7 Weighted Sobolev spaces on flat space

7.1 Abstract localized norms

Definition 7.1.1 *Let A be a Banach space with norm $\|\cdot\|_A$. A* Paley-Littlewood partition of identity *(PL partition for short) is a sequence $\pi = \{\pi_j\}_{j\geq 0}$ of bounded operators on A such that: the series $\sum_{j\geq 0}\pi_j$ converges strongly (i.e., pointwise) to the identity operator on A, and in addition there exists an integer $N \geq 1$ such that*

(7.1.1) $$\pi_j\pi_k = 0 \quad \textit{for} \quad |j-k| \geq N.$$

Remark 7.1.1 *We shall frequently encounter the following situation: we have two real valued functions $\phi(x)$ and $\psi(x)$ defined on some domain D, so that there exists a constant $C > 0$ such that for all $x \in D$*

(7.1.2) $$C^{-1}\phi(x) \leq \psi(x) \leq C\phi(x).$$

In such cases we shall say that ϕ and *ψ* are equivalent *on D, and we shall write*

(7.1.3) $$\phi(x) \sim \psi(x)$$

for $x \in D$.

Example 7.1.1 *Let $\{\phi_j\}_{j\geq 0}$ be a Paley-Littlewood partition of unity on $\mathbb{R}^n$, i.e., a sequence $\phi_j \in C_c^\infty(\mathbb{R}^n)$ such that $\phi_j \geq 0$, $\sum \phi_j = 1$, and*

(7.1.4) $$\operatorname{supp}\phi_0 \subseteq \{|x| \leq 2\}, \qquad \operatorname{supp}\phi_j \subseteq \{2^{j-1} \leq |x| \leq 2^{j+1}\} \quad j \geq 1.$$

More precisely, fix an arbitrary nonnegative $\psi \in C_c^\infty(\mathbb{R}^n)$, $0 \leq \psi \leq 1$, equal to 1 on the ball $B(0,1/2)$ and vanishing outside $B(0,1)$, and define

(7.1.5) $$\phi(x) = \psi(x/2) - \psi(x), \quad \phi_0(x) = \psi(x/2), \quad \phi_j(x) = \phi(2^{-j}x), \ j \geq 1.$$

This gives a partition of unity satisfying 7.1.4, and we shall call it a (standard) Paley-Littlewood partition of unity (PL partition for short).

We remark that if we choose $A = L^p(\mathbb{R}^n)$, $p \in [1,\infty]$, and define $\pi_j : A \to A$ as the multiplication operator by ϕ_j then $\pi = \{\pi_j\}$ is a PL partition of identity in the sense of Definition 7.1.1. Moreover, it satisfies the following important property, which will be used several times in the sequel: for any $1 \leq p < \infty$

(7.1.6) $$\|u\|^p_{L^p(\mathbb{R}^n)} \sim \sum_{j\geq 0} \|\phi_j u\|^p_{L^p(\mathbb{R}^n)}$$

and similarly

$$\|u\|_{L^\infty(\mathbb{R}^n)} \sim \sup_{j\geq 0} \|\phi_j u\|_{L^\infty(\mathbb{R}^n)}.$$

The second relation is obvious. On the other hand, for $p < \infty$ we have

$$\frac{1}{2^{p-1}} \leq \sum_{j\geq 0} \phi_j(x)^p \leq 1$$

since, at each $x \in \mathbb{R}^n$, at most 2 of the functions ϕ_j do not vanish. This implies

$$\frac{1}{2^{p-1}} \int |u|^p dx \leq \int \sum |\phi_j u|^p dx \leq \int |u|^p dx$$

and noticing that

$$\int \sum |\phi_j u|^p dx = \sum \int |\phi_j u|^p dx$$

by monotone convergence, we obtain 7.1.6.

In the sequel we shall need the following technical

Lemma 7.1.1 *Assume $\{\lambda_j\}_{j=-\infty}^{+\infty}$ is a two sided sequence of nonnegative real numbers, and let $\{\Lambda_j\}_{j\geq 0}$ be a sequence of positive real numbers such that for some $C_0 > 0$ and all $j, k \geq 0$*

$$\sum_{h\geq 0} \Lambda_h \lambda_{k-h} \leq C_0 \Lambda_k, \qquad \sum_{h\geq 0} \frac{\lambda_{h-j}}{\Lambda_h} \leq \frac{C_0}{\Lambda_j}. \tag{7.1.7}$$

Then for all $q \in [1, \infty[$, for any sequence $\{a_j\}_{j\geq 0}$ of complex numbers,

$$\sum_{j\geq 0} \Lambda_j^q \Big|\sum_{k\geq 0} \lambda_{k-j} a_k\Big|^q \leq C_0^q \sum_{j\geq 0} \Lambda_j^q |a_j|^q \tag{7.1.8}$$

and also ("$q = \infty$")

$$\sup_{j\geq 0} \Lambda_j \Big|\sum_{k\geq 0} \lambda_{k-j} a_k\Big| \leq C_0 \sup_{j\geq 0} \Lambda_j |a_j|. \tag{7.1.9}$$

Proof. Let T be the operator acting on sequences of $\mathbf{C}$

$$T(\{a_k\}) = \{b_j\}, \quad b_j = \Lambda_j \sum_{k\geq 0} \frac{\lambda_{k-j}}{\Lambda_k} a_k, \quad j \geq 0. \tag{7.1.10}$$

The operator T is easily seen to be bounded on ℓ^∞; indeed, by the second property in (7.1.7),

$$\|T(\{a_k\})\|_{\ell^\infty} \sim \sup_{j\geq 0} \Lambda_j \Big|\sum_{k\geq 0} \frac{\lambda_{k-j}}{\Lambda_k} a_k\Big| \leq$$

$$\leq \sup_{k\geq 0} |a_k| \cdot \sup_{j\geq 0} \Lambda_j \sum_{k\geq 0} \frac{\lambda_{k-j}}{\Lambda_k} \leq C_0 \sup_{k\geq 0} |a_k|. \tag{7.1.11}$$

Notice that, when applied to the sequence $\Lambda_k a_k$, this proves (7.1.9). Moreover, T is bounded on ℓ^1; indeed, using the first property in (7.1.7) we have

$$\|T(\{a_k\})\|_{\ell^1} \sim \sum_{j\geq 0} \Lambda_j \Big| \sum_{k\geq 0} \frac{\lambda_{k-j}}{\Lambda_k} a_k \Big| \leq$$
$$\leq \sum_{k\geq 0} \frac{|a_k|}{\Lambda_k} \sum_{j\geq 0} \Lambda_j \lambda_{k-j} \leq C_0 \sum_{k\geq 0} |a_k|.$$

Now, by the Riesz-Thorin interpolation theorem (see e.g. [2]), we see that T is a bounded operator on ℓ^q for all q $(1 \leq q < \infty)$, with norm not greater than C_0; this gives the inequality

$$\|T(\{a_k\})\|_{\ell^q}^q \sim \sum_{j\geq 0} \Lambda_j^q \Big| \sum_{k\geq 0} \frac{\lambda_{k-j}}{\Lambda_k} a_k \Big|^q \leq C_0^q \sum_{k\geq 0} |a_k|^q. \tag{7.1.12}$$

If we apply (7.1.12) to the sequence $\Lambda_k a_k$ we obtain (7.1.8).

We are now ready to prove an abstract localization lemma, which in the next section will be applied to produce several equivalent norms on weighted Sobolev spaces.

Lemma 7.1.2 (localization lemma) *Let A, B be Banach spaces with norms $\|\cdot\|_A$, $\|\cdot\|_B$, endowed with PL partitions of identity $\{\pi_j\}$ and $\{p_k\}$ respectively, with the same integer N from Definition 2.1, and assume $F : A \to B$ is an invertible isometry. Let $\{\lambda_j\}_{j=-\infty}^{+\infty}$, $\{\Lambda_j\}_{j\geq 0}$ be two nonnegative sequences satisfying (7.1.7) and the following additional property: for some $C_1 > 0$*

$$\lambda_j \leq C_1 \lambda_k \quad \text{for} \quad |j-k| \leq N. \tag{7.1.13}$$

Finally, assume that for some $C_2 > 0$ and all j, k

$$\|F\pi_j F^{-1} p_k\|_{L(B)} + \|p_j F \pi_k F^{-1}\|_{L(B)} \leq C_2 \lambda_{k-j}. \tag{7.1.14}$$

Then the following equivalencies of norms hold on A:

$$\left(\sum_{k\geq 0} \Lambda_k^q \|p_k F u\|_B^q \right)^{1/q} \sim \left(\sum_{j\geq 0} \Lambda_j^q \|F\pi_j u\|_B^q \right)^{1/q}, \qquad q \in [1, \infty[, \tag{7.1.15}$$

$$\sup_{k\geq 0} \|p_k F u\|_B \sim \sup_{j\geq 0} \|F \pi_j u\|_B. \tag{7.1.16}$$

Proof. Using $\sum p_k = I$ and the property (7.1.1) we can write

$$F\pi_j u = \sum_k F\pi_j F^{-1} p_k F u = \sum_k \sum_{|\ell-k|\leq N} F\pi_j F^{-1} p_\ell p_k F u. \tag{7.1.17}$$

Hence, by (7.1.14) and (7.1.13),

$$\|F\pi_j u\|_B \le C_2 \sum_k \sum_{|\ell-k|\le N} \lambda_{\ell-j}\|p_k Fu\|_B \le (2N+1)C_1C_2 \sum_k \lambda_{k-j}\|p_k Fu\|_B.$$

Thus we can apply Lemma 7.1.1 to the sequence $a_k = \|p_k Fu\|_B$, and we obtain easily

$$\sum \Lambda_j^q \|F\pi_j u\|_B^q \le C \sum \Lambda_k^q \|p_k Fu\|_B^q, \tag{7.1.18}$$

which is the first inequality to prove (the case $q = \infty$ is analogous). The reverse inequality is proved in a similar way, writing

$$p_k Fu = \sum_j \sum_{|j-\ell|\le N} p_k F\pi_j F^{-1} F\pi_\ell u. \tag{7.1.19}$$

7.2 Localized Sobolev norms and weighted spaces

Notation 1 *In the following we shall frequently use the operators $\Lambda^s = (1-\Delta)^{s/2}$, $s \in \mathbf{R}$, defined as*

$$\Lambda^s u = \mathrm{F}^{-1}\langle\xi\rangle^s \mathrm{F}u, \tag{7.2.1}$$

where $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$ and $\mathrm{F} : \mathrm{L}^2(\mathbb{R}^n) \to \mathrm{L}^2(\mathbb{R}^n)$ is the Fourier transform.

Remark 7.2.1 *In the sequel we shall exclusively use the complex interpolation in the sense of Chapter 4 of [2]. We recall briefly the definition already introduced in the previous Chapter 6. Given a couple $A = (A_0, A_1)$ of Banach spaces embedded continuously in a common Hausdorff topological vector space, let Ω be the complex strip $0 < \mathrm{Re}\ z < 1$, and denote by $F(A)$ the space of functions bounded and continuous on $\overline{\Omega}$ and holomorphic on Ω, with values in $A_0 + A_1$, such that $\|F(iy)\|_{A_0}$ and $\|F(1+iy)\|_{A_1}$ are bounded for $y \in \mathbf{R}$. $F(A)$ is a Banach space with the norm*

$$\|f\|_F = \sup_y \left[\|F(iy)\|_{A_0} + \|F(1+iy)\|_{A_1}\right].$$

Then $A_\theta = (A_0, A_1)_\theta$, $0 < \theta < 1$, is defined as the Banach space of values $\{f(\theta)\}$ with $f \in F(A)$, endowed with the norm

$$\|u\|_{A_\theta} = \inf\{\|f\|_F : f \in F, f(\theta) = u\}.$$

7.3 The generalized Sobolev spaces

To give a first example of localized norms we shall consider the spaces

$$H_p^s = H_p^s(\mathbb{R}^n), \qquad s \in \mathbf{R}, \quad \mathbf{p} \in [1, \infty], \tag{7.3.1}$$

also denoted by $\mathrm{L}_\mathrm{p}^\mathrm{s}(\mathbb{R}^n)$, whose norm is defined as follows:

$$\|u\|_{H_p^s} = \|\Lambda^s u\|_{L^p}. \tag{7.3.2}$$

As usual H_p^s is defined as the space of all tempered distributions u such that $\Lambda^s u \in L^p$ and the above norm is finite. These spaces are well studied; see e.g. [2], [59], [62]. We list a few properties of these spaces, whose proofs can be found in the given references or in the previous Chapter 6.

1. If $s \geq 0$ is an integer and $1 < p < \infty$, then H_p^s coincides with the usual Sobolev space $W^{s,p}(\mathbb{R}^n)$.

2. Λ^s is an isomorphism of H_p^σ onto $H_p^{\sigma-s}$, $s, \sigma \in \mathbf{R}$, $1 \leq p \leq \infty$.

3. We have the Sobolev type continuous embeddings

$$H_p^s \subseteq C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n), \qquad s > \frac{n}{p}, \quad 1 < p < \infty; \tag{7.3.3}$$

$$H_p^s \subseteq L^q(\mathbb{R}^n), \qquad s \geq \frac{n}{p} - \frac{n}{q}, \quad 1 < p \leq q < \infty; \tag{7.3.4}$$

$$H_p^s \subseteq H_p^\sigma, \qquad s \geq \sigma, \quad 1 \leq p \leq \infty. \tag{7.3.5}$$

4. If $s \in \mathbf{R}$ and $1 \leq p < \infty$, then

$$(H_p^s)' = H_q^{-s}, \qquad \frac{1}{p} + \frac{1}{q} = 1; \tag{7.3.6}$$

moreover, $C_c^\infty(\mathbb{R}^n)$ and $\mathcal{S}$ are dense in H_p^s.

5. Probably the most useful property of these spaces is their behavior with respect to interpolation: for all real $s_0 \neq s_1$ and all $p_0, p_1 \in]1, \infty[$ we have

$$(H_{p_0}^{s_0}, H_{p_1}^{s_1})_\theta = H_p^s, \tag{7.3.7}$$

where

$$0 < \theta < 1, \tag{7.3.8}$$

$$s = (1-\theta)s_0 + \theta s_1, \tag{7.3.9}$$

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}. \tag{7.3.10}$$

Remark 7.3.1 *The following property will be used frequently in the sequel. Let $\phi(x)$ be a smooth function such that*

$$\|\partial_x^\alpha \phi\|_{L^\infty} \leq C_N \quad \text{for} \quad |\alpha| \leq N. \tag{7.3.11}$$

Then the multiplication operator by ϕ is a bounded operator on H_p^s, for all $s \in \mathbf{R}$ with $|s| \leq N$ and $1 < p < \infty$. This is trivial when $s \geq 0$ is an integer (Leibnitz' rule), hence is true for real $s \geq 0$ by interpolation property (7.3.7), and follows easily by duality for negative s.

We show now how it is possible to localize the H^s_p norm. A first localization is trivial, and follows immediately by the equivalence (7.1.6):

$$\|u\|^p_{H^s_p} \sim \sum_{j\geq 0} \|\phi_j \Lambda^s u\|^p_{L^p}, \tag{7.3.12}$$

where $\phi_j(x)$ is a PL partition of unity as in Example 7.1.1. The following result is more subtle:

Lemma 7.3.1 *For $s \in \mathbf{R}$, $1 < p < \infty$, we have*

$$\|u\|^p_{H^s_p} \sim \sum_{j\geq 0} \|\phi_j \Lambda^s u\|^p_{L^p} \sim \sum_{j\geq 0} \|\Lambda^s(\phi_j u)\|^p_{L^p}. \tag{7.3.13}$$

Proof. Taking into account (7.3.12), we need only to prove the equivalence of the last two quantities. We shall apply Lemma 7.1.2 with the choices $A = H^s_p$, $B = L^p$, while the partitions of identity p_j, π_j are both defined as multiplication by ϕ_j as in Example 7.1.1; we can take $N = 2$. Moreover we choose

$$\Lambda_j = 1, \qquad \lambda_j = 2^{-m|j|} \tag{7.3.14}$$

where $m > 1$ will be precised in the following. It is trivial to verify that assumptions (7.1.7), (7.1.13) are satisfied. Finally we take $F = \Lambda^s$ which is an invertible isometry of A onto B. With these choices, (7.3.13) is exactly (7.1.15) (with $q = p$), thus the result will follow as soon as we verify that (7.1.14) is satisfied. Hence we must prove that for some C independent of $u \in H^s_p$

$$\|\Lambda^s \phi_j \Lambda^{-s} \phi_k u\|_{L^p} \leq \frac{C}{2^{|j-k|m}} \|u\|_{L^p}, \tag{7.3.15}$$

$$\|\phi_k \Lambda^s \phi_j \Lambda^{-s} u\|_{L^p} \leq \frac{C}{2^{|j-k|m}} \|u\|_{L^p}. \tag{7.3.16}$$

Actually, it is possible to choose any $m > 1$, as it will be clear at the end of the proof. Notice that (7.3.16) is a consequence of (7.3.15), since the operator $\Lambda^s \phi_j \Lambda^{-s} \phi_k$ is dual to $\phi_k \Lambda^{-s} \phi_j \Lambda^s$ in the pairing $\langle L^p, L^{p'} \rangle$ (with s arbitrary real and $1 < p < \infty$). To prove (7.3.15), we begin by remarking that

$$\|\Lambda^s \phi_j \Lambda^{-s} \phi_k\|_{\mathrm{L}(\mathrm{L}^p)} \leq C \tag{7.3.17}$$

with C independent of j, k; this follows from Remark (7.3.1):

$$\|\Lambda^s \phi_j \Lambda^{-s} \phi_k u\|_{L^p} = \|\phi_j \Lambda^{-s} \phi_k u\|_{H^s_p} \leq C \|\Lambda^{-s} \phi_k u\|_{H^s_p} \leq C \|\phi_k u\|_{L^p} \leq C \|u\|_{L^p}$$

(we have used the fact that $\Lambda^{-s} : H^s_p \to L^p$ is an isometry and that $|\partial^\alpha \phi_j| \leq C_\alpha$ with C_α independent of j). Thus it is sufficient to prove (7.3.15) for $|j - k| \geq 3$, i.e., when the supports of ϕ_j and ϕ_k are disjoint.

Let $u \in C_0^\infty(\mathbb{R}^n)$. By the standard computation ($\tilde{d}\xi = (2\pi)^{-n}d\xi$, $D = \partial/i$)

$$\iint e^{i(x-y)\xi}\langle\xi\rangle^s u(y)(x-y)^\alpha dy\tilde{d}\xi = \int D_\xi^\alpha \left(\int e^{i(x-y)\xi}u(y)dy\right)\langle\xi\rangle^s\tilde{d}\xi$$
$$= \iint e^{i(x-y)\xi}u(y)(-D_\xi)^\alpha\langle\xi\rangle^s dy\tilde{d}\xi,$$

we see that the kernel $K_s(x-y)$ of the operator Λ^s, defined by

$$\Lambda^s u(x) = \langle K_s(x-\cdot), u(\cdot)\rangle, \tag{7.3.18}$$

satisfies for any α

$$(x-y)^\alpha K_s(x-y) = \int e^{i(x-y)\xi}(-D_\xi)^\alpha\langle\xi\rangle^s\tilde{d}\xi \tag{7.3.19}$$

which is an ordinary (not oscillatory) integral as soon as $|\alpha| > s+n$. So $K_s(z)$ is smooth for $z \neq 0$ and we have

$$|z^\alpha K_s(z)| \le C(\alpha, s) \ \text{ for any } \ |\alpha| > s+n.$$

In a similar way,

$$|D_z^\beta z^\alpha K_s(z)| \le C(\alpha, \beta, s) \ \text{ for any } \ |\alpha| - |\beta| > s+n.$$

Consequently,

$$|z^\alpha D_z^\beta K_s(z)| \le C(\alpha, \beta, s) \ \text{ for any } \ |\alpha| - |\beta| > s+n.$$

So we arrive at

$$|D_z^\beta K_s(z)| \le \frac{C(\beta, s, M)}{|z|^M} \ \text{ for any } \ M - |\beta| > s+n. \tag{7.3.20}$$

Since $|j-k| \ge 3$ the supports of ϕ_j, ϕ_k are disjoint and more precisely

$$x \in \operatorname{supp}\phi_j,\ y \in \operatorname{supp}\phi_k \quad \Rightarrow \quad |x-y| \ge \frac{1}{4}2^{|j-k|} \tag{7.3.21}$$

as it is readily seen. Thus the operator $\phi_j\Lambda^s\phi_k$ has the kernel

$$K_{ij}(x,y) = \phi_j(x)K_s(x-y)\phi_k(y) \tag{7.3.22}$$

which is a smooth function. Since

$$D_x^\alpha(\phi_j\Lambda^s\phi_k u) = D_x^\alpha \int \phi_j(x)K_s(x-y)\phi_k(y)u(y)dy, \tag{7.3.23}$$

by Leibnitz' rule and using (7.3.20), (7.3.21), we obtain

$$|D_x^\alpha(\phi_j\Lambda^s\phi_k u)| \leq$$

$$C\sum_{\beta\leq\alpha}|D^\beta\phi_j(x)|\cdot\int\phi_k(y)|u(y)|dy\cdot 2^{-|j-k|M}. \tag{7.3.24}$$

This implies easily for $p=1$ or $p=\infty$

$$\|D^\alpha(\phi_j\Lambda^s\phi_k u)\|_{L^p}\leq\frac{C(\alpha,s,M)}{2^{|j-k|M}}\|u\|_{L^p} \tag{7.3.25}$$

with a little bit larger constant $C(\alpha,s,M)$ and hence for any $p\in[1,\infty]$ by interpolation. In particular we have proved that

$$\|\Lambda^{2\ell}\phi_j\Lambda^{-s}\phi_k u\|_{L(L^p)}\leq C(\ell,s,M)\cdot 2^{-|j-k|M} \tag{7.3.26}$$

for any nonnegative integer ℓ, $1\leq p\leq\infty$, and any $M,j,k\geq 0$. From this, (7.3.15) follows easily, for $1<p<\infty$, by the well known L^p boundedness of the operator $\Lambda^{s-2\ell}$ for $s\leq 2\ell$ (and in fact of any operator in $OPS^0_{1,0}$).

7.4 The weighted Sobolev spaces $H_p^s(\rho)$

Definition 7.4.1 *Let $\chi(x)\in C^\infty(\mathbb{R}^n)$ be a smooth, strictly positive, radial function $\chi(x)=\rho(|x|)$. We shall say that $\chi(x)$ (or $\rho(R)$) is a* weight function, *or simply a* weight, *if for all $k\geq 0$*

$$\left|\rho^{(k)}(R)\right|\leq C_k\rho(R), \tag{7.4.1}$$

and for any $\delta>0$ there exists $C=C(\delta)>0$ such that

$$C^{-1}\rho(R_1)\rho(R_2)\leq\rho(R_1R_2)\leq C\rho(R_1)\rho(R_2)\quad\textit{for any}\quad R_1,R_2>\delta. \tag{7.4.2}$$

In some cases it is useful to require the stronger property

$$\left|\rho^{(k)}(R)\right|\leq C_k\langle R\rangle^{-k}\rho(R); \tag{7.4.3}$$

we shall call such a ρ a *strong weight.*

The most typical example of a weight corresponds to the choice

$$\rho(R)=\langle R\rangle^s \tag{7.4.4}$$

for any $s\in\mathbf{R}$; notice this is also a strong weight.

We notice two consequences of this definition. There exists $C>0$ independent of j such that

$$2^{j-1}\leq|x|\leq 2^{j+1}\quad\Rightarrow\quad C^{-1}\rho(2^j)\leq\rho(|x|)\leq C\rho(2^j). \tag{7.4.5}$$

Moreover, the reciprocal of a weight is still a weight, and in particular

$$\left|\left(\frac{1}{\rho}\right)^{(k)}\right| \leq C_k \frac{1}{\rho}. \tag{7.4.6}$$

In a similar way, the reciprocal of a strong weight is a strong weight. This is easily proved using the formula

$$\left(\frac{1}{\rho}\right)^{(k)} = \sum_{\nu=1}^{k} \sum_{j_1+\cdots+j_\nu=k} \binom{k}{j_1 \cdots j_\nu} (-1)^\nu \frac{\rho^{(j_1)} \cdots \rho^{(j_\nu)}}{\nu \rho^{\nu+1}} \tag{7.4.7}$$

together with 7.4.1.

We are now ready to introduce the weighted Sobolev space $H_p^s(\rho)$, whose norm is defined, for any $s \in \mathbf{R}$ and $1 < p < \infty$, by

$$\|u\|_{H_p^s(\rho)} = \|\Lambda^s[\rho(|x|)u(x)]\|_{L^p(\mathbf{R}^n)} = \|\Lambda^s(\rho u)\|_{L^p} = \|\rho u\|_{H_p^s}. \tag{7.4.8}$$

The properties of the spaces H_p^s can be extended to the case of weighted spaces $H_p^s(\rho)$. In particular we have the complex interpolation property: for all real $s_0 \neq s_1$ and all $p_0, p_1 \in]1, \infty[$

$$(H_{p_0}^{s_0}(\rho_0), H_{p_1}^{s_1}(\rho_1))_\theta = H_p^s(\rho), \tag{7.4.9}$$

where

$$0 < \theta < 1, \tag{7.4.10}$$

$$\rho = \rho_0^{1-\theta} \rho_1^\theta \tag{7.4.11}$$

$$s = (1-\theta)s_0 + \theta s_1, \tag{7.4.12}$$

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}. \tag{7.4.13}$$

This is an immediate consequence of the corresponding property for the spaces H_p^s. Indeed, the operator

$$\phi(z) \mapsto \rho_0^{1-z} \rho_1^z \phi(z) \tag{7.4.14}$$

is evidently an isomorphism of $F(H_{p_0}^{s_0}(\rho_0), H_{p_1}^{s_1}(\rho_1))$ onto $F(H_{p_0}^{s_0}, H_{p_1}^{s_1})$ (see Remark 7.2.1 for notations).

Moreover, we have for any $s \in \mathbf{R}$ and $1 < p < \infty$

$$(H_p^s(\rho))' = H_q^{-s}(1/\rho), \qquad \frac{1}{q} + \frac{1}{p} = 1 \tag{7.4.15}$$

(consequence of the duality property of H_p^s).

Finally, one can obtain from (7.3.3) – (7.3.5) corresponding embedding properties for weighted spaces. In particular we notice

$$\|\rho u\|_{L^\infty} \le C(s,n,\rho)\ \|u\|_{H_p^s(\rho)} \tag{7.4.16}$$

valid for any real $s > n/p, 1 < p < \infty$.

Moreover, we have

$$\|\rho f\|_{L^q} \le C\|f\|_{H_p^s(\rho)}, \tag{7.4.17}$$

provided $1 < p, q < \infty$ and

$$\frac{s}{n} \ge \frac{1}{p} - \frac{1}{q}.$$

We can see that some basic properties of $H^s(\mathbb{R}^n)$ spaces holds also in $H^s(\rho) = H_2^s(\rho)$.

Theorem 7.4.1 *Let s, λ be positive real numbers such that $\lambda > 1$ and $s < \lambda$. If ρ_0, ρ_1 are strong weights and*

$$\rho = \rho_0 \rho_1^{\lambda-1},$$

then

$$\||f|^\lambda\|_{H^s(\rho)} \le C\|f\|_{H^s(\rho_0)}\|\rho_1 f\|_{L^\infty(\mathbb{R}^n)}^{\lambda-1} \tag{7.4.18}$$

for all f such that the norms on the right side are finite.

This result follows in a trivial way from the corresponding result on flat space without weights. In the same way we obtain the following.

Theorem 7.4.2 *Suppose ρ_0, ρ_1 are strong weights. Let $\rho = \rho(|x|)$ be such that*

$$\rho = \rho_0 \rho_1.$$

For

$$f, g \in H^s(\rho) \cap L^\infty(\mathbb{R}^n)$$

and any non-negative s we have

$$\begin{aligned} \|fg\|_{H^s(\rho)} \le C(&\|f\|_{H^s(\rho_0)}\|\rho_1 g\|_{L^\infty(\mathbb{R}^n)} + \\ &+\|\rho_0 f\|_{L^\infty(\mathbb{R}^n)}\|g\|_{H^s(\rho_1)}). \end{aligned} \tag{7.4.19}$$

We give now several equivalent localizations of the weighted norm.

Lemma 7.4.1 *Let $\{\phi_j\}$ be a PL partition of unity, $1 < p < \infty$, $s \in \mathbf{R}$ and $\rho(|x|)$ be a weight. Then the following norms raised to power p are equivalent on $H^s_p(\rho)$:*

$$I = \|u\|^p_{H^s_p(\rho)} \sim \|\Lambda^s(\rho u)\|^p_{L^p}, \tag{7.4.20}$$

$$II = \sum_{j\geq 0} \|\phi_j \Lambda^s(\rho u)\|^p_{L^p}, \tag{7.4.21}$$

$$III = \sum_{j\geq 0} \|\Lambda^s(\phi_j \rho u)\|^p_{L^p}, \tag{7.4.22}$$

$$IV = \sum_{j\geq 0} \rho(2^j)^p \|\Lambda^s(\phi_j u)\|^p_{L^p}, \tag{7.4.23}$$

$$V = \|\rho\Lambda^s u\|^p_{L^p}. \tag{7.4.24}$$

Proof. $I \sim II$ is a consequence of (7.1.6). $II \sim III$ follows by (7.3.13) of Lemma 7.3.1 applied to the function $\rho(|x|)u(x)$. To prove $III \sim IV$ we remark that, by properties (7.4.1) and (7.4.5) (resp. (7.4.6) and (7.4.5)), the functions

$$\psi_j(x) = (\phi_{j-1} + \phi_j + \phi_{j+1})\frac{\rho(|x|)}{\rho(2^j)},$$
$$\chi_j(x) = (\phi_{j-1} + \phi_j + \phi_{j+1})\frac{\rho(2^j)}{\rho(|x|)},$$

(set $\phi_{-1} \sim 0$) satisfy for all α

$$|\partial^\alpha_x \psi_j(x)| + |\partial^\alpha_x \chi_j(x)| \leq C_\alpha \tag{7.4.25}$$

with constants C_α independent of j. Hence by Remark 7.3.1 multiplication by ψ_j or χ_j is a bounded operator on H^s_p, $s \in \mathbf{R}$, $1 < p < \infty$, with norm uniformly bounded in j; equivalently,

$$\|\Lambda^s \psi_j \Lambda^{-s}\|_{L(L^p)} + \|\Lambda^s \chi_j \Lambda^{-s}\|_{L(L^p)} \leq C(s) \tag{7.4.26}$$

with $C(s)$ independent of j. Notice that $\phi_j\rho = \rho(2^j)\psi_j\phi_j$, because $\phi_{j-1} + \phi_j + \phi_{j+1} \sim 1$ on the support of ϕ_j. Thus writing

$$\|\Lambda^s(\phi_j \rho u)\|_{L^p} = \rho(2^j)\|(\Lambda^s\psi_j\Lambda^{-s})\Lambda^s(\phi_j u)\|_{L^p} \leq C\rho(2^j)\|\Lambda^s(\phi_j u)\|_{L^p},$$

we get $III \leq C \cdot IV$, and similarly for the reverse inequality writing $\rho(2^j)\Lambda^s(\phi_j u) = \Lambda^s\chi_j\Lambda^{-s}\Lambda^s\rho\phi_j u$.

Finally, $IV \sim V$ is a consequence of Lemma 7.1.2. Indeed, we choose A, B, p_j, π_j, λ_j exactly as in the proof of Lemma 7.3.1 (recall in particular (7.3.15), (7.3.16) already proved there), the only difference consisting in the choice

$$\Lambda_j = \rho(2^j); \tag{7.4.27}$$

assumption (7.1.7) is readily verified. Indeed, by property (7.4.1) it follows that, for a suitable $C > 1$,

$$C^{-j}\rho(1) \leq \rho(2^j) \leq C^j \rho(1); \tag{7.4.28}$$

hence it is clear that, choosing m large enough in (7.3.15), (7.3.16), we obtain (7.1.7). Thus by Lemma 7.1.2 we get

$$IV \sim \sum_{j\geq 0} \rho(2^j)^p \|\phi_j \Lambda^s u\|_{L^p}^p \tag{7.4.29}$$

and the last quantity is clearly equivalent to V by (7.4.5) and (7.1.6).

Remark 7.4.1 *When s is a nonnegative integer, $1 < p < \infty$, we may use the identity $H_p^s \sim W^{s,p}$ (classical Sobolev spaces) in connection with Lemma 7.4.1 to give further equivalent representations for the $H_p^s(\rho)$ norm (on power p):*

$$\|u\|_{H_p^s(\rho)}^p, \sum_{|\alpha|\leq s} \|\rho\partial^\alpha u\|_{L^p}^p, \tag{7.4.30}$$

$$\sum_{j\geq 0,\ |\alpha|\leq s} \|\phi_j \partial^\alpha(\rho u)\|_{L^p}^p, \quad \sum_{j\geq 0,\ |\alpha|\leq s} \|\partial^\alpha(\phi_j \rho u)\|_{L^p}^p, \tag{7.4.31}$$

$$\sum_{j\geq 0,\ |\alpha|\leq s} \rho(2^j)^p \|\partial^\alpha \phi_j u\|_{L^p}^p, \quad \sum_{j\geq 0,\ |\alpha|\leq s} \rho(2^j)^p \|\phi_j \partial^\alpha u\|_{L^p}^p. \tag{7.4.32}$$

7.5 Sobolev spaces associated to Lie algebras

Let Z be an N-tuple of smooth vector fields on $\mathbb{R}^n$

$$Z = (Z_1, \ldots, Z_N), \tag{7.5.1}$$

such that their commutators satisfy

$$[Z_j, Z_k] = \sum_{m=1}^{N} c_{jk}^m(x) Z_m \tag{7.5.2}$$

for suitable $c_{jk}^m \in C^\infty(\mathbb{R}^n)$. It is convenient to require also that

$$|Z^\alpha c_{jk}^m(x)| \leq C_\alpha \tag{7.5.3}$$

for all x, α. Moreover, let $\rho(|x|)$ be a weight function. Then one can define, for any integer $s \geq 0$ and $1 \leq p \leq \infty$, the Sobolev spaces generated by Z, written $H_p^s(\rho, Z)$ through the norm

$$\|u\|_{H_p^s(\rho,Z)}^p = \sum_{|\alpha|\leq s} \|\rho\, Z^\alpha u\|_{L^p}^p. \tag{7.5.4}$$

In the following we shall consider only the following choice of Z:

$$Z = \langle x\rangle \partial_x = (\langle x\rangle \partial_1, \dots, \langle x\rangle \partial_n). \tag{7.5.5}$$

Of special importance are the weight functions

$$\rho(|x|) = \langle x\rangle^{\delta}, \quad \delta \in \mathbf{R}; \tag{7.5.6}$$

we shall denote the corresponding spaces by $H_p^{s,\delta}$:

$$H_p^{s,\delta} \sim H_p^s(\langle x\rangle^{\delta}, \langle x\rangle \partial_x), \tag{7.5.7}$$

and in particular we shall omit p when $p = 2$:

$$H^{s,\delta} \sim H_2^{s,\delta} \sim H_2^s(\langle x\rangle^{\delta}, \langle x\rangle \partial_x). \tag{7.5.8}$$

The $H^{s,\delta}$ spaces where introduced in [5] for integer s, in connection with elliptic systems. These spaces are especially well suited to estimate solutions of the wave equation; in order to obtain optimal results, it will be necessary to extend the definition to any real s. The simplest way would be to use interpolation and duality arguments, but the abstract spaces thus obtained are not easy to handle. Instead, we prefer to give explicit representations of the norms as in the following definition, and to recover *a posteriori* the interpolation and duality properties.

To motivate our definition, let us first rephrase the definition in the integer case in a suitable way:

Remark 7.5.1 *Let $s \geq 0$ be a positive integer, $1 \leq p < \infty$, $\delta \in \mathbf{R}$. According to Definition 7.5.7, the norm of the space $H_p^{s,\delta}$, which we shall denote by X for short, has (on power p)*

$$\|u\|_X^p \sim \sum_{|\alpha| \leq s} \|(\langle x\rangle D)^{\alpha}(\langle x\rangle^{\delta} u)\|_{L^p}^p. \tag{7.5.9}$$

Noticing that

$$(\langle x\rangle D)^{\alpha} = \sum_{\beta \leq \alpha} \psi_{\beta}(x) D^{\beta} \quad \textit{with} \quad |\psi_{\beta}(x)| \leq C_{\beta}\langle x\rangle^{|\beta|},$$

and an identical property for $D^{\alpha}\langle x\rangle^{|\alpha|}$, $(D\langle x\rangle)^{\alpha}$, it is clear that the following equivalencies hold:

$$\|u\|_X^p \sim \sum_{|\alpha| \leq s} \|\langle x\rangle^{|\alpha|} D^{\alpha}\langle x\rangle^{\delta} u\|_{L^p}^p \sim \sum_{|\alpha| \leq s} \|D^{\alpha}(\langle x\rangle^{|\alpha|+\delta} u)\|_{L^p}^p \tag{7.5.10}$$
$$\sim \sum_{|\alpha| \leq s} \|(D\langle x\rangle)^{\alpha}\langle x\rangle^{\delta} u\|_{L^p}^p.$$

We use now a PL partition of identity (recall (7.1.6)) to obtain

$$\|u\|_X^p \sim \sum_{\substack{j\geq 0\\ |\alpha|\leq s}} \|\langle x\rangle^{|\alpha|} D^\alpha(\phi_j\langle x\rangle^\delta u)\|_{L^p}^p, \tag{7.5.11}$$

and by (7.1.4) we get

$$\|u\|_X^p \sim \sum_{\substack{j\geq 0\\ |\alpha|\leq s}} \|(2^j D)^\alpha(\phi_j\langle x\rangle^\delta u)\|_{L^p}^p. \tag{7.5.12}$$

We introduce now the dilation operators S, $\lambda > 0$, *defined by*

$$(\mathrm{S}_\lambda \mathrm{u})(\mathrm{x}) = \mathrm{u}(\lambda \mathrm{x}),$$

and we notice the following properties:

$$\|\mathrm{S}_\lambda \mathrm{u}\|_{\mathrm{L}^\mathrm{p}} = \lambda^{-\mathrm{n}/\mathrm{p}}\|\mathrm{u}\|_{\mathrm{L}^\mathrm{p}}, \tag{7.5.13}$$
$$D^\alpha \mathrm{S}_\lambda \mathrm{u} = \lambda^{|\alpha|}\mathrm{S}_\lambda \mathrm{D}^\alpha \mathrm{u} = \mathrm{S}_\lambda((\lambda \mathrm{D})^\alpha \mathrm{u}), \tag{7.5.14}$$
$$S_{1/\lambda} D^\alpha \mathrm{S}_\lambda \mathrm{u} = (\lambda \mathrm{D})^\alpha \mathrm{u}, \tag{7.5.15}$$
$$\mathrm{F}\mathrm{S}_\lambda \mathrm{u} = \lambda^{-\mathrm{n}}\mathrm{S}_{1/\lambda}\widehat{\mathrm{u}}. \tag{7.5.16}$$

Thus using (7.5.15) we may write for any even integer $s \geq 0$

$$\|u\|_{H_p^{s,\delta}} \sim \sum_{\substack{j\geq 0\\ |\alpha|\leq s}} \|S_{2^{-j}} D^\alpha S_{2^j}(\phi_j\langle x\rangle^\delta u)\|_{L^p}^p \sim$$
$$\sum_{j\geq 0} \|S_{2^{-j}}(1-\Delta)^{s/2} S_{2^j}(\phi_j\langle x\rangle^\delta u)\|_{L^p}^p. \tag{7.5.17}$$

This suggests the following definition.

Definition 7.5.1 *Let* $s \in \mathbf{R}$, $1 < p < \infty$, *let* $\{\phi_j\}$ *be a PL partition of unity, and let* $\rho(|x|)$ *be a strong weight (see (7.4.3)). The* $H_p^s(\rho, \langle x\rangle\partial)$ *norm raised to power p is defined as*

$$\|u\|_{H_p^s(\rho,\langle x\rangle\partial)}^p = \sum_{j\geq 0} \|S_{2^{-j}}\Lambda^s S_{2^j}(\rho\phi_j u)\|_{L^p}^p. \tag{7.5.18}$$

and $H_p^s(\rho, \langle x\rangle\partial)$ *is the Banach space of all tempered distributions such that the above norm is (defined and) finite. We shall also write*

$$\Lambda_j^s = S_{2^{-j}}\Lambda^s S_{2^j}; \tag{7.5.19}$$

it is trivial to verify that Λ_j^s *is a pseudodifferential operator, and more precisely*

$$\Lambda_j^s \ \textit{has symbol}\ \langle 2^j\xi\rangle^s = (1 + 2^{2j}|\xi|^2)^{s/2}. \tag{7.5.20}$$

Thus we may write also

$$\|u\|^p_{H^s_p(\rho,\langle x\rangle\theta)} = \sum_{j\geq 0} \|\Lambda^s_j(\rho\phi_j u)\|^p_{L^p}. \tag{7.5.21}$$

For integer $s \geq 0$, this is equivalent to the norms (7.5.10) and (7.5.9).

The next lemma gives an equivalent form of the norm:

Lemma 7.5.1 *For any $1 < p < \infty$, $s \in \mathbf{R}$ and $\rho(|x|)$ strong weight, we have the equivalence*

$$\|u\|^p_{H^s_p(\rho,\langle x\rangle\theta)} = \sum_{j\geq 0} \|\Lambda^s_j(\rho\phi_j u)\|^p_{L^p} \sim \sum_{j\geq 0} \rho(2^j)^p \|\Lambda^s_j(\phi_j u)\|^p_{L^p}. \tag{7.5.22}$$

Proof. The equivalence of the terms with $j = 0$ is obvious; for $j \geq 1$ we shall prove that

$$\|\Lambda^s_j(\rho\phi_j u)\|_{L^p} \leq C\rho(2^j)\|\Lambda^s_j(\phi_j u)\|_{L^p} \tag{7.5.23}$$

with a constant independent of j, and a similar reverse inequality, from which (7.5.22) follows immediately.

We recall that, for $j \geq 1$, $\phi_j(x) = \phi(2^{-j}x)$ (see (7.1.5)). Now, let $\psi \in C^\infty_c(\mathbf{R}^n)$ be equal to 1 on $\operatorname{supp}\phi \subset \{1/2 \leq |x| \leq 2\}$, and set

$$\psi_j(x) = \frac{\rho(2^j|x|)}{\rho(2^j)}\psi(x).$$

Then it is trivial to verify that

$$\Lambda^s_j(\rho\phi_j u) = S_{2^{-j}}\Lambda^s\psi_j\Lambda^{-s}S_{2^j}\Lambda^s_j(\phi_j u)\cdot\rho(2^j),$$

and in order to prove (7.5.23) it is sufficient to prove that the operators

$$S_{2^{-j}}\Lambda^s\psi_j\Lambda^{-s}S_{2^j}$$

are bounded on L^p uniformly in j. Since $S_{2^j}, S_{2^{-j}}$ are isomorphisms of L^p onto itself, with norms $2^{-jn/p}, 2^{jn/p}$ respectively (see (7.5.13)), it is sufficient to prove that

$$\|\Lambda^s\psi_j\Lambda^{-s}\|_{\mathrm{L}(\mathrm{L}^p)} \leq C, \tag{7.5.24}$$

with C independent of j, or equivalently that multiplication by ψ_j is bounded on H^s_p, with uniform bound in j. Thus we may use again Remark 7.3.1, and we are reduced to prove that

$$|D^\alpha\psi_j(x)| \leq C_\alpha \ \text{ independent of } \ j;$$

but this is a simple consequence of property (7.4.3) of the strong weight ρ.

The proof of the reverse inequality is similar, using a function of the form

$$\chi_j(x) = \frac{\rho(2^j)}{\rho(2^j x)}\psi(x)$$

and recalling that $1/\rho$ is also a strong weight.

Remark 7.5.2 *An equivalent characterization of the $H_p^s(\rho, \langle x\rangle\partial)$ spaces can be given using the selfadjoint operator*

$$A = D(\langle x\rangle^2 D).$$

In fact, A is a selfadjoint (see Section 4.2 in Chapter IV). Indeed, we have

$$\|u\|^p_{H_p^s(\rho,\langle x\rangle\partial)} \sim \|A^s(\rho u)\|_{L^p}$$

(compare (7.4.8)). Here we shall not use this equivalent norm.

In the sequel we shall restrict ourselves to the spaces $H_p^{s,\delta}$ defined in (7.5.7), with norm on power p

$$\|u\|^p_{H_p^{s,\delta}} = \sum_{j\geq 0} \|\Lambda_j^s(\langle x\rangle^\delta \phi_j u)\|^p_{L^p} \sim \sum_{j\geq 0} 2^{j\delta p}\|\Lambda_j^s(\phi_j u)\|^p_{L^p}.$$

The following lemma collects a few properties of these spaces:

Lemma 7.5.2 *Let $p, p_0, p_1 \in]1,\infty[$, $a, s, s_0, s_1, \delta, \delta_0, \delta_1 \in \mathbf{R}$.*

1. *The following duality relation holds:*

(7.5.25) $$(H_p^{s,\delta})' = H_q^{-s,-\delta}, \qquad \frac{1}{p}+\frac{1}{q} = 1.$$

Moreover, the complex interpolation property holds:

(7.5.26) $$(H_{p_0}^{s_0,\delta_0}, H_{p_1}^{s_1,\delta_1})_\theta = H_p^{s,\delta},$$

where

(7.5.27) $$0 < \theta < 1,$$

(7.5.28) $$\delta = (1-\theta)\delta_0 + \theta\delta_1,$$

(7.5.29) $$s = (1-\theta)s_0 + \theta s_1,$$

(7.5.30) $$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

2. *The following Sobolev type embeddings hold: for any* $1 < p < \infty$, $\delta \in \mathbf{R}$, $s > n/p$,

$$\langle x \rangle^{\delta+n/p} |u(x)| \leq C \|u\|_{H_p^{s,\delta}} \tag{7.5.31}$$

with $C = C(p,s,\delta,n)$ *independent of* $u \in H_p^{s,\delta}$*; and for any* $1 < p \leq q < \infty$, $\delta \in \mathbf{R}$, $s \geq n/p - n/q$,

$$\|\langle x \rangle^{\delta+n/p-n/q} u\|_{L^q} \leq C \|u\|_{H_p^{s,\delta}} \tag{7.5.32}$$

with $C = C(p,q,s,\delta,n)$ *independent of* $u \in H_p^{s,\delta}$*. Moreover, if* $s_0 \geq s_1$ *and* $\delta_0 \geq \delta_1$,

$$H_p^{s_0,\delta_0} \subseteq H_p^{s_1,\delta_1}$$

3. *Multiplication by a function* $\psi \in C_c^\infty(\mathbb{R}^n)$ *is a bounded operator on* $H_p^{s,\delta}$*. More generally, let* $\psi \in C^\infty(\mathbb{R}^n)$ *be a smooth function such that*

$$|D^\alpha \psi| \leq C_\alpha \quad \text{for } |\alpha| \leq N.$$

Then multiplication by ψ *is a bounded operator on* $H_p^{s,\delta}$ *provided* $|s| \leq N$*:*

$$\|\psi u\|_{H_p^{s,\delta}} \leq C \|u\|_{H_p^{s,\delta}} \tag{7.5.33}$$

with C *depending only on* s, δ, p *and on* C_α *for* $|\alpha| \leq N$*.*

4. *The multiplication operator by* $\langle x \rangle^a$ *is an isometry of* $H_p^{s,\delta}$ *onto* $H_p^{s,\delta-a}$*; moreover, for any multiindex* α,

$$x^\alpha : H_p^{s,\delta} \to H_p^{s,\delta-|\alpha|}, \qquad D^\alpha : H_p^{s,\delta} \to H_p^{s-|\alpha|,\delta+|\alpha|}, \tag{7.5.34}$$

are bounded operators. Thus in particular

$$\langle x \rangle^{|\alpha|} D^\alpha, x^\alpha D^\alpha : H_p^{s,\delta} \to H_p^{s-|\alpha|,\delta} \tag{7.5.35}$$

are bounded.

Proof. We begin by introducing the auxiliary spaces $A_p^{s,\delta}$, defined as follows: $A_p^{s,\delta}$ is the space of all sequences $\{u_j\}_{j\geq 0}$ with $u_j \in H_p^s$, such that the norm on power p

$$\|\{u_j\}\|_{A_p^{s,\delta}}^p = \sum_{j\geq 0} 2^{pj\delta} \|\Lambda_j^s u_j\|_{L^p}^p \tag{7.5.36}$$

is finite. Notice that

$$\|u\|_{H_p^{s,\delta}} = \|\{\phi_j u\}\|_{A_p^{s,\delta}}; \tag{7.5.37}$$

we shall return on this below. The space $A_p^{s,\delta}$ can be regarded as a space of type $\ell^p(A_j)$ of ℓ^p sequences with values in a sequence of Banach spaces; indeed, it is sufficient to define A_j as the Banach space of $u \in H_p^s$ with norm

$$\|u\|_{A_j} = 2^{j\delta} \|\Lambda_j^s u\|_{L^p}$$

and then

$$\|\{u_j\}\|^p_{A_p^{s,\delta}} \sim \sum_{j\geq 0} \|u_j\|^p_{A_j},$$

as required.

1. To prove the duality property, we remark that

$$(A_p^{s,\delta})' \sim A_q^{-s,-\delta}, \qquad \frac{1}{p} + \frac{1}{q} = 1, \tag{7.5.38}$$

meaning that a $T \in (A_p^{s,\delta})'$ can be identified to a sequence $\{v_j\} \in A_q^{-s,-\delta}$ through the identity

$$T(\{u_j\}) \sim \sum_{j\geq 0} v_j(u_j) \qquad \forall \{u_j\} \in A_p^{s,\delta},$$

(and of course $v_j(u_j) = \langle v_j, u_j\rangle$ is the usual duality pairing $\langle S', S\rangle$). The proof of (7.5.38) is standard. Now, let $T \in (H_p^{s,\delta})'$ and define an element $T_1 \in (A_p^{s,\delta})'$ according to the rule

$$T_1(\{u_j\}) = T\Big(\sum_{j\geq 0} \widetilde{\phi}_j u_j\Big),$$

where

$$\widetilde{\phi}_j = \phi_{j-1} + \phi_j + \phi_{j+1}, \qquad \phi_{-1} \sim 0; \tag{7.5.39}$$

notice that $\widetilde{\phi}_j \sim 1$ on the support of ϕ_j. We know T_1 can be identified with a sequence $\{v_j\} \in A_q^{-s,-\delta}$, and

$$\sum_{j\geq 0} v_j(u_j) = T_1(\{u_j\}) = T\Big(\sum_{j\geq 0} \widetilde{\phi}_j u_j\Big)$$

for any $\{u_j\} \in A_p^{s,\delta}$. Thus, in particular, for a fixed $u \in H_p^{s,\delta}$ we can write

$$T(u) = T\Big(\sum \widetilde{\phi}_j \phi_j u\Big) = \sum v_j(\phi_j u) = v(u)$$

where

$$v = \sum \phi_j v_j;$$

notice the last sum is locally finite, and gives an element $v \in H_q^{-s,-\delta}$. This proves the embedding $(H_p^{s,\delta})' \subseteq H_q^{-s,-\delta}$; the reverse embedding is trivial.

To prove (7.5.26), we start from the interpolation property

$$(A_{p_0}^{s_0,\delta_0}, A_{p_1}^{s_1,\delta_1})_\theta = A_p^{s,\delta}, \tag{7.5.40}$$

with indices as for $H_p^{s,\delta}$ spaces above (for a proof, see e.g. Section 1.18.1 of [62]).

We notice now that $H_p^{s,\delta}$ can be regarded as a *retract* of $A_p^{s,\delta}$, meaning that there exist two bounded maps

$$R : A_p^{s,\delta} \to H_p^{s,\delta}, \quad S : H_p^{s,\delta} \to A_p^{s,\delta}$$

with the property

$$RS = I \ \text{ on } \ H_p^{s,\delta};$$

R and S are called *retraction* and *coretraction* respectively (*belonging* to each other). Notice that S is an isomorphism of B with a subspace of A.

We recall the following general property of complex (and real) interpolation with respect to retractions. Assume A_j, B_j, $j = 0, 1$ are Banach spaces, embedded in some common Hausdorff vector topological space. Moreover, let R be a bounded operator from $A_0 + A_1$ to $B_0 + B_1$ whose restriction is bounded from A_j to B_j, $j = 0, 1$; similarly, let S be bounded from $B_0 + B_1$ to $A_0 + A_1$ and from B_j to A_j, $j = 0, 1$. Finally, let R be a retraction of A_j on B_j, $j = 0, 1$, with coretraction S. Then S is an isomorphism of the complex interpolation space $(B_0, B_1)_\theta$, $0 < \theta < 1$, onto a complemented subspace of $(A_0, A_1)_\theta$; this subspace is exactly the range of SR restricted to $(A_0, A_1)_\theta$, and SR is a projection onto it. For a proof see Section 1.2.4 of [62]; see also Section 6.4 of [2].

In the present case, we can define

$$R(\{u_j\}) = \sum \widetilde{\phi}_j u_j, \qquad S(u) = \{\phi_j u\}. \tag{7.5.41}$$

It is trivial to prove that $S : H_p^{s,\delta} \to A_p^{s,\delta}$ is bounded, actually it is an isometry onto its image, in view of (7.5.37). To prove that $R : A_p^{s,\delta} \to H_p^{s,\delta}$ is bounded, we notice that

$$\|R(\{u_j\})\|^p_{H_p^{s,\delta}} = \sum_{j\geq 0} 2^{j\delta p} \|\Lambda_j^s \phi_j \sum_{k\geq 0} \widetilde{\phi}_k u_k\|^p_{L^p};$$

now the products $\widetilde{\phi}_k \phi_j$ are different from zero only for $|j-k| \leq 2$, so that it is sufficient to estimate the sums

$$\Sigma_\epsilon = \sum_{j\geq 0} 2^{j\delta p} \|\Lambda_j^s \phi_j \widetilde{\phi}_{j+\epsilon} u_{j+\epsilon}\|^p_{L^p},$$

with $\epsilon = \pm 2, \pm 1, 0$. We show e.g. how to estimate Σ_{-1}, the others are identical. Let $\psi_j = \widetilde{\phi}_{j-1}\phi_j$; we have, for $j \geq 2$, $\psi_j = S_{2^j}\psi$ for a fixed function ψ with compact support (the terms for $j = 0, 1, 2$ are treated by a similar argument). Then

$$\|\Lambda_j^s \psi_j u_{j-1}\|_{L^p} = 2^{jn/p} \|\Lambda^s \psi S_{2^j} u_{j-1}\|_{L^p},$$

and noticing that multiplication by ψ is a bounded operator on H_p^s we obtain

$$\|\Lambda_j^s \psi_j u_{j-1}\|_{L^p} \leq C2^{jn/p}\|\Lambda^s S_{2^j} u_{j-1}\|_{L^p} \tag{7.5.42}$$

$$C\|\Lambda_j^s u_{j-1}\|_{L^p} = C\|\Lambda_j^s \Lambda_{j-1}^{-s} \Lambda_{j-1}^s u_{j-1}\|_{L^p}. \tag{7.5.43}$$

If we can prove that $\Lambda_j^s \Lambda_{j-1}^{-s}$ is bounded on L^p with norm independent of j, we obtain

$$\Sigma_{-1} \leq \sum_{j\geq 0} 2^{j\delta p} C\|\Lambda_{j-1}^s u_{j-1}\|_{L^p} \leq C\|\{u_j\}\|_{A_p^{s,\delta}}$$

i.e., the thesis. Now, $\Lambda_j^s \Lambda_{j-1}^{-s}$ has symbol

$$\left(\frac{1+2^{2j}|\xi|^2}{1+2^{2(j-1)}|\xi|^2}\right)^{s/2} = \chi(2^j\xi)^{s/2}, \quad \chi(\xi) = \frac{1+|\xi|^2}{1+|\xi|^2/4}.$$

To prove the L^p-boundedness we can use the Mikhlin theorem (see e.g. [60]), and we need only to verify that

$$|\xi|^{|\alpha|}|D_\xi^\alpha \chi(2^j\xi)| \leq C_\alpha \tag{7.5.44}$$

for $|\alpha| \leq [n/2]+1$, with C_α independent of j; but (7.5.44) follows from the condition

$$|\xi|^{|\alpha|}|D_\xi^\alpha \chi(\xi)| \leq C_\alpha \tag{7.5.45}$$

and (7.5.45) is obvious by the definition of $\chi(\xi)$.

Now, let H be the interpolation space $(H_{p0}^{s_0,\delta_0}, H_{p_1}^{s_1,\delta_1})_\theta$; by the above general result S is an isomorphism of H onto a subspace of $A_p^{s,\delta}$, which can be characterized as the range of SR restricted to $A_p^{s,\delta}$. Thus given $u \in H$ we know that $S(u) = \{\phi_j u\} \in A_p^{s,\delta}$, and this implies $u \in H_p^{s,\delta}$ at once by the definition; conversely, if $u \in H_p^{s,\delta}$ then it is easy to see that $\{\tilde{\phi}_j u\} \in A_p^{s,\delta}$, hence $R(\{\tilde{\phi}_j u\}) \in H$, but $R(\{\tilde{\phi}_j u\}) = \sum \phi_j \tilde{\phi}_j u = u$ and this concludes the proof.

2. Recalling (7.3.3) we have, for $s > n/p$, $1 < p < \infty$,

$$\|v\|_{L^\infty} \leq C\|\Lambda^s v\|_{L^p}$$

with $C = C(s,n,p)$ independent of v. By (7.5.13) we get

$$\|v\|_{L^\infty} \leq C2^{-jn/p}\|S_{2^{-j}}(\Lambda^s v)\|_{L^p}$$

and if we apply this to $v = S_{2^j}(\phi_j u)$ we obtain

$$\|\phi_j u\|_{L^\infty} \sim \|S_{2^j}(\phi_j u)\|_{L^\infty} \leq C2^{-jn/p}\|\Lambda_j^s(\phi_j u)\|_{L^p}$$

with a constant independent of j. This implies

$$\|\phi_j u\|_{L^\infty}^p \leq C\sum 2^{-jn}\|\Lambda_j^s(\phi_j u)\|_{L^p}^p \sim C\|u\|_{H_p^{s,-n/p}}^p$$

with C independent of j, and using the fact that

$$\|u\|_{L^\infty} \leq \sup_{j\geq 0} \|\phi_j u\|_{L^\infty}$$

we obtain

$$\|u\|_{L^\infty} \leq C\|u\|_{H_p^{s,-n/p}}.$$

This gives (7.5.31) at once, using the definition of the $H_p^{s,\delta}$ norm.

The other properties are proved in a similar way, starting from the corresponding properties of the H_p^s spaces.

3. The property is trivial for $s \geq 0$ integer and follows from Leibnitz rule (recall (7.5.9)). Thus it can be extended to $s \geq 0$ real using the interpolation property (7.5.26). Finally, it holds also for $s \leq 0$ using a duality argument and (7.5.25).

4. The first property is an immediate consequence of the definition of the $H_p^{s,\delta}$ norm. Properties (7.5.34) are trivial for s integer and nonnegative, extend to real s by interpolation, and to negative values of s by duality. The last property (7.5.35) is a consequence of (7.5.34).

Using the estimate (6.6.1) in combination with Lemma 7.5.1, we obtain the following.

Theorem 7.5.1 *Let δ_1, δ be real numbers. For any non-negative s we have*

$$\|fg\|_{H_p^{s,\delta}} \leq C(\|f\|_{H_p^{s,\delta-\delta_1}}\| < x >^{\delta_1} g\|_{L^\infty(\mathbf{R}^n)} + \tag{7.5.46}$$
$$+\| < x >^{\delta_1} f\|_{L^\infty(\mathbf{R}^n)}\|g\|_{H_p^{s,\delta-\delta_1}}.$$

Similarly from Moser type estimate in the flat space, established in Corollary 6.6.1, we obtain via localization Lemma 7.5.1 the following weighted estimate.

Theorem 7.5.2 *Let s, λ positive real numbers such that $\lambda > 1$ and $s < \lambda$. If α, β, γ are real numbers satisfying the relation*

$$\alpha = \beta + \gamma(\lambda - 1),$$

then we have

$$\||f|^\lambda\|_{H^{s,\alpha}} \leq C\|f\|_{H^{s,\beta}}\| < x >^\gamma f\|_{L^\infty(\mathbf{R}^n)}^{\lambda-1} \tag{7.5.47}$$

for all f such that the norms on the right side are finite.

7.6 The spaces $H^{s,-s}$ spaces

Of special interest are the spaces $H^{s,-s}$, whose norm on power 2 is equivalent to

$$\|u\|_{H^{s,-s}}^2 \sim \sum_{j\geq 0} 2^{-2js}\|\Lambda_j^s(\phi_j u)\|_{L^2}^2.$$

Lemma 7.6.1 *The spaces $H^{s,-s}$ have the following properties.*

1. *For any $s \geq 0$, we have the equivalence on $H^{s,-s}$*

$$\|u\|_{H^{s,-s}} \sim \|\langle x\rangle^{-s}u\|_{L^2} + \||\xi|^s\widehat{u}\|_{L^2}. \tag{7.6.1}$$

If in addition $0 \leq s < n/2$, we have the equivalence

$$\|u\|_{H^{s,-s}} \sim \||\xi|^s\widehat{u}\|_{L^2}. \tag{7.6.2}$$

2. *For any $\lambda > 0$, $0 \leq s < n/2$, we have*

$$C^{-1}\|u\|_{H^{s,-s}} \leq \lambda^{n/2-s}\|S_\lambda u\|_{H^{s,-s}} \leq C\|u\|_{H^{s,-s}} \tag{7.6.3}$$

with $C = C(s,n)$ independent of λ and $u \in H^{s,-s}$.

3. *For any $s \geq 0$ we have*

$$\|u\|_{H^{-s,s}} \leq C\|\langle x\rangle^s u\|_{L^2} \tag{7.6.4}$$

with $C = C(s,n)$ independent of $u \in H^{-s,s}$.

4. *For any $s > -n/2$ we have*

$$\||\xi|^s\widehat{u}\|_{L^2} \leq C\|u\|_{H^{s,-s}} \tag{7.6.5}$$

with $C = C(s,n)$ independent of $u \in H^{s,-s}$.

Proof. 1. Thanks to the interpolation property (7.5.26), it is sufficient to prove (7.6.1) only when $s \geq 0$ is an even integer. Notice that for integer s

$$\||\xi|^s\widehat{u}\|^2_{L^2} \sim \sum_{|\alpha|=s} \|D^\alpha u\|^2_{L^2}.$$

We begin by showing

$$\Sigma_{1\leq|\alpha|\leq m-1}\|D^\alpha u\|_{L^2(U)} \leq C(\|u\|_{L^2(U)} + \Sigma_{|\alpha|=m}\|D^\alpha u\|_{L^2(U)}), \tag{7.6.6}$$

for any positive integer m and for any open set $U \subset \mathbb{R}^n$ with smooth boundary. For simplicity, we write

$$\|u\|_{\dot{H}^j(U)} = \Sigma_{|\alpha|=j}\|D^\alpha u\|_{L^2(U)}, \qquad \|u\|_{H^m(U)} = \Sigma_{0\leq j\leq m}\|u\|_{\dot{H}^j(U)}.$$

By Theorem 9.6 in [35] (see also Chapter 3, Section 3.6) we have

$$\begin{aligned}\|u\|_{\dot{H}^j(U)} &\leq C_j\|u\|^{1-j/m}_{L^2(U)}\|u\|^{j/m}_{H^m(U)}\\ &\leq C_j\epsilon\|u\|_{H^m(U)} + C_{j,\epsilon}\|u\|_{L^2(U)}\end{aligned}$$

for any $\epsilon > 0$. Setting $S = \Sigma_{1\le j\le m-1}\|u\|_{\dot H^j(U)}$, we get

$$S \le C_m\epsilon S + C_m\epsilon\|u\|_{\dot H^m(U)} + C_{m,\epsilon}\|u\|_{L^2(U)},$$

because $\|u\|_{H^m(U)} \le \|u\|_{L^2(U)} + S + \|u\|_{\dot H^m(U)}$. If we choose ϵ sufficiently small, we obtain (7.6.6).

Now recall that, for $j \ge 1$, $\phi_j = \phi(2^{-j}x)$ with ϕ defined in (7.1.5), and let $\widetilde\phi(\xi) = \phi(\xi) + \phi(2\xi) + \phi(\xi/2) + \phi(\xi/4)$. By analogy we write for $j \ge 2$

$$\widetilde\phi_j = \widetilde\phi(2^{-j}x) = \phi_{j-1}(x) + \phi_j(x) + \phi_{j+1}(x) + \phi_{j+2}(x),$$

and we define

$$\widetilde\phi_1 = \phi_0(x) + \phi_1(x) + \phi_2(x) + \phi_3(x), \qquad \widetilde\phi_0 = \phi_0(x) + \phi_1(x) + \phi_2(x).$$

Notice that $\widetilde\phi(\xi) = 1$ on $U = \{1/2 \le |\xi| \le 4\}$ and that $\operatorname{supp}\phi \subset \{1/2 \le |\xi| \le 2\}$. Then we can write (recall s is an even integer)

$$\|\Lambda^s(\phi w)\|_{L^2}^2 \le c(s)\sum_{|\alpha|\le s}\|D^\alpha(\phi w)\|_{L^2}^2 \tag{7.6.7}$$

$$\le c(s)\sum_{|\alpha|\le s}\|D^\alpha w\|_{L^2(U)}^2 \tag{7.6.8}$$

$$\le c(s)\Big(\|w\|_{L^2(U)}^2 + \sum_{|\alpha|=s}\|D^\alpha w\|_{L^2(U)}^2\Big) \tag{7.6.9}$$

$$\le c(s)\Big(\|\widetilde\phi w\|_{L^2}^2 + \sum_{|\alpha|=s}\|\widetilde\phi D^\alpha w\|_{L^2}^2\Big). \tag{7.6.10}$$

Hence for $j \ge 2$, using property (7.5.13) and (7.5.14),

$$\|\Lambda_j^s(\phi_j u)\|_{L^2}^2 = 2^{nj}\|\Lambda^s\phi S_{2^j}u\|_{L^2}^2 \tag{7.6.11}$$

$$\le C2^{nj}\Big(\|\widetilde\phi S_{2^j}u\|_{L^2}^2 + \sum_{|\alpha|=s}\|\widetilde\phi D^\alpha S_{2^j}u\|_{L^2}^2\Big) \tag{7.6.12}$$

$$= C\Big(\|\widetilde\phi_j u\|_{L^2}^2 + 2^{2sj}\sum_{|\alpha|=s}\|\widetilde\phi_j D^\alpha u\|_{L^2}^2\Big); \tag{7.6.13}$$

the same estimate holds true for $j = 0, 1$ by an almost identical proof. Since

$$\|u\|_{H^{s,-s}}^2 \sim \sum_{j\ge 0} 2^{-2js}\|\Lambda_j^s(\phi_j u)\|_{L^2}^2,$$

we obtain

$$\|u\|_{H^{s,-s}}^2 \le C\sum_{j\ge 0}2^{-2sj}\|\widetilde\phi_j u\|_{L^2}^2 + C\sum_{\substack{|\alpha|=s\\ j\ge 0}}\|\widetilde\phi_j D^\alpha u\|_{L^2}^2 \tag{7.6.14}$$

$$\le C\|\langle x\rangle^{-s}u\|_{L^2}^2 + C\sum_{|\alpha|=s}\|D^\alpha u\|_{L^2}^2. \tag{7.6.15}$$

Conversely, we have (for $s \geq 0$)

$$\|\Lambda_j^s w\|_{L^2}^2 = 2^{nj}\|\Lambda^s S_{2^j} w\|_{L^2}^2 \tag{7.6.16}$$

$$\sim 2^{nj}\Big(\|S_{2^j} w\|_{L^2}^2 + \sum_{|\alpha|=s} \|D^\alpha S_{2^j} w\|_{L^2}^2\Big) \tag{7.6.17}$$

$$= \|w\|_{L^2}^2 + 2^{2sj} \sum_{|\alpha|=s} \|D^\alpha w\|_{L^2}^2, \tag{7.6.18}$$

hence

$$\|u\|_{H^{s,-s}}^2 \sim \sum_{j\geq 0} 2^{-2js}\|\phi_j u\|_{L^2}^2 + \sum_{\substack{|\alpha|=s \\ j\geq 0}} \|D^\alpha(\phi_j u)\|_{L^2}^2.$$

The first term is equivalent to $\|\langle x\rangle^{-s}u\|_{L^2}^2$, and to handle the second it is sufficient to write

$$\sum_{|\alpha|=s} \Big\|\sum_{j\geq 0} D^\alpha(\phi_j u)\Big\|_{L^2}^2 \leq 2 \sum_{|\alpha|=s}\sum_{j\geq 0} \|D^\alpha(\phi_j u)\|_{L^2}^2$$

since $\phi_j\phi_k \sim 0$ for $|j-k| \geq 2$. This give the second inequality need to prove (7.6.1).

To prove (7.6.2), it is sufficient to show the inequality

$$\|u\|_{H^{s,-s}} \leq C\||\xi|^s\widehat{u}\|_{L^2}, \tag{7.6.19}$$

in view of (7.6.1). Indeed, for nonnegative s we have

$$\|\langle x\rangle^{-s}u\|_{L^2} \leq \||x|^{-s}u\|_{L^2} \leq C\||\xi|^s\widehat{u}\|_{L^2}$$

where the last inequality is true for $s < n/2$, thanks to the extended Hardy inequality (7.7.1). By (7.6.1), we conclude the proof.

2. By (7.5.16) and (7.5.13), we have

$$\||\xi|^s \mathrm{F} \mathrm{S}_\lambda \mathrm{u}\|_{\mathrm{L}^2} = \lambda^{-n}\||\xi|^s \mathrm{S}_{1/\lambda}\widehat{\mathrm{u}}\|_{\mathrm{L}^2} = \lambda^{s-n}\|\mathrm{S}_{1/\lambda}|\xi|^s\widehat{\mathrm{u}}\|_{\mathrm{L}^2} = \lambda^{s-n/2}\||\xi|^s\widehat{\mathrm{u}}\|_{\mathrm{L}^2}.$$

Recalling (7.6.2), we obtain

$$\|S_\lambda u\|_{H^{s,-s}} \sim \lambda^{s-n/2}\|u\|_{H^{s,-s}}.$$

3. Recall that

$$\|u\|_{H^{-s,s}}^2 \sim \sum_{j\geq 0} 2^{2js}\|\Lambda_j^{-s}(\phi_j u)\|_{L^2}^2.$$

For $s \geq 0$ we have $\|\Lambda^{-s}v\|_{L^2} \leq \|v\|_{L^2}$, so that

$$\|u\|_{H^{-s,s}}^2 \leq C\sum_{j\geq 0} 2^{2js}\|\phi_j u\|_{L^2}^2 \sim \sum_{j\geq 0} \|\phi_j\langle x\rangle^s u\|_{L^2}^2 \sim \|\langle x\rangle^s u\|_{L^2}^2.$$

4. For $s \geq 0$ the inequality is a consequence of (7.6.1). Assume now $s < 0$, and define the Hilbert spaces

$$A = L^2(\mathbf{R}^n_{\mathbf{x}}, \langle \mathbf{x} \rangle^s \mathbf{dx}), \tag{7.6.20}$$
$$B_1 = L^2(\mathbf{R}^n_\xi, |\xi|^{-s} \mathbf{d}\xi). \tag{7.6.21}$$

If $s > -n/2$, we have

$$B_1 \subseteq L^1_{loc} \subseteq S',$$

since on any compact set K

$$\int_K |v| d\xi = \int_K |v||\xi|^s |\xi|^{-s} d\xi \leq \|v\|_{B_1} \left(\int_K |\xi|^{2s} d\xi \right)^{1/2}$$

and the last integral is finite for $s > -n/2$. Thus we can define the Hilbert space

$$B = \mathrm{F}^{-1}(\mathrm{B}_1).$$

Formula (7.6.1) with s replaced by $-s$ (since now $-s \geq 0$) can be written

$$\|u\|_{H^{-s,s}} \sim \|u\|_A + \|u\|_B,$$

i.e., we have the isomorphism of Hilbert spaces

$$H^{-s,s} \sim A \cap B.$$

Hence

$$H^{s,-s} = (H^{-s,s})' \sim (A \cap B)' \sim A' + B'$$

and by general properties of Hilbert spaces we can write

$$\|u\|_{A'+B'} \sim \inf_{u=u_1+u_2} (\|u_1\|_A + \|u_2\|_B)$$

where the infimum is taken over all decompositions $u = u_1 + u_2$ with $u_1 \in A'$ and $u_2 \in B'$. This means, for $0 \geq s > -n/2$,

$$\|u\|_{H^{s,-s}} \sim \inf_{u=u_1+u_2} (\|\langle x \rangle^{-s} u_1\|_{L^2} + \||\xi|^s \widehat{u}_2\|_{L^2}). \tag{7.6.22}$$

Now take $u \in H^{s,-s}$; for any decomposition $u = u_1 + u_2$, by the extended Hardy inequality (7.7.1) proved in the Appendix,

$$\||\xi|^s \widehat{u}\|_{L^2} \leq \||\xi|^s \widehat{u}_1\|_{L^2} + \||\xi|^s \widehat{u}_2\|_{L^2} \tag{7.6.23}$$
$$\leq C\||x|^{-s} u_1\|_{L^2} + \||\xi|^s \widehat{u}_2\|_{L^2} \tag{7.6.24}$$
$$\leq C\|\langle x \rangle^{-s} u_1\|_{L^2} + \||\xi|^s \widehat{u}_2\|_{L^2}, \tag{7.6.25}$$

and, by (7.6.22), this implies (7.6.5).

Theorem 1 (Special Hardy inequality) *Let $s \in [0, 1/2[$, $\lambda \geq 0$. Then*

$$\left\| \frac{u}{||x| - \lambda|^s} \right\|_{L^2} \leq C\|u\|_{H^{s,-s}} \tag{7.6.26}$$

with $C = C(s,n)$ independent of $u \in H^{s,-s}$, λ.

Proof. When $\lambda = 0$, (7.6.26) is a consequence of the extended Hardy inequality (Theorem 2) and of property (7.6.5). Thus we shall consider $\lambda > 0$.

Assume first $\lambda = 1$. Let $\psi_0, \psi_1, \ldots, \psi_{2n+1}$ be C^∞ functions on $\mathbb{R}^n$ such that $\sum_{j=0}^{2n+1} \psi_j = 1$, the support of ψ_0 is the closed ball $\overline{B(0,1/2)}$, the support of ψ_{2n+1} is $\mathbb{R}^n \setminus B(0,2)$, and the supports of ψ_j for $1 \leq j \leq 2n$ are compact and contained in one of the open half spaces $\pm x_j > 0$. We can write $u = \sum_{j=0}^{2n+1} u_j$, $u_j = \psi_j u$. We have trivially

$$\left\| \frac{u_0}{||x| - 1|^s} \right\|_{L^2} + \left\| \frac{u_{2n+1}}{||x| - 1|^s} \right\|_{L^2} \leq C(s)\|\langle x\rangle^{-s} u\|_{L^2} \leq C(s)\|u\|_{H^{s,-s}} \tag{7.6.27}$$

by property (7.6.1).

Now, consider the u_j for $j = 1, \ldots, 2n$. We can assume, e.g., that $\operatorname{supp} u_j = K$ is contained in $x_n > 0$. Consider the map $x = \Phi(y)$ defined by

$$x_1 = y_1, \ldots, x_{n-1} = y_{n-1},$$

and

$$x_n = \left[(1 + y_n)^2 - (y_1^2 + \ldots + y_{n-1}^2)\right]^{1/2}.$$

Writing $K' = \Phi^{-1}(K)$, it is clear that Φ is a diffeomorphism of a neighborhood of K' onto a neighborhood of K; notice that Φ maps $K' \cap \{y_n = 0\}$ onto $K \cap S^{n-1}$. We can modify Φ outside K' in such a way that $\Phi = I$ (the identity map of $\mathbb{R}^n$) outside a compact set, Φ is C^∞ and globally invertible on $\mathbb{R}^n$. Hence, writing

$$v = u_j \circ \Phi,$$

we have

$$\left\| \frac{u_j}{||x| - 1|^s} \right\|_{L^2} \leq C(\Phi) \left\| \frac{v}{|y_n|^s} \right\|_{L^2}.$$

Since $s < 1/2$, we can apply the extended Hardy inequality (Theorem 2 in the Appendix) with respect to the variable y_n, Denoting by $\mathrm{F_n}$ the partial Fourier transform with respect to y_n, we have

$$\left\| \frac{v}{|y_n|^s} \right\|_{L^2} \leq C\| |\xi_n|^s \mathrm{F_n v}\|_{L^2} = \mathrm{C}\| |\xi_n|^s \widehat{\mathrm{v}}\|_{\mathrm{L}^2}$$

by Plancherel's identity (with respect to $y_1, \ldots, y_{n-1}$). We thus obtain

$$\left\| \frac{u_j}{||x|-1|^s} \right\|_{L^2} \leq C \| |\xi_n|^s \widehat{v} \|_{L^2} \leq C \| \langle \xi \rangle^s \widehat{v} \|_{L^2} = C \|v\|_{H^s}.$$

Remark now that the linear operator

$$T : H^s(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$$

defined as

$$T(g) = g \circ \Phi$$

is bounded for all real $s \geq 0$. Indeed, for integer s this follows by standard differentiation of composite functions, and for real s by interpolation. This implies

$$\|v\|_{H^s} = \|T(u_j)\|_{H^s} \leq C(s, \Phi) \|u_j\|_{H^s}.$$

Since ψ_j has compact support (independent of u) we have

$$\|u_j\|_{H^s} \leq C(\|u_j\|_{L^2} + \| |\xi|^s \widehat{u}_j \|_{L^2}) \leq C(\| \langle x \rangle^{-s} u_j \|_{L^2} + \| |\xi|^s \widehat{u}_j \|_{L^2})$$

and by property (7.6.1) we obtain

$$\|u_j\|_{H^s} \leq C \|u_j\|_{H^{s,-s}}.$$

Summing up, we have proved that

$$\left\| \frac{u_j}{||x|-1|^s} \right\|_{L^2} \leq C \|u_j\|_{H^{s,-s}},$$

and using the fact that multiplication by ψ_j is a bounded operator on $H^{s,\delta}$ (Lemma 7.5.2), we obtain

$$\left\| \frac{u_j}{||x|-1|^s} \right\|_{L^2} \leq C \|u\|_{H^{s,-s}}.$$

Together with (7.6.27), this proves the thesis for $\lambda = 1$.

For general $\lambda > 0$, we can write

$$\left\| \frac{u}{||x|-\lambda|^s} \right\|_{L^2} = \left\| S_{1/\lambda} S_\lambda \frac{u}{||x|-\lambda|^s} \right\|_{L^2} = \left\| S_{1/\lambda} \frac{S_\lambda u}{\lambda^s ||x|-1|^s} \right\|_{L^2}$$

and by property (7.5.13) we obtain

$$\left\| \frac{u}{||x|-\lambda|^s} \right\|_{L^2} = \lambda^{n/2-s} \left\| \frac{S_\lambda u}{||x|-1|^s} \right\|_{L^2} \leq C \lambda^{n/2-s} \|S_\lambda u\|_{H^{s,-s}}$$

using the thesis for $\lambda = 1$ already proved. Recalling property (7.6.3), we conclude the proof.

7.7 Appendix

In this appendix we shall establish the following.

Theorem 2 (Extended Hardy inequality) *For any real $a \in [0, n/2[$ and any $f \in C_c^\infty(\mathbb{R}^n)$, we have*

$$\left\| \frac{\widehat{f}(\xi)}{|\xi|^a} \right\|_{L^2} \le C \| |x|^a f \|_{L^2}, \tag{7.7.1}$$

with $C = C(n, a)$ independent of f.

Proof. Inequality (7.7.1) is a special case of a result of Muckenhoupt (see Theorem 1 in [37]). For sake of completeness, we give here the proof.

We must prove that

$$\int \left| \int u(x) e^{-ix\cdot\xi} dx \right|^2 |\xi|^{-2a} d\xi \le C \int |u(x)|^2 |x|^{2a} dx.$$

Split the first integral as $I + II$, with

$$I = \sum_{j \in \mathbf{Z}} \int_{2^j < |\xi|^{-a} \le 2^{j+1}} \left| \int_{|x|^a > 2^j} u(x) e^{-ix\cdot\xi} dx \right|^2 |\xi|^{-2a} d\xi$$

and

$$II = \sum_{j \in \mathbf{Z}} \int_{2^j < |\xi|^{-a} \le 2^{j+1}} \left| \int_{|x|^a < 2^j} u(x) e^{-ix\cdot\xi} dx \right|^2 |\xi|^{-2a} d\xi.$$

We can write

$$\begin{aligned} I &\le \sum_{j \in \mathbf{Z}} \int_{2^j < |\xi|^{-a} \le 2^{j+1}} \left| \int_{|x|^a > 2^j} u(x) e^{-ix\cdot\xi} dx \right|^2 2^{2j+2} d\xi \\ &= \sum_{j \in \mathbf{Z}} 2^{2j+2} \|\mathrm{F}(\mathrm{u} \cdot \chi_{\{|\mathrm{x}|^a > 2^j\}})\|_{\mathrm{L}^2}^2, \end{aligned} \tag{7.7.2}$$

where F is the Fourier transform and χ_A is the characteristic function of the set A. Thus, by Plancherel's theorem,

$$I \le \sum_{j \in \mathbf{Z}} 2^{2j+2} \|u \cdot \chi_{\{|x|^a > 2^j\}}\|_{L^2}^2 \le \|u \cdot h\|_{L^2}^2$$

where the function $h(x)$ is defined by

$$h(x) = \sum_{j \in \mathbf{Z}} \chi_{\{|x|^a > 2^j\}} 2^{j+1}$$

and hence satisfies

$$h(x) \leq 2|x|^a.$$

This concludes the estimate for I.

To estimate II, we begin by writing

$$II \leq \int \left(\int_{|x| \leq 1/|\xi|} |u(x)| dx \right)^2 |\xi|^{-2a} d\xi.$$

Now, consider the lowest integer $J \in \mathbf{Z}$ such that $2^J \geq \|u\|_{L^1}$; then define $r_J = \infty$ and, for $j < J$, choose any nondecreasing sequence of positive numbers r_j such that

$$\int_{|x|^a \leq r_j} |u| dx = 2^j;$$

finally, define the sets for $-\infty < j \leq J$

$$A_j = \{x : r_{j-2} < |x|^a \leq r_{j-1}\}, \quad B_j = \{x : r_{j-1} < |\xi|^{-a} \leq r_j\}.$$

We notice the following property:

$$\int_{|x|^a \leq r_j} |u| dx \leq 2^j = 4 \int_{|x|^a \leq r_{j-2}} |u| dx = 4 \int_{A_j} |u| dx;$$

hence, for $\xi \in B_j$,

$$\left(\int_{|x| \leq 1/|\xi|} |u(x)| dx \right)^2 \leq \left(\int_{|x|^a \leq r_j} |u| dx \right)^2 \leq \tag{7.7.3}$$

$$\leq \left(4 \int_{A_j} |u| dx \right)^2 \leq 16 \int_{A_j} |u|^2 |x|^{2a} dx \cdot \int_{A_j} |x|^{-2a} dx \tag{7.7.4}$$

$$\leq c_n r_{j-1}^{-2+n/a} \int_{A_j} |u|^2 |x|^{2a} dx \tag{7.7.5}$$

by Cauchy-Schwarz inequality and the explicit computation (valid for $a < n/2$)

$$\int_{A_j} |x|^{-2a} dx \leq c_n r_{j-1}^{-2+n/a}.$$

Thus, writing

$$II = \sum_{j=-\infty}^{J} \int_{B_j} \left(\int_{|x| \leq 1/|\xi|} |u(x)| dx \right)^2 |\xi|^{-2a} d\xi$$

we have easily

$$II \leq c_n \sum_{j=-\infty}^{J} r_{j-1}^{-2+n/a} \int_{A_j} |u|^2 |x|^{2a} dx \int_{B_j} |\xi|^{-2a} d\xi$$

whence

$$II \leq c_n \sum_{j=-\infty}^{J} \int_{A_j} |u|^2 |x|^{2a} dx$$

by the explicit computation

$$\int_{B_j} |\xi|^{-2a} d\xi \leq c_n r_{j-1}^{-n/a+2}.$$

Since the A_j are disjoint sets, the proof is concluded.