some $\mathbb{Q}_p$-algebra isomorphic to $\mathbb{C}_p[[(2i\pi)_p]]$, and which we shall now describe in the case of abelian varieties.

4.3. The Fontaine ring $\mathrm{B_{dR}}$.

4.3.1. *The ring* R. Let $\mathcal{O}_{\mathbb{C}_p}$ be the ring of integers of $\mathbb{C}_p$, *i.e.* $\{x \in \mathbb{C}_p, |x|_p \leq 1\}$. Let us set

$$\mathsf{R} := \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_p},$$

i.e. R is the set of all series $\left(x^{(n)}\right)_{n \in \mathbb{N}}$ such that $\left(x^{(n+1)}\right)^p = x^{(n)}$. This is in fact a ring of characteristic p with

$$\left(\left(x^{(n)}\right) + \left(y^{(n)}\right)\right)^{(n)} = \lim_{m \to \infty} \left(x^{(n+m)} + y^{(n+m)}\right)^{p^m}$$

and

$$\left(\left(x^{(n)}\right) \cdot \left(y^{(n)}\right)\right)^{(n)} = x^{(n)} \cdot y^{(n)}.$$

Let $\mathrm{W}(\mathsf{R})$ be the Witt ring with coefficients in R. for $x \in \mathsf{R}$, let $[x]$ denote the Teichmüller representative in $\mathrm{W}(\mathsf{R})$.

4.3.2. *The ring* $\mathrm{B}_{\mathrm{dR}}^{+}$. To any element $(x_0, x_1, \ldots, x_n, \ldots) = \sum p^n [x_n^{p^{-n}}]$ in $\mathrm{W}(\mathrm{R})$, we associate

$$\theta((x_n)) = \sum_{n=0}^{\infty} p^n x_n^{(n)}.$$

This defines a surjective homomorphism $\theta \colon \mathrm{W}(\mathrm{R}) \to \mathcal{O}_{\mathbb{C}_p}$, whose kernel is principal. It extends to a homomorphism $\theta : \mathrm{W}(\mathrm{R})\left[\frac{1}{p}\right] \to \mathbb{C}_p$, and $\mathrm{B}_{\mathrm{dR}}^{+}$ is defined as the $(\operatorname{Ker}\theta)$-adic completion

$$\mathrm{B}_{\mathrm{dR}}^{+} := \varprojlim \mathrm{W}(\mathrm{R})\left[\tfrac{1}{p}\right] \Big/ (\operatorname{Ker}\theta)^n.$$

By continuity, θ further extends to a homomorphism $\theta \colon \mathrm{B}_{\mathrm{dR}}^{+} \to \mathbb{C}_p$. Then $\mathrm{B}_{\mathrm{dR}}^{+}$ is a complete discrete valuation ring with maximal ideal $\operatorname{Ker}\theta$ and residue field $\mathbb{C}_p$. Moreover, the Galois group $\operatorname{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ acts on $\mathrm{B}_{\mathrm{dR}}^{+}$ in such a way that θ is equivariant with respect to this Galois action,

$$\operatorname{Gr}^{\bullet} \mathrm{B}_{\mathrm{dR}}^{+} \simeq \bigoplus_{r \in \mathbb{N}} \mathbb{C}_p(r),$$

where $\operatorname{Gr}^{\bullet}$ refers to the filtration by the powers of $\operatorname{Ker}\theta$, and where the "twist" (r) indicates that the $\operatorname{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$-action is twisted by the rth power of the cyclotomic character.

It turns out that $\mathrm{B}_{\mathrm{dR}}^{+}$ contains naturally a copy of $\overline{\mathbb{Q}}_p$ (which θ maps isomorphically to $\overline{\mathbb{Q}}_p \subset \mathbb{C}_p$). More precisely, P. Colmez [**Co94**] has shown that $\mathrm{B}_{\mathrm{dR}}^{+}$ is the separated *completion* of $\overline{\mathbb{Q}}_p$ with respect to the topology defined by taking $\left(p^n \mathcal{O}_{\overline{\mathbb{Q}}_p}^{(k)}\right)_{n,k}$ as a basis of neighborhoods of 0, where $\mathcal{O}_{\overline{\mathbb{Q}}_p}^{(k)}$ denotes the subring of $\overline{\mathbb{Z}}_p$ defined inductively as

$$\mathcal{O}_{\overline{\mathbb{Q}}_p}^{(0)} = \overline{\mathbb{Z}}_p,\ \mathcal{O}_{\overline{\mathbb{Q}}_p}^{(k)} = \operatorname{Ker}\Big(d : \mathcal{O}_{\overline{\mathbb{Q}}_p}^{(k-1)} \to \Omega^1_{\mathcal{O}_{\overline{\mathbb{Q}}_p}^{(k-1)}/\mathbb{Z}_p} \otimes \overline{\mathbb{Z}}_p\Big).$$

It is easy to deduce from this description that $\mathrm{B}_{\mathrm{dR}}^{+}$ contains $\widehat{\mathbb{Z}_p^{\mathrm{ur}}}$.

4.3.3. *Some remarkable elements of* $\mathrm{B}_{\mathrm{dR}}^{+}$. For any $x \in \mathrm{B}_{\mathrm{dR}}^{+}$ such that $|\theta(x) - 1|_p < 1$, the series $\log(x) = -\sum_{n>0} \frac{(1-x)^n}{n}$ converges in $\mathrm{B}_{\mathrm{dR}}^{+}$. In particular, let $\underline{z} = (\ldots, z^{(1)}, z^{(0)})$ be an element of R such that $z^{(0)} \in \overline{\mathbb{Z}}_p$ or $\widehat{\mathbb{Z}_p^{\mathrm{ur}}}$. Then one can define the element $\operatorname{Log}\underline{z} = \log(\frac{z^{(0)}}{[\underline{z}]})$[(4)]. Note that $\theta(\operatorname{Log}\underline{z}) = 0$.

In the special case where each $z^{(n)} = \zeta_{p^n}$ is a primitive p^nth root of unity, $\operatorname{Log}(\ldots, \zeta_{p^n}, \ldots, 1)$ is the element $(2i\pi)_p$ we were looking for. This is a generator of $\operatorname{Ker}\theta$, and $\operatorname{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ acts on it through the cyclotomic character; in other words, we have a $\operatorname{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-equivariant isomorphism

$$\operatorname{Gr}^{\bullet} \mathrm{B}_{\mathrm{dR}}^{+} \simeq \mathbb{C}_p[(2i\pi)_p].$$

[(4)] we follow Colmez' sign convention; Fontaine's LOG is log − Log.

Note that another choice of $(\dots, \zeta_{p^n}, \dots, 1)$ changes $(2i\pi)_p$ by multiplication by a unit in $\mathbb{Z}_p$. Similarly, up to addition of an element of $(2i\pi)_p\mathbb{Z}_p$, Log $\underline{z}$ depends only on $z^{(0)}$; it is sometimes simply denoted by Log $z^{(0)}$.

By choosing a double embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}$ and $\overline{\mathbb{Q}}_p$, one obtains a canonical element $(2i\pi)_p$, attached to the sequence $(\dots, e^{2i\pi/p^n}, \dots, 1)$. Similarly, if $z^{(0)} \in \overline{\mathbb{Q}}$, then $\operatorname{Log} z^{(0)}$ is well-defined up to addition of an element of $(2i\pi)_p\mathbb{Z}$.

4.4. Colmez' construction of abelian p-adic periods.

Let A be an abelian variety defined over a p-adic local field K. The Fontaine-Messing p-adic period pairing is a pairing

$$\int^{(p)} \; : \; \mathrm{H}^1_{\mathrm{dR}}(A) \otimes T_p(A_{\overline{K}}) \longrightarrow \mathrm{B}^+_{\mathrm{dR}}$$

where $T_p(A_{\overline{K}}) = \varprojlim \operatorname{Ker}([p^n] : A_{\overline{K}} \to A_{\overline{K}})$ is the Tate module (a $\mathbb{Z}_p$-module of rank $2\dim A$ with $\operatorname{Gal}(\overline{K}/K)$-action).

We present Colmez' construction, which is parallel to 4.1.

Let ω be a differential form on A of second kind, and F^3_ω the function on A^3 determined as in 4.1. Then:

Proposition 4.4.1 ([**Co92**, 4.1]). *There exists a locally meromorphic function F_ω on $A(\mathrm{B}_{\mathrm{dR}})$, unique up to constant, such that:*

(1) $dF_\omega = \omega$.
(2) $F_\omega(z_1+z_2+z_3) - F_\omega(z_1+z_2) - F_\omega(z_1+z_3) + F_\omega(z_1) = F^3_\omega(z_1, z_2, z_3)$.
(3) *If* $\omega = dF$, *then* $F_\omega = F$.

Moreover, if $\alpha\colon A_1 \to A_2$ is a morphism of abelian varieties and ω is a differential form of second kind on A_2, then $F_{\alpha^*\omega} = \alpha^* F_\omega$.

Note that the function F_ω is *not* multivalued; this fact comes from the following lemma specific to the p-adic case:

Lemma 4.4.2 ([**Co91**, 4.3]). *For any neighborhood V of 0 in $A(\mathrm{B}^+_{\mathrm{dR}})$, there exists an open subgroup U of $A(\mathrm{B}^+_{\mathrm{dR}})$ contained in V such that $A(\mathrm{B}^+_{\mathrm{dR}})/U$ is a torsion group.*

Take a proper model $\mathcal{A}$ of A over $\mathcal{O}_K$. Let $\gamma = (\cdots, u_n, \cdots, u_2, u_1 = 0) \in \mathrm{T}_p(A_{\overline{K}})$ with each $u_n \in \mathcal{A}(\mathcal{O}_{\mathbb{C}_p})$, and choose $a_n \in \mathcal{A}(\mathrm{B}^+_{\mathrm{dR}})$ so that neither a_n nor $a_n +_A \widehat{u}_n$ is close to a pole of ω. For suitable liftings $\widehat{u}_n \in \mathcal{A}(\mathrm{B}^+_{\mathrm{dR}})$ of u_n (*i.e.* $\theta(\widehat{u}_n) = u_n$), the following holds.

Theorem 4.4.3 ([**Co91**, 5.2]). *The limit*

$$\oint_\gamma^{(p)} \omega := \lim_{n\to\infty} p^n\big(F_\omega(a_n) - F_\omega(a_n +_A \widehat{u}_n)\big)$$

converges to an element in $\mathrm{B}^+_{\mathrm{dR}}$, and it defines a non degenerate bilinear pairing

$$\oint^{(p)} \; : \; \mathrm{H}^1_{\mathrm{dR}}(A) \otimes \mathrm{T}_p(A_{\overline{K}}) \longrightarrow \mathrm{B}^+_{\mathrm{dR}},$$

compatible with the Galois action and the filtrations.

Remark 4.4.4. On composing the period pairing $\int^{(p)}$ with θ, one gets a bilinear map $\mathrm{H}^1_{\mathrm{dR}}(A) \otimes T_p(A_{\overline{K}}) \longrightarrow \mathbb{C}_p$ which sends $\Omega^1(A) \subset \mathrm{H}^1_{\mathrm{dR}}(A)$ to 0. This degenerate pairing describes "half" of the (Hodge)-Tate-Raynaud decomposition

$$\mathrm{H}^1_{\mathrm{et}}(A_{\overline{K}}, \mathbb{Q}_p) \otimes \mathbb{C}_p \simeq \Omega^1(A) \otimes \mathbb{C}_p(-1) \oplus \mathrm{H}^1(\mathcal{O}(A)) \otimes \mathbb{C}_p$$

of $\mathrm{Gal}(\overline{K}/K)$-modules.

4.5. Some computations of elliptic p-adic periods.

4.5.1. *Tate elliptic curves.* In this case, $T_p(A_{\overline{K}})$ sits in an exact sequence

$$0 \to \mathbb{Z}_p(1) \to T_p(A_{\overline{K}}) \to \mathbb{Z}_p \to 0.$$

Let us take for γ_1 the image of $(2i\pi)_p \in \mathbb{Z}_p(1)$ in $T_p(A_{\overline{K}})$, and for γ_2 any lifting of $1 \in \mathbb{Z}_p$. For concreteness, let $A = X_z$ be in Legendre form as in 4.2.1. Let $\eta \in \mathrm{H}^1_{\mathrm{dR}}$ satisfy $\langle \omega, \eta \rangle = 1$ (*e.g.* $\frac{xdx}{4y}$). We set $\omega_i^{(p)} = \int_{\gamma_i}^{(p)} \omega, \eta_i^{(p)} = \int_{\gamma_i}^{(p)} \eta$. Then one has the "Legendre relation"

$$\omega_1^{(p)} \eta_2^{(p)} - \eta_1^{(p)} \omega_2^{(p)} = (2i\pi)_p$$

and the formulas (*cf.* [**And90**], [**And96**])

$$\frac{\omega_1^{(p)}}{(2i\pi)_p} = \sqrt{-1} F\Big(\frac{1}{2}, \frac{1}{2}, 1; z\Big) = 1/\Theta(z), \;\; \omega_2^{(p)} = \sqrt{-1} F\Big(\frac{1}{2}, \frac{1}{2}, 1; z\Big) \operatorname{Log} q$$

(compare with the complex case in 3.4.1, and also [**Car61**, p.406]). Note that the "period" we found in 4.2.1 by integration along the Berkovich loop is *not* a Fontaine-Messing period (there is a log instead of Log).

4.5.2. *Elliptic curves with ordinary reduction.* We now assume that $z \in \mathbb{Z}_p^{\mathrm{ur}}$ and that $A = X_z$ has good ordinary reduction (with the notation of §§3.1, this means that $z(1-z)h_p(z) \neq 0$). Again, $T_p(A_{\overline{\mathbb{Q}}_p})$ sits in an exact sequence ([**Ser68a**, A.2.4])

$$0 \to \mathbb{Z}_p \gamma_1 \to T_p(A_{\overline{\mathbb{Q}}_p}) \to T_p(A \bmod p) \to 0.$$

Then one still has the formula $\Theta(z) = \frac{(2i\pi)_p}{\omega_1^{(p)}}$, where $\Theta(z) \in \widehat{\mathbb{Z}_p^{\mathrm{ur}}}$ is the Tate constant discussed in 3.4.1 (compare [**And90**], [**dSh87**, 4.3]). In particular, $\frac{\omega_1^{(p)}}{(2i\pi)_p}$ is given by the evaluation at z of "the" extension of $F(\frac{1}{2}, \frac{1}{2}, 1; ?)$ discussed in §§3.4.

On the other hand, let $\gamma_2 \in T_p(A_{\overline{\mathbb{Q}}_p})$ be such that the Weil pairing $\langle \gamma_1, \gamma_2 \rangle \in \mathbb{Z}_p(1) = \mathbb{Z}_p(2i\pi)_p$ is the chosen generator $(2i\pi)_p$. Then one has the "Legendre relation", and $\omega_2^{(p)} = \omega_1^{(p)} \cdot \frac{\operatorname{Log} q}{(2i\pi)_p}$, where $q \in \widehat{\mathbb{Z}_p^{\mathrm{ur}}}$ denotes here the so-called *Dwork(-Serre-Tate) parameter* of A introduced in 3.4.2. If $q = 1$, *i.e.* when A is the *canonical lifting* of A (mod p), then one can choose γ_2 such the corresponding value of Log 1 is 0, and then $\omega_2^{(p)} = 0$.

4.6. Periods of CM elliptic curves and values of the Gamma function.

4.6.1. The case of an elliptic curve A with *supersingular reduction* is more delicate. If A does not have complex multiplication (CM), one can show, using [**Ser68a**, A.2.2], that the four basic periods $\omega_1^{(p)}, \omega_2^{(p)}, \eta_1^{(p)}, \eta_2^{(p)}$ are algebraically independent over $\overline{\mathbb{Q}}_p$, so that one cannot expect "formulas" for the periods.

On the other hand, it is well-known that the complex periods of an elliptic curve with complex multiplication can be expressed in terms of special values of the Γ function. So one may ask for a p-adic analogue involving Γ_p (*cf.* 2.6). Such a formula has been found by A. Ogus, actually not for the periods, but for the action of Frobenius.

4.6.2. *The Lerch-Chowla-Selberg-Ogus formulas.* Let A be an elliptic curve with complex multiplication by $\mathbb{Q}(\sqrt{-d})$ over $\overline{\mathbb{Q}}$ ($-d$ denotes a fundamental discriminant). Let $\epsilon = (\frac{-d}{\cdot})$ be the quadratic character $(\mathbb{Z}/d)^\times \to \mathbb{Z}/2$ induced by the embedding $\mathbb{Q}(\sqrt{-d}) \subset \mathbb{Q}(\zeta_d)$, $h =$ the class number of $\mathbb{Q}(\sqrt{-d})$, and $w =$ the number of roots of unity in $\mathbb{Q}(\sqrt{-d})$. For any $u \in (\mathbb{Z}/d)^\times$, we denote by $\langle \frac{u}{d} \rangle$ the unique rational number in $]0, 1]$ such that $d\langle \frac{u}{d} \rangle \equiv u$.

Let v be a place of $\overline{\mathbb{Q}}$ of residue characteristic p, with associated embedding $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_p$. To any lifting ψ_v to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ of the Frobenius element in $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)$, one attaches a ψ_v-linear endomorphism Ψ_v of $\mathrm{H}^1_{\mathrm{dR}}(A/\overline{\mathbb{Q}}) \otimes \overline{\mathbb{Q}}_p$ [(5)]. On the other hand, it is well-known that A has supersingular reduction at v if and only if $\epsilon(p) = -1$ or 0.

Theorem 4.6.3. *There exists a basis (ω, η) of $\mathrm{H}^1_{\mathrm{dR}}(A/\overline{\mathbb{Q}})$ of eigenvectors under the action of $\sqrt{-d}$ (ω being the class of a regular differential), and an element $\gamma \in \mathrm{H}_1(A(\mathbb{C}), \mathbb{Q})$, such that*

$$\int_\gamma \omega = \sqrt{2i\pi} \prod_{u \in (\mathbb{Z}/d)^\times} \left(\Gamma\left\langle \frac{u}{d} \right\rangle\right)^{\epsilon(u)w/4h}$$

$$\int_\gamma \eta = \sqrt{2i\pi} \prod_{u \in (\mathbb{Z}/d)^\times} \left(\Gamma\left\langle -\frac{u}{d} \right\rangle\right)^{\epsilon(u)w/4h}$$

and such that for every place of $\overline{\mathbb{Q}}$ of residue characteristic p satisfying $\epsilon(p) = -1$,

$$\Psi_v^*(\omega) = p \prod_{u \in (\mathbb{Z}/d)^\times} \left(\Gamma_p\left\langle p\frac{u}{d} \right\rangle\right)^{-\epsilon(u)w/4h} \eta$$

$$\Psi_v^*(\eta) = \prod_{u \in (\mathbb{Z}/d)^\times} \left(\Gamma_p\left\langle -p\frac{u}{d} \right\rangle\right)^{-\epsilon(u)w/4h} \omega$$

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(5) using its crystalline interpretation, *cf.* [**BeO83**, 4]; we refer to [**Chl98**] for a nice recent survey of crystalline cohomology.

up to multiplication by some root of unity.

This is a concatenation of [**O90**, 3.15, 3.9], taking into account the formula $2hd/w = -\sum_1^d \epsilon(u)u$, *cf.* [**H81**, VII]).

Remark. The last two formulas extend to the case when $\epsilon(p) = +1$ if one interchanges ω and η in the right hand sides (for $d = 4$ and $p \equiv 1 \pmod 4$), this is compatible with Young's formula in §§3.3, since $\Gamma_p(1/2)^4 = 1$, and with the formula $F(\frac{1}{2}, \frac{1}{2}, 1; -1) = \frac{1}{2}\frac{\Gamma(1/4)}{\Gamma(1/2)\Gamma(3/4)}$.).

In that case, the expressions

$$\prod\left(\Gamma_p\left\langle p\frac{u}{d}\right\rangle\right)^{-\epsilon(u)w/4h}, \prod\left(\Gamma_p\left\langle p\frac{u}{d}\right\rangle\right)^{-\epsilon(u)w/4h}$$

are algebraic numbers: indeed, let r be the order of the subgroup $\langle p\rangle$ of $(\mathbb{Z}/d)^\times$ generated by p, and $k = p^r - 1/d$; then

$$\prod_{u\in(\mathbb{Z}/d)^\times}\left(\Gamma_p\left\langle p\frac{u}{d}\right\rangle\right)^{-\epsilon(u)} = \prod_{w\in(\mathbb{Z}/d)^\times/\langle p\rangle}\prod_1^r \Gamma_p\left(\frac{p^i wk}{p^r-1}\right)^{-\epsilon(w)}$$

and one concludes by the Gross-Koblitz formula.

4.6.4. *The ramified case.* In the ramified case, *i.e.* when $\epsilon(p) = 0$, there is again an analogue of the last two formulas for $p \neq 2$. This relies on Coleman's computation of the Frobenius matrix of Fermat curves of degree divisible by p — which have arboreal reduction modulo p. The result takes the same form as in the case $\epsilon(p) = -1$: Ψ is now attached to an element ψ of degree one in the Weil group (*i.e.* a lifting of the Frobenius element in $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$), the expressions $p\frac{u}{d}$ have to be replaced by $\psi\frac{u}{d}$, and Γ_p must be extended to $\mathbb{Q}_p$ [**Col90**, 6.5].

Coleman's complicated expressions have been simplified by F. Urfels in his thesis (Strasbourg, 1998; unpublished). If one passes to the Iwasawa logarithm $\log_p$ (at the cost of rational powers of p), the result is that one should replace $\log_p \Gamma_p\langle\psi\frac{u}{d}\rangle$ in Ogus' formula by $G_p(\langle\psi\frac{u}{d}\rangle) - G_p(\langle\frac{u}{d}\rangle)$, where G_p denotes Diamond's LogGamma function

$$G_p(x) = \lim_{m\to\infty}\frac{1}{p^m}\sum_{n=0,\dots,p^m-1}(x+n)\log_p(x+n) - (x+n).$$

Example. We take $p = 3$. Let $\mathbb{Q}[\sqrt{-3n}]$ be an imaginary quadratic field with fundamental discriminant $-3n$, n prime to 3. Let A be an elliptic curve with complex multiplication by some order in $\mathbb{Q}[\sqrt{-3n}]$.

Let ψ be an element of the Weil group which acts by -1 on $\mathbb{Q}_3/\mathbb{Z}_3 \cong \mu_{3^\infty}$. One has a formula $\Psi_v^*(\omega) = \kappa\eta$ with

$$\log_3\kappa = -(\epsilon(u)w/4h)\sum_{u\in(\mathbb{Z}/3n)^\times}\left(G_3\left(\left\langle\psi\frac{u}{3n}\right\rangle\right) - G_3\left(\left\langle\frac{u}{3n}\right\rangle\right)\right)$$

We notice that for $u \in (\mathbb{Z}/3n)^\times$, $\epsilon\Big(3n\Big\langle \psi \frac{u}{3n}\Big\rangle\Big) = -\epsilon(u)$, whence

$$\log_3 \kappa = (w/2h) \sum_{u \in (\mathbb{Z}/3n)^\times} \epsilon(u) G_3\Big(\Big\langle \frac{u}{3n}\Big\rangle\Big).$$

On the other hand, for $p = 3$, the Teichmüller character is nothing but the Legendre symbol $(\frac{-3}{\cdot})$. Using this, the latter expression can be rewritten, by the Ferrero-Greenberg formula (*cf.* [**La90**, chap. 17]), as

$$\begin{aligned}\log_3 \kappa &= (w/2h).(L_3'(0,\epsilon) + L_3(0,\epsilon)\log_3 n) \\ &= (w/2h). \sum_{v \in (\mathbb{Z}/n)^\times} \Big(\frac{n}{v}\Big) \log_3 \Gamma_3 \Big\langle \frac{v}{n} \Big\rangle.\end{aligned}$$

When $(\frac{n}{3}) = 1$, then $L_3(0,\epsilon) = 0$ and the Gross-Koblitz formula shows that $L_3'(0,\epsilon) = \sum (\frac{n}{v}) \log_3 \Gamma_3 \langle \frac{v}{n} \rangle$ is the Iwasawa logarithm of an algebraic number (a Gauss sum, *cf. loc. cit.*), hence $\kappa \in \overline{\mathbb{Q}}$.
A particularly simple case is $n = 1$: one finds $\log_3 \kappa = 0$, thus $\kappa \in 3^{\mathbb{Q}}.\mu_\infty \subset \overline{\mathbb{Q}}$.
For $n = 8$, a contrario, one has $(\frac{n}{3}) = -11$: one finds

$$\log_3 \kappa = (1/2).\Big[\log_3 \Gamma_3 \Big\langle \frac{1}{8} \Big\rangle - \log_3 \Gamma_3 \Big\langle \frac{3}{8} \Big\rangle - \log_3 \Gamma_3 \Big\langle \frac{5}{8} \Big\rangle + \log_3 \Gamma_3 \Big\langle \frac{7}{8} \Big\rangle\Big] = 0$$

by the functional equation of Γ_3, so that again $\kappa \in 3^{\mathbb{Q}}.\mu_\infty \subset \overline{\mathbb{Q}}$.

4.6.5. *Colmez' product formula.* There is a natural extension of $|\ \ |_p$ on $\overline{\mathbb{Q}}_p$ to $\mathrm{B}_{\mathrm{dR}}^+$ (however not as an absolute value), such that $|(2i\pi)_p|_p = p^{-\frac{1}{p-1}}$ *cf.* [**Co91**].

Colmez has remarked that the logarithm of the product $|2i\pi| \prod_p |(2i\pi)_p|_p$ $= 2\pi \prod_p p^{-\frac{1}{p-1}}$ is formally equal to $\log(2\pi) + \zeta'(1)/\zeta(1)$, a divergent sum which can be renormalized using the functional equation of ζ: setting $\zeta'(1)/\zeta(1)$ $= -\zeta'(0)/\zeta(0) = -\log(2\pi)$, the renormalized product is

$$|2i\pi| {\prod_p}' |(2i\pi)_p|_p = 1.$$

He has given an amazing generalization of this product formula to periods of CM elliptic curves and many other CM abelian varieties (loc. cit.).

5. Periods as solutions of the Gauss-Manin connection.

ABSTRACT: The variation of the periods in a family of complex algebraic varieties is controlled by the Picard-Fuchs differential equation. We discuss the p-adic situation, present Dwork's general viewpoint on p-adic periods. We then discuss the question of the existence of an arithmetic structure on the space of solutions, which would give an intrinsic meaning to the period modulo $\overline{\mathbb{Q}}^\times$. We present such an arithmetic structure (analogous to the Betti lattices in the complex situation) in the case of abelian varieties with either multiplicative reduction or supersingular reduction. In the latter case, we relate the periods, in the presence of complex multiplication, to special values of the p-adic Gamma function.

5.1. Stokes.

We come back to the basic question of the meaning of *integration over a loop*, already discussed in §§4.2. A different approach is based on the Stokes lemma: integrating *exact* differentials on a loop gives zero. This is the approach favored by K. Aomoto in the complex case and by B. Dwork in the p-adic case.

5.1.1. In order to see how this idea can be implemented, let us consider the Hankel expression for the gamma function

$$\frac{1}{\Gamma(1-\alpha)} = \frac{1}{2i\pi}\int_\gamma x^\alpha e^x \frac{dx}{x}$$

where γ is the following loop based at $-\infty$.

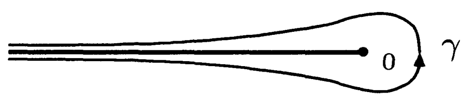

FIGURE 9

Stokes' lemma suggests to attach to this integral the following complex

$$x^\alpha e^x \mathcal{O} \xrightarrow{xd/dx} x^\alpha e^x \mathcal{O}$$

where $\mathcal{O}$ is a suitable ring of analytic functions. From the point of view of index theory (Malgrange-Ramis), a natural choice is

$$\mathcal{O} = \mathbb{C}[[x]]_{-1,1^-} = \left\{ \sum_{n\geq 0} a_n x^n \;\middle|\; \exists\kappa > 0, \exists r \in]0,1[,\ |a_n| \leq \kappa r^n/n! \right\}$$

identified with the ring of entire functions of *exponential order* $O(e^{r|x|})$ for some $r < 1$[(6)]. Using the formula $x^\alpha e^x x^k = \frac{-1}{\alpha+k} x^\alpha e^x x^{k+1} + xd/dx(\frac{1}{\alpha+k}x^\alpha e^x x^k)$, a simple computation then shows that $\mathrm{H}^0 = 0$ and that H^1 is of dimension 1; it is generated by $[\frac{1}{2i\pi}x^\alpha e^x]$, whose integration along γ gives $\frac{1}{\Gamma(1-\alpha)}$.

(6) we use the traditional notation for rings of Gevrey series; the value $(-1, 1^-)$ of the index is "characteristic", *i.e.* an extremal value for which H^1 is of dimension one.

The p-adic analogue

$$\mathbb{C}_p[[x]]_{-1,1^-} = \left\{ \sum_{n\geq 0} a_n x^n \;\middle|\; \exists \kappa > 0, \exists r \in]0,1[,\ |a_n| \leq \kappa r^n / |n!|_p \right\}$$

is nothing but the ring of *overconvergent* analytic functions on $\mathrm{D}(0, |\pi|_p)$ (here π is Dwork's constant). By change of variable $z = \pi x$, we thus recover the complex $M_\alpha^\dagger \xrightarrow{zd/dz} M_\alpha^\dagger$ of 2.6, where $\Gamma_p(\alpha)$ $\left(= \pm \frac{1}{\Gamma_p(1-\alpha)}\right)$ appeared.

5.1.2. This approach inspired by Stokes' lemma is well-suited for studying many kinds of hypergeometric functions (confluent or not). Basically, the integral will satisfy difference equations with respect to the exponents (α in the previous example), and differential algebraically on parameters. In the complex situation, one of the main problems is the construction of nice loops (such as γ); this is the object of a so-called "topological intersection theory" generalizing the usual Betti homology and period pairing. In the p-adic case, the focus has been more on the construction and properties of the Frobenius structure, in particular its analyticity with respect to the exponents ("Boyarsky principle" [**Dw83**]).

5.1.3. When no exponential is involved and when the exponents are rational, the integrand is algebraic; if it depends algebraically on parameters, the period integral is a solution of a *Gauss-Manin connection.*

Let us say a few words about the classical case of $F(a, b, c; z)$. We assume that $a, b, c \in \mathbb{Z}_p$, and that $c - a, c - b, b, a$ all lie outside $\mathbb{Z}$. We set

$$f_{a,b,c;z} = x^b (1-x)^{c-b} (1-zx)^{-a}$$

where, for simplicity, z is limited to $|z(1-z)| = 1$. Let Z be the affinoid $\mathrm{D}(0, 1^+) \setminus (\mathrm{D}(0, 1^-) \cup \mathrm{D}(1, 1^-) \cup \mathrm{D}(1/z, 1^-))$. The relevant complex is

$$f_{a,b,c;z} \mathcal{H}^\dagger(Z) \xrightarrow{xd/dx} f_{a,b,c;z} . \mathcal{H}^\dagger(Z).$$

Its $\mathrm{H}^1 = \mathrm{H}^1_{a,b,c}$ is of dimension 2; it is generated by $[f_{a,b,c;z}]$ and $[\frac{f_{a,b,c;z}}{1-x}]$. The Gauss-Manin connection is

$$\nabla\left(\frac{d}{dz}\right) \begin{pmatrix} [f_{a,b,c;z}] \\ [\frac{f_{a,b,c;z}}{1-x}] \end{pmatrix} = \begin{pmatrix} -\frac{c}{z} & \frac{c-b}{z} \\ \frac{c-a}{1-z} & \frac{a+b-c}{1-z} \end{pmatrix} \begin{pmatrix} [f_{a,b,c;z}] \\ [\frac{f_{a,b,c;z}}{1-x}] \end{pmatrix}$$

cf. [**Dw82**, 3.1, 1.2]. The Frobenius structure relates $\mathrm{H}^1_{a,b,c}$ to $\mathrm{H}^1_{a',b',c'}$, where a', b', c' are the successors of a, b, c respectively (*cf.* 2.6). For all these hypergeometric series, there is a story similar to that of $F(\frac{1}{2}, \frac{1}{2}, 1; z)$. In this context, it is fruitful to combine the Boyarsky principle and the contiguity relations.

5.1.4. In this sketch of Dwork's approach, "periods" are just analytic solutions of the Gauss-Manin connection. One can ask more: namely, one can ask for a dual theory of p-adic cycles and a period pairing as in the complex case[(7)]. At best, we can expect these "*p-adic Betti lattices*" to be locally horizontal, functorial with respect to the endomorphisms of the geometric fibres, and defined over $\mathbb{Z}$ or at least over some number field.

It turns out that this can be done in some cases, *e.g.* in the cases of abelian varieties with multiplicative reduction, and of abelian varieties with supersingular reduction ([**And90**], [**And95**]). Let us outline the results.

5.2. p-adic Betti lattices for abelian varieties with multiplicative reduction.

5.2.1. *Over* $\mathbb{C}$. We first recall the "multiplicative uniformization" of complex abelian varieties. Let A be an abelian variety of dimension g over $\mathbb{C}$. One has the analytic representation

$$A(\mathbb{C}) = T(\mathbb{C})/M,$$

where T is a torus of dimension g, and M is a lattice of rank g. Set $M' := \mathrm{Hom}(T, \mathbb{G}_m)\ (= M_{A^\vee})$. Then we have the exact sequence

$$0 \longrightarrow 2i\pi M'^\vee \longrightarrow \Lambda \longrightarrow M \longrightarrow 0,$$

where Λ is the period lattice (of rank $2g$), and this sequence splits by choosing a branch of log. The inclusion $M \hookrightarrow T$ defines a pairing $M \times M' \to \mathbb{C}^\times$. A polarization, on the other hand, induces $M \to M'$. These data give rise to the pairing (*"multiplicative period"*)

$$q = (q_{ij})\colon M \otimes M \longrightarrow \mathbb{C}^\times$$

and $-\log|q|$ is a scalar product on $M_\mathbb{R}$.

5.2.2. *Over the p-adics.* Let K be a p-adic local field and A an abelian variety over K having (split) multiplicative reduction. One has a similar analytic representation [**Mu72a**]

$$A(K) = T(K)/M$$

Set again $M' := \mathrm{Hom}(T, \mathbb{G}_m)$. We have the following identifications

$$M \simeq \mathrm{H}_1(A^{\mathrm{rig}}, \mathbb{Z}) \quad \text{and} \quad M' \simeq \mathrm{H}_1(A^{\vee\,\mathrm{rig}}, \mathbb{Z}),$$

where A^{rig} denotes the associated rigid analytic variety over K to A. Then, just as in the complex case, one can construct a natural exact sequence:

$$0 \longrightarrow (2i\pi)_p M'^\vee \longrightarrow \Lambda \longrightarrow M \longrightarrow 0,$$

split by the choice of a branch of the p-adic logarithm $\log_p$, and a non-degenerate pairing

$$\int^p : \mathrm{H}^1_{\mathrm{dR}}(A) \times \Lambda \longrightarrow K[(2i\pi)_p].$$

(7) Dwork has developed a "dual theory" and a pairing given by residues, which play somehow the role of "p-adic cycles" (*cf.* [**Ro86**, 5]); however, this dual theory is an avatar of de Rham cohomology with proper supports and does not enjoy the properties of discreteness and horizontality that we are looking for.

(The construction involves one-motives and Frobenius [**And90**]). Restricted to $(2i\pi)_p M'^{\vee}$ which is in a canonical way a subgroup of $T_p(A_{\overline{K}})$, this pairing coincides with the Fontaine-Messing pairing.

5.2.3. *Degenerating abelian pencils.* Let $\mathcal{A} \to \mathbb{A}^1 \setminus \{0, \zeta_1, \ldots, \zeta_r\}$ be a pencil of polarized abelian varieties defined over a number field k. We consider an element ω of the relative $\mathrm{H}^1_{\mathrm{dR}}$ and its associated Picard-Fuchs differential equation $\mathcal{L}\omega = 0$. We assume that the connected component of identity of the special fibre at 0 of the Neron model of $\mathcal{A}$ is a split torus T over k. The dual abelian pencil then has the same property, with a torus T'. We set $M = \mathrm{Hom}(T', \mathbb{G}_m)$, $M' = \mathrm{Hom}(T, \mathbb{G}_m)$.

Let us fix $k \hookrightarrow \mathbb{C}$. The constant subsheaf of $R_1(f^{\mathrm{an}}_{\mathbb{C}})_*\mathbb{Z}$ in the neighborhood of 0 identifies with $2i\pi M'^{\vee}$: its fibre at any $z(\neq 0, \zeta_i)$ is the lattice $2i\pi M_z'^{\vee}$ attached to $\mathcal{A}_z$ (5.2.1). Similarly for M. On the other hand, the choice of a branch of log identifies $\mathrm{H}_{1\mathrm{B}}(\mathcal{A}_z, \mathbb{Z}) \simeq (R_1(f^{\mathrm{an}}_{\mathbb{C}})_*\mathbb{Z})_z$ with $\Lambda = 2i\pi M_z'^{\vee} \oplus M_z$. Then for any $\gamma \in 2i\pi M'^{\vee}$, $y(z) = \frac{1}{2i\pi}\int_{\gamma_z}\omega_z$ is a solution of $\mathcal{L}$ in $k[[z]]$.

More generally, for any $\gamma^* \in \Lambda$, $\frac{1}{2i\pi}\int_{\gamma_z^*}\omega_z$ is a solution of the Picard-Fuchs differential equation of the form $y(z)\frac{\log az^n}{2i\pi} + y^*(z)$, with $y^*(z) \in k[[z]]$, $a \in k$, $n \in \mathbb{Z}$. (In the case of the Legendre pencil, we have already met this situation with $y(z) = iF(\frac{1}{2}, \frac{1}{2}, 1; z), y^*(z) = 2F^*(\frac{1}{2}, \frac{1}{2}, 1; z), a = 2^{-8}, n = 2$).

Let now fix an embedding $k \hookrightarrow K \subset \mathbb{C}_p$. For any $z \neq 0$ close enough to 0, M and M' identify respectively with the lattices M_z, M'_z attached to $\mathcal{A}_z$ (5.2.2), and the choice of a determination of the p-adic logarithm identifies Λ with Λ_z. It turns out that the p-adic pairing $\int^p$ of 5.2.2 extends to a horizontal pairing over a punctured disk around 0, which is given by the "same" formula as in the complex case

$$\frac{1}{(2i\pi)_p}\int^p_{\gamma_z^*}\omega_z = y(z)\frac{\log_p az^n}{(2i\pi)_p} + y^*(z),$$

where $y(z), y^*(z)$ are the *same formal series as above, evaluated p-adically* (loc. cit.). The computation of 4.2.1 can be considered as a special case. This is also closely related to the work of T. Ichikawa on "universal periods" for Mumford curves [**Ic97**].

5.2.4. *Relation to the Fontaine-Messing periods.* Since the restriction of $\int^p$ to $(2i\pi)_p M_z'^{\vee} \subset T_p(\mathcal{A}_{z,\overline{K}})$ is the Fontaine-Messing pairing, we obtain $(2i\pi)_p y(z)$ as a Fontaine-Messing period of $\mathcal{A}_z$. More generally, $y(z)\,\mathrm{Log}_p\, az^n + (2i\pi)_p y^*(z)$ appears as a period (Log instead of log). We conclude that in this degenerating case, the Fontaine-Messing periods behave relatively well with respect to the Gauss-Manin connection. A similar observation applies to the pairing composed with θ (Hodge-Tate periods).

5.3. p-adic Betti lattices for abelian varieties with supersingular reduction.

5.3.1. Any supersingular abelian variety over $\overline{\mathbb{F}}_p$ is isogenous to a power of a supersingular elliptic curve. We refer to [**Wa69**, 4] for a detailed study of supersingular elliptic curves over finite fields (including the classification of isogeny classes and isomorphism classes, refining Deuring's classical work), and to [**LiO98**] for the higher dimensional case. Let us simply recall a few basic facts:

- there is exactly one $\overline{\mathbb{F}}_p$-isogeny class of supersingular elliptic curves.
- If A_0 is any supersingular elliptic curve over $\mathbb{F}_{p^n}$, then $\mathcal{D} := \mathrm{End}(A_0)_{\overline{\mathbb{F}}_p}$ is a maximal order in a quaternion algebra over $\mathbb{Q}$, ramified exactly at p, ∞; we set $\boldsymbol{D} = \mathcal{D}_{\mathbb{Q}}$.
- A_0 is $\overline{\mathbb{F}}_p$-isomorphic (but not necessarily $\mathbb{F}_{p^n}$-isomorphic) to an elliptic curve defined over $\mathbb{F}_{p^2}$.
- If $n = 1$, $\mathrm{End}(A_0) \otimes \mathbb{Q} = \mathbb{Q}[Fr_p]$, an imaginary quadratic field; if $p \neq 2$, this is $\mathbb{Q}[\sqrt{-p}]$. More precisely, the order $\mathrm{End}(A_0)$ can be either $\mathbb{Z}[Fr_p]$ or the maximal order in $\mathrm{End}(A_0) \otimes \mathbb{Q}$ (which coincide if $p \equiv 3 \pmod 4$). If $p = 2$, there is another possibility, namely $\mathrm{End}(A_0) = \mathbb{Z}[Fr_2] \simeq \mathbb{Z}[\sqrt{-1}]$.
- There exists a supersingular elliptic curve over $\mathbb{F}_p$ whose Frobenius endomorphism satisfies $(Fr_p)^2 = -p.\mathrm{id}$ and whose endomorphism algebra is the maximal order in $\mathbb{Q}[Fr_p]$. Such elliptic curves belong to a single $\mathbb{F}_p$-isogeny class.
- If $p \neq 2, 3$, there is exactly one $\mathbb{F}_p$-isogeny class of supersingular elliptic curves over $\mathbb{F}_p$ (but several $\overline{\mathbb{F}}_p$-isomorphism classes in general).

5.3.2. Let A_0 be a supersingular abelian variety of dimension g over $\mathbb{F}_{p^n}$. By functoriality, the elements of $\mathrm{End}\big((A_0)_{\overline{\mathbb{F}}_p}\big) \otimes \mathbb{Q} \simeq M_g(\boldsymbol{D})$ act linearly on the crystalline cohomology $\mathrm{H}^*_{\mathrm{crys}}\big((A_0)_{\overline{\mathbb{F}}_p}/\overline{\mathbb{Q}}_p\big) = \mathrm{H}^*_{\mathrm{crys}}\big((A_0)_{\overline{\mathbb{F}}_p}/\widehat{\mathbb{Z}_p^{\mathrm{ur}}}\big) \otimes_{\widehat{\mathbb{Z}_p^{\mathrm{ur}}}} \overline{\mathbb{Q}}_p$. In degree one, this provides an embedding of $M_g(\boldsymbol{D})$ into the endomorphism ring of $\mathrm{H}^1_{\mathrm{crys}}\big((A_0)_{\overline{\mathbb{F}}_p}/\overline{\mathbb{Q}}_p\big)$.

On the other hand, one has a canonical isomorphism

$$\bigwedge^2 \mathrm{H}^1_{\mathrm{crys}}\big((A_0)_{\overline{\mathbb{F}}_p}/\overline{\mathbb{Q}}_p\big) \simeq \mathrm{H}^2_{\mathrm{crys}}\big((A_0)_{\overline{\mathbb{F}}_p}/\overline{\mathbb{Q}}_p\big)$$

and a canonical $\mathbb{Q}$-structure in $\mathrm{H}^2_{\mathrm{crys}}\big((A_0)_{\overline{\mathbb{F}}_p}/\overline{\mathbb{Q}}_p\big) \simeq \overline{\mathbb{Q}}_p^{g(2g-1)}$ coming from the fact that the whole cohomology in degree 2 is generated by algebraic cycles (since $(A_0)_{\overline{\mathbb{F}}_p}$ is isogenous to the gth power of a supersingular elliptic curve, one reduces easily to the case $g = 2$, in which case this fact is well-known).

Let F be either $\overline{\mathbb{Q}}$ or a splitting number field for $\boldsymbol{D}$: $\boldsymbol{D} \otimes_{\mathbb{Q}} F \simeq M_2(F)$. In the latter case, this amounts to saying that F is totally imaginary, and for any place v above p, $[F_v : \mathbb{Q}_p]$ is even. We fix an embedding of F in $\overline{\mathbb{Q}}_p$.

Via this embedding, the $M_g(\boldsymbol{D})$-action extends to a F-linear $M_{2g}(F)$-action on $\mathrm{H}^1_{\mathrm{crys}}((A_0)_{\overline{\mathbb{F}}_p}/\overline{\mathbb{Q}}_p)$.

We consider an irreducible cyclic sub-$M_{2g}(F)$-module $M_{2g}(F).u \subset \mathrm{H}^1_{\mathrm{crys}}((A_0)_{\overline{\mathbb{F}}_p}/\overline{\mathbb{Q}}_p)$. Obviously, such submodules exist, and have F-dimension $2g$. We may and shall choose u in such a way that its exterior square of this F-space coincides with the canonical F-structure on $\mathrm{H}^2_{\mathrm{crys}}((A_0)_{\overline{\mathbb{F}}_p}/\overline{\mathbb{Q}}_p)$ (this can be achieved by replacing u by a suitable multiple).

Proposition 5.3.3. *Up to a homothety by a factor in $\sqrt{F^\times}$, the normalized $M_{2g}(F)$-submodule $M_{2g}(F)\cdot u \subset \mathrm{H}^1_{\mathrm{crys}}((A_0)_{\overline{\mathbb{F}}_p}/\overline{\mathbb{Q}}_p)$ depends only on $F \subset \overline{\mathbb{Q}}_p$, not on the choice of u. In particular, for $F = \overline{\mathbb{Q}}$, this defines a canonical $\overline{\mathbb{Q}}$-structure in $\mathrm{H}^1_{\mathrm{crys}}((A_0)_{\overline{\mathbb{F}}_p}/\overline{\mathbb{Q}}_p)$, stable under $\mathrm{End}((A_0)_{\overline{\mathbb{F}}_p})$.*

PROOF. Indeed, two such $M_{2g}(F)$-submodules are related by some $h \in GL_{2g}(\overline{\mathbb{Q}}_p)$, such that $\bigwedge^2(h) \in GL_{g(2g-1)}(F)$. Now $M_{2g}(F)\cdot hu = h(M_{2g}(F)\cdot u)$ implies that h normalizes $GL_{2g}(F)$. It follows that the image of h in $PGL_{2g}(\overline{\mathbb{Q}}_p)$ lies in $PGL_{2g}(F)$.

(When $g > 1$, the proposition is not surprising, since $\bigwedge^2 V$ is a faithful representation of $PSL(V)$ for any space V of dimension > 1). $\square$

Remarks 5.3.4. (a) A minimal choice for F is a splitting quadratic field $\mathbb{Q}(\sqrt{-d})$ (the splitting property amounts to saying that $d > 0$ and p ramifies or remains prime in $\mathbb{Q}(\sqrt{-d})$. A natural choice is $d = p$).

(b) Let $\mathcal{O}_F$ be the ring of integers of F. Then $\mathrm{End}((A_0)_{\overline{\mathbb{F}}_p}) \otimes \mathcal{O}_F$ is an order in $M_{2g}(F)$, and there is a full $(\mathrm{End}((A_0)_{\overline{\mathbb{F}}_p}) \otimes \mathcal{O}_F)$-lattice in $(M_{2g}(F)\cdot u)$. As an $\mathcal{O}_F$-module, it is projective of rank $2g$. If $F = \overline{\mathbb{Q}}$, *i.e.* $\mathcal{O}_F = \overline{\mathbb{Z}}$, then this module is *canonically determined* by the normalization of its top exterior power, taking into account the canonical isomorphism

$$\bigwedge^{2g} \mathrm{H}^1_{\mathrm{crys}}((A_0)_{\overline{\mathbb{F}}_p}/\overline{\mathbb{Z}}_p) \simeq \mathrm{H}^{2g}_{\mathrm{crys}}((A_0)_{\overline{\mathbb{F}}_p}/\overline{\mathbb{Z}}_p) \simeq \overline{\mathbb{Z}}_p,$$

the latter isomorphism coming from the trace map.

Hence there is a *canonical $\overline{\mathbb{Z}}$-structure* in the crystalline cohomology of supersingular abelian varieties (not used in the sequel).

5.3.5. Let A be an abelian variety defined over a subfield $k \subset \mathbb{C}_p$, which has good *supersingular reduction* A_0 over the residue field of k. There is a canonical embedding $\mathrm{End}(A) \hookrightarrow \mathrm{End}((A_0)_{\overline{\mathbb{F}}_p})$.

The canonical $\overline{\mathbb{Q}}$-subspace of $\mathrm{H}^1_{\mathrm{crys}}((A_0)_{\overline{\mathbb{F}}_p}/\overline{\mathbb{Q}}_p)$ defined in 5.3.3 will be denoted by

$$\mathrm{H}^1_{\mathrm{B}}(A_{\mathbb{C}_p}, \overline{\mathbb{Q}}) \subset \mathrm{H}^1_{\mathrm{crys}}((A_0)_{\overline{\mathbb{F}}_p}/\overline{\mathbb{Q}}_p)$$

because, as we shall see, it shares many properties with the Betti space $\mathrm{H}^1_{\mathrm{B}}(A_{\mathbb{C}}, \overline{\mathbb{Q}}) = \mathrm{H}^1_{\mathrm{B}}(A_{\mathbb{C}}, \mathbb{Z}) \otimes \overline{\mathbb{Q}}$ of a complex abelian variety $A_{\mathbb{C}}$). Its dual will be denoted by $\mathrm{H}_{1,\mathrm{B}}(A_{\mathbb{C}_p}, \overline{\mathbb{Q}})$.

Similarly, we shall write $\mathrm{H}^1_{\mathrm{B}}(A_{\mathbb{C}_p}, F)$ for the F-subspace described 5.3.3 (well-defined up to a homothety in $\sqrt{F^\times}$), and $\mathrm{H}_{1,\mathrm{B}}(A_{\mathbb{C}_p}, F)$ for its F-dual.

Assume that $k \subset \overline{\mathbb{Q}}_p$. There is a functorial isomorphism [**BeO83**]

$$\mathrm{H}^1_{\mathrm{dR}}(A) \otimes_k \overline{\mathbb{Q}}_p \simeq \mathrm{H}^1_{\mathrm{crys}}\big((A_0)_{\overline{\mathbb{F}}_p}/\overline{\mathbb{Q}}_p\big),$$

from which one derives a canonical isomorphism

$$\mathrm{H}^1_{\mathrm{dR}}(A) \otimes_k \overline{\mathbb{Q}}_p \simeq \mathrm{H}^1_{\mathrm{B}}(A_{\mathbb{C}_p}, \overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}_p,$$

which is functorial with respect to homomorphisms of abelian varieties with supersingular reduction
This isomorphism, which is far from being tautological, as we shall see, can be translated into a $\overline{\mathbb{Q}}_p$-valued pairing between $\mathrm{H}_{1\mathrm{B}}(A_{\mathbb{C}_p}, F)$ and $\mathrm{H}^1_{\mathrm{dR}}(A)$. This pairing is conveniently expressed in the form of a *"period matrix"* $\Omega = (\omega_{ij})$ with entries in $\overline{\mathbb{Q}}_p$, depending on the choice of a basis $(\gamma_i)_{i=1,\dots,2g}$ of $\mathrm{H}_{1\mathrm{B}}(A_{\mathbb{C}_p}, F)$ and a basis $(\omega_j)_{j=1,\dots,2g}$ of $\mathrm{H}^1_{\mathrm{dR}}(A)$ (if A is principally polarized, we choose symplectic bases).
Warning: these p-adic periods attached to abelian varieties with supersingular reduction have little to do with Fontaine-Messing periods (a motivic interpretation of these periods is proposed in [**And95**]).

5.3.6. *Horizontality.* If we let A move in a family A_z, $\mathrm{H}^1_{\mathrm{crys}}\big((A_0)_{\overline{\mathbb{F}}_p}/\overline{\mathbb{Q}}_p\big)$ may be identified with a space of horizontal sections of the de Rham cohomology localized in the disk parameterizing the liftings A_z of $(A_0)_{\overline{\mathbb{F}}_p}$ [**BeO83**]. Thus the period matrix $\Omega(z)$ is a fundamental matrix of solutions of the Gauss-Manin connection.

In the sequel of this subsection, we assume for simplicity that $g = 1$, *i.e.* A is an elliptic curve.

5.3.7. *CM periods, Γ_p-values, and transcendence.*

• Let A_0 be as before a supersingular elliptic curve over $\mathbb{F}_{p^n}$. Let A be a CM-lifting of A_0, *i.e.* an elliptic curve defined over some number field $k \subset \overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_p$ with residue field $\mathbb{F}_{p^n}$, such that $E = \mathrm{End}(A) \otimes \mathbb{Q}$ is quadratic. Note that E is ipso facto a subfield of $\boldsymbol{D}$. We denote by c the conductor of the order $\mathrm{End}(A)$ (in particular, $ce \in \mathrm{End}(A)$).

• Let (γ_1, γ_2) be any symplectic basis of $\mathrm{H}_{1\mathrm{B}}(A, F)$. We can write $ce.\gamma_1 = a\gamma_1 + b\gamma_2$ with $a, b \in F$, so that

$$\tau := \frac{\omega_{12}}{\omega_{11}} = \frac{-a + c\sqrt{-d}}{b} \in \mathbb{P}^1(FE).$$

In particular, $\tau \in \mathbb{P}^1(\overline{\mathbb{Q}})$. We conjecture the following converse, which is a p-adic analogue of T. Schneider's theorem [**Schn37**] (in the complex case):

Conjecture 5.3.8. *Assume that A is a lifting of A_0 defined over $\overline{\mathbb{Q}}$, and that $\tau = \frac{\omega_{12}}{\omega_{11}} \in \mathbb{P}^1(\overline{\mathbb{Q}})$. Then A has complex multiplication.*

• Let $E = \mathbb{Q}[\sqrt{-d}]$ and $E' = \mathbb{Q}[\sqrt{-d'}]$ be two distinct subfields of $\boldsymbol{D}$ $(d, d'$ squarefree; we do not exclude the case $d = d'$). According to [**Vig80**, II 1.4], $E \cap \boldsymbol{\mathcal{D}} = \mathcal{O}_E$, $E' \cap \boldsymbol{\mathcal{D}} = \mathcal{O}_{E'}$. The elements $e = \sqrt{-d} \in E$ and $e' = \sqrt{-d'} \in E'$ generate the algebra $\boldsymbol{D}$. We remark that $ee' + e'e$ commutes to e and e', hence is an integer $m \in \mathbb{Z}$ (an even integer if d or $d' \equiv 3 \pmod 4$, which is divisible by 4 if both $d, d' \equiv 3 \pmod 4$). We have $(ee' - e'e)^2 = m^2 - 4dd'$. Since $ee' - e'e$ anticommutes with e, it is not an integer, hence $|m| < 2\sqrt{dd'}$. On the other hand, the images of e and e' in $\boldsymbol{\mathcal{D}} \otimes \mathbb{F}_p$ commute ($ee' - e'e$ acts trivially on regular differentials in characteristic p), therefore p divides $4dd' - m^2$.

• On the other hand, there exists a unique symplectic basis of eigenvectors for e in $\mathrm{H}^1_{\mathrm{dR}}(A)$ of the form $(\omega_1, \omega_2 + \sigma\omega_1)$, with $\sigma \in k$. We get the relation

$$\omega_{22} + \sigma\omega_{12} = \tilde{\tau}(\omega_{21} + \sigma\omega_{11}), \text{ with } \tilde{\tau} = \frac{-a - c\sqrt{-d}}{b}.$$

Let now $E' = \mathbb{Q}[\sqrt{-d'}] \neq E$ be another subfield of $\boldsymbol{D}$, and assume that (γ_1, γ_2) is a basis of eigenvectors for E': $e'.\gamma_1 = \sqrt{-d'}\gamma_1$. We get $(ee' + e'e)\gamma_1 = \frac{2a\sqrt{-d'}}{c}\gamma_1 = m\gamma_1$, so that

$$\frac{\tilde{\tau}}{\tau} = \frac{m + 2\sqrt{dd'}}{m - 2\sqrt{dd'}} \in \mathbb{Q}[\sqrt{dd'}]$$

the sign of the square root being chosen in such a way that $p \mid 2\sqrt{dd'} - m$ (in the unramified case).

• The previous relations, together with $\det \Omega = 1$, show that the transcendence degree over $\overline{\mathbb{Q}}$ of the entries of Ω is at most one in the CM case.

• We now describe the p-adic number ω_{11} modulo $\overline{\mathbb{Q}}^\times$, keeping the notation of 4.6.2 and Theorem 4.6.3 (ϵ is the Dirichlet character, w the number of roots of unity, h the class number).

Theorem 5.3.9. *In case p does not ramify in the field of complex multiplications $\mathbb{Q}[\sqrt{-d}] = \mathrm{End}(A) \otimes \mathbb{Q}$, one has*

$$\omega_{11} \sim \prod_{u \in (\mathbb{Z}/d)^\times} \left(\Gamma_p\left\langle p\frac{u}{d}\right\rangle\right)^{-\epsilon(u)w/8h} \quad in\ \overline{\mathbb{Q}}_p^\times/\overline{\mathbb{Q}}^\times.$$

Examples. If A is the Legendre curve $X_{1/2}$ $(y^2 = x(x-1)(x-1/2))$ and $p \equiv 3 \pmod 4$, we find $\omega_{11} \sim \Gamma_p(1/4)$. Similarly, if $\mathrm{End}(A) \otimes \mathbb{Q} = \mathbb{Q}(\sqrt{-3})$ and $p \equiv 2 \pmod 3$, then $\omega_{11} \sim \Gamma_p(1/3)^{3/2}$.

PROOF. We may assume that the number field k $(\subset \overline{\mathbb{Q}}_p)$ is Galois over $\mathbb{Q}$. We denote by $k' = k \cap \mathbb{Q}_p$ the subfield of k fixed by the local Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. Let us consider the Weil restriction $B := \mathcal{R}_{k/k'}(A)$ (*cf.* [**BLR90**, 7.6]). This is an abelian variety over $k' \subset \mathbb{Q}_p$, with good, supersingular reduction at p (like A), and we have $B_k \simeq \prod_{\tau \in \mathrm{Gal}(k/k')} A^\tau$.

On the other hand, let ψ be any element of degree one in the Weil group (*i.e.* any lifting of the Frobenius element in $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$), and let Ψ_{A^τ} (*resp.* Ψ_B) be the corresponding semilinear endomorphism of $\mathrm{H}^1_{\mathrm{dR}}(A^\tau/k)\otimes\overline{\mathbb{Q}}_p$ (*resp.* $\mathrm{H}^1_{\mathrm{dR}}(B/k')\otimes\overline{\mathbb{Q}}_p$). If Φ_B denotes the canonical linear action induced by $Fr_{\mathbb{F}_p}$ on $\mathrm{H}^1_{\mathrm{crys}}(B_0/\mathbb{Q}_p)\otimes\overline{\mathbb{Q}}_p\simeq\mathrm{H}^1_{\mathrm{dR}}(B)_{\overline{\mathbb{Q}}_p}$, one has the formula $\Psi_B=\Phi_B\circ(\mathrm{id}\otimes\psi)$ (*cf.* [**BeO83**, 4]). Moreover, $\Psi_B=\bigoplus\Psi_{A^\tau}$ with respect to the decomposition $\mathrm{H}^1_{\mathrm{dR}}(B,\overline{\mathbb{Q}}_p)\simeq\bigoplus_\tau\mathrm{H}^1_{\mathrm{dR}}(A^\tau,\overline{\mathbb{Q}}_p)$.
If (ω_1,ω_2) is any symplectic basis of eigenvectors for $\mathbb{Q}[\sqrt{-d}]$ in $\mathrm{H}^1_{\mathrm{dR}}(A)_{\overline{\mathbb{Q}}}$, with $\omega_1\in\Omega^1(A)_{\overline{\mathbb{Q}}}$, we can write $\Psi_A(\omega_1)=\kappa.\omega_2$, where κ modulo $\overline{\mathbb{Q}}^\times$ is given by Ogus' formula in Theorem 4.6.3

$$\kappa\sim\prod_{u\in(\mathbb{Z}/d)^\times}\left(\Gamma_p\left\langle p\frac{u}{d}\right\rangle\right)^{-\epsilon(u)w/4h}\quad\text{in }\overline{\mathbb{Q}}_p^\times/\overline{\mathbb{Q}}^\times.$$

Because the conjugates A^τ are isogenous to each other over k (and since isogenies preserve the regular differential forms), we derive that the constant κ, resp. the "period" ω_{11}, is the same modulo $\overline{\mathbb{Q}}^\times$ for each of them. Therefore, for any $\omega\in\Omega^1(B)\subset\mathrm{H}^1_{\mathrm{crys}}(B_0/\mathbb{Q}_p)$, we can write $\Phi_B(\omega)=\Psi_B(\omega)=\kappa.\eta$ for some $\eta\in\mathrm{H}^1_{\mathrm{dR}}(B)_{\overline{\mathbb{Q}}}$ whose pairing with any element of $\mathrm{H}_{1\mathrm{B}}(B,\overline{\mathbb{Q}})$ belongs to $\omega_{21}.\overline{\mathbb{Q}}$. Since $\omega_{11}^{-1}\omega\in\mathrm{H}^1_{\mathrm{B}}(B,\overline{\mathbb{Q}})$, $\omega_{21}^{-1}\eta\in\mathrm{H}^1_{\mathrm{B}}(B,\overline{\mathbb{Q}})$, and since Φ_B respects $\mathrm{H}_{1\mathrm{B}}(B,\overline{\mathbb{Q}})$, we get $\kappa\sim\frac{\omega_{11}}{\omega_{21}}\sim\omega_{11}^2$, whence the result. □

Conjecture 5.3.10. *Assume that A is a lifting of A_0 defined over $\overline{\mathbb{Q}}$, and that p does not ramify* $\mathrm{End}(A)\otimes\mathbb{Q}$ *(which is obviously the case if A does not have complex multiplication). Then* $\mathrm{H}^1_{\mathrm{dR}}(B)_{\overline{\mathbb{Q}}}\neq\mathrm{H}^1_{1\mathrm{B}}(B,\overline{\mathbb{Q}})$.

In the absence of complex multiplication, this follows from the previous conjecture. In the presence of complex multiplication, it amounts to the transcendence of ω_{11}.
Taking into account 5.3.9, the conjecture would imply, for instance, the transcendence of the adic numbers $\Gamma_3(1/4)$, $\Gamma_7(1/4)$, $\Gamma_2(1/3)$, $\Gamma_5(1/3)$ (note, in contrast, that $\Gamma_5(1/4)$ and $\Gamma_7(1/3)$ are algebraic numbers, according to the Gross-Koblitz formula (2.6.4)).
• Actually, it may happen, in the ramified CM case, that $\mathrm{H}^1_{\mathrm{dR}}(A)_{\overline{\mathbb{Q}}}=\mathrm{H}^1_{1\mathrm{B}}(A,\overline{\mathbb{Q}})$.
Let us first notice that the argument given in the proof of theorem 5.3.9 still works in the ramified case, and allows to conclude that $\kappa\sim\omega_{11}^2$. The computation of κ (mod $\overline{\mathbb{Q}}^\times$) (for p odd) according to Ogus-Coleman-Urfels was explained in 4.6.4. In the example of an elliptic curve with complex multiplication by $\mathbb{Q}[\sqrt{-3}]$, we have seen that $\kappa\in\overline{\mathbb{Q}}$, hence $\omega_{11}\in\overline{\mathbb{Q}}$ and $\mathrm{H}^1_{\mathrm{dR}}(A)_{\overline{\mathbb{Q}}}=\mathrm{H}^1_{1\mathrm{B}}(A,\overline{\mathbb{Q}})$.

This is also the case for an elliptic curve with complex multiplication by $\mathbb{Q}[\sqrt{-3n}]$ if the Legendre symbol $(\frac{n}{3})$ is 1, but probably not in general if $(\frac{n}{3})=-1$ (although this happens for $n=8$, after the last example in 4.6.4).

5.3.11. $L_{\frac{1}{2},\frac{1}{2},1}$ *in a supersingular disk.* For concreteness, we consider the special case of the supersingular Legendre elliptic curve with parameter $z = 1/2$ in characteristic $p \equiv 3 \pmod 4$. A basis of solutions of the hypergeometric differential operator $L_{\frac{1}{2},\frac{1}{2},1} = HGDO(\frac{1}{2},\frac{1}{2},1)$ in the supersingular disk $\mathrm{D}(1/2, 1^-)$ is given by

$$F\Big(\frac{1}{4},\frac{1}{4},\frac{1}{2};(1-2z)^2\Big),\quad (1-2z)F\Big(\frac{3}{4},\frac{3}{4},\frac{3}{2};(1-2z)^2\Big).$$

- Let us consider the symplectic basis $\omega_1 = [\frac{dx}{2y}], \omega_2 = [\frac{(2x-1)dx}{4y}]$ of $\mathcal{M}$ (de Rham cohomology). At $z = 1/2$, this is a basis of eigenvectors for the action of $E' := \mathrm{End}(X_{1/2}) \otimes \mathbb{Q} = \mathbb{Q}(\sqrt{-1})$. The Gauss-Manin connection satisfies

$$\omega_2 = 2z(z-1)\nabla\Big(\frac{d}{dz}\Big)\omega_1 + \frac{4z-5}{6}\omega_1.$$

The fundamental solution matrix Y of the Gauss-Manin connection (expressed in the basis (ω_1, ω_2)), normalized by $Y(1/2) = \mathrm{id}$, is then given by

$$y_{11} = F\Big(\frac{1}{4},\frac{1}{4},\frac{1}{2};(1-2z)^2\Big) + \frac{1}{2}(1-2z)F\Big(\frac{3}{4},\frac{3}{4},\frac{3}{2};(1-2z)^2\Big),$$
$$y_{12} = (1-2z)F\Big(\frac{3}{4},\frac{3}{4},\frac{3}{2};(1-2z)^2\Big),$$
$$y_{2i} = 2z(z-1)\frac{dy_{1i}}{dz} + \frac{4z-5}{6}y_{1i}, \quad i = 1, 2.$$

- We choose a symplectic basis (γ_1, γ_2) of eigenvectors for E' in $\mathrm{H}^1_{\mathrm{1B}}(X_{1/2}, F)$ (note that $\mathbb{Q}(\sqrt{-1})$ is always contained in F). We have $\mathrm{H}^1_{\mathrm{1B}}(X_{1/2}, F) = \mathrm{H}^1_{\mathrm{1B}}(X_z, F)$ for any point z in the supersingular disk $\mathrm{D}(1/2, 1^-)$. Because the period matrix $\Omega(z)$ is another solution matrix of the Gauss-Manin connection, we have the relation $Y(z) = \Omega(z).\Omega(1/2)^{-1}$.

  Let us consider a CM-point ζ in the supersingular disk $\mathrm{D}(1/2, 1^-)$: $\mathrm{End}(X_\zeta) = E = \mathbb{Q}(\sqrt{-d})$ (and $E \neq E'$ in $\boldsymbol{D}$).

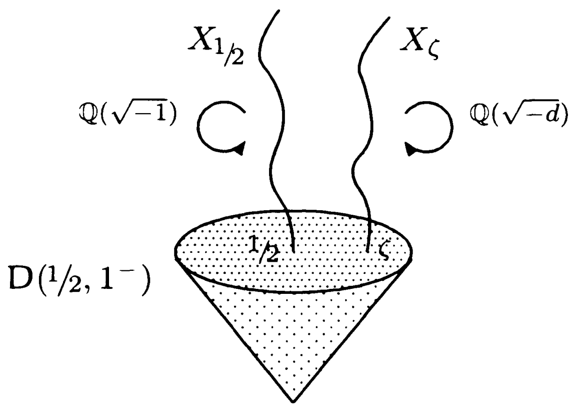

FIGURE 10

Let σ and m be as before (in the present case, the integer m is even: $m = 2n$). Our period relations can be easily expressed in terms of $Y(\zeta)$. Apart from the obvious relation $\det Y(\zeta) = 1$, we find a relation [8]

$$\left.\frac{y_{11}(y_{22}+\sigma y_{12})}{y_{12}(y_{21}+\sigma y_{11})}\right|_{z=\zeta} = \frac{\tilde{\tau}}{\tau} = \frac{n+\sqrt{d}}{n-\sqrt{d}}.$$

Changing the viewpoint, one can consider ζ as fixed, and vary the prime $p \neq 2$ (or more precisely the place of $\mathbb{Q}(\zeta, \sqrt{-1}, \sqrt{-d})$). This relation between values of p-adic hypergeometric functions at ζ holds whenever $|\zeta - 1/2|_p < 1$; it depends on p via the integer n ($|n| < \sqrt{d}$, $p \mid d - n^2$). A relation of the same kind holds at the places at infinity (derived along similar lines, using the usual Betti lattices). As a specific example, we can take $p = 3, \zeta =$ a primitive 6th root of unity. Then X_ζ has complex multiplication by $\mathbb{Z}[\zeta]$, $\sigma = \frac{-\zeta^2}{2(1+\zeta)}$, $n = 0$ and $\frac{\tilde{\tau}}{\tau} = -1$.

- On the other hand, one can combine Theorem 5.3.9 and the formula $Y(\zeta) = \Omega(\zeta).\Omega(1/2)^{-1}$ in order to express the value mod $\overline{\mathbb{Q}}^\times$ of the p-adic hypergeometric functions y_{ij} in terms of Γ_p (*supersingular avatar of Young's formulas*).
  For instance, if $p = 7, \zeta = 2(\sqrt{2} - 1)$ (complex multiplication by $\mathbb{Z}[\sqrt{-2}]$), one has $\omega_{11}(\zeta) \sim (\Gamma_7(1/8)\Gamma_7(3/8))^{1/2}$, $\omega_{11}(1/2) \sim \Gamma_7(1/4)$, from which one derives the 7-adic evaluation

$$F\left(\frac{3}{4}, \frac{3}{4}, \frac{3}{2}; 5 - 4\sqrt{2}\right) \sim \left(\Gamma_7(1/8)\Gamma_7(3/8)\right)^{1/2}\Gamma_7(1/4)^{-1}.$$

5.4. Conclusion and vista.

We have seen many instances of the following scenario: some familiar notion or phenomenon from the complex realm shows itself in two different aspects; each of these aspects has a natural p-adic counterpart; these counterparts are not complementary aspects of the same p-adic entity, but belong to totally different theories.

In II, we shall see that the theory of p-adic period mappings is a privileged field where this semantic splitting process integrates into an harmonious picture, where the complementarity between the p-adic counterparts is restored at a deeper level.

The p-adic period mappings which we shall deal with relate some deformation spaces of p-divisible groups to certain grassmannians. It will turn out that they can be described by quotients of analytic solutions of the Gauss-Manin connection, just as in the complex case. In particular, at the *modular level*, the Fontaine-Messing periods (which links up directly the étale and crystalline representations of p-divisible groups) go offstage or remain at the background.

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[8] F. Beukers has also found these relations independently, by a different method.

On the other hand, we shall see that the differential equations studied in section 1 (*resp.* 2) arise in the context of deformations of supersingular (resp. ordinary) p-divisible groups.