4 Main hyperbolic equations and energy type estimates

4.1 Linear wave and Klein-Gordon equations

Our first step in this chapter is to formulate some of the most important hyperbolic equations in mathematical physics.

In these lectures we shall focus our attention mainly to the wave and Klein-Gordon equations as the basic examples of hyperbolic equations in mathematical physics.

The wave equation is an important problem in continuum mechanics. A derivation of this equation in the model of vibrating string can be found in [59] Chapter 2.

The same equation plays a crucial role in relativistic quantum mechanics, since it is connected with a model of a massless relativistic field $u = u(t, x)$, where t is the time variable and

$$x = (x_1, ..., x_n) \in \mathbf{R}^n$$

are the space variables. The wave equation satisfied by the field u has the form

$$(-\partial_t^2 + \Delta)u = F, \tag{4.1.1}$$

where

$$\Delta = \partial_{x_1}^2 + ... + \partial_{x_n}^2$$

is the Laplace operator and $F = F(t, x)$ is a given known function. Usually, the operator

$$\Box = -\partial_t^2 + \Delta$$

is called D'Alembert operator.

For the case, when a scalar relativistic field has a mass, the corresponding equation is called Klein-Gordon equation and this equation has the form

$$(-\partial_t^2 + \Delta - M^2)u = F, \tag{4.1.2}$$

where $M > 0$ is the mass of the field.

In general we can consider the wave equation as a partial case of Klein-Gordon equation with mass zero.

The first important physical law for these equation is the conservation of energy, when the external force F is identically zero.

Indeed, let us assume the solution is smooth and for any fixed t has a compact support. Then multiplying (4.1.2) by $\partial_t u$ we see that the energy

$$E(t) = \frac{1}{2} \int |\partial_t u(t, x)|^2 +$$
$$+ |\nabla_x u(t, x)|^2 + M|u(t, x)|^2 dx \tag{4.1.3}$$

is a constant independent of the time variable t.

We shall rewrite the Klein-Gordon equation (and therefore the wave equation) as abstract evolution equation of the form

$$\partial_t v = Av, \tag{4.1.4}$$

where A is a skew-selfadjoint operator in a suitable Hilbert space H. We refer to [31] for complete information about this reduction.

For simplicity we shall consider here only the case of positive mass M.

It is clear that we can define the operator $M^2 - \Delta$ on the space of smooth compactly supported functions in $\mathbf{R}^n$. Then this operator is a symmetric with respect to the scalar product

$$(f, g)_{L^2} = \int_{\mathbf{R}^n} f(x)\overline{g(x)}dx.$$

Moreover, this is a strictly monotone operator in the Hilbert space $L^2(\mathbf{R})$.

To show that the closure of this operator is a self-adjoint one, we use Theorem 2.3.3, so it is sufficient to show that any weak solution $u \in H^1$ of

$$(M^2 - \Delta)u = f \in L^2$$

is also strong, i.e. there exists a sequence u_k of smooth compactly supported functions, so that

$$u_k \to u \text{ in } H^1,$$

$$(M^2 - \Delta)u_k \to f \text{ in } L^2.$$

To construct such an approximation sequence, let $\varphi(s)$ be a smooth compactly supported function on $\mathbf{R}$ with $\varphi(s) = 1$ for $|s| \le 1$ and $\varphi(s) = 0$ for $|s| \ge 2$. The key point is to construct an approximation sequence v_k in $H^1 \cap E'$, where E' denotes the space of distributions with compact support. Approximation sequence in $H^1 \cap E'$ means that there exists a sequence $v_k \in H^1 \cap E'$ so that

$$v_k \to u \text{ in } H^1,$$

$$(M^2 - \Delta)v_k \to f \text{ in } L^2.$$

Once this sequence is constructed, we can use Friedrich's molifiers j_ε (see (3.3.18)) and define

$$u_{\varepsilon,k} = j_\varepsilon * v_k.$$

Since v_k is compactly supported, we see that $u_{\varepsilon,k}$ is a smooth compactly supported function. Then using the properties

$$u_{\varepsilon,k} \to v_k \text{ in } H^1,$$

$$(M^2 - \Delta)u_{\varepsilon,k} \to (M^2 - \Delta)v_k \text{ in } L^2,$$

we see that $u \in H^1$ is a strong solution.

Therefore it remains to construct v_k. Set

$$v_k(x) = \varphi(k^{-1}|x|)u(x).$$

Then v_k satisfies the equation

$$(M^2 - \Delta)v_k = f + r_k(x),$$

where

$$r_k(x) = (1 - \varphi(k^{-1}|x|))f(x) + [\Delta, \varphi(k^{-1}|.|)]u.$$

It is clear that r_k tends to 0 in L^2 and this shows we have an approximation sequence in $H^1 \cap E'$. As we have seen this is sufficient to concluded that the closure of the operator $(M^2 - \Delta)$ with dense domain C_0^∞ is a self-adjoint operator.

Setting

$$v = \begin{pmatrix} u \\ \partial_t u \end{pmatrix} \ , \ A = \begin{pmatrix} 0 & 1 \\ \Delta - M^2 & 0 \end{pmatrix} \ ,$$

we see that the nonlinear Klein-Gordon equation (4.1.2) takes the form (4.1.4). The form of the energy in (4.1.3) suggests us to consider the Hilbert space $H = H^1(\mathbf{R}^n) \times L^2(\mathbf{R}^n)$. For any couple $v = (v_1, v_2) \in H$ the corresponding norm is defined by

$$\|v\|_H^2 = \int |\nabla v_1|^2 + M^2|v_1|^2 + |v_2|^2 dx \tag{4.1.5}$$

Denote by $(,)_H$ is the corresponding Hilbert norm.

A dense domain for the operator A can be choosen as

$$D(A) = C_0^\infty(\mathbf{R}^n) \times C_0^\infty(\mathbf{R}^n). \tag{4.1.6}$$

Then we see that A can be extended to a skew-selfadjoint operator with a dense domain

$$D(A) = H^2(\mathbf{R}^n) \times H^1(\mathbf{R}^n).$$

Applying the Stone theorem we see that A is a generator of an unitary group $U(t)$ acting in the Hilbert space H.

The fact that A is a generator of the group $U(t)$ means that

$$\lim_{t \to 0} \frac{U(t)f - f}{t} = A(f)$$

for $f \in D(A)$. The fact that A is skew-selfadjoint assures that $U(t)$ is a unitary operator

$$\|U(t)f\|_H = \|f\|_H \tag{4.1.7}$$

The abstract Cauchy problem associated with the generator A can be written in the form

$$\partial_t v = Av,$$
$$v(0) = f.$$

The unique solution of this linear Cauchy problem can be represented as $v = U(t)f$.

Turning back to our original formulation of the Klein-Gordon equation we can state the corresponding Cauchy problem as follows

$$(-\partial_t^2 + \Delta - M^2)u = 0,$$
$$u(0,x) = f_0(x)\ ,\ \partial_t u(0,x) = f_1(x).$$

Here $f = (f_0, f_1) \in H$.

For the case of nontrivial external force F we have the Cauchy problem

$$(-\partial_t^2 + \Delta - M^2)u = F,$$
$$u(0,x) = f_0(x)\ ,\ \partial_t u(0,x) = f_1(x).$$

The energy conservation law for the linear wave equation is represented in (4.1.7) so the norm in the Hilbert space H has an interpretation as energy.

4.2 Self - adjoint generators

The operator $M^2 - \Delta$, considered in the previous section, is a typical example of a symmetric strictly monotone operator with dense domain $C_0^\infty(\mathbf{R}^n)$. We have seen that its closure is a self-adjoint operator in the Hilbert space $H = L^2(\mathbf{R}^n)$.

Another important case is connected with the case when $\mathbf{R}^n$ is equipped with a global Riemannian metric

$$ds^2 = \sum_{j,k=1}^{n} g_{jk}(x)dx^j dx^k,$$

where g_{jk} are smooth functions defined in $\mathbf{R}^n$ such that the matrix $G(x) = (g_{jk}(x))_{j,k=1}^n$ is a symmetric uniformly positive matrix, i.e. there exists a positive constant $\mu > 0$, such that

$$G(x) \geq \mu I$$

in the sense of symmetric matrices. Set

$$g(x) = detG(x).$$

Let $(g^{jk}(x))_{jk=1}^n = G^{-1}(x)$ be the inverse matrix. The Laplace-Beltrami operator associated with the above metric is

$$\Delta_g = \frac{1}{\sqrt{g}} \sum_{j,k=1}^{n} \partial_{x_j} \sqrt{g} g^{jk} \partial_{x_k}.$$

This is a symmetric operator in the Hilbert space $L^2(\sqrt{g}dx)$ with inner product

$$(f,h)_{L^2(\sqrt{g}dx)} = \int_{\mathbf{R}^n} f(x)\overline{h(x)}\sqrt{g(x)}dx.$$

A more general situation is connected with the operator

$$P(x,\partial_x) = \frac{1}{b(x)} \sum_{j,k=1}^{n} \partial_{x_j} a^{jk}(x)\partial_{x_k}, \tag{4.2.1}$$

where $b(x) > 0$ and $a^{jk}(x)$ are smooth functions defined in $\mathbf{R}^n$ such that the matrix $A(x) = (a^{jk}(x))_{j,k=1}^n$ is a symmetric uniformly positive matrix, i.e. there exists a positive constant $\mu > 0$, such that

$$A(x) \geq \mu I \tag{4.2.2}$$

in the sense of symmetric matrices.

The corresponding Hilbert space is $L^2(b(x)dx)$ with inner product

$$(f,h)_{L^2(bdx)} = \int_{\mathbf{R}^n} f(x)\overline{h(x)}b(x)dx.$$

It is clear that the operator P in (4.2.1) can be defined on $C_0^\infty(\mathbf{R}^n)$.

Lemma 4.2.1 *The operator $1-P$ with dense domain $C_0^\infty(\mathbf{R}^n)$ is a symmetric strictly monotone operator in $L^2(bdx)$.*

Proof. Given $f, g \in C_0^\infty(\mathbf{R}^n)$ we have

$$((1-P)f,g)_{L^2(bdx)} = (f,g)_{L^2(bdx)} + \int (A(x)\nabla f(x), \nabla g(x))dx, \tag{4.2.3}$$

where $(\cdot,\cdot)$ denotes the inner product in $\mathbf{C}^n$. In the same way we get

$$(f,(1-P)g)_{L^2(bdx)} = (f,g)_{L^2(bdx)} + \int (A(x)\nabla f(x), \nabla g(x))dx. \tag{4.2.4}$$

Hence, $1-P$ is a symmetric operator. From (4.2.3) it follows

$$((1-P)f,f)_{L^2(bdx)} \geq \|f\|^2_{L^2(bdx)}.$$

This completes the proof of the Lemma.

Now we can define the norm in the energetic space

$$H_E = \{f; ((1-P)f,f)_{L^2(bdx)} < \infty\}.$$

More precisely, the energetic space is the closure of $C_0^\infty(\mathbf{R}^n)$ with respect the norm

$$\|f\|^2_{H_E} = ((1-P)f,f)_{L^2(bdx)}.$$

Note that the inner product in H_E is

$$(4.2.5) \qquad (f,g)_{H_E} = (f,g)_{L^2(bdx)} + \int (A(x)\nabla f(x), \nabla g(x))dx.$$

due to (4.2.3).

Our goal is to show that any weak solution $u \in H_E$ of the equation

$$(1-P)u = f$$

for $f \in L^2(bdx)$ is also a strong one, i.e. there exists a sequence $v_m, m = 1,2,...,$ so that

a) $v_m \in C_0^\infty(\mathbf{R}^n)$,
b) $v_m \to u$ in H_E,
c) $(1-P)v_m \to f \in L^2(bdx)$ in $L^2(bdx)$.

Having in mind the treatment of the case of usual Laplacian, we shall make an intermediate step, namely we shall construct an approximation sequence $u_k, k = 1,2,...$ in $H_E \cap E'(\mathbf{R}^n)$, where $E'(\mathbf{R}^n)$ denotes the space of distributions with compact support. This means the sequence satisfies the properties

$$(4.2.6) \qquad u_k \in H_E \cap E'(\mathbf{R}^n),$$

$$(4.2.7) \qquad u_k \to u \ \text{ in } \ H_E,$$

$$(4.2.8) \qquad (1-P)u_k \to f \in L^2(bdx) \ \text{ in } \ L^2(bdx).$$

Once the approximation sequence of this type is constructed, one can use Friedrich's molifiers j_ε (see (3.3.18)) to approximate any u_k by $j_\varepsilon * u_k$. Fixing k and taking $\varepsilon \to 0$, we have $j_\varepsilon * u_k$ is a smooth function. We can see that

$$(4.2.9) \qquad j_\varepsilon * u_k \to u_k$$

in H_E as $\varepsilon \to 0$. Indeed, for $x \in K$, where $K \subset \mathbf{R}^n$ is a compact, there exists a positive constant $C = C(K)$ such that

$$C^{-1} \le b(x) \le C \ , \ C^{-1}I \le A(x) \le CI.$$

This means that the norms $\|v\|_{H_E}$ and $\|v\|_{H^1(\mathbf{R}^n)}$ are equivalent when $v \in H_E$ has a compact support in K. Since $j_\varepsilon * u_k \to u_k$ in the Sobolev space H^1, we conclude that (4.2.9) is fulfilled.

Note that for a fixed compact K on the linear space $E'(K) \cap L^2(K)$ the norms $L^2(dx)$ and $L^2(bdx)$ are equivalent. Also it is easy to see that

$$(4.2.10) \qquad b(1-P)(j_\varepsilon * u_k) - j_\varepsilon * ((b - bP)u_k) \to 0$$

in L^2. In fact, denoting by $[A,B] = AB - BA$ the commutator between the operators A, B, we see that the term in (4.2.10) is

$$[j_\varepsilon *, bP]u_k + [b, j_\varepsilon *]u_k$$

Since $[j_\varepsilon *, \partial_{x_k}] = 0$, we have

$$[j_\varepsilon *, bP] = \sum_{m,k=1}^{n} \partial_{x_m}[j_\varepsilon *, a^{mk}]\partial_{x_k}.$$

On the other hand, for $x \in K$ with K being a compact set, we have

$$|[j_\varepsilon *, a]f(x)| \leq C\varepsilon |j_\varepsilon * |f|(x)|$$

provided the derivatives of $a(x)$ are bounded.

Since u_k have compact support, we get

$$\|[j_\varepsilon *, b(1-P)]u_k\|_{L^2} \leq C\varepsilon \|u_k\|_{H^2}.$$

Applying the Gärding inequality, we have

$$\|(1-P)u_k\|_{L^2} \geq C(\|u_k\|_{H^2} - \|u_k\|_{H_1}).$$

Since u_k have compact support, the norms $\|u_k\|_{H^1}$ and $\|u_k\|_{H_E}$ are equivalent so we arrive at

$$\|[j_\varepsilon *, (b - bP)]u_k\|_{L^2} \leq C\varepsilon (\|(1-P)u_k\|_{L^2} + \|u_k\|_{H_E}).$$

Taking $\varepsilon \to 0$, we get (4.2.10).

Therefore, it remains only to localize the problem, i.e. to show the existence of a sequence $u_k, k = 1, 2, ...,$ satisfying the properties (4.2.6)-(4.2.8).

To do this we shall make an additional assumption about the asymptotic behavior of $b(x)$ and the matrix $A(x)$ at infinity. Namely , we shall assume that there exists a smooth positive function $\chi(x)$ and a constant $C > 0$ so that

$$\lim_{x \to \infty} \chi(x) = \infty, \tag{4.2.11}$$

$$|(A(x)\nabla\chi(x), \nabla\chi(x))| \leq Cb(x)\chi^2(x), \tag{4.2.12}$$

$$|P(\chi)(x)| \leq C\chi(x) \tag{4.2.13}$$

for any $x \in \mathbf{R}^n$.

At the end of this section we shall see various examples, when the above assumptions can be checked easily.

To construct the sequence u_k take a non-negative smooth compactly supported function $\varphi(s)$ defined on $\mathbf{R}$ and such that $\varphi(s) = 1$ for $|s| \leq 1$ and $\varphi(s) = 0$ for $|s| \geq 2$. Set

$$u_k = \varphi(k^{-1}\chi(x))u(x).$$

Applying the Lebesgue convergence theorem and using the fact that $u \in L^2(bdx)$ we see that u_k tends to u in $L^2(bdx)$. To show that u_k tends to u in H_E, it is sufficient to show that the function

$$(A(x)\nabla(u_k(x) - u(x)), \nabla(u_k(x) - u(x)))$$

tends to 0 in $L^1(\mathbf{R}^n)$. The above term can be represented as (for the case of real-valued function u)

$$\begin{gathered}(1-\varphi(k^{-1}\chi(x)))^2(A\nabla u,\nabla u)-\\ 2k^{-1}(A\nabla\chi,\nabla u)u\varphi'(k^{-1}\chi(x))(1-\varphi(k^{-1}\chi(x)))\\ +k^{-2}(A\nabla\chi,\nabla\chi)u^2(\varphi'(k^{-1}\chi(x)))^2.\end{gathered}$$

Note that on the support of $\varphi'(k^{-1}\chi(x))$ the parameter k and the function $\chi(x)$ are equivalent. Moreover, we have the Cauchy inequality

$$(A\nabla\chi,\nabla u)^2\leq(A\nabla\chi,\nabla\chi)(A\nabla u,\nabla u). \tag{4.2.14}$$

Therefore, using the assumption (4.2.12) and Lebesgue convergence theorem we see that u_k tends to $u\in H_E$ in H_E. Finally, it remains to show that $(1-P)u_k$ tends to f in $L^2(bdx)$. For the purpose it is sufficient to show that $[P,\varphi_k(x)]u$ tends to 0 in $L^2(bdx)$. Here

$$\varphi_k(x)=\varphi(k^{-1}\chi(x)).$$

A direct computation shows that the above commutator is

$$[P,\varphi_k]u=uP(\varphi_k)+2\frac{1}{b}(A\nabla\varphi_k,\nabla u).$$

Our assumptions (4.2.12) and (4.2.13) show that

$$|P(\varphi_k)(x)|\leq C$$

so the Lebesgue convergence theorem implies that $uP(\varphi_k)$ tends to 0 in $L^2(bdx)$.

For the term $2\frac{1}{b}(A\nabla\varphi_k,\nabla u)$ we use the Cauchy inequality (4.2.14) and see that this term also tends to 0 in $L^2(bdx)$. Thus we have established the following.

Theorem 4.2.1 *Suppose the elliptic operator P from (4.2.1) satisfies the assumptions (4.2.2), (4.2.12) and (4.2.13). Then the operator $1-P$ with dense domain $C_0^\infty(\mathbf{R}^n)$ has a self - adjoint closure. (i.e. $1-P$ is essentially self-adjoint operator)*

Finally, we consider few examples.

Example 1. The operator

$$\sum_{j=1}^n \partial_{x_j} < x >^a \partial_{x_j} \quad ,a\leq 2,$$

in the Hilbert space $L^2(\mathbf{R}^n)$. For this case we take

$$A=< x >^a,\quad b=1\quad,\ \chi(x)=< x >.$$

Then the assumptions of Theorem 4.2.1 are fulfilled.

Example 2. The operator

$$\sum_{j=1}^{n} < x > \partial_{x_j} < x >^a \partial_{x_j} \quad ,a \leq 1,$$

in the Hilbert space $L^2(< x >^{-1}\, dx)$. For this case we take

$$A =< x >^a, \quad b =< x >^{-1} \quad , \quad \chi(x) =< x > .$$

Again the assumptions of Theorem 4.2.1 are fulfilled.

Example 3. The operator

$$\Delta + \Delta_{\mathbf{S}^{n-1}},$$

where $\Delta_{\mathbf{S}^{n-1}}$ is the standard Laplace-Beltrami operator on the unit sphere in $\mathbf{R}^n$. Since

$$\Delta_{\mathbf{S}^{n-1}} = \frac{1}{2} \sum_{j,k} Y_{jk}^2,$$

where $Y_{j,k} = x_j \partial_{x_k} - x_k \partial_{x_j}$ are the generators of rotations in the space, we obtain

$$\Delta_{\mathbf{S}^{n-1}} = \sum_{k=1}^{n} \partial_{x_k} |x|^2 \partial_{x_k} - \sum_{k,j} \partial_{x_k} x_j x_k \partial_{x_j}.$$

For this case we take

$$A(x) =< x >^2 I - x \otimes x \ , \quad b = 1 \ , \ \chi(x) =< x >$$

and see again that the assumptions of Theorem 4.2.1 are fulfilled. In this way we conclude that the operator

$$1 - \Delta - \Delta_{\mathbf{S}^{n-1}},$$

with dense domain $C_0^\infty(\mathbf{R}^n)$ has a self-adjoint closure.

4.3 Energy estimate for Klein - Gordon equation

In this section we shall derive some simple L^2- estimate of the solution of Klein – Gordon equation

(4.3.1) $$\Box u - u = -F.$$

We shall assume that the supports of u, F are contained in $\{|x| \leq t - 1\}$, the initial data

(4.3.2) $$u_{|t=t_0} = g_0, \quad \partial_t u_{|t=t_0} = g_1$$

with $t_0 > 1$ are such that their supports are contained in $\{|x| \leq t_0 - 1\}$, and they have small energy norm

$$\|g_0\|_{H^1(\mathbb{R}^n)} + \|g_1\|_{L^2(\mathbb{R}^n)} \leq \varepsilon.$$

The classical energy estimate for $t \geq t_0$ gives

$$\|\nabla_{t,x}u(t,\cdot)\|_{L^2(\mathbb{R}^n)} + \|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \leq C\varepsilon + C\int_{t_0}^{t} \|F(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}d\tau. \tag{4.3.3}$$

For any fixed $\rho > t_0$ we consider the hyperboloid

$$X_\rho = \{t^2 - |x|^2 = \rho^2, t > 0\}$$

and our goal is to control the energy over X_ρ. Multiplying (4.3.1) by $-\partial_t u$, we obtain the identity

$$\sum_{\mu=0}^{n} \partial_\mu P^\mu = 2F\partial_t u, \tag{4.3.4}$$

where

$$\partial_0 = \partial_t, \quad \partial_j = \partial_{x_j}, \quad j = 1,\dots,n; \tag{4.3.5}$$

$$P^0 = |\nabla_{t,x}u|^2 + |u|^2, \quad P^j = -2\partial_t u \partial_j u, \quad j = 1,\dots,n, \tag{4.3.6}$$

P^0, P^j are the components of the energy-momentum tensor. We start integrating (4.3.4) into domain

$$D_\rho = \{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n; t > t_0, t^2 - |x|^2 < \rho^2\}.$$

Having in mind that

$$D_\rho \cap \{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n; |x| \leq t - 1\}$$

is bounded ($t + |x| \leq \rho^2$), we see from the assumption about the supports of u, F, g_0, g_1 that

$$\int_{\partial D_\rho} \sum_{\mu=0}^{n} \nu_\mu P^\mu(t,x) d\Sigma_{t,x} = \int_{D_\rho} 2F\partial_t u(t,x)dxdt, \tag{4.3.7}$$

where ∂D_ρ is the boundary of the domain D_ρ, ν is the outward normal and $\Sigma_{t,x}$ is the surface element with respect to the Riemannian metric in $\mathbb{R} \times \mathbb{R}^n$. The boundary ∂D_ρ for $\rho > t_0$ consists of the hyperboloid X_ρ and the plane $t = t_0$. For $(t,x) \in X_\rho$ we have

$$\nu = \nu(t,x) = (\rho^2 + 2|x|^2)^{-1/2}(\sqrt{|x|^2 + \rho^2}, -x) \tag{4.3.8}$$

while

$$d\Sigma_{t,x} = \frac{(\rho^2 + 2|x|^2)^{1/2}}{\sqrt{|x|^2 + \rho^2}} dx.$$

Lemma 4.3.1 *For $\rho > t_0$ and $(t,x) \in X_\rho$ we have*

$$|\nabla_{t,x}u(t,x)|^2 + 2\frac{r\partial_t u \partial_r u}{(\rho^2+r^2)^{1/2}} \geq C\frac{\rho^2}{\rho^2+r^2}|\nabla_{t,x}u|^2.$$

Here $r = |x|$.

Proof. Indeed, using

$$|\partial_r u| \leq |\nabla_x u|$$

and the inequality

$$|\nabla_{t,x}u|^2 \geq 2|\partial_t u||\nabla_x u|,$$

we get

$$|\nabla_{t,x}u(t,x)|^2 + 2\frac{r\partial_t u \partial_r u}{(\rho^2+r^2)^{1/2}} \geq \left(1 - \frac{r}{\sqrt{\rho^2+r^2}}\right)|\nabla_{t,x}u|^2.$$

On the other hand, the identity

$$1 - \frac{r}{\sqrt{\rho^2+r^2}} = \frac{\rho^2}{\sqrt{\rho^2+r^2}(r+\sqrt{\rho^2+r^2})}$$

shows that this weight is equivalent to

$$\frac{\rho^2}{\rho^2+r^2}.$$

This completes the proof of the Lemma. □

By the aid of this Lemma and of identity (4.3.7) we find

$$\int_{\mathbb{R}^n} |u(\sqrt{\rho^2+|x|^2},x)|^2 dx + \int_{\mathbb{R}^n} \frac{\rho^2}{\rho^2+|x|^2}|\nabla_{t,x}u(\sqrt{\rho^2+|x|^2},x)|^2 dx \leq$$

$$\leq C(\|g_0\|_{H^1} + \|g_1\|_{L^2}) + 2\int_{D_\rho} |\partial_t u(t,x)||F(t,x)|dxdt. \tag{4.3.9}$$

On the other hand for any $\rho_0 \geq t_0$ and for any L^1−non-negative function $H(t,x)$, having support in $\{|x| \leq t-1\}$, we have

$$\int_{D_\rho} H(t,x)dxdt \leq \int_{\rho_0}^{\rho}\int_{\mathbb{R}^n} H(\sqrt{\sigma^2+|x|^2},x)\frac{\sigma dx}{\sqrt{\sigma^2+|x|^2}}d\sigma +$$

$$+ \int_{t_0}^{t_1}\int_{\mathbb{R}^n} H(\tau,x)dxd\tau,$$

where t_1 is determined so that the plain $t = t_1$ contains the intersection $\{|x| = t-1\} \cap X_{\rho_0}$; this gives

$$t_1 = \frac{1+\rho_0^2}{2}.$$

Combining this relation with (4.3.3), (4.3.9) and Gronwall inequality, we get (for $\rho \geq \rho_0$)

$$\int_{\mathbf{R}^n} |u(\sqrt{\rho^2+|x|^2},x)|^2 dx + \int_{\mathbf{R}^n} \frac{\rho^2}{\rho^2+|x|^2} |\nabla_{t,x} u(\sqrt{\rho^2+|x|^2},x)|^2 dx \leq$$

$$\leq C(\|g_0\|^2_{H^1} + \|g_1\|^2_{L^2}) + C\left(\int_{\rho_0}^{\rho}\left(\int_{\mathbf{R}^n} |F(\sqrt{\sigma^2+|x|^2},x)|^2 dx\right)^{1/2} d\sigma\right)^2 +$$

$$+C\left(\int_{t_0}^{t_1}\left(\int_{\mathbf{R}^n} |F(\tau,x)|^2 dx\right)^{1/2} d\tau\right)^2. \tag{4.3.10}$$

4.4 Some other hyperbolic problems of mathematical physics

Another important hyperbolic problem is the Dirac system

$$i\gamma_\mu \partial_\mu \psi = 0. \tag{4.4.1}$$

Here $\psi(t,x)$ is a function defined in the Minkowski space $\mathbf{R}^{1+3}$ with values in $\mathbf{C}^4$. Usually, ψ is called a spinor. Moreover, γ_μ are the Dirac matrices defined as follows

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, k = 1,2,3.$$

The Pauli matrices σ_k are determined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

The initial data are determined by

$$\psi(0,x) = f(x)$$

The Dirac matrices satisfy the relations

$$\gamma^\mu \gamma^\ni + \gamma^\ni \gamma^\mu = -2\eta^{\mu\ni} \tag{4.4.2}$$

A simple reduction of the Dirac equation to the wave equation can be done by applying the operator $i\gamma_\mu\partial_\mu$ to the Dirac equation in (4.4.1). We use the relations (4.4.2) and find

$$\partial_\mu \partial^\mu \psi = 0.$$

From (4.4.1) we have

$$(\partial_t + \alpha^j \partial_j)\psi = 0. \tag{4.4.3}$$

Here $\alpha_j = \gamma_0\gamma_j$ are selfadjoint matrices. Making the Fourier transform in x we get

$$(\partial_t + i\alpha(\xi))\hat{\psi} = 0$$
$$\hat{\psi}(0,\xi) = \hat{f}(\xi).$$

Here $\alpha(\xi) = \sum \alpha_j\xi_j$ is a selfadjoint matrix. Then the solution of the Cauchy problem for the linear Dirac equation has the form

$$\psi(t,x) = \sum_{\pm}(2\pi)^{-3}\int e^{ix\xi \pm |\xi| t}\pi_{\pm}(\xi)\hat{f}(\xi)d\xi, \tag{4.4.4}$$

where π_+ (respectively π_-) is the positive (respectively negative) eigenspace of the matrix $\alpha(\xi)$.

The Maxwell equations in vacuum have the form

$$\begin{aligned} \partial_t E &= \mathrm{rot} H, \\ \partial_t H &= -\mathrm{rot} E, \\ \mathrm{div} E &= \mathrm{div} H = 0, \end{aligned} \tag{4.4.5}$$

where E (resp. H) is the electric (resp. magnetic) field. Recall that $E(t,x), H(t,x)$ are vector-valued functions from Minkowski space in $\mathbf{R}^3$.

To pose correctly the Cauchy problem for the Maxwell equations we take the initial conditions

$$E(0,x) = e(x) \quad , \quad H(0,x) = h(x). \tag{4.4.6}$$

Then the equations $\mathrm{div}E = \mathrm{div}H = 0$ in (4.4.5) show that the initial data have to satisfy the constraint conditions

$$\mathrm{div} e = \mathrm{div} h = 0. \tag{4.4.7}$$

Taking the evolution part

$$\begin{aligned} \partial_t E &= \mathrm{rot} H, \\ \partial_t H &= -\mathrm{rot} E, \end{aligned} \tag{4.4.8}$$

of the Maxwell equations, we see that we can solve the Cauchy problem for (4.4.8) with initial data (4.4.6) satisfying the constraint conditions (4.4.7). Then taking the div operator in the equations (4.4.8), we see that

$$\partial_t \mathrm{div} E = \partial_t \mathrm{div} H = 0 \tag{4.4.9}$$

so the constraint conditions (4.4.7) assure the elliptic part $\mathrm{div}E = \mathrm{div}H = 0$ in Maxwell equations (4.4.5). Setting $\psi = (E, H)$, we see that the equations (4.4.8) can be written in the form (4.4.3) of a Dirac system so we can use the representation (4.4.4) to solve this system by the aid of the Fourier transform.

Again a simple reduction to the wave equation can be done. In fact taking the time derivative in the first equation in (4.4.5) and using the relation $\mathrm{rotrot}E = -\Delta E$ provided $\mathrm{div}E = 0$, we get

$$(\partial_t^2 - \Delta)E = 0.$$

In a similar way one can see that H also satisfies the wave equation.

In order to write the system (4.4.5) in relativistic form a natural procedure is to construct the following matrix (called usually electromagnetic tensor).

$$F_{km} = \varepsilon_{kml} H_l \quad , \quad F_{0k} = -F_{k0} = E_k, \tag{4.4.10}$$

where $\varepsilon_{kml} = 1$ if (kml) is an even permutation of (123), $\varepsilon_{kml} = -1$ if (kml) is an odd permutation of (123) and $\varepsilon_{kml} = 0$ otherwise. Moreover, in (4.4.10) we use the summation convention for repeated Latin indices varying from 1 to 3. It is clear that $F_{\mu\ni}$ is skew-symmetric, i.e. $F_{\mu\ni} = -F_{\ni\mu}$.

By the aid of the metric $\{\eta^{\alpha\beta}\} = diag\{-1, 1, ..., 1\}$ one can freely raise the indices

$$F^{\alpha\beta} = \eta^{\alpha\mu}\eta^{\beta\ni}F_{\mu\ni}.$$

The corresponding dual tensor $\tilde{F}_{\mu\ni}$ can be defined as follows

$$\tilde{F}_{\mu\ni} = \varepsilon_{\mu\ni\alpha\beta}F^{\alpha\beta}, \tag{4.4.11}$$

where $\varepsilon_{\mu\ni\alpha\beta} = 1$ if $(\mu \ni \alpha\beta)$ is an even permutation of (0123), $\varepsilon_{\mu\ni\alpha\beta} = -1$ if $(\mu \ni \alpha\beta)$ is an odd permutation of (0123) and $\varepsilon_{\mu\ni\alpha\beta} = 0$ otherwise. Moreover, in (4.4.11) we use the summation convention for repeated Greek indices varying from 0 to 3.

Then the Maxwell equations (4.4.5) take the simple form

$$\partial_\mu F^{\mu\ni} = 0 \quad , \quad \partial_\mu \tilde{F}^{\mu\ni} = 0. \tag{4.4.12}$$

One can show that if

$$F^{\mu\ni}(t, x)$$

are smooth functions satisfying (4.4.12), then there exist functions $A_\mu(t, x)$, such that

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$$

Remark. The vector $A_\mu(t, x)$ is called electromagnetic potential. This potential is not unique. Namely, we can take $\tilde{A}_\mu(t, x) = A_\mu(t, x) - \partial_\mu\varphi(t, x)$, where $\varphi(t, x)$ is arbitrary.

Problem 4.4.1 *(Maxwell equations in the form of Dirac equations)*

Let E, H satisfy the Maxwell equations (4.4.5) in vacuum. Consider the vector

$$\chi = \begin{pmatrix} 0 \\ E_1 - iH_1 \\ E_2 - iH_2 \\ E_3 - iH_3 \end{pmatrix}.$$

Find three selfadjoint (4×4) *matrices* $\alpha_1, \alpha_2, \alpha_3$ *so that* χ *satisfies the Dirac equation (4.4.3) and the relation*

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk}$$

4.5 Examples of nonlinear hyperbolic equations

One of the simplest nonlinear hyperbolic equation is the equation of a scalar self-interacting field, that is

$$(-\partial_t^2 + \Delta)u - M^2 u = u^3. \tag{4.5.1}$$

In order to prove the existence of global in time solution even in the case of large initial data we shall use the conservation law of the energy. Indeed, multiplying (4.5.1) by $\partial_t u$ we see that the energy

$$E(t) = \frac{1}{2}\int |\partial_t u(t,x)|^2 dx +$$

$$+\frac{1}{2}\int \left(|\nabla_x u(t,x)|^2 + M|u(t,x)|^2 + \frac{1}{2}|u(t,x)|^4\right) dx \tag{4.5.2}$$

is a constant. As usual the initial data are given by

$$u(0,x) = f_0(x), \partial_t u(0,x) = f_1(x) \tag{4.5.3}$$

To establish the existence of global solution we shall make two steps.

First step. We shall rewrite (4.5.1) in abstract evolution equation of the form

$$\partial_t v = Av + K(v), \tag{4.5.4}$$

where A is a selfadjoint operator in a suitable Hilbert space H and $K(v)$ is an operator in this Hilbert space.

Second step. We shall prove for (4.5.4) a suitable continuation principle. Combining the existence of local solution with this principle we shall establish the existence of global solution.

For simplicity we shall consider here only the case of positive mass M. Setting

$$v = \begin{pmatrix} u \\ \partial_t u \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ \Delta - M & 0 \end{pmatrix},$$

$$K(v) = \begin{pmatrix} 0 \\ -v_1^3 \end{pmatrix}, \tag{4.5.5}$$

we see that the nonlinear wave equation (4.5.1) takes the form (4.5.4). The form of the energy in (4.5.2) suggests us to consider the Hilbert space $H = H^1(\mathbf{R}^n) \times L^2(\mathbf{R}^n)$. For any couple $v = (v_1, v_2) \in H$ the corresponding norm is defined by

$$\|v\|_H^2 = \int |\nabla v_1|^2 + M|v_1|^2 + |v_2|^2 dx \tag{4.5.6}$$

Denote by $(,)_H$ is the corresponding Hilbert norm.

The operator A with dense domain

$$D(A) = H^2(\mathbf{R}^n) \times H^1(\mathbf{R}^n). \tag{4.5.7}$$

is skew – selfadjoint.

Turning to the continuation principle we consider the abstract nonlinear evolution problem (4.5.4) assuming A is skew-selfadjoint and K is Lipschitz continuos, i.e. for any ball $B(R)$ of radius R in H there exists a constant $C = C(R)$, such that

$$\|K(u) - K(v)\|_H \leq C\|u - v\|_H. \tag{4.5.8}$$

The nonlinear problem (4.5.4) can be written in integral form in the same manner as it is done for ordinary differential equations.

$$v(t) = U(t)f + \int_0^t U(t-s)K(v(s))ds. \tag{4.5.9}$$

Now we are in position to state the continuation principle

Theorem 4.5.1 *Under the above assumptions there exists a maximum interval* $[0, \bar{t})$ *of existence of a unique solution* $v(t) \in C([0, \bar{t}); H)$ *of the integral equation (4.5.9). Then either* $\bar{t} = \infty$, *or else* $\|v(t)\|_H \to \infty$ *as* $t \to \bar{t}$.

Proof.

For any $t_0 \geq 0$ consider the following local problem

$$v(t) = U(t-t_0)f + \int_{t_0}^t U(t-s)K(v(s))ds. \tag{4.5.10}$$

To establish the existence of solution in the interval $[t_0, t_0 + \varepsilon]$ we define inductively the sequence $v_k(t)$ in the Banach space $B = C([t_0, t_0 + \varepsilon]; H)$ as follows $v_0(t) = f$,

$$v_{k+1}(t) = U(t-t_0)f + \int_{t_0}^t U(t-s)K(v_k(s))ds \tag{4.5.11}$$

Using the fact that $U(t)$ is a unitary operator and K is Lipschitz continuous, we obtain the estimate

$$\|v_{k+1} - v_k\|_B \leq C\varepsilon\|v_k - v_{k-1}\|_B \tag{4.5.12}$$

with some constant C independent of ε. When f varies in a ball of radius R in H the constant C in (4.5.12) depends on R, but is independent of f.

The contraction mapping principle shows that a unique solution exists, when $C(R)\varepsilon < 1$.

This means that for $\varepsilon < 1/C(r)$ the life span ε of the solution depends only on R, but it is independent of the concrete choice of f in

$$\{h \in H : \ |h|_H \leq R\}.$$

To finish the proof let us consider the maximal interval $[0, \bar{t})$ of existence of solution with finite $\bar{t}$ and $\|v(t)\|_H \leq C$ for $t \in [0, \bar{t})$. Then taking $R = 2C$ and applying the above argument based on the contraction mapping principle we see that one can find $\varepsilon >$ depending only on R so that the local problem (4.5.10) with initial data at t_0 very close to $\bar{t}$ (more precisely our choice is determined by $\bar{t} - \varepsilon < t_0 < \bar{t}$), can be solved in the interval $[t_0, t_0 + \varepsilon]$. Since $t_0 + \varepsilon > \bar{t}$ this contradicts the fact that $[0, \bar{t})$ is the maximal interval of existence of solution and completes the proof.

We shall prove that the semi linear problem (4.5.1) for the wave equation has a global solution in case of space dimension $n = 3$. To do this it remains to show that the nonlinear operator K defined in (4.5.5) is Lipschitz continuous. The definition of the norm in H and the Hölder inequality imply that

$$\|K(u) - K(v)\|_H = \|u_1^3 - v_1^3\|_{L^2} \leq C\|u_1 - v_1\|_{L^6}(\|u_1\|_{L^6} + \|v_1\|_{L^6})^2$$

for any two couples $u = (u_1, u_2)^t, v = (v_1, v_2)^t$ in H. Applying the Sobolev inequality

$$\|f\|_{L^6(\mathbf{R}^3)} \leq C\|f\|_{H^1(\mathbf{R}^3)}$$

and the definition of the Hilbert space H we arrive at

$$\|K(u) - K(v)\|_H \leq C\|u - v\|_H(\|u\|_H + \|v\|_H)^2.$$

Thus K is a Lipschitz operator and the continuation principle assures the existence and the uniqueness of the solution.