CHAPTER 5

Log del Pezzo surfaces

In the present chapter we discuss some properties of log del Pezzo surfaces.

5.1. Definitions and examples

DEFINITION 5.1.1. A projective log surface (X, D) is called

- a *log del Pezzo surface* if $K_X + D$ is lc and $-(K_X + D)$ is nef and big;
- a *log Enriques surface* if $K_X + D$ is lc and $K_X + D \equiv 0$.

Higher dimensional analogs of these are called *log Fano* and *log Calabi-Yau* varieties, respectively. Usually we omit D if $D = 0$.

If (X, D) is a log del Pezzo, then by Proposition 11.1.1 there exists some $\mathbb{Q}$-complement $K_X + D^+$ of $K_X + D$. The pair (X, D^+) is a log Enriques surface.

Examples of log del Pezzo surfaces are the classical ones, weighted projective planes $\mathbb{P}(a_1, a_2, a_3)$ with boundary $D = \sum d_i D_i$, where $D_i := \{x_i = 0\}$ and $\sum d_i < 3$, Hirzebruch surfaces $\mathbb{F}_n$ with boundary $\alpha\Sigma_0$, where Σ_0 is the negative section and $(n-2)/n \le \alpha \le 1$.

Let $f\colon (X', D') \to (X, D)$ be a birational log crepant morphism; that is,

$$K_{X'} + D' = f^*(K_X + D), \quad \text{with} \quad f_*D' = D.$$

Then (X, D) is a log del Pezzo if and only if so is (X', D') (see 1.1.5). Conversely, if $f\colon X' \to X$ is a birational morphism and (X', D') is a log del Pezzo then so is (X, f_*D'). Many examples can also be obtained by taking finite quotients; see 1.2.

EXAMPLE 5.1.2. Let $G \subset \mathrm{PGL}_2(\mathbb{C})$ be a finite subgroup, $X := \mathbb{P}^2/G$ and $f\colon \mathbb{P}^2 \to X$ the natural projection. As in 1.2, we define a boundary D on X by the condition $K_{\mathbb{P}^2} = f^*(K_X + D)$, where $D = \sum(1 - 1/r_i)D_i$, all the D_i are images of lines on $\mathbb{P}^2$, and r_i is the ramification index over D_i. For example, if G is the symmetric group $\mathfrak{S}_3$, acting on $\mathbb{P}^2$ by permutations of coordinates, X is the weighted projective plane $\mathbb{P}(1,2,3) = \mathrm{Proj}\,\mathbb{C}[\sigma_1, \sigma_2, \sigma_3]$, where the σ_i are the symmetric functions on coordinates on $\mathbb{P}^2$. The divisor D has exactly one component D_1 with coefficient $1/2$, where D_1 is determined by the equation

$$\sigma_1^2\sigma_2^2 - 4\sigma_2^3 - 4\sigma_1^3\sigma_3 - 27\sigma_3^2 + 18\sigma_1\sigma_2\sigma_3 = 0$$

(the equation of the discriminant). The surface X has exactly two singular points which are Du Val of types A_1 and A_2. Therefore X is a Gorenstein del Pezzo surface of degree $K_X^2 = 6$. The curve D_1 is contained in the smooth locus and has a unique singularity at $(1, 1/3, 1/27)$, which is a cusp.

LEMMA 5.1.3. *Let (X, D) be a log del Pezzo surface. Assume additionally either $K_X + D$ is klt or $K_X + D$ is dlt and $-(K_X + D)$ is ample. Then*

(i) Pic(X) *is a finitely generated free abelian group;*
(ii) *the numerical equivalence in* Pic(X) *coincides with linear one;*
(iii) *group of classes of Weil divisors* Weil$_{\text{lin}}(X)$ *is finitely generated.*

Recall that two-dimensional log terminal singularities are automatically $\mathbb{Q}$-factorial.

SKETCH OF PROOF. From the exponential sequence and Kawamata-Viehweg vanishing we have Pic$(X) \simeq H^2(X, \mathbb{Z})$. Assume that $D \in$ Pic(X) is n-torsion. Then again Kawamata-Viehweg vanishing and by Riemann-Roch, $|D| \neq \varnothing$. Therefore $D \sim 0$. (iii) follows by [**K**, Lemma 1.1]. □

5.2. Boundedness of log del Pezzos

It is well known that the degree K_X^2 of classical del Pezzo surfaces is bounded by 9. This however is not true for log del Pezzo surfaces. Indeed, we can take $(\mathbb{F}_n, (1-2/n)\Sigma_0)$, where $\mathbb{F}_n$ is the Hirzebruch surface and Σ_0 is the negative section. Then $-(K + (1-2/n)\Sigma_0)$ is nef and big. It is easy to compute that

$$(K + (1-2/n)\Sigma_0)^2 = n + 4 + 4/n,$$

so it is unbounded. However, results of Alexeev and Nikulin (see [**A**]) show that the degree $(K_X + D)^2$ of log del Pezzo surfaces is bounded by a constant Const(ε) if $K_X + D$ is ε-lt. More precisely we have

THEOREM 5.2.1 ([**A**], see also [**KeM**, Sect. 9]). *Fix $\varepsilon > 0$. Let $(X, D = \sum d_i D_i)$ be a projective log surface such that $-(K_X + D)$ is nef, $K_X + D$ is ε-lt and $d_i < 1 - \varepsilon$. Then the class $\{X\}$ is bounded in the algebraic moduli sense except for the case when $D = 0$ and $K_X \sim 0$.*

In the case when $D = 0$ and $-K_X$ is ample we have more effective Nikulin's results.

THEOREM 5.2.2 ([**N2**]). *Let X be a projective surface with only klt singularities such that $-K_X$ is ample and $X^{\min} \to X$ a minimal resolution. Then*

$$\rho(X^{\min}) < \begin{cases} 3141 & \text{if } e = 2, \\ 5317 & \text{if } e = 3, \\ 17735 & \text{if } e = 4, \\ \frac{192e(e-3)(6e-27)}{\varepsilon(e)} + 1536e^2(e-3) + 1820e + 69 & \text{if } e \geq 5, \end{cases}$$

where e is the maximal multiplicity of the singularities of X and $\varepsilon(e)$ is some function of e.

Nikulin [**N1**] obtained also a better bound (linear in e)

$$\rho(X^{\min}) \le 352e + 1284$$

in the case when all the discrepancies of X satisfy the inequalities

$$a(E, 0) \le -1/2 \qquad \text{or} \qquad a(E, 0) \ge 0.$$

Proofs of theorems 5.2.1, 5.2.2 use weighted graph technique and Nikulin's diagram method. In the case $\rho(X) = 1$ Theorem 5.2.2 was proved also in [**KeM**, Sect. 9] by using nonnegativity of $\hat{c}_2(\hat{\Omega}^1_X(\log D))$ and Bogomolov type inequality $\hat{c}_1^2(\hat{\Omega}^1_X(\log D)) \le 3\hat{c}_2(\hat{\Omega}^1_X(\log D))$ (see also [**Ut**, Ch. 10] and [**K1**]). As an easy consequence of this theory we have the following

THEOREM 5.2.3 ([**KeM**, 9.2]). *Let (X, C) be a log surface with $\rho(X) = 1$ such that $K_X + C$ is lc, K_X is klt and $C = \sum C_i$ is reduced. Then*

$$\sum_{P \in (X \setminus C)} \frac{m_P - 1}{m_P} \le \chi_{top}(X) - \chi_{top}(C)$$

where m_P is the order of the local fundamental group $\pi_1(U_P \setminus \{P\})$ (U_P is a sufficiently small neighborhood of P). If X is rational and $p_a(C) = 0$, then

$$\sum_{P \in (X \setminus C)} \frac{m_P - 1}{m_P} \begin{cases} \le 3 & \text{if } C = 0 \\ \le 1 & \text{if } \#\{C_i\} = 1 \\ = 0 & \text{if } \#\{C_i\} = 2. \end{cases} \tag{5.1}$$

Using this fact one can easily show the following:

COROLLARY 5.2.4. *Let X be a log del Pezzo surface with $\rho(X) = 1$ such that K_X is klt. Then the number of singular points of X is at most 5.*

PROOF. By Theorem 5.2.3 the number of singular points is ≤ 6. Assume that X has exactly six singular points $P_1, \dots, P_6$. Then by inequality (5.1) we have $m_{P_1} = \cdots = m_{P_6} = 2$. This means that $P_1, \dots, P_6$ are ordinary double points. In particular, K_X is Cartier. Applying Noether's formula to the minimal resolution $\tilde{X}$ of X, we obtain $K_X^2 = K_{\tilde{X}}^2 = 10 - \rho(\tilde{X}) = 10 - 1 - 6 = 3$. Let $\tilde{L} \subset \tilde{X}$ be a -1-curve and $L \subset X$ its image. Then $-K_X \cdot L = -K_{\tilde{X}} \cdot \tilde{L} = 1$. Since $\rho(X) = 1$, we have $L \equiv -\frac{1}{3}K_X$, so $L^2 = \frac{1}{3}$. On the other hand, $2L$ is Cartier, a contradiction. □

5.3. On the existence of regular complements

PROPOSITION 5.3.1 (Inductive Theorem, Weak Form [**Sh3**]). *Let (X, D) be a log del Pezzo surface. If $K_X + D$ is not klt, then there exists a regular complement of $K_X + D$ (i.e. n-complement with $n \in \mathcal{R}_2$). Moreover, if $K_X + D$ is not 1 or 2-complementary, then there is at most one divisor of $\mathcal{K}(X)$ with discrepancy $a(\cdot, D) = -1$.*

In [**KeM**] such a log divisor $K_X + D$ was called a *tiger*. This is a sort of antithesis to Reid's general elephant (see 4.1.1).

PROOF. Replacing (X, D) with a log terminal modification, we may assume that $K_X + D$ is dlt. Then $\lfloor D \rfloor \neq 0$. In this situation we can apply Proposition 4.4.3 and 4.1.10. The last statement follows by Connectedness Lemma. $\square$

COROLLARY 5.3.2. *Let (X, D) be a log del Pezzo surface with $D \in \Phi_{\mathbf{m}}$. Write, as usual, $D = C + B$, where $C := \lfloor D \rfloor$, $B := \{D\}$. Assume that $|-nK_X - nC - \lfloor (n+1)B \rfloor| \neq \varnothing$ for some $n \in \mathcal{R}_2$. Then $K_X + D$ has a regular complement.*

Note that the inverse implication follows by (4.1).

DEFINITION 5.3.3 ([**Ut**, 18.2]). A log divisor $K_X + D + \sum b_i B_i$ is said to be *maximally log canonical* if $K_X + D + \sum b'_i B_i$ is not lc, where $b'_i \geq b_i$ with inequality holding for at least one index i. Note that this definition depends on the decomposition $D + \sum b_i B_i$, not only on the sum $D + \sum b_i B_i$.

PROOF. If $K_X + D$ is not klt, the assertion follows by Proposition 5.3.1. Thus we may assume that $K_X + D$ is klt (in particular, $C = 0$). Let

$$\overline{D} \in |-nK_X - \lfloor (n+1)B \rfloor|$$

As in (4.1) and (4.2) put

$$D' := \frac{1}{n} \left(\lfloor (n+1)B \rfloor + \overline{D} \right).$$

Note that $D' \geq D$ (because $D \in \Phi_{\mathbf{m}}$, see 4.2.8). If $K_X + D'$ is lc, then this is a regular complement. Assume that $K_X + D'$ is not lc. Take α so that $K_X + D + \alpha(D' - D)$ is maximally lc. It is clear that $0 < \alpha < 1$ and $-(K_X + D + \alpha(D' - D))$ is nef and big. Now we can apply Proposition 5.3.1 to $K_X + D + \alpha(D' - D)$ to get the desired regular complement. $\square$

COROLLARY 5.3.4 ([**Sh3**], cf. Corollary 8.4.3). *Let (X, D) be a log del Pezzo surface. If $(K_X + D)^2 > 4$, then it is nonexceptional. In this case, there exists a regular complement of $K_X + D$. Moreover, there exists such a complement which is not klt.*

PROOF. Riemann-Roch gives that $\dim |-n(K_X + D)|$ is sufficiently large, where n is divisible enough and $n \gg 0$. Then standard arguments show that $K_X + D + \alpha H$ is not klt for some $H \in |-n(K_X + D)|$ and $\alpha < 1/n$ (see e.g. [**Ko2**, Lemma 6.1] or the proof of Corollary 8.4.3). $\square$

COROLLARY 5.3.5. *Let X be a log del Pezzo surface. Assume that $K_X^2 > 4$. Then $|-nK_X| \neq \varnothing$ for some $n \in \mathcal{R}_2$.*

EXERCISE 5.3.6. Let $G \subset \mathrm{PGL}_2(\mathbb{C})$ be a finite subgroup. Then G acts naturally on $\mathbb{P}^1 \times \mathbb{P}^1$ so that the action is free in codimension one. Prove that the quotient $X := (\mathbb{P}^1 \times \mathbb{P}^1)/G$ is a log del Pezzo and K_X is 1, 2, 3, 4, or 6-complementary. *Hint.* Apply Proposition 5.3.1 to $K_X + \Delta$, where Δ is the image of the diagonal.

5.3.7. Log del Pezzo surfaces can be classified in terms of complements. A log del Pezzo (or log Enriques) surface (X, D) is said to be *regular* if $K_X + D$ is r-complementary for some $r \in \mathcal{R}_2$. Shokurov [**Sh3**] proposed the following rough classification of them.

Let (X, D) be a log surface having a regular r-complement $K_X + G$ (i.e., with $r \in \mathcal{R}_2$). Let $\mathcal{A}(X, G)$ be the set of divisors with discrepancy $a(\cdot, G) = -1$.

We say that $K_X + G$ is of type

$\mathbb{A}^n_m$: if $r = 1$ and $\mathcal{A}(X, G)$ is infinite;
$\mathbb{E}1^n_m$: if $r = 1$ and $\mathcal{A}(X, G)$ is finite;
$\mathbb{D}^n_m$: if $r = 2$ and $\mathcal{A}(X, G)$ is infinite;
$\mathbb{E}2^n_m$: if $r = 2$ and $\mathcal{A}(X, G)$ is finite;
$\mathbb{E}3^n_m$, $\mathbb{E}4^n_m$, $\mathbb{E}6^n_m$: if $r = 3, 4, 6$, respectively,

where n is the number of components of $\lfloor G \rfloor$ and m is the number of exceptional divisors with $a(\cdot, G) = -1$ on a minimal log terminal modification. Thus $n + m = 0$ if and only if $K_X + G$ is klt. In cases $\mathbb{E}1\text{-}6^n_m$ we always have $n + m \leq 2$. Moreover, $n + m = 2$ only in the dipole case. For example, a weighted projective plane $\mathbb{P}(a, b, c)$ has a natural structure of toric 1-complement of type $\mathbb{A}^3_m$. More general, any toric surface has a complement of type $\mathbb{A}^n_m$. Note that this division into cases gives is very rough classification, more delicate invariant of a nonexceptional log variety is the simplicial topological space introduced in [**Sh3**, Sect. 7], see also [**I**].

EXERCISE 5.3.8 ([**Sh3**], cf. 2.2.18). Let (X, D) be a log del Pezzo surface such that $K_X + D$ is dlt and $-(K_X + D)$ is ample. Prove that $\lfloor D \rfloor$ has at most two components. Moreover, if $\lfloor D \rfloor$ has exactly two components, then $K_X + D$ is 1 or 2-complementary. If $K_X + D$ is 1-complementary, then $(X, \lfloor D^+ \rfloor)$ is a toric pair (see 2.2.18). *Hint.* Use Adjunction and 4.4.3.

5.4. Nonrational log del Pezzo surfaces

Of course we cannot expect to get a reasonable classification of all log del Pezzos. Below we describe nonrational ones. Results of Ch. 9.3 shows that exceptional log del Pezzos (see 4.5.1) at least in principle can be classified. By Kawamata-Viehweg vanishing we have

LEMMA 5.4.1. *Let (X, D) be a log del Pezzo surface. Assume that $K_X + D$ is dlt. Then X is rational if and only if $H^1(X, \mathcal{O}_X) = 0$. Moreover, X is rational if either $K_X + D$ is klt or $-(K_X + D)$ is ample.*

PROPOSITION 5.4.2. *Let (X, D) be a log del Pezzo surface such that $K_X + D$ is dlt. Put $C := \lfloor D \rfloor$ and $B := \{D\}$. Assume that X is nonrational. Then*

(i) $\rho(X) \geq 2$;
(ii) *if $\rho(X) = 2$, then X is smooth, $X \simeq \mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is a rank two vector bundle on an elliptic curve, all components of D are horizontal and $C = \Sigma_0$ is the negative section, which does not intersect other components of D;*
(iii) *if $\rho(X) \geq 3$, then there exists a contraction with rational fibers $g\colon X \to X'$ onto a log del Pezzo $(X', D' = C' + B')$ with $\rho(X') = 2$. Moreover, C' is a*

smooth elliptic curve contained in the smooth locus of X (and then it is a section of the composition map $f\colon X \to X' \to Z$ and it does not intersects other components of D);

(iv) $H^1(X, \mathcal{O}_X) \simeq \mathbb{C}$.

PROOF. First note that $C \neq 0$, because $K_X + D$ is not klt and C is connected by 2.3.1.

The assertion (i) follows by Lemma 5.4.1.

To prove (ii) we note that there exists an extremal $(K_X + D)$-negative contraction $f\colon X \to Z$. By (i), f is not birational. Hence Z is a curve of genus $g(Z) \geq 1$ and fibers of f are irreducible. If C is contained in fibers of f, then $-(K_X + D - \varepsilon C)$ is nef and big. By Lemma 5.4.1, X is rational in this case. So we assume that there is a component $C' \subset C$ such that $f(C') = Z$. Thus $p_a(C') \geq 1$. Let F be a general fiber. Then $C' \cdot F \leq D \cdot F < 2$. It follows that $C' \cdot F = 1$. Further,

$$0 \leq 2p_a(C') - 2 + \deg \mathrm{Diff}_{C'}(0) = (K_X + C') \cdot C' \\ \leq (K_X + D) \cdot C' \leq 0.$$

From this we have that C' is smooth elliptic curve, $g(Z) = 1$, C' is contained in the smooth part of X and does not intersect other components of D. In particular, $C' = C$. Since C is the section, $f\colon X \to Z$ has no multiple fibers. Therefore X is smooth and $X \simeq \mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is a rank two vector bundle on Z. From $K_X^2 = 0$ and $(K_X + C)^2 > 0$ we have $C^2 < 0$.

As for (iii), run $(K_X + D)$-MMP $g\colon X \to X'$. At the end we obtain (X', D') with an extremal contraction $X' \to Z$ as in (ii). By Lemma 5.4.1 $C' := g(C) \neq 0$ cannot be contracted to a point on X'. As in (ii), by Adjunction we have

$$0 \geq (K_X + D) \cdot C \geq (K_X + C) \cdot C = 2p_a(C) - 2 + \deg \mathrm{Diff}_C(0) \geq 0.$$

This yields $p_a(C) = p_a(Z) = 1$ and $\mathrm{Diff}_C(0) = 0$. Hence C is a smooth elliptic curve, X is smooth along C and C does not intersect other components of D.

Finally, (iv) follows by (iii) because $R^1 f_* \mathcal{O}_X = 0$. □

COROLLARY 5.4.3. *Let (X, D) be a log del Pezzo surface. Then X is rational or is birationally isomorphic to a ruled surface over an elliptic curve.*

COROLLARY 5.4.4. *Let $(X, D = C + B)$ be a log del Pezzo surface. Assume that X is nonrational. If $\rho(X) = 1$, then X is a generalized cone over an elliptic curve (the contraction of the negative section on $\mathbb{P}(\mathcal{E})$; see Proposition 5.4.2).*

EXERCISE 5.4.5. Let (X, D) be a log del Pezzo surface. Assume that X is nonrational. Prove that $K_X + D$ is 1-complementary. *Hint.* Apply Proposition 4.4.3.