CHAPTER 9

Boundedness of exceptional complements

9.1. The main construction

In this chapter we discuss some boundedness results for exceptional log surfaces. The main results are Theorems 9.1.7, 9.1.11 and 9.1.12. Fix the following notation. Let (X, B) be a projective log surface such that

(i) $K_X + B$ is lc;

(ii) $-(K_X + B)$ is nef;

(iii) the coefficients of B are standard or $d_i \geq 6/7$ (i.e. $B \in \Phi_m$);

(iv) (X, B) is exceptional, i.e., any $\mathbb{Q}$-complement of $K_X + B$ is klt;

(v) there is a boundary $B^\triangledown \leq B$ such that $K_X + B^\triangledown$ klt and $-(K_X + B^\triangledown)$ is nef and big.

By Corollary 8.4.2, (iv) is equivalent to

(iv)' there are no regular nonklt complements.

Note also that by Theorem 8.3.1 and Corollary 8.3.2, $K_X + B$ is klt.

9.1.1. If $K_X + B$ is $(1/7)$-lt and $b_i < 6/7$, $\forall i$, then by Theorem 5.2.1, X belongs to a finite number of families. In this case $b_i \in \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{6}{7}\}$, so (X, B) is bounded. Therefore we have a finite number of exceptional values of $\mathrm{compl}(X, B)$ in this case.

9.1.2. Now we assume that there is at least one (exceptional or not) divisor with $a(\cdot, B) \leq -1 + 1/7$. Following Shokurov [**Sh3**] we construct a model of (X, B) with $\rho = 1$. Similar birational modifications were used in [**KeM**] and was called the *hunt for a tiger*. Let $\mu \colon \widehat{X} \to X$ be the blowup of all exceptional divisors with $a(\cdot, B) \leq -6/7$ (see Lemma 3.1.9 and Proposition 3.1.2). Consider the crepant pull back

$$K_{\widehat{X}} + \widehat{B} = \mu^*(K_X + B), \quad \text{with} \quad \mu_* \widehat{B} = B.$$

Then $K_{\widehat{X}} + \widehat{B}$ is $(1/7)$-lt. By construction, $\widehat{B} \in \Phi_m$. As in 8.3.3 we can construct a boundary $\widehat{B}^\triangledown \leq \widehat{B}$ on $\widehat{X}$ such that $K_{\widehat{X}} + \widehat{B}^\triangledown$ is klt and $-(K_{\widehat{X}} + \widehat{B}^\triangledown)$ is nef and big. So on $\widehat{X}$ all our assumptions (i) – (v) hold and moreover,

(i)' $K_{\widehat{X}} + \widehat{B}$ is $(1/7)$-lt and $\left\lfloor \frac{7}{6} \widehat{B} \right\rfloor \neq 0$.

By Lemma 8.3.4 the Mori cone $\overline{NE}(\widehat{X})$ is polyhedral and generated by contractible extremal curves. Moreover, if an extremal ray R on $\widehat{X}$ is birational, then the contraction preserves assumptions (i)–(iii), (v) (see 1.1.6).

Write $\widehat{B} = \sum b_i \widehat{B}_i$. Define the boundary $\widehat{D} = \sum d_i \widehat{B}_i$, where

$$(9.1) \qquad\qquad d_i = \begin{cases} 1 & \text{if } b_i \geq 6/7; \\ b_i & \text{otherwise.} \end{cases}$$

In particular, $\mathrm{Supp}\,\widehat{D} = \mathrm{Supp}\,\widehat{B}$. Put

$$\widehat{C} := \left\lfloor \frac{7}{6}\widehat{B} \right\rfloor = \sum_{b_i \geq 6/7} \widehat{B}_i \qquad \text{and} \qquad \widehat{F} := \widehat{D} - \widehat{C} = \sum_{b_i < 6/7} b_i \widehat{B}_i.$$

Then $\widehat{D} = \widehat{C} + \widehat{F}$, $\widehat{C} = \left\lfloor \widehat{D} \right\rfloor$ and $\widehat{F} \in \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}\}$. Since $\widehat{D} \in \Phi_{\mathbf{sm}}$, so $\widehat{D} \in \mathcal{P}_n$ for all $n \in \mathbb{N}$ (see 4.2.1).

LEMMA 9.1.3. $K_{\widehat{X}} + \widehat{D}$ *is lc.*

PROOF. By Theorem 6.0.6 the log divisor $K_{\widehat{X}} + \widehat{B}$ has a regular n-complement $\widehat{B}^+$ near each point $x \in \widehat{X}$. Then $K_{\widehat{X}} + \widehat{B}^+$ is lc and $\widehat{B}^+ \geq \widehat{B}$ (see 4.2.8). Moreover, if $b_i \geq 6/7$, then by definition of complements $b_i^+ \geq \frac{1}{n}\lfloor (n+1)b_i \rfloor \geq 1$. Therefore $\widehat{B}^+ \geq \widehat{D}$. This gives that $K_{\widehat{X}} + \widehat{D}$ is also lc at x. $\qquad\square$

Recall that we have assumed $\left\lfloor \widehat{D} \right\rfloor \neq 0$. By the Inductive Theorem 8.3.1, $-(K_{\widehat{X}} + \widehat{D})$ is not nef. Assume that $\rho(\widehat{X}) > 1$. By Lemma 8.3.4 the cone $\overline{NE}(\widehat{X})$ is polyhedral and generated by contractible curves. Therefore there exists an extremal ray R which is positive with respect to $K_{\widehat{X}} + \widehat{D}$. Recall that $-(K_{\widehat{X}} + \widehat{B})$ is nef, so $(K_{\widehat{X}} + \widehat{B}) \cdot R \leq 0$.

LEMMA 9.1.4. *The contraction of R is birational.*

PROOF. Indeed, otherwise $\rho(\widehat{X}) = 2$ and we have an extremal contraction $\widehat{X} \to Z$ onto a curve, positive with respect to $K_{\widehat{X}} + \widehat{D}$. Let F be a general fiber. Then

$$\deg(K_F + \widehat{B}|_F) = (K_{\widehat{X}} + \widehat{B}) \cdot F \leq 0,$$

and by 4.1.11 and by 4.1.12, $K_F + \widehat{B}|_F$ is 1, 2, 3, 4, or 6-complementary. But then (as in the proof of Lemma 9.1.3) $(\widehat{B}|_F)^+ \geq \widehat{D}|_F$. Hence

$$0 = \deg(K_F + \widehat{B}^+|_F) \geq \deg(K_F + \widehat{D}|_F) = (K_{\widehat{X}} + \widehat{D}) \cdot F > 0,$$

which is a contradiction. $\qquad\square$

Thus R is of birational type and generated by a (rational) curve, say E.

LEMMA 9.1.5. E *is not a component of* $\left\lfloor \widehat{D} \right\rfloor$.

PROOF. By Theorem 6.0.6 and by Corollary 7.0.10 there exists a regular complement of $K_{\widehat{X}} + \widehat{B}$ near E. As in the proof of Lemma 9.1.4, $\widehat{B}^+ \geq \widehat{D}$. If E is a component of $\lfloor \widehat{D} \rfloor$, then $0 = (K_{\widehat{X}} + \widehat{B}^+) \cdot E \geq (K_{\widehat{X}} + \widehat{D}) \cdot E > 0$, a contradiction. $\square$

By Lemma 9.1.6 below we have that if the contraction $\widehat{X} \to \widehat{X}'$ of E is birational, then $K_{\widehat{X}'} + \widehat{B}'$ is also $(1/7)$-lt. Thus we can make a birational contraction $\widehat{X} \to \widehat{X}'$. On $\widehat{X}'$ all our assumptions (i)', (ii)–(v) hold. We replace $\widehat{X}$ with $\widehat{X}'$.

LEMMA 9.1.6 (cf. e.g., [**KM**, 3.38]). *Let* (X, B) *be an* ε-lt *pair and* $f \colon X \to X'$ *a divisorial contraction of an* $(K_X + B)$-*nonpositive extremal ray. Assume that* X *is* $\mathbb{Q}$-*factorial and* f *does not contract components of* B *with coefficients* $\geq 1 - \varepsilon$. *Then* $K_{X'} + f(B)$ *is again* ε-lt.

PROOF. Denote $B' := f(B)$ and let E be the exceptional divisor. Then we can write $K_X + B - \alpha E = f^*(K_{X'} + B')$, where $\alpha \geq 0$. Since $K_X + B - \alpha E$ is ε-lt, by 1.1.6 it is sufficient to show only that f does not contract components of $B - \alpha E$ with coefficients $\geq 1 - \varepsilon$. $\square$

After a number of such birational contractions $\varphi \colon \widehat{X} \to \widetilde{X}$ we obtain the diagram

$$
\begin{array}{ccc}
 & \widehat{X} & \\
{}^{\mu}\swarrow & & \searrow^{\varphi} \\
X & & \widetilde{X}
\end{array}
$$

(9.2)

where $\rho(\widetilde{X}) = 1$.

By construction, $K_{\widetilde{X}} + \widetilde{B}$ is klt and $-(K_{\widetilde{X}} + \widetilde{B})$ is nef. Moreover, there is a boundary $\widetilde{B}^{\triangledown} \leq \widetilde{B}$ such that $K_{\widetilde{X}} + \widetilde{B}^{\triangledown}$ is klt and $-\left(K_{\widetilde{X}} + \widetilde{B}^{\triangledown}\right)$ is nef and big. By the Inductive Theorem 8.3.1, $K_{\widetilde{X}} + \widetilde{D}$ is ample. Since $\rho(\widetilde{X}) = 1$, $-K_{\widetilde{X}}$ is ample. Applying Alexeev's Theorem 5.2.1 to $(\widetilde{X}, 0)$, we obtain that families of such $\widetilde{X}$ are bounded. Further, let $\widetilde{B}_i$ be a component of $\widetilde{B} = \sum b_i \widetilde{B}_i$. Then $H \cdot \widetilde{B}_i \leq -2K_{\widetilde{X}} \cdot H$ for any very ample H. This gives that $\widetilde{B}_i$ lies in a finite number of algebraic families. Therefore $\operatorname{Supp}\widetilde{B}$ is bounded and we may assume that $(\widetilde{X}, \operatorname{Supp}\widetilde{B})$ is fixed. Since $\widetilde{B} \in \Phi_{\mathbf{m}}$, there is only a finite number of possibilities for coefficients $b_i \leq 6/7$. Therefore we have the following.

THEOREM 9.1.7 ([**Sh3**]). *Let* (X, B) *be a projective log surface such that* $K_X + B$ *is lc and* $-(K_X + B)$ *is nef. Assume that the coefficients of* B *are standard or* $d_i \geq 6/7$ *(i.e.,* $B \in \Phi_{\mathbf{m}}$). *Furthermore, assume that there is a boundary* $B^{\triangledown} \leq B$ *such that* $K_X + B^{\triangledown}$ *klt and* $-(K_X + B^{\triangledown})$ *is nef and big. Then either*

(i) (X, B) *is nonexceptional, and then there exists a regular nonklt complement of* $K_X + B$, *or*

(ii) $(X, \operatorname{Supp}B)$ *belongs to a finite number of algebraic families.*

LEMMA 9.1.8. *Let* $(S \ni o, \Delta = \sum \delta_i \Delta_i)$ *be a log surface germ. Assume that* $K_S + \Delta$ *is* $(1/7)$*-lt,* $\delta_i \geq 6/7$, $\forall i$ *and* $\mathrm{Supp}\Delta$ *is singular at* o. *Then* $(S \ni o)$ *is smooth,* Δ *has two (analytic) components at* o *and* $\delta_1 + \delta_2 < 13/7$.

PROOF. By Theorem 6.0.6 there is a regular complement $K_S + \Delta^+$. Clearly, $\Delta^+ \geq \lceil \Delta \rceil$. Therefore $K_S + \lceil \Delta \rceil$ is lc. By Theorem 2.1.3 $\lceil \Delta \rceil$ has exactly two (analytic) components at o and we have an analytic isomorphism

$$(S, \lceil \Delta \rceil) \simeq (\mathbb{C}^2, \{xy = 0\})/\mathbb{Z}_m(1, q), \qquad \gcd(q, m) = 1, \quad 1 \leq q \leq m - 1.$$

Taking the corresponding weighted blowup, one can compute

$$-1 + \frac{1}{7} < a(E, \Delta) = -1 + \frac{1+q}{m} - \frac{\delta_1 + q\delta_2}{m}$$

(see Lemma 3.2.1). This yields

$$(9.3) \qquad \frac{1}{7} < \frac{(1 - \delta_1) + q(1 - \delta_2)}{m} \leq \frac{1+q}{7m} \leq \frac{1}{7},$$

a contradiction. Hence $m = 1$ and $S \ni o$ is smooth. The rest is obvious. □

REMARK 9.1.9. Notation as in Lemma 9.1.8. If we replace $(1/7)$-lt condition with $(1/7)$-lc one, then in (9.3) equalities may hold. Then we have two cases

   (i) $(S \ni o)$ is smooth, Δ has two (analytic) components at o and $\delta_1 + \delta_2 \leq 13/7$;
   (ii) $(S \ni o)$ is Du Val of type A_{m-1}, Δ has two (analytic) components at o and $\delta_1 = \delta_2 = 6/7$.

We claim that the coefficients of $\widetilde{B}$ are bounded from above by an absolute constant $c < 1$ (cf. [Kol]).

LEMMA 9.1.10. *Let* $(S, \Delta = \sum \delta_i \Delta_i)$ *be a projective log surface with* $\rho(S) = 1$ *and* $\Theta_1, \ldots, \Theta_m$ *irreducible curves on* X. *Assume that* $\delta_i \in \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}, \frac{6}{7}\}$. *Then there is a constant* $c < 1$ *such that* $\theta_i < c$ *for all* $\theta_1, \ldots, \theta_m$ *whenever*

   (i) $\theta_i \geq 6/7$;
   (ii) $K_S + \Delta + \sum \theta_i \Theta_i$ *is* $(1/7)$*-lc;*
   (iii) $-(K_S + \Delta + \sum \theta_i \Theta_i)$ *is nef;*
   (iv) $K_S + \Delta + \sum \theta_i \Theta_i$ *has no regular complements.*

PROOF. Assume that there is a sequence $\Theta^{(k)}$ of boundaries as in our lemma such that $\theta_i^{(k)} \longrightarrow 1$ for some (fixed) i. By Lemma 9.1.8, Remark 9.1.9 and because $\rho(S) = 1$ we have $\theta_i + \theta_j \leq 13/7$, for all $j \neq i$. Then $\theta_j^{(k)} \longrightarrow 6/7$ for $j \neq i$. Denote $\Theta^0 := \lim \Theta^{(k)}$. By the above, Θ^0 is a $\mathbb{Q}$-boundary and $\lfloor \Theta^0 \rfloor \neq 0$. Moreover $K_S + \Delta + \Theta^0$ is lc and $-(K_S + \Delta + \Theta^0)$ is nef (see e.g., [Ut, 2.17]). By the Inductive Theorem 8.3.1 there is a regular n-complement $K_S + \Delta^+ + \Theta^{0+}$ of $K_S + \Delta + \Theta^0$ and this is an n-complement of $K_S + \Delta + \Theta^{(k)}$ if $\|\Theta^{(k)} - \Theta^0\| \ll 1$ (see Proposition 4.1.7). A contradiction with our assumptions. □

The above lemma gives that the coefficients of $\widetilde{B}$ are bounded from above by an absolute constant $c < 1$. So are coefficients of $\widehat{B}$ because $\varphi \colon \widehat{X} \to \widetilde{X}$ does not

contract components of $\widehat{B}$ with coefficients $\geq 6/7$. Now we can apply Theorem 5.2.1 to $(\widehat{X}, \widehat{B})$. As above we obtain that $\{(\widehat{X}, \mathrm{Supp}\widehat{B})\}$ is bounded. Finally, $(\widehat{X}, \widehat{B}^{\triangledown})$ is a log del Pezzo and $K_{\widehat{X}} + \widehat{B}^{\triangledown}$ is klt. By Lemma 8.3.4 the Mori cone $\overline{NE}(\widehat{X})$ is polyhedral. Therefore there is only a finite number of contractions $\widehat{X} \to X$ and $\{(X, \mathrm{Supp}B)\}$ is bounded. Moreover, if $B \in \Phi_{\mathbf{sm}}$, then by Lemma 9.1.10 there are only a finite number of possibilities for coefficients of B. This shows the following results. Note that Shokurov [**Sh3**] proved them under weaker assumptions.

THEOREM 9.1.11 ([**Sh3**]). *Let (X, B) be a projective log surface such that $K_X + B$ is lc and $-(K_X + B)$ is nef. Assume that the coefficients of B are standard (i.e., $B \in \Phi_{\mathbf{sm}}$). Furthermore, assume that there is a boundary $B^{\triangledown} \leq B$ such that $K_X + B^{\triangledown}$ klt and $-(K_X + B^{\triangledown})$ is nef and big. Then either*

(i) *(X, B) is nonexceptional, and then there exists a regular nonklt complement of $K_X + B$, or*

(ii) *(X, B) belongs to a finite number of algebraic families.*

THEOREM 9.1.12 ([**Sh3**]). *Notation as in 9.1.11. There is an absolute constant Const such that $K_X + B$ is n-complementary for some $n \leq$ Const.*

9.2. Corollaries: Case of log Enriques surfaces

Note in many applications of the above theorems we do not need condition (v) of 9.1:

THEOREM 9.2.1 (cf. [**Bl**], [**Z**], [**Z1**]). *Let (X, B) be a log Enriques surface. Assume that $B \in \Phi_{\mathbf{m}}$. Then there is an absolute constant Const such that $n(K_X + B) \sim 0$ for some $n \leq$ Const.*

PROOF. If X is rational, then we can omit (v) of 9.1 by Proposition 8.4.6. Otherwise there is a regular complement by Proposition 9.2.2 below. $\square$

PROPOSITION 9.2.2 (cf. Th. 8.2.1). *Let (X, B) be a log Enriques surface. Assume that $B \in \Phi_{\mathbf{m}}$ and X is nonrational. Then $n(K_X + B) \sim 0$ for some $n \in \mathfrak{R}_2$. In particular, nB is integral and $B \in \Phi_{\mathbf{sm}}$.*

PROOF. By Theorem 8.2.1 we may assume that $K_X + B$ is klt. First consider the case when $B \neq 0$. By Corollary 8.2.3 the pair (X, B) has at worst canonical singularities. Replace (X, B) with the minimal resolution and the crepant pull back. It is easy to see that this preserves all the assumptions. Run K_X-MMP. Clearly, whole B cannot be contracted. At the end we get an extremal contraction $f : X \to Z$ to a smooth curve Z with $p_a(Z) \geq 1$. Moreover X is smooth, so X is a ruled surface. Thus $h^1(X, \mathcal{O}_X) = p_a(Z) \geq 1$. On each step we can pull back regular complement, so it is sufficient to prove our statement for this new X. By Lemma 8.2.2 all the components of B are horizontal. Further, if $B_i^2 > 0$ for some component $B_i \subset \mathrm{Supp}B$, then $-(K_X + B - \varepsilon B_i)$ is nef and big, a contradiction (see Lemma 5.4.1). On the other hand, if $B_i^2 < 0$, then $(K_X + B + \varepsilon B_i) \cdot B_i < 0$. Thus B_i is an extremal rational curve, a contradiction with $p_a(Z) \geq 1$. Therefore

$B_i^2 = 0$ for all components $B_i \subset \operatorname{Supp} B$. Since $\rho(X) = 2$, this gives that all the B_i are numerically proportional and $B_i \cap B_j = \varnothing$ for $i \neq j$. As in (8.1) we have $(K_X + B_i) \cdot B_i = (K_X + B) \cdot B_i = 0$. Hence $p_a(B_i) = p_a(Z) = 1$ and all B_i and Z are smooth elliptic curves. Moreover, $K_X^2 = K_X \cdot B_i = 0$. Since $B_i^2 = 0$, B_i generate an extremal ray of $\overline{NE}(X) \subset \mathbb{R}^2$ (see Proposition 11.2.1). In particular, X contains no curves with negative self-intersections. Restricting $B = \sum b_i B_i$ to a general fiber F of the rulling $f \colon X \to Z$ we get a numerically trivial divisor $K_F + B|_F$ with $B|_F \in \Phi_{\mathbf{m}}$. Obviously, $B|_F \in \Phi_{\mathbf{sm}}$ and $B \in \Phi_{\mathbf{sm}}$. Thus we can write

$$B = \sum_i (1 - 1/m_i) B_i, \quad \text{where} \quad m_i \in \mathbb{N}.$$

CLAIM. *There is a structure of an elliptic fibration* $g \colon X \to \mathbb{P}^1$. *All components of B are contained in fibers of g.*

PROOF. If B has at least two components B_i and B_j, then $0 \equiv K_X + B \equiv K_X + B + \varepsilon B_i - \varepsilon' B_j$ is klt and numerically trivial for some small positive $\varepsilon, \varepsilon' \in \mathbb{Q}$. By the Log Abundance Theorem [**Ut**, Ch. 11], we have $K_X + B \sim_{\mathbb{Q}} K_X + B + \varepsilon B_i - \varepsilon' B_j \sim_{\mathbb{Q}} 0$, i.e., $\varepsilon B_i \sim_{\mathbb{Q}} \varepsilon' B_j$. This gives an elliptic pencil. Assume that $B = (1 - 1/m_1) B_1$, where B_1 is an irreducible smooth elliptic curve. Clearly, $3 \leq B_1 \cdot F \leq 4$. If the rulling $X \to Z$ corresponds to a vector bundle of splitting type, then there is a section B_0 such that $B_0^2 = 0$. Again we can apply the Log Abundance Theorem to $K_X + B + \varepsilon B_0 - \varepsilon' B_1$.

Finally, consider the case when X corresponds to an indecomposable vector bundle $\mathcal{E}$. By Example 8.1.1 the degree of $\mathcal{E}$ is odd. Then up multiplication by an invertible sheaf, $\mathcal{E}$ is a nontrivial extension

$$0 \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_Z(P) \longrightarrow 0,$$

where P is a point on Z (see e.g., [**Ha**, Ch. V, §2]). Let C be a section corresponding to the above exact sequence. Denote also $F_P := f^{-1}(P)$. It is easy to see that $-K_X \sim 2C - F_P$ is nef but not big (see [**Ha**, Ch. V, 2.10, 2.21]). It is sufficient to show the existence of an effective divisor $B_0 \equiv -K_X \equiv 2C - F$. Indeed, then B_0 is irreducible, $B_0 \neq B_1$ and the same arguments as above gives an elliptic pencil. To find such B_0 we consider the linear system $2C$. Since $C^2 = 1$, C is ample. By Riemann-Roch and Kodaira vanishing, we have

$$h^i(X, \mathcal{O}_X(C)) = h^i(X, \mathcal{O}_X(2C)) = 0, \quad i > 0,$$

$$h^0(X, \mathcal{O}_X(C)) = 1, \qquad h^0(X, \mathcal{O}_X(2C)) = 3.$$

Thus the surjection

$$H^0(X, \mathcal{O}_X(2C)) \longrightarrow H^0(C, \mathcal{O}_C(2C))$$

is surjective. Therefore the linear system $|2C|$ is base point free and determines a finite morphism $\phi \colon X \to \mathbb{P}^2$ of degree 4. Since $2C \cdot F = 2$, the images of the fibers F form a pencil of curves of degree ≤ 2. Since any pencil of conics contains a degenerate member, we derive that $\phi(F_0)$ is a line for at least one fiber

$F_0 = f^{-1}(P_0)$. Then the residual curve $\phi^{-1}(F_0) - F_0$ belongs to the linear system $|2C - F_0|$. The claim is proved. $\square$

EXERCISE 9.2.3 ([**Sh3**, §2]). As above, let $X = \mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is an indecomposable vector bundle of odd degree. Describe the structure of elliptic fibration on X and multiple fibers explicitly.

Going back to the proof of Proposition 9.2.2, assume that $\sum B_i$ contains also all multiple fibers of the fibration g (we alow $m_i = 1$). By the canonical bundle formula (see e.g. [**BPV**, Ch. V, §12])

$$K_X = -2L + \sum(r_i - 1)B_i,$$

where L is a general fiber and r_i is the multiplicity of B_i. This gives us

$$(9.4) \qquad K_X + B = -2L + \sum\left(1 - \frac{1}{r_i m_i}\right) r_i B_i.$$

Since $r_i B_i \sim L$,

$$\sum\left(1 - \frac{1}{r_i m_i}\right) = 2.$$

Hence the collection $(r_1 m_1, r_2 m_2, \dots)$ is one of the following (see 4.1.12):

$$(2,2,2,2), \quad (3,3,3), \quad (2,4,4), \quad (2,3,6).$$

By (9.4) we see that $n(K_X + B) \sim 0$ for $n = 2, 3, 4,$ and 6, respectively.

Finally, let $B = 0$. Replace X with its minimal resolution and let B be the crepant pull back of K_X. If again $B = 0$, then by the classification of smooth surfaces of Kodaira dimension $\kappa = 0$ we have $nK_X \sim 0$ for $n \in \mathfrak{R}_2$. If $B \neq 0$, then we run K_X-MMP (i.e. contract -1-curves step by step). As above, at the end we get a contraction $f \colon X \to Z$ to a smooth curve Z with $p_a(Z) \geq 1$. On the other hand, all components of B are rational (because so are singularities of our original X) and at least one component of B is horizontal, a contradiction. The proposition is proved. $\square$

Note that Proposition 9.2.2 can be proved by using log canonical covers (cf. 8.3.9 and [**I1**]).

In the case of log Enriques surfaces with a trivial boundary K_X is n-complementary if and only if $nK_X \sim 0$. It is known that this n can be taken ≤ 21 [**Bl**], [**Z**]. Note that here the lc condition of K_X cannot be removed (as well as in the case of log del Pezzos): there are examples of series of normal surfaces with rational singularities and numerically trivial K_X such that their index tends to infinity [**S3**].

9.3. On the explicit bound of exceptional complements

PROBLEM 9.3.1. Notation as in Theorem 9.1.12. Describe the set of all $\mathrm{compl}(X, B)$. In particular, find the precise value of Const.

The proof of Theorem 9.1.12 shows that we can hope to classify exceptional del Pezzos (at least modulo some birational modifications). We describe below an explicit method to reconstruct our original pair (X, B) from a model $(\widetilde{X}, \widetilde{B})$ (see (9.2)). Let (X, B) be an exceptional log del Pezzo. Assume that $B \in \Phi_{\mathbf{m}}$ and $K_X + B$ has no regular complements. Shokurov defined the following invariant (which is finite by Lemma 3.1.9):

$$\delta = \delta(X, B) := \#\{ \text{ divisors } E \text{ such that } a(E, B) \leq -6/7\}.$$

An analog of the invariant δ was considered in [**Z1**].

9.3.2. Case $\delta(X, B) = 0$. In this case, $K_X + B$ is $(1/7)$-lt and coefficients of B are contained in $\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}\}$. By Theorem 5.2.1 there is only a finite number of families of such (X, B), but very few classification results are known.

We give alternative proof of the boundedness in this case using more effective Nikulin's theorem 5.2.2. Assume that $-K_X$ is not ample. We set

$$\alpha := \begin{cases} 0 & \text{if } -K_X \text{ is nef}, \\ \min\{t \mid -(K_X + tB) \text{ is nef}\} & \text{otherwise}. \end{cases}$$

Then $0 \leq \alpha \leq 1$, $-(K_X + \alpha B)$ is nef and not ample. Moreover, $K_X + \alpha B$ is $(1/7)$-lt. Further, there is an extremal ray, say R, on X such that $(K_X + \alpha B) \cdot R = 0$. Let $\varphi \colon X \to X'$ be its contraction. If φ is not birational, then $0 \leq (1 - \alpha)B \cdot R = (K_X + B) \cdot R \leq 0$. This yields $(K_X + B) \cdot R = 0$ and this contradicts to that $-(K_X + B)$ is big. Therefore φ is birational. Put $B' := \varphi_* B$. By Lemma 9.1.6, $K_{X'} + B'$ is $(1/7)$-lt and all our assumptions hold on X'. Note also that for the exceptional divisor E one has $a(E, B') \leq a(E, \alpha B') = 0$. Continuing the process, we obtain a pair (X'', B'') such that $-K_{X''}$ is ample, $K_{X''} + B''$ is $(1/7)$-lt and $-(K_{X''} + B'')$ is nef and big. By Theorem 5.2.2 and by Lemma 9.3.3 below, $\{X''\}$ is bounded and so is B'' (because $B'' \in \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}\}$). By Lemma 3.1.9 there are only a finite number of divisors with discrepancies $a(E, B'') \leq 0$. Therefore there is only a finite number of extractions $X \to X''$ (and all of them are dominated by the terminal modification $X^t \to X$; see 3.1.8). Finally, $B \in \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}\}$ gives the boundedness of $\{(X, B)\}$.

LEMMA 9.3.3. *Let $X \ni o$ be a germ of two-dimensional ε-lt singularity. Then the multiplicity $e(X)$ is bounded by $C(\varepsilon)$.*

PROOF. By Corollary 6.1.4 it is sufficient to check it for two series of nonexceptional klt singularities. Consider, for example, singularities of type $\mathbb{A}_n$, i.e., it has the following dual graph of the minimal resolution:

$$\overset{-b_1}{\bigcirc} \underline{\qquad} \overset{-b_2}{\bigcirc} \underline{\qquad} \cdots \underline{\qquad} \overset{-b_{n-1}}{\bigcirc} \underline{\qquad} \overset{-b_n}{\bigcirc}.$$

Then the discrepancies $a_1, \ldots, a_n$ can be found from the system of linear equations (see [**Il**], [**Ut**, Ch. 3]):

$$b_1 - 2 = -b_1 a_1 + a_2$$
$$b_2 - 2 = a_1 - b_2 a_2 + a_3$$
$$\cdots \quad \cdots$$
$$b_{n-1} - 2 = a_{n-2} - b_{n-1} a_{n-1} + a_n$$
$$b_n - 2 = a_{n-1} - b_n a_n.$$

This yields

$$\varepsilon \sum (b_i - 2) < \sum (b_i - 2)(a_i + 1) = -a_1 - a_n < 2 - 2\varepsilon.$$

Finally, by [**Br**],

$$e(X) = 2 + \sum (b_i - 2) < 2 + \frac{2 - 2\varepsilon}{\varepsilon} = \frac{2}{\varepsilon}.$$

We left computations in case $\mathbb{D}_n$ (with the dual graph as in Fig. 6.2) to the reader. $\square$

9.3.4. Case $\delta \geq 1$. We can argue similar to 9.3.2. Notation as in (9.2). Take

$$\alpha := \max\{t \mid -(K_{\widehat{X}} + \widehat{B} + t(\widehat{D} - \widehat{B})) \text{ is nef}\}.$$

Clearly, $\alpha < 1$. Then there is an extremal rational curve E such that $(K_{\widehat{X}} + \widehat{B} + \alpha(\widehat{D} - \widehat{B})) \cdot E = 0$ and $(\widehat{D} - \widehat{B}) \cdot E > 0$. Hence $E \cdot (K_{\widehat{X}} + \widehat{D}) > 0$. By Lemma 9.1.4 E is of birational type. We contract it. This contraction is $(K_{\widehat{X}} + \widehat{B} + \alpha(\widehat{D} - \widehat{B}))$-crepant. Repeating this process, we obtain a sequence of contractions $\widehat{X} \to \widetilde{X}$ and the sequence of rational numbers $\alpha = \alpha_1 \leq \alpha_2 \leq \cdots \alpha_n < 1$ such that on each step $K + B$ is $(1/7)$-lt, $-(K + B)$ is nef and big, $K + B + \alpha(D - B)$ is klt and $-(K + B + \alpha(D - B))$ is nef. On the last step we have $\rho(\widetilde{X}) = 1$ and $K_{\widetilde{X}} + \widetilde{B} + \alpha_n(\widetilde{D} - \widetilde{B})$ is klt. Since $\widetilde{X}$ is $(1/7)$-lt, $\{\widetilde{X}\}$ is bounded by Theorem 5.2.2. As above $\text{Supp} \widetilde{B} = \text{Supp} \widetilde{D}$ is also bounded (because $\widetilde{B}$ has coefficients $\geq 1/2$).

First assume that $\delta = 1$, i.e., $\widetilde{C}$ is irreducible. Since both $-(K_{\widetilde{X}} + \widetilde{B})$ and $K_{\widetilde{X}} + \widetilde{D}$ are ample, there is α', $\alpha_n < \alpha' < 1$ such that $K_{\widetilde{X}} + \widetilde{B} + \alpha'(\widetilde{D} - \widetilde{B})$ is numerically trivial (and klt). Write

$$K_{\widetilde{X}} + \widetilde{B} + \alpha'(\widetilde{D} - \widetilde{B}) = K_{\widetilde{X}} + \widetilde{B} + \beta \widetilde{C}, \qquad \beta > 0.$$

Clearly, we can find this β even we do not know the coefficient of $\widetilde{C}$ in $\widetilde{B}$. Thus we may assume that $\widetilde{B} + \beta \widetilde{C}$ is fixed. As in the case $\delta = 0$ there are only a finite number of divisors with discrepancies $a(E, \widetilde{B} + \beta \widetilde{C}) \leq 0$ (see Lemma 3.1.9). Therefore there is only a finite number of extractions $\widehat{X} \to \widetilde{X}$.

In the case $\delta \geq 2$ we have to be more careful. Again we can take α', $\alpha_n \leq \alpha' < 1$ such that $K_{\widetilde{X}} + \widetilde{B} + \alpha'(\widetilde{D} - \widetilde{B})$ is numerically trivial (and klt). If this $K_{\widetilde{X}} + \widetilde{B} + \alpha'(\widetilde{D} - \widetilde{B})$ is $(1/7)$-lc, then by Lemma 9.1.10 the coefficients of $\widetilde{B} + \alpha'(\widetilde{D} - \widetilde{B})$ are

bounded from above by $c < 1$ (and we can find this constant on fixed $\widetilde{X}$). Then the log divisor $K_{\widetilde{X}} + \widetilde{\Delta} + c\sum \widetilde{C}_i$ is klt, where $\widetilde{\Delta} := \left\{\widetilde{D}\right\}$ (see 1.1.4). Clearly,

$$\widetilde{\Delta} + c\sum \widetilde{C}_i \geq \widetilde{B} + \alpha_n(\widetilde{D} - \widetilde{B})$$

Arguing as above, we find only a finite number of extractions $\widehat{X} \to \widetilde{X}$. More precisely, the discrepancy of any exceptional divisor E of $\widehat{X} \to \widetilde{X}$ with respect to $K_{\widetilde{X}} + \widetilde{B} + \alpha_n(\widetilde{D} - \widetilde{B})$ must be nonpositive (i.e., $a(E, \widetilde{B} + \alpha_n(\widetilde{D} - \widetilde{B})) \leq 0$). In particular, the discrepancy of E with respect to $K_{\widetilde{X}} + \widetilde{\Delta} + c\sum \widetilde{C}_i$ must be nonpositive. By our assumptions, $\widetilde{X}$, $\widetilde{\Delta}$, $\widetilde{C}_i$ and c are fixed. Then Lemma 3.1.9 gives that there is only a finite number of extractions $\widehat{X} \to \widetilde{X}$.

If $K_{\widetilde{X}} + \widetilde{B} + \alpha'(\widetilde{D} - \widetilde{B})$ is not $(1/7)$-lc, then $K_{\widetilde{X}} + \widetilde{B} + \alpha''(\widetilde{D} - \widetilde{B})$ is not $(1/7)$-lc (and antiample) for some $\alpha'' < \alpha'$. Then this new log divisor has

$$\delta(\widetilde{X}, \widetilde{B} + \alpha''(\widetilde{D} - \widetilde{B})) > \delta(\widetilde{X}, \widetilde{B}) \geq 2$$

and $\widetilde{B} + \alpha''(\widetilde{D} - \widetilde{B}) \in \Phi_{\mathbf{m}}$. This is impossible by Corollary 10.1.2 below.