11 A priori estimates for the wave equation

11.1 Statement of the main weighted estimates for inhomogeneous wave equation

The representation of solution of the inhomogeneous problem (10.2.1) have been expressed by the formula (10.2.21) involving the operator $T_{\rho,\sigma}$. This is an operator acting on functions f on X by the formula

$$T_{\rho,\sigma}(f)(\Omega) = \frac{\sigma^{(n+1)/2}}{\rho^{(n-1)/2}} \int_0^\infty \frac{\sin(\lambda \ln(\rho/\sigma))}{\lambda} P_\lambda(f)(\Omega) d\lambda. \tag{11.1.1}$$

Our main estimate for this operator is given in the following.

Theorem 11.1.1 *Let $\rho \geq 4\sigma \geq 1$ and $f \in S(X)$. Then for*

$$\frac{n-1}{2(n+1)} \leq \frac{1}{q} \leq \frac{1}{2}\ , \ \frac{1}{p} = 1 - \frac{1}{q}$$

we have the estimate

$$\|T_{\rho,\sigma}(f)\|_{L^q(X)} \leq C \ln(\rho/\sigma) \frac{\sigma^A}{\rho^B} \|f\|_{L^p(X)}, \tag{11.1.2}$$

where

$$A = 1 + B\ , \ B = \frac{n-1}{p}.$$

Assuming the supports of u and F are in the light cone, we can use the coordinates

$$\rho = (t^2 - |x|^2)^{1/2}, \Omega = (t,x)/\rho \in X,$$

and we can represent the L^q−norm in the form have

$$\|\rho^\alpha u\|_{L^q(\mathbf{R}^{n+1})} = \left(\int_0^\infty \|\rho^\alpha u(\rho.)\|^q_{L^q(X)} \rho^n d\rho \right)^{1/q}. \tag{11.1.3}$$

Next step is to use the trivial inequality

$$\|F\|_{L^q(\mathbf{R}^{n+1})} \leq C \sup_{\rho>0} \rho^{(n+1)/q+\varepsilon} \|F(\rho\ .)\|_{L^q(X)} \tag{11.1.4}$$

Thus the right side of (11.1.3) can be estimated from above by constant times

$$\sup_{\rho>0} \rho^{\alpha+(n+1)/q+\varepsilon} \|u(\rho\ .)\|_{L^q(X)}$$

From the representation formula (10.2.21) we see that the estimate (11.1.2) guarantees that

$$\|\rho^{\alpha}u\|_{L^{q}(\mathbf{R}^{n+1})} \leq C\int_{0}^{\infty} \|\sigma^{A+\varepsilon}F(\sigma.)\|_{L^{p}(X)}d\sigma. \tag{11.1.5}$$

provided $\alpha < B-(n+1)/q = n-1-2n/q$. Now application of the estimate

$$\int_{1}^{\infty} \|F(\rho\ .)\|_{L^{p}(X)}d\rho \leq C\|\rho^{1-(n+1)/p+\varepsilon}F\|_{L^{p}(\mathbf{R}^{n+1})} \tag{11.1.6}$$

leads to the following estimate.

Theorem 11.1.2 *Let u be a solution of the Cauchy problem (10.2.1) and let the assumption*

$$\operatorname{supp} F(s,y) \subset \{(s,y); |y| \leq s-1\}$$

be fulfilled.

Then the solution of (10.2.1) satisfies the estimate

$$\|\rho^{\alpha}u\|_{L^{q}(\mathbf{R}^{n+1})} \leq C\|\rho^{\beta}F\|_{L^{p}(\mathbf{R}^{n+1})}$$

for

$$\frac{n-1}{2(n+1)} \leq \frac{1}{q} \leq \frac{1}{2}\ , \ \frac{1}{p} = 1-\frac{1}{q}$$

$$\alpha < n-1-\frac{2n}{q}\ , \beta > \frac{2}{q}.$$

Remark: The above estimate was established in [14] by using Fourier integral techniques. The proof of this estimate, we represent here, is based on the Fourier transform on the hyperboloid. A similar idea was followed in [61].

11.2 Analytic family of operators associated with inhomogeneous wave equation

The solution of the inhomogeneous problem (10.2.1) can be expressed by the formula (10.2.22), where $T_{\rho,\sigma}$ is an operator acting on functions f on X by the formula

$$T_{\rho,\sigma}(f)(\Omega) = \frac{\sigma^{(n+1)/2}}{\rho^{(n-1)/2}}\int_{0}^{\infty} \frac{\sin(\lambda\ \ln(\rho/\sigma))}{\lambda}P_{\lambda}(f)(\Omega)d\lambda. \tag{11.2.1}$$

Here P_{λ} is the spectral projection

$$P_{\lambda}(f)(\Omega) = |c(\lambda)|^{-2}\int_{\mathbf{S}^{n-1}} [\Omega, \Lambda(\omega)]^{-i\lambda-(n-1)/2}\hat{f}(\lambda,\omega)d\omega. \tag{11.2.2}$$

In order to have a meaningful definition of the expression above we can choose the function f to be compactly supported. Then the Fourier transform $\hat{f}(\lambda, \omega)$ shall be smooth in λ, ω. As we have seen in Theorem 3.2.2 this is not an essential restriction for the application of this interpolation Theorem. To be sure that the definition of this operator is correct we need also an uniform convergence of the integral in λ. To assure this convergence we can use the other generalization of the Stein interpolation theorem represented in Theorem 3.3.2. Therefore we can introduce the approximation operators

$$T_{\rho,\sigma,\varepsilon}(f)(\Omega) =$$

$$\frac{\sigma^{(n+1)/2}}{\rho^{(n-1)/2}} \int_0^\infty \frac{\sin(\lambda \ln(\rho/\sigma))}{\lambda} P_\lambda(f)(\Omega) e^{-\varepsilon\lambda} d\lambda. \tag{11.2.3}$$

If we obtain the corresponding $L^p - L^q$ estimates with constant independent of $\varepsilon > 0$, then the density argument represented in the proof of Theorem 3.3.2 will complete the proof.

From (11.2.1) and the inverse formula for the Fourier transform on X we get

$$\widehat{T_{\rho,\sigma,\varepsilon}} f(\lambda, \omega) =$$

$$\frac{\sigma^{(n+1)/2}}{\rho^{(n-1)/2}} \frac{\sin(\lambda \ln(\rho/\sigma))}{\lambda} \hat{f}(\lambda, \omega) e^{-\varepsilon\lambda}. \tag{11.2.4}$$

To obtain $L^p - L^q$ estimate we consider a suitable extention of the operator $T = T_{\rho,\sigma,\varepsilon}$ to a family of operators $T_z = T_{z,\rho,\sigma,\varepsilon}$ defined by

$$T_z(f) = T\left(\chi_z\left(\sqrt{-\left(\frac{n-1}{2}\right)^2 - \Delta_X}\right) f\right). \tag{11.2.5}$$

Here $\chi_z(\lambda)$ is a function satisfying the properties

a) for any fixed $\lambda > 0$ the function

$$z \to \chi_z(\lambda)$$

is analytic in the strip $\mathrm{Re} z \in [0, (n+1)/2]$,

b) we have the relation

$$\chi_1(\lambda) = 1, \tag{11.2.6}$$

c) one can find positive constant $D = D(\rho, \sigma)$, independent of $\varepsilon > 0$ so that

$$\left| \frac{\chi_z(\lambda)}{\lambda} \sin(\lambda \ln(\rho/\sigma)) \right| \le D(\rho, \sigma), \tag{11.2.7}$$

for $\mathrm{Re} z = 0$ and for $d(\Omega, \Omega') \ge \delta > 0$ we have

$$\left| \int_0^\infty \frac{\chi_z(\lambda)}{\lambda} \varphi_\lambda(\Omega, \Omega') \sin(\lambda \ln(\rho/\sigma)) e^{-\varepsilon\lambda} d\lambda \right| \le$$

$$\le D(\rho, \sigma) e^{-r(n-1)/2}, \tag{11.2.8}$$

for $\mathrm{Re} z = (n+1)/2$. Here $r = d(\Omega, \Omega')$.

Later on we shall see that these requirements are sufficient to derive suitable $L^p - L^q$ estimate on the hyperboloid X for the operator $T = T_{\rho,\sigma}$. Now we can see how to choose the function $\chi_z(\lambda)$. For this we need more information about the asymptotic behavior of the spherical function

$$\varphi_\lambda(\Omega, \Omega') =$$

$$(11.2.9) \quad = |c(\lambda)|^{-2} \int_{S^{n-1}} [\Omega, \Lambda(\omega)]^{-i\lambda-(n-1)/2} [\Omega', \Lambda(\omega)]^{i\lambda-(n-1)/2} d\omega.$$

A suitable change of ω-variables in (11.2.9) shows that

$$\varphi_\lambda(\Omega, \Omega') = \varphi_\lambda(g\Omega, g\Omega')$$

for any $g \in SO(1,n)$ so

$$\varphi_\lambda(\Omega, \Omega')$$

is a function of the distance $d(\Omega, \Omega')$ between the points Ω and Ω'. Then formula (8.3.9) imply with $r = d(\Omega, \Omega')$

$$(11.2.10) \qquad \varphi_\lambda(r) = c_n |c(\lambda)|^{-2} L^n_\lambda(r),$$

where

$$L^n_\sigma = (\mathrm{sh} r)^{-(n-2)/2} P^{-(n-2)/2}_{-1/2-i\sigma}(\mathrm{ch} r).$$

Further, lemma 8.5.2 shows that for $n \geq 3$ odd we have

$$(11.2.11) \qquad \varphi_\lambda(r) = c_n \left(\frac{1}{\mathrm{sh} r}\partial_r\right)^{(n-1)/2} \cos \lambda r.$$

This representation formula for $r \geq \delta > 0$ gives

$$\begin{aligned} &|\varphi_\lambda(r) - \frac{1}{\mathrm{sh}^{(n-1)/2} r} \lambda^{(n-1)/2} (c_+ e^{i\lambda r} + c_- e^{-i\lambda r})| \\ (11.2.12) \qquad &\leq C e^{-r(n-1)/2} \lambda (1+\lambda)^{(n-5)/2}. \end{aligned}$$

On the other hand, for $n \geq 2$ even Lemma 8.5.3 guarantees that a similar estimate is valid too. In fact, we have the following representation of the spherical function

$$\varphi_\lambda(r) =$$

$$\sum_{k=1}^{n/2} \int_r^\infty \frac{\mathrm{sh} s}{\sqrt{\mathrm{ch} s - \mathrm{ch} r}} P_{k,r}(s) \partial_s^k (\cos(\lambda s)) ds,$$

where the functions $P_{k,r}(s)$ and all derivatives of these functions with respect to s are bounded from above by constant times $e^{-rn/2}$. It is clear that the spherical function is a linear combination of integrals of type

$$\lambda^k \int_r^\infty \frac{\mathrm{sh} s}{\sqrt{\mathrm{ch} s - \mathrm{ch} r}} P_{k,r}(s) e^{\pm i\lambda s} ds$$

with $k = 1, ..., n/2$. Take a cut-off function $\psi(s) \in C_0^\infty$ so that $\psi(s) = 1$ for $|s| \le 1$. Then using an integration by parts, we reduce the analysis of the spherical function to the following oscillatory integrals

$$\lambda^k \int_r^\infty \frac{\mathrm{sh}s}{\sqrt{\mathrm{ch}s - \mathrm{ch}r}} \psi(s-r) P_{k,r}(s) \mathrm{e}^{\pm i\lambda s} ds.$$

Making the change of variables $s \to \tau = \sqrt{s-r}$ and applying the stationary phase method, we obtain

$$\begin{aligned} |\varphi_\lambda(r) - \mathrm{sh}^{-(n-1)/2} r \lambda^{(n-1)/2} (c_+(r)\mathrm{e}^{i\lambda r} + c_-(r)\mathrm{e}^{-i\lambda r})| \\ \le C\mathrm{e}^{-r(n-1)/2} \lambda(1+\lambda)^{(n-5)/2}. \end{aligned} \tag{11.2.13}$$

Here the functions $c_\pm(r)$ are uniformly bounded.

Now choosing

$$\chi_z(\lambda) = d(z - (n+1)/2)\ \mathrm{e}^{z^2} (1+\lambda)^{1-z}, \tag{11.2.14}$$

we see that the requirement (11.2.6) is fulfilled for

$$d = -\frac{1}{\mathrm{e}(n-1)/2}.$$

Further, the inequality (11.2.7) is satisfied with

$$D(\rho, \sigma) = C\,(1 + \ln(\rho/\sigma)).$$

Finally, from (11.2.12) and (11.2.13) we conclude that the inequality (11.2.8) is fulfilled too.

It is clear that

$$\begin{aligned} \widehat{T_{z,\rho,\sigma,\varepsilon} f}(\lambda, \omega) = \\ \frac{\sigma^{(n+1)/2}}{\rho^{(n-1)/2}} \frac{\sin(\lambda \ln(\rho/\sigma))}{\lambda} \mathrm{e}^{-\varepsilon\lambda} \chi_z(\lambda) \hat{f}(\lambda, \omega). \end{aligned} \tag{11.2.15}$$

For $\mathrm{Re}z = 0$ we have

$$\left|\widehat{T_{z,\rho,\sigma,\varepsilon} f}(\lambda, \omega)\right| \le D(\rho,\sigma) \frac{\sigma^{(n+1)/2}}{\rho^{(n-1)/2}} |\hat{f}(\lambda,\omega)|, \tag{11.2.16}$$

where $D(\rho, \sigma)$ is the constant from (11.2.7) (modulo multiplier independent of $\rho, \sigma, \varepsilon$). Now the Plancherel identity (8.4.19) implies that

$$\|T_{z,\rho,\sigma,\varepsilon} f\|_{L^2(X)} \le D(\rho,\sigma) \frac{\sigma^{(n+1)/2}}{\rho^{(n-1)/2}} \|f\|_{L^2(X)}. \tag{11.2.17}$$

For Re$z = (n+1)/2$ we shall derive estimate for the kernel $K_{z,\rho,\sigma}(\Omega,\Omega')$ defined by the identity

$$T_{z,\rho,\sigma}(f)(\Omega) = \int_X K_{z,\rho,\sigma}(\Omega,\Omega')f(\Omega')d\Omega'. \tag{11.2.18}$$

Then we have

$$K_{z,\rho,\sigma}(\Omega,\Omega') =$$

$$= \frac{\sigma^{(n+1)/2}}{\rho^{(n-1)/2}} \int_0^\infty \frac{\sin(\lambda \ln(\rho/\sigma))}{\lambda} \chi_z(\lambda)\varphi_\lambda(\Omega,\Omega')d\lambda. \tag{11.2.19}$$

Since the estimates (11.2.12) and (11.2.13) require $d(\Omega,\Omega') \geq \delta > 0$, we need a suitable decomposition of the operator T_z of type

$$T_z = S_z + \Sigma_z,$$

where the kernel of the operator S_z is supported into

$$\{(\Omega,\Omega') \in X \times X; d(\Omega,\Omega') \geq \delta\}.$$

11.3 Partition of unity on the hyperboloid

Our starting point in this section is the following property of the flat Euclidean space $\mathbf{R}^n$. For any space dimension $n \geq 1$ one can find an universal constant $N = N(n)$ so that for any positive $R > 0$ one can find a covering $\{U_\alpha\}$ of $\mathbf{R}^n$ with the properties:

a) diam$U_\alpha \leq R$,

b) any family of $N+1$ elements of the covering has empty intersection.

The index α in the above covering is varying in a metric space A. For example, we can take $A \subseteq \mathbf{Z}^k$ with usual distance $d(\alpha,\beta) = |\alpha - \beta|$ between any two multiindices α,β.

A covering $\{U_\alpha\}$, satisfying the property a) is called $R-$ covering. So the above property means the existence of a finite integer $N = N(n)$ so that for any $R > 0$ one can find $R-$covering so that any family of $N+1$ elements of the covering has empty intersection.

For example, we can take $N(n) = n$. Sometimes the above property has the meaning that the topological dimension(defined as the minimal number N in the above property) of $\mathbf{R}^n$ is n.

The above property has the following consequence.

Lemma 11.3.1 *There exists a number $b(n) > 0$ so that for any real number $R > 0$ there is a covering of $\mathbf{R}^n$ formed by a family $\{U_\alpha\}$ of open connected domains in $\mathbf{R}^n$, so that*

$$d(U_\alpha, U_\beta) \geq R \text{ when } \rho(\alpha,\beta) \geq b, \tag{11.3.1}$$

$$\text{diam}(U_\alpha) \leq R \tag{11.3.2}$$

Here

$$d(Y_1, Y_2) = \inf_{y_1 \in Y_1, y_2 \in Y_2} d(y_1, y_2)$$

for any couple of subsets Y_1 and Y_2 of $\mathbf{R}^n$, $d(y_1, y_2)$ is the usual distance , while

$$\operatorname{diam}(Y) = \sup_{y_1, y_2 \in Y} d(y_1, y_2).$$

Our main goal in this section is to verify a corresponding property for the case of manifold of constant negative curvature -1. It is clear that the main difficulty is the fact that the metric is a Riemannian metric different from Euclidean.

Our construction of the covering shall be explicit.

Thus we want to construct a suitable partition of unity on the hyperboloid

$$X = \{\Omega \in \mathbf{R}^{n+1};\ [\Omega,\ \Omega] = 1,\ \Omega_0 > 0\}.$$

having the same properties a) and b).

The covering we are looking for is $\{D_\alpha\}$ and again the index α is varying in a metric space A.

Then we want to find an integer b so that for any real number $R > 0$ there is a covering of X formed by the family $\{D_\alpha\}$ of open connected domains in X, so that

$$d(D_\alpha, D_\beta) \geq R \ \text{ when } \ \rho(\alpha, \beta) \geq b, \tag{11.3.3}$$

$$\operatorname{diam}(D_\alpha) \leq R \tag{11.3.4}$$

Here

$$d(Y_1, Y_2) = \inf_{y_1 \in Y_1, y_2 \in Y_2} d(y_1, y_2)$$

for any couple of subsets Y_1 and Y_2 of the hyperboloid X, $d(y_1, y_2)$ is the distance on the hyperboloid X, while

$$\operatorname{diam}(Y) = \sup_{y_1, y_2 \in Y} d(y_1, y_2).$$

If the above conditions are checked, then one can choose suitable $N = N(b)$, independent of R, so that any family of $N+1$ elements of the covering has empty intersection.

A more explicit construction of a covering satisfying the above requirements is given below.

Namely, given any real $R > 0$ and any nonnegative integer m we consider the set

$$D_m(R) = \{\Omega \in X; (m-1)R/4 < d(\Omega, \Omega^*) < (m+1)R/4\}, \tag{11.3.5}$$

where $\Omega^* = (1, 0, ..., 0)$. Then it is clear that these open sets form a covering of X so that the property (11.3.3) is fulfilled and at most 3 sets of the covering have nonempty intersection. It is clear also that the diameter of $D_0(R)$ and $D_1(R)$ is not greater than R so (11.3.4) is fulfilled for these sets. To arrange the property

(11.3.4) for the sets $D_m(R)$ with $m \geq 2$, we shall make a more refined covering of each $D_m(R)$ by means of balls of type

$$D_{m,l}(R) = \{\Omega \in X; d(\Omega, \Omega_{m,l}) < R/2\}, \tag{11.3.6}$$

where

$$\Omega_{m,l} = (\mathrm{ch}(mR/4), \omega_l \mathrm{sh}(mR/4)), \tag{11.3.7}$$

and ω_l are suitable points on the unit sphere $\mathbf{S}^{n-1}$.

Our goal is to find an universal number $b = b(n)$ so that for any $R > 0$ one can choose the points ω_l, so that

$$d(D_{m,l}, D_{m,l'}) \geq R \text{ when } |l - l'| \geq b, \tag{11.3.8}$$

$$\mathrm{diam}(D_{m,l}) \leq R \tag{11.3.9}$$

Let us formulate more precisely the corresponding assertion.

Lemma 11.3.2 *Let $n \geq 2$. There exists an integer $b = b(n)$, so that for any real number $R > 0$ and any integer $m \geq 2$ one can find a real number $\delta = \delta(R, m) > 0$ and a finite number $M(R, m)$ of points $\omega_l, l = 1, ..., M$ on the unit sphere so that the balls $D_{m,l}(R)$ defined in (11.3.6) cover*

$$D_m(R) = \{\Omega \in X; (m-1)R/4 < d(\Omega, \Omega^*) < (m+1)R/4\} \tag{11.3.10}$$

and the properties (11.3.8), (11.3.9) are fulfilled for this choice of b and R.

Proof. Any point on $D_m(R)$ can be represented as

$$\Omega = (\mathrm{ch}r, \omega \mathrm{sh}r), \tag{11.3.11}$$

where $\omega \in \mathbf{S}^{n-1}$ and r satisfies the inequalities

$$(m-1)R/4 < r < (m+1)R/4. \tag{11.3.12}$$

Having in mind the assertion of Lemma 11.3.1, we see that our goal is to find so small $\delta = \delta(R, m) > 0$ and a finite number of points $\omega_l, l = 1, ..., M$ on the unit sphere so that the balls $D_{m,l}(R)$ with radius R and centers

$$\Omega_{m,l} = (\mathrm{ch}(mR/4), \omega_l \mathrm{sh}(mR/4)), \tag{11.3.13}$$

with $l = 1, ..., M$ form a covering of $D_m(R)$. This means that for any $\Omega \in X$ of the form (11.3.11), satisfying (11.3.12), one can find

$$\omega_l, \ l = 1, ..., M,$$

so that

$$d(\Omega, \Omega_{m,l}) \leq R$$

or equivalently

(11.3.14) $$[\Omega, \Omega_{m,l}] \leq \mathrm{ch} R.$$

Since

$$[\Omega, \Omega_{m,l}] = \mathrm{ch}(r - mR/4) + \mathrm{sh} r \mathrm{sh}(mR/4) \frac{|\omega - \omega_l|^2}{2},$$

from (11.3.12) we see that (11.3.14) follows from

$$|\omega - \omega_l|^2 \leq \frac{2}{\mathrm{sh}((m+1)R/4)\mathrm{sh}(mR/4)} (\mathrm{ch} R - \mathrm{ch}(R/4)).$$

This observation and Lemma 11.3.1 show that for suitable small $\delta = \delta(R, m) > 0$ we can find $\omega_l, l = 1, ..., M$ on the unit sphere so that the conclusions of the Lemma are true.

Remark. One can also verify that

$$M(R,m) = O\left(\frac{1}{(\delta(R,m))^{n-1}}\right).$$

Therefore, the integer $M(R,m)$ can grow to ∞ as R grows to ∞.

The above Lemma shows that the set $A = A(R)$ of indices for the domains D_α from (11.3.3) is included in $\mathbf{Z} \times \mathbf{Z}$ so the corresponding metric is induced by the Euclidean metric on $\mathbf{Z} \times \mathbf{Z}$. More precisely, we have

$$A(R) = \{(m,l); l = 1, ..., M(R,m)\},$$

where $M(R,m)$ is chosen according to Lemma 11.3.2. Moreover, for $\alpha = (m,l) \in A(R)$ and $\alpha' = (m', l') \in A(R)$ we have

$$\rho(\alpha, \alpha') = |m - m'| + |l - l'|.$$

For the covering $\{D_\alpha; \alpha \in A(R)\}$ of X we can consider the corresponding partition of unity

(11.3.15) $$1 = \sum_{\alpha \in A(R)} \kappa_\alpha(\Omega),$$

where $\kappa_\alpha(\Omega)$ are smooth non-negative functions supported in D_α.

Further, we make the following decomposition of the operator T_z

(11.3.16) $$T_z = S_z + \Sigma_z,$$

where

(11.3.17) $$S_z = \sum_{\rho(\alpha,\alpha') \geq b} \kappa_\alpha T_z \kappa_{\alpha'}.$$

Here the number $b > 0$ is chosen according to (11.3.3). The operator Σ_z evidently is defined by

(11.3.18) $$\Sigma_z = \sum_{\rho(\alpha,\alpha') < b} \kappa_\alpha T_z \kappa_{\alpha'}.$$

Our next step is to localize the estimate (11.2.17). Namely, using the fact that $\|f\|^2_{L^2(X)}$ is equivalent to

$$\sum_{\alpha\in A(R)} \|\kappa_\alpha f\|^2_{L^2(X)},$$

from (11.2.17) and the definition (11.3.18) of the operator Σ_z we get for $\mathrm{Re}z = 0$

$$\|\Sigma_z f\|_{L^2(X)} \leq D(\rho,\sigma)\frac{\sigma^{(n+1)/2}}{\rho^{(n-1)/2}}\|f\|_{L^2(X)} \tag{11.3.19}$$

and hence

$$\|S_z f\|_{L^2(X)} \leq D(\rho,\sigma)\frac{\sigma^{(n+1)/2}}{\rho^{(n-1)/2}}\|f\|_{L^2(X)} \tag{11.3.20}$$

for $\mathrm{Re}z = 0$.

On the line $\mathrm{Re}z = (n+1)/2$ we use the representation formula (11.2.19) for the kernel K_z of the operator T_z. Therefore, applying the estimate (11.2.8) for the kernel

$$\sum_{\rho(\alpha,\alpha')\geq b} \kappa_\alpha K_z(\Omega,\Omega')\kappa_{\alpha'}$$

of the operator S_z we obtain

$$\|S_z f\|_{L^\infty(X)} \leq D(\rho,\sigma)\frac{\sigma^n}{\rho^{n-1}}\|f\|_{L^1(X)} \tag{11.3.21}$$

for $\mathrm{Re}z = (n+1)/2$.

Making interpolation (see Theorem 3.3.2) between (11.3.20) and (11.3.21) , we find

$$\|S_1 f\|_{L^q(X)} \leq D(\rho,\sigma)\frac{\sigma^A}{\rho^B}\|f\|_{L^p(X)} \tag{11.3.22}$$

with

$$\frac{1}{q} = \frac{n-1}{2(n+1)}, \ \frac{1}{p} = 1 - \frac{1}{q} = \frac{n+3}{2(n+1)}$$

and

$$A = B+1, \quad B = \frac{(n-1)(n+3)}{2(n+1)} = \frac{n-1}{p}.$$

To obtain $L^q - L^p$ estimate for the operator Σ_1 we use the representation formula (10.2.2) and conclude that the kernel of the operator Σ_1 is

$$\sum_{\rho(\alpha,\alpha')<b} \kappa_\alpha(\Omega)\kappa_{\alpha'}(\Omega')E_{(n-1)/2}(\rho\Omega - \sigma\Omega')\sigma^n, \tag{11.3.23}$$

where $E_z(t,x)$ is the distribution introduced in (10.1.13).

Now we can choose the parameter $R > 0$ used in the construction of the partition of unity $\{\kappa_\alpha\}$. Namely, let the integers N and b from the beginning of this section are fixed. Since for fixed α the set of indices α' satisfying

$$\rho(\alpha, \alpha') < b$$

is finite, one can find $C = C(b)$ so that

$$d(D_\alpha, D_{\alpha'}) \leq C(b)R$$

for $\rho(\alpha, \alpha') < b$. Now for $\rho \geq 4\sigma \geq 1$ we take $R = R(\rho, \sigma)$ so that

$$C(b)R = \ln\left(\frac{\rho}{3\sigma}\right)$$

so

$$d(D_\alpha, D_{\alpha'}) \leq \ln\left(\frac{\rho}{3\sigma}\right) \tag{11.3.24}$$

for $\rho(\alpha, \alpha') < b$.

First, we consider the case $n \geq 3$ odd. In this case the representation formula (10.1.23) and (10.1.22) show that the distribution $E_{(n-1)/2}(t, x)$ is supported in the cone $t^2 - |x|^2 = 0$. On the other hand, from (11.3.24) we have

$$\begin{aligned}[\rho\Omega - \sigma\Omega', \rho\Omega - \sigma\Omega'] &= \rho^2 + \sigma^2 - 2\rho\sigma[\Omega, \Omega'] \geq \\ &\geq \rho^2 + \sigma^2 - 2\rho\sigma\frac{\rho}{3\sigma} \geq \frac{\rho^2}{3}\end{aligned} \tag{11.3.25}$$

for $\rho(\alpha, \alpha') < b$. Hence, the kernel in (11.3.23) is identically 0 for the case $n \geq 3$ odd. For the case $n \geq 2$ even the above observation shows that the kernel in (11.3.23) is a classical function with absolute value dominated by constant times

$$\sum_{\rho(\alpha,\alpha')<b} \kappa_\alpha(\Omega)\kappa_{\alpha'}(\Omega')\rho^{-(n-1)}\sigma^n \tag{11.3.26}$$

so we have

$$\|\Sigma_1 f\|_{L^\infty(X)} \leq C\frac{\sigma^n}{\rho^{n-1}}\|f\|_{L^1(X)}. \tag{11.3.27}$$

On the other hand, from (11.2.4), the estimate

$$\left|\frac{\sin(\lambda \ln(\rho/\sigma))}{\lambda}\right| \leq C\ln(\rho/\sigma)$$

and the Plancherel identity on X we obtain

$$\|T_1 f\|_{L^2(X)} \leq C\ln(\rho/\sigma)\frac{\sigma^{(n+1)/2}}{\rho^{(n-1)/2}}\|f\|_{L^2(X)}. \tag{11.3.28}$$

Using the argument to extract the localized version (11.3.20) of the estimate (11.2.17), we can write the following local version of (11.3.28)

$$\|\Sigma_1 f\|_{L^2(X)} \leq C \ln(\rho/\sigma) \frac{\sigma^{(n+1)/2}}{\rho^{(n-1)/2}} \|f\|_{L^2(X)}. \tag{11.3.29}$$

Making an interpolation between (11.3.27) and (11.3.29) we get

$$\|\Sigma_1 f\|_{L^q(X)} \leq D(\rho,\sigma) \frac{\sigma^A}{\rho^B} \|f\|_{L^p(X)} \tag{11.3.30}$$

with

$$\frac{1}{q} = \frac{n-1}{2(n+1)} \ , \ \frac{1}{p} = 1 - \frac{1}{q} = \frac{n+3}{2(n+1)}$$

and

$$A = \frac{n^2+4n-1}{2(n+1)} = B+1 \ , \ B = \frac{(n-1)(n+3)}{2(n+1)} = \frac{n-1}{p}.$$

From this estimate, (11.3.22) and the decomposition $T_z = S_z + \Sigma_z$ we arrive at

$$\|T_1 f\|_{L^q(X)} \leq D(\rho,\sigma) \frac{\sigma^A}{\rho^B} \|f\|_{L^p(X)} \tag{11.3.31}$$

Finally, an interpolation between this estimate and (11.3.28) gives the conclusion of Theorem 11.1.1.