10 Fundamental solutions

10.1 Cauchy problem for homogeneous wave equation

Our purpose in this section is to construct a solution of the problem

$$\Box E = 0,$$

$$E(0,x) = 0\ ,\ \partial_t E(0,x) = \delta(x), \tag{10.1.1}$$

where

$$\Box = -\partial_t^2 + \Delta.$$

Once the solution $E = E(t,x)$ is found, one can represent the solution of

$$\Box u = 0,$$

$$u(0,x) = 0\ ,\ \partial_t u(0,x) = f(x) \tag{10.1.2}$$

by

$$u = E(t,.) * f. \tag{10.1.3}$$

Since

$$\hat{u}(t,\xi) = \hat{E}(t,\xi)\hat{f}(\xi),$$

comparing the representation of u with (3.3.7), we see that

$$\hat{E}(t,\xi) = \frac{\sin(t|\xi|)}{|\xi|}. \tag{10.1.4}$$

To construct fundamental solution we define s_+^{-z} for any complex number z with $\mathrm{Re}z < 1$ by

$$s_+^{-z} = \begin{cases} s^{-z} & \text{if } s > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{10.1.5}$$

It is clear that this is a classical function in $L^1_{loc}(\mathbf{R})$ for $\mathrm{Re}z < 1$. Note that

$$\frac{d}{ds} s_+^{-z} = -z\ s_+^{-z-1}, \quad \text{for Re } z < 0. \tag{10.1.6}$$

The above relation enables one to extend the definition of s_+^{-z} for $1 \le \mathrm{Re}z < 2$. Namely, we define (for $1 \le \mathrm{Re}z < 2$)

$$s_+^{-z} = \frac{1}{(-z+1)} \frac{d}{ds} \left(s_+^{-z+1}\right), \tag{10.1.7}$$

where the derivative in the right side is taken in the sense of distributions. Moreover for $k \le \mathrm{Re}z < k+1$ we define s_+^{-z} by the relation

$$\begin{aligned} s_+^{-z} &= \frac{1}{(-z+1)...(-z+k)} \left(\frac{d}{ds}\right)^k \left(s_+^{-z+k}\right) \\ &= \frac{\Gamma(-z+1)}{\Gamma(-z+k+1)} \left(\frac{d}{ds}\right)^k s_+^{-z+k}, \end{aligned} \tag{10.1.8}$$

where the derivatives are taken in distribution sense. Take

$$E_z(s) = \frac{c_n}{\Gamma(1-z)} s_+^{-z} \tag{10.1.9}$$

for $z \neq \{1,2,3,...\}$. Here the constant $c_n > 0$ will be chosen later on. We can rewrite (10.1.8) as

$$E_z(s) = \frac{d^k}{ds^k} E_{z-k}(s). \tag{10.1.10}$$

It is not difficult to establish the relation

$$\lim_{z \to k} E_z(s) = c_n \delta^{(k-1)}(s) \tag{10.1.11}$$

for any integer $k \geq 1$. Here the limit is taken in distribution sense. In fact, the relation (see (8.5.4)

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

implies that

$$\lim_{z \to k} \Gamma(1-z)(z-k) = \frac{(-1)^k}{\Gamma(k)}. \tag{10.1.12}$$

On the other hand, for any $\varphi \in C_0^\infty(\mathbf{R})$ and $z = k + \varepsilon$ we see that the quantity

$$\varepsilon(s_+^{-z}, \varphi) = \frac{\varepsilon}{(-z+1)...(-z+k)} (\frac{d^k}{dz^k}(s_+^{-\varepsilon}), \varphi) =$$
$$= \frac{(-1)^k \varepsilon}{(-z+1)...(-z+k)} \int_0^\infty s^{-\varepsilon} \frac{d^k}{ds^k} \varphi(s) ds$$

tends to

$$\frac{1}{\Gamma(k)} \int_0^\infty \frac{d^k}{ds^k} \varphi(s) ds = \frac{-1}{\Gamma(k)} \frac{d^{k-1}}{ds^{k-1}} \varphi(0)$$

as ε tends to 0 and from (10.1.12) we obtain the needed relation (10.1.11).

Further, we consider the following family of distributions depending on $z \in \mathbf{C}$

$$E_z(t,x) = \frac{c_n}{\Gamma(1-z)} (t^2 - x^2)_+^{-z}. \tag{10.1.13}$$

As before, this is a classical function for $\text{Re} z < 1$. Using the relation

$$-\Box E_z(t,x) = 4(\frac{n-1}{2} - z) E_{z+1}, \tag{10.1.14}$$

one can extend the definition of $E_z(t,x)$ for $\text{Re} z \neq k + (n-1)/2$, $k = 0,1,2,...$ as a distribution in $D'(\mathbf{R}^{n+1})$. Our next step is to compute the partial Fourier transform of the distribution $D'(\mathbf{R}^n)$. First, we start with the case $\text{Re} z < 1$, when $E_z(t,x)$ is a classical function.

Lemma 10.1.1 *For* Re $z < 1$ *we have*

$$\int_{\mathbf{R}^n} e^{-ix\xi}(t^2 - x^2)_+^{-z} \frac{dx}{\Gamma(1-z)} =$$

$$= \frac{(2\pi)^{n/2}}{2^z} |t|^{-z+n/2} |\xi|^{z-n/2} J_{-z+n/2}(|t\xi|). \tag{10.1.15}$$

Proof. Since the scalar product $x.\xi$ is invariant under the action of the group $SO(n)$ of rotations, we see that the left side of the needed identity is a spherical function in ξ. For this we lose no generality assuming $\xi = (|\xi|, 0, ..., 0)$. Then the integral in the left side of (10.1.15) takes the form

$$I = \int_{\mathbf{R}} e^{-ix_1|\xi|} \int_{\mathbf{R}^{n-1}} (t^2 - x_1^2 - |x'|^2)_+^{-z} dx' \frac{dx_1}{\Gamma(1-z)}. \tag{10.1.16}$$

For $n \geq 3$ one can use polar coordinates $r = |x'|$, $\omega' = x'/r \in \mathbf{S}^{n-2}$. So we have

$$\int_{\mathbf{R}^{n-1}} (t^2 - x_1^2 - |x'|^2)_+^{-z} dx' =$$

$$= \mu(\mathbf{S}^{n-2})(t^2 - x_1^2)_+^{-z+(n-1)/2} \int_0^1 (1 - r^2)^{-z} r^{n-2} dr.$$

Choosing $\mu(\mathbf{S}^0) = 2$, we see that this identity holds also for $n = 2$. The relations (8.5.6) and (8.5.7) enable one to compute explicitly the integral

$$\int_0^1 (1 - r^2)^{-z} r^{n-2} dr = \frac{1}{2} \int_0^1 (1 - \rho)^{-z} \rho^{(n-3)/2} dr =$$

$$= \frac{1}{2} B(1 - z, \frac{n-1}{2}) = \frac{1}{2} \frac{\Gamma(1-z)\Gamma((n-1)/2)}{\Gamma((n+1)/2 - z)}.$$

So the integral in (10.1.16) becomes

$$I = \frac{\mu(\mathbf{S}^{n-2})\Gamma((n-1)/2)}{2\Gamma((n+1)/2 - z)} \int_{\mathbf{R}} e^{-ix_1|\xi|}(t^2 - x_1^2)_+^{-z+(n-1)/2} dx_1 =$$

$$= \frac{\mu(\mathbf{S}^{n-2})\Gamma((n-1)/2)}{2\Gamma((n+1)/2 - z)} t^{-2z+n} \int_{-1}^1 \cos(tx_1|\xi|)(1 - x_1^2)^{-z+(n-1)/2} dx_1 =$$

$$\frac{\mu(\mathbf{S}^{n-2})\Gamma((n-1)/2)}{\Gamma((n+1)/2 - z)} t^{-2z+n} \int_0^1 \cos(tx_1|\xi|)(1 - x_1^2)^{-z+(n-1)/2} dx_1.$$

Combining the formula (8.5.9) for the surface of the unit sphere with the Poisson integral representation (10.4.7) of the Bessel function, we get

$$I = (2\pi)^{n/2} 2^{-z} t^{-z+n/2} |\xi|^{z-n/2} J_{-z+n/2}(t|\xi|).$$

This completes the proof of the Lemma.

Remark. Note that the action (E_z, φ) of the distribution $E_z(t,x)$ on any test function $\varphi(t,x) \in C_0^\infty(\mathbf{R}^{n+1})$ is analytic function for $z \neq k + (n-1)/2$ for $k = 0, 1, 2, \ldots$ in view of the recurrence relation (10.1.14).

Since the right side of (10.1.15) is analytic function of z for $|t| \neq 0, |\xi| \neq 0$, we see that (10.1.15) is valid in the sense of distributions for $z \neq k + (n-1)/2$ and $k = 0, 1, 2, \ldots$

Given any function $\varphi(t,x) \in C_0^\infty(\mathbf{R}^{n+1})$ we denote by

$$\tilde{\varphi}(t,\xi) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{ix\xi} \varphi(t,x)dx$$

its inverse partial Fourier transform and then the action of E_z on φ satisfies

$$\frac{(2\pi)^{-n/2}}{c_n}(E_z, \varphi) = \tag{10.1.17}$$
$$\int_{\mathbf{R}} \int_{\mathbf{R}^n} \frac{1}{2^z} |t|^{-z+n/2} |\xi|^{z-n/2} J_{-z+n/2}(|t\xi|) \tilde{\varphi}(t,\xi) d\xi dt.$$

Since

$$J_{1/2}(s) = \sqrt{\frac{2}{s\pi}} \sin s,$$

we have

$$\frac{2^{-1}\pi^{(-n+1)/2}}{c_n}(E_{(n-1)/2}, \varphi) =$$
$$\int_{\mathbf{R}} \int_{\mathbf{R}^n} \frac{\sin |t||\xi|}{|\xi|} \tilde{\varphi}(t,\xi) d\xi dt. \tag{10.1.18}$$

Then we choose c_n so that

$$\frac{2^{-1}\pi^{(-n+1)/2}}{c_n} = 1,$$

i.e.

$$c_n = \frac{1}{2\pi^{(n-1)/2}}.$$

Thus we get

$$(E_{(n-1)/2}, \varphi) =$$
$$\int_{\mathbf{R}} \int_{\mathbf{R}^n} \frac{\sin |t||\xi|}{|\xi|} \tilde{\varphi}(t,\xi) d\xi dt. \tag{10.1.19}$$

Comparing with (10.1.4), we see that $E(t,x) = E_{(n-1)/2}(t,x)$ is a solution of our problem (10.1.1) and this is the fundamental solution of the initial problem (10.1.1) for the wave equation.

The recurrence relation (10.1.14) shows that the fundamental solution $E_{(n-1)/2}$ can be expressed by the aid of $E_1(t,x)$, $E_0(t,x)$ or $E_{1/2}(t,x)$. For this our next step will be an explicit representation formula for these solutions. In fact, we have

$$(E_{1/2}(t,.),f) = c\int_{|y|<t}(t^2-|y|^2)^{-1/2}f(y)dy =$$

$$= ct^{n-1}\int_0^1\int_{\mathbf{S}^{n-1}}f(tr\omega)(1-r^2)^{-1/2}r^{n-1}d\omega dr \tag{10.1.20}$$

for $f \in C_0^\infty(\mathbf{R}^n)$. Here

$$c = \frac{c_n}{\Gamma(1/2)} = \frac{1}{2\pi^{n/2}}.$$

Further, we have

$$(E_0(t,.),f) = c\int_{|y|<t}f(y)dy =$$

$$= ct^n\int_0^1\int_{\mathbf{S}^{n-1}}f(tr\omega)r^{n-1}d\omega dr \tag{10.1.21}$$

with

$$c = c_n = \frac{1}{2\pi^{(n-1)/2}}.$$

Taking advantage of (10.1.11), we find

$$(E_1(t,.),f) = \frac{c}{t}\int_{|y|=t}f(y)dS_y =$$

$$= ct^{n-2}\int_{\mathbf{S}^{n-1}}f(t\omega)d\omega, \tag{10.1.22}$$

where

$$c = c_n = \frac{1}{2\pi^{(n-1)/2}}.$$

From recurrence relation (10.1.14) we see that

$$E_{(n-1)/2} = C_1\Box^{(n-3)/2}E_1 = C_2\Box^{(n-1)/2}E_0 \tag{10.1.23}$$

for $n \geq 3$ odd. Here

$$C_1 = \frac{(-1)^{(n-3)/2}}{2^{n-3}\Gamma((n-1)/2)}\,,\ C_2 = \frac{(-1)^{(n-1)/2}}{2^{n-1}\Gamma((n+1)/2)}.$$

For $n \geq 2$ even we have

$$E_{(n-1)/2} = C\Box^{(n-2)/2}E_{1/2} \tag{10.1.24}$$

with

$$C = \frac{(-1)^{(n-2)/2}}{2^{n-2}\Gamma(n/2)}.$$

So we conclude that the expression $\Box^k E_z(t,x)$ for some particular values of k, z will appear in the representation formula for the fundamental solution of (10.1.1). For this we shall establish the following representation of term of this type.

Lemma 10.1.2 *Let $z \in \mathbf{C}$ satisfy*

$$z \neq k + (n-1)/2 \quad , \quad k = 0, 1, 2, \ldots$$

and let $l \geq 1$ be an integer such that

$$z + l \neq k + (n-1)/2 \quad , \quad k = 0, 1, 2, \ldots$$

Then we have the relation (in sense of distributions)

$$\Box^l E_z(t,x) = \sum_{|\alpha| \leq l} \partial^\alpha_{t,x}(c_{l,\alpha}(t,x)E_z(t,x)), \tag{10.1.25}$$

where $c_{l,\alpha}(t,x)$ are smooth functions in $\mathbf{R}^{n+1} \setminus 0$ and satisfy for any multiindex β the estimate

$$|\partial^\beta_{t,x} c_{l,\alpha}(t,x) \leq C_{\alpha,\beta,l}(|t| + |x|)^{-|\beta|-2l+|\alpha|} \tag{10.1.26}$$

Proof. It is sufficient to establish (10.1.25) for $l = 1$. Our starting point is the representation

$$E_z(t,x) = \varphi\left(\frac{|x|}{\sqrt{t^2+|x|^2}}\right) E_z(t,x) + \left[1 - \varphi\left(\frac{|x|}{\sqrt{t^2+|x|^2}}\right)\right] E_z(t,x),$$

where $\varphi(s) \in C_0^\infty(\mathbf{R})$ is a cut-off function, such that $\varphi(s) = 1$ for $|s| \leq 1/4$ and $\varphi(s) = 0$ for $|s| \geq 1/2$. Note that we have the estimate

$$\left|\partial^\beta_{t,x}\varphi\left(\frac{|x|}{\sqrt{t^2+|x|^2}}\right)\right| \leq C_\beta(|t| + |x|)^{-|\beta|}. \tag{10.1.27}$$

Then it is sufficient to establish that

$$\Box\left(\varphi\left(\frac{|x|}{\sqrt{t^2+|x|^2}}\right) E_z(t,x)\right) = \sum_{|\alpha| \leq 1} \partial^\alpha_{t,x}(d_\alpha(t,x)E_z(t,x)) \tag{10.1.28}$$

and

$$\square\left(\left(1-\varphi\left(\frac{|x|}{\sqrt{t^2+|x|^2}}\right)\right)E_z(t,x)\right) =$$

$$\sum_{|\alpha|\le 1} \partial^\alpha_{t,x}(e_\alpha(t,x)E_z(t,x)), \tag{10.1.29}$$

where the coefficients $d_\alpha(t,x), e_\alpha(t,x)$ satisfy (10.1.26) with $l=1$.

The relation $\square E_z = c(z)(t^2-|x|^2)^{-1}E_z$ was established in (10.1.14). Then the Leibnitz rule implies that

$$\square(\varphi E_z) = -E_z\square\varphi + 2\sum_{\mu=0}^{n}\partial_\mu(E_z\partial^\mu\varphi) + \frac{\varphi c(z)E_z}{t^2-|x|^2}. \tag{10.1.30}$$

Note that we have

$$|x| \le \frac{|t|}{\sqrt{3}} \le \frac{2t}{3} \tag{10.1.31}$$

on the support of $\varphi(|x|/\sqrt{t^2+|x|^2})$. Combining this fact, the representation formula (10.1.30) and the estimate (10.1.27) we arrive at (10.1.28).

To verify (10.1.29) we represent the operator $\square$ in polar coordinates

$$-\square = \partial_t^2 - \partial_r^2 - \frac{n-1}{r}\,\partial_r - \frac{1}{r^2}\,\Delta_{\mathbf{S}^{n-1}}.$$

Since

$$\left(1-\varphi\left(\frac{|x|}{\sqrt{t^2+|x|^2}}\right)\right)E_z(t,x)$$

is invariant under any rotation, we have

$$-\square\left(\left(1-\varphi\left(\frac{|x|}{\sqrt{t^2+|x|^2}}\right)\right)E_z(t,x)\right) =$$

$$\left(\partial_t^2-\partial_r^2-\frac{n-1}{r}\partial_r\right)\left[\left(1-\varphi\left(\frac{|x|}{\sqrt{t^2+|x|^2}}\right)\right)E_z(t,x)\right].$$

Since

$$\partial_t E_z(t,x) = -\frac{2zt}{t^2-|x|^2}\,E_z(t,x),$$

$$\partial_{x_j}E_z(t,x) = \frac{2zx_j}{t^2-|x|^2}\,E_z(t,x),$$

we have the relation

$$\left(\partial_t^2 - \partial_r^2\right) E_z(t,r) = -2z(\partial_t - \partial_r)\left(\frac{1}{t+r}E_z(t,r)\right) = \frac{4z^2}{t^2-|x|^2}E_z. \tag{10.1.32}$$

Now we are in position to apply the following variant of Leibniz rule

$$\Box\left(\left(1-\varphi\left(\frac{|x|}{\sqrt{t^2+|x|^2}}\right)\right)E_z\right) =$$

$$= E_z\Box\varphi\left(\frac{|x|}{\sqrt{t^2+|x|^2}}\right) - 2\sum_{\mu=0}^{n}\partial_\mu\left(E_z\partial^\mu\varphi\left(\frac{|x|}{\sqrt{t^2+|x|^2}}\right)\right) +$$

$$+\frac{\left(1-\varphi\left(\frac{|x|}{\sqrt{t^2+|x|^2}}\right)\right)c(z)E_z}{t^2-|x|^2}. \tag{10.1.33}$$

The relation (10.1.32) implies that

$$-2z\frac{\left(1-\varphi\left(\frac{|x|}{\sqrt{t^2+|x|^2}}\right)\right)E_z}{t^2-|x|^2} = (\partial_t - \partial_r)\left(\frac{\left(1-\varphi\left(\frac{|x|}{\sqrt{t^2+|x|^2}}\right)\right)E_z}{t+|x|}\right) +$$

$$+\frac{E_z}{t+|x|}(\partial_t - \partial_r)\left(\varphi\left(\frac{|x|}{\sqrt{t^2+|x|^2}}\right)\right).$$

Using the fact that on the support of

$$1-\varphi\left(\frac{|x|}{\sqrt{t^2+|x|^2}}\right)$$

we have

$$r = |x| \geq \frac{t}{\sqrt{15}},$$

while on the support of

$$\varphi'\left(\frac{|x|}{\sqrt{t^2+|x|^2}}\right)$$

the weights $|x|$ and t are equivalent, we see that (10.1.29) is valid.

From (10.1.28) and (10.1.29) we get the desired representation (10.1.25) and the lemma is proved.

Now we can obtain an explicit formula representing the solution of (10.1.2). From (10.1.22) and

$$\left(\frac{d^k}{(dt)^k}\right)\int_{\mathbf{S}^{n-1}} f(x+t\omega)d\omega = \sum_{|\alpha|=k}\int_{\mathbf{S}^{n-1}} \omega^\alpha \partial^\alpha f(x+t\omega)d\omega$$

we can compute the time derivative of

$$E_1(t,.) * f(x).$$

For $n \geq 3$ odd we combine the representation formula (10.1.23) together with (10.1.22) and applying the above lemma, we get

$$u(t,x) = E_{(n-1)/2}(t,.) * f(x) = \tag{10.1.34}$$

$$= \sum_{l=0}^{(n-3)/2}\sum_{|\alpha|=l}\frac{C}{t^{n-2-l}}\int_{|x-y|=t} c_{l,\alpha}\left(\frac{x-y}{|x-y|}\right)\partial_y^\alpha f(y)dS_y,\ t>0, \tag{10.1.35}$$

where $c_{l,\alpha}(\omega)$ are smooth functions on $\mathbf{S}^{n-1}$. Moreover, from

$$E_{(n-1)/2} = c\Box^{(n-1)/2}E_0$$

we have

$$u(t,x) = E_{(n-1)/2}(t,.) * f(x) = \tag{10.1.36}$$

$$= \sum_{l=0}^{(n-1)/2}\sum_{|\alpha|=l}\frac{C}{t^{n-1-l}}\int_{|x-y|<t} c_{l,\alpha}\left(\frac{x-y}{t}\right)\partial_y^\alpha f(y)dy,\ t>0,$$

where $c_{l,\alpha}(y)$ are smooth functions in the unit ball. For $n \geq 2$ even we can use the relation (10.1.24) and in this way we get

$$u(t,x) = E_{(n-1)/2}(t,.) * f(x) = \tag{10.1.37}$$

$$= \sum_{l=0}^{(n-2)/2}\sum_{|\alpha|=l}\frac{C}{t^{n-3/2-l}}\int_{|x-y|<t} c_{l,\alpha}\left(\frac{x-y}{t}\right)\partial_y^\alpha f(y)\frac{dy}{\sqrt{t-|x-y|}},\ t>0,$$

where $c_{l,\alpha}(y)$ are smooth functions in the unit ball.

Now we can turn to the representation of the solution the the Cauchy problem

$$\Box u = 0,$$
$$u(0,x) = f_0(x)\ ,\ \partial_t u(0,x) = f_1(x). \tag{10.1.38}$$

It is clear that

$$u(t,x) = \frac{d}{dt}E_{(n-1)/2}(t,.) * f_0(x) + E_{(n-1)/2}(t,.) * f_1(x).$$

For $n = 1$ we have the D'Alembert formula

(10.1.39)
$$u(t,x) = \\ = \frac{f_0(t+x) + f_0(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} f_1(y)dy, \quad t > 0.$$

For $n = 2$ we have the Poisson formula

(10.1.40)
$$u(t,x) = \\ = \partial_t \left(\frac{1}{2\pi} \int_{|x-y|<t} \frac{f_0(y)}{\sqrt{t^2 - |x-y|^2}} dy \right) + \\ + \frac{1}{2\pi} \int_{|x-y|<t} \frac{f_1(y)}{\sqrt{t^2 - |x-y|^2}} dy, \ t > 0,$$

For $n = 3$ we have the Kirchhoff formula

(10.1.41)
$$u(t,x) = \\ = \partial_t \left(\frac{1}{4\pi t} \int_{|x-y|=t} f_0(y) dS_y \right) + \\ + \frac{1}{4\pi t} \int_{|x-y|=t} f_1(y) dS_y, \ t > 0,$$

For $n \geq 3$ odd the solution of (10.1.38) takes the form

(10.1.42)
$$u(t,x) = \\ = \sum_{l=0}^{(n-1)/2} \sum_{|\alpha|=l} \frac{1}{t^{n-1-l}} \int_{|x-y|=t} c_{l,\alpha}\left(\frac{x-y}{|x-y|}\right) \partial_y^\alpha f_0(y) dS_y + \\ + \sum_{l=0}^{(n-3)/2} \sum_{|\alpha|=l} \frac{1}{t^{n-2-l}} \int_{|x-y|=t} d_{l,\alpha}\left(\frac{x-y}{|x-y|}\right) \partial_y^\alpha f_1(y) dS_y, \ t > 0,$$

where $c_{l,\alpha}(\omega)$, $d_{l,\alpha}(\omega)$ are smooth functions on $\mathbf{S}^{n-1}$. Moreover, for $n \geq 1$ odd from (10.1.36) we get

(10.1.43)
$$u(t,x) = \\ = \sum_{l=0}^{(n+1)/2} \sum_{|\alpha|=l} \frac{1}{t^{n-l}} \int_{|x-y|<t} c_{l,\alpha}\left(\frac{x-y}{t}\right) \partial_y^\alpha f_0(y) dy + \\ + \sum_{l=0}^{(n-1)/2} \sum_{|\alpha|=l} \frac{1}{t^{n-1-l}} \int_{|x-y|<t} d_{l,\alpha}\left(\frac{x-y}{t}\right) \partial_y^\alpha f_1(y) dy, \ t > 0,$$

where $c_{l,\alpha}(y)$, $d_{l,\alpha}(y)$ are smooth functions on the unit ball. Finally, for $n \geq 2$ even from (10.1.37) we deduce

(10.1.44)
$$u(t,x) =$$
$$= \sum_{l=0}^{n/2} \sum_{|\alpha|=l} \frac{1}{t^{n-1/2-l}} \int_{|x-y|<t} c_{l,\alpha}\left(\frac{x-y}{t}\right) \partial_y^\alpha f_0(y) \frac{dy}{\sqrt{t-|x-y|}} +$$
$$+ \sum_{l=0}^{(n-2)/2} \sum_{|\alpha|=l} \frac{1}{t^{n-3/2-l}} \int_{|x-y|<t} d_{l,\alpha}\left(\frac{x-y}{t}\right) \partial_y^\alpha f_1(y) \frac{dy}{\sqrt{t-|x-y|}},$$

where $c_{l,\alpha}(y)$, $d_{l,\alpha}(y)$ are smooth functions on the unit ball.

10.2 Cauchy problem for inhomogeneous wave equation

The solution of the inhomogeneous wave equation

$$\Box u = F,$$
$$u(0,x) = \partial_t u(0,x) = 0 \tag{10.2.1}$$

can be found as follows

$$u(t,x) = \int_0^t E_{(n-1)/2}(t-s,.) * F(s,.)(x) ds. \tag{10.2.2}$$

Since we shall use the recurrence relation (10.1.14), our first step is to find sufficient conditions so that all boundary terms after integration by parts with respect to time variable in (10.2.2) are identically zero.

The terms at $s=0$ can be neglected if we impose the assumption

$$\operatorname{supp}_s F(s,y) \subset (0,\infty), \tag{10.2.3}$$

because the function $F(s,y)$ together with all space-time derivatives of arbitrary order are zero at $s=0$. To be sure that all terms at $s=t$ vanish we need the following.

Lemma 10.2.1 *If* Re$z < n/2$, *then*

$$\lim_{t \to 0_+} E_z(t,x) = 0 \tag{10.2.4}$$

in the sense of distributions in $\mathbf{R}^n$.

If Re$z < (n-1)/2$, *then we also have*

$$\lim_{t \to 0_+} \partial_t E_z(t,x) = 0. \tag{10.2.5}$$

Proof. Given any $\varphi \in C_0^\infty(\mathbf{R}^n)$ we have

$$(E_z(t,.),\varphi) = \tag{10.2.6} c2^{-z}t^{-z+n/2}\int_{\mathbf{R}^n}|\xi|^{z-n/2}J_{-z+n/2}(t|\xi|)\hat{\varphi}(\xi)d\xi$$

for any $t > 0$ according to (10.1.15). In fact, for Re$z < 1$ this follows from (10.1.15). Then using the fact that both sides of (10.2.6) are analytic functions for $z \neq k + (n-1)/2$, $k = 0, 1, 2, ...$, we establish (10.2.6).

Then the series expansion (10.4.2) together with the asymptotic expansion for the Bessel function (see [1]) show that for Re$\nu > -1/2$ we have

$$|J_\nu(s)| \leq C_\nu s^{\mathrm{Re}\,\nu}\ ,\ s > 0. \tag{10.2.7}$$

Thus we get

$$|(E_z(t,.),\varphi)| \leq c2^{|\mathrm{Re}z|}t^{n-2\mathrm{Re}z}\int_{\mathbf{R}^n}|\hat{\varphi}(\xi)|d\xi$$

and we see that the property (10.2.4) is fulfilled.

To establish the property (10.2.4) for the time derivative of $E_z(t,x)$ we use the recurrence relation (10.4.3) for the Bessel functions and find

$$\partial_t(t^\nu J_\nu(t|\xi|)) = t^\nu|\xi|J_{\nu-1}(t|\xi|)$$

so we have the following variant of (10.2.6)

$$\partial_t(E_z(t,.),\varphi) = c(z)t^{-z+n/2}\int_{\mathbf{R}^n}e^{ix\xi}|\xi|^{z-n/2+1}J_{-1-z+n/2}(t|\xi|)\hat{\varphi}(\xi)d\xi.$$

Applying the estimate (10.2.7) with $\nu = -1 - z - n/2$, we see that Re $\nu > -1/2$ so

$$|\partial_t(E_z(t,.),\varphi)| \leq c(z)t^{n-2\mathrm{Re}z-1}\int_{\mathbf{R}^n}|\hat{\varphi}(\xi)|d\xi$$

and the assumption Re$z < (n-1)/2$ implies that (10.2.5) is true.

This proves the Lemma.

After this preparation we can obtain the following representation formula.

Proposition 10.2.1 *If $n \geq 3$ is odd and the inclusion (10.2.3) is fulfilled, then the solution of (10.2.1) is*

$$u(t,x) = c_n\int_0^t\frac{1}{t-s}\int_{|x-y|=t-s}\Box^{(n-3)/2}F(s,y)dS_y ds.$$

Proof. It is sufficient to apply the recurrence relation

$$E_{(n-1)/2} = c_n \Box^{(n-3)/2} E_1$$

from the previous section in combination with Lemma 10.2.1 and the representation formula (10.1.22) of the distribution E_1.

In the same way we arrive at

Proposition 10.2.2 *If $n \geq 2$ is even and the inclusion (10.2.3) is fulfilled, then the solution of (10.2.1) is*

$$u(t,x) = c_n \int_0^t \int_{|x-y|<t-s} \frac{1}{\sqrt{(t-s)^2 - |x-y|^2}} \Box^{(n-2)/2} F(s,y)\, dy ds.$$

The rest of this section is devoted to another representation of the solution $u(t,x)$ of (10.2.1) closely related to the Fourier transform on manifolds with constant negative curvature, discussed before.

Namely, we shall consider the case

$$\text{supp } F(s,y) \subset \{(s,y); |y| \leq s-1\}. \tag{10.2.8}$$

In the interior of the light cone $K = \{(t,x); |x| < t\}$ we introduce coordinates

$$\rho = \sqrt{t^2 - |x|^2} \ , \ \ \Omega = \left(\frac{t}{\rho}, \frac{x}{\rho}\right). \tag{10.2.9}$$

Applying Lemma 8.2.1, we represent the D'Alembert operator

$$\Box = -\partial_{x_0}^2 + \partial_{x_1}^2 + \dots + \partial_{x_n}^2$$

as follows

$$\begin{aligned} \rho^2 \Box &= -(\rho\partial_\rho)^2 - (n-1)\rho\partial_\rho + \Delta_X, \\ \Box &= -\partial_\rho^2 - \frac{n}{\rho}\partial_\rho + \frac{1}{\rho^2}\Delta_X, \end{aligned} \tag{10.2.10}$$

where Δ_X is the Laplace-Beltrami operator on the hyperboloid

$$X = \{\Omega; [\Omega, \Omega] = 1\}.$$

Then the equation (10.2.1) takes the form

$$\left((\rho\partial_\rho)^2 + (n-1)\rho\partial_\rho - \Delta_X\right) u = -\rho^2 F. \tag{10.2.11}$$

Next step is the application of the Fourier transform on X defined in Definition 8.2.2 by

$$\hat{f}(\lambda,\omega) = \int_X [\Omega, \Lambda(\omega)]^{(i\lambda-(n-1)/2)} f(\Omega) d\Omega, \tag{10.2.12}$$

The inverse Fourier transform on X is determined in (8.4.2) by the relation

$$f(\Omega) = \int_{-\infty}^{\infty}\int_{\mathbf{S}^{n-1}} [\Omega, \Lambda(\omega)]^{(-i\lambda-(n-1)/2)} \hat{f}(\lambda,\omega) |c(\lambda)|^{-2} d\omega d\lambda, \tag{10.2.13}$$

where

$$c(\lambda) = \sqrt{2}(2\pi)^{n/2} \frac{\Gamma(i\lambda)}{\Gamma((n-1)/2+i\lambda)}. \tag{10.2.14}$$

Having in mind that the exponential function

$$[\Omega, \Lambda(\omega)]^{(i\lambda-(n-1)/2)}$$

satisfies according to (8.2.15) the relation

$$-\Delta_X [\Omega, \Lambda(\omega)]^{(i\lambda-(n-1)/2)} = \left(\lambda^2 + \left(\frac{n-1}{2}\right)^2\right) [\Omega, \Lambda(\omega)]^{(i\lambda-(n-1)/2)}, \tag{10.2.15}$$

one can make Fourier transform in (10.2.11) and in this way we get

$$\left((\rho\partial_\rho)^2 + (n-1)\rho\partial_\rho + \left(\lambda^2 + \left(\frac{n-1}{2}\right)^2\right)\right) \hat{u}(\rho;\lambda,\omega) = -\rho^2 \hat{F}(\rho;\lambda,\omega), \tag{10.2.16}$$

where

$$\hat{u}(\rho;\lambda,\omega) = \int_X [\Omega, \Lambda(\omega)]^{(i\lambda-(n-1)/2)} u(\rho\Omega) d\Omega \tag{10.2.17}$$

is the partial Fourier transform in $\Omega-$ coordinates. Let us make change of variables

$$\rho \to \tau = \ln\rho. \tag{10.2.18}$$

Then $\rho\partial_\rho = \partial_\tau$ and we get

$$\left((\partial_\tau)^2 + (n-1)\partial_\tau + \left(\lambda^2 + \left(\frac{n-1}{2}\right)^2\right)\right) \hat{u} = -e^{2\tau}\hat{F}, \tag{10.2.19}$$

Since the solution of

$$\left((\partial_\tau)^2 + (n-1)\partial_\tau + \left(\lambda^2 + \left(\frac{n-1}{2}\right)^2\right)\right) v = 0,$$

(10.2.20)
$$v(0) = 0 \ , \ \partial_\tau v(0) = 1$$

is

$$e^{-\tau(n-1)/2}\frac{\sin(\lambda\tau)}{\lambda},$$

from the Duhamel principle we find

$$\hat{u}(e^\tau; \lambda, \omega) =$$
$$= -\int_0^\tau e^{-(\tau-\kappa)(n-1)/2}\frac{\sin(\lambda(\tau-\kappa))}{\lambda}e^{2\kappa}\hat{F}(e^\kappa; \lambda, \omega)d\kappa$$

or

$$\hat{u}(\rho; \lambda, \omega) =$$
$$= -\int_1^\rho \rho^{-(n-1)/2}\sigma^{(n+1)/2}\frac{\sin(\lambda \ln(\rho/\sigma))}{\lambda}\hat{F}(\sigma; \lambda, \omega)d\sigma.$$

Then the inverse formula (10.2.13) for the Fourier transform on X gives

(10.2.21)
$$u(\rho\Omega) = -\int_1^\rho T_{\rho,\sigma}(F(\sigma, .))d\sigma,$$

where $T_{\rho,\sigma}$ is an operator acting on functions f on X by the formula

$$T_{\rho,\sigma}(f)(\Omega) =$$

(10.2.22)
$$\frac{\sigma^{(n+1)/2}}{\rho^{(n-1)/2}}\int_{-\infty}^{\infty}\frac{\sin(\lambda \ln(\rho/\sigma))}{\lambda}P_\lambda(f)(\Omega)d\lambda,$$

where P_λ is the spectral projection

$$P_\lambda(f)(\Omega) =$$

(10.2.23)
$$= |c(\lambda)|^{-2}\int_{S^{n-1}}[\Omega, \Lambda(\omega)]^{-i\lambda-(n-1)/2}\hat{f}(\lambda, \omega)d\omega.$$

10.3 Cauchy problem for the Klein-Gordon equation

The Klein-Gordon equation can be associated with a description of a relativistic scalar field with a fixed positive mass M. The corresponding Cauchy problem can be written in the form

$$(\Box - M^2)u = g,$$

(10.3.1)
$$u(0, x) = f_0(x) \ , \ \partial_t u(0, x) = f_1(x),$$

where $f_0, f_1 \in S(\mathbf{R}^n)$, $F(t,x) \in S(\mathbf{R}^{n+1})$ and $\Box = -\partial_t^2 + \Delta$. Introducing coordinates

$$X = (x_1, ..., x_n, x_{n+1}) = (x, x_{n+1}),$$

we set

$$U(t,X) = u(t,x)e^{iMx_{n+1}},$$
$$G(t,X) = g(t,x)e^{iMx_{n+1}},$$
$$F_j(X) = f_j(x)e^{iMx_{n+1}},\ j = 0,1,$$
$$\Box_{n+1} = -\partial_t^2 + \Delta_{n+1} = -\partial_t^2 + \partial_{x_1}^2 + ... + \partial_{x_n}^2 + \partial_{x_{n+1}}^2. \tag{10.3.2}$$

Then the Cauchy problem (10.3.1) is reduced to the following problem for the wave equation in $\mathbf{R} \times \mathbf{R}^{n+1}$.

$$\Box_{n+1} U = G,$$
$$U(0,X) = F_0(X)\ ,\ \partial_t U(0,X) = F_1(X), \tag{10.3.3}$$

Now we are in situation to apply the results from the previous section. More precisely, first we consider the family of operators

$$E_z(t,X) = \frac{c_n}{\Gamma(1-z)}(t^2 - X^2)_+^{-z}.$$

Since the solution of (10.3.3) can be expressed as a convolution in the space of variables, we start with the following preliminary

Lemma 10.3.1 *Let*

$$H(X) = h(x)e^{iMx_{n+1}}.$$

Then for Re$z < 1$ *we have*

$$E_z(t,.) * H(x,0) = c_n\sqrt{\pi}(M/2)^{z-1/2} \times \tag{10.3.4}$$
$$\times \int_{|x-y|\le t} (t^2 - |x-y|^2)^{1/4-z/2} J_{-z+1/2}(M\sqrt{t^2 - |x-y|^2}) h(y) dy.$$

Proof. Taking $X = (x,0)$, we have

$$E_z(t,.) * H(x,0) =$$
$$= \frac{c_n}{\Gamma(1-z)} \int_{|X-Y|<t} (t^2 - |X-Y|^2)^{-z} h(y) e^{iMy_{n+1}} dY =$$
$$= \frac{c_n}{\Gamma(1-z)} \int_{|x-y|<t} K(x,y) h(y) dy,$$

where

$$K(x,y) = \left(2 \int_0^{\sqrt{t^2-|x-y|^2}} (t^2 - |x-y|^2 - y_{n+1}^2)^{-z} \cos(My_{n+1}) dy_{n+1} \right) =$$

$$= 2 \left(\int_0^1 (1-\tau^2)^{-z} \cos(M\tau\sqrt{t^2 - |x-y|^2}) d\tau \right) (t^2 - |x-y|^2)^{-z+1/2}.$$

Applying now the Poisson integral representation (10.4.7), we get the relation (10.3.4).

This completes the proof.

When $n \geq 2$ is even, the fundamental solution for the problem (10.3.3) is $E_{n/2}(t,X)$ and from the recurrence relation (10.1.14) we find

$$E_{n/2}(t,X) = c_n \Box_{n+1}^{n/2} E_0(t,X).$$

Applying (10.1.36) , we get with $X = (x, 0)$

$$E_{n/2}(t,.) * H(x,0) =$$

$$= \sum_{|\alpha| \leq n/2} \int_{|X-Y| \leq t} \frac{c_\alpha(t, X-Y)}{t^{n-|\alpha|}} \partial_Y^\alpha H(Y) dY,$$

where $c_\alpha(t, X-Y)$ are bounded functions. With $H(Y) = h(y)e^{iMy_{n+1}}$ we see that

$$\partial_Y^\alpha H(Y) = \sum_{|\beta| \leq |\alpha|} c_\beta \partial_y^\beta h(y) e^{iMy_{n+1}}.$$

Then the argument of the proof of Lemma 10.3.1 guarantees that

$$E_{n/2}(t,.) * H(x,0) =$$

$$= \sum_{l \leq n/2} \sum_{|\beta| \leq l} \int_{|x-y| \leq t} \frac{c_\beta(t, x-y)}{t^{n-l}} \partial_y^\beta h(y) \sin(M\sqrt{t^2 - |x-y|^2}) dy.$$

This calculation shows that for $n \geq 2$ even the solution of (10.3.1) is

(10.3.5)
$$u(t,x) =$$

$$= \sum_{l \leq (n+2)/2} \sum_{|\beta| \leq l} \int_{|x-y| \leq t} \frac{c_\beta(t, x-y)}{t^{n+1-l}} \partial_y^\beta f_0(y) K(t, |x-y|) dy +$$

$$+ \sum_{l \leq n/2} \sum_{|\beta| \leq l} \int_{|x-y| \leq t} \frac{d_\beta(t, x-y)}{t^{n-l}} \partial_y^\beta f_1(y) K(t, |x-y|) dy +$$

$$+ c_n \int_0^t \int_{|x-y| \leq t-s} (\Box - M^2)^{n/2} g(s,y) K(t-s, |x-y|) dy ds.$$

Here $c_\beta(t, x-y)$ and $d_\beta(t, x-y)$ are bounded functions and

$$K(t, |x|) = \sin(M\sqrt{t^2 - |x|^2}).$$

When $n \geq 3$ is odd we can use the relation

$$E_{n/2}(t, X) = c_n \Box^{(n+1)/2} E_{-1/2}(t, X).$$

Then repeating the argument given above and using the fact that for $z = -1/2$

$$\left|(t^2 - |x-y|^2)^{1/4 - z/2} J_{-z+1/2}(M\sqrt{t^2 - |x-y|^2})\right| \leq$$
$$\leq C(1 + \sqrt{t^2 - |x-y|^2})^{1/2} \leq C(1+t)^{1/2}.$$

we arrive at

$$u(t, x) = \tag{10.3.6}$$
$$= \sum_{l \leq (n+3)/2} \sum_{|\beta| \leq l} \int_{|x-y| \leq t} \frac{c_\beta(t, x-y)}{t^{n+3/2-l}} \partial_y^\beta f_0(y) K(t, |x-y|) dy +$$
$$+ \sum_{l \leq (n+1)/2} \sum_{|\beta| \leq l} \int_{|x-y| \leq t} \frac{d_\beta(t, x-y)}{t^{n+1/2-l}} \partial_y^\beta f_1(y) K(t, |x-y|) dy +$$
$$+ c_n \int_0^t \int_{|x-y| \leq t-s} K(t-s, x-y)(\Box - M^2)^{(n+1)/2} g(s, y) dy ds$$

for $n \geq 3$ odd. Here the kernel $K(t, x)$ is given by

$$K(t, x) = \sqrt{t^2 - |x|^2}) J_1\left(M\sqrt{(t)^2 - |x|^2}\right).$$

10.4 Some properties of Bessel function

(see [1].)

The Bessel function $J_\nu(z)$ is a solution of the ordinary differential equation

$$z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} + (z^2 - \nu^2) w = 0. \tag{10.4.1}$$

For $z \in \mathbf{C}$ close to 0 we have the following series expansion

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{\nu+2m}}{m!\, \Gamma(m+\nu+1)}. \tag{10.4.2}$$

The Bessel function satisfies the following recurrence relations

$$z J'_\nu(z) + \nu J_\nu(z) = z J_{\nu-1}(z) \tag{10.4.3}$$

or equivalently

$$\frac{d}{dz}\,[z^{\nu}J_{\nu}(z)] = z^{\nu}J_{\nu-1}(z). \tag{10.4.4}$$

Moreover, we have

$$zJ_{\nu}'(z) - \nu J_{\nu}(z) = -zJ_{\nu+1}(z). \tag{10.4.5}$$

The Wronskian of two Bessel functions is given by

$$W(w_1, w_2) = w_1 w_2' - w_2 w_1'.$$

For the case,when $w_1 = J_{\nu}$, $w_2 = J_{-\nu}$ the corresponding Wronskian is

$$W[J_{\nu}(z), J_{-\nu}(z)] = -\frac{2}{\pi z}\,\sin(\pi\nu). \tag{10.4.6}$$

The following integral representations by Poisson's integral shall be of special interest in our considerations

$$\Gamma(\nu + 1/2)J_{\nu}(z) = \frac{2z^{\nu}}{\sqrt{\pi}2^{\nu}}\int_0^1 (1-t^2)^{\nu-1/2}\cos(zt)dt. \tag{10.4.7}$$

For the special cases $\nu = \pm 1/2$ we have

$$J_{1/2}(z) = \frac{\sqrt{2}}{\sqrt{\pi z}}\sin z,\ J_{-1/2}(z) = \frac{\sqrt{2}}{\sqrt{\pi z}}\cos z. \tag{10.4.8}$$