CHAPTER 7

Contractions onto curves

In this chapter we discuss complements on log surfaces over curves. The main result is Theorem 7.2.11. From Theorem 6.0.6 we have

COROLLARY 7.0.10. *Let $f\colon X \to Z \ni o$ be a contraction from a normal surface X onto a smooth curve Z. Let D be a boundary on X. Assume that $K_X + D$ is lc and $-(K_X + D)$ is f-nef and f-big. Then there exists a nonklt* 1, 2, 3, 4, *or* 6*-complement of $K_X + D$ near $f^{-1}(o)$. Moreover, if there are no nonklt* 1 *or* 2*-complements of $K_X + D$, then $f\colon X \to Z \ni o$ is exceptional.*

Below we give generalization of this result for the case when $K_X + D \equiv 0$ and classify two-dimensional log conic bundles.

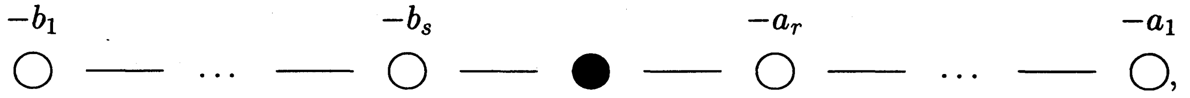
7.1. Log conic bundles

7.1.1. Assumptions. Let $(X \supset C)$ be a germ of normal surface X with only klt singularities along a reduced curve C, and $(Z \ni o)$ a smooth curve germ. Let $f\colon (X, C) \to (Z, o)$ be a K_X-negative contraction such that $f^{-1}(o)_{\mathrm{red}} = C$. Then it is easy to prove that $p_a(C) = 0$ and each irreducible component of C is isomorphic to $\mathbb{P}^1$. Everywhere in this paragraph if we do not specify the opposite, we assume that C is irreducible (or, equivalently, $\rho(X/Z) = 1$, i.e., f is extremal). Let $X_{\min} \to X$ be the minimal resolution. Since the composition map $f_{\min}\colon X_{\min} \to Z$ is flat, the fiber of $f_{\min}^{-1}(o)$ is a tree of rational curves. Therefore it is possible to define the dual graph of $f_{\min}^{-1}(o)$. We draw it in the following way: ● denotes the proper transform of C, while ○ denotes the exceptional curve. We attach the selfintersection number to the corresponding vertex. By construction, the proper transform of C is the only -1-curve in $f_0^{-1}(o)$, so we usually omit -1 over ●.

EXAMPLE 7.1.2. Let $\mathbb{P}^1 \times \mathbb{C}^1 \to \mathbb{C}^1$ be the natural projection. Consider the following action of $\mathbb{Z}_m$ on $\mathbb{P}^1_{x,y} \times \mathbb{C}^1_u$:

$$(x, y; u) \longrightarrow (x, \varepsilon^q y; \varepsilon u), \qquad \varepsilon = \exp 2\pi i/m, \qquad \gcd(m, q) = 1.$$

Then the morphism $f\colon X = (\mathbb{P}^1 \times \mathbb{C}^1)/\mathbb{Z}_m \to \mathbb{C}^1/\mathbb{Z}_m$ satisfies the conditions above. The surface X has exactly two singular points which are of types $\frac{1}{m}(1, q)$ and $\frac{1}{m}(1, -q)$. The morphism f is toric, so K_X is 1-complementary. One can check that the minimal resolution of X has the dual graph



where $(b_1, \dots, b_s)$ and $(a_r, \dots, a_1)$ are defined by (2.1).

PROPOSITION 7.1.3 (see also [**KeM**, (11.5.12)]). *Let $f\colon (X, C) \to (Z, o)$ be a contraction as in 7.1.1, but not necessarily extremal (i.e., C may be reducible). Assume that X singular and has only Du Val singularities. Then X is analytically isomorphic to a surface in $\mathbb{P}^2_{x,y,z} \times \mathbb{C}^1_t$ which is given by one of the following equations:*

(i) $x^2 + y^2 + t^n z^2 = 0$, *then the central fiber is a reducible conic and X has only one singular point, which is of type A_{n-1};*
(ii) $x^2 + ty^2 + tz^2 = 0$, *then the central fiber is a nonreduced conic and X has exactly two singular points, which are of type A_1;*
(iii) $x^2 + ty^2 + t^2 z^2 = 0$, *then the central fiber is a nonreduced conic and X has only one singular point, which is of type A_3;*
(iv) $x^2 + ty^2 + t^n z^2 = 0$, $t \geq 3$ *then the central fiber is a nonreduced conic and X has only one singular point, which is of type D_{n+1}.*

SKETCH OF PROOF. One can show that the linear system $|-K_X|$ is very ample and determines an embedding $X \subset \mathbb{P}^2 \times Z$. Then X must be given by the equation $x^2 + t^k y^2 + t^n z^2 = 0$. □

7.1.4. Construction. Notation and assumptions as in 7.1.1. Let d be the index of C on X, i. e. the smallest positive integer such that $dC \sim 0$. If $d = 1$, then C is a Cartier divisor and X must be smooth along C, because so is C. If $d > 1$, then there exists the following commutative diagram:

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{\;g\;} & X \\ {\scriptstyle \widehat{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \widehat{Z} & \xrightarrow{\;h\;} & Z, \end{array}$$

where $\widehat{X} \to X$ is a cyclic étale outside Sing$X$ cover of degree d defined by C and $\widehat{X} \to \widehat{Z} \to Z$ is the Stein factorization. Then $\widehat{f}\colon \widehat{X} \to \widehat{Z}$ is also a $K_{\widehat{X}}$-negative contraction but not necessarily extremal. By construction, the central fiber $\widehat{C} := \widehat{f}^{-1}(\widehat{o})$ is a reducible Cartier divisor. Note that $p_a(\widehat{C}) = 0$. Therefore $\widehat{X}$ is smooth outside Sing$\widehat{C}$. We distinguish two cases.

7.1.5. Case: $\widehat{C}$ is irreducible. Then $\widehat{X}$ is smooth and $\widehat{X} \simeq \mathbb{P}^1 \times \widehat{Z}$. Thus $f\colon X \to Z$ is analytically isomorphic to the contraction from Example 7.1.2.

7.1.6. Case: $\widehat{C}$ is reducible. Then the group $\mathbb{Z}_d$ permutes components of $\widehat{C}$ transitively. Since $p_a(\widehat{C}) = 0$, this gives that all the components of $\widehat{C}$ passes through one point, say $\widehat{P}$, and they do not intersect each other elsewhere. The surface $\widehat{X}$ is smooth outside $\widehat{P}$. Note that in this case $K_X + C$ is not plt, because neither is $K_{\widehat{X}} + \widehat{C}$.

COROLLARY 7.1.7. *Notation as in 7.1.4. Then X has at most two singular points on C.*

PROOF. In Case 7.1.6 any nontrivial element $a \in \mathbb{Z}_d$ have $\widehat{P}$ as a fixed point. It can have at most one more fixed point $\widehat{P}_i$ on each component $\widehat{C}_i \subset \widehat{C}$. Moreover, $\mathbb{Z}_d$ permutes points $\widehat{P}_1, \ldots$. Then X can be singular only at images of $\widehat{P}$ and $\widehat{P}_1, \ldots$. □

7.1.8. Additional notation. In Case 7.1.6 we denote $P := g(\widehat{P})$. If X has two singular points, let Q be another singular point. To distinguish exceptional divisors over P and Q in the corresponding Dynkin graph we reserve the notation ◯ for exceptional divisors over P and ⊖ for exceptional divisors over Q.

COROLLARY 7.1.9. *In the above conditions, $K_X + C$ is plt outside of P.*

LEMMA 7.1.10. *Notation as in 7.1.1, 7.1.4 and 7.1.8. Let $X' \to X$ be a finite étale in codimension one cover. Then there exists the decomposition $\widehat{X} \to X' \to X$. In particular, $X' \to X$ is cyclic and the preimage of P on X' consists of one point.*

PROOF. Let X'' be the normalization of $X' \times_X \widehat{X}$. Consider the Stein factorization $X'' \to Z'' \to Z$. Then $X'' \to Z''$ is flat and a generically $\mathbb{P}^1$-bundle. Therefore for the central fiber C'' one has $(-K_{X''} \cdot C'') = 2$, where C'' is reduced and it is the preimage of $\widehat{C}$. On the other hand,

$$(-K_{X''} \cdot C'') = n(-K_{\widehat{X}} \cdot \widehat{C}) = 2n,$$

where n is the degree of $X'' \to \widehat{X}$. Whence $n = 1$, $X'' \simeq \widehat{X}$. This proves the assertion. □

LEMMA 7.1.11. *Let $f\colon X \to (Z \ni o)$ be an extremal contraction as in 7.1.1 (with irreducible C). Assume that $K_X + C$ is plt. Then*

(i) *$f\colon X \to (Z \ni o)$ is analytically isomorphic to the contraction from Example 7.1.2 (so it is toroidal). In particular, X has exactly two singular points on C which are of types $\frac{1}{m}(1,q)$ and $\frac{1}{m}(1,-q)$;*
(ii) *$K_X + C$ is 1-complementary.*

PROOF. In the construction 7.1.4 we have Case 7.1.5. Then

$$\mathrm{Diff}_C(0) = (1 - 1/d)P_1 + (1 - 1/d)P_2,$$

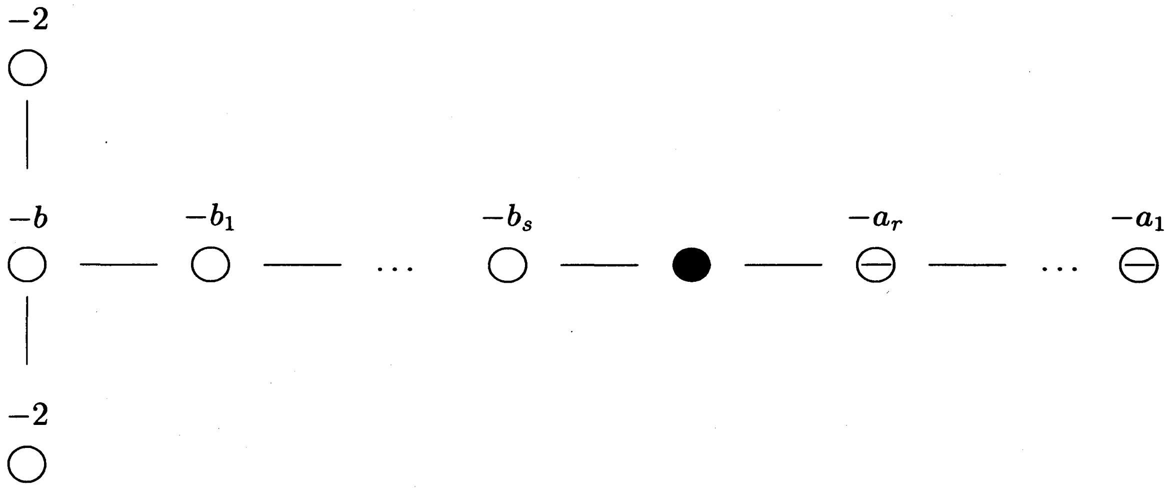
where P_1, P_2 are singular points of X and d is the index of C. By Corollary 4.1.11 and by Proposition 4.4.3, $K_X + C$ is 1-complementary. □

The following result gives the classification of surface log terminal contractions of relative dimension one. For applications to three-dimensional case and generalizations we refer to [**P2**], [**P3**].

THEOREM 7.1.12 ([**P3**]). *Let $f\colon (X \supset C) \to (Z \ni o)$ be an extremal contraction as in 7.1.1 (with irreducible C). Then K_X is 1, 2 or 3-complementary. Moreover, there are the following cases:*

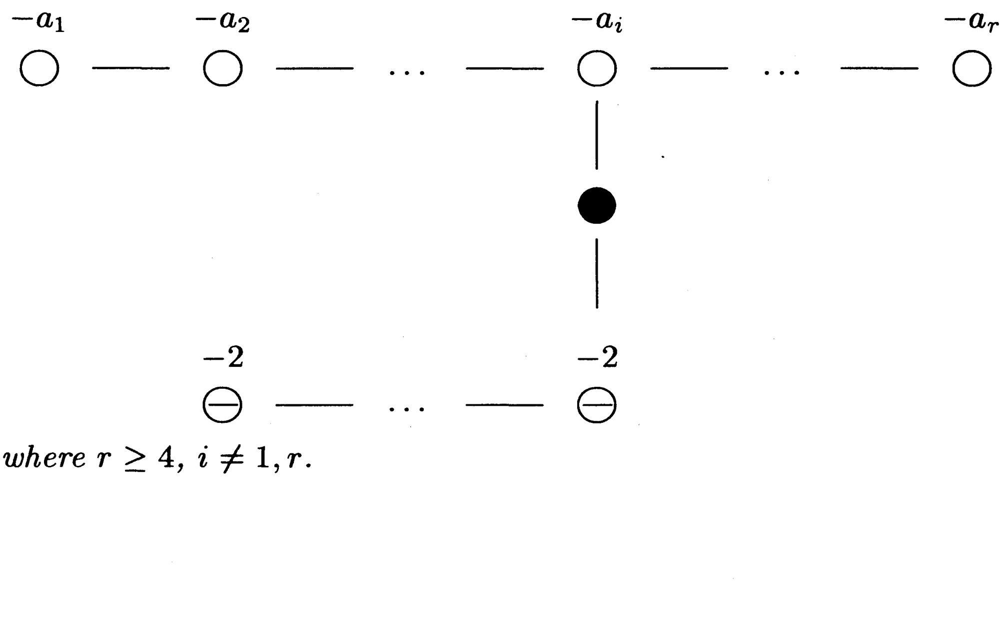
Case A^*: $K_X + C$ is plt, then $K_X + C$ is 1-complementary and f is toroidal (see Example 7.1.2, cf. Conjecture 2.2.18);

Case D^*: $K_X + C$ is lc, but not plt, then $K_X + C$ is 2-complementary and f is a quotient of a conic bundle of type (i) of Proposition 7.1.3 by a cyclic group $\mathbb{Z}_{2m}$ which permutes components of the central fiber and acts on X freely in codimension one. The minimal resolution of X is



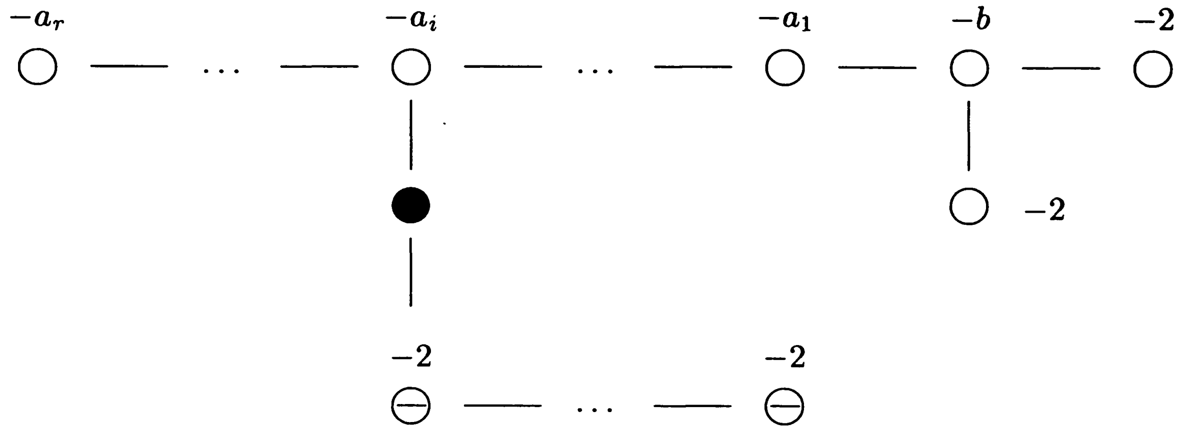
where $s, r \geq 0$ (recall that X can be smooth outside P, so $r = 0$ is also possible).

Case A^{**}: K_X is 1-complementary, but $K_X + C$ is not lc. The minimal resolution of X is



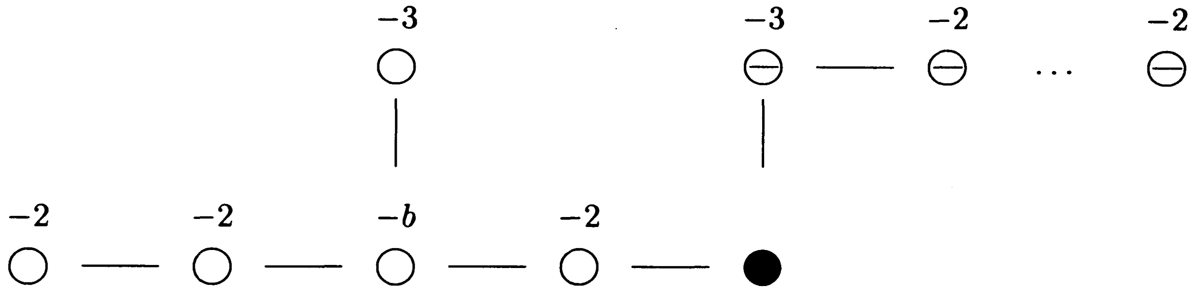
where $r \geq 4$, $i \neq 1, r$.

Case D^{**}: K_X *is 2-complementary, but not 1-complementary and* $K_X + C$ *is not lc. The minimal resolution of* X *is*



where $r \geq 2$, $i \neq r$.

Case E_6^* **(exceptional case):** K_X *is 3-complementary, but not 1- or 2-complementary. The minimal resolution of* X *is*



Here the number of $\ominus$*-vertices is* $b-2$ *(it is possible that* $b = 2$ *and* $Q \in X$ *is smooth).*

REMARK 7.1.13. (i) In the case D^* the canonical divisor K_X can be 1-complementary:
  a) if $P \in X$ is Du Val (see 7.1.3 (ii)), or
  b) if $s = 0$, $a_1 = \cdots = a_r = 2$, $b = r + 2$.

(ii) In cases D^*, A^{**} and D^{**} there are additional restrictions on the graph of the minimal resolution. For example, in the case A^{**} one easily can check that

$$\left(\sum_{j=1}^{i-1} a_j\right) - (i-1) = \left(\sum_{j=i+1}^{r} a_j\right) - (r-i)$$

and

$$a_i = (\text{number } \ominus\text{-vertices}) + 2.$$

PROOF. If $K_X + C$ is plt, then by Lemma 7.1.11 we have Case A^*. Thus we may assume that $K_X + C$ is not plt.

We claim that K_X is 1, 2 or 3-complementary. Assume that K_X is not 1-complementary. For some $\alpha \leq 1$ the log divisor $K_X + \alpha C$ is lc, but not plt (so, $K_X + \alpha C$ is maximally lc). Consider a minimal log terminal modification $\varphi\colon (\check{X}, \sum E_i + \alpha\check{C}) \to (X, \alpha C)$, where $\sum E_i$ is the reduced exceptional divisor, $\check{C}$ is the proper transform of C and $K_{\check{X}} + \sum E_i + \alpha\check{C} = \varphi^*(K_X + \alpha C)$ is dlt. As in 3.1.4, applying the $(K_{\check{X}} + \sum E_i)$-MMP to $\check{X}$ at the last step we obtain the blowup $\sigma\colon \widetilde{X} \to X$ with irreducible exceptional divisor E. Moreover, $\sigma^*(K_X + \alpha C) = K_{\widetilde{X}} + E + \alpha\widetilde{C}$ is lc, where $\widetilde{C}$ is the proper transform of C and $K_{\widetilde{X}} + E$ is plt and negative over X. Since $K_{\widetilde{X}} + E + (\alpha - \varepsilon)\widetilde{C}$ is antiample for $0 < \varepsilon \ll 1$, the curve $\widetilde{C}$ can be contracted in the appropriate log MMP over Z and this gives a contraction $(\overline{X}, \overline{E}) \to Z$ with purely log terminal $K_{\overline{X}} + \overline{E}$. By Lemma 7.1.11 $(\overline{X}, \overline{E}) \to Z$ is as in Example 7.1.2. If $K_{\widetilde{X}} + E$ in nonnegative on $\widetilde{C}$, then by Proposition 4.3.2 we can pull back 1-complements from $\overline{X}$ on $\widetilde{X}$ and then push-down them on X (see 4.3.1). Thus we obtain 1-complement of K_X, a contradiction. From now on we assume that $-(K_{\widetilde{X}} + E)$ is ample over Z. Then by Proposition 4.4.3 complements for $K_E + \mathrm{Diff}_E(0)$ can be extended on $\widetilde{X}$. According to 4.1.11, $\mathrm{Diff}_E(0) = \sum_{i=1}^{3}(1 - 1/m_i)P_i$, where for (m_1, m_2, m_3) there are the following possibilities:

$$(2,2,m),\ (2,3,3),\ (2,3,4),\ (2,3,5).$$

Further, $\overline{X}$ has exactly two singular points and these are of type $\frac{1}{m}(1,q)$ and $\frac{1}{m}(1,-q)$, respectively (see Lemma 7.1.11). Since $\widetilde{C}$ intersects E at only one point, this point must be singular and there are two more points with $m_i = m_j$. We get two cases:

7.1.14. $(2,2,m)$, $\widetilde{C} \cap E = \{P_3\}$, there is a 2-complement;

7.1.15. $(2,3,3)$, $\widetilde{C} \cap E = \{P_1\}$, there is a 3-complement.
This proves the claim.

If $K_X + C$ is lc (but not plt), then in Construction 7.1.4 $K_{\widehat{X}} + \widehat{C}$ is also lc but not plt (see Proposition 1.2.1). Since $\widehat{C}$ is a Cartier divisor, $K_{\widehat{X}}$ is canonical. Hence $\widehat{f}$ is as in Proposition 7.1.3, (i). We get the case D^*.

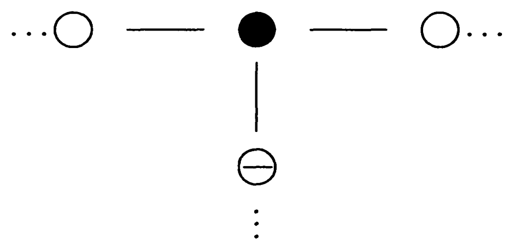
To prove that note that $\alpha = 1$ and $K_{\widetilde{X}} + E + \widetilde{C}$ is lc. Hence $f\colon X \to Z$ is not exceptional and K_X is 1- or 2-complementary by Corollary 7.0.10.

Assume that K_X is 1-complementary, but $K_X + C$ is not lc. Then there exists a reduced divisor D such that $K_X + D$ is lc and linearly trivial. By our assumption and by Propositions 2.1.2 and 2.1.3, $C \not\subset D$. Let $P \in X$ be a point of index > 1. Then $P \in C \cap D$ and again by Propositions 2.1.2 and 2.1.3 there are two components $D_1, D_2 \subset D$ passing through P. But since $D \cdot L = 2$, where L is a generic fiber of f, $D = D_1 + D_2$, $P \in D_1 \cap D_2$ and P is the only point of index > 1 on X.

Now assume that K_X is 2-complementary, but not 1-complementary and $K_X + C$ is not lc. Then we are in the case 7.1.14. Therefore

$$(\widetilde{X} \ni P_1) \simeq (\widetilde{X} \ni P_2) \simeq (\mathbb{C}^2, 0)/\mathbb{Z}_2(1,1),$$
$$(\widetilde{X} \ni P_3) \simeq (\mathbb{C}^2, 0)/\mathbb{Z}_m(1,q), \quad \gcd(m,q) = 1.$$

Take the minimal resolution $X_{\min} \to \widetilde{X}$ of $P_1, P_2, P_3 \in \widetilde{X}$. Over P_1 and P_2 we have only single -2-curves and over P_3 we have a chain which must intersect the proper transform of $\widetilde{C}$, because $\widetilde{C}$ passes through P_3. Since the fiber of $\widetilde{X}_{\min} \to Z$ over o is a tree of rational curves, there are no three of them passing through one point. Whence proper transforms of E and $\widetilde{C}$ on $\widetilde{X}_{\min}$ are disjoint. Moreover, the proper transform of E cannot be a -1-curve. Indeed, otherwise contracting it we get three components of the fiber over $o \in Z$ passing through one point. It gives that $\widetilde{X}_{\min}$ coincides with the minimal resolution $X_{\min}$ of X. Therefore configuration of curves on $X_{\min}$ looks like that in Case D^{**}. We have to show only that all the curves in the down part have selfintersections -2. Indeed, contracting -1-curves over Z we obtain a $\mathbb{P}^1$-bundle. Each time, we contract a -1-curve, we have the configuration of the same type. If there is a vertex with selfintersection < -2, then at some step we get the configuration



It is easy to see that this configuration cannot be contracted to a smooth point over $o \in Z$, because contraction of the central -1-curve gives configuration curves which is not a tree. This completes Case D^{**}.

Case E_6^* is very similar to D^{**}. We omit it. □

From Corollary 6.1.4 we have

COROLLARY 7.1.16 (cf. [**P2**]). *Fix $\varepsilon > 0$. There is only a finite number of exceptional (i.e., of type E_6^*) log conic bundles $f\colon X \to Z$ as in Theorem 7.1.12 with ε-lt X.*

EXERCISE 7.1.17 (cf. 2.2.18, 6.2.9). Let $f\colon X \to Z \ni o$ be a contraction from a surface onto a curve and $D = \sum d_i D_i$ a boundary on X such that $K_X + D$ is lc and $-(K_X + D)$ is nef over Z. Prove that

$$\rho_{\mathrm{num}}(X/Z) + 2 \geq \sum d_i.$$

Moreover, the equality holds only if $(X/Z \ni o, \lfloor D \rfloor)$ is a toric pair.

7.2. Elliptic fibrations

As an application of complements we obtain Kodaira's classification of degenerate of elliptic fibers (see [**Sh3**]).

DEFINITION 7.2.1. *An elliptic fibration* is a contraction from a surface to a curve such that its general fiber is a smooth elliptic curve. An elliptic fibration $f\colon X \to Z$ is said to be *minimal* if X is smooth and $K_X \equiv 0$ over Z.

REMARK 7.2.2. (i) Note that any elliptic fibration obtained from minimal one by contracting curves in fibers has only Du Val singularities.

(ii) Let $K_X + B$ be a $\mathbb{Q}$-complement on an elliptic fibration $f\colon X \to Z \ni o$ with $K_X \equiv 0$. Then $B \equiv 0$. By Zariski's lemma, $pB \sim qf^*o$ for some $p, q \in \mathbb{N}$. In particular, there exists exactly one complement $K_X + B$ which is not klt.

Recall also that a minimal model is unique up to isomorphisms.

PROPOSITION-DEFINITION 7.2.3. Let $f\colon X \to Z \ni o$ be a minimal elliptic fibration. Then there exists a birational model $\overline{f}\colon \overline{X} \to Z$ such that $K_{\overline{X}} + \overline{F}$ is dlt and numerically trivial near $\overline{f}^{-1}(o)$, where $\overline{F} := \overline{f}^{-1}(o)_{\mathrm{red}}$. Such a model is called a *dlt model* of f. Moreover, if $K_{\overline{X}} + \overline{F}$ is n-complementary, then K_X is n-complementary. More precisely, $\mathrm{compl}'(X) \le \mathrm{compl}(\overline{X}, \overline{F})$. If $(X/Z \ni o)$ is exceptional, then $\overline{F}$ is irreducible, $K_{\overline{X}} + \overline{F}$ is plt and a dlt model is unique.

PROOF. First take the maximal $c \in \mathbb{Q}$ such that $K_X + cf^*o$ is lc. Put $B := cf^*o$. Next we consider a minimal log terminal modification $g\colon Y \to X$ of (X, B) (if $K_X + B$ is dlt, we put $Y = X$). Thus we can write $g^*(K_X + B) = K_Y + C + B_Y \equiv 0$, where C is reduced and nonempty, $\lfloor B_Y \rfloor = 0$ and $\mathrm{Supp}(C + B_Y)$ is contained in the fiber over o. Run $(K_Y + C + (1+\varepsilon)B_Y)$-MMP over Z:

$$
\begin{array}{ccccc}
 & & Y & & \\
 & \swarrow^{g} & & \searrow^{\overline{g}} & \\
X & & & & \overline{X} \\
 & \searrow_{f} & & \swarrow_{\overline{f}} & \\
 & & Z & &
\end{array}
\tag{7.1}
$$

If $B_Y \neq 0$, then $B_Y^2 < 0$ and we can contract a component of B_Y. At the end we get the situation when $B_Y = 0$. Taking $\overline{F} := \overline{g}(C)$ we see the first part of the proposition. The second part follows by 4.3.2 and the fact that all contractions $Y \to \overline{X}$ are positive with respect to $K + C$.

Finally, assume that $(X/Z \ni o)$ is exceptional. Then by Remark 7.2.2, there is exactly one nonklt complement $K_X + B$ (where $B = cf^*o$). Clearly, C is irreducible in this case. Contractions g and $\overline{g}$ are crepant with respect to $K_Y + C + B_Y$. By Proposition 1.1.6 $K_{\overline{X}} + \overline{F}$ is plt. Assume that there are two dlt models $(\overline{X}/Z \ni o, \overline{F})$

and $(\overline{X}'/Z \ni o, \overline{F}')$. Consider the diagram

$$\begin{array}{ccccc} & & X_{\min} & & \\ & \overset{h}{\swarrow} & & \overset{\overline{h}}{\searrow} & \\ X & & & & \overline{X} \\ & \underset{f}{\searrow} & & \underset{\overline{f}}{\swarrow} & \\ & & Z & & \end{array}$$

where $\overline{h}\colon X_{\min} \to \overline{X}$ is the minimal resolution and $h\colon X_{\min} \to X$ is a composition of contractions of -1-curves. Let $K_{\overline{X}} + \overline{F} + \overline{D}$ be a $\mathbb{Q}$-complement and

$$K_{X_{\min}} + F_{\min} + D_{\min} = \overline{h}^*(K_{\overline{X}} + \overline{F} + \overline{D})$$

the crepant pull back, where $F_{\min}$ is the proper transform of $\overline{F}$ and $D_{\min}$ is a boundary. Clearly,

$$-1 = a(F^i_{\min}, F_{\min} + D_{\min}) = a(F^i_{\min}, h_*(F_{\min} + D_{\min}))$$

for any irreducible component $F^i_{\min}$ of $F_{\min}$. Hence $K_X + h_*(F_{\min} + D_{\min})$ is a nonklt $\mathbb{Q}$-complement, so $h_*(F_{\min} + D_{\min}) = B$ and $a(\overline{F}^i, B) = -1$. Similarly, we get $a(\overline{F'}^j, B) = -1$. By exceptionality, $\overline{F}$ and $\overline{F}'$ are irreducible and $\overline{F} \approx \overline{F}'$ (as discrete valuations of $\mathcal{K}(X)$). Then $\overline{X} \dashrightarrow \overline{X}'$ is an isomorphism in codimension one, hence it is an isomorphism. □

REMARK 7.2.4. Let $\overline{f}\colon (\overline{X}, \overline{F}) \to Z \ni o$ be a dlt model of an elliptic fibration and $K_{\overline{X}} + \overline{F} + \overline{B}$ a $\mathbb{Q}$-complement. As in Remark 7.2.2 we have $\mathrm{Supp}\overline{B} \subset \overline{F}$, hence $\overline{B} = 0$.

COROLLARY 7.2.5. *Under notation of 7.2.3 the following are equivalent:*

(i) *$(X/Z \ni o)$ is exceptional;*

(ii) *$(\overline{X}/Z \ni o, \overline{F})$ is exceptional;*

(iii) *$K_{\overline{X}} + \overline{F}$ is plt.*

PROOF. The implication (ii)$\Longrightarrow$(iii) is obvious (because $\overline{F}$ is reduced, see 2.2.6). If $K_{\overline{X}} + \overline{F}$ is plt, then by Remark 7.2.4 $K_{\overline{X}} + \overline{F}$ is the only nonklt complement and $\overline{F}$ is the only divisor with $a(\overline{F}, \overline{F}) = -1$. This shows (iii)$\Longrightarrow$(ii). (i) $\Longrightarrow$ (ii) follows by 7.2.3.

Let us prove the implication (ii) $\Longrightarrow$ (i). Assume that $(X/Z \ni o)$ is nonexceptional. By Remark 7.2.2 there are two different divisors E_1, E_2 such that $a(E_1, B) = a(E_2, B) = -1$. Then in (7.1) we have $a(E_1, C+B_Y) = a(E_2, C+B_Y) = -1$. Since $K_Y + C + B_Y \equiv 0$, $a(E_1, \overline{F}) = a(E_2, \overline{F}) = -1$, i.e., $(\overline{X}/Z \ni o, \overline{F})$ is nonexceptional. □

Similar to Theorem 6.1.6 we have the following

PROPOSITION 7.2.6. *Let $\overline{f}\colon \overline{X} \to Z \ni o$ be dlt model of an elliptic fibration and $\overline{F} := \overline{f}^{-1}(o)_{\mathrm{red}}$. Then one of the following holds:*

Ell-$\widetilde{A}_n$: $p_a(\overline{F}) = 1$, $\overline{X}$ *is smooth and* $\overline{F}$ *is either*

Ell: *a smooth elliptic curve, or*

$\widetilde{A}_n$: *a wheel of smooth rational curves;*

$\widetilde{D}_n$, $n \geq 5$: $\overline{F}$ *is a chain of smooth rational curves, and it is as in Lemma 6.1.9 and Fig. 6.6 (here* $n-3$ *is the number of components of* $\overline{F}$*);*

Exc: $K_{\overline{X}} + \overline{F}$ *is plt (therefore it is exceptional), then* $\mathrm{Diff}_{\overline{F}}(0) = \sum_{i=1}^{r}(1 - 1/m_i)$ *where for* $(m_1, \dots, m_r)$ *there are possibilities as in 4.1.12:*

$\widetilde{D}_4$: $(2,2,2,2)$;

$\widetilde{E}_6$: $(3,3,3)$;

$\widetilde{E}_7$: $(2,4,4)$;

$\widetilde{E}_8$: $(2,3,6)$.

PROOF. Follows by 6.1.7 and 6.1.9. □

COROLLARY 7.2.7. *Notation as in Proposition 7.2.6. Then the index of* $K_{\overline{X}} + \overline{F}$ *is equal to* 1, 2, 3, 4, *or* 6, *in cases* $\widetilde{A}_n$ *(and Ell),* $\widetilde{D}_n$ $(n \geq 4)$, $\widetilde{E}_6$, $\widetilde{E}_7$ *and* $\widetilde{E}_8$, *respectively.*

SKETCH OF PROOF. Applying Zariski's lemma on the minimal resolution we get $\overline{F} \sim_{\mathbb{Q}} 0$. Let r be the index of $\overline{F}$, i.e., the smallest positive integer such that $r\overline{F} \sim 0$. By taking the corresponding cyclic cover (cf. 1.3)

$$X' := \mathrm{Spec}\left(\bigoplus_{i=0}^{r-1} \mathcal{O}_{\overline{X}}(-i\overline{F})\right) \to \overline{X}$$

we obtain an elliptic fibration $f' \colon X' \to Z' \ni o'$ such that F' is linearly trivial and log canonical. Since $\overline{X}$ is smooth at singular points of $\overline{F}$, we have that $K_{X'} + F'$ is dlt (see Theorem 2.1.3 or [**Sz**]). Again by Theorem 2.1.3 X' is smooth along F' (because F' is Cartier). Hence the elliptic fibration $f' \colon X' \to Z' \ni o'$ must be of type *Ell* or $\widetilde{A}_k$. By the canonical bundle formula, $K_{X'} + F' \sim 0$ (see e.g. [**BPV**, Ch. V, §12]). Therefore, $m(K_{\overline{X}} + \overline{F}) \sim 0$ for some m. Again let m be the index of $K_{\overline{X}} + \overline{F}$. Now we consider the log canonical cover (see 1.3)

$$X'' := \mathrm{Spec}\left(\bigoplus_{i=0}^{m-1} \mathcal{O}_{\overline{X}}(-iK_{\overline{X}} - i\overline{F})\right) \to \overline{X}$$

As above, $K_{X''} + F''$ is dlt and the elliptic fibration $f'' \colon X'' \to Z'' \ni o''$ is of type *Ell* or $\widetilde{A}_k$.

If f'' is of type $\widetilde{A}_k$, then the group $\mathrm{Gal}(X''/\overline{X})$ acts on F'' so that the stabilazer of every singular point is trivial. If $m > 1$, then the only possibility is $m = 2$ and f is of type $\widetilde{D}_n$, $n \geq 5$.

Assume that f'' is of type *Ell*. Note that $\mathrm{Gal}(X''/\overline{X})$ contains no subgroups G acting freely on F'' (otherwise the quotient $X''/G \to Z''/G$ is again of type *Ell*). In particular, $\mathrm{Gal}(X''/\overline{X}) \subset \mathrm{Aut}(F'')$ and this group contains no translations of the elliptic curve F''. It is well known (see e.g., [**Ha**]) that, in this situation, the order of Gal$(X''/\overline{X})$ can be 2, 3, 4 or 6. Moreover, it is easy to see that the ramification indices are such as in $\widetilde{D}_4$, $\widetilde{E}_6$, $\widetilde{E}_7$, or $\widetilde{E}_8$ of 4.1.12. □

COROLLARY 7.2.8. *Notation as in Proposition 7.2.6. Assume that $\overline{f}$ is exceptional and not of type Ell. Then $\overline{f}$ is a quotient of a smooth elliptic fibration of type Ell by a cyclic group of order* 2, 3, 4, 6 *in cases $\widetilde{D}_4$, $\widetilde{E}_6$, $\widetilde{E}_7$, and $\widetilde{E}_8$, respectively.*

COROLLARY 7.2.9. *Let $f\colon X \to Z \ni o$ be a minimal elliptic fibration. Then there exists a regular complement of K_X.*

For convenience we recall Kodaira's classification of singular elliptic fibers and give a new proof of it using birational techniques (cf. e.g. [**BPV**, Ch. V, §7]).

THEOREM 7.2.10. *Let $f\colon X \to Z \ni o$ be a minimal elliptic fibration (X is smooth) and $F = (f^*o)_{\mathrm{red}}$, $o \in Z$ the special fiber. Then there is one of the following possibilities for F (in the graphs all vertices correspond to -2-curves which are components of F):*

I_b: *a smooth elliptic curve ($b = 0$);*
a rational curve with one node ($b = 1$);
a wheel of smooth rational curves ($b \geq 2$);

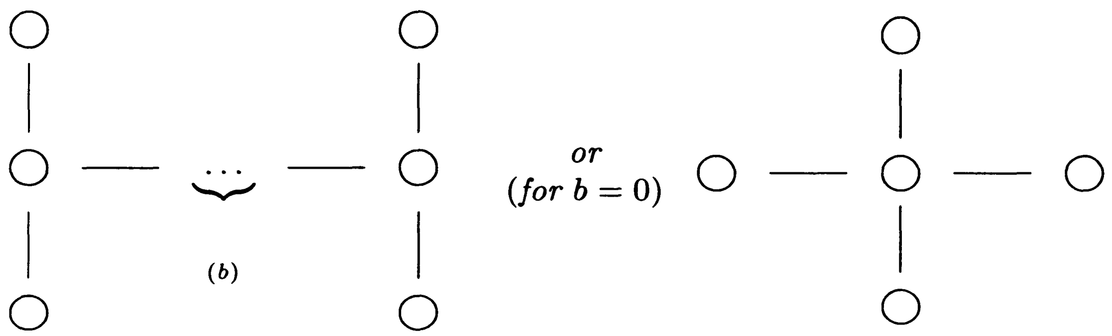
${}_mI_b$: *multiple I_b;*

II: *a rational curve with a simple cusp;*

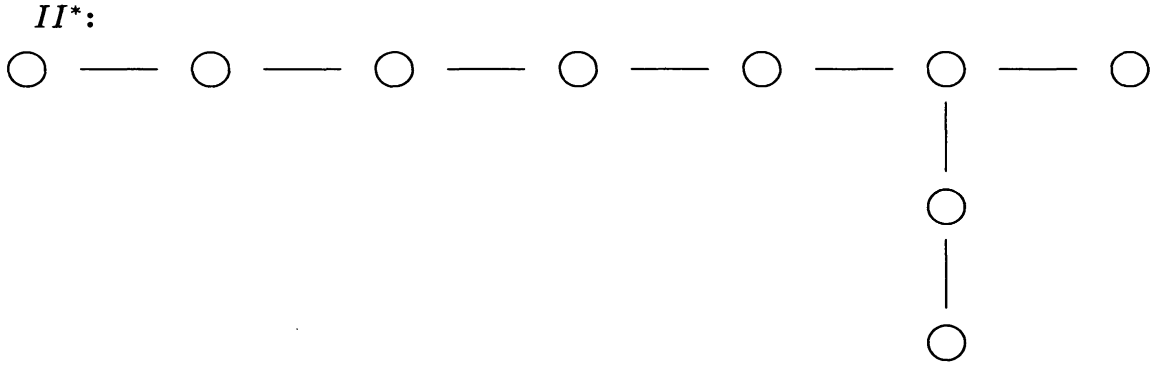
III: *$F = F_1 + F_2$ is a pair of smooth rational, tangent each other curves;*

IV: *$F = F_1 + F_2 + F_3$ is a union of three smooth rational curves passing through one point;*

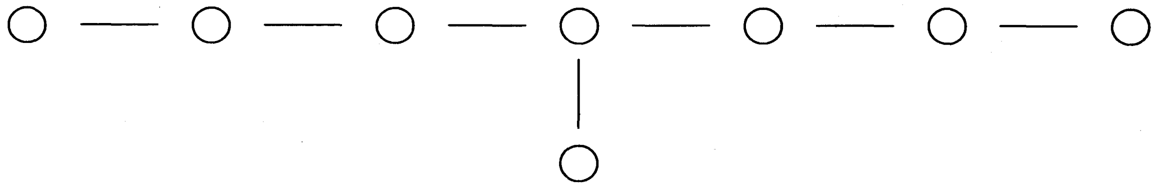
I_b^*:



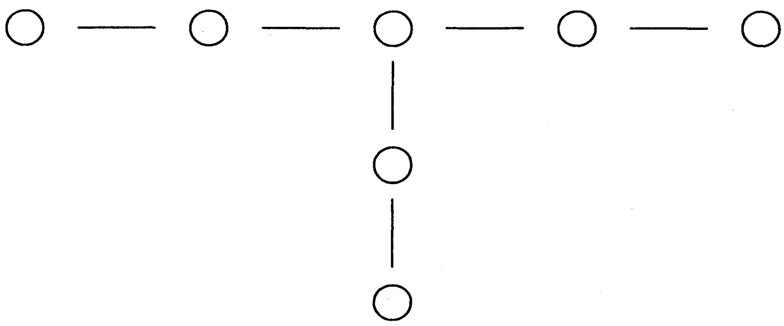
II^*:



III^*:

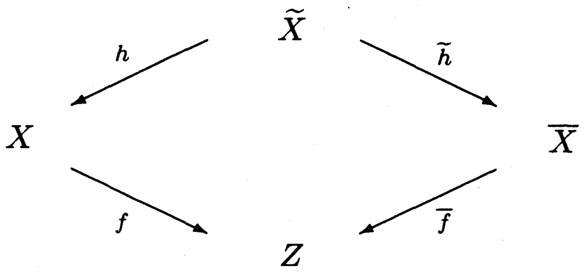


IV^*:



The proof is very similar to that of Theorem 6.1.6.

PROOF. We are going to apply Proposition 7.2.6. So we consider a dlt model $\overline{f}\colon \overline{X} \to Z \ni o$ and $\widetilde{h}\colon \widetilde{X} \to \overline{X}$ the minimal resolution of singularities of $\overline{X}$. Then we have the following diagram:



where $h\colon \widetilde{X} \to X$ is a sequence of contractions of -1-curves. If $p_a(\overline{C}) = 1$, then $\overline{X} = \widetilde{X}$ and C is a smooth elliptic curve or a wheel of smooth rational curves. Contracting, if necessary, -1-curves we obtain case ${}_mI_b$. Further, we assume that $p_a(\overline{C}) = 0$. Then $\overline{X}$ is singular, so $\widetilde{X} \neq \overline{X}$. Consider the crepant pull back

$$\widetilde{h}^*(K_{\overline{X}} + \overline{C}) = K_{\widetilde{X}} + \widetilde{C} + \widetilde{B},$$

where $\widetilde{C}$ is the proper transform of $\overline{C}$, $\widetilde{h}_*\widetilde{B} = 0$, and $\widetilde{B} \geq 0$. Since $K_{\overline{X}} + \overline{C}$, it is easy to see that $\lfloor \widetilde{B} \rfloor = 0$. It is clear also that the set $\operatorname{Supp}(\overline{C} + \widetilde{B})$ coincides with the fiber over o. By construction, $\operatorname{Supp}\widetilde{B}$ contains no -1-curves.

First we consider the case when $\operatorname{Supp}\widetilde{C}$ also contains no -1-curves. Then $X = \widetilde{X}$ is exactly the minimal resolution of $\overline{X}$. By 7.2.2 singular points of $\overline{X}$ are Du Val. Cases $\widetilde{D}_n$ ($n \geq 5$), $\widetilde{D}_4$, $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$ of Proposition 7.2.6 gives cases I_b^* (with $b \geq 1$), I_0^*, IV^*, III^*, and II^*, respectively. For example, if $\overline{C}$ is irreducible and

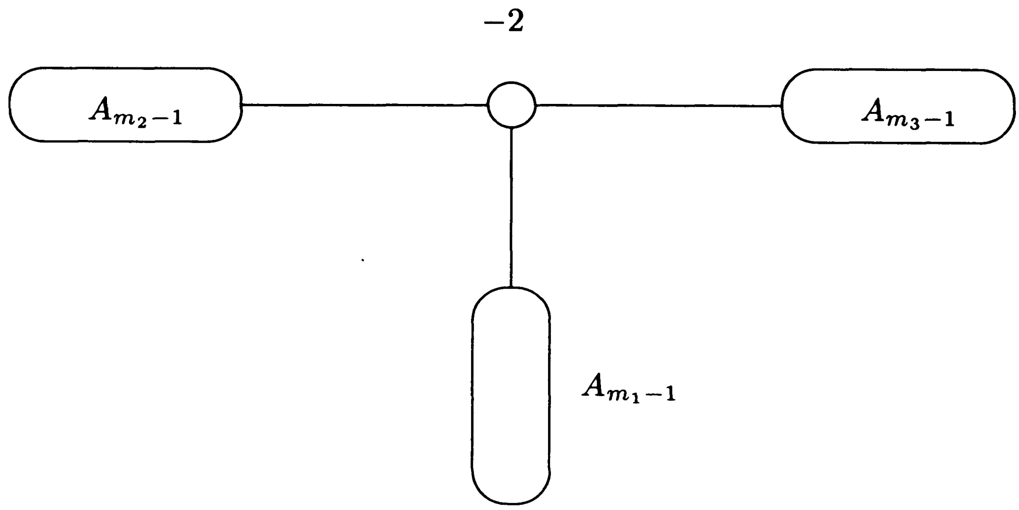


FIGURE 7.1

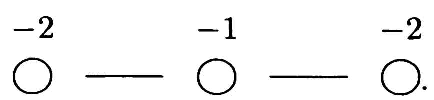
there are exactly three singular points of $\overline{X}$, then similar to 6.1 the graph of the minimal resolution $\widetilde{h}\colon \widetilde{X} \to \overline{X}$ must be as in Fig. 7.1.

By 4.1.12 we have the following possibilities for (m_1, m_2, m_3):

$$\begin{array}{llll} \widetilde{E}_6: & (m_1,m_2,m_3) = (3,3,3) & \Longrightarrow & \text{case } IV^*, \\ \widetilde{E}_7: & (m_1,m_2,m_3) = (2,4,4) & \Longrightarrow & \text{case } III^*, \\ \widetilde{E}_8: & (m_1,m_2,m_3) = (2,3,6) & \Longrightarrow & \text{case } II^*. \end{array}$$

Now, we consider the case when Supp$\widetilde{C}$ contains a -1-curve. Since $\widetilde{h}\colon \widetilde{X} \to \overline{X}$ is a minimal resolution, all -1-curves are contained in $\widetilde{C}$, the proper transform of $\overline{C}$. Using the negative semidefiniteness for the fiber $\widetilde{F} \subset \widetilde{X}$ over o one can show that the dual graph of $\widetilde{F}$ cannot contain proper subgraphs of the form

and



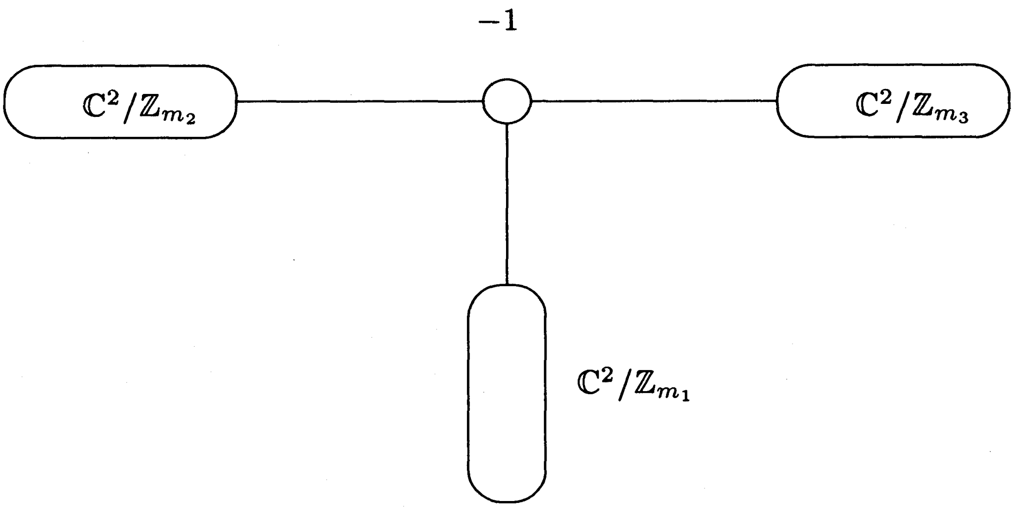
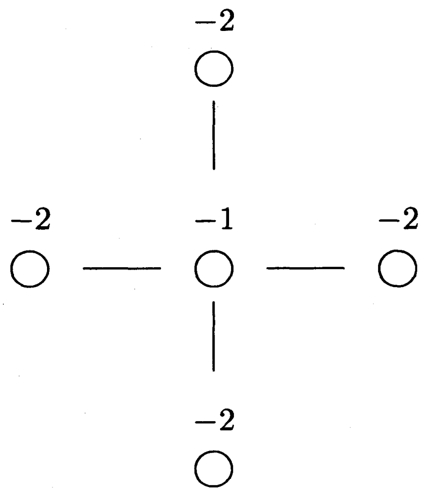


FIGURE 7.2

Suppose that $\overline{C}$ is irreducible. Then $K_{\overline{X}} + \overline{C}$ is plt and $\widetilde{C}$ is the only a -1-curve. Thus in the case $\widetilde{D}_4$ we obtain the dual graph for a fiber of $\widetilde{X} \to Z$ as below



By the above this is impossible. In other cases we have the dual graphs as in Fig. 7.2. For $(m_1, m_2, m_3) = (3, 3, 3)$, $(2, 4, 4)$ and $(2, 3, 6)$ we obtain cases IV, III and II, respectively. Similarly Case $\widetilde{D}_n$, $n \geq 5$ of Proposition 7.2.6 gives Case I_b^*.

Non-simply connected fibers are only of type I_b, so only they can be multiple. This proves the theorem. □

The following table shows correspondence between fibers of minimal smooth elliptic fibrations and their dlt models:

$\overline{X}$	Ell	$\widetilde{A}_n,\ n \geq 1$	$\widetilde{D}_4$	$\widetilde{D}_n,\ n \geq 5$	$\widetilde{E}_6$	$\widetilde{E}_7$	$\widetilde{E}_8$
$\widetilde{X} = X$	$_mI_0$	$_mI_n,\ n \geq 2$	I_0^*	I_{n-4}^*	IV^*	III^*	II^*
$\widetilde{X} \neq X$	–	$_mI_b,\ b \leq n-1$	–	$I_b^*,\ b \leq n-5$	IV	III	II
$\operatorname{compl}(\overline{X}, \overline{F})$	1	1	2	2	3	4	6

THEOREM 7.2.11 ([**Sh3**], cf. Theorem 6.0.6). *Let $f\colon X \to Z \ni o$ be a contraction from a normal surface X onto a smooth curve Z. Let $D = \sum d_i D_i$ be a boundary on X. Assume that $K_X + D$ is lc and $-(K_X + D)$ is f-nef. Then there exists a regular complement of $K_X + D$. This complement $K_X + D^+$ can be taken so that $a(E, D) = -1$ implies $a(E, D^+) = -1$ for any divisor E of $\mathcal{K}(X)$. Moreover, if there are no 1, or 2-complements, then $(X/Z \ni o, D)$ is exceptional.*

PROOF. By Corollaries 7.0.10 and 7.2.9 we may assume that $K_X + D \equiv 0$ over Z and a general fiber of f is rational. First, as in the proof of Theorem 6.0.6, we replace the boundary D with $D + \alpha f^*o$ so that $K_X + D + \alpha f^*o$ is maximally lc. Replacing X with its log terminal modification, we may assume that X is smooth and the reduced part $C := \lfloor D \rfloor$ of the boundary is nonempty. Next we blow up a sufficiently general point on $C := \lfloor D \rfloor$. We get a new model such that some component E of $F = f^{-1}(o)$ is -1-curve and it is not contained in $\operatorname{Supp} D$. Moreover, $E \cap \lfloor D \rfloor$ is a point which is nonsingular for $\operatorname{Supp} D$. Let $C_1 \subset \lfloor D \rfloor$ be a (unique) component passing through $E \cap \operatorname{Supp} D$. Then the curve $\operatorname{Supp} F \setminus E$ can be contracted to a point, say Q:

$$f\colon X \xrightarrow{g} Y \to Z.$$

The central fiber $g(E)$ of $Y \to Z$ is irreducible. Since $K_X + D \equiv 0/Y$, the point $Q \in Y$ is lc. Apply Theorem 6.0.6 to the birational contraction $g\colon X \to Y$. We get a regular n-complement $K_X + D^+$ in a neighborhood of $g^{-1}(Q) = \operatorname{Supp}(F - E)$. We claim that this complement extends to a complement in a neighborhood of the whole fiber F. We need to check only that $nD^+ \sim -nK_X$ in a neighborhood of F. But in our situation the numerical equivalence over Z coincides with linear one. Therefore the last is equivalent to $D^+ \equiv -K_X$. Obviously, both sides have the same intersection numbers with all components of F different from E. For E we have $1 = -K_X \cdot E$, $E \cdot D^+ = E \cdot C_1 = 1$ (because the coefficients of C_1 in D and D^+ are equal to 1). This proves the theorem. □