2 Preliminaries from functional analysis

2.1 Overview

In this chapter we shall make a review of some basic facts from functional analysis and we shall focus our attention to two main points.

On one hand, we shall give suitable sufficient conditions that assure that a symmetric strictly monotone operator in a Hilbert space is self-adjoint. More precisely, we consider Friedrich's extention of a symmetric strictly monotone operator. The criterion to assure that its closure is self-adjoint operator is of type: weak solution $\Rightarrow$ strong solution. We shall apply this criterion in the next chapters.

On the other hand, we represent some of the basic interpolation theorems for the Lebesgue spaces L^p.

To get a complete information on the subject one can use [42], [43] , [65].

2.2 Linear operators in Banach spaces

Given any couple A, B of Banach spaces we denote their corresponding norms by

$$\|a\|_A \quad , \quad \|b\|_B$$

for $a \in A, b \in B$. A linear operator

$$F : A \to B$$

is called bounded (or continuous) if there is a constant $C > 0$ such that

$$\|Fa\|_B \leq C\|a\|_A.$$

The space $L(A, B)$ is the set of bounded linear operators

$$F : A \to B$$

with norm

$$\|F\| = \sup_{\|a\|_A=1} \|Fa\|_B.$$

In case $A = B$ we shall denote by $L(A)$ the corresponding linear space of bounded linear operators from A in A. It is easy to see that $L(A, B)$ equipped with the above norm is a Banach space.

If B is the field $\mathbf{C}$ of complex numbers, then the elements in $L(A, \mathbf{C})$ are called functionals and $L(A, \mathbf{C})$ itself is called dual space of A and is denoted by A'.

For any $v' \in A'$ we denote by

$$< v', v >$$

the action of the linear functional v' on $v \in A$. There is a natural embedding

$$J : A \to A',$$

defined by the identity

$$< J(v), v' >=< v', v > .$$

In dominant part of applications we work with Banach spaces that are reflexive ones, i.e. $J(A) = A''$.

For the typical case of Hilbert space H with inner product $(\cdot, \cdot)_H$ for any element $h' \in H'$ there exists an element $h_0 \in H$ so that

$$< h', h >= (h, h_0)_H$$

for any $h \in H$. This is the classical Riesz representation theorem. On the basis of this theorem there is an isometry

$$h' \in H' \to h_0 \in H.$$

We shall denote this isometry by

$$H' \sim_{(,)_H} H.$$

It is clear that the isometry depends on the choice of the product $(\cdot, \cdot)_H$.

Sometimes it is possible to define the linear operator only on a dense domain $D \subset A$ so that

$$F : D \to B.$$

Then $D = D(F)$ is called a domain for F. The range of the operator F is

$$R(F) = \{b : b = F(a), a \in D(F)\}.$$

A linear operator

$$F : D(F) \to B$$

is an extension of the operator

$$G : D(G) \to B$$

if $D(G) \subset D(F)$ and $Ga = Fa$ for $a \in D(G)$. The operator $G : D(G) \to B$ is called closed, if the conditions

$$a_n \to a \ , \ a_n \in D(G), \ G(a_n) \to b$$

imply $a \in D(G)$ and $b = Ga$.

Let

$$F : D(F) \to B$$

be a linear operator with dense domain $D(F)$. On the product

$$A \times B$$

one can define a norm by

$$\|a\|_A + \|b\|_B$$

for $a \in A, b \in B$. Then F is a closed operator if and only if its graph

$$\Gamma(F) = \{(a, F(a)); a \in D(F)\}$$

is a closed subset in $A \times B$.

Theorem 2.2.1 *(closed graph theorem) Let $F : D(F) \to B$ be a linear operator with $D(F) = A$. If the operator is closed, then the operator is bounded, i.e. there exists a constant $C > 0$ such that*

$$\|Fa\|_B \leq C\|a\|_A$$

for $a \in D(F) = A$.

If F has a dense domain $D(F) \subset A$

$$F : D(F) \to B,$$

then the dual operator F' is an operator between B' and A' and this operator has a domain $D(F')$ defined as follows: $b' \in D(F')$ if and only if there exists an element $a' \in A'$ so that

$$< b', Fa > = < a', a > \tag{2.2.1}$$

for any $a \in D(F)$. We put $F'(b') = a'$.

Let $b' \in D(F')$. Then there is a unique $a' \in A'$, satisfying (2.2.1).

Given any Banach space A we call

$$T : A \to \mathbf{C}$$

a conjugate linear functional if

$$T(\alpha_1 a_1 + \alpha_2 a_2) = \overline{\alpha_1}\, T(a_1) + \overline{\alpha_2}\, T(a_2).$$

Moreover, we shall say that the conjugate linear functional T is bounded, if there exists a constant $C > 0$ such that

$$|T(a)| \leq C\|a\|_A$$

for any $a \in D(F)$.

We denote by A^* the vector space of linear conjugate functionals on A.

Then A^* is a Banach space and one can see that there is a natural isomorphism between A^* and A'.

For any $a^* \in A^*$ we denote by

$$< a^*, a >$$

the action of the linear functional a^* on $a \in A$.

Let F be an operator with a dense domain $D(F) \subset A$ and

$$F : D(F) \to B.$$

The conjugate operator F^* is an operator between B^* and A^* and has a domain $D(F^*)$ defined as follows: $b^* \in D(F^*)$ if and only if there exists an element $a^* \in A^*$ so that

$$< b^*, Fa >=< a^*, a > \tag{2.2.2}$$

for any $a \in D(F)$.

Let $b^* \in D(F^*)$. Then there is a unique $a^* \in A^*$, satisfying (2.2.2).

By definition $F^*(b^*) = a^*$, where the element a^* is the unique element satisfying (2.2.2). In general the fact that F has dense domain does not guarantee that $D(F^*)$ is dense in A^*. However, if the spaces A, B are reflexive ones one can show (see Theorem III.21 in [4] for example) that the space $D(F^*)$ is dense in B.

The operator F^* with dense domain $D(F^*)$ is closed operator.

Further, we turn again to the situation of a Hilbert space H. An operator F with dense domain $D(F) \subset H$ is called symmetric if

$$(Fh, g)_H = (h, Fg)_H$$

for any $h, g \in D(F)$. Using the definition of the adjoint operator F^* we see that F^* is an extention of the operator F, when F is symmetric.

We shall say that F is self-adjoint if

$$F = F^*.$$

The following criterion for self-adjointness plays an important role.

Theorem 2.2.2 *(see [43], [44]) Suppose F is symmetric operator on a Hilbert space H with dense domain $D(F)$ and*

$$R(F - \lambda) = R(F - \bar{\lambda}) = H \tag{2.2.3}$$

for some complex number λ. Then F is self-adjoint.

The condition (2.2.3) with $\lambda = i$ is equivalent to

$$\mathrm{Ker}(F^* - i) = \mathrm{Ker}(F^* + i) = 0.$$

Let F be a symmetric operator with a dense domain $D(F) \subset H$.

A natural way to extend this operator to a closed operator is to take the closure $\overline{\Gamma(F)}$ of the graph

$$\Gamma(F) = \{(h, Fh); h \in D(F)\}$$

in $H \times H$.

If F is a symmetric operator with a dense domain $D(F)$ in H, then there exists an operator $\bar{F}$ such that

$$\overline{\Gamma(F)} = \Gamma(\bar{F}).$$

We call $\bar{F}$ a closure of F.

The importance of self-adjoint operators is connected with the possibility to use the spectral theorem. (see [42])

Theorem 2.2.3 *(Spectral theorem - functional calculus) Let F be a self-adjoint operator in a Hilbert space H. Then there is a unique map $\hat{\phi}$ from the bounded Borel functions on $\mathbf{R}$ into $L(H)$ so that*

a) $\hat{\phi}$ is an algebric $*-$ homomorphism, i.e.

$$\hat{\phi}(fg) = \hat{\phi}(f)\hat{\phi}(g)\ ,\hat{\phi}(\lambda f) = \lambda\hat{\phi}(f)\ ,\hat{\phi}(f_1 + f_2) = \hat{\phi}(f_1) + \hat{\phi}(f_2),$$

$$\hat{\phi}(1) = I\ ,\ \hat{\phi}(\bar{f}) = (\hat{\phi}(f))^*.$$

b) $\|\hat{\phi}(f)\|_{L(H)} \leq \|f\|_{L^\infty}$,

c) let $h_n(x)$ be a sequence of bounded Borel functions with

$$\lim_{n\to\infty} h_n(x) = x$$

for each x and $|h_n(x)| \leq |x|$ for all x and n. Then for any $\psi \in D(F)$ we have

$$\lim \hat{\phi}(h_n)\psi = F\psi.$$

d) if $h_n(x) \to h(x)$ pointwise and if the sequence $\|h_n\|_{L^\infty}$ is bounded, then

$$\hat{\phi}(h_n) \to \hat{\phi}(h)$$

strongly.

e) if $F\psi = \lambda\psi$ then

$$\hat{\phi}(h)\psi = h(\lambda)\psi.$$

f)if $h \geq 0$, then $\hat{\phi}(h) \geq 0$.

This spectral theorem gives us a possibility to define the function of the operator F by means of the identity

$$f(F) = \hat{\phi}$$

for any measurable function f on $\mathbf{R}$.

The above spectral theorem can be rewritten in projection valued measure form (see [42]).

Given any Borel set $\Omega \subset \mathbf{R}$, we denote by χ_Ω the corresponding characteristic function for the set Ω. Then the functional calculus for the self-adjoint operator F enables one to consider the projection:

$$P_\Omega = \chi_\Omega(F) = \hat{\phi}(\chi_\Omega).$$

The family $\{P_\Omega\}$ satisfies the properties:

a) P_Ω is an orthogonal projection,

b) $P_\emptyset = 0$, $P_{(-\infty,\infty)} = I$,

c) If Ω is a countable disjoint union of Borel sets $\Omega_m, m = 1, 2, ...$, then for any $h \in H$ we have

$$P_\Omega h = \lim_{N\to\infty} \sum_{m=1}^{N} P_{\Omega_n} h,$$

d) $P_{\Omega_1} P_{\Omega_2} = P_{\Omega_1 \cap \Omega_2}$.

Given any $h \in H$, we see that

$$\mu(\Omega) = (h, P_\Omega h)_H$$

is a classical measure. By $d(h, P_\lambda h)$ we shall denote the corresponding volume element needed for integration with respect to this measure so we have

$$\int_{-\infty}^{\infty} \chi_\Omega(\lambda) d(h, P_\lambda h) = (h, P_\Omega h)_H$$

Now for any (eventually unbounded) Borel function g on $(-\infty, \infty)$ we consider the domain

$$D_g = \{h \in H; \int_{\mathbf{R}} |g(\lambda)|^2 d(h, P_\lambda h) < \infty\}$$

and then define the operator (eventually unbounded) $h \in D_g \to g(F)h$ by means of the identity

$$(h, g(F)h)_H = \int_{\mathbf{R}} g(\lambda) d(h, P_\lambda h).$$

Then we have the following assertion.

Theorem 2.2.4 *For any real-valued Borel function $g(\lambda)$ defined on $(-\infty, \infty)$ the operator $g(F)$ with dense domain D_g is self-adjoint.*

The functional calculus enables one to define the exponential $U(t) = e^{itF}$.

Theorem 2.2.5 *(see [42]) If F is a self-adjoint operator in the Hilbert space H, then $U(t) = e^{itF}$ satisfies the properties:*

a) $U(t)$ is a bounded unitary operator for any $t \in \mathbf{R}$.

b) $U(t)U(s) = U(t + s)$ for any real numbers t, s,

c) $\lim_{t\to 0} U(t)h = h$ for any $h \in H$.

d) $h \in D(F)$ if and only if

$$\lim_{t\to o} \frac{U(t)h - h}{t}$$

exists in H.

Remarks A. The property a) in the above theorem means that

$$\|U(t)h\|_H = \|h\|_H.$$

Remark B. An operator-valued function $U(t)$ satisfying the above properties a),b) and c) is called a strongly continuous one-parameter unitary group.

Theorem 2.2.6 *(Stone's theorem, see [43]) If $U(t)$ is a strongly continuous one-parameter unitary group, then we can define its generator G so that $h \in D(G)$ if and only if the limit*

$$\lim_{t \to 0} \frac{U(t)h - h}{t}$$

exists. The above limit shall be denoted Gh for $h \in D(G)$. One has

$$G = iF,$$

where F is a self-adjoint operator in H.

2.3 Symmetric strictly monotone operators on Hilbert space

In this section we shall consider the special case when a symmetric operator B is defined on a dense domain $D(B) \subset H$, where H is a real Hilbert space. For simplicity we take Hilbert space over $\mathbf{R}$, but the results are valid also for Hilbert spaces over $\mathbf{C}$. We shall denote by

$$(\cdot,\cdot)_H \quad , \quad \|\cdot\|_H$$

the inner product and the norm in H respectively.

Our main assumption is that B is strictly monotone, i.e. there exists a constant $C > 0$, so that

(2.3.1) $$(Bu, u) \geq C\|u\|_H^2$$

for $u \in D(B)$.

First we consider the case, when the range $R(B)$ is dense in H.

Lemma 2.3.1 *If B is a symmetric strictly monotone operator with dense range $R(B)$, then the closure $\bar{B}$ is a self-adjoint operator.*

Proof. The operator $\bar{B}$ is also symmetric and strictly monotone. Then the inequality

$$\|\bar{B}u\|_H^2 \geq C\|u\|_H^2$$

shows that $R(\bar{B})$ is closed . Since $R(B) \subset R(\bar{B})$ and $R(B)$ is dense in H, we see that $R(\bar{B}) = H$. Applying Theorem 2.2.2, we see that $\bar{B}$ is self-adjoint.

The next step is to introduce the corresponding "energetic" space (see [65]).

For the purpose for any $u, v \in D(B)$ we define the corresponding energy inner product

$$(u, v)_E = (Bu, v)_H. \tag{2.3.2}$$

The corresponding norm is

$$\|u\|_E = \sqrt{(u, u)_E}.$$

Definition 2.3.1 *The space H_E consists of all $u \in H$ such that there exists a sequence $\{u_n\}_{n=1}^{\infty}$ with the properties:*

a) $u_n \in D(B)$,

b) $u_n \to u$ in H,

c) u_n is a Cauchy sequence for the norm $\|\cdot\|_E$, i.e. for any $\varepsilon > 0$ there exists an integer $N \geq 1$, such that

$$\|u_n - u_m\|_E \leq \varepsilon$$

for $n, m \geq N$.

We shall call the sequence $\{u_n\}$, satisfying the above properties, admissible for u. Given any $u \in H_E$, we can define its norm by

$$\|u\|_E = \lim_{n\to\infty} \|u_n\|_E. \tag{2.3.3}$$

Our first step is to show that this definition is independent of the concrete choice of admissible sequence $\{u_n\}$.

Lemma 2.3.2 *Suppose $\{u_n\}$ is an admissible sequence of* 0. *Then*

$$\lim_{n\to\infty} \|u_n\|_E = 0.$$

Proof. Assume the assertion of Lemma is not true. Choosing a subsequences we can reduce the proof of a contradiction to the case

$$a < \|u_n\|_E < a^{-1} \tag{2.3.4}$$

with some $a > 0$. Given any $\varepsilon > 0$, we can choose N depending on $\varepsilon > 0$ according to property c) of Definition 2.3.1. Then for any $n \geq N$ we have the inequalities

$$\|u_n\|_E^2 \leq |(u_n, u_N)_E| + |(u_n, u_n - u_N)_E| \leq |(u_n, u_N)_E| + a^{-1}\varepsilon.$$

On the other hand, we have the identity

$$(u_n, u_N)_E = (u_n, Bu_N)_H,$$

according to our definition of the inner product $(\cdot,\cdot)_E$ on $D(B)$. Since $\{u_n\}$ is admissible sequence for 0, we have $\lim_{n\to\infty} \|u_n\|_H = 0$. Therefore, we can find $n \geq N$ so large that

$$|(u_n, u_N)_E| \leq \varepsilon.$$

Thus, for any $\varepsilon > 0$ we can find n so that

$$\|u_n\|_E^2 \leq \varepsilon(1 + a^{-1})$$

It is clear that this inequality is in contradiction with the left inequality in (2.3.4), when $\varepsilon > 0$ is sufficiently small.

Therefore we have a contradiction and this completes the proof of the lemma.

The above lemma enables one to introduce a norm in H_E as follows

$$\|u\|_E = \lim_{n\to\infty} \|u_n\|_E, \tag{2.3.5}$$

where $\{u_n\}$ is an admissible sequence for $u \in H_E$.

Also it is easy to define the inner product in H_E. For $u_n, v_n \in D(B)$ such that $\{u_n\}, \{v_n\}$ are admissible sequences for $u, v \in H_E$ we have the polarization identity

$$(u_n, v_n)_E = \frac{1}{4}(\|u_n + v_n\|_E^2) - \frac{1}{4}(\|u_n - v_n\|_E^2).$$

Then from (2.3.5) we see that the limit

$$\lim_{n\to\infty} (u_n, v_n)_E$$

exists and it is independent of the concrete choice of admissible sequences. For this we can introduce the inner product in H_E as follows

$$(u, v)_E = \lim_{n\to\infty} (u_n, v_n)_E.$$

The next step is of special importance to verify the fact that the space H_E is a Hilbert space.

Lemma 2.3.3 *If $\{u_n\}$ is an admissible sequence for $u \in H_E$, then*

$$\lim_{n\to\infty} \|u_n - u\|_E = 0. \tag{2.3.6}$$

Proof. For any integer $m \geq 1$ the sequence

$$u_n - u_m$$

is admissible for $u - u_m$. The fact that $\{u_n\}$ is a Cauchy sequence in H_E means that for any positive number ε there exists an integer $N \geq 1$, so that

$$\|u_n - u_m\|_E \leq \varepsilon$$

for $n, m \geq N$. Then definition (2.3.5) shows that

$$\|u - u_m\|_E \leq \varepsilon$$

for $m \geq N$. This completes the proof.

It is clear that the definition (2.3.5) guarantees that

$$\|u\|_E^2 \geq C\|u\|_H^2 \tag{2.3.7}$$

This estimate shows that $(u,u)_E = 0$ implies $u = 0$, so H_E is a pre - Hilbert space. Also it is a trivial fact that $D(B)$ is a dense subset in H_E, since any element u in H_E by the definition of H_E is such that there exists an admissible sequence $\{u_n\}$ with $u_n \in D(B)$.

Our next step is to study the space H_E.

Theorem 2.3.1 *The space H_E is a Hilbert space.*

Proof. Let $\{u_n\}$ be a Cauchy sequence in H_E. Since $D(B)$ is dense in H_E, for any integer $n \geq 1$ one can find $v_n \in D(B)$, so that

$$\|v_n - u_n\|_E \leq \frac{1}{n}. \tag{2.3.8}$$

Then the estimate $\|v_n\|_E^2 \geq C\|v_n\|_H^2$ shows that $\{v_n\}$ is a Cauchy sequence in H so there exists $u \in H$, so that

$$v_n \to u \text{ in } H.$$

Applying Lemma 2.3.3, we conclude that

$$\lim_{n\to\infty} \|u - v_n\|_E = 0$$

and from (2.3.8) we get

$$\lim_{n\to\infty} \|u - u_n\|_E = 0.$$

This completes the proof.

Further, we turn to the dual space H_E^*. As usual for any linear continuous functional $f \in H_E^*$ and any $g \in H_E$ we denote by

$$< f, g >$$

the action of the functional f on g. The inclusion $H \subset H_E^*$ is such that

$$< f, g > = (f, g)_H$$

for $f \in H, g \in H_E$. The norm in H_E^* is

$$\|f\|_{H_E^*} = \sup_{g \in H_E, \|g\|_E = 1} < f, g > .$$

Then H_E^* is clearly a Banach space. Later on we shall introduce on H_E^* a structure of a Hilbert space. The main preparation for this is the following

Lemma 2.3.4 *The symmetric strictly monotone operator $B : D(B) \to H$ can be extended to an invertible isometry*

$$B_E : H_E \to H_E^*,$$

i.e. we have the properties

a) $B_E u = Bu$ *for* $u \in D(B)$,

b) B_E *maps* H_E *onto* H_E^*,

c) $\|B_E u\|_{H_E^*} = \|u\|_{H_E}$.

Proof. For any $u \in H_E$ we take an admissible sequence $\{u_n\}$, such that

$$\lim_{n\to\infty} \|u_n - u\|_E = 0.$$

On the other hand, we have the relation

$$\|Bu\|_{H_E^*} = \|u\|_E \tag{2.3.9}$$

for $u \in D(B)$. Indeed, for $u \in D(B), v \in H_E$ we have

$$| < Bu, v > | = |(Bu, v)_H| = |(u, v)_E| \le \|u\|_E \|v\|_E. \tag{2.3.10}$$

Hence,

$$\|Bu\|_{H_E^*} \le \|u\|_E.$$

To establish inequality in the opposite direction we choose $v = u$ in (2.3.10) and get

$$\|u\|_E^2 \le \|Bu\|_{H_E^*} \|u\|_E.$$

Once, the relation (2.3.9) is established, we can conclude that $\{Bu_n\}$ is a Cauchy sequence in H_E^* so it is convergent in H_E^* to an element $v \in H_E^*$ so by definition

$$B_E u = v.$$

It is clear that the element v is independent of the concrete choice of the admissible sequence $\{u_n\}$ for u. Also (2.3.9) can be extended to $u \in H_E$.

Therefore, it remains to show that B_E maps the energetic space H_E onto its dual H_E^*. To do this take $v \in H_E^*$ and consider the linear continuos functional

$$h \in H_E \to < v, h > \in \mathbf{R}.$$

According to Riesz representation theorem, there exists $u \in H_E$ so that

$$< v, h > = (u, h)_E.$$

Taking an admissible sequence for u we can see that

$$(u_n, h)_E = (Bu_n, h)_H \to < B_E u, h >$$

Hence, $< B_E u, h >=< v, h >$ so $B_E u = v$. This completes the proof.

Using the fact that $B_E : H_E \to H_E^*$ is an invertible isometry, we can define via the polarization identity inner product on H_E^* and conclude that this is a Hilbert space.

In fact starting with the relations

$$\|Bu\|^2_{H_E^*} = \|u\|^2_E = (Bu, u)_H$$

for $u \in D(B)$ and using the previous Lemma, we see that we can introduce the inner product in H_E^* by means of

$$(B_E u, B_E v)_{H_E^*} = (u, v)_E =< B_E u, v > .$$

The above relations show that B_E is a symmetric operator. It is easy to see that B_E is a strictly monotone operator on H_E^* with dense domain H_E. Applying the first Lemma of this section, we conclude that

Lemma 2.3.5 *The operator B_E is self-adjoint.*

Our main result in this section is the following.

Theorem 2.3.2 *(see [65]) If B is a symmetric strictly monotone operator, then the operator A with dense domain*

$$D(A) = \{u \in H_E, B_E u \in H\}$$

defined with $Au = B_E u$ for $u \in D(A)$ is a self-adjoint extention of B.

Proof.

Given any $f \in H$, we can find $u \in H_E$ so that $f = B_E u$.

It is not difficult to see that the operator

$$F : f \in H \to u = F(f) \in H_E$$

is well - defined bounded, symmetric and

$$F(Bh) = h, h \in D(B).$$

In fact F is a restriction of the isometry

$$B_E^{-1} : H_E^* \to H_E$$

to H. Moreover, F is a symmetric bounded operator from H into H. Then the symmetric bounded operator F is self-adjoint. Applying the spectral theorem in the form of Theorem 2.2.4 with $g(\lambda) = 1/\lambda$, we see that the operator $A = F^{-1}$ with dense domain $D(A)$ is selfadjoint.

It is an open problem if the closure of the graph of B is the graph of A. For this we introduce the following.

Definition 2.3.2 *Given any $f \in H$, we shall say that $u \in H_E$ is a weak solution of the equation $Bu = f$, if*

$$(u, Bv)_H = (f, v)_H$$

for any $v \in D(B)$.

The above identity ca be rewritten in the form

$$< B_E u, v >= (f, v)_H$$

for any $v \in D(B)$. Since $D(B)$ is dense H_E, we see that any weak solution satisfies

$$B_E u = f.$$

On the other hand, we introduce the following

Definition 2.3.3 *Given any $f \in H$, we shall say that $u \in H_E$ is a strong solution of $Bu = f$, if there exists a sequence $\{u_k\}$ such that*
a) $u_k \in D(B)$,
b) $u_k \to u$ in H_E,
c) Bu_k tends to f in H.

One can show that any strong solution of $Bu = f$ is also a weak one.
For the applications of special importance is the following result.

Theorem 2.3.3 *Suppose in addition to assumptions of Theorem 2.3.2 that any weak solution of $Bu = f$ for $f \in H$ is also a strong solution. Then the closure of the operator B is self-adjoint.*

Proof. The result follows from Theorem 2.3.2 and the fact that the assumption "weak implies strong" guarantees that the closures of the graphs of the operators A and B coincide.

2.4 Basic interpolation theorems

Let L^q denote the Lebesgue space $L^q(\mathbf{R}^n)$.

The first important interpolation theorem is the Riesz-Thorin interpolation theorem. To state this theorem we start with some notations.

Given any positive real numbers p_0, p_1 with $1 \le p_0 < p_1 \le \infty$, we denote by $L^{p_0}(\mathbf{R}^n) + L^{p_1}(\mathbf{R}^n)$ the linear space

$$\{f : f = f_0 + f_1, f_0 \in L^{p_0}(\mathbf{R}^n), f_1 \in L^{p_1}(\mathbf{R}^n)\}.$$

The norm in this space we define as follows

$$\|f\|_{L^{p_0}+L^{p_1}} = \inf_{f=f_0+f_1} \|f_0\|_{L^{p_0}} + \|f_1\|_{L^{p_1}}.$$

Here the infimum is taken over all representations $f = f_0 + f_1$, where $f_0 \in L^{p_0}(\mathbf{R}^n)$ and $f_1 \in L^{p_1}(\mathbf{R}^n)$.

It is easy to see that $L^{p_0} + L^{p_1}$ is a Banach space.

Theorem 2.4.1 *Suppose T is a linear bounded operator from $L^{p_0}+L^{p_1}$ into $L^{q_0}+L^{q_1}$ satisfying the estimates*

$$\begin{aligned} &\|Tf\|_{L^{q_0}} \le M_0\|f\|_{L^{p_0}}, \ f \in L^{p_0}, \\ &\|Tf\|_{L^{q_1}} \le M_0\|f\|_{L^{p_1}}, \ f \in L^{p_1}. \end{aligned} \tag{2.4.1}$$

Then for any $t \in (0,1)$ we have

$$\|Tf\|_{L^{q_t}} \le M_0\|f\|_{L^{p_t}}, \tag{2.4.2}$$

where

$$1/p_t = t/p_1 + (1-t)/p_0 \quad , \quad 1/q_t = t/q_1 + (1-t)/q_0. \tag{2.4.3}$$

Applying this interpolation theorem, one can derive (see [43]) the Young inequality

$$\|f * g\|_{L^q} \le \|f\|_{L^1}\|g\|_{L^q} \tag{2.4.4}$$

for $1 \le q \le \infty$. Here

$$f * g(x) = \int f(x-y)g(y)dy.$$

It is not difficult to derive the following more general variant of (2.4.4)

$$\|f * g\|_{L^s} \le \|f\|_{L^r}\|g\|_{L^p} \tag{2.4.5}$$

for $1/p + 1/r = 1 + 1/s$.

Further, we turn to a weighted variant of Young inequality. For simplicity, we consider only the continuous case. Let $w(x), w_1(x)$ and $w_2(x)$ be smooth positive functions satisfying the assumption

$$w(x+y) \le Cw_1(x)w_2(y). \tag{2.4.6}$$

Then the argument of the proof of Young inequality leads to

$$\|w(f * g)\|_{L^q} \le C\|w_1 f\|_{L^1}\|w_2 g\|_{L^q} \tag{2.4.7}$$

Indeed, we have the inequality

$$|w(x)(f * g)(x)| \le C(|w_1 f| * |w_2 g|)(x)$$

and (2.4.7) follows from the classical Young inequality.

Two typical examples of weights satisfying the assumption (2.4.6) are considered below.

Example 1. let $w(x) =< x >^s$ with $s > 0$. Then we can choose $w_1 = w_2 = w$ and the assumption (2.4.6) is fulfilled.

Example 2. Let $w(x) =< x >^s$ with $s < 0$. Then we take $w_1(x) =< x >^{-s}$ and $w_2(x) =< x >^s$. Again (2.4.6) is fulfilled.

To prove the Sobolev inequality we need more fine interpolation theorems concerning the weak L^p spaces. To define these weak spaces we shall denote by μ the Lebesgue measure. Given any measurable function f we shall say that $f \in L^p_w$ if the quantity

$$\|f\|_{L^p_w} = \sup_t \left(t^p \mu\{x : |f(x)| > t\}\right)^{1/p} \tag{2.4.8}$$

is finite. Note that the quantity in (2.4.8) is not a norm. We have the inclusion $L^p \subset L^p_w$ in view of the inequality $\|f\|_{L^p_w} \leq \|f\|_{L^p}$.

Example. The function $|x|^{-n/p}$ is in L^p_w, but not in L^p.

The following two theorems play crucial role in the interpolation theory.

Theorem 2.4.2 (**Marcinkiewicz interpolation theorem**) *Suppose T is a linear operator satisfying the estimates*

$$\begin{aligned} \|Tf\|_{L^{q_0}_w} &\leq M_0 \|f\|_{L^{p_0}} \\ \|Tf\|_{L^{q_1}_w} &\leq M_0 \|f\|_{L^{p_1}} \end{aligned} \tag{2.4.9}$$

with $p_0 \neq p_1$, $1 \leq p_0 \neq p_1 \leq \infty$ and $1 \leq q_0 \neq q_1 \leq \infty$.

Then we have

$$\|Tf\|_{L^q} \leq M_0 \|f\|_{L^p}, \tag{2.4.10}$$

provided

$$1/p = t/p_1 + (1-t)/p_0 \quad , \quad 1/q = t/q_1 + (1-t)/q_0 \tag{2.4.11}$$

for some $t \in (0,1)$ and $p \leq q$.

Theorem 2.4.3 (**Hunt interpolation theorem**) *Suppose T is a linear operator satisfying the inequalities*

$$\begin{aligned} \|Tf\|_{L^{q_0}} &\leq M_0 \|f\|_{L^{p_0}} \\ \|Tf\|_{L^{q_1}} &\leq M_0 \|f\|_{L^{p_1}} \end{aligned} \tag{2.4.12}$$

with $1 \leq p_1 < p_0 \leq \infty$ and $1 \leq q_1 < q_0 \leq \infty$. Then for any $t \in (0,1)$ we have

$$\|Tf\|_{L^{q_t}_w} \leq M_0 \|f\|_{L^{p_t}_w}, \tag{2.4.13}$$

where

$$1/p_t = t/p_1 + (1-t)/p_0 \quad , \quad 1/q_t = t/q_1 + (1-t)/q_0. \tag{2.4.14}$$

As an application of the above interpolation theorems one can prove (see [43]) the following generalization of the Young inequality

$$\|f * g\|_{L^s} \le \|f\|_{L^p} \|g\|_{L^r_w} \tag{2.4.15}$$

for $1/p + 1/r = 1 + 1/s, 1 < p, r, s < \infty$.

After this preparation we can turn to the proof of the following Sobolev estimate.

Lemma 2.4.1 *Suppose* $0 < \lambda < n, f \in L^p(\mathbf{R}^n), g \in L^r(\mathbf{R}^n)$, *where* $1/p + 1/r + \lambda/n = 2$ *and* $1 < p, r < \infty$. *Then we have*

$$\int \int \frac{|f(x)||g(y)|}{|x - y|^\lambda} dx dy \le C \|f\|_{L^p} \|g\|_{L^r} \tag{2.4.16}$$

Proof of Lemma 2.4.1 We know that (2.4.15) is fulfilled. Then for the left hand side of the Sobolev inequality (2.4.16) we can apply the Hölder inequality so we get

$$\int \int \frac{|f(x)||g(y)|}{|x - y|^\lambda} dx dy \le C \|f\|_{L^p} \|g * h\|_{L^{p'}} \tag{2.4.17}$$

with $h(x) = |x|^{-|\lambda|}$. Now the application of (2.4.15) yields

$$\|g * h\|_{L^{p'}} \le \|g\|_{L^r} \|h\|_{L^l_w} \tag{2.4.18}$$

provided

$$\frac{1}{p'} + 1 = \frac{1}{r} + \frac{1}{l} \tag{2.4.19}$$

The example considered after the definition of the weak L^p spaces shows that the quantity $\|h\|_{L^l_w}$ is bounded when $\lambda l = n$. From this relation and (2.4.19) we see that for $2 = 1/p + 1/r + \lambda/n$ we have the Sobolev inequality.