

CHAPTER 3

Log terminal modifications

3.1. Log terminal modifications

Many results of this chapter holds in arbitrary dimension modulo log MMP.

PROPOSITION-DEFINITION 3.1.1 (cf. [Sh2, 9.1]; see also [Ut, 6.16]). Let (X, D) be a log variety of dimension ≤ 3 . Assume that $K_X + D$ is lc. Then there exists a *log terminal modification* of (X, D) ; that is, a birational contraction $g: X' \rightarrow X$ and a boundary D' on X' such that

- (i) $K_{X'} + D' \equiv g^*(K_X + D)$;
- (ii) $K_{X'} + D'$ is dlt;
- (iii) X' is \mathbb{Q} -factorial.

Moreover, if $\dim X = 2$, it is possible to choose X' smooth.

PROOF. Consider a log resolution $h: Y \rightarrow X$. We have

$$(3.1) \quad K_Y + D_Y = h^*(K_X + D) + E^{(+)} - E^{(-)},$$

where D_Y is the proper transform D on Y and $E^{(+)}, E^{(-)}$ are effective exceptional \mathbb{Q} -divisors without common components. Then $D_Y + E^{(-)}$ is a boundary and $K_Y + D_Y + E^{(-)}$ is dlt. Apply log MMP to $(Y, D_Y + E^{(-)})$ over X . We get a birational contraction $g: X' \rightarrow X$ from a normal \mathbb{Q} -factorial variety X' . Denote by D' the proper transform of $D_Y + E^{(-)}$ on X' . Then $K_{X'} + D'$ is dlt and g -nef. It is also obvious that $g_*D' = D$. We prove (i). Since the inverse to the birational map $h: Y \dashrightarrow X'$ does not contract divisors,

$$K_{X'} + D' = h_* (K_Y + D_Y + E^{(-)}) = h_* (f^*(K_X + D) + E^{(+)}) = g^*(K_X + D) + h_*E^{(+)}.$$

On the other hand, by numerical properties of contractions (see e.g., [Sh2, 1.1]) in the last formula all the coefficients of $h_*E^{(+)}$ should be nonpositive, i.e., all of them are equal to zero.

Finally, we consider the case $\dim X = 2$. If $E^{(+)} \neq 0$, then $(E^{(+)})^2 < 0$. From this $E^{(+)} \cdot E < 0$ for some E . Then $E^2 < 0$ and by (3.1), $K_Y \cdot E < 0$ and $(K_Y + D_Y + E^{(-)}) \cdot E < 0$. Hence E is a -1 -curve and steps of log MMP over X are contractions of such curves. Continuing the process, we obtain a smooth surface X' . This proves the statement. \square

PROPOSITION 3.1.2 ([Sh4, 3.1], [Ut, 21.6.1]). *Notation as in Theorem 3.1.1. Let $h: Y \rightarrow X$ be any log resolution. Consider a set $\mathcal{E} = \{E_i\}$ of exceptional divisors on Y such that*

- a) *if $a(E_i, D) = -1$, then $E_i \in \mathcal{E}$;*
- b) *if $E_i \in \mathcal{E}$, then $a(E_i, D) \leq 0$.*

Then there exists a blowup $g: X' \rightarrow X$ and a boundary D' on X' such that (i), (ii) and (iii) of 3.1.1 holds and moreover,

- (iv) *the exceptional divisors of g are exactly the elements of \mathcal{E} (i.e., they give the same discrete valuations of the field $\mathcal{K}(X)$).*

PROOF. Take a sufficiently small $\varepsilon > 0$ and put

$$d_i = \begin{cases} -a(E_i, D) & \text{if } E_i \in \mathcal{E}, \\ \max\{-a(E_i, D) + \varepsilon, 0\} & \text{otherwise.} \end{cases}$$

Then

$$(3.2) \quad K_Y + D_Y + \sum d_i E_i \equiv h^*(K_X + D) + \sum_{E_j \notin \mathcal{E}} (d_j + a(E_j, D)) E_j.$$

Next, run $(K_Y + D_Y + \sum d_i E_i)$ -MMP over X . By (3.2), each extremal ray is negative with respect to the proper transform of $\sum_{E_j \notin \mathcal{E}} (d_j + a(E_j, D)) E_j$, an effective divisor. Such a divisor can be nef only if it is trivial. Hence the process terminates when the proper transform of $\sum_{E_j \notin \mathcal{E}} (d_j + a(E_j, D)) E_j$ becomes zero. \square

COROLLARY-DEFINITION 3.1.3 ([Sh4, 3.1], [Ut, 21.6.1]). *Notation as in Theorem 3.1.1. Then there exists a blowup $g: X' \rightarrow X$ and a boundary D' on X' such that (i), (ii) and (iii) of 3.1.1 holds and moreover,*

- (iv)' *if $K_X + D$ is dlt, then f can be taken small; if $K_X + D$ is not dlt, then for all exceptional divisors E_i of g we have $a(E_i, D) = -1$.*

We call $g: X' \rightarrow X$ a *minimal log terminal modification*.

We generalize slightly the last result:

PROPOSITION 3.1.4. *Let X be a normal \mathbb{Q} -factorial variety of dimension ≤ 3 and D a boundary on X such that $K_X + D$ is lc, but is not plt. Assume also that X has only klt singularities. Then there exists a blowup $f: Y \rightarrow X$ such that*

- (i) *Y is \mathbb{Q} -factorial, $\rho(Y/X) = 1$ and the exceptional locus of f is an irreducible divisor, say E ;*
- (ii) *$K_Y + E + D_Y = f^*(K_X + D)$ is lc, where D_Y is the proper transform of D ;*
- (iii) *$K_Y + E + (1 - \varepsilon)D_Y$ is plt and is negative over X for any $\varepsilon > 0$.*

PROOF. As a first approximation to Y we take a minimal log terminal modification $g: X' \rightarrow X$ as in 3.1.3. Then $g^*(K_X + D) = K_{X'} + E' + D'$, where $E' \neq 0$ is an integral reduced divisor and D' is the proper transform of D . In particular, $\rho(X'/X)$ is the number of components of E' . Since X has only klt singularities, $K_{X'} = g^*K_X + \sum a_i E'_i$, where the E'_i are components of E' and $a_i > -1$ for all i .

Therefore $K_{X'} + E' = g^*K_X + \sum(a_i + 1)E'_i$ cannot be g -nef by numerical properties of contractions [Sh2, 1.1]. Run $(K_{X'} + E')$ -MMP over X . At each step, as above, $K + E$ cannot be nef over X . Hence at the last step we get a divisorial extremal contraction $f: Y \rightarrow X$, negative with respect to $K_Y + E$ and such that $\rho(Y/X) = 1$. Since at each step $K + E + D$ is numerically trivial over X , the log divisor $f^*(K_X + D) = K_Y + E + D_Y$ is lc (see Corollary 1.1.7). Obviously, E is irreducible and $\rho(Y/X) = 1$. Then $K_Y + E$ is plt and $K_Y + E + D_Y$ is lc. By Proposition 1.1.4, $K_Y + E + (1 - \varepsilon)D$ is plt. \square

DEFINITION 3.1.5. Let X be a normal variety and $f: Y \rightarrow X$ a blowup such that the exceptional locus of f contains only one irreducible divisor, say S . Assume that $K_Y + S$ is plt and $-(K_Y + S)$ is f -ample. Then $f: (Y \supset S) \rightarrow X$ is called a *purely log terminal (plt) blowup* of X . The blowup $f: (Y, E + D_Y) \rightarrow (X, D)$ constructed in 3.1.4 is called an *inductive blowup* of (X, D) . Note that it not necessarily unique (cf. 6.1.3).

EXAMPLE 3.1.6. Let $f: (Y, S) \rightarrow X$ be a plt blowup and $\varphi: X' \rightarrow X$ a finite étale in codimension one cover. Consider the following commutative diagram

$$(3.3) \quad \begin{array}{ccc} Y' & \xrightarrow{\psi} & Y \\ f' \downarrow & & \downarrow f \\ X' & \xrightarrow{\varphi} & X, \end{array}$$

where Y' is the normalization of $X' \times_X Y$. Then $f': Y' \rightarrow X'$ is a plt blowup. Indeed, f' is a contraction and ψ is a finite morphism, étale in codimension one outside of S . Set $S' := \psi^{-1}(S)$. The ramification formula (1.5), gives $K_{Y'} + S' = \psi^*(K_Y + S)$. Therefore $K_{Y'} + S'$ is plt (see Proposition 1.2.1). By Connectedness Lemma and Theorem 2.2.6, S' is irreducible.

EXAMPLE 3.1.7. Let $f': Y' \rightarrow X'$ be a plt blowup and $G \times Y' \rightarrow Y'$ an equivariant action of a finite group such that the induced action $G \times X' \rightarrow X'$ is free in codimension one. Put $Y := Y'/G$, $X := X'/G$ and consider the commutative diagram (3.3). As in 3.1.6 we obtain that $f: Y \rightarrow X$ is a plt blowup.

PROPOSITION-DEFINITION 3.1.8 ([K2], [Ut]). Let (X, D) be a klt log variety of dimension ≤ 3 . Then there exists a blowup $f: X^t \rightarrow X$ and a boundary D^t on Y such that

- (i) X^t is \mathbb{Q} -factorial;
- (ii) $K_{X^t} + D^t = f^*(K_X + D)$;
- (iii) $K_{X^t} + D^t$ is terminal.

This blowup is called a *terminal blowup* of (X, D) .

The proof uses Proposition 3.1.2 and the following simple lemma.

LEMMA 3.1.9 ([Sh1], [Ut, 2.12.2]). *Let (X, D) be a klt log variety. Then the number of divisors E of the function field $\mathcal{K}(X)$ with $a(E, D) \leq 0$ is finite.*

SKETCH OF PROOF. Let $f: Y \rightarrow X$ be a log resolution and B a crepant pull back of D :

$$f^*(K_X + D) = K_Y + B, \quad \text{with} \quad f_*B = D.$$

Write $B = B^{(+)} - B^{(-)}$, where $B^{(+)}, B^{(-)}$ are effective and have no common components. We assume that $\text{Supp} B^{(+)}$ contains also all f -exceptional divisors E with discrepancy $a(E, D) = 0$. By Hironaka it is sufficient to construct f so that all components of $B^{(+)}$ are disjoint. Let B_i, B_j be two components of B such that $B_i \cap B_j \neq \emptyset$. Put $b_{i,j} := a(B_i, D) + a(B_j, D)$. We want to change $f: Y \rightarrow X$ so that $b_{i,j} > 0$ whenever $B_i \cap B_j \neq \emptyset$. Let $a := 1 + \inf_E \{a(E, D)\}$. Since (X, D) is klt, $a > 0$. By blowing-up $B_i \cap B_j$ we obtain a new log resolution such that the proper transforms of B_i and B_j are disjoint and new exceptional divisor B_k has the discrepancy

$$a(B_k, D) = 1 + a(B_i, D) + a(B_j, D).$$

Then we have

$$b_{i,k} = a(B_i, D) + 1 + a(B_i, D) + a(B_j, D) \geq b_{i,j} + a.$$

Similarly, $b_{j,k} \geq b_{i,j} + a$. Thus after a finite number of such blowing ups we get the situation when $b_{i,k} > 0$ whenever $B_i \cap B_k \neq \emptyset$. In particular, all components of $B^{(+)}$ are disjoint. \square

One can see that if (X_1^t, D_1^t) is another terminal blowup, then the induced map $X^t \dashrightarrow X_1^t$ is an isomorphism in codimension one. In particular, the terminal blowup is unique in the surface case.

3.2. Weighted blowups

Consider a cyclic quotient singularity $X := \mathbb{C}^n / \mathbb{Z}_m(a_1, \dots, a_n)$, where $a_i \in \mathbb{N}$ and $\gcd(a_1, \dots, a_n) = 1$ (the case $m = 1$, i.e., $X \simeq \mathbb{C}^n$, is also possible). Let x_1, \dots, x_n be eigen-coordinates in \mathbb{C}^n , for \mathbb{Z}_m . The *weighted blowup* of X with weights a_1, \dots, a_n is a projective birational morphism $f: Y \rightarrow X$ such that Y is covered by affine charts U_1, \dots, U_n , where

$$U_i = \mathbb{C}_{y_1, \dots, y_n}^n / \mathbb{Z}_{a_i}(-a_1, \dots, \underset{\substack{\uparrow \\ i}}{m}, \dots, -a_n).$$

The coordinates in X and in U_i are related by

$$x_i = y_i^{a_i/m}, \quad x_j = y_j y_i^{a_j/m}, \quad j \neq i.$$

The exceptional locus E of f is an irreducible divisor and $E \cap U_i = \{y_i = 0\} / \mathbb{Z}_{a_i}$. The morphism $f: Y \rightarrow X$ is toric, i.e., there is an equivariant natural action of $(\mathbb{C}^*)^n$. It is easy to show that E is the *weighted projective space* $\mathbb{P}(a_1, \dots, a_n)$ and $\mathcal{O}_E(bE) = \mathcal{O}_{\mathbb{P}}(-mb)$, if b is divisible by $\text{lcm}(a_1, \dots, a_n)$ (and then bE is a Cartier divisor).

Note that the blowup constructed above depends on a choice of numbers a_1, \dots, a_n , and not just on their values mod m .

LEMMA 3.2.1. *In the above conditions we have*

- (i) $K_Y = f^*K_X + (-1 + \sum a_i/m)E$;
- (ii) if $D = \{x_i = 0\}/\mathbb{Z}_m$ and D_Y is the proper transform of D , then $D_Y = f^*D - \frac{a_i}{m}E$.

PROOF. The relation in (i) follows from the equality

$$dx_1 \wedge \cdots \wedge dx_n = y_1^{(\sum a_i/m - 1)} dy_1 \wedge \cdots \wedge dy_n.$$

The assertion (ii) can be proved similarly. \square

Any weighted blowup $f: Y \rightarrow \mathbb{C}^n/\mathbb{Z}_m$ of a cyclic quotient singularity is a plt blowup.

EXAMPLE 3.2.2. Let $X \subset \mathbb{C}^3$ be a Du Val singularity given by one of the equations

$$\begin{aligned} D_n (n \geq 4) : & \quad x^2 + y^2z + z^{n-1} = 0, \\ E_6 : & \quad x^2 + y^3 + z^4 = 0, \\ E_7 : & \quad x^2 + y^3 + yz^3 = 0, \\ E_8 : & \quad x^2 + y^3 + z^5 = 0. \end{aligned}$$

Let $f: Y \rightarrow X$ be the weighted blowup with weights $(n-1, n-2, 2)$, $(6, 4, 3)$, $(9, 6, 4)$, $(15, 10, 6)$ in cases D_n , E_6 , E_7 , E_8 , respectively. Then f is a plt blowup. We will see below that it is unique. In the case D_n any weighted blowup with weights $(w+1, w, 2)$, where $w = 1, \dots, n-2$ gives a blowup with irreducible exceptional divisor C such that $K_Y + C$ is lc. It is proved in [IP] that for any hypersurface canonical singularity given in \mathbb{C}^n by a nondegenerate function, there exists a weighted blowup which gives a plt blowup.

3.3. Generalizations of Connectedness Lemma

Now we generalize Connectedness Lemma to the case nef anticanonical divisor. We prove them in dimension two. However there are similar results in arbitrary dimension (modulo log MMP) [F].

PROPOSITION 3.3.1 ([Sh2, 6.9]). *Let $(X/Z \ni o, D)$ be a log surface, where Z is a curve. Assume that $K_X + D$ is lc and $-(K_X + D)$ is nef over Z . Then in a neighborhood of the fiber over o the locus of lc singularities of (X, D) has at most two connected components. Moreover, if $\text{LCS}(X, D)$ has exactly two connected components, then (X, D) is plt and $\text{LCS}(X, D) = \lfloor D \rfloor$ is a disjoint union of two sections (and a general fiber of $X \rightarrow Z$ is \mathbb{P}^1).*

PROOF. Let $(Y, D_Y) \rightarrow (X, D)$ be a minimal log terminal modification (see 3.1.3). Since the fibers of g are connected and $\text{LCS}(X, D) = g(\text{LCS}(Y, D_Y))$, it is sufficient to prove the assertion for (Y, D_Y) . Let $h: Y \rightarrow Z$ be the composition map. Set $C_Y := \lfloor D_Y \rfloor$ and $B_Y := \{D_Y\}$. Then $\text{LCS}(Y, D_Y) = C_Y$. Assume that

C_Y is disconnected. Run $(K_Y + B_Y)$ -MMP over Z . If the fiber $h^{-1}(o)$ is reducible, then there is its component $F \not\subset C_Y$ meeting C_Y . Then F is an extremal curve. Let $Y \rightarrow Y_1$ be its contraction. Since $F \cdot (K_Y + D_Y) = 0$ and $F \not\subset C_Y$, the dlt property of $K_Y + D_Y$ is preserved (see 1.1.7). On the other hand, by Connectedness Lemma 2.3.1, F meets only one connected component of C_Y . Hence the number of connected components of C_Y remains the same. Continuing the process, we obtain a contraction $\bar{h}: \bar{Y} \rightarrow Z$ with irreducible fiber $\bar{h}^{-1}(o)$. Since $K_{\bar{Y}} + D_{\bar{Y}}$ is nef, for a general fiber L of \bar{h} we have $L \cdot C_{\bar{Y}} \leq L \cdot D_{\bar{Y}} = -K_{\bar{Y}} \cdot L \leq 2$. By our assumption, the fiber $\bar{h}^{-1}(o)$ does not contain $C_{\bar{Y}}$. Hence $C_{\bar{Y}}$ has exactly two connected components, which are sections of $\bar{Y} \rightarrow Z$. It is also clear that $(K_{\bar{Y}}, C_{\bar{Y}})$ is plt. The components of C_Y cannot be contractible over Z . Hence $Y \rightarrow X$ is the identity map. This proves the statement. \square

Similarly we have

PROPOSITION 3.3.2 ([Sh2, 6.9]). *Let (X, D) be a projective log surface such that $K_X + D$ is lc and $-(K_X + D)$ is nef. Then the locus of log canonical singularities of (X, D) has at most two connected components. Moreover, if $\text{LCS}(X, D)$ has exactly two connected components and $K_X + D$ is dlt, then (X, D) is plt and there exists a contraction $f: X \rightarrow Z$ with a general fiber \mathbb{P}^1 onto a curve Z of genus 0 or 1 such that $\text{LCS}(X, D) = [D]$ is a disjoint union of two sections.*

PROOF. As in the proof of 3.3.1, $g: (Y, D_Y) \rightarrow (X, D)$ a minimal log terminal modification. Again set $C_Y := [D_Y]$ and $B_Y := \{D_Y\}$. Assume that C_Y is disconnected. Run $(K_Y + B_Y)$ -MMP. All intermediate contractions is $(K_Y + D_Y)$ -nonpositive. Therefore the log canonical property of $K_Y + D_Y$ is preserved (see 1.1.6). Since at each step $K_Y + B_Y$ is klt, $K_Y + C_Y + B_Y$ is klt outside of C_Y and $\text{LCS}(Y, D_Y) = C_Y$. By Connectedness Lemma 2.3.1, each contractible curve meets only one connected component of C_Y . Therefore the number of connected components of $\text{LCS}(Y, D_Y)$ is preserved. At the last step there are two possibilities:

- 1) $\rho(\bar{Y}) = 1$, then irreducible components of $\text{LCS}(\bar{Y}, D_{\bar{Y}})$ are intersect each other and gives only one connected component of $\text{LCS}(X, D)$;
- 2) $\rho(\bar{Y}) = 2$ and there is a nonbirational contraction $\bar{h}: \bar{Y} \rightarrow Z$ onto a curve. Here we can apply Proposition 3.3.1.

\square

Log surfaces $(X/Z, D)$, such that $K_X + D$ is lc and numerically trivial are called *monopoles* if $\text{LCS}(X, D)$ is connected and *dipoles* if $\text{LCS}(X, D)$ has two connected components. From 3.3.2 we can see that dipoles have a simpler structure.

EXAMPLE 3.3.3. Let Z be a rational or elliptic curve and $X := \mathbb{P}(\mathcal{O}_Z \oplus \mathcal{F})$, where \mathcal{F} is an invertible sheaf of degree $d \geq 0$. There are two nonintersecting sections C_1, C_2 . If $g(Z) = 1$, then $K_X + C_1 + C_2 = 0$ (see [Ha, Ch. 5, §2]) and $\text{LCS}(X, C_1 + C_2)$ has two connected components, i.e. $(X, C_1 + C_2)$ is a dipole. Similarly, in the case of a rational curve Z , we can take the log divisor $K_X + C_1 +$

$C_2 + \sum b_i F_i$, where F_i are different fibers of $X \rightarrow Z$, $\sum b_i = 2$, $b_i < 1$, $\forall i$. Then $(X, C_1 + C_2 + \sum b_i F_i)$ is also a dipole. We may construct many examples of dipoles by blowing up points on C_i or blowing down the negative section of $X \rightarrow Z$. For example, we can take a cone over a projectively normal elliptic curve $Z_d \subset \mathbb{P}^{d-1}$, and its general hyperplane section as boundary.

EXAMPLE 3.3.4 (see [Bl], cf. [Um]). Let X be a log Enriques surface (i.e., K_X is lc and numerically trivial, see 5.1.1). Then X has at most two nonklt points. Moreover, if X has exactly two nonklt points, then they are simple elliptic singularities (see Theorem 6.1.6).