

CHAPTER 4

Definition of complements and elementary properties

4.1. Introduction

The following conjecture is called *Reid's general elephant conjecture*

CONJECTURE 4.1.1. *Let $f: X \rightarrow Z \ni o$ be a K_X -negative contraction from a threefold with only terminal singularities. Then near the fiber over o the linear system $-K_X$ contains a divisor having only Du Val singularities.*

At the moment it is known that this conjecture is true (only in analytic situation) in the following cases:

- $X = Z \ni o$ is an isolated singularity [RY], moreover, this is equivalent to the classification of three-dimensional terminal singularities;
- $f: X \rightarrow Z$ is an extremal flipping or divisorial small contraction [Mo], [KoM], this is a sufficient condition for the existence of flips [K].

Some particular results are known in the case when $f: X \rightarrow Z$ is an extremal contraction to a surface [P]. This case is interesting for applications to rationality problem of conic bundles.

However, at the moment it is not so clear how one can prove Reid's conjecture in the algebraic situation. Moreover, it fails for the case $Z = \text{pt}$ (there are examples of \mathbb{Q} -Fano threefolds with empty $|-K_X|$). Shokurov proposed the notion of complements, which is weaker than "general elephant" but much more easier to work with.

DEFINITION 4.1.2. Let (X, D) be a log pair, where D is a subboundary. Then a \mathbb{Q} -complement of $K_X + D$ is a log divisor $K_X + D'$ such that $D' \geq D$, $K_X + D'$ is lc and $n(K_X + D') \sim 0$ for some $n \in \mathbb{N}$.

DEFINITION 4.1.3 ([Sh2]). Let X be a normal variety and $D = S + B$ a subboundary on X , such that B and S have no common components, S is an effective integral divisor and $\lfloor B \rfloor \leq 0$. Then we say that $K_X + D$ is n -complementary, if there is a \mathbb{Q} -divisor D^+ such that

- (i) $n(K_X + D^+) \sim 0$ (in particular, nD^+ is integral divisor);
- (ii) $K_X + D^+$ is lc;
- (iii) $nD^+ \geq nS + \lfloor (n+1)B \rfloor$.

In this situation the n -complement of $K_X + D$ is $K_X + D^+$. If moreover $K_X + D^+$ is plt, then we say that $K_X + D$ is *strongly n -complementary*.

Note that an n -complement is not necessarily a \mathbb{Q} -complement because of condition (iii). We need this condition for technical reasons (see 4.4.1). If $B = 0$, then (iii) holds automatically. In applications this is the most interesting case. We give also a generalization of this definition for the case of nodal curves.

DEFINITION 4.1.4. Let X be a reduced (not necessarily irreducible) curve. Then X is said to be *nodal* if all its singularities are normal crossing points. A subboundary $D = \sum d_i D_i$ on a nodal curve is said to be *semilog canonical (slc)* if $\text{Supp} D \cap \text{Sing} X = \emptyset$ and $d_i \leq 1$ for all i .

Let X be a nodal curve and $D = S + B$ a subboundary on X , such that B and S have no common components, S is an effective integral divisor and $\lfloor B \rfloor \leq 0$. Assume that $\text{Supp} D \cap \text{Sing} X = \emptyset$. Then an n -semicomplement of $K_X + D$ is a log divisor $K_X + D^+$ such that conditions (i), (iii) of 4.1.3 and the following (ii') below holds.

(ii') $K_X + D^+$ is slc.

The last definition can be generalized to the higher dimensional case (see [Ut]).

REMARK 4.1.5. Assume that on a variety X the canonical divisor K_X is strongly 1-complementary. Let $K_X + B$ be this complement. Then B is an integral divisor, $B \in |-K_X|$ and $K_X + B$ is plt (and even canonical because $K_X + B \sim 0$). By 2.2.4, $\text{Diff}_B(0) = 0$ and by Inversion of Adjunction, K_B is klt. Since $K_B \sim 0$, B has only canonical Gorenstein singularities. This shows that K_X is strongly 1-complementary if and only if Reid's general elephant conjecture holds for X/Z .

The following conjecture seems to be more realistic than Conjecture 4.1.1:

CONJECTURE 4.1.6. Let $f: X \rightarrow Z \ni o$ be a contraction from a threefold with only terminal singularities such that $-K_X$ is f -nef and f -big. Then near the fiber over o the canonical divisor K_X is 1, 2, 3, 4, or 6-complementary.

Note that the condition that $K_X + D$ is n -complementary implies the existence of an integral effective divisor

$$(4.1) \quad \overline{D} \in |-nK_X - nS - \lfloor (n+1)B \rfloor|$$

related to D^+ by the equality

$$(4.2) \quad D^+ := S + \frac{1}{n} (\lfloor (n+1)B \rfloor + \overline{D}).$$

It is also easy to see that if D is a boundary, then so is D^+ . As an immediate consequence of the definition we have

PROPOSITION 4.1.7. Let X be a normal variety.

(i) Fix $n \in \mathbb{N}$. Let $D = \sum d_i D_i$ and $D' = \sum d'_i D_i$ be subboundaries on X such that the following conditions hold

- a) $d'_i \geq d_i - \varepsilon$ for $0 < \varepsilon \ll 1$;
- b) $d'_i \geq d_i$ whenever $(n+1)d_i$ is an integer $\leq n$.

Assume that $K_X + D'$ is n -complementary. Then $K_X + D$ is n -complementary.

- (ii) Fix $n \in \mathbb{N}$. Let $D = \sum d_i D_i$ be a subboundary. Assume that $K_X + D$ is n -complementary. Then $K_X + D'$ is n -complementary for any subboundary $D' = \sum d'_i D_i$ such that $|d'_i - d_i| < \frac{1}{(n+1)q_i}$, $\forall i$, where $q_i \geq 1$ is the denominator of d_i .

SKETCH OF PROOF. We show, for example, (i). Let $D'^+ = \sum d_i^+ D_i$ be an n -complement of $K_X + D'$. Put $D^+ := D'^+$. It is sufficient to verify (iii) of 4.1.3, i.e.

$$(4.3) \quad d_i^+ \geq \begin{cases} \frac{1}{n} \lfloor (n+1)d_i \rfloor & \text{if } d_i < 1 \\ 1 & \text{otherwise.} \end{cases}$$

On the other hand, we have

$$d_i^+ \geq \begin{cases} \frac{1}{n} \lfloor (n+1)d'_i \rfloor & \text{if } d'_i < 1 \\ 1 & \text{otherwise.} \end{cases}$$

If $n+1$ is a denominator of d_i , then $d_i \leq d'_i$ and (4.3) is obvious. If $n+1$ is not a denominator of d_i , then $d'_i \geq d_i - \varepsilon$. Hence $\lfloor (n+1)d_i \rfloor = \lfloor (n+1)d'_i \rfloor$ for small positive ε . Again we obtain (4.3). Finally, if $d_i = 1$, then $d_i^+ \geq \frac{1}{n} \lfloor (n+1)(1-\varepsilon) \rfloor = 1$ for $\varepsilon < 1/(n+1)$. \square

COROLLARY 4.1.8. Let X be a normal variety and $D = \sum d_i D_i$ a subboundary on X . Fix $n \in \mathbb{N}$. Let $D' = \sum d'_i D_i$, where $d'_i \geq \min\{d_i, \frac{n}{n+1}\}$. Assume that $K_X + D'$ is n -complementary. Then so is $K_X + D$.

- EXAMPLE 4.1.9. (i) Let X be a toric variety and $S = \sum S_i$ be the toric boundary. Then $K_X + S \sim 0$ and $K_X + S$ is lc. Hence $K_X + S$ is 1-complementary.
- (ii) Let $(X \ni P)$ be an analytic germ of a three-dimensional terminal singularity. Then K_X is strongly 1-complementary (see 2.2.12). Conversely, if there is a strong nontrivial 1-complement near an isolated three-dimensional \mathbb{Q} -Gorenstein singularity $(X \ni P)$, then $(X \ni P)$ is terminal. A three-dimensional Gorenstein canonical singularity is (nontrivially) 1-complementary if and only if it is cDV (see 2.2.14).
 - (iii) Consider the cyclic quotient singularity $X := \mathbb{C}^3/\mathbb{Z}_9(1,4,7)$. By 1.2.4 it is canonical. Since X is not terminal, K_X is not strongly 1-complementary. However, K_X is strongly 2-complementary. Indeed, $(x^2y + y^2z + z^2x)(dx \wedge dy \wedge dz)^{-2}$ is an invariant form. Hence this gives us a member of $|-2K_X|$. It is easy to check that $x^2y + y^2z + z^2x = 0$ defines a log canonical singularity $F \subset \mathbb{C}^3$. Let $F' := F/\mathbb{Z}_9 \subset X$. By Corollary 1.2.2 $K_X + \frac{1}{2}F'$ is klt. Note that K_X is 1-complementary in this case (see (i)).

- (iv) According to [MMM] there are four-dimensional terminal cyclic quotient singularities which have no strong 1 or 2-complements. However, it is expected that there are only a finite number of such exceptions. For example, the singularity $\mathbb{C}^4/\mathbb{Z}_{83}(3, 14, 23, 44)$ has no strong 1 or 2-complements. As above, the invariant $(dx \wedge dy \wedge dz)^{-3}$ gives us a strong 3-complement.
- (v) Let $f: X \rightarrow Z \ni o$ be an analytic germ of a three-dimensional flipping extremal contraction. Then K_X is strongly 1-complementary [Mo], [KoM].
- (vi) Let X be a Fano threefold with Gorenstein canonical singularities. Then K_X is strongly 1-complementary [Sh], [R].
- (vii) Let X be a variety with log canonical singularities and numerically trivial canonical divisor K_X . Then K_X is n -complementary if and only if there exists n such that $nK_X \sim 0$. For example, in the case of a smooth surface of Kodaira dimension 0 the canonical divisor is either 1, 2, 3, 4 or 6-complementary (see e.g. [BPV]).
- (viii) Let $g: X \rightarrow \mathbb{P}^1$ be a relatively minimal elliptic fibration, where X is a smooth surface of Kodaira dimension $\kappa(X) \leq 0$. Then X is n -complementary for some $n \in \{1, 2, 3, 4, 6\}$. Indeed, we have the canonical bundle formula (see e.g. [BPV, Ch. V, §12])

$$K_X \sim (\chi(\mathcal{O}_X) - 2)L + \sum_{i=1}^s (r_i - 1)E_i,$$

where E_i are multiple fibers of multiplicities r_i and L is a general fiber. Consider, for example, case $\kappa(X) = -\infty$. Then X is a ruled surface over an elliptic curve. Let F be a general fiber of the ruling and denote $\delta := L \cdot F$. Clearly, $K_X \cdot F = -2$ and $E_i \cdot F = \delta/r_i$. This gives us

$$-2 = -2\delta + \sum_{i=1}^s (r_i - 1)\delta/r_i, \quad 2 - 2/\delta = \sum_{i=1}^s (1 - 1/r_i).$$

It is easy to see that $2 \leq s \leq 3$ and $K_X \sim (s - 1)L + \sum E_i$. There are only the following possibilities:

s	2	3	3	3	3
(r_1, \dots, r_s)	(r_1, r_2)	$(2, 2, r)$	$(2, 3, 3)$	$(2, 3, 4)$	$(2, 3, 5)$
δ	$2(r_1^{-1} + r_2^{-1})^{-1}$	$2r$	12	24	60
n	1	2	3	4	6
n -complement	$E_1 + E_2$	E_3	$\frac{1}{3}E_1$	$\frac{1}{4}E_2$	$\frac{1}{6}E_3$

This shows that $n \in \{1, 2, 3, 4, 6\}$.

Further, in the two-dimensional case 1, 2, 3, 4 and 6-complement we call *regular* and define

$$\mathcal{R}_2 := \{1, 2, 3, 4, 6\}.$$

In the higher-dimensional case we should replace the set \mathcal{R}_2 with bigger one \mathcal{R}_n (see [PSh]).

A very important question is:

when does some n -complement of $K_X + D$ exist?

Obviously, these exist for some $n \gg 0$ when $-(K_X + D)$ is ample (or even semiample) [Sh2, 5.5]. By Base Point Free Theorem (see [KMM, 3-12]), n -complements exist for some $n \gg 0$ if $K_X + D$ is klt and $-(K_X + D)$ is nef and big. It is expected also that we can remove klt condition on lc and $D \in \Phi_{\text{sm}}$ (see Proposition 11.1.1).

In general, only the nef condition is not sufficient for the existence of complements (see Example 8.1.1).

THEOREM 4.1.10 (see [Sh2, 5.2], [Ut, 19.4]). *Let X be a nodal connected (but not necessarily compact) curve. Let D be a boundary on X contained in the smooth and compact part of X . Assume that the degree of $-(K_X + D)$ is nonnegative (on each compact component of X). Then*

- (i) $K_X + D$ is n -semicomplementary for $n \in \mathcal{R}_2$;
- (ii) if $K_X + D$ is not 1 or 2-semicomplementary, then $X \simeq \mathbb{P}^1$ and $[D] = [D^+] = 0$;
- (iii) if X contains a noncomplete component and $K_X + D$ is not 1-semicomplementary, then the compact components of X form a chain $\sum_{i=1}^r X_i$, a (unique) noncomplete component X' intersect an end X_1 of $\sum X_i$, $\text{Supp} D$ is contained in another one X_r and $D = 1/2P_1 + 1/2P_2$ (the case $r = 1$, $X_1 = X_r$ is also possible).

For each log pair $(X/Z \ni o, D)$ we define the minimal complementary number by

$$(4.4) \quad \text{compl}(X, D) := \min\{m \mid K_X + D \text{ is } m\text{-complementary}\}.$$

This is an invariant to “measure” how singular a log pair is. We also define

$$(4.5) \quad \text{compl}'(X, D) := \min\{m \mid \exists m\text{-complement of } K_X + D \text{ which is not klt}\}.$$

By definition, $\text{compl}(X, D), \text{compl}'(X, D) \in \mathbb{N} \cup \{\infty\}$. Take a subset $\Phi \subset [0, 1]$. For example, consider cases $\Phi = \Phi_{\text{sm}}$ (see 2.2.5), $\Phi = [0, 1]$, or $\Phi = \Phi_{\text{m}}$. For a \mathbb{Q} -divisor D we write simply $D \in \Phi$ if all the coefficients of D belong to Φ .

Define the set of natural numbers $\mathcal{N}_n(\Phi)$ by

$$\mathcal{N}_n(\Phi) := \{m \in \mathbb{N} \mid \exists \text{ a log Fano variety } (X, D) \text{ of dimension } n \\ \text{such that } D \in \Phi \text{ and } \text{compl}(X, D) = m\}.$$

Thus Theorem 4.1.10 and Corollary 4.1.11 below give us $\mathcal{N}_1([0, 1]) = \mathcal{R}_2 := \{1, 2, 3, 4, 6\}$. Obviously, $\mathcal{N}_n(\Phi') \subset \mathcal{N}_n(\Phi'')$ if $\Phi' \subset \Phi''$. Theorem 4.1.10 and Corollary 4.1.11 show that $\mathcal{N}_1(\Phi_{\text{sm}}) = \mathcal{N}_1([0, 1]) = \mathcal{R}_2$. We will see below that $\mathcal{N}_2(\Phi_{\text{sm}})$ is bounded (Theorem 9.1.12).

COROLLARY 4.1.11. *Notation as in Theorem 4.1.10. Assume that $X \simeq \mathbb{P}^1$, $[D] = 0$, $-(K_X + D)$ is ample and $D \in \Phi_{\text{sm}}$. Then $D = \sum_{i=1}^r (1 - 1/m_i)D_i$, where for (m_1, \dots, m_r) there is only one of the following possibilities (up to permutations):*

- A_n : (m) , (m_1, m_2) , $K_X + D$ is 1-complementary;
 D_n : $(2, 2, m)$, $K_X + D$ is 2-complementary;
 E_6 : $(2, 3, 3)$, $K_X + D$ is 3-complementary;
 E_7 : $(2, 3, 4)$, $K_X + D$ is 4-complementary;
 E_8 : $(2, 3, 5)$, $K_X + D$ is 6-complementary.

Relations between our notation A_n , D_n , E_6 , E_7 , E_8 and two-dimensional Du Val singularities will be explained in Ch. 6.

EXERCISE 4.1.12. Let $X \simeq \mathbb{P}^1$ and $D \in \Phi_{\text{sm}} \cap [0, 1)$. Assume also that $\deg D = 2$. Show that $D = \sum_{i=1}^r (1 - 1/m_i)D_i$, where for (m_1, \dots, m_r) there is only one of the following possibilities:

- \tilde{D}_4 : $(2, 2, 2, 2)$, $K_X + D$ is 2-complementary;
 \tilde{E}_6 : $(3, 3, 3)$, $K_X + D$ is 3-complementary;
 \tilde{E}_7 : $(2, 4, 4)$, $K_X + D$ is 4-complementary;
 \tilde{E}_8 : $(2, 3, 6)$, $K_X + D$ is 6-complementary.

4.2. Monotonicity

We noticed that the inequality $D^+ \geq D$ does not hold in general. However, under some additional restrictions on coefficients we can expect $D^+ \geq D$ to be true.

4.2.1. Fix $n \in \mathbb{N}$ and define the set \mathcal{P}_n by

$$\alpha \in \mathcal{P}_n \iff 0 \leq \alpha \leq 1 \quad \text{and} \quad \lfloor (n+1)\alpha \rfloor \geq n\alpha.$$

COROLLARY 4.2.2. Let (X, D) be a log pair such that $D \in \mathcal{P}_n$ and $K_X + D^+$ any n -complement. Then $D^+ \geq D$.

LEMMA 4.2.3 (Monotonicity of the integral part). (i) Let $r \in \mathbb{Q}$ such that $r < 1$ and $nr \in \mathbb{Z}$. Then

$$\lfloor (n+1)r \rfloor \leq nr.$$

(ii) Let $r = 1 - 1/m$, $n \in \mathbb{N}$. Then for any $n \in \mathbb{N}$

$$\lfloor (n+1)r \rfloor \geq nr.$$

PROOF. Let us proof, for example, (ii). Write $nr = q + k/m$, where $q = \lfloor nr \rfloor$ and $k/m = \{nr\}$, $k \in \mathbb{Z}$, $0 \leq k \leq m - 1$. Then

$$\lfloor (n+1)r \rfloor = \lfloor q + k/m + 1 - 1/m \rfloor = \begin{cases} q & \text{if } k = 0, \\ q + 1 & \text{otherwise.} \end{cases}$$

In both cases $\lfloor (n+1)r \rfloor \geq q + k/m = nr$. □

By Monotonicity Lemma, $\mathcal{P}_n \supset \Phi_{\text{sm}}$ for any $n \in \mathbb{N}$. Moreover, we have

COROLLARY 4.2.4.

$$\Phi_{\text{sm}} = \bigcap_{n \in \mathbb{N}} \mathcal{P}_n.$$

PROOF. Let $\alpha \notin \Phi_{sm}$. Then $1 - 1/m < \alpha < 1 - 1/(m+1)$ for some $m \in \mathbb{N}$. This yields $\lfloor (m+1)\alpha \rfloor \leq m-1$ and $m\alpha > m-1$. Hence $\alpha \notin \mathcal{P}_m$. \square

EXAMPLE 4.2.5. Let (X, D) be a log variety with a standard boundary (i.e. $D \in \Phi_{sm}$). Assume that $K_X + D$ is numerically trivial. Let $K_X + D^+$ be some n -complement. Then $D^+ \geq D$ and $D^+ \equiv D$. If X is projective, then this yields $D^+ = D$. In this case, n is any natural such that nD is an integral divisor. In general case, we say that a complement $K_X + D^+$ of $K_X + D$ is *trivial* if $D = D^+$.

It is easy to check that

$$\mathcal{P}_n = \{0\} \cup \left[\frac{1}{n+1}, \frac{1}{n} \right] \cup \left[\frac{2}{n+1}, \frac{2}{n} \right] \cup \dots \\ \cup \left[\frac{k}{n+1}, \frac{k}{n} \right] \cup \dots \cup \left[\frac{n}{n+1}, 1 \right].$$

This gives

- LEMMA 4.2.6. (i) If $\alpha_1, \alpha_2 \in \mathcal{P}_n$, then either $\alpha_1 + \alpha_2 \in \mathcal{P}_n$ or $\alpha_1 + \alpha_2 > 1$.
(ii) Let $m \in \mathbb{N}$, $k_j \in \mathbb{N} \cup \{0\}$ and $b_j \in \mathcal{P}_n$, $j = 1, \dots, r$. Assume that

$$\alpha := \frac{m-1}{m} + \frac{1}{m} \sum_{j=1}^r k_j b_j \leq 1.$$

Then $\alpha \in \mathcal{P}_n$.

PROOF. (i) is trivial. As for (ii) we notice that $\sum k_j b_j \leq 1$. Hence by (i), $\sum k_j b_j \in \mathcal{P}_n$ and we may assume that $r = 1$ and $k_1 = 1$. Put $b := b_1$. It is sufficient to show that there exists $q \in \mathbb{N}$ such that

$$\frac{q}{n+1} \leq \frac{m-1+b}{m} \leq \frac{q}{n}.$$

This is equivalent to

$$n(m-1+b) \leq mq \leq (n+1)(m-1+b).$$

Taking into account that $b \in \mathcal{P}_n$, we have $l/(n+1) \leq b \leq l/n$ for some $l \in \mathbb{N}$. So there exists $q \in \mathbb{N}$ such that

$$n(m-1+b) \leq n(m-1) + l \leq mq \leq \\ (n+1)(m-1) + l \leq (n+1)(m-1+b).$$

This proves the lemma. \square

From Corollary 2.2.8 we have

COROLLARY 4.2.7. Let $(X, S + B)$ be a log variety, where S is reduced and B is effective. Assume that $K_X + S + B$ is plt and $B \in \mathcal{P}_n$. Then $\text{Diff}_S(B) \in \mathcal{P}_n$.

REMARK 4.2.8. It is easy to see that $\Phi_m \subset \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3 \cup \mathcal{P}_4 \cup \mathcal{P}_6$. Therefore if $D \in \Phi_m$ and $K_X + D^+$ is an 1, 2, 3, 4 or 6-complement of $K_X + D$, then $D^+ \geq D$.

4.3. Birational properties of complements

Now we will see that complements have good birational properties.

PROPOSITION 4.3.1 ([Sh2]). *Let $f: X \rightarrow Y$ be a birational contraction and D a subboundary on X . Assume that $K_X + D$ is n -complementary for some $n \in \mathbb{N}$. Then $K_Y + f(D)$ is also n -complementary.*

PROOF. Take $f(D)^+ := f_*(D^+)$ and apply 1.1.6. \square

Under additional assumptions we have the inverse implication:

PROPOSITION 4.3.2 ([Sh3, 2.13]). *Fix $n \in \mathbb{N}$. Let $f: Y \rightarrow X$ be a birational contraction and D a subboundary on Y such that*

- (i) $K_Y + D$ is nef over X ;
- (ii) $f(D) \in \mathcal{P}_n$ (in particular, $f(D)$ is a boundary).

Assume that $K_X + f(D)$ is n -complementary. Then $K_Y + D$ is also n -complementary.

PROOF. Consider the crepant pull back

$$K_Y + D' = f^*(K_X + f(D)^+), \quad \text{with} \quad f_*D' = f(D)^+.$$

Write $D' = S' + B'$, where S' is reduced, S', B' have no common components, and $[B'] \leq 0$. We claim that $K_Y + D'$ is an n -complement of $K_Y + D$. The only thing we need to check is that $nB' \geq [(n+1)\{D\}]$. From (ii) we have $f(D)^+ \geq f(D)$. This gives that $D' \geq D$ (because $D - D'$ is f -nef; see [Sh2, 1.1] or [KM, 3.39]). Finally, by Monotonicity Lemma 4.2.3 and because nD' is an integral divisor, we have

$$nD' \geq nS' + [(n+1)B'] \geq n[D] + [(n+1)\{D\}].$$

\square

REMARK 4.3.3. (i) By Monotonicity Lemma 4.2.3, the condition (ii) holds if all the coefficients of $f(D)$ are standard, i.e., $f(D) \in \Phi_{\text{sm}}$. By 4.2.8 (ii) also holds if $n \in \mathcal{R}_2$ and $f(D) \in \Phi_{\text{m}}$.

- (ii) The above proof shows that the proposition holds under the following weaker assumption instead of (ii):
- (ii)' for each nonexceptional component D_i of $D = \sum d_i D_i$ meeting the exceptional divisor of f we have $d_i \in \mathcal{P}_n$.

4.4. Inductive properties of complements

PROPOSITION 4.4.1 (cf. [Sh2, Proof of 5.6], [Ut, 19.6]). *Let $(X/Z \ni o, D = S + B)$ be a log variety. Set $S := [D]$ and $B := \{D\}$. Assume that*

- (i) $K_X + D$ is plt;
- (ii) $-(K_X + D)$ is nef and big over Z ;
- (iii) $S \neq 0$ near $f^{-1}(o)$;
- (iv) $D \in \mathcal{P}_n$ for some $n \in \mathbb{N}$.

Further, assume that near $f^{-1}(o) \cap S$ there exists an n -complement $K_S + \text{Diff}_S(B)^+$ of $K_S + \text{Diff}_S(B)$. Then near $f^{-1}(o)$ there exists an n -complement $K_X + S + B^+$ of $K_X + S + B$ such that $\text{Diff}_S(B)^+ = \text{Diff}_S(B^+)$.

This proposition should be true in the case when $K_X + D$ is dlt. We need only good definitions of complements on nonnormal varieties (see [Ut]).

PROOF. Let $g: Y \rightarrow X$ be a log resolution. Write $K_Y + S_Y + A = g^*(K_X + S + B)$, where S_Y is the proper transform of S on Y and $[A] \leq 0$. By Inversion of Adjunction, S is normal and $K_S + \text{Diff}_S(B)$ is plt. In particular, $g_S: S_Y \rightarrow S$ is a birational contraction. Therefore we have

$$K_{S_Y} + \text{Diff}_{S_Y}(A) = g_S^*(K_S + \text{Diff}_S(B)).$$

Note that $\text{Diff}_{S_Y}(A) = A|_{S_Y}$, because Y is smooth. By Corollary 4.2.7 we see that $\text{Diff}_S(B) \in \mathcal{P}_n$. So we can apply Proposition 4.3.2 to g_S . We get an n -complement $K_{S_Y} + \text{Diff}_{S_Y}(A)^+$ of $K_{S_Y} + \text{Diff}_{S_Y}(A)$. In particular, by (4.1), there exists

$$\Theta \in |-nK_{S_Y} - \lfloor (n+1)\text{Diff}_{S_Y}(A) \rfloor|$$

such that

$$n\text{Diff}_{S_Y}(A)^+ = \lfloor (n+1)\text{Diff}_{S_Y}(A) \rfloor + \Theta.$$

By Kawamata-Viehweg Vanishing,

$$\begin{aligned} R^1 h_* (\mathcal{O}_Y(-nK_Y - (n+1)S_Y - \lfloor (n+1)A \rfloor)) = \\ R^1 h_* (\mathcal{O}_Y(K_Y + \lceil -(n+1)(K_Y + S_Y + A) \rceil)) = 0. \end{aligned}$$

From the exact sequence

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_Y(-nK_Y - (n+1)S_Y - \lfloor (n+1)A \rfloor) \\ \longrightarrow \mathcal{O}_Y(-nK_Y - nS_Y - \lfloor (n+1)A \rfloor) \\ \longrightarrow \mathcal{O}_{S_Y}(-nK_{S_Y} - \lfloor (n+1)A \rfloor|_{S_Y}) \longrightarrow 0 \end{aligned}$$

we get surjectivity of the restriction map

$$\begin{aligned} H^0(Y, \mathcal{O}_Y(-nK_Y - nS_Y - \lfloor (n+1)A \rfloor)) \longrightarrow \\ H^0(S_Y, \mathcal{O}_{S_Y}(-nK_{S_Y} - \lfloor (n+1)A \rfloor|_{S_Y})). \end{aligned}$$

Therefore there exists a divisor

$$\Xi \in |-nK_Y - nS_Y - \lfloor (n+1)A \rfloor|$$

such that $\Xi|_{S_Y} = \Theta$. Set

$$A^+ := \frac{1}{n}(\lfloor (n+1)A \rfloor + \Xi).$$

Then $n(K_Y + S_Y + A^+) \sim 0$ and $(K_Y + S_Y + A^+)|_{S_Y} = K_{S_Y} + \text{Diff}_{S_Y}(A)^+$. Note that we cannot apply Inversion of Adjunction on Y because A^+ can have negative coefficients. So we put $B^+ := g_* A^+$. Again we have $n(K_X + S + B^+) \sim 0$ and $(K_X + S + B^+)|_S = K_S + \text{Diff}_S(B)^+$. We have to show only that $K_X + S + B^+$ is

lc. Assume that $K_X + S + B^+$ is not lc. Then $K_X + S + B + \alpha(B^+ - B)$ is also not lc for some $\alpha < 1$. It is clear that $-(K_X + S + B + \alpha(B^+ - B))$ is nef and big over Z . By Inversion of Adjunction, $K_X + S + B + \alpha(B^+ - B)$ is plt near $S \cap f^{-1}(o)$. Hence $\text{LCS}(X, B + \alpha(B^+ - B)) = S$ near $S \cap f^{-1}(o)$. On the other hand, by Connectedness Lemma, $\text{LCS}(X, B + \alpha(B^+ - B))$ is connected near $f^{-1}(o)$. Thus $K_X + S + B + \alpha(B^+ - B)$ is plt. This contradiction proves the proposition. \square

REMARK 4.4.2. It follows from the proof that we can replace (iv) in Proposition 4.4.1 with

(iv)' $\text{Diff}_S(B) \in \mathcal{P}_n$ for some n .

In the two-dimensional case we have a stronger result.

PROPOSITION 4.4.3 (cf. [Sh2, Proof of 5.6], [Ut, 19.6]). *Let $(X/Z \ni o, D = S + B)$ be a log surface such that*

- (i) $K_X + D$ is dlt;
- (ii) $-(K_X + D)$ is nef and big over Z ;
- (iii) $S := \lfloor D \rfloor \neq 0$ near $f^{-1}(o)$.

Assume that near $f^{-1}(o) \cap S$ there exists an n -semicomplement $K_S + \text{Diff}_S(B)^+$ of $K_S + \text{Diff}_S(B)$. Then near $f^{-1}(o)$ there exists an n -complement $K_X + S + B^+$ of $K_X + S + B$ such that $\text{Diff}_S(B)^+ = \text{Diff}_S(B^+)$.

PROOF. Similar to the proof of 4.4.1. By Propositions 2.1.2 and 2.1.3, the curve S is nodal. Further, we can take a log resolution $g: Y \rightarrow X$ so that $S_Y \simeq S$. \square

EXERCISE 4.4.4 ([Sh3]). Let $(X \ni P)$ be a germ of a two-dimensional normal singularity, let $C \neq 0$ be a reduced divisor on X , and $B = \sum b_i B_i \neq 0$ a boundary on X such that $K_X + C + B$ is plt. Assume that $b_i \geq 1/2$ for all i . Show that $K_X + C + \lceil B \rceil$ is lc and $\text{Supp} B$ is irreducible. Moreover,

$$\text{Diff}_C(B) = \left(1 - \frac{1}{n} + \frac{b_1}{n}\right) P, \quad \text{where } (X \ni P) \simeq \mathbb{C}^2/\mathbb{Z}_n(1, q).$$

If $B \in \Phi_{\text{sm}}$ (i.e., $B = (1 - 1/m_1)B_1$, $m_1 \in \mathbb{N}$), then

$$(K_X + C + B)|_C = (1 - 1/m)P, \quad m = m_1 n.$$

Hint. Show that $K_X + C + B$ is 1-complementary (using 4.1.10 and 4.4.3).

4.5. Exceptionality

DEFINITION 4.5.1. Let $(X/Z \ni o, D)$ be a log variety such that there is at least one \mathbb{Q} -complement of $K_X + D$ near the fiber over o .

- Assume that Z is not a point (local case). Then $(X/Z \ni o, D)$ is said to be *exceptional* over o if for any \mathbb{Q} -complement of $K_X + D$ near the fiber over o there exists at most one (not necessarily exceptional) divisor S such that $a(S, D) = -1$.
- Assume that Z is a point (global case). Then (X, D) is said to be *exceptional* if every \mathbb{Q} -complement of $K_X + D$ is klt.

The main advantage of this definition is Shokurov's conjecture that exceptional log varieties are bounded in some sense (see 4.5.2, 6.1.4, 6.1.10 (ii), 7.1.16, 7.2.6, 9.1.7, 9.1.11, [Sh3, §7], [KeM], [P2]). On the contrary, nonexceptional ones has "regular" complements (i.e., n -complements with small n). This phenomena was discovered by Shokurov [Sh3]. In [KeM] exceptional log del Pezzo surfaces are called *del Pezzo surfaces without tiger*. Studying of such surfaces is closely related to the uniruledness of affine surfaces [KeM, 6.1].

EXAMPLE 4.5.2. Let D be a boundary on a curve X . If (X, D) is nonexceptional, then by Theorem 4.1.10, $K_X + D$ is 1 or 2-complementary. Assume additionally that $D \in \Phi_{\text{sm}}$ and $X = \mathbb{P}^1$. By 4.1.11 and 4.1.12, (X, D) is exceptional only in the following cases: $E_6, E_7, E_8, \tilde{D}_4, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$.

We discuss two-dimensional generalizations of this fact in Ch. 8 and 9.

EXAMPLE 4.5.3. A log canonical singularity (X, o) is exceptional if and only if for every boundary B such that $K_X + B$ is lc there exists at most one divisor S (not necessarily exceptional) such that $a(S, B) = -1$. We see in Ch. 6 that a two-dimensional klt singularity is exceptional if and only if it is of type $\mathbb{E}_6, \mathbb{E}_7$ or \mathbb{E}_8 . Note that they are bounded. In contrary, nonexceptional klt singularities belong to two infinite series \mathbb{A}_n and \mathbb{D}_n . Refer to [I], [MP], [IP] for generalizations of this observation.

An isolated log canonical nonklt singularity (X, o) is exceptional if and only if there is exactly one divisor with discrepancy $a(\cdot, 0) = -1$. Under the assumption that X is Gorenstein such singularities are called *simple elliptic* in dimension two and *simple K3* in higher dimensions [IW].

EXAMPLE 4.5.4. Let $C \subset \mathbb{C}^2$ be a curve given by $x^2 = y^3$. Then $K_{\mathbb{C}^2} + \frac{5}{6}C$ is lc and not klt. Simple computations show that there exists only one divisor with discrepancy -1 . Therefore $(\mathbb{C}^2, \frac{5}{6}C)$ is exceptional.

The following proposition gives a nice relationship between local and global exceptional objects.

PROPOSITION 4.5.5 ([P1, Theorem 5]). *Let $(X \ni P)$ be a klt singularity and $f: (Y, S) \rightarrow X$ a plt blowup of P . Then the following are equivalent:*

- (i) $(X \ni P)$ is exceptional;
- (ii) $f(S) = P$ and $(S, \text{Diff}_S(0))$ is exceptional;
- (iii) $(S, \text{Diff}_S(0))$ is exceptional.

PROPOSITION 4.5.6 ([PSh], see also [MP], [IP], [P2]). *Let $(X/Z \ni o, D)$ be an exceptional log variety of local type. Then there exists a divisor S of $\mathcal{K}(X)$ such that $a(S, D^+) = -1$ for any nonklt \mathbb{Q} -complement of $K_X + D$ (i.e. S does not depend on the choice of D^+).*

COROLLARY 4.5.7 ([P1]). *Let $(X \ni P)$ be a \mathbb{Q} -factorial exceptional lc singularity. Then a plt blowup is unique up to isomorphisms.*

EXAMPLE 4.5.8 ([MP], cf. [P1, Theorem 5]). Let G be a finite group acting on \mathbb{C}^n freely in codimension one. Then the quotient singularity \mathbb{C}^n/G is exceptional if and only if so is the log Fano $(\mathbb{P}^{n-1}/G, D)$, where D is given by the formula (1.4). In dimension two there are exactly three types of exceptional groups: tetrahedral, octahedral and icosahedral (up to conjugation and scalar multiplication). In dimension three there are four types of them: F, G, I, J , in the classical notation.