1 Introduction

The main purpose of these lecture notes is to represent some new ideas for the study of semi linear hyperbolic equations .

A fruitful classical tool to establish existence and to study the qualitative properties of the solutions of hyperbolic equations are the Sobolev spaces $W_p^s(\mathbf{R}^n)$. For integer values of $s \in \mathbf{R}$ the norm in this space is defined by

$$\|f\|_{W_p^s(\mathbf{R}^n)} = \sum_{|\alpha| \leq s} \|\partial_x^\alpha f\|_{L^p(\mathbf{R}^n)}, \tag{1.0.1}$$

while for $s \geq 0$ fractional the definition of W_p^s needs interpolation methods.

To obtain existence of a local solution to corresponding nonlinear problems one can combine classical Sobolev embedding with the energy conservation law. For the important problem, when one looks for local solutions with minimal regularity properties, it is necessary to use Sobolev spaces of fractional order ([3], [16], [17], [40]).

In these lectures we shall focus our attention to two basic hyperbolic equations - wave equation and Klein – Gordon equation. They have an additional symmetry, namely they are invariant under space - time rotations. This fact implies that we can use the usual derivatives

$$\partial_t, \partial_{x_1}, ..., \partial_{x_n} \tag{1.0.2}$$

in combination with the following vector fields

$$\begin{aligned} Y_{jk} &= x_j \partial_{x_k} - x_k \partial_{x_j}, \ j, k = 1, ..., n, \\ Y_{0j} &= t\partial_{x_j} + x_j \partial_t, \ j = 1, ..., n. \end{aligned} \tag{1.0.3}$$

Modifying the definition of norm in Sobolev space (1.0.1) as follows

$$\|f\|_{W_p^s(\mathbf{R}^n, Y)} = \sum_{|\alpha|+|\beta| \leq s} \|\partial_x^\alpha Y^\beta f\|_{L^p(\mathbf{R}^n)}, \tag{1.0.4}$$

for the case $s \geq 0$ an integer, one can treat the problem of existence of global solutions with small initial data. This approach was developed successfully by S.Klainerman [29] , L.Hörmander [24] and was applied to other important equations and systems of mathematical physics.

As far as we know, there is no satisfactory theory of these modified spaces for fractional values of s.

Having in mind the importance of classical Sobolev spaces of fractional order, we shall represent here an approach to introduce and study Sobolev spaces associated with vector fields (1.0.3) for the case of fractional order s.

This first new idea is based on the application of Sobolev and weighted Sobolev spaces on the manifold

$$M = \{t^2 - |x|^2 = \pm 1\}.$$

For this manifold the vector fields Y in (1.0.3) form a basis of the tangent space $T(M)$. For example, the upper branch

$$X = \{t^2 - |x|^2 = 1, t > 0\} \tag{1.0.5}$$

of the manifold M is a connected Riemannian manifold of constant negative curvature -1. Using the projection

$$(t, x) \in X \to x \in \mathbf{R}^n,$$

we shall see that weighted Sobolev spaces on X can be connected with weighted Sobolev spaces on $\mathbf{R}^n$, associated with vector fields

$$< x > \partial_{x_j}, j = 1, ..., n, \tag{1.0.6}$$

where $< x >^2 = 1 + |x|^2$. Weighted Sobolev spaces, associated with vector fields (1.0.6) have been studied in [5] for the case of integer values of s.

Studying systematically the properties of corresponding Sobolev spaces of fractional order, we shall be able to treat a concrete application connected with semi linear Klein - Gordon equation with nonlinearity having a singularity at the origine.

To realize this idea we use harmonic analysis results for the Laplace - Beltrami operator on X, i.e. we need a suitable modification of the Fourier transform on a manifold of curvature -1. The other tools we apply are classical interpolation methods in combination with the techniques of pseudodifferential operators in $\mathbf{R}^n$. For this we assume that the reader has some knowledge of the basis of differential geometry, harmonic analysis, functional analysis (interpolation theory) and the techniques of pseudodifferential operators in the theory of partial differential equations.

Nevertheless, since our purpose is to present lectures useful also for graduate students, we prepared few introductory chapters, where these tools are introduced in a relatively self - contained way. For more complete and detailed treatment the reader can consult the following books: Y. Choquet - Bruhat, C. De Witt - Morette and M. Dillard-Bleick, [6], R. Racke [41], T. Runst and W. Sinckel [46], J. Shatah, M. Struwe [48], L. Hörmander [24].

The second main new idea is connected with one of the basic a priori estimates for hyperbolic equations - Strichartz estimate. Recently, a suitable modification of this estimate was announced in [12] and established in [14]. This weighted Strichartz estimate plays a crucial role in the proof of Strauss conjecture [14] for semilinear wave equation. The proof of weighted Strichartz estimate, represented in [14], is based on the use of theory of Fourier integral operators in $\mathbf{R}^n$. We shall see that the proof can be simplified essentially, if we use Fourier transform on X in the place of Fourier integral operators in $\mathbf{R}^n$.

Therefore, we see that this idea continues the same direction of application of harmonic analysis results on manifolds of constant negative curvature to nonlinear hyperbolic equations of mathematical physics.

As we mentioned above, we hope that these lecture notes shall be useful for graduate students, but we hope something new shall be found by specialists in the field of partial differential equations too. We hope also that the tools developed in these lectures can be applied to other unsolved problems for nonlinear hyperbolic equations.

For example, it is an open problem to establish a variant of von Wahl's estimate [63] for the wave equation using weighted Sobolev spaces associated with vector fields (1.1.6). Therefore, the possibility to use Sobolev spaces of fractional order, enables us to obtain more fine estimates about the regularity properties of the solution.

Another possible direction is to obtain weighted estimates for some other problems of mathematical physics as Dirac and Maxwell equations. Since these equations have translation and rotation symmetry, there are corresponding analogues of the vector fields Y in (1.1.3). This means it is reasonable to study the weighted Sobolev spaces associated with these vector fields and get a priori estimates for Dirac and Maxwell's equations. Applying these estimates to the corresponding nonlinear Cauchy problems, we shall obtain eventually existence of a global in time low regularity solutions with small initial data.

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