Chapter 3

Quasilinear strictly hyperbolic systems

In this chapter, we consider quasilinear strictly hyperbolic systems and study systematically classical solutions in the large to their Cauchy problems. Throughout this chapter, we always assume that system (1.1) is strictly hyperbolic in a neighbourhood of $u = 0$.

§3.1. Matching condition

Consider system (1.1) and assume that

$$\lambda_1(0) < \cdots < \lambda_n(0). \tag{3.1.1}$$

Definition 3.1. The i-th characteristic $\lambda_i(u)$ is *weakly linearly degenerate*, if, along the i-th characteristic trajectory $u = u^{(i)}(s)$ passing through $u = 0$, defined by

$$\begin{cases} \frac{du}{ds} = r_i(u), \\ s = 0: \quad u = 0, \end{cases} \tag{3.1.2}$$

we have

$$\nabla\lambda_i(u)\,r_i(u) \equiv 0, \quad \forall\, |u| \text{ small}, \tag{3.1.3}$$

namely,

$$\lambda_i\left(u^{(i)}(s)\right) \equiv \lambda_i(0), \quad \forall\, |s| \text{ small}. \tag{3.1.4}$$

If all characteristics are weakly linearly degenerate, then system (1.1) is called to be *weakly linearly degenerate.* □

Definition 3.1 and the following Lemma 3.1 can be found in [LZK1].

Lemma 3.1. Suppose that $A(u) \in C^k$, where $k \geq 1$ is an integer. Then there exists an invertible C^{k+1} transformation $u = u(\tilde{u})$ $(u(0) = 0)$ such that in $\tilde{u}$-space, for each $i = 1, \cdots, n$, the i-th characteristic trajectory passing through $\tilde{u} = 0$ coincides with the $\tilde{u}_i$-axis at least for $|\tilde{u}_i|$ small, namely,

$$\tilde{r}_i(\tilde{u}_i e_i) \equiv e_i, \quad \forall\, |\tilde{u}_i| \text{ small} \quad (i = 1, \cdots, n), \tag{3.1.5}$$

where

$$e_i = (0, \cdots, 0, \overset{(i)}{1}, 0, \cdots, 0)^T. \tag{3.1.6}$$

□

Such a transformation is called the *normalized transformation* and the corresponding unknown variables $\tilde{u} = (\tilde{u}_1, \cdots, \tilde{u}_n)^T$ are called the *normalized variables* or *normalized coordinates.* Obviously, in the normalized coordination (3.1.4) simply reduces to

$$\lambda_i(u_i e_i) \equiv \lambda_i(0), \quad \forall\, |u_i| \text{ small}. \tag{3.1.7}$$

Proof of Lemma 3.1. Let

$$u^{(1)} = \bar{u}^{(1)}(\widetilde{u}_1)$$

be the 1st characteristic trajectory passing through the origin $u = 0$, where $\widetilde{u}_1$ is a variable parameter;

$$u^{(2)} = u^{(2)}\left(u^{(1)}, \widetilde{u}_2\right) = \bar{u}^{(2)}(\widetilde{u}_1, \widetilde{u}_2)$$

be the 2nd characteristic trajectory passing through $u^{(1)}$, where $\widetilde{u}_2$ is a variable parameter; $\cdots\cdots$;

$$u^{(n)} = u^{(n)}\left(u^{(n-1)}, \widetilde{u}_n\right) = \bar{u}^{(n)}\left(\widetilde{u}_1, \cdots, \widetilde{u}_n\right)$$

be the n-th characteristic trajectory passing through $u^{(n-1)}$, where $\widetilde{u}_n$ is a variable parameter. By (3.1.2), $\widetilde{u} = (\widetilde{u}_1, \cdots, \widetilde{u}_n)$ can be taken as normalized coordinates and $u = \bar{u}^{(n)}(\widetilde{u})$ is a normalized transformation. Q.E.D.

For the normalized transformation $u = u(\tilde{u})$ $(u(0) = 0)$, we have

$$\frac{\partial u}{\partial \tilde{u}_i}(0) \,//\, r_i(0) \quad (i = 1, \cdots, n). \tag{3.1.8}$$

Since the property that coordinates are the normalized ones or not is invariant under any invertible smooth transformation in the following form:

$$\tilde{u}_i = f_i\left(\tilde{\tilde{u}}_i\right) \quad (i = 1, \cdots, n), \tag{3.1.9}$$

in which $f_i(0) = 0$ and $f_i'(0) \neq 0$, we can always choose suitable normalized coordinates $\tilde{u} = (\tilde{u}_1, \cdots, \tilde{u}_n)^T$ such that

$$\frac{\partial u}{\partial \tilde{u}_i}(0) = r_i(0) \quad (i = 1, \cdots, n), \tag{3.1.10}$$

namely,

$$\frac{\partial u}{\partial \tilde{u}}(0) = R(0), \tag{3.1.11}$$

where $R(u) = (r_{ij}(u))$ is the matrix composed by the right eigenvectors $r_i(u)$ $(i = 1, \cdots, n)$. Therefore, noting (1.4) we have

$$\frac{\partial \tilde{u}_i}{\partial u}(0) = l_i(0) \quad (i = 1, \cdots, n), \tag{3.1.12}$$

namely,

$$\frac{\partial \tilde{u}}{\partial u}(0) = L(0), \tag{3.1.13}$$

where $L(u) = (l_{ij}(u))$ is the matrix composed by the left eigenvectors $l_i(u)$ $(i = 1, \cdots, n)$.

We now introduce the concept of matching condition.

Definition 3.2. The inhomogeneous term $B(u)$ is called to satisfy the *matching condition*, if, along any characteristic trajectories $u = u^{(i)}(s)$ $(i = 1, \cdots, n)$ passing through $u = 0$ (see (3.1.2)), we have

$$B\left(u^{(i)}(s)\right) \equiv 0 \quad (i = 1, \cdots, n), \quad \forall\, |s| \text{ small.} \tag{3.1.14}$$

□

In the normalized coordinates, (3.1.14) simply reduces to

$$B\left(u_i e_i\right) \equiv 0 \quad (i = 1, \cdots, n), \quad \forall\, |u_i| \text{ small.} \tag{3.1.15}$$

Definition 3.3. The functions $b_i(u)$ $(i = 1, \cdots, n)$ are called to satisfy the matching condition, if $B(u)$ satisfies the matching condition, where $b_i(u)$ are defined by (2.2.3). □

§3.2. Some relations in the normalized coordinates

In this section we establish some relations on the decomposition of waves in the normalized coordinates. These relations play an important role in our proof.

Noting (3.1.5), from (2.2.10) and (2.2.11) we observe that in the normalized coordinates

$$\beta_{ijj}\left(u_j e_j\right) \equiv 0, \quad \forall\, |u_j| \text{ small}, \quad \forall\, j \tag{3.2.1}$$

and

$$\nu_{ijj}\left(u_j e_j\right) \equiv 0, \quad \forall\, |u_j| \text{ small}, \quad \forall\, j. \tag{3.2.2}$$

When $B(u)$ satisfies the matching condition, it follows from (1.6), (2.2.3) and (3.1.14) or (3.1.15) that in the normalized coordinates

$$b_i(u) = \sum_{j \neq k} b_{ijk}(u) u_j u_k, \quad \forall\, i, \tag{3.2.3}$$

where $b_{ijk}(u)$ are continuous functions of u, which are produced by Taylor's formula.

Noting (3.2.1), from (2.2.16) we see that in the normalized coordinates

$$\tilde{\beta}_{ijj}\left(u_j e_j\right) \equiv 0, \quad \forall\, |u_j| \text{ small}, \quad \forall\, j \neq i \tag{3.2.4}$$

and, when the i-th characteristic $\lambda_i(u)$ is weakly linearly degenerate, by (3.1.4) or (3.1.7) we get

$$\tilde{\beta}_{iii}\left(u_i e_i\right) \equiv 0, \quad \forall\, |u_i| \text{ small}. \tag{3.2.5}$$

Moreover, when $\lambda_i(u)$ is weakly linearly degenerate, it follows from (2.2.22) and (3.1.4) or (3.1.7) that in the normalized coordinates

$$\gamma_{iii}\left(u_i e_i\right) \equiv 0, \quad \forall\, |u_i| \text{ small}. \tag{3.2.6}$$

Noting (2.2.6), we have

$$\left(b_i(u)\right)_x = \sum_{k=1}^{n} \tilde{b}_{ik}(u) w_k, \tag{3.2.7}$$

where

$$\tilde{b}_{ik}(u) = \sum_{l=1}^{n} \frac{\partial b_i(u)}{\partial u_l} r_{kl}(u). \tag{3.2.8}$$

In the normalized coordinates, by (3.1.5) we get

$$\tilde{b}_{ik}(u_k e_k) = \frac{\partial b_i(u_k e_k)}{\partial u_k}. \tag{3.2.9}$$

When $B(u)$ satisfies the matching condition, in the normalized coordinates, by (2.2.3) and (3.1.14) or (3.1.15) we have

$$b_i(u_k e_k) \equiv 0, \quad \forall\, |u_k| \text{ small}, \quad \forall\, i, k, \tag{3.2.10}$$

and then

$$\frac{\partial b_i(u_k e_k)}{\partial u_k} \equiv 0, \quad \forall\, |u_k| \text{ small}, \quad \forall\, i, k. \tag{3.2.11}$$

Thus, when $B(u)$ satisfies the matching condition, in the normalized coordinates, by (3.2.9) and (3.2.11) we obtain

$$\tilde{b}_{ik}(u_k e_k) \equiv 0, \quad \forall\, |u_k| \text{ small}, \quad \forall\, i, k. \tag{3.2.12}$$

§3.3. Main results

Consider Cauchy problem

$$u_t + A(u)u_x = B(u), \tag{3.3.1}$$

$$t = 0: \quad u = \varphi(x), \tag{3.3.2}$$

where $A(u) = (a_{ij}(u))$ is an $n \times n$ matrix with suitably smooth elements $a_{ij}(u)$, $B(u) = (B_1(u), \cdots, B_n(u))^T$ is a suitably smooth vector function of u, and $\varphi(x)$ is a C^1 vector function of x. Suppose that in a neighbourhood of $u = 0$, system (3.3.1) is strictly hyperbolic:

$$\lambda_1(0) < \cdots < \lambda_n(0) \tag{3.3.3}$$

and the following normalized conditions hold:

$$l_i(u)r_j(u) \equiv \delta_{ij} \quad (i, j = 1, \cdots, n) \tag{3.3.4}$$

and

$$r_i^T(u)r_i(u) \equiv 1 \quad (i = 1, \cdots, n). \tag{3.3.5}$$

Obviously, all $\lambda_i(u)$, $l_i(u)$ and $r_i(u)$ $(i = 1, \cdots, n)$ have the same regularity as $A(u)$. Furthermore, suppose that $B(u)$ satisfies

$$B(0) = 0 \quad \text{and} \quad \nabla B(0) = 0. \tag{3.3.6}$$

Finally, suppose that $\varphi(x)$ satisfies that there exists a constant $\mu > 0$ such that

$$\theta \triangleq \sup_{x \in \mathbf{R}} \left\{ (1 + |x|)^{1+\mu} \left(|\varphi(x)| + |\varphi'(x)| \right) \right\} < \infty. \tag{3.3.7}$$

We shall prove in §3.4 the following.

Theorem 3.1. Under the hypotheses mentioned above, suppose that $A(u)$ and $B(u)$ are C^2 in a neighbourhood of $u = 0$. Suppose furthermore that system (3.3.1) is weakly linearly degenerate and $B(u)$ satisfies the matching conditon. Then there exists $\theta_0 > 0$ so small that for any given $\theta \in [0, \theta_0]$, the Cauchy problem (3.3.1)-(3.3.2) admits a unique global C^1 solution $u = u(t, x)$ on $t \geq 0$. □

In particular, we have

Corollary 3.1. If, in a neighbourhood of $u = 0$, system (3.3.1) is linearly degenerate in the sense of P.D.Lax and $B(u)$ satisfies the matching condition, then the conclusion of Theorem 3.1 holds. □

A discussion on the large time behaviour of the global classical solution will be given in §3.5.

In the case that system (3.3.1) is not weakly linearly degenerate, we will show that, for a quite large class of initial data, the first order derivatives of the C^1 solution to the Cauchy problem (3.3.1)-(3.3.2) must blow up in a finite time, and give a sharp estimate of life span of the C^1 solution.

When system (3.3.1) is not weakly linearly degenerate, there exists a nonempty set $J \subseteq \{1, 2, \cdots, n\}$ such that $\lambda_i(u)$ is not weakly linearly degenerate if and only if $i \in J$.

Noting (3.1.4), we observe that for any fixed $i \in J$, either there exists an integer $\alpha_i \geq 0$ such that

$$\left.\frac{d^l \lambda_i\left(u^{(i)}(s)\right)}{ds^l}\right|_{s=0} = 0 \quad (l = 1, \cdots, \alpha_i) \quad \text{but} \quad \left.\frac{d^{\alpha_i+1} \lambda_i\left(u^{(i)}(s)\right)}{ds^{\alpha_i+1}}\right|_{s=0} \neq 0 \tag{3.3.8}$$

or

$$\left.\frac{d^l \lambda_i\left(u^{(i)}(s)\right)}{ds^l}\right|_{s=0} = 0 \quad (l = 1, 2, \cdots), \tag{3.3.9}$$

where $u = u^{(i)}(s)$ is defined by (3.1.2). In the case that (3.3.9) holds, we define $\alpha_i = +\infty$.

In the normalized coordinates, conditions (3.3.8)-(3.3.9) simply reduce to

$$\frac{\partial^l \lambda_i}{\partial u_i^l}(0) = 0 \quad (l = 1, \cdots, \alpha_i) \quad \text{but} \quad \frac{\partial^{\alpha_i+1} \lambda_i}{\partial u_i^{\alpha_i+1}}(0) \neq 0 \tag{3.3.10}$$

and

$$\frac{\partial^l \lambda_i}{\partial u_i^l}(0) = 0 \quad (l = 1, 2, \cdots) \tag{3.3.11}$$

respectively.

Definition 3.4. The i-th characteristic $\lambda_i(u)$ is *critical*, if it is not weakly linearly degenerate but satisfies (3.3.9).

If all characteristics are critical, then the system is called to be *critical*. □

The following Theorem will be proved in §3.6.

Theorem 3.2. Under the assumptions mentioned at the beginning of this section, suppose that $A(u)$ is suitably smooth and $B(u) \in C^2$ in a neighbourhood of $u = 0$. Suppose furthermore that $\phi(x) = \varepsilon\psi(x)$, where $\varepsilon > 0$ is a small parameter and $\psi(x)$ is a C^1 vector function satisfying that there exists a constant $\mu > 0$ such that

$$\sup_{x\in\mathbf{R}} \left\{ (1+|x|)^{1+\mu} \left(|\psi(x)| + |\psi'(x)| \right) \right\} < \infty. \tag{3.3.12}$$

Suppose finally that $B(u)$ satisfies the matching condition, system (3.3.1) is not weakly linearly degenerate and

$$\alpha = \min\{\alpha_i \mid i \in J\} < \infty, \tag{3.3.13}$$

where α_i is defined by (3.3.8)-(3.3.9). Let

$$J_1 = \{i \mid i \in J, \alpha_i = \alpha\}. \tag{3.3.14}$$

If there exists $i_0 \in J_1$ such that

$$l_{i_0}(0)\psi(x) \not\equiv 0, \tag{3.3.15}$$

where $l_{i_0}(u)$ stands for the i_0-th left eigenvector, then there exists $\varepsilon_0 > 0$ so small that for any fixed $\varepsilon \in (0, \varepsilon_0]$ the first order derivatives of the C^1 solution $u = u(t,x)$ to the Cauchy problem (3.3.1)-(3.3.2) must blow up in a finite time and the life span $\tilde{T}(\varepsilon)$ of $u = u(t,x)$ satisfies

$$\begin{aligned} &\lim_{\varepsilon\to 0} \left(\varepsilon^{\alpha+1}\tilde{T}(\varepsilon)\right)^{-1} = \\ &\max_{i\in J_1}\sup_{x\in\mathbf{R}} \left\{ -\frac{1}{\alpha!} \left.\frac{d^{\alpha+1}\lambda_i(u^{(i)}(s))}{ds^{\alpha+1}}\right|_{s=0} [l_i(0)\psi(x)]^{\alpha}\, l_i(0)\psi'(x) \right\}, \end{aligned} \tag{3.3.16}$$

where $u = u^{(i)}(s)$ is defined by (3.1.2). □

Remark 3.1. Noting (3.3.12) and (3.3.15), we observe that the right-hand side of (3.3.16) must be a positive number. For the quasilinear strictly hyperbolic systems, each left (resp. right) eigenvector possesses a degree of freedom with an arbitrary non-zero factor for any given u on the domain under consideration. However, using (3.3.4) and (3.1.2), we see that the constant defined by the right-hand side of (3.3.16) is invariant under such a change of eigenvectors. □

Remark 3.2. By the definition of J_1, $\max_{i\in J_1}$ can be replaced by $\max_{i=1,\cdots,n}$ in (3.3.16). □

Remark 3.3. By §3.1, there exists a suitable normalized transformation $u = u(\tilde{u})$ ($u(0) = 0$) such that (3.1.13) holds. In the normalized coordinates $\tilde{u}$, let

$$\tilde{\lambda}_i(\tilde{u}) = \lambda_i(u) \quad (i = 1, \cdots, n). \tag{3.3.17}$$

Noting (3.3.8), for each $i \in J_1$ we can prove

$$\frac{\partial^l \tilde{\lambda}_i}{\partial \tilde{u}_i^l}(0) = 0 \quad (l = 1, \cdots, \alpha) \quad \text{and} \quad \frac{\partial^{\alpha+1} \tilde{\lambda}_i}{\partial \tilde{u}_i^{\alpha+1}}(0) = \left. \frac{d^{\alpha+1} \lambda_i\left(u^{(i)}(s)\right)}{ds^{\alpha+1}} \right|_{s=0}. \tag{3.3.18}$$

Thus, under the hypotheses of Theorem 3.2, in the normalized coordinates $\tilde{u}$ (3.3.16) simply reduces to

$$\begin{aligned} &\lim_{\varepsilon\to 0} \left(\varepsilon^{\alpha+1}\tilde{T}(\varepsilon)\right)^{-1} = \\ &\quad \max_{i\in J_1} \sup_{x\in \boldsymbol{R}} \left\{ -\frac{1}{\alpha!} \frac{\partial^{\alpha+1}\tilde{\lambda}_i}{\partial \tilde{u}_i^{\alpha+1}}(0) \left(l_i(0)\psi(x)\right)^{\alpha} l_i(0)\psi'(x) \right\}. \end{aligned} \tag{3.3.19}$$

□

Remark 3.4. Introduce

$$\bar{u} = L(0)u \tag{3.3.20}$$

and let

$$\bar{\lambda}_i(\bar{u}) = \lambda_i(u) \quad (i = 1, \cdots, n), \tag{3.3.21}$$

where $L(u) = (l_{ij}(u))$ is the $n \times n$ matrix composed by the left eigenvectors $l_i(u)$ $(i = 1, \cdots, n)$. Under the hypotheses of Theorem 3.2, we furthermore assume that $\alpha = 0$, where α is defined by (3.3.13). Similar to (3.3.18), for each $i \in J_1$

we can easily prove

$$\frac{\partial \bar{\lambda}_i}{\partial \bar{u}_i}(0) = \left.\frac{d\lambda_i\left(u^{(i)}(s)\right)}{ds}\right|_{s=0}. \tag{3.3.22}$$

In the present situation, similar to (3.3.19), in the coordinates $\bar{u}$ (3.3.16) simply reduces to

$$\lim_{\varepsilon\to 0}\left(\varepsilon\tilde{T}(\varepsilon)\right)^{-1} = \max_{i\in J_1}\sup_{x\in R}\left\{-\frac{\partial \bar{\lambda}_i}{\partial \bar{u}_i}(0)l_i(0)\psi'(x)\right\}. \tag{3.3.23}$$

□

Remark 3.5. In the special case that $B(u) \equiv 0$, Theorem 3.1 and Theorem 3.2 give the results presented in [LZK1]-[LZK2] as well as the limit formula on the life span of the C^1 solution to the Cauchy problem (3.3.1)-(3.3.2). In particular, when $B(u) \equiv 0$ and $\alpha = 0$, Theorem 3.2 goes back to the corresponding result given by F.John [Jo], T.P.Liu [Lu] and L.Hörmander [Ho1]. □

The results mentioned above are obtained under the assumption that $B(u)$ satisfies the matching condition. If this condition fails, then the question becomes quite complicated.

In the case that system (3.3.1) is weakly linearly degenerate but $B(u)$ does not satisfy the matching condition, the Cauchy problem (3.3.1)-(3.3.2) might admit a unique global C^1 solution on $t \geq 0$; on the other hand, it is also possible that the Cauchy problem (3.3.1)-(3.3.2) does not have any global C^1 solution on $t \geq 0$ even if (3.3.1) is linearly degenerate in the sense of P.D.Lax[1]. For instance, consider the following Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t} = u^2, \\ t = 0:\ u = u_0(x), \end{cases} \tag{3.3.24}$$

where $u_0(x)$ is a C^1 function with bounded C^1 norm. It is easy to see that the first equation in (3.3.24) is linearly degenerate in the sense of P.D.Lax but the inhomogeneous term u^2 does not satisfy the matching condition. We can easily

[1] Up to now, we do not have a systematic result yet even for the case of ordinary differential equations.

prove that the Cauchy problem (3.3.24) admits a unique C^1 solution on $t \geq 0$ if and only if $u_0(x) \leq 0$.

In the case that system (3.3.1) is not weakly linearly degenerate and $B(u)$ does not satisfy the matching condition, we shall prove in §3.7 the following.

Theorem 3.3. Under the assumptions mentioned at the beginning of this section, suppose that $A(u)$ and $B(u)$ are suitably smooth in a neighbourhood of $u = 0$ and $\varphi(x) = \varepsilon\psi(x)$, where $\varepsilon > 0$ is a small parameter and $\psi(x)$ is a C^1 vector function satisfying (3.3.12). Suppose furthermore that system (3.3.1) is not weakly linearly degenerate and $\alpha < \infty$, where α is defined by (3.3.13). Suppose finally that $B(u)$ satisfies

$$B(u) = O\left(|u|^p\right) \tag{3.3.25}$$

in a neighbourhood of $u = 0$, where $p > \alpha + 2$ is an integer. If there exists $i_0 \in J_1$ such that (3.3.15) holds, where J_1 is defined by (3.3.14), then there exists $\varepsilon_0 > 0$ so small that for any fixed $\varepsilon \in (0, \varepsilon_0]$ the first order derivatives of the C^1 solution $u = u(t, x)$ to the Cauchy problem (3.3.1)-(3.3.2) must blow up in a finite time and the life span $\tilde{T}(\varepsilon)$ of $u = u(t, x)$ satisfies (3.3.16). □

Remark 3.6. The condition that $p > \alpha + 2$ is essential. If this condition does not hold, then the Cauchy problem (3.3.1)-(3.3.2) might admit a unique global C^1 solution on $t \geq 0$. Moreover, $\alpha + 2$ is a critical power. □

To illustrate the above fact, we consider the following Cauchy problem

$$\frac{\partial u}{\partial t} + u^{\alpha+1}\frac{\partial u}{\partial x} = -|u|^{p-1}u, \tag{3.3.26}$$

$$t = 0: \quad u = \varepsilon\psi(x), \tag{3.3.27}$$

where $\alpha \geq 0$ and $p \geq 1$ are two integers, $\varepsilon > 0$ is a small parameter and $\psi(x)$ is a nontrivial C^1 function with compact support. It is easy to prove the following proposition by means of a classical argument of the standard characteristic method (see [ZK]).

Proposition 3.1. Suppose that $p < \alpha + 2$. Then there exists $\varepsilon_0 > 0$ so small that for any given $\varepsilon \in [0, \varepsilon_0]$, the Cauchy problem (3.3.26)-(3.3.27) admits a unique global C^1 solution $u = u(t, x)$ on $t \geq 0$.

Suppose that $p = \alpha + 2$. If there exists a point $x_0 \in R$ such that

$$|\psi(x_0)|^{\alpha+1} + (\psi(x_0))^{\alpha}\, \psi'(x_0) < 0, \tag{3.3.28}$$

then there exists $\varepsilon_0 > 0$ so small that for any given $\varepsilon \in (0, \varepsilon_0]$ the first order derivatives of the C^1 solution $u = u(t, x)$ to the Cauchy problem (3.3.26)-(3.3.27) must blow up in a finite time and the life span $\tilde{T}(\varepsilon)$ of $u = u(t, x)$ satisfies

$$\varepsilon^{\alpha+1}\tilde{T}(\varepsilon) = \frac{1}{\sup\limits_{x \in \boldsymbol{R}} \left\{ -(\alpha+1)\left(|\psi(x)|^{\alpha+1} + (\psi(x))^{\alpha}\, \psi'(x) \right) \right\}}. \tag{3.3.29}$$

If

$$|\psi(x)|^{\alpha+1} + (\psi(x))^{\alpha}\, \psi'(x) \geq 0, \quad \forall\, x \in \boldsymbol{R}, \tag{3.3.30}$$

then, for any given $\varepsilon \geq 0$, the Cauchy problem (3.3.26)-(3.3.27) admits a unique global C^1 solution $u = u(t, x)$ on $t \geq 0$.

Suppose that $p > \alpha + 2$. Then there exists $\varepsilon_0 > 0$ so small that for any fixed $\varepsilon \in (0, \varepsilon_0]$ the first order derivatives of the C^1 solution $u = u(t, x)$ to the Cauchy problem (3.3.26)-(3.3.27) must blow up in a finite time and the life span $\tilde{T}(\varepsilon)$ of $u = u(t, x)$ satisfies

$$\lim_{\varepsilon \to 0} \left(\varepsilon^{\alpha+1}\tilde{T}(\varepsilon) \right) = \frac{1}{\sup\limits_{x \in \boldsymbol{R}} \left\{ -(\alpha+1)\, (\psi(x))^{\alpha}\, \psi'(x) \right\}}. \tag{3.3.31}$$

□

Remark 3.7. Proposition 3.1 shows that, in the case that $p \leq \alpha + 2$, the Cauchy problem (3.3.1)-(3.3.2) might admit a unique global C^1 solution on $t \geq 0$ for small initial data with compact support. On the other hand, when $p \leq \alpha + 2$, it is easy to give another kind of examples to show that the Cauchy problem (3.3.1)-(3.3.2) might not have any global C^1 solution on $t \geq 0$ even for small initial data with compact support. For example, consider the following Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t} + u^{\alpha+1}\frac{\partial u}{\partial x} = u^2, \\ t = 0: \quad u = \varepsilon u_0(x), \end{cases} \tag{3.3.32}$$

where $\alpha \geq 0$ is an integer, $\varepsilon > 0$ is a small parameter and $u_0(x) \geq 0$ is a nontrivial C^1 function with compact support. It is easy to show that the Cauchy problem (3.3.32) does not have any global C^1 solution on $t \geq 0$.

Moreover, when $p > \alpha + 2$, (3.3.31) is nothing but (3.3.16) in the present situation. □

Applying Theorem 3.2 and Theorem 3.3, in §3.8 we shall show the following.

Theorem 3.4. Under the hypotheses of Theorem 3.2 or Theorem 3.3, on the existence domain $0 \leq t < \tilde{T}(\varepsilon)$ of the C^1 solution $u = u(t,x)$ to the Cauchy problem (3.3.1)-(3.3.2), the solution itself remains bounded, but the first order derivatives of $u = u(t,x)$ tend to the infinity as $t \nearrow \tilde{T}(\varepsilon)$. Moreover, singularities just occur at the starting point of the envelope of characteristics of the same family, i.e., the point with minimum t-value on the envelope. Particularly, for each $i \notin J_1$, the family of i-th characteristics never forms any envelope on the domain $0 \leq t \leq \tilde{T}(\varepsilon)$. □

Corollary 3.2. Under the hypotheses of Theorem 3.2 or Theorem 3.3, each family of weakly linearly degenerate characteristics and then each family of linearly degenerate characteristics never form any envelope on the domain $0 \leq t \leq \tilde{T}(\varepsilon)$. □

Corollary 3.3. In the case that

$$J_1 = J = \{1, \cdots, n\} \quad \text{and} \quad \alpha < +\infty, \tag{3.3.33}$$

for any nontrivial initial data (3.3.2) (in which $\varphi(x) = \varepsilon\psi(x)$ with $\psi(x) \not\equiv 0$), the conclusion of Theorem 3.2, 3.3 and 3.4 holds. More particularly, when system (3.3.1) is genuinely nonlinear, i.e., $\alpha = 0$ in (3.3.33), we have the same conclusion. □

Some remarks on the critical case will be carried out in §3.9.

We finally point out that our theory is established under the assumption that the initial data (3.3.2) satisfies the decay property (3.3.7). The condition that $\mu > 0$ is essential; otherwise the conclusion may be false (see Appendix 1, also see [K5] for a detailed discussion). By the way, we point out that this condition is also essential in Chapter 4 and Chapter 5.

§3.4. Global existence of C^1 solution — Proof of Theorem 3.1

For simplicity and without loss of generality, we may suppose that

$$0 < \lambda_1(0) < \lambda_2(0) < \cdots < \lambda_n(0). \tag{3.4.1}$$

By the existence and uniqueness of local C^1 solution to the Cauchy problem (see Chap. 1 in [LY]), in order to prove Theorem 3.1 it suffices to establish a uniform *a priori* estimate on the C^0 norm of u and $\frac{\partial u}{\partial x}$ on the existence domain of the C^1 solution $u = u(t,x)$.

By (3.4.1), there exist positive constants δ and δ_0 so small that

$$\lambda_{i+1}(u) - \lambda_i(v) \geq 4\delta_0, \quad \forall\, |u|,\, |v| \leq \delta \quad (i = 1, \cdots, n-1) \tag{3.4.2}$$

and

$$|\lambda_i(u) - \lambda_i(v)| \leq \frac{\delta_0}{2}, \quad \forall\, |u|,\, |v| \leq \delta \quad (i = 1, \cdots, n). \tag{3.4.3}$$

For the time being it is supposed that on the existence domain of the C^1 solution $u = u(t,x)$ we have

$$|u(t,x)| \leq \delta. \tag{3.4.4}$$

At the end of the proof of Lemma 3.4, we shall explain that this hypothesis is reasonable. Thus, to prove Theorem 3.1 we only need to establish a uniform *a priori* estimate on the C^0 norm of v and w defined by (2.2.1)-(2.2.2) on the existence domain of the C^1 solution $u = u(t,x)$.

By (3.4.1) and (3.4.4), on the existence domain of C^1 solution we have

$$0 < \lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u), \tag{3.4.5}$$

provided that δ is suitably small.

For any fixed $T > 0$, let

$$D_-^T = \{(t,x) \,|\, 0 \le t \le T,\ x \le -t\}, \tag{3.4.6}$$

$$D_0^T = \{(t,x) \,|\, 0 \le t \le T,\ -t \le x \le (\lambda_1(0) - \delta_0)\,t\}, \tag{3.4.7}$$

$$D_+^T = \{(t,x) \,|\, 0 \le t \le T,\ x \ge (\lambda_n(0) + \delta_0)\,t\}, \tag{3.4.8}$$

$$D^T = \{(t,x) \,|\, 0 \le t \le T,\ (\lambda_1(0) - \delta_0)\,t \le x \le (\lambda_n(0) + \delta_0)\,t\} \tag{3.4.9}$$

and for $i = 1, \cdots, n$,

$$\begin{aligned} D_i^T \ = \ & \{(t,x) \,|\, 0 \le t \le T, \\ & -[\delta_0 + \eta(\lambda_i(0) - \lambda_1(0))]\,t \le x - \lambda_i(0)\,t \le [\delta_0 + \eta(\lambda_n(0) - \lambda_i(0))]\,t\}, \end{aligned} \tag{3.4.10}$$

where $\eta > 0$ is suitably small, see Figure 2.

Noting that $\eta > 0$ is small, from (3.4.2) we see that

$$D_i^T \cap D_j^T = \emptyset, \quad \forall\, i \ne j \tag{3.4.11}$$

and

$$\bigcup_{i=1}^{n} D_i^T \subset D^T. \tag{3.4.12}$$

Let

$$V(D_\pm^T) = \max_{i=1,\cdots,n} \|\,(1+|x|)^{1+\mu}\, v_i(t,x)\,\|_{L^\infty(D_\pm^T)}, \tag{3.4.13}$$

$$V(D_0^T) = \max_{i=1,\cdots,n} \|\,(1+t)^{1+\mu}\, v_i(t,x)\,\|_{L^\infty(D_0^T)}, \tag{3.4.14}$$

$$W(D_\pm^T) = \max_{i=1,\cdots,n} \|\,(1+|x|)^{1+\mu}\, w_i(t,x)\,\|_{L^\infty(D_\pm^T)}, \tag{3.4.15}$$

$$W(D_0^T) = \max_{i=1,\cdots,n} \|\,(1+t)^{1+\mu}\, w_i(t,x)\,\|_{L^\infty(D_0^T)}, \tag{3.4.16}$$

$$U_\infty^c(T) = \max_{i=1,\cdots,n} \sup_{(t,x)\in D^T \setminus D_i^T} (1 + |x - \lambda_i(0)\,t|)^{1+\mu}\, |u_i(t,x)|, \tag{3.4.17}$$

$$V_\infty^c(T) = \max_{i=1,\cdots,n} \sup_{(t,x)\in D^T \setminus D_i^T} (1 + |x - \lambda_i(0)\,t|)^{1+\mu}\, |v_i(t,x)|, \tag{3.4.18}$$

$$W_\infty^c(T) = \max_{i=1,\cdots,n} \sup_{(t,x)\in D^T \setminus D_i^T} (1 + |x - \lambda_i(0)\,t|)^{1+\mu}\, |w_i(t,x)|, \tag{3.4.19}$$

$$U_1(T) = \max_{i=1,\cdots,n} \sup_{0\le t\le T} \int_{D_i^T(t)} |u_i(t,x)|\,dx, \tag{3.4.20}$$

$$V_1(T) = \max_{i=1,\cdots,n} \sup_{0\le t\le T} \int_{D_i^T(t)} |v_i(t,x)|\,dx, \tag{3.4.21}$$

$$W_1(T) = \max_{i=1,\cdots,n} \sup_{0\le t\le T} \int_{D_i^T(t)} |w_i(t,x)|\,dx, \tag{3.4.22}$$

$$V_\infty(T) = \max_{i=1,\cdots,n} \sup_{\substack{0\le t\le T\\ x\in \mathbf{R}}} |v_i(t,x)| \tag{3.4.23}$$

and

$$W_\infty(T) = \max_{i=1,\cdots,n} \sup_{\substack{0\le t\le T\\ x\in \mathbf{R}}} |w_i(t,x)|, \tag{3.4.24}$$

where $D_i^T(t)$ $(t \ge 0)$ denotes the t-section of D_i^T:

$$D_i^T(t) = \{(\tau,x) \mid \tau = t, (\tau,x) \in D_i^T\}.$$

Obviously, $V_\infty(T)$ is equivalent to

$$U_\infty(T) = \max_{i=1,\cdots,n} \sup_{\substack{0\le t\le T\\ x\in \mathbf{R}}} |u_i(t,x)|. \tag{3.4.25}$$

By the definitions of D_i^T, D^T and $D_\pm^T$, it is easy to get the following.

Lemma 3.2. The following inequalities hold:

$$\begin{cases} ct \le |x - \lambda_i(0)\,t| \le Ct, \\ cx \le |x - \lambda_i(0)\,t| \le Cx \end{cases} \quad (i = 1,\cdots,n), \quad \forall\,(t,x) \in D^T \backslash D_i^T \tag{3.4.26}$$

and

$$0 \le t \le C_0|x|, \quad \forall\,(t,x) \in D_-^T \cup D_+^T, \tag{3.4.27}$$

where c, C and C_0 are positive constants independent of (t,x) and T. □

Lemma 3.3. Suppose that (3.3.3) and (3.3.6) hold, and $A(u), B(u) \in C^2$ in a neighbourhood of $u = 0$. Suppose furthermore that $\phi(x)$ is a C^1 vector function satisfying (3.3.7). There exists $\theta_0 > 0$ so small that for any fixed $\theta \in [0,\theta_0]$, on any given existence domain $0 \le t \le T$ of the C^1 solution $u = u(t,x)$ to the Cauchy problem (3.3.1)-(3.3.2) there exist positive constants k_1 and k_2 independent of θ and T, such that the following uniform a priori estimates hold:

$$V\left(D_{\pm}^{T}\right),\ W\left(D_{\pm}^{T}\right) \le k_1\theta \tag{3.4.28}$$

and

$$V\left(D_{0}^{T}\right),\ W\left(D_{0}^{T}\right) \le k_2\theta. \tag{3.4.29}$$

□

Proof. Noting (3.4.4), from (3.3.7) we see that for $i = 1, \cdots, n$,

$$|v_i(0,x)|,\ |w_i(0,x)| \le C_1\theta(1+|x|)^{-(1+\mu)}, \quad \forall\, x \in \boldsymbol{R}, \tag{3.4.30}$$

henceforth C_j $(j = 1, 2, \cdots)$ will denote positive constants independent of θ and T.

According to the local existence and uniqueness theorem, there exists $\tau_0 > 0$ such that the Cauchy problem (3.3.1)-(3.3.2) admits a unique C^1 solution $u = u(t,x)$ on $0 \le t \le \tau_0$.

Let

$$V(0,\tau) = \max_{i=1,\cdots,n} \|(1+|x|)^{1+\mu} v_i(t,x)\|_{L^\infty(\{0\le t\le\tau\}\times\boldsymbol{R})}, \tag{3.4.31}$$

$$W(0,\tau) = \max_{i=1,\cdots,n} \|(1+|x|)^{1+\mu} w_i(t,x)\|_{L^\infty(\{0\le t\le\tau\}\times\boldsymbol{R})}, \tag{3.4.32}$$

in which $\tau \in [0, \tau_0]$.

For each $i = 1, \cdots, n$, suppose that on the domain $0 \le t \le \tau_0$ the i-th characteristic passing through any given point (t,x) intersects the x-axis at a point $(0,y)$. This characteristic is denoted by $\xi = x_i(s,y)$, where (s,ξ) denote the coordinates of variable point on this characteristic. It satisfies

$$\begin{cases} \frac{dx_i(s,y)}{ds} = \lambda_i\left(u\left(s, x_i(s,y)\right)\right), & 0 \le s \le t \le \tau_0, \\ x_i(t,y) = x. \end{cases} \tag{3.4.33}$$

Noting (3.4.4), from (3.4.33) we get

$$C_2(1+|y|) \le 1 + |x_i(s,y)| \le C_3(1+|y|), \quad \forall\, s \in [0,\tau_0]. \tag{3.4.34}$$

By (2.2.13) and (2.2.24), we have

$$\begin{cases} v_i(t,x) = v_i(0,y) + \displaystyle\int_0^t \left[\sum_{j,k=1}^n \beta_{ijk}(u) v_j w_k + \right. \\ \qquad \left. \displaystyle\sum_{j,k=1}^n \nu_{ijk}(u) v_j b_k(u) + b_i(u) \right] (s, x_i(s,y))\, ds, \quad 0 \le t \le \tau_0, \\ w_i(t,x) = w_i(0,y) + \displaystyle\int_0^t \left[\sum_{j,k=1}^n \gamma_{ijk}(u) w_j w_k + (b_i(u))_x \right] (s, x_i(s,y))\, ds, \\ \qquad\qquad 0 \le t \le \tau_0. \end{cases} \tag{3.4.35}$$

Noting (3.3.6) and using Taylor's formula, by (2.2.3) and (2.2.5)-(2.2.6) we have

$$b_i(u) = \sum_{j,k=1}^n \tilde{b}_{ijk}(u) v_j v_k \quad (i = 1, \cdots, n) \tag{3.4.36}$$

and

$$\begin{aligned} (b_i(u))_x &= (l_i(u))_x B(u) + l_i(u)(B(u))_x \\ &= B^T(u) \nabla l_i^T(u) u_x + l_i(u) \nabla B(u) u_x \\ &= \sum_{j,k=1}^n \bar{b}_{ijk}(u) v_j w_k \quad (i = 1, \cdots, n), \end{aligned} \tag{3.4.37}$$

where $\tilde{b}_{ijk}(u)$ and $\bar{b}_{ijk}(u)$ are continuous functions of u.

Multiplying both sides of (3.4.35) by $(1+|x|)^{1+\mu}$ and noting (3.4.34), (3.4.30), (3.4.36)-(3.4.37) and (3.4.4), we get

$$Z(t) \le C_4 \theta + C_5 \int_0^t Z^2(s)\, ds, \quad \forall\, t \in [0, \tau_0], \tag{3.4.38}$$

where

$$Z(t) = V(0,t) + W(0,t). \tag{3.4.39}$$

Hence, we can choose τ_0 so small that

$$Z(t) \le C_6 \theta, \quad \forall\, t \in [0, \tau_0]. \tag{3.4.40}$$

Thus, if

$$k_1 \ge C_6, \tag{3.4.41}$$

then we get

$$V(0,\tau_0), \ W(0,\tau_0) \le k_1\theta. \tag{3.4.42}$$

Particularly, we have

$$V\left(D_{\pm}^T;0,\tau_0\right), \ W\left(D_{\pm}^T;0,\tau_0\right) \le k_1\theta, \tag{3.4.43}$$

where

$$V\left(D_{\pm}^T;T_1,T_2\right) = \max_{i=1,\cdots,n} \|\left(1+|x|\right)^{1+\mu} v_i(t,x)\|_{L^\infty(D_{\pm}^T\cap\{T_1\le t\le T_2\})} \tag{3.4.44}$$

etc.

We now prove (3.4.28) for D_+^T. The proof of (3.4.28) for D_-^T is similar.

For each $i=1,\cdots,n$, let $\xi = x_i(s,y)$ be the i-th characteristic defined by (3.4.33) as before. Noting (3.4.4) and (3.4.2)-(3.4.3), we see that the whole characteristic $\xi = x_i(s,y)$ $(0\le s\le t)$ is included in D_+^T (see Figure 3, where L_n^0 (resp. L_i^y) stands for the line $x = (\lambda_n(0)+\delta_0)\,t$ (resp. $x = y + \left(\lambda_i(0)+\frac{\delta_0}{2}\right)t$)).

Noting (3.4.4), (3.4.5) and (3.4.3), we get

$$y \le x_i(s,y) \le y + \left(\lambda_i(0) + \frac{\delta_0}{2}\right)s, \quad \forall\, s\in[0,t], \tag{3.4.45}$$

then

$$s \le t \le t_0 \triangleq \frac{1}{\lambda_n(0)-\lambda_i(0)+\frac{\delta_0}{2}}y. \tag{3.4.46}$$

Hence, it follows from (3.4.45) that

$$1+y \le 1+x_i(s,y) \le C_7(1+y) \quad (i=1,\cdots,n), \quad \forall\, s\in[0,t]. \tag{3.4.47}$$

Noting (3.4.43), in order to prove (3.4.28) we only need to show that we can choose k_1 in such a way that for any fixed T_0 $(0<T_0\le T)$ such that

$$V\left(D_+^T;0,T_0\right), \ W\left(D_+^T;0,T_0\right) \le 2k_1\theta, \tag{3.4.48}$$

we have

$$V\left(D_+^T;0,T_0\right), \ W\left(D_+^T;0,T_0\right) \le k_1\theta. \tag{3.4.49}$$

In fact, for $t \le T_0$, by (2.2.13) we have

$$v_i(t,x) = v_i(0,y) + \int_0^t \left[\sum_{j,k=1}^n \beta_{ijk}(u)\, v_j w_k + \sum_{j,k=1}^n \nu_{ijk}(u)\, v_j b_k(u) + b_i(u) \right] (s, x_i(s,y))\, ds. \tag{3.4.50}$$

Noting (3.4.4) and using (3.4.30), (3.4.36), (3.4.48) and (3.4.46), when $\theta_0 > 0$ is suitably small, from (3.4.50) we have

$$\begin{aligned} |v_i(t,x)| &\le C_1\theta(1+y)^{-(1+\mu)} + 4C_8k_1^2\theta^2(1+y)^{-2(1+\mu)}t \\ &\le C_1\theta(1+y)^{-(1+\mu)} + 4C_9k_1^2\theta^2(1+y)^{-1-2\mu} \\ &\le 2C_1\theta(1+y)^{-(1+\mu)}, \end{aligned} \tag{3.4.51}$$

then, noting (3.4.47) we get

$$(1+x)^{1+\mu}|v_i(t,x)| \le 2C_1C_7^{1+\mu}\theta. \tag{3.4.52}$$

Thus, if

$$k_1 \ge 2C_1C_7^{1+\mu}, \tag{3.4.53}$$

then we get the first inequality in (3.4.49). The second inequality in (3.4.49) can be obtained in a similar way. This proves (3.4.28).

It remains to prove (3.4.29).

In order to prove (3.4.29), similar to (3.4.44) we introduce

$$V(D_0^T; T_1, T_2) = \max_{i=1,\cdots,n} \|(1+t)^{1+\mu} v_i(t,x)\|_{C^0(D_0^T \cap \{T_1 \le t \le T_2\})} \tag{3.4.54}$$

etc.

Noting (3.4.40), we can choose τ_0 so small that

$$V(D_0^T; 0, \tau_0),\ W(D_0^T; 0, \tau_0) \le C_{10}\theta. \tag{3.4.55}$$

Thus, if

$$k_2 \ge C_{10}, \tag{3.4.56}$$

then we have

$$V(D_0^T; 0, \tau_0),\ W(D_0^T; 0, \tau_0) \le k_2\theta. \tag{3.4.57}$$

For any given $(t,x) \in D_0^T$, let $\xi = x_i(s,y)$ $(i = 1,\cdots,n)$ be the i-th characteristic defined by (3.4.33) as before. Noting (3.4.4), by (3.4.2)-(3.4.3) we see that the characteristic $\xi = x_i(s,y)$ intersects the line L_-^0 : $\xi + s = 0$ at a point denoted by $P_i \triangleq (\tau_i, -\tau_i)$ and $\xi = x_i(s,y)$ $(\tau_i \le s \le t)$ is included in D_0^T, where (s,ξ) denote the coordinates of variable point on this characteristic and the line L_-^0 (see Figure 4, where L_1^0 stands for the line $x = (\lambda_1(0) - \delta_0)\, t$). Moreover, we have

$$\tau_i^0 \le \tau_i \le t, \tag{3.4.58}$$

where τ_i^0 is the t-coordinate of the intersection point P_i^0 of the line $\xi + s = 0$ with the line $\xi - (\lambda_1(0) - \delta_0)t = \left(\lambda_i(0) - \frac{\delta_0}{2}\right)(s - t)$, where (s,ξ) denote the coordinates of variable point on these lines (see Figure 4). It is easy to see that

$$\tau_i^0 = \frac{\lambda_i(0) - \lambda_1(0) + \frac{\delta_0}{2}}{\lambda_i(0) + 1 - \frac{\delta_0}{2}} t. \tag{3.4.59}$$

Hence, by (3.4.58) we get

$$a_i t \le \tau_i \le t, \tag{3.4.60}$$

where

$$a_i = \frac{\lambda_i(0) - \lambda_1(0) + \frac{\delta_0}{2}}{\lambda_i(0) + 1 - \frac{\delta_0}{2}} > 0.$$

Noting (3.4.57), in order to prove (3.4.29) we only need to show that we can choose k_2 in such a way that for any fixed T_0 $(0 < T_0 \le T)$ such that

$$V\left(D_0^T; 0, \tau_0\right), \ W\left(D_0^T; 0, T_0\right) \le 2k_2\theta, \tag{3.4.61}$$

we have

$$V\left(D_0^T; 0, \tau_0\right), \ W\left(D_0^T; 0, T_0\right) \le k_2\theta. \tag{3.4.62}$$

In fact, for $t \le T_0$, by (2.2.13) we have

$$\begin{aligned} v_i(t,x) \ &= \ v_i(\tau_i, -\tau_i) + \int_{\tau_i}^t \left[\sum_{j,k=1}^n \beta_{ijk}(u)\, v_j w_k + \right. \\ &\left. \sum_{j,k=1}^n \nu_{ijk}(u)\, v_j b_k(u) + b_i(u) \right] (s, x_i(s,y))\, ds. \end{aligned} \tag{3.4.63}$$

Noting (3.4.4) and using (3.4.28), (3.4.36) and (3.4.60)-(3.4.61), when $\theta_0 > 0$ is suitably small, from (3.4.63) we have

$$\begin{aligned} |v_i(t,x)| &\le k_1\theta(1+\tau_i)^{-(1+\mu)} + 4C_{11}k_2^2\theta^2(1+s)^{-2(1+\mu)}(t-\tau_i) \\ &\le k_1\theta(1+\tau_i)^{-(1+\mu)} + C_{12}\theta^2(1+\tau_i)^{-1-2\mu} \\ &\le 2k_1\theta(1+\tau_i)^{-(1+\mu)}, \end{aligned}$$

then, noting (3.4.60) again we get

$$(1+t)^{1+\mu}|v_i(t,x)| \le C_{13}k_1\theta.$$

Thus, if

$$k_2 \ge C_{13}k_1,$$

then we get (3.4.62). The second inequality in (3.4.62) can be obtained in a similar way. This proves (3.4.29). The proof of Lemma 3.3 is completed. Q.E.D.

Remark 3.8. By the definition of D_0^T, for any given $(t,x) \in D_0^T$ we have

$$|x| \le C_{14}t. \tag{3.4.64}$$

Let

$$V\left(D_-^T \cup D_0^T\right) = \max_{i=1,\cdots,n} \|(1+|x|)^{1+\mu} v_i(t,x)\|_{L^\infty(D_-^T \cup D_0^T)}$$

and

$$W\left(D_-^T \cup D_0^T\right) = \max_{i=1,\cdots,n} \|(1+|x|)^{1+\mu} w_i(t,x)\|_{L^\infty(D_-^T \cup D_0^T)}.$$

By (3.4.28)-(3.4.29) and (3.4.64) we get

$$V\left(D_-^T \cup D_0^T\right), \quad W\left(D_-^T \cup D_0^T\right) \le k_3\theta, \tag{3.4.65}$$

where k_3 is a positive constant independent of θ and T. □

Lemma 3.4. Suppose that (3.3.3) and (3.3.6) hold, $A(u), B(u) \in C^2$ in a neighbourhood of $u = 0$, and (3.3.4)-(3.3.5) hold. Suppose furthermore that system (3.3.1) is weakly linearly degenerate and $B(u)$ satisfies the matching condition. Suppose finally that $\varphi(x)$ is a C^1 vector function satisfying (3.3.7). In the normalized coordinates there exists $\theta_0 > 0$ so small that for any fixed $\theta \in [0,\theta_0]$, on any given existence domain $0 \le t \le T$ of the C^1 solution $u = u(t,x)$ to the Cauchy problem (3.3.1)-(3.3.2) there exist positive constants k_i $(i = 4, \cdots, 10)$ independent of θ and T, such that the following uniform *a priori* estimates hold:

$$U_\infty^c(T) \le k_4\theta, \tag{3.4.66}$$

$$V_\infty^c(T) \le k_5\theta, \tag{3.4.67}$$

$$W_\infty^c(T) \le k_6\theta, \tag{3.4.68}$$

$$V_1(T) \le k_7\theta, \tag{3.4.69}$$

$$W_1(T) \le k_8\theta, \tag{3.4.70}$$

$$V_\infty(T) \le k_9\theta \tag{3.4.71}$$

and

$$W_\infty(T) \le k_{10}\theta. \tag{3.4.72}$$

□

The key idea of the proof of Lemma 3.4 is as follows: (I) noting that the system is weakly linearly degenerate and using Hadamard's Lemma and some relations given in Subsection 3.2, we observe that there are only some transversal terms such as v_jv_k, v_jw_k and w_jw_k with $j,k = 1,2,\cdots,n$ and $j \ne k$, but no non-transversal terms such as v_j^2, v_jw_j and w_j^2 in the right-hand sides of (2.2.9), (2.2.19) and (2.2.28)-(2.2.29); (II) integrating (2.2.9), (2.2.19) and (2.2.28)-(2.2.29) along the i-th characteristic, then using the fact mentioned in (I) and noting the notations of the norms introduced at the beginning of this subsection, we may obtain a complete system on the inequalities of those norms; (III) by *continuous induction*, from the complete system we can successfully get the estimates given in Lemma 3.4. In what follows we give the details of the proof.

Proof of Lemma 3.4. We first prove that

$$U_\infty^c(T) \le C_{15}V_\infty^c(T) + C_{16}V_\infty(T)U_\infty^c(T) \tag{3.4.73}$$

and

$$U_1(T) \le C_{17}V_1(T) + C_{18}V_\infty^c(T). \tag{3.4.74}$$

In fact, for any given point $(t,x) \in D^T \backslash D_i^T$, by (2.2.5) we have

$$u_i(t,x) = u^T(t,x)e_i = \sum_{k=1}^{n} v_k r_k^T(u)e_i, \tag{3.4.75}$$

where e_i is defined by (3.1.6). If $(t,x) \notin D_k^T$ $(k = 1, \cdots, n)$, then, noting (3.4.26) and (3.3.5), we get

$$(1 + |x - \lambda_i(0)t|)^{1+\mu} |u_i(t,x)| \le C_{19}V_\infty^c(T). \tag{3.4.76}$$

On the other hand, if there exists some $j(\neq i)$ such that $(t,x) \in D_j^T$, then, noting (3.4.11) we see that $(t,x) \notin D_k^T$ $(k \neq j)$, and using (3.1.5), in the normalized coordinates we can rewrite (3.4.75) as

$$u_i(t,x) = \sum_{k \neq j} v_k r_k^T(u)e_i + v_j \left(r_j^T(u) - r_j^T(u_j e_j)\right) e_i. \tag{3.4.77}$$

By Hadamard's formula, we have

$$r_j(u) - r_j(u_j e_j) = \int_0^1 \sum_{k \neq j} \frac{\partial r_j}{\partial u_k}(su_1, \cdots, su_{j-1}, u_j, su_{j+1}, \cdots, su_n)u_k ds, \tag{3.4.78}$$

then it follows from (3.4.77) that

$$(1 + |x - \lambda_i(0)t|)^{1+\mu} |u_i(t,x)| \le C_{20}V_\infty^c(T) + C_{21}V_\infty(T)U_\infty^c(T). \tag{3.4.79}$$

The combination of (3.4.76) and (3.4.79) leads to (3.4.73).

Noting (3.4.4) and (3.4.26), by (2.2.5) we have

$$\begin{aligned} \int_{D_i^T(t)} |u_i(t,x)|dx &= \int_{D_i^T(t)} \left| \sum_{j=1}^{n} v_j r_{ji}(u) \right| dx \\ &\le C_{22}V_1(T) + C_{23}V_\infty^c(T) \int_0^\infty (1+x)^{-(1+\mu)}dx \\ &\le C_{22}V_1(T) + C_{23}V_\infty^c(T). \end{aligned} \tag{3.4.80}$$

(3.4.80) gives (3.4.74) immediately.

We now estimate

$$\tilde{W}_1(T) = \max_{i=1,\cdots,n} \max_{j\neq i} \sup_{\tilde{C}_j} \int_{\tilde{C}_j} |w_i(t,x)|dt, \tag{3.4.81}$$

where $\tilde{C}_j$ $(j \neq i)$ stands for any given j-th characteristic in D_i^T, see Figure 5.

Let

$$\tilde{C}_j : x = x_j(t) \quad (t_1 \leq t \leq t_2), \tag{3.4.82}$$

where $0 \leq t_1 \leq t_2 \leq T$. By (3.4.3), the whole i-th characteristic passing through $O \triangleq (0,0)$ is included in D_i^T. Let $P_0 \triangleq (t_0, x_j(t_0))$ be the intersection point of this characteristic with $\tilde{C}_j$. Passing through the point $P_1 \triangleq (t_1, x_j(t_1))$ (resp. $P_2 \triangleq (t_2, x_j(t_2))$), we draw the i-th characteristic which intersects $x = (\lambda_1(0) - \delta_0)t$ (resp. $x = (\lambda_n(0) + \delta_0)t$) at $A_1 \triangleq \left(\frac{y_1}{\lambda_1(0)-\delta_0}, y_1\right)$ (resp. $A_2 \triangleq \left(\frac{y_2}{\lambda_n(0)+\delta_0}, y_2\right)$). We have

$$\int_{\tilde{C}_j} |w_i(t,x)|dt = \int_{t_1}^{t_0} |w_i(t,x_j(t))|dt + \int_{t_0}^{t_2} |w_i(t,x_j(t))|dt. \tag{3.4.83}$$

In order to estimate $\int_{t_0}^{t_2} |w_i(t,x_j(t))|dt$, using (2.2.31) on the domain $P_0OA_2P_2$, we get

$$\begin{aligned}
&\int_{t_0}^{t_2} |w_i(t,x_j(t))|\, |\lambda_j(u(t,x_j(t))) - \lambda_i(u(t,x_j(t)))|dt \\
&\leq \int_0^{\frac{y_2}{\lambda_n(0)+\delta_0}} |w_i(t,(\lambda_n(0)+\delta_0)t)|(\lambda_n(0)+\delta_0-\lambda_i(t,(\lambda_n(0)+\delta_0)t))dt \\
&\quad + \iint\limits_{P_0OA_2P_2} \left| \sum_{j,k=1}^{n} \tilde{\gamma}_{ijk}(u)w_jw_k + (b_i(u))_x \right| dtdx.
\end{aligned} \tag{3.4.84}$$

Noting (3.2.12) and using Hadamard's formula, by (3.2.7) we get

$$\begin{aligned}
(b_i(u))_x &= \sum_{k=1}^{n} \left(\tilde{b}_{ik}(u) - \tilde{b}_{ik}(u_ke_k)\right) w_k \\
&= \sum_{\substack{j,k=1 \\ j\neq k}}^{n} \left[\int_0^1 \frac{\partial \tilde{b}_{ik}}{\partial u_j}(su_1,\cdots,su_{k-1},u_k,su_{k+1},\cdots,su_n)ds\right] u_jw_k.
\end{aligned} \tag{3.4.85}$$

Then, noting (3.4.2), (3.4.4), (3.4.28), (2.2.27), (3.4.85) and (3.4.26), from (3.4.84) we obtain

$$\begin{aligned}
&\int_{t_0}^{t_2} |w_i(t, x_j(t))| dt \leq \\
&\quad C_{25}\left\{\theta + (W_\infty^c(T))^2 \int_0^\infty \int_0^T (1+s)^{-(1+\mu)}(1+x)^{-(1+\mu)} ds dx \right. \\
&\qquad + W_\infty^c(T) W_1(T) \int_0^T (1+s)^{-(1+\mu)} ds \\
&\qquad + U_\infty^c(T) W_\infty^c(T) \int_0^\infty \int_0^T (1+s)^{-(1+\mu)}(1+x)^{-(1+\mu)} ds dx \\
&\qquad + W_\infty^c(T) U_1(T) \int_0^T (1+s)^{-(1+\mu)} ds \\
&\qquad \left. + U_\infty^c(T) W_1(T) \int_0^T (1+s)^{-(1+\mu)} ds \right\} \leq \\
&\quad C_{26}\left\{\theta + (W_\infty^c(T))^2 + W_\infty^c(T) W_1(T) + U_\infty^c(T) W_\infty^c(T) \right. \\
&\qquad \left. + W_\infty^c(T) U_1(T) + U_\infty^c(T) W_1(T)\right\}.
\end{aligned} \tag{3.4.86}$$

Hence, using (3.4.74) we have

$$\begin{aligned}
\int_{t_0}^{t_2} |w_i(t, x_j(t))| dt \leq\ & C_{27}\left\{\theta + (W_\infty^c(T))^2 + W_\infty^c(T) W_1(T) + U_\infty^c(T) W_\infty^c(T) \right. \\
& \left. + W_\infty^c(T) V_\infty^c(T) + W_\infty^c(T) V_1(T) + U_\infty^c(T) W_1(T)\right\}.
\end{aligned} \tag{3.4.87}$$

In a similar manner, we can estimate $\int_{t_1}^{t_0} |w_i(t, x_j(t))| dt$, where we use (3.4.29) instead of (3.4.28). Thus we get

$$\begin{aligned}
\tilde{W}_1(T) \leq\ & C_{28}\left\{\theta + (W_\infty^c(T))^2 + W_\infty^c(T) W_1(T) + U_\infty^c(T) W_\infty^c(T) \right. \\
& \left. + W_\infty^c(T) V_\infty^c(T) + W_\infty^c(T) V_1(T) + U_\infty^c(T) W_1(T)\right\}.
\end{aligned} \tag{3.4.88}$$

Similar to (3.4.81), let

$$\tilde{V}_1(T) = \max_{i=1,\cdots,n} \max_{j \neq i} \sup_{\tilde{C}_j} \int_{\tilde{C}_j} |v_i(t, x)| dt. \tag{3.4.89}$$

Similarly, in order to estimate $\tilde{V}_1(T)$, it suffices to estimate $\int_{t_1}^{t_0} |v_i(t, x_j(t))| dt$ and $\int_{t_0}^{t_2} |v_i(t, x_j(t))| dt$.

Using (2.2.30) on the domain $P_0OA_2P_2$ (see Figure 5) and noting (3.2.4)-(3.2.5) and (3.2.3), we get

$$\begin{aligned}
\int_{t_0}^{t_2} & |v_i(t, x_j(t))| \, |\lambda_j(u(t,x_j(t))) - \lambda_i(u(t,x_j(t)))| dt \\
& \le \int_0^{\frac{y_2}{\lambda_n(0)+\delta_0}} |v_i(t, (\lambda_n(0)+\delta_0)t)| (\lambda_n(0)+\delta_0 - \lambda_i(t, (\lambda_n(0)+\delta_0)t)) dt \\
& + \iint_{P_0OA_2P_2} \left| \sum_{j \neq k} \tilde{\beta}_{ijk}(u) v_j w_k \right| dtdx \\
& + \iint_{P_0OA_2P_2} \left| \sum_{j=1}^{n} \left(\tilde{\beta}_{ijj}(u) - \tilde{\beta}_{ijj}(u_j e_j) \right) v_j w_j \right| dtdx \\
& + \iint_{P_0OA_2P_2} \left| \sum_{\substack{j,k,l,m=1 \\ l \neq m}}^{n} \nu_{ijk}(u) b_{klm}(u) v_j u_l u_m \right| dtdx \\
& + \iint_{P_0OA_2P_2} \left| \sum_{j \neq k} b_{ijk}(u) u_j u_k \right| dtdx.
\end{aligned} \tag{3.4.90}$$

By Hadamard's formula, we have

$$\tilde{\beta}_{ijj}(u) - \tilde{\beta}_{ijj}(u_j e_j) = \int_0^1 \sum_{l \neq j} \frac{\partial \tilde{\beta}_{ijj}}{\partial u_l} (\tau u_1, \cdots, \tau u_{j-1}, u_j, \tau u_{j+1}, \cdots, \tau u_n) \, u_l d\tau. \tag{3.4.91}$$

Then, noting (3.4.2), (3.4.4), (3.4.28), (3.4.74) and (3.4.26), similar to (3.4.87), from (3.4.90) we obtain

$$\begin{aligned}
\int_{t_0}^{t_2} |v_i(t, x_j(t))| dt \le\ & C_{29} \{\theta + V_\infty^c(T) W_\infty^c(T) + V_1(T) W_\infty^c(T) \\
& + V_\infty^c(T) W_1(T) + U_\infty^c(T) W_1(T) + U_1(T) W_\infty^c(T) \\
& + U_\infty^c(T) W_\infty^c(T) + (U_\infty^c(T))^2 + U_1(T) U_\infty^c(T) \Big\} \\
\le\ & C_{30} \{\theta + V_\infty^c(T) W_\infty^c(T) + V_1(T) W_\infty^c(T) \\
& + V_\infty^c(T) W_1(T) + U_\infty^c(T) W_1(T) + U_\infty^c(T) W_\infty^c(T) \\
& + (U_\infty^c(T))^2 + U_\infty^c(T) V_1(T) + U_\infty^c(T) V_\infty^c(T) \Big\}.
\end{aligned} \tag{3.4.92}$$

In a similar way, we can estimate $\int_{t_1}^{t_0} |v_i(t, x_j(t))| dt$. Hence we get

$$\begin{aligned}\tilde{V}_1(T) \le\ & C_{31}\{\theta + V_\infty^c(T)W_\infty^c(T) + V_1(T)W_\infty^c(T) \\ & + V_\infty^c(T)W_1(T) + U_\infty^c(T)W_1(T) + U_\infty^c(T)W_\infty^c(T) \\ & + (U_\infty^c(T))^2 + U_\infty^c(T)V_1(T) + U_\infty^c(T)V_\infty^c(T)\Big\}.\end{aligned} \tag{3.4.93}$$

We next estimate $W_\infty^c(T)$.

For any given point $(t,x) \in D^T$ but $(t,x) \notin D_i^T$, by the definition of D_i^T, for fixing the idea we may suppose that

$$x - \lambda_i(0)t > [\delta_0 + \eta(\lambda_n(0) - \lambda_i(0))]\, t, \tag{3.4.94}$$

which implies $i < n$. Let $\xi = \tilde{x}_i(s;t,x)$ be the i-th characteristic passing through (t,x), which intersects the boundary $x = (\lambda_n(0) + \delta_0)t$ of D^T at a point (t_0, y), see Figure 6. It follows from (3.4.3) that

$$x - \left(\lambda_i(0) + \frac{\delta_0}{2}\right) t \le y - \left(\lambda_i(0) + \frac{\delta_0}{2}\right) t_0. \tag{3.4.95}$$

Since

$$y = (\lambda_n(0) + \delta_0)\, t_0, \tag{3.4.96}$$

noting (3.4.94) and the fact that $t \ge t_0$, from (3.4.95) we get

$$t \ge t_0 \ge \eta t. \tag{3.4.97}$$

Noting (2.2.24), we have

$$w_i(t,x) = w_i(t_0,y) + \int_{t_0}^{t_2} \left[\sum_{j,k=1}^{n} \gamma_{ijk}(u) w_j w_k + (b_i(u))_x\right] (s, x_i(s;t,x)) ds. \tag{3.4.98}$$

Using Lemma 3.3 and noting (3.4.96)-(3.4.97), we obtain

$$\begin{aligned}|w_i(t_0,y)| &\le k_1\theta(1+y)^{-(1+\mu)} \le C_{32}\theta(1+t_0)^{-(1+\mu)} \\ &\le C_{33}\theta(1+t)^{-(1+\mu)}.\end{aligned} \tag{3.4.99}$$

Similar to (3.4.81), let

$$\tilde{U}_1(T) = \max_{i=1,\cdots,n} \max_{j \ne i} \sup_{\tilde{C}_j} \int_{\tilde{C}_j} |u_i(t,x)| dt. \tag{3.4.100}$$

Similar to (3.4.74), it is easy to show that

$$\tilde{U}_1(T) \leq C_{34}\tilde{V}_1(T) + C_{35}V_\infty^c(T). \tag{3.4.101}$$

Noting (2.2.21), (3.4.85), (3.4.97), (3.4.99) and (3.4.26), from (3.4.98) we have

$$\begin{aligned} W_\infty^c(T) \leq\ & C_{36}\Big\{\theta + (W_\infty^c(T))^2 + W_\infty^c(T)\tilde{W}_1(T) + U_\infty^c(T)W_\infty^c(T) \\ & +\tilde{U}_1(T)W_\infty^c(T) + U_\infty^c(T)\tilde{W}_1(T)\Big\}. \end{aligned} \tag{3.4.102}$$

Substituting (3.4.101) into the right-hand side of (3.4.102) gives

$$\begin{aligned} W_\infty^c(T) \leq\ & C_{37}\Big\{\theta + (W_\infty^c(T))^2 + W_\infty^c(T)\tilde{W}_1(T) + U_\infty^c(T)W_\infty^c(T) \\ & +\tilde{V}_1(T)W_\infty^c(T) + V_\infty^c(T)W_\infty^c(T) + U_\infty^c(T)\tilde{W}_1(T)\Big\}. \end{aligned} \tag{3.4.103}$$

In a similar manner, we can show

$$\begin{aligned} V_\infty^c(T) \leq\ & C_{38}\Big\{\theta + V_\infty^c(T)W_\infty^c(T) + V_\infty^c(T)\tilde{W}_1(T) + \tilde{V}_1(T)W_\infty^c(T) \\ & +U_\infty^c(T)W_\infty^c(T) + U_\infty^c(T)\tilde{W}_1(T) + \tilde{V}_1(T)W_\infty^c(T) \\ & + (U_\infty^c(T))^2 + U_\infty^c(T)\tilde{V}_1(T) + U_\infty^c(T)V_\infty^c(T)\Big\}. \end{aligned} \tag{3.4.104}$$

Moreover, similar to (3.4.88) and (3.4.93), we can prove

$$\begin{aligned} W_1(T) \leq\ & C_{39}\Big\{\theta + (W_\infty^c(T))^2 + W_\infty^c(T)W_1(T) + U_\infty^c(T)W_\infty^c(T) \\ & +W_\infty^c(T)V_\infty^c(T) + W_\infty^c(T)V_1(T) + U_\infty^c(T)W_1(T)\} \end{aligned} \tag{3.4.105}$$

and

$$\begin{aligned} V_1(T) \leq\ & C_{40}\{\theta + V_\infty^c(T)W_\infty^c(T) + V_1(T)W_\infty^c(T) \\ & +V_\infty^c(T)W_1(T) + U_\infty^c(T)W_1(T) + U_\infty^c(T)W_\infty^c(T) \\ & + (U_\infty^c(T))^2 + U_\infty^c(T)V_1(T) + U_\infty^c(T)V_\infty^c(T)\Big\}. \end{aligned} \tag{3.4.106}$$

We finally estimate $U_\infty(T), V_\infty(T)$ and $W_\infty(T)$.

For any given point $(t,x) \in D^T$, we have

$$u(t,x) = u(t,x_0) + \int_{x_0}^{x} u_\xi(t,\xi)d\xi, \tag{3.4.107}$$

where (t, x_0) is located on the left boundary of D^T. Noting (2.2.6), (3.4.4) and (3.4.29), we obtain

$$U_\infty(T) \le C_{41}\left\{\theta + W^c_\infty(T) + W_1(T)\right\}. \tag{3.4.108}$$

By the equivalence of $U_\infty(T)$ and $V_\infty(T)$, we get immediately

$$V_\infty(T) \le C_{42}\left\{\theta + W^c_\infty(T) + W_1(T)\right\}. \tag{3.4.109}$$

We now estimate $W_\infty(T)$.

The i-th characteristic $\xi = x_i(s; t, x)$ passing through any given point $(t, x) \in D_i^T$ intersects one of the boundaries of D^T at one point. For fixing the idea, suppose that this characteristic intersects $x = (\lambda_n(0)+\delta_0)t$ at a point $\left(\frac{y}{\lambda_n(0)+\delta_0}, y\right)$. Noting (2.2.21), (3.2.6)-(3.2.7) and (3.2.12), in normalized coordinates, by (2.2.24) we have

$$\begin{aligned} w_i(t,x) &= w_i\left(\tfrac{y}{\lambda_n(0)+\delta_0}, y\right) + \int_{\frac{y}{\lambda_n(0)+\delta_0}}^t \sum_{j\neq k}\gamma_{ijk}(u)\, w_j w_k(s, x_i(s;t,x))ds \\ &+ \int_{\frac{y}{\lambda_n(0)+\delta_0}}^t \left(\gamma_{iii}(u) - \gamma_{iii}(u_i e_i)\right) w_i^2(s, x_i(s;t,x))ds \\ &+ \int_{\frac{y}{\lambda_n(0)+\delta_0}}^t \sum_{k=1}^n \left(\tilde{b}_{ik}(u) - \tilde{b}_{ik}(u_k e_k)\right) w_k(s, x_i(s;t,x))ds. \end{aligned} \tag{3.4.110}$$

Using Hadamard's formula and noting (3.4.26) and (3.4.28), by (3.4.110) we get

$$\begin{aligned} |w_i(t,x)| \le\ & C_{43}\Big\{\theta + (W^c_\infty(T))^2 + W^c_\infty(T)\, W_\infty(T) \\ & + U_\infty(T)\,(W^c_\infty(T))^2 + U^c_\infty(T)\,(W_\infty(T))^2 + U^c_\infty(T)\, W^c_\infty(T) \\ & + U_\infty(T)\, W^c_\infty(T) + U^c_\infty(T) W_\infty(T)\Big\}. \end{aligned} \tag{3.4.111}$$

On the other hand, for any given point $(t, x) \notin D_i^T$ $(i = 1, \cdots, n)$, $|w_i(t, x)|$ can be controlled by $W^c_\infty(T)$ or $W\left(D_\pm^T\right)$ or $W\left(D_0^T\right)$. Thus, using (3.4.4) and Lemma 3.3, we have

$$\begin{aligned} W_\infty(T) \le\ & C_{44}\Big\{\theta + W^c_\infty(T) + (W^c_\infty(T))^2 + W^c_\infty(T)\, W_\infty(T) \\ & + U_\infty(T)\,(W^c_\infty(T))^2 + U^c_\infty(T)\,(W_\infty(T))^2 + U^c_\infty(T)\, W^c_\infty(T) \\ & + U_\infty(T)\, W^c_\infty(T) + U^c_\infty(T) W_\infty(T)\Big\}. \end{aligned} \tag{3.4.112}$$

We now prove (3.4.66)-(3.4.71),

$$\tilde{W}_1(T) \leq k_8\theta \tag{3.4.113}$$

and

$$\tilde{V}_1(T) \leq k_7\theta. \tag{3.4.114}$$

Noting (3.4.30), evidently we have

$$U_\infty^c(0),\ V_\infty^c(0),\ W_\infty^c(0),\ V_\infty(0) \leq C_{45}\theta \tag{3.4.115}$$

and

$$V_1(0) = W_1(0) = \tilde{W}_1(0) = \tilde{V}_1(0) = 0. \tag{3.4.116}$$

Hence, by continuity there exist positive constants k_i $(i = 4, \cdots, 9)$ independent of θ and T, such that (3.4.66)-(3.4.71) and (3.4.113)-(3.4.114) hold at least for $0 \leq T \leq \tau_0$, where τ_0 is a small positive number. Thus, in order to prove (3.4.66)-(3.4.71) and (3.4.113)-(3.4.114) it suffices to show that we can choose k_i $(i = 4, \cdots, 9)$ in such a way that for any fixed T_0 $(0 \leq T_0 \leq T)$ such that

$$U_\infty^c(T_0) \leq 2k_4\theta, \tag{3.4.117}$$

$$V_\infty^c(T_0) \leq 2k_5\theta, \tag{3.4.118}$$

$$W_\infty^c(T_0) \leq 2k_6\theta, \tag{3.4.119}$$

$$V_1(T_0) \leq 2k_7\theta, \tag{3.4.120}$$

$$W_1(T_0) \leq 2k_8\theta, \tag{3.4.121}$$

$$\tilde{W}_1(T_0) \leq 2k_8\theta, \tag{3.4.122}$$

$$\tilde{V}_1(T_0) \leq 2k_7\theta, \tag{3.4.123}$$

$$V_\infty(T_0) \leq 2k_9\theta, \tag{3.4.124}$$

we have

$$U_\infty^c(T_0) \leq k_4\theta, \tag{3.4.125}$$

$$V_\infty^c(T_0) \leq k_5\theta, \tag{3.4.126}$$

$$W^c_\infty(T_0) \le k_6\theta, \tag{3.4.127}$$

$$V_1(T_0) \le k_7\theta, \tag{3.4.128}$$

$$W_1(T_0) \le k_8\theta, \tag{3.4.129}$$

$$\tilde{W}_1(T_0) \le k_8\theta, \tag{3.4.130}$$

$$\tilde{V}_1(T_0) \le k_7\theta, \tag{3.4.131}$$

$$V_\infty(T_0) \le k_9\theta. \tag{3.4.132}$$

Substituting (3.4.117)-(3.4.124) into the right-hand sides of (3.4.73), (3.4.88), (3.4.93), (3.4.103)-(3.4.106) and (3.4.109) (in which we take $T = T_0$) yields that, when $\theta_0 > 0$ is suitably small, we have

$$U^c_\infty(T_0) \le 3C_{15}k_5\theta, \tag{3.4.133}$$

$$\tilde{W}_1(T_0) \le 2C_{28}\theta, \tag{3.4.134}$$

$$\tilde{V}_1(T_0) \le 2C_{31}\theta, \tag{3.4.135}$$

$$W^c_\infty(T_0) \le 2C_{37}\theta, \tag{3.4.136}$$

$$V^c_\infty(T_0) \le 2C_{38}\theta, \tag{3.4.137}$$

$$W_1(T_0) \le 2C_{39}\theta, \tag{3.4.138}$$

$$V_1(T_0) \le 2C_{40}\theta, \tag{3.4.139}$$

$$V_\infty(T_0) \le C_{42}\left\{1 + 2k_6 + 2k_8\right\}\theta. \tag{3.4.140}$$

Hence, if

$$\begin{cases} k_4 \ge 3C_{15}k_5, \ k_5 \ge 2C_{38}, \ k_6 \ge 2C_{37}, \\ k_7 \ge 2\max\left\{C_{31}, C_{40}\right\}, \ k_8 \ge 2\max\left\{C_{28}, C_{39}\right\}, \\ k_9 \ge C_{42}\left\{1 + 2k_6 + 2k_8\right\}, \end{cases} \tag{3.4.141}$$

then we get (3.4.125)-(3.4.132). Thus, we proves (3.4.66)-(3.4.71) and (3.4.113)-(3.4.114).

We now estimate $U_\infty(T)$.

Using (3.4.68) and (3.4.70), from (3.4.108) we have

$$U_\infty(T) \le C_{41}\{1 + k_6 + k_8\}\,\theta. \tag{3.4.142}$$

By the way, we point out that when $\theta_0 > 0$ is suitably small

$$\begin{aligned} U_\infty(T) &\le C_{41}\{1 + k_6 + k_8\}\,\theta \\ &\le C_{41}\{k_1 + k_6 + k_8\}\,\theta_0 \le \tfrac{1}{2}\delta. \end{aligned} \tag{3.4.143}$$

This implies the validity of hypothesis (3.4.4).

Finally, we prove (3.4.72).

Noting (3.4.66), (3.4.68) and (3.4.142), from (3.4.112) we get

$$W_\infty(T) \le C_{46}\left\{1 + W_\infty(T) + (W_\infty(T))^2\right\}\theta. \tag{3.4.144}$$

It follows from (3.4.42) that if $k_{10} \ge k_1$, then there exists $\tau_0 > 0$ such that

$$W_\infty(\tau_0) \le k_{10}\theta. \tag{3.4.145}$$

Hence, in order to prove (3.4.72) it suffices to show that we can choose k_{10} in such a way that for any fixed τ_1 $(0 < \tau_1 < T)$ such that

$$W_\infty(\tau_1) \le 2k_{10}\theta, \tag{3.4.146}$$

we have

$$W_\infty(\tau_1) \le k_{10}\theta. \tag{3.4.147}$$

Substituting (3.4.146) into the right-hand side of (3.4.144) (in which we take $T = \tau_1$) gives that if

$$k_{10} \ge 2C_{46},$$

then we get (3.4.147) and then (3.4.72). This completes the proof of Lemma 3.4. Q.E.D.

Proof of Theorem 3.1. It suffices to prove Theorem 3.1 in the normalized coordinates. Under the assumptions of Theorem 3.1, Lemma 3.4 holds. By (3.4.71) and (3.4.72) we know that if $\theta_0 > 0$ is suitably small, then for any fixed $\theta \in [0, \theta_0]$, on any given existence domain $0 \leq t \leq T$ of the C^1 solution $u = u(t,x)$ to the Cauchy problem (3.3.1)-(3.3.2), we have the following uniform *a priori* estimate on the C^1 norm of the solution

$$\|u(t,\cdot)\|_{C^1} \triangleq \|u(t,\cdot)\|_{C_0} + \|u_x(t,\cdot)\|_{C^0} \leq K_1\theta, \tag{3.4.148}$$

where K_1 is a positive constant independent of θ and T. Then we immediately get the conclusion of Theorem 3.1. Q.E.D.

§3.5. Large time behaviour of the global classical solution

We first recall the definitions of D_-^T, D_0^T, D_+^T, D^T and D_i^T (see (3.4.6)-(3.4.10)). Taking $T = +\infty$ in (3.4.6)-(3.4.10), in the present situation we denote them by $D_-^\infty, D_0^\infty, D_+^\infty, D^\infty$ and D_i^∞ respectively. From the proof of Theorem 3.1, we observe

Lemma 3.5. Under the assumptions of Theorem 3.1, if $\theta > 0$ is suitably small, then in the normalized coordinates

$$|u(t,x)|,\ |u_x(t,x)| \leq \tilde{K}_1\theta(1+t)^{-(1+\mu)}, \quad \forall\, (t,x) \in D_c^\infty, \tag{3.5.1}$$

henceforth $\tilde{K}_j$ $(j = 1, 2, \cdots)$ will denote positive constants independent of t, x and θ, and where D_c^∞ is defined by

$$D_c^\infty = D_-^\infty \cup D_0^\infty \cup D_+^\infty \cup \left(D^\infty \setminus \bigcup_{i=1}^{n} D_i^\infty \right). \tag{3.5.2}$$

□

Proof. In fact, noting (3.4.27), from (3.4.28) we have

$$(1+t)^{1+\mu}\left(|v_i(t,x)| + |w_i(t,x)|\right) \leq \tilde{K}_2\theta \quad (i = 1, \cdots, n), \quad \forall\, (t,x) \in D_-^\infty \cup D_+^\infty. \tag{3.5.3}$$

Hence, noting (2.2.5)-(2.2.6) and employing (3.4.29), (3.4.67)-(3.4.68) and (3.5.3), we obtain (3.5.1) immediately. The proof is completed. Q.E.D.

By §3.1, there exists a suitable normalized transformation $u = u(\tilde{u})$ $(u(0) = 0)$ such that (3.1.13) holds. In the normalized coordinates $\tilde{u}$, let

$$\tilde{v}_i = \tilde{l}_i(\tilde{u})\tilde{u} \quad (i = 1, \cdots, n), \tag{3.5.4}$$

$$\tilde{w}_i = \tilde{l}_i(\tilde{u})\tilde{u}_x \quad (i = 1, \cdots, n) \tag{3.5.5}$$

and

$$\tilde{\lambda}_i(\tilde{u}) = \lambda_i(u) \quad (i = 1, \cdots, n). \tag{3.5.6}$$

Lemma 3.6. Under the hypotheses of Theorem 3.1, if $\theta > 0$ is suitably small, then in the normalized coordinates $\tilde{u}$, for any $y \in \boldsymbol{R}$

$$|\tilde{v}_i(t, x_i(t, y)) - l_i(0)\varphi(y)| \leq \overline{K}_1\theta^2 \quad (i = 1, \cdots, n), \quad \forall\, t \geq 0 \tag{3.5.7}$$

and

$$|\tilde{w}_i(t, x_i(t, y)) - l_i(0)\varphi'(y)| \leq \overline{K}_2\theta^2 \quad (i = 1, \cdots, n), \quad \forall\, t \geq 0, \tag{3.5.8}$$

where $\xi = x_i(t, y)$ stands for the i-th characteristic passing through the point $(0, y)$ and henceforth $\overline{K}_j$ $(j = 1, 2, \cdots)$ will denote positive constants independent of t, y and θ. □

Proof. By §3.1, in the normalized coordinates $\tilde{u}$, the initial data (3.3.2) reduces to

$$t = 0: \quad \tilde{u} = \Phi(x) \tag{3.5.9}$$

with

$$|\Phi(x) - L(0)\varphi(x)|, \quad \left|\frac{\partial \Phi(x)}{\partial x} - L(0)\varphi'(x)\right| \leq \tilde{K}_3\theta^2, \tag{3.5.10}$$

where $L(0)$ is given by (3.1.13). Thus, we have

$$|\tilde{v}_i(0, y) - l_i(0)\varphi(y)| \leq \overline{K}_3\theta^2 \quad (i = 1, \cdots, n), \quad \forall\, y \in \boldsymbol{R} \tag{3.5.11}$$

and

$$|\tilde{w}_i(0, y) - l_i(0)\varphi'(y)| \leq \overline{K}_4\theta^2 \quad (i = 1, \cdots, n), \quad \forall\, y \in \boldsymbol{R}. \tag{3.5.12}$$

Hence, in order to prove (3.5.7) and (3.5.8) it suffices to show

$$|\tilde{v}_i(t, x_i(t,y)) - \tilde{v}_i(0,y)| \le \overline{K}_5 \theta^2 \quad (i = 1, \cdots, n), \quad \forall\, y \in \mathbf{R} \tag{3.5.13}$$

and

$$|\tilde{w}_i(t, x_i(t,y)) - \tilde{w}_i(0,y)| \le \overline{K}_6 \theta^2 \quad (i = 1, \cdots, n), \quad \forall\, y \in \mathbf{R}. \tag{3.5.14}$$

We only prove (3.5.14). The proof of (3.5.13) is similar.

Similar to (3.4.111), we have[2]

$$\begin{aligned} |\tilde{w}_i(t, x_i(t,y)) - \tilde{w}_i(0,y)| \le\ & \overline{K}_7 \left[\left(V(D^t_\pm) + V(D^t_0)\right)\left(W(D^t_\pm) + W(D^t_0)\right) \right. \\ & + \left(W(D^t_\pm) + W(D^t_0)\right)^2 + \left(W^c_\infty(t)\right)^2 \\ & + W^c_\infty(t) W_\infty(t) + U_\infty(t)\left(W^c_\infty(t)\right)^2 \\ & + U^c_\infty(t)\left(W_\infty(t)\right)^2 + U^c_\infty(t) W^c_\infty(t) \\ & \left. + U_\infty(t) W^c_\infty(t) + U^c_\infty(t) W_\infty(t) \right]. \end{aligned} \tag{3.5.15}$$

Making use of Lemma 3.3 and Lemma 3.4, from (3.5.15) we get (3.5.14) immediately. The proof is finished. Q.E.D.

Lemma 3.7. Under the hypotheses of Theorem 3.1, if $\theta > 0$ is suitably small, then in the normalized coordinates $\tilde{u}$, for any $y \in \mathbf{R}$

$$|\tilde{u}_i(t, x_i(t,y)) - l_i(0)\varphi(y)| \le \overline{K}_8 \theta^2 \quad (i = 1, \cdots, n), \quad \forall\, t \ge 0 \tag{3.5.16}$$

and

$$\left| \frac{\partial \tilde{u}_i}{\partial x}(t, x_i(t,y)) - l_i(0)\varphi'(y) \right| \le \overline{K}_9 \theta^2 \quad (i = 1, \cdots, n), \quad \forall\, t \ge 0. \tag{3.5.17}$$

□

Proof. We only show (3.5.17). The proof of (3.5.16) is similar.

Noting that in the normalized coordinates

$$\frac{\partial \tilde{u}_i}{\partial x} - \tilde{w}_i = \sum_{j=1}^{n} \tilde{w}_j \left(r_j(\tilde{u}) - r_j(\tilde{u}_j e_j)\right) e_i \tag{3.5.18}$$

[2]Here $V(D^t_\pm), V(D^t_0)$, etc. are defined by (3.4.13)-(3.4.17),(3.4.19) and (3.4.24)-(3.4.25) (in which $T = t$) in the normalized coordinates $\tilde{u}$.

and employing (3.4.71)-(3.4.72), we obtain

$$\left|\frac{\partial \tilde{u}_i}{\partial x}(t,x) - \tilde{w}_i(t,x)\right| = \tilde{K}_4\theta^2. \tag{3.5.19}$$

Thus, the combination of (3.5.8) and (3.5.19) gives (3.5.17) immediately. The proof is completed. Q.E.D.

Introduce the following Cauchy problem for a linear system

$$\frac{\partial \overline{u}_i}{\partial t} + \tilde{\lambda}_i(0)\frac{\partial \overline{u}_i}{\partial x} = 0 \quad (i = 1, \cdots, n), \tag{3.5.20}$$

$$t = 0 : \overline{u}_i = l_i(0)\varphi(x) \quad (i = 1, \cdots, n). \tag{3.5.21}$$

The solution to the Cauchy problem (3.5.20)-(3.5.21) is given by

$$\overline{u}_i(t,x) = l_i(0)\varphi(x - \tilde{\lambda}_i(0)t) \quad (i = 1, \cdots, n). \tag{3.5.22}$$

Theorem 3.5. Under the assumptions of Theorem 3.1, suppose that $\varphi(x) \in C^2$ and

$$|\varphi''(x)| \leq \tilde{K}_5\theta, \quad \forall\, x \in \boldsymbol{R}. \tag{3.5.23}$$

If $\theta > 0$ is suitably small, then in the normalized coordinates $\tilde{u}$, the C^1 solution $\tilde{u} = \tilde{u}(t,x)$ to the Cauchy problem (3.3.1)-(3.3.2) exists globally on $t \geq 0$; moreover, the following estimates hold:

$$|\tilde{u}_i(t,x) - \overline{u}_i(t,x)| \leq \tilde{K}_6\theta^2 \quad (i = 1, \cdots, n), \quad \forall\, (t,x) \in \boldsymbol{R}^+ \times \boldsymbol{R}, \tag{3.5.24}$$

$$\left|\frac{\partial \tilde{u}_i}{\partial x} - \frac{\partial \overline{u}}{\partial x}(t,x)\right| \leq \tilde{K}_7\theta^2 \quad (i = 1, \cdots, n), \quad \forall\, (t,x) \in \boldsymbol{R}^+ \times \boldsymbol{R}, \tag{3.5.25}$$

particularly, on the domain D_c^∞ (see (3.5.2))

$$|\tilde{u}(t,x)|,\ |\tilde{u}_x(t,x)| \leq \tilde{K}_8\theta(1+t)^{-(1+\mu)}, \tag{3.5.26}$$

where $\overline{u} = \overline{u}(t,x)$ is defined by (3.5.22). □

Proof. (3.5.26) follows from (3.5.1) directly.

We only show (3.5.24). The proof of (3.5.25) is similar.

Let $\xi_i = \xi_i(\tau; t, x)$ be the i-th characteristic passing through the point (t, x). Denote the space coordinate of the intersection point of this characteristic with x-axis by y. On the other hand, let

$$\overline{y} = x - \tilde{\lambda}_i(0)t. \tag{3.5.27}$$

Then

$$|\tilde{u}_i(t,x) - \overline{u}_i(t,x)| = |\tilde{u}_i(t,x) - l_i(0)\varphi(y)| + |l_i(0)\varphi(y) - l_i(0)\varphi(\overline{y})|. \tag{3.5.28}$$

Noting (3.5.16) and (3.3.7), from (3.5.28) we get

$$|\tilde{u}_i(t,x) - \overline{u}_i(t,x)| \leq \tilde{K}_9\theta^2 + \tilde{K}_{10}\theta|y - \overline{y}|. \tag{3.5.29}$$

We now estimate $|y - \overline{y}|$.

By the definitions of y and $\overline{y}$, we have

$$|y - \overline{y}| = \left| \int_0^t \left[\tilde{\lambda}_i\left(\tilde{u}(\tau, \xi_i(\tau; t, x))\right) - \tilde{\lambda}_i(0) \right] d\tau \right|. \tag{3.5.30}$$

Noting the fact that $\tilde{u}$ is the normalized coordinates and system (3.3.1) is weakly linearly degenerate, we have

$$\tilde{\lambda}_i(\tilde{u}_i e_i) \equiv \tilde{\lambda}_i(0) \quad (i = 1, \cdots, n), \quad \forall\, |\tilde{u}_i| \text{ small}. \tag{3.5.31}$$

Making use of Hadamard's formula, we get

$$\tilde{\lambda}_i(\tilde{u}) - \tilde{\lambda}_i(0) = \sum_{j \neq i} \left[\int_0^1 \frac{\partial \tilde{\lambda}_i}{\partial \tilde{u}_j}(s\tilde{u}_1, \cdots, s\tilde{u}_{i-1}, \tilde{u}_i, s\tilde{u}_{i+1}, \cdots, s\tilde{u}_n) ds \right] \tilde{u}_j. \tag{3.5.32}$$

Then, noting (3.5.30) and (3.5.32), and using (3.4.101), we obtain

$$\begin{aligned} |y - \overline{y}| &\leq \tilde{K}_{11} \left[V\left(D_{\pm}^t\right) + V(D_0^t) + U_\infty^c(t) + \tilde{U}_1(t) \right] \\ &\leq \tilde{K}_{12} \left[V\left(D_{\pm}^t\right) + V(D_0^t) + U_\infty^c(t) + \tilde{V}_1(t) + V_\infty^c(t) \right], \end{aligned} \tag{3.5.33}$$

where $V\left(D_{\pm}^t\right), V(D_0^t)$, etc. are defined as in (3.5.15). Noting (3.4.28)-(3.4.29), (3.4.66)-(3.4.67) and (3.4.114), from (3.5.33) we get easily

$$|y - \overline{y}| \leq \tilde{K}_{13}\theta. \tag{3.5.34}$$

Substituting (3.5.34) into (3.5.29) leads to (3.5.24). Thus, the proof is finished. Q.E.D.

Remark 3.9. We do not require (3.5.23) in the proof of (3.5.24), however we need it in the proof of (3.5.25). □

§3.6. Blow-up phenomenon and life span of C^1 solution (I) — Proof of Theorem 3.2

In order to precisely estimate the life span of C^1 solution, we consider the following Cauchy problem

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = B(u), \tag{3.6.1}$$

$$t = 0: \ u = \varepsilon\psi(x), \tag{3.6.2}$$

where $\varepsilon > 0$ is a small parameter and $\psi(x)$ is a C^1 vector fundtion satisfying

$$\sup_{x\in \boldsymbol{R}}\left\{(1+|x|)^{(1+\mu)}(|\psi(x)|+|\psi'(x)|)\right\} < \infty \quad (\mu > 0, \text{ constant}). \tag{3.6.3}$$

In the present situation, Lemma 3.3 is still valid and can be stated as the following.

Lemma 3.8. Suppose that (3.3.3) and (3.3.6) hold, and $A(u), B(u) \in C^2$ in a neighbourhood of $u = 0$. There exists $\varepsilon_0 > 0$ so small that for any fixed $\varepsilon \in [0, \varepsilon_0]$, on any given existence domain $0 \le t \le T$ of the C^1 solution $u = u(t,x)$ to the Cauchy problem (3.6.1)-(3.6.2) there exists a positive constant k_1 independent of ε and T, such that the following uniform *a priori* estimates hold:

$$V\left(D_0^T\right), \ V\left(D_\pm^T\right), \ W\left(D_0^T\right), \ W\left(D_\pm^T\right) \le k_1\varepsilon. \tag{3.6.4}$$

□

Lemma 3.9. Under the assumptions of Theorem 3.2, in the normalized coordinates there exists $\varepsilon_0 > 0$ so small that for any fixed $\varepsilon \in (0, \varepsilon_0]$, on any given existence domain $0 \le t \le T$ of the C^1 solution $u = u(t,x)$ to the Cauchy problem (3.6.1)-(3.6.2) there exist positive constants k_i $(i = 2, \cdots, 10)$ independent of ε and T, such that the following uniform *a priori* estimates hold:

$$U_\infty^c(T) \le k_2 \varepsilon, \tag{3.6.5}$$

$$V_\infty^c(T) \le k_3 \varepsilon, \tag{3.6.6}$$

$$W_\infty^c(T) \le k_4 \varepsilon, \tag{3.6.7}$$

$$W_1(T) \le k_5 \varepsilon, \tag{3.6.8}$$

$$V_1(T) \le k_6 \varepsilon + k_7 \varepsilon^{2+\alpha} T, \tag{3.6.9}$$

$$V_\infty(T) \le k_8 \varepsilon, \tag{3.6.10}$$

where

$$T \varepsilon^{\frac{3}{2}+\alpha} \le 1. \tag{3.6.11}$$

Moreover,

$$W_\infty(T) \le k_9 \varepsilon, \tag{3.6.12}$$

where

$$T \varepsilon^{1+\alpha} \le k_{10}. \tag{3.6.13}$$

□

Proof. This lemma will be proved in a way similar to the proof of Lemma 3.4. In what follows we only point out the essentially different part in the proof and $\varepsilon_0 > 0$ is always supposed to be suitably small.

For $i \notin J$, we can estimate (3.4.90) just as in the proof of Lemma 3.4; while, for $i \in J$, noting (3.2.4), instead of (3.4.90) we have

$$\begin{aligned}
&\int_{t_0}^{t_2} |v_i\,(t, x_j(t))\,|\;|\lambda_j(u(t, x_j(t))) - \lambda_i(u(t, x_j(t)))|dt \\
&\leq \int_0^{\frac{y_2}{\lambda_n(0)+\delta_0}} |v_i(t, (\lambda_n(0) + \delta_0)t)|(\lambda_n(0) + \delta_0 - \lambda_i(t, (\lambda_n(0) + \delta_0)t))dt \\
&+ \iint\limits_{P_0OA_2P_2} \left|\sum_{j\neq k}\tilde{\beta}_{ijk}(u)v_jw_k\right| dtdx \\
&+ \iint\limits_{P_0OA_2P_2} \left|\sum_{j=1}^{n}\left(\tilde{\beta}_{ijj}(u) - \tilde{\beta}_{ijj}(u_je_j)\right) v_jw_j\right| dtdx \\
&+ \iint\limits_{P_0OA_2P_2} \left|\sum_{\substack{j,k,l,m=1\\ l\neq m}}^{n} \nu_{ijk}(u)b_{klm}(u)v_ju_lu_m\right| dtdx \\
&+ \iint\limits_{P_0OA_2P_2} \left|\sum_{j\neq k} b_{ijk}(u)u_ju_k\right| dtdx + \iint\limits_{P_0OA_2P_2} \left|\tilde{\beta}_{iii}(u_ie_i)v_iw_i\right| dtdx,
\end{aligned} \tag{3.6.14}$$

hence, we only need to estimate the last term of the right-hand side of (3.6.14). Noting the fact that in the normalized coordinates

$$\tilde{\beta}_{iii}\,(u_ie_i) = \frac{\partial\lambda_i\,(0, \cdots, 0, u_i, 0, \cdots, 0)}{\partial u_i} \tag{3.6.15}$$

and the difinition of α_i and α, we have

$$|\tilde{\beta}_{iii}\,(u_ie_i)\,| \leq C_1|u_i|^\alpha, \tag{3.6.16}$$

henceforth C_j $(j = 1, 2, \cdots)$ will denote positive constants independent of ε and T. Thus, noting (3.4.26) we get

$$\iint\limits_{P_0OA_2P_2} \left|\tilde{\beta}_{iii}(u_ie_i)v_iw_i\right| dtdx \leq C_2\,(V_\infty\,(T))^{1+\alpha}\,(W_\infty^c\,(T) + W_1\,(T))\,T, \tag{3.6.17}$$

then, noting Lemma 3.8, instead of (3.4.93) we have

$$\begin{aligned}\tilde{V}_1(T) \le\ & C_3\{\varepsilon + V^c_\infty(T)W^c_\infty(T) + V_1(T)W^c_\infty(T) \\ & + V^c_\infty(T)W_1(T) + U^c_\infty(T)W_1(T) + U^c_\infty(T)W^c_\infty(T) \\ & + (U^c_\infty(T))^2 + U^c_\infty(T)V_1(T) + U^c_\infty(T)V^c_\infty(T) \\ & + (V_\infty(T))^{1+\alpha}(W^c_\infty(T) + W_1(T))T\Big\}.\end{aligned} \tag{3.6.18}$$

Similar to (3.4.106), we obtain

$$\begin{aligned}V_1(T) \le\ & C_4\{\varepsilon + V^c_\infty(T)W^c_\infty(T) + V_1(T)W^c_\infty(T) \\ & + V^c_\infty(T)W_1(T) + U^c_\infty(T)W_1(T) + U^c_\infty(T)W^c_\infty(T) \\ & + (U^c_\infty(T))^2 + U^c_\infty(T)V_1(T) + U^c_\infty(T)V^c_\infty(T) \\ & + (V_\infty(T))^{1+\alpha}(W^c_\infty(T) + W_1(T))T\Big\}.\end{aligned} \tag{3.6.19}$$

Moreover, similar to (3.4.73), (3.4.88), (3.4.103)-(3.4.105) and (3.4.109), we have

$$U^c_\infty(T) \le C_5 V^c_\infty(T) + C_6 V_\infty(T)U^c_\infty(T), \tag{3.6.20}$$

$$\begin{aligned}\tilde{W}_1(T) \le\ & C_7\Big\{\varepsilon + (W^c_\infty(T))^2 + W^c_\infty(T)W_1(T) + U^c_\infty(T)W^c_\infty(T) \\ & + W^c_\infty(T)V^c_\infty(T) + W^c_\infty(T)V_1(T) + U^c_\infty(T)W_1(T)\},\end{aligned} \tag{3.6.21}$$

$$\begin{aligned}W^c_\infty(T) \le\ & C_9\Big\{\varepsilon + (W^c_\infty(T))^2 + W^c_\infty(T)\tilde{W}_1(T) + U^c_\infty(T)W^c_\infty(T) \\ & + \tilde{V}_1(T)W^c_\infty(T) + V^c_\infty(T)W^c_\infty(T) + U^c_\infty(T)\tilde{W}_1(T)\Big\},\end{aligned} \tag{3.6.22}$$

$$\begin{aligned}V^c_\infty(T) \le\ & C_{10}\Big\{\varepsilon + V^c_\infty(T)W^c_\infty(T) + V^c_\infty(T)\tilde{W}_1(T) + \tilde{V}_1(T)W^c_\infty(T) \\ & + U^c_\infty(T)W^c_\infty(T) + U^c_\infty(T)\tilde{W}_1(T) + \tilde{V}_1(T)W^c_\infty(T) \\ & + (U^c_\infty(T))^2 + U^c_\infty(T)\tilde{V}_1(T) + U^c_\infty(T)V^c_\infty(T)\Big\},\end{aligned} \tag{3.6.23}$$

$$\begin{aligned}W_1(T) \le\ & C_8\Big\{\varepsilon + (W^c_\infty(T))^2 + W^c_\infty(T)W_1(T) + U^c_\infty(T)W^c_\infty(T) \\ & + W^c_\infty(T)V^c_\infty(T) + W^c_\infty(T)V_1(T) + U^c_\infty(T)W_1(T)\},\end{aligned} \tag{3.6.24}$$

$$V_\infty(T) \le C_{11}\{\varepsilon + W^c_\infty(T) + W_1(T)\}. \tag{3.6.25}$$

It follows from (3.6.3) that

$$U^c_\infty(0),\ V^c_\infty(0),\ W^c_\infty(0),\ V_\infty(0) \le C_{12}\varepsilon \tag{3.6.26}$$

and

$$W_1(0) = \tilde{W}_1(0) = V_1(0) = \tilde{V}_1(0) = 0. \tag{3.6.27}$$

Then, by continuity there exist constants k_i $(i = 2, \cdots, 8)$ independent of ε and T, such that (3.6.5)-(3.6.10),

$$\tilde{W}_1(T) \leq k_5\varepsilon \tag{3.6.28}$$

and

$$\tilde{V}_1(T) \leq k_6\varepsilon + k_7\varepsilon^{2+\alpha}T \tag{3.6.29}$$

hold at least for $0 \leq T \leq \tau_0$, where τ_0 is a small positive number.

Thus, in order to prove (3.6.5)-(3.6.10) it suffices to show that we can choose k_i $(i = 2, \cdots, 8)$ in such a way that for any fixed T_0 $(0 < T_0 \leq T)$ with $T_0\varepsilon^{\frac{3}{2}+\alpha} \leq 1$ such that

$$U^c_\infty(T_0) \leq 2k_2\varepsilon, \tag{3.6.30}$$

$$V^c_\infty(T_0) \leq 2k_3\varepsilon, \tag{3.6.31}$$

$$W^c_\infty(T_0) \leq 2k_4\varepsilon, \tag{3.6.32}$$

$$W_1(T_0),\ \tilde{W}_1(T_0) \leq 2k_5\varepsilon, \tag{3.6.33}$$

$$V_1(T_0),\ \tilde{V}_1(T_0) \leq 2k_6\varepsilon + 2k_7\varepsilon^{2+\alpha}T_0, \tag{3.6.34}$$

$$V_\infty(T_0) \leq 2k_8\varepsilon, \tag{3.6.35}$$

we have

$$U^c_\infty(T_0) \leq k_2\varepsilon, \tag{3.6.36}$$

$$V^c_\infty(T_0) \leq k_3\varepsilon, \tag{3.6.37}$$

$$W^c_\infty(T_0) \leq k_4\varepsilon, \tag{3.6.38}$$

$$W_1(T_0),\ \tilde{W}_1(T_0) \leq k_5\varepsilon, \tag{3.6.39}$$

$$V_1(T_0),\ \tilde{V}_1(T_0) \leq k_6\varepsilon + k_7\varepsilon^{2+\alpha}T_0, \tag{3.6.40}$$

$$V_\infty(T_0) \leq k_8\varepsilon. \tag{3.6.41}$$

Substituting (3.6.30)-(3.6.35) into the right-hand sides of (3.6.18)-(3.6.25) (in which we take $T = T_0$) and noting the fact that $T_0\varepsilon^{\frac{3}{2}+\alpha} \leq 1$, we get

$$U_\infty^c(T_0) \leq 3C_5k_3\varepsilon, \tag{3.6.42}$$

$$\tilde{W}_1(T_0) \leq 2C_7\varepsilon, \tag{3.6.43}$$

$$W_1(T_0) \leq 2C_8\varepsilon, \tag{3.6.44}$$

$$W_\infty^c(T_0) \leq 2C_9\varepsilon, \tag{3.6.45}$$

$$V_\infty^c(T_0) \leq 2C_{10}\varepsilon, \tag{3.6.46}$$

$$V_\infty(T_0) \leq C_{11}\{1 + 2k_4 + 2k_5\}\varepsilon, \tag{3.6.47}$$

$$\tilde{V}_1(T_0) \leq C_3\left\{2\varepsilon + 2^{2+\alpha}(k_8)^{\alpha+1}(k_4 + k_5)\varepsilon^{2+\alpha}T_0\right\} \tag{3.6.48}$$

and

$$V_1(T_0) \leq C_4\left\{2\varepsilon + 2^{2+\alpha}(k_8)^{\alpha+1}(k_4 + k_5)\varepsilon^{2+\alpha}T_0\right\}. \tag{3.6.49}$$

Hence, if

$$\begin{cases} k_2 \geq 3C_5k_3, \quad k_3 \geq 2C_{10}, \quad k_4 \geq 2C_9, \quad k_5 \geq 2\max\{C_7, C_8\}, \\ k_6 \geq 2\max\{C_3, C_4\}, \quad k_7 \geq 2^{2+\alpha}\max\{C_3, C_4\}(k_8)^{\alpha+1}(k_4 + k_5), \\ k_8 \geq C_{11}\{1 + 2(k_4 + k_5)\}, \end{cases} \tag{3.6.50}$$

then we have (3.6.36)-(3.6.41). This proves (3.6.5)-(3.6.10).

Finally, we prove (3.6.12).

For any given point $(t, x) \in D_i^T$, similar to (3.4.110), it follows that

$$\begin{aligned} w_i(t,x) &= w_i\left(\tfrac{y}{\lambda_n(0)+\delta_0}, y\right) + \int_{\frac{y}{\lambda_n(0)+\delta_0}}^t \sum_{j\neq k}\gamma_{ijk}(u)\, w_j w_k(s, x_i(s;t,x))ds \\ &+ \int_{\frac{y}{\lambda_n(0)+\delta_0}}^t (\gamma_{iii}(u) - \gamma_{iii}(u_ie_i))\, w_i^2(s, x_i(s;t,x))ds \\ &+ \int_{\frac{y}{\lambda_n(0)+\delta_0}}^t \sum_{k=1}^n \left(\tilde{b}_{ik}(u) - \tilde{b}_{ik}(u_ke_k)\right) w_k(s, x_i(s;t,x))ds \\ &+ \int_{\frac{y}{\lambda_n(0)+\delta_0}}^t \gamma_{iii}(u_ie_i)\, w_i^2(s, x_i(s;t,x))ds. \end{aligned} \tag{3.6.51}$$

Noting that in the normalized coordinates

$$\gamma_{iii}(u_i e_i) = -\frac{\partial \lambda_i(0,\cdots,0,u_i,0,\cdots,0)}{\partial u_i}, \tag{3.6.52}$$

similar to (3.6.16), we have

$$|\gamma_{iii}(u_i e_i)| \le C_{13}|u_i|^{\alpha}. \tag{3.6.53}$$

Hence, similar to (3.4.111), it follows from (3.6.51) that

$$\begin{aligned} |w_i(t,x)| \le\ & C_{14}\{\varepsilon + (W_\infty^c(T))^2 + W_\infty^c(T)W_\infty(T) \\ & + U_\infty(T)(W_\infty^c(T))^2 + U_\infty^c(T)(W_\infty(T))^2 \\ & + U_\infty^c(T)W_\infty^c(T) + U_\infty(T)W_\infty^c(T) \\ & + U_\infty^c(T)W_\infty(T) + (V_\infty(T))^{\alpha}(W_\infty(T))^2 T\Big\}. \end{aligned} \tag{3.6.54}$$

Then, substituting (3.6.5), (3.6.7) and (3.6.10) into (3.6.54) and noting (3.6.4) and (3.6.7), similar to (3.4.144), we get

$$W_\infty(T) \le C_{15}\left\{\varepsilon\left(1 + W_\infty(T) + (W_\infty(T))^2\right) + \varepsilon^{\alpha}T(W_\infty(T))^2\right\}, \tag{3.6.55}$$

where T satisfies (3.6.11).

Noting (3.4.42), we see that there exists a positive constant k_9 independent of ε and T, such that (3.6.12) holds at least for $0 \le T \le \tau_0$, where τ_0 is a small positive number. Hence, in order to prove (3.6.12) it suffices to show that we can choose k_9 and k_{10} in such a way that for any fixed T_0 $(0 < T_0 \le T)$ with $T_0\varepsilon^{1+\alpha} \le k_{10}$ such that

$$W_\infty(T_0) \le 2k_9\varepsilon, \tag{3.6.56}$$

we have

$$W_\infty(T_0) \le k_9\varepsilon. \tag{3.6.57}$$

For this purpose, substituting (3.6.56) into the right-hand side of (3.6.55) (in which we take $T = T_0$)[3], we obtain

$$W_\infty(T_0) \le C_{16}\left\{1 + k_9^2 k_{10}\right\}\varepsilon.$$

[3]Noting (3.6.11), we observe that (3.6.55) always holds for the case that $T_0\varepsilon^{1+\alpha} \le k_{10}$, provided that $\varepsilon_0 > 0$ is small enough.

Hence, if

$$k_9 \geq 2C_{16} \quad \text{and} \quad k_9^2 k_{10} = 1,$$

then we have (3.6.57). This proves (3.6.12). The proof of Lemma 3.9 is finished. Q.E.D.

Remark 3.10. By (3.6.10) and (3.6.12), when $\varepsilon_0 > 0$ is suitably small, the Cauchy problem (3.6.1)-(3.6.2) admits a unique C^1 solution $u = u(t, x)$ on $0 \leq t \leq T$, where T satisfies (3.6.13). Hence, we get the following lower bound on the life span of the C^1 solution

$$\tilde{T}(\varepsilon) \geq K_* \varepsilon^{-(1+\alpha)},$$

where K_* $(= k_{10})$ is a positive constant independent of ε. □

Proof of Theorem 3.2. By Remark 3.3, in order to prove Theorem 3.2 it suffices to show that, in the normalized coordinates $\tilde{u}$, (3.3.19) holds, where the corresponding normalized transformation $u = u(\tilde{u})$ ($u(0) = 0$)) satisfies (3.1.13). In what follows, we will consider the problem in the normalized coordinates $\tilde{u}$ mentioned above. For the sake of the simplicity of statement, we still denote the normalized coordinates $\tilde{u}$ and the functions $\tilde{\lambda}_i(\tilde{u})$ (see (3.3.17)) by u and $\lambda_i(u)$ respectively. Thus, by Remark 3.1 and Remark 3.2, (3.3.19) can be rewritten as

$$\lim_{\varepsilon \to 0} \left(\varepsilon^{\alpha+1} \tilde{T}(\varepsilon) \right) = M_0, \tag{3.6.58}$$

where

$$\begin{aligned} M_0 &= \left\{ \max_{i \in J_1} \sup_{x \in \mathbf{R}} \left\{ -\frac{1}{\alpha!} \frac{\partial^{1+\alpha} \lambda_i}{\partial u_i^{1+\alpha}}(0) \left(l_i(0) \psi(x) \right)^{\alpha} l_i(0) \psi_i'(x) \right\} \right\}^{-1} \\ &= \left\{ \max_{i=1,\cdots,n} \sup_{x \in \mathbf{R}} \left\{ -\frac{1}{\alpha!} \frac{\partial^{1+\alpha} \lambda_i}{\partial u_i^{1+\alpha}}(0) \left(l_i(0) \psi_i(x) \right)^{\alpha} l_i(0) \psi_i'(x) \right\} \right\}^{-1} > 0. \end{aligned} \tag{3.6.59}$$

Hence we only need to show (3.6.58).

In order to prove (3.6.58) it suffices to show that

(I) for any fixed constant M^* satisfying $M^* > M_0$, there exists a constant $\varepsilon_0 > 0$ so small that $\tilde{T}(\varepsilon) \leq M^* \varepsilon^{-(\alpha+1)}$ for any $\varepsilon \in (0,\ \varepsilon_0]$, namely,

$$\overline{\lim_{\varepsilon \to 0}} \left(\varepsilon^{\alpha+1} \tilde{T}(\varepsilon) \right) \leq M_0;$$

(II) for any fixed constant M_* satisfying $0 < M_* < M_0$, there exists a constant $\varepsilon_0 > 0$ so small that $\tilde{T}(\varepsilon) \geq M_* \varepsilon^{-(\alpha+1)}$ for any $\varepsilon \in (0,\ \varepsilon_0]$, namely,

$$\underline{\lim}_{\varepsilon \to 0} \left(\varepsilon^{\alpha+1} \tilde{T}(\varepsilon) \right) \geq M_0.$$

Thus, in the proof it is sufficient to discuss solutions defined in the strip $0 \leq t \leq M\varepsilon^{-(\alpha+1)}$ for some fixed M.

In what follows, we always suppose that $\varepsilon_0 > 0$ is suitably small.

In the present situation, the initial data (3.3.2) reduces to

$$t = 0: \quad u = \varepsilon \Psi(x, \varepsilon), \tag{3.6.60}$$

where

$$\Psi(x, \varepsilon) = L(0)\psi(x) + O(\varepsilon) \quad \text{and} \quad \frac{\partial \Psi(x, \varepsilon)}{\partial x} = L(0)\psi'(x) + O(\varepsilon), \tag{3.6.61}$$

in which $L(0)$ is the same as the one in (3.1.13). Noting (2.2.1) and the fact that the present variables u are the normalized coordinates, from (3.6.60)-(3.6.61) we get

$$t = 0: \quad v_i = \varepsilon l_i(0)\psi(x) + O(\varepsilon^2) \quad (i = 1, \cdots, n). \tag{3.6.62}$$

On the other hand, noting (2.2.6) we have

$$\frac{\partial u_i}{\partial x}(0, x) - w_i(0, x) = \sum_{k=1}^{n} w_k(0, x) \left(r_k(u(0, x)) - r_k(u_k(0, x)e_k) \right)^T e_i. \tag{3.6.63}$$

Then it follows from (3.6.61) that

$$|w_i(0, x) - \varepsilon l_i(0)\psi'(x)| \leq C_{17}\varepsilon^2 \quad (i = 1, \cdots, n). \tag{3.6.64}$$

Let

$$T_0 = \varepsilon^{-(\frac{1}{2}+\alpha)}, \quad T_* = K_* \varepsilon^{-(1+\alpha)} \quad \text{and} \quad T^* = M^* \varepsilon^{-(1+\alpha)}, \tag{3.6.65}$$

where K_* is the positive constant given in Remark 3.10 and M^* is an arbitrary fixed constant satisfying that $M^* > M_0$. It is easy to see that

$$T_0 < T^* < \varepsilon^{\alpha+\frac{3}{2}} \quad \text{and} \quad T_0 < T_* < \varepsilon^{\alpha+\frac{3}{2}}, \tag{3.6.66}$$

provided that $\varepsilon > 0$ is suitably small.

By Lemma 3.9, on the existence domain $0 \le t \le T$ $(\le T^*)$ of the C^1 solution $u = u(t,x)$, we have

$$|u(t,x)|\,,\ |v(t,x)| \le C_{18}\varepsilon. \tag{3.6.67}$$

On the other hand, by Remark 3.10, the C^1 solution $u = u(t,x)$ to the Cauchy problem (3.6.1)-(3.6.2) exists at least on the strip $0 \le t \le T_*$; moreover, by (3.6.12) we have

$$|w(t,x)|\,,\ \left|\frac{\partial u}{\partial x}(t,x)\right| \le C_{19}\varepsilon, \quad \forall\, t \in [0, T_*]. \tag{3.6.68}$$

Let $x = x_i(t,y)$ $(i = 1,\cdots,n)$ be the i-th characteristic passing through an arbitrary given point $(0,y)$.

We now estimate $|u_i(t,x_i(t,y)) - \varepsilon l_i(0)\psi(y)|$ $(i = 1,\cdots,n)$ on the existence domain $0 \le t \le T$ $(\le T^*)$ of the C^1 solution $u = u(t,x)$. To do so, we first estimate $|v_i(t,x_i(t,y)) - v_i(0,y)|$. By (2.2.13) we get

$$\begin{aligned}
&|v_i\,(t,x_i\,(t,y)) - v_i\,(0,y)\,| \le \\
&\left|\int_0^t \left[\sum_{k\neq i}\beta_{ijk}\,(u)\,v_j w_k + \sum_{j,k=1}^n \nu_{ijk}\,(u)\,v_j b_k(u) + b_i(u)\right](s,x_i\,(s,y))\,ds\right| \le \\
&\left|\int_0^{t_y} \left[\sum_{k\neq i}\beta_{ijk}\,(u)\,v_j w_k + \sum_{j,k=1}^n \nu_{ijk}\,(u)\,v_j b_k(u) + b_i(u)\right](s,x_i\,(s,y))\,ds\right| \\
&+\left|\int_{t_y}^{\tilde t_y} \left[\sum_{k\neq i}\beta_{ijk}\,(u)\,v_j w_k + \sum_{j,k=1}^n \nu_{ijk}\,(u)\,v_j b_k(u) + b_i(u)\right](s,x_i\,(s,y))\,ds\right| \\
&+\left|\int_{\tilde t_y}^{t} \left[\sum_{k\neq i}\beta_{ijk}\,(u)\,v_j w_k + \sum_{j,k=1}^n \nu_{ijk}\,(u)\,v_j b_k(u) + b_i(u)\right](s,x_i\,(s,y))\,ds\right| \\
&\triangleq I + II + III,
\end{aligned} \tag{3.6.69}$$

where t_y (resp. $\tilde t_y$) stands for the t-coordinate of the intersection point of the characteristic $x = x_i(t,y)$ with the boundary of D^T (resp. D_i^T). It is possible that the characteristic $x = x_i(s,y)$ does not intersect the boundary of D^T (resp. D_i^T), in this case, we take $t_y = \tilde t_y = t$ (resp. $\tilde t_y = t$). Noting (3.4.3), we see that

$$(s,\ x_i(s,y)) \in D_-^T \cup D_0^T \cup D_+^T \quad \forall\, t \in [0,\ t_y]\,, \tag{3.6.70}$$

$$(s,\, x_i(s,y)) \in D^T \setminus D_i^T \quad \forall\, t \in [t_y,\, \tilde{t}_y] \tag{3.6.71}$$

$$(s,\, x_i(s,y)) \in D_i^T \quad \forall\, t \in [\tilde{t}_y,\, t]\,. \tag{3.6.72}$$

Noting (3.2.3), (3.6.70), (3.4.27) and Lemma 3.8, we have

$$\begin{aligned} |I| &\le C_{20} \left(V\left(D_\pm^T\right) + V\left(D_0^T\right)\right)\left(V\left(D_\pm^T\right) + V\left(D_0^T\right) + W\left(D_\pm^T\right) + W\left(D_0^T\right)\right) \\ &\le C_{21}\varepsilon^2, \end{aligned} \tag{3.6.73}$$

henceforth C_j $(j = 20, 21, \cdots)$ will denote the positive constants independent of ε, y and T. Noting (2.2.12) and (3.2.1), we get

$$\begin{aligned} |II| = \int_{t_y}^{\tilde{t}_y} &\Bigg[\sum_{j\neq k}\beta_{ijk}(u)v_j w_k + \sum_{j\neq k}\left(\beta_{ijj}(u) - \beta_{ijj}(u_j e_j)\right)v_j w_j \\ &+ \sum_{j,k=1}^{n} \nu_{ijk}\,(u)\, v_j b_k(u) + b_i(u)\Bigg]\, (s, x_i\,(s,y))\, ds. \end{aligned} \tag{3.6.74}$$

Applying Hadamard's formula to $\beta_{ijj}(u) - \beta_{ijj}(u_j e_j)$ and noting (3.2.3), (3.6.71) and (3.4.26), from (3.6.74), we obtain

$$\begin{aligned} |II| \le\ & C_{22}\Big\{V_\infty^c(T)W_\infty^c(T) + \tilde{V}_1(T)W_\infty^c(T) + V_\infty^c(T)\tilde{W}_1(T) \\ &+ V_\infty(T)\left(U_\infty^c(T)W_\infty^c(T) + \tilde{U}_1(T)W_\infty^c(T) + U_\infty^c(T)\tilde{W}_1(T)\right) \\ &+ (1 + V_\infty(T))\left(\tilde{U}_1(T)U_\infty^c(T) + (U_\infty^c(T))\right)\Big\}. \end{aligned} \tag{3.6.75}$$

Noting the fact that $T \le T^* \triangleq M^*\varepsilon^{-(1+\alpha)}$ and using (3.4.101), (3.6.5)-(3.6.7), (3.6.10) and (3.6.28)-(3.6.29), from (3.6.75) we get immediately

$$|II| \le C_{23}\varepsilon^2. \tag{3.6.76}$$

Similarly, we have

$$\begin{aligned} |III| &\le C_{24}\left\{V_\infty(T)W_\infty^c(T) + (1 + V_\infty(T))\left(\tilde{U}_1(T)U_\infty^c(T) + (U_\infty^c(T))^2\right)\right\} \\ &\le C_{25}\varepsilon^2. \end{aligned} \tag{3.6.77}$$

Thus, from (3.6.69), (3.6.73), (3.6.76) and (3.6.77) we obtain

$$|v_i\,(t, x_i\,(t,y)) - v_i\,(0,y)\,| \le C_{26}\varepsilon^2, \quad \forall\, t \in [0,T]\,, \tag{3.6.78}$$

where T satisfies that $T \leq T^*$.

Noting that in the normalized coordinates

$$u_i - v_i = \sum_{k=1}^{n} v_k \left(r_k(u) - r_k(u_k e_k)\right)^T e_i \tag{3.6.79}$$

and using (3.6.67), on the existence domain $0 \leq t \leq T$ $(\leq T^*)$ of the C^1 solution we have

$$|u_i(t,x) - v_i(t,x)| \leq C_{27}\varepsilon^2. \tag{3.6.80}$$

In particular, it follows from (3.6.62) that

$$|v_i(0,y) - \varepsilon l_i(0)\psi(y)| \leq C_{28}\varepsilon^2. \tag{3.6.81}$$

Hence, noting (3.6.78) and (3.6.80)-(3.6.81), on the existence domain $0 \leq t \leq T$ $(\leq T^*)$ of the C^1 solution we obtain

$$|u_i(t, x_i(t,y)) - \varepsilon l_i(0)\psi(y)| \leq C_{29}\varepsilon^2 \quad (i = 1, \cdots, n). \tag{3.6.82}$$

We next estimate $|w_i(t, x_i(t,y)) - w_i(0,y)|$ $(i = 1, \cdots, n)$ on the strip $0 \leq t \leq T_*$.

Similar to (3.6.69), noting (2.2.21), (3.2.7) and (3.2.12), from (2.2.24) we have

$$\begin{aligned}
w_i(t, x_i(t,y)) - w_i(0,y) &= \int_0^t \left[\sum_{j,k=1}^{n} \gamma_{ijk}(u) w_j w_k + (b_i(u))_x\right] L_i^y(s)ds \\
&= \int_0^{t_y} \left[\sum_{j,k=1}^{n} \gamma_{ijk}(u) w_j w_k + (b_i(u))_x\right] L_i^y(s)ds \\
&\quad + \int_{t_y}^{\bar{t}_y} \left[\sum_{j,k=1}^{n} \gamma_{ijk}(u) w_j w_k + (b_i(u))_x\right] L_i^y(s)ds \\
&\quad + \int_{\bar{t}_y}^{t} \left[\sum_{j,k=1}^{n} \gamma_{ijk}(u) w_j w_k + (b_i(u))_x\right] L_i^y(s)ds \\
&\triangleq \widehat{I} + \widehat{II} + \widehat{III}, \quad t \in [0, T_*],
\end{aligned} \tag{3.6.83}$$

where $L_i^y(s) = (s, x_i(s,y))$. Noting (3.4.85), (3.6.70), (3.4.27) and (3.6.4), we get

$$\begin{aligned} \left|\widehat{I}\right| &\le C_{30}\left(W\left(D_{\pm}^{T_*}\right)+W\left(D_0^{T_*}\right)\right) \\ &\quad \times\left(W\left(D_{\pm}^{T_*}\right)+W\left(D_0^{T_*}\right)+V\left(D_{\pm}^{T_*}\right)+V\left(D_0^{T_*}\right)\right) \\ &\le C_{31}\varepsilon^2. \end{aligned} \tag{3.6.84}$$

Noting (2.2.21), (3.4.85), (3.6.71) and (3.4.26), we obtain

$$\begin{aligned} \left|\widehat{II}\right| &= \left|\int_{t_y}^{\tilde{t}_y}\left[\sum_{j\neq k}\gamma_{ijk}(u)\,w_j w_k + \gamma_{ijk}(u)w_i^2 + (b_i(u))_x\right](s, x_i(s,y))ds\right| \\ &\le C_{32}\left\{\tilde{W}_1(T_*)\,W_\infty^c(T_*) + (W_\infty^c(T_*))^2 + \tilde{U}_1(T_*)\,W_\infty^c(T_*)\right. \\ &\quad \left. + \tilde{W}_1(T_*)\,U_\infty^c(T_*) + U_\infty^c(T_*)W_\infty^c(T_*)\right\}. \end{aligned} \tag{3.6.85}$$

Noting (3.4.101) and using (3.6.5)-(3.6.7) and (3.6.28)-(3.6.29) (in which we take $T = T_* = K_*\varepsilon^{-(1+\alpha)}$), from (3.6.85) we get

$$\left|\widehat{II}\right| \le C_{33}\varepsilon^2. \tag{3.6.86}$$

Noting (2.2.21), we have

$$\begin{aligned} \left|\widehat{III}\right| &= \left|\int_{\tilde{t}_y}^{t}\left[\sum_{j,k=1}^{n}\gamma_{ijk}(u)w_j w_k + (b_i(u))_x\right](s, x_i(s,y))ds\right| \\ &= \left|\int_{\tilde{t}_y}^{t}\left[\sum_{j\neq k}\gamma_{ijk}(u)w_j w_k + (\gamma_{iii}(u) - \gamma_{iii}(u_i e_i))\,w_i^2\right.\right. \\ &\quad \left.\left. + \gamma_{iii}(u_i e_i)w_i^2 + (b_i(u))_x\right](s, x_i(s,y))ds\right|. \end{aligned} \tag{3.6.87}$$

Applying Hadamard's formula to $\gamma_{iii}(u) - \gamma_{iii}(u_i e_i)$ and using (3.6.53) and (3.4.85), from (3.6.80) we get

$$\begin{aligned} \left|\widehat{III}\right| &\le C_{34}\left\{W_\infty(T_*)W_\infty^c(T_*) + (W_\infty(T_*))^2\,U_\infty^c(T_*)\right. \\ &\quad + (W_\infty(T_*))^2\,(V_\infty(T_*))^\alpha\,t + \tilde{U}_1(T_*)W_\infty(T_*) + \tilde{W}_1(T_*)V_\infty(T_*) \\ &\quad \left. + U_\infty^c(T_*)W_\infty^c(T_*)\right\}. \end{aligned} \tag{3.6.88}$$

Then, noting (3.4.101) and using (3.6.5)-(3.6.7), (3.6.10), (3.6.12) and (3.6.28)-(3.6.29) (in which we take $T = T_*$), from (3.6.88) we obtain

$$\left|\widehat{III}\right| \le C_{35}\left\{\varepsilon^2 + \varepsilon^{2+\alpha}t\right\}, \quad \forall\, t \in [0, T_*]. \tag{3.6.89}$$

Thus, by (3.6.84), (3.6.86) and (3.6.89), from (3.6.83) it follows that

$$|w_i(t, x_i(t, y)) - w_i(0, y)| \le C_{36}\left\{\varepsilon^2 + \varepsilon^{2+\alpha} t\right\} \quad (i = 1, \cdots, n), \quad \forall\, t \in [0, T_*]. \tag{3.6.90}$$

Particularly, noting (3.6.65)-(3.6.66), from (3.6.90) we have

$$|w_i(t, x_i(t, y)) - w_i(0, y)| \le C_{37}\varepsilon^{\frac{3}{2}} \quad (i = 1, \cdots, n), \quad \forall\, t \in [0, T_0]. \tag{3.6.91}$$

Noting (3.6.64), from (3.6.91) we obtain

$$w_i(t, x_i(t, y)) = \varepsilon l_i(0)\psi'(y) + O\left(\varepsilon^{\frac{3}{2}}\right) \quad (i = 1, \cdots, n), \quad \forall\, y \in \boldsymbol{R}, \quad \forall\, t \in [0, T_0]. \tag{3.6.92}$$

The above arguments can be summarized as

Lemma 3.10. On any given existence domain $0 \le t \le T$ $(\le T^*)$ of the C^1 solution $u = u(t, x)$, it follows that

$$|u_i(t, x_i(s, y)) - \varepsilon l_i(0)\psi'(y)| \le \tilde{k}_1\varepsilon^2 \quad (i = 1, \cdots, n), \quad \forall\, y \in \boldsymbol{R}, \quad \forall\, t \in [0, T], \tag{3.6.93}$$

henceforth $\tilde{k}_i$ $(i = 1, 2, 3)$ will denote the positive constants independent of ε, y and T. Moreover, on the strip $0 \le t \le T_*$ the following estimates hold:

$$\begin{aligned} |w_i(t, x_i(t, y)) - \varepsilon l_i(0)\psi'(y)| \;\le\; & \tilde{k}_2\left\{\varepsilon^2 + \varepsilon^{2+\alpha} t\right\} \quad (i = 1, \cdots, n), \\ & \forall\, y \in \boldsymbol{R}, \quad \forall\, t \in [0, T_*]. \end{aligned} \tag{3.6.94}$$

Particularly,

$$|w_i(t, x_i(t, y)) - \varepsilon l_i(0)\psi'(y)| \le \tilde{k}_3\varepsilon^{\frac{3}{2}} \quad (i = 1, \cdots, n), \quad \forall\, y \in \boldsymbol{R}, \quad \forall\, t \in [0, T_0]. \tag{3.6.95}$$

□

On any given existence domain $0 \le t \le T$ $(\le T^*)$ of the C^1 solution $u = u(t, x)$, we consider (2.2.19) along the i-th characteristic $x = x_i(s, y)$. Noting (2.2.21), (3.2.7) and (3.2.12), we can rewrites (2.2.19) as

$$\frac{dw_i}{d_i t} = a_0(t; i, y)w_i^2 + a_1(t; i, y)w_i + a_2(t; i, y), \tag{3.6.96}$$

where

$$a_0(t;i,y) = \gamma_{iii}(u), \tag{3.6.97}$$

$$a_1(t;i,y) = \sum_{j\neq i} (\gamma_{iij}(u) + \gamma_{iji}(u))\, w_j + \left(\tilde{b}_{ii}(u) - \tilde{b}_{ii}(u_i e_i)\right), \tag{3.6.98}$$

$$a_2(t;i,y) = \sum_{j,k\neq i} \gamma_{ijk}(u) w_j w_k + \sum_{k\neq i} \left(\tilde{b}_{ik}(u) - \tilde{b}_{ik}(u_k e_k)\right) w_k, \tag{3.6.99}$$

in which $u = u(t, x_i(t,y))$ and $w_j = w_j(t, x_i(t,y))$ $(j = 1, \cdots, n)$.

Lemma 3.11. On any given existence domain $0 \le t \le T$ $(\le T^*)$ of the C^1 solution $u = u(t,x)$, there exist positive constants $\tilde{k}_4, \tilde{k}_5$ and $\tilde{k}_6$ independent of ε, y and T such that the following estimates hold:

$$\int_0^T |a_1(t;i,y)|dt \le \tilde{k}_4 \varepsilon, \tag{3.6.100}$$

$$\int_0^T |a_2(t;i,y)|dt \le \tilde{k}_5 \varepsilon^2 \tag{3.6.101}$$

and

$$K(i,y;0,T) \triangleq \int_0^T |a_2(t;i,y)|dt \cdot \exp\left(\int_0^T |a_1(t;i,y)|dt\right) \le \tilde{k}_6 \varepsilon^2. \tag{3.6.102}$$

□

Proof. Similar to (3.6.69), we have

$$\int_0^T |a_1(t;i,y)|dt = \int_0^{T_y} |a_1(t;i,y)|dt + \int_{T_y}^{\tilde{T}_y} |a_1(t;i,y)|dt + \int_{\tilde{T}_y}^T |a_1(t;i,y)|dt, \tag{3.6.103}$$

where the definition of T_y (resp. $\tilde{T}_y$) is similar to that of t_y (resp. $\tilde{t}_y$) in (3.6.69). Similarly, by (3.6.98) we get

$$\begin{aligned}\int_0^T |a_1(t;i,y)|dt \le\ & C_{38} \{W(D_\pm^T) + W(D_0^T) + V(D_\pm^T) + V(D_0^T) \\ & + \tilde{W}_1(T) + W_\infty^c(T) + \tilde{U}_1(T) + U_\infty^c(T)\}.\end{aligned} \tag{3.6.104}$$

Noting (3.4.101) and using (3.6.4)-(3.6.7) and (3.6.28)-(3.6.29), from (3.6.104) we obtain

$$\int_0^T |a_1(t;i,y)|dt \le C_{39}\varepsilon. \tag{3.6.105}$$

Similarly, we have

$$\begin{aligned}\int_0^T |a_2(t;i,y)|dt \le\ & C_{40}\Big\{\left(W\left(D_\pm^T\right)+W\left(D_0^T\right)\right)^2\\ &+\left(W\left(D_\pm^T\right)+W\left(D_0^T\right)\right)\left(V\left(D_\pm^T\right)+V\left(D_0^T\right)\right)\\ &+\left(W_\infty^c(T)\right)^2+W_\infty^c(T)\tilde{W}_1(T)+U_\infty^c(T)W_\infty^c(T)\\ &+W_\infty^c(T)\tilde{U}_1(T)+U_\infty^c(T)\tilde{W}_1(T)+V_\infty(T)W_\infty^c(T)\Big\}\\ \le\ & C_{41}\varepsilon^2.\end{aligned} \tag{3.6.106}$$

The combination of (3.6.105) and (3.6.106) gives

$$K(i,y;0,T) \le C_{42}\varepsilon^2. \tag{3.6.107}$$

Taking

$$\tilde{k}_4 = C_{39}, \quad \tilde{k}_5 = C_{41} \quad \text{and} \quad \tilde{k}_6 = C_{42}, \tag{3.6.108}$$

we see that (3.6.105)-(3.6.107) are just the desired (3.6.100)-(3.6.102). The proof is completed. Q.E.D.

(I) Upper bound of the life span — Estimate on $\overline{\lim\limits_{\varepsilon\to 0}}\left(\varepsilon^{1+\alpha}\tilde{T}(\varepsilon)\right)$

Noting that the initial data satisfies (3.3.12) and (3.3.15), we observe that there exist an index $i_0 \in J_1$ and a point $x_0 \in R$ such that

$$M_0 = \left[-\frac{1}{\alpha!}\frac{\partial^{1+\alpha}\lambda_{i_0}}{\partial u_{i_0}^{1+\alpha}}(0)\left(l_{i_0}(0)\psi(x_0)\right)^\alpha l_{i_0}(0)\psi'(x_0)\right]^{-1}, \tag{3.6.109}$$

where M_0 is defined by (3.6.59).

By (3.4.3), it follows that

$$x_0+\left(\lambda_{i_0}(0)-\frac{1}{2}\delta_0\right)t \le x_{i_0}(t,x_0) \le x_0+\left(\lambda_{i_0}(0)+\frac{1}{2}\delta_0\right)t. \tag{3.6.110}$$

Hence, by the definition of $D_{i_0}^T$, the characteristic $x = x_{i_0}(t,x_0)$ must enter $D_{i_0}^T$ at a finite time $t_0 \le \frac{2|x_0|}{\delta_0}$ and stay in $D_{i_0}^T$ for $t > t_0$. Clearly, when $\varepsilon_0 > 0$ is suitably small we have

$$T_0 \triangleq \varepsilon^{-(\alpha+\frac{1}{2})} > t_0, \quad \forall\, \varepsilon \in (0,\, \varepsilon_0].$$

In what follows, we always suppose that $\varepsilon_0 > 0$ is so small that $T_0 > t_0$.

Noting (3.6.52) and (3.3.10), we have

$$\frac{\partial^l \gamma_{i_0 i_0 i_0}}{\partial u_{i_0}^l}(0) = 0 \quad (l = 0, 1, \cdots, \alpha - 1) \quad \text{but} \quad \frac{\partial^\alpha \gamma_{i_0 i_0 i_0}}{\partial u_{i_0}^\alpha}(0) = -\frac{\partial^{1+\alpha} \lambda_{i_0}}{\partial u_{i_0}^{1+\alpha}}(0) \neq 0. \tag{3.6.111}$$

Then (3.6.109) becomes

$$M_0 = \left[\frac{1}{\alpha!}\frac{\partial^\alpha \gamma_{i_0 i_0 i_0}}{\partial u_{i_0}^\alpha}(0)\,(l_{i_0}(0)\psi(x_0))^\alpha\, l_{i_0}(0)\psi'(x_0)\right]^{-1}. \tag{3.6.112}$$

Let

$$a = \frac{1}{\alpha!}\frac{\partial^\alpha \gamma_{i_0 i_0 i_0}}{\partial u_{i_0}^\alpha}(0) \quad \text{and} \quad b = a\,(l_{i_0}(0)\psi(x_0))^\alpha. \tag{3.6.113}$$

Without loss of generality, we may suppose that

$$b > 0 \quad \text{and} \quad l_{i_0}(0)\psi'(x_0) > 0. \tag{3.6.114}$$

Otherwise, changing the sign of u, we can draw the same conclusion.

Noting (3.6.111), we get

$$\gamma_{i_0 i_0 i_0}(u_{i_0} e_{i_0}) = a u_{i_0}^\alpha + O\left(|u_{i_0}|^{1+\alpha}\right), \quad \forall\ |u_{i_0}| \text{ small}. \tag{3.6.115}$$

We further rewrite (3.6.115) as

$$\begin{aligned}\gamma_{i_0 i_0 i_0}(u_{i_0} e_{i_0}) \;=\; & a\,(\varepsilon l_{i_0}(0)\psi(x_0))^\alpha + a\left[(u_{i_0}(t, x_{i_0}(t, x_0)))^\alpha - (\varepsilon l_{i_0}(0)\psi(x_0))^\alpha\right] \\ & + O\left(|u_{i_0}|^{1+\alpha}\right), \quad \forall\ |u_{i_0}| \text{ small}.\end{aligned} \tag{3.6.116}$$

Noting (3.6.113), (3.6.67) and (3.6.93), on any given existence domain $0 \le t \le T\ (\le T^*)$ of the C^1 solution $u = u(t, x)$ we obtain

$$\gamma_{i_0 i_0 i_0}(u_{i_0} e_{i_0}) = b\varepsilon^\alpha + O\left(\varepsilon^{1+\alpha}\right), \quad \forall\, t \in [0, T]. \tag{3.6.117}$$

Noting (3.6.5) and the fact that the characteristic $x = x_{i_0}(t, x_0)$ must stay in $D_{i_0}^T$ for any $t > T_0$, on $x = x_{i_0}(t, x_0)$ we have

$$\begin{aligned}|\gamma_{i_0 i_0 i_0}(u) - \gamma_{i_0 i_0 i_0}(u_{i_0} e_{i_0})| \;\le\; & C_{43}(1+t)^{-(1+\mu)} U_\infty^c(T) \\ \le\; & C_{44}\varepsilon(1+t)^{-(1+\mu)} \\ \le\; & C_{45}\varepsilon(1+T_0)^{-(1+\mu)} \\ \le\; & C_{46}\varepsilon^{\frac{1}{2}+\alpha+\frac{1}{2}\mu+\mu\alpha} \\ \le\; & C_{47}\varepsilon^{\frac{1}{2}+\alpha}, \quad \forall\, t \in [T_0, T].\end{aligned} \tag{3.6.118}$$

Thus, on $x = x_{i_0}(t, x_0)$ it follows that

$$a_0(t; i_0, x_0) = \gamma_{i_0 i_0 i_0}(u) = b\varepsilon^\alpha + O\left(\varepsilon^{\frac{1}{2}+\alpha}\right) > 0, \quad \forall\, t \in [T_0, T], \tag{3.6.119}$$

provided that $\varepsilon > 0$ is small enough, where T satisfies that $T \le T^*$.

From (3.6.114) and (3.6.95) it follows that

$$w_{i_0}(T_0,\ x_{i_0}(T_0, x_0)) = \varepsilon l_{i_0}(0)\psi'(x_0) + O\left(\varepsilon^{\frac{3}{2}}\right) > 0, \tag{3.6.120}$$

provided that $\varepsilon > 0$ is small enough.

Finally, noting (3.6.102) and (3.6.120), we get immediately

$$w_{i_0}(T_0, x_{i_0}(T_0, x_0)) = \varepsilon l_{i_0}(0)\psi'(x_0) + O\left(\varepsilon^{\frac{3}{2}}\right) > \tilde{k}_6\varepsilon^2 \ge K(i_0, x_0; T_0, T) \tag{3.6.121}$$

provided that $\varepsilon > 0$ is small enough, where

$$K(i_0, x_0; T_0, T) \triangleq \int_{T_0}^{T} |a_2(t; i_0, x_0)|dt \cdot \exp\left(\int_{T_0}^{T} |a_1(t; i_0, x_0)|dt\right).$$

Hence, noting (3.6.119) and (3.6.121), we observe that Lemma 2.1 can be applied to the initial value problem for (3.6.96) (in which we take $i = i_0$ and $y = x_0$) with the following initial condition

$$t = T_0: \quad w_{i_0} = w_{i_0}(T_0, x_{i_0}(T_0, x_0)) \triangleq w_{i_0}(T_0), \tag{3.6.122}$$

and then we obtain

$$\int_{T_0}^{T} a_0(t; i_0, x_0)dt \cdot \exp\left(-\int_{T_0}^{T} |a_1(t; i_0, x_0)|dt\right) < \left(w_{i_0}(T_0) - K(i_0, x_0; T_0, T)\right)^{-1}, \tag{3.6.123}$$

namely,

$$\begin{aligned} &\exp\left(-\int_{T_0}^{T} |a_1(t; i_0, x_0)|dt\right) \times \\ &\int_{T_0}^{T} \gamma_{i_0 i_0 i_0}\left(u\left(t, x_{i_0}(t, x_0)\right)\right)\left(w_{i_0}(T_0) - K(i_0, x_0; T_0, T)\right) dt < 1. \end{aligned} \tag{3.6.124}$$

Noting the fact that $T \le T^* = M^*\varepsilon^{-(1+\alpha)}$ and using (3.6.119)-(3.6.120) and Lemma 3.11, from (3.6.124) we obtain

$$\overline{\lim_{\varepsilon \to 0}} \left\{\varepsilon^{1+\alpha} T \cdot \frac{1}{\alpha!} \frac{\partial^\alpha \gamma_{i_0 i_0 i_0}}{\partial u_{i_0}^\alpha}(0)\left[l_{i_0}(0)\psi(x_0)\right]^\alpha l_{i_0}(0)\psi'(x_0)\right\} \le 1. \tag{3.6.125}$$

Noting (3.6.112) and taking $T=\tilde{T}(\varepsilon)-1$, from (3.6.125) we get immediately

$$\overline{\lim_{\varepsilon\to 0}}\left(\varepsilon^{1+\alpha}\tilde{T}(\varepsilon)\right)\le M_0. \qquad (3.6.126)$$

(3.6.126) gives an upper bound of the life span $\tilde{T}(\varepsilon)$.

(II) Lower bound of the life span — Estimate on $\underline{\lim}_{\varepsilon\to 0}\left(\varepsilon^{1+\alpha}\tilde{T}(\varepsilon)\right)$

By (3.6.126), in order to prove (3.3.19) it remains to show that

$$\underline{\lim}_{\varepsilon\to 0}\left(\varepsilon^{1+\alpha}\tilde{T}(\varepsilon)\right)\ge M_0. \qquad (3.6.127)$$

To do so, it suffices to prove that, for any fixed constant M_* satisfying that

$$0<M_*<M_0, \qquad (3.6.128)$$

we have

$$\tilde{T}(\varepsilon)\ge M_*\varepsilon^{-(1+\alpha)}, \qquad (3.6.129)$$

provided that $\varepsilon>0$ is small enough. Hence, we only need to establish a uniform *a priori* estimate on C^1 norm of the C^1 solution $u=u(t,x)$ on any given existence domain $0\le t\le T\le M_*\varepsilon^{-(1+\alpha)}$. The uniform *a priori* estimate on the C^0 norm of $u=u(t,x)$ has been established in Lemma 3.9. It remains to establish a uniform *a priori* estimate on the C^0 norm of the first derivatives of $u=u(t,x)$, namely, a uniform *a priori* estimate on the C^0 norm of $w=(w_1(t,x),\cdots,w_n(t,x))^T$.

In order to estimate $w_i=w_i(t,x)$ on the existence domain $0\le t\le T$ (where T satisfies $T\le M_*\varepsilon^{-(1+\alpha)}$) of the C^1 solution $u=u(t,x)$, we still consider (3.6.96) along the i-th characteristic $x=x_i(t,y)$ passing through an arbitrary fixed point $(0,y)$. Without loss of generality, we may suppose that

$$w_i(0,y)\ge 0. \qquad (3.6.130)$$

Otherwise, changing the sign of w_i, we can draw the same conclusion.

Using Hadamard's formula, from (3.6.97) we get

$$\begin{aligned} a_0(t;i,y) &= \gamma_{iii}(u)=(\gamma_{iii}(u)-\gamma_{iii}(u_ie_i))+\gamma_{iii}(u_ie_i)=\gamma_{iii}(u_ie_i)+\\ &\sum_{j\ne i}\left[\int_0^1\frac{\partial\gamma_{iii}}{\partial u_j}(su_1,\cdots,su_{i-1},u_i,su_{i+1},\cdots,su_n)ds\right]u_j. \end{aligned} \qquad (3.6.131)$$

On the other hand, noting (3.6.52), (3.3.10) (or (3.3.11)), (3.3.13) and (3.6.93), similar to (3.6.115), we have

$$\begin{aligned}\gamma_{iii}(u_i e_i) &= -\tfrac{1}{\alpha!}\tfrac{\partial^{1+\alpha}\lambda_i}{\partial u_i^{1+\alpha}}(0)\,(u_i)^\alpha + O\left(|u_i|^{1+\alpha}\right)\\ &= -\tfrac{1}{\alpha!}\tfrac{\partial^{1+\alpha}\lambda_i}{\partial u_i^{1+\alpha}}(0)\,(\varepsilon l_i(0)\psi(y))^\alpha + O\left(\varepsilon^{1+\alpha}\right).\end{aligned} \tag{3.6.132}$$

Substituting (3.6.132) into (3.6.131) gives

$$\begin{aligned}a_0(t;i,y) &= -\tfrac{1}{\alpha!}\tfrac{\partial^{1+\alpha}\lambda_i}{\partial u_i^{1+\alpha}}(0)\,(\varepsilon l_i(0)\psi(y))^\alpha + O\left(\varepsilon^{1+\alpha}\right)\\ &\quad+\sum_{j\neq i}\left[\int_0^1 \tfrac{\partial\gamma_{iii}}{\partial u_j}(su_1,\cdots,su_{i-1},u_i,su_{i+1},\cdots,su_n)ds\right]u_j.\end{aligned} \tag{3.6.133}$$

Let

$$a_0^+(t;i,y) = \max\{a_0(t;i,y),0\}. \tag{3.6.134}$$

We now calculate $w_i(0,y)\int_0^T a_0^+(t;i,y)dt$.

Noting the fact that $T \leq M_*\varepsilon^{-(1+\alpha)}$ and using (3.6.64), (3.4.101), (3.6.4)-(3.6.6), (3.6.29), (3.6.59) and (3.6.128), we obtain

$$\begin{aligned}&w_i(0,y)\int_0^T a_0^+(t;i,y)dt\\ &\leq \left(\varepsilon l_i(0)\psi'(y) + C_{17}\varepsilon^2\right)\Big\{\max\Big\{-\tfrac{1}{\alpha!}\tfrac{\partial^{1+\alpha}\lambda_i}{\partial u_i^{1+\alpha}}(0)\,(\varepsilon l_i(0)\psi(y))^\alpha\,,\ 0\Big\}T\\ &\quad+C_{48}\left[\varepsilon^{1+\alpha}T + \sum_{j\neq i}\left(\int_0^{T_y}|u_j|dt + \int_{T_y}^{\tilde T_y}|u_j|dt + \int_{\tilde T_y}^{T}|u_j|dt\right)\right]\Big\}\\ &\leq \max\Big\{-\tfrac{1}{\alpha!}\tfrac{\partial^{1+\alpha}\lambda_i}{\partial u_i^{1+\alpha}}(0)\,(l_i(0)\psi(y))^\alpha\,,\ 0\Big\}\left(l_i(0)\psi'(y)M_* + M_*C_{17}\varepsilon\right)\\ &\quad+C_{49}\varepsilon\left\{M_* + V\left(D_\pm^T\right) + V\left(D_0^T\right) + \tilde U_1(T) + U_\infty^c(T)\right\}\\ &\leq M_0^{-1}M_* + C_{50}\varepsilon < 1,\end{aligned} \tag{3.6.135}$$

provided that $\varepsilon > 0$ is small enough, where T_y (resp. $\tilde T_y$) is defined as before. In (3.6.135) we have made use of Lemma 3.2. On the other hand, noting (3.6.133), we get similarly

$$\begin{aligned}\int_0^T |a_0(t,y)|dt &\leq C_{51}\varepsilon^\alpha T + C_{52}\left\{\left(V\left(D_\pm^T\right) + V\left(D_0^T\right)\right) + \tilde U_1(T) + U_\infty^c(T)\right\}\\ &\leq C_{51}M_*\varepsilon^{-1} + C_{53}\varepsilon \leq C_{54}\varepsilon^{-1},\end{aligned} \tag{3.6.136}$$

provided that $\varepsilon > 0$ is suitably small. Then, noting (3.6.135)-(3.6.136) and Lemma 3.11, we obtain

$$\int_0^T a_0^+(t;i,y)dt \cdot \exp\left(\int_0^T |a_1(t;i,y)|\, dt\right) < (w_i(0,y)+K(i,y;0,T))^{-1} \quad (3.6.137)$$

and

$$\int_0^T |a_0(t;i,y)|dt \cdot \exp\left(\int_0^T |a_1(t;i,y)|\, dt\right) < (K(i,y;0,T))^{-1}, \quad (3.6.138)$$

provided that $\varepsilon > 0$ is small enough, where $T \leq M_* \varepsilon^{-(1+\alpha)}$.

Noting (3.6.130) and (3.6.137)-(3.6.138), we observe that Lemma 2.2 can be applied to the initial value problem for equation (3.6.96) with the following initial conditon

$$t = o: \; w_i = w_i(0,y).$$

Then it follows from (2.1.9) and (2.1.10) that

$$\begin{aligned}(w_i(T,x_i(T,y)))^{-1} \;\geq\; & (w_i(0,y)+K(i,y;0,T))^{-1} - \\ & \int_0^T a_0^+(t;i,y)dt \cdot \exp\left(\int_0^T |a_1(t;i,y)|dt\right), \quad (3.6.139)\\ & \text{if} \quad w_i(T,x_i(T,y)) > 0\end{aligned}$$

and

$$\begin{aligned}|w_i(T,x_i(T))|^{-1} \geq (K(i,y;0,T))^{-1} - \int_0^T |a_0(t;i,y)|dt \exp\left(\int_0^T |a_1(t;i,y)|dt\right), \\ \text{if} \quad w_i(T,x_i(T,y)) > 0. \\ (3.6.140)\end{aligned}$$

Noting (3.6.135)-(3.6.136) and Lemma 3.11, from (3.6.139)-(3.6.140) we get respectively

$$\begin{aligned}(w_i(T,x_i(T,y)))^{-1} \geq \tfrac{1}{2}\left(1 - \tfrac{M_*}{M_0}\right)(w_i(0,y)+K(i,y;0,T))^{-1}, \\ \text{if} \quad w_i(T,x_i(T,y)) > 0\end{aligned} \quad (3.6.141)$$

and

$$|w_i(T,x_i(T,y))|^{-1} \geq \frac{1}{2}(K(i,y;0,T))^{-1}, \quad \text{if} \quad w_i(T,x_i(T,y)) < 0, \quad (3.6.142)$$

provided that $\varepsilon > 0$ is small enough. It follows from (3.6.141)-(3.6.142) that

$$|w_i(T, x_i(T, y))| \leq C_{55}\varepsilon. \tag{3.6.143}$$

For each $i = 1, \cdots, n$ and any $t \in [0, T]$, we can prove similarly that $w_i(t, x_i(t, y))$ satisfies the same estimate. On the other hand, noting the fact that $(0, y)$ is arbitrary, then we obtain

$$||w(t, x)||_{C^0[0,T]\times \mathbf{R}} \leq C_{56}\varepsilon, \tag{3.6.144}$$

where T satisfies that $T \leq M_* \varepsilon^{-(1+\alpha)}$. Hence, (3.6.129) holds and then (3.6.127) is valid.

The combination of (3.6.126) and (3.6.127) gives (3.6.58). Thus, the proof of Theorem 3.2 is completed. Q.E.D.

From the proof of (3.6.127), we can easily obtain the following.

Remark 3.11. Suppose that there exist $y \in \mathbf{R}$, $i \in \{1, \cdots, n\}$ and a positive constant B_0 such that

$$-\frac{1}{\alpha!}\frac{\partial^{1+\alpha}\lambda_i}{\partial u_i^{1+\alpha}}(0)\,(l_i(0)\psi(y))^{\alpha}\, l_i(0)\psi'(y) \leq \frac{1}{B_0}. \tag{3.6.145}$$

For any given positive constant $B < B_0$, if the Cauchy problem (3.3.1)-(3.3.2) admits a unique C^1 solution $u = u(t, x)$ on the domain $0 \leq t \leq T$ with $0 < T \leq B\varepsilon^{-(1+\alpha)}$, then, for suitably small $\varepsilon > 0$ we have

$$|w_i(t, x_i(t, y))| \leq K_0\varepsilon, \quad \forall\, t \in [0, T], \tag{3.6.146}$$

where $x = x_i(t, y)$ stands for the i-th characteristic passing through the point $(0, y)$, and K_0 is a positive constant independent of i, T, y and ε. □

§3.7. Blow-up phenomenon and life span of C^1 solution (II) — Proof of Theorem 3.3

Theorem 3.3 will be proved in a way similar to the proof of Theorem 3.2. In what follows, we only point out the essentially different part in the proof and $\varepsilon_0 > 0$

is always assumed to be suitably small. As in §3.6, in order to precisely estimate the life span of the C^1 solution, we consider the Cauchy problem (3.6.1)-(3.6.2).

In the present situation, Lemma 3.8 is still valid. Moreover, similar to Lemma 3.9, we have

Lemma 3.12. Under the assumptions of Theorem 3.3, in the normalized coordinates there exists $\varepsilon_0 > 0$ so small that for any fixed $\varepsilon \in (0, \varepsilon_0]$, on any given existence domain $0 \le t \le T$ of the C^1 solution $u = u(t, x)$ to the Cauchy problem (3.6.1)-(3.6.2) there exist positive constants k_i $(i = 2, \cdots, 10)$ independent of ε and T, such that the following uniform *a priori* estimates hold:

$$U_\infty^c(T) \le k_2 \varepsilon, \tag{3.7.1}$$

$$V_\infty^c(T) \le k_3 \varepsilon, \tag{3.7.2}$$

$$W_\infty^c(T) \le k_4 \varepsilon, \tag{3.7.3}$$

$$W_1(T) \le k_5 \varepsilon, \tag{3.7.4}$$

$$V_1(T) \le k_6 \varepsilon + k_7 \varepsilon^{2+\alpha} T, \tag{3.7.5}$$

$$V_\infty(T) \le k_8 \varepsilon, \tag{3.7.6}$$

where

$$T \varepsilon^{\frac{7}{6}+\alpha} \le 1. \tag{3.7.7}$$

Moreover

$$W_\infty(T) \le k_9 \varepsilon, \tag{3.7.8}$$

where

$$T \varepsilon^{1+\alpha} \le k_{10}. \tag{3.7.9}$$

□

Proof. This Lemma can be shown in a way similar to the proof of Lemma 3.9. In what follows, we only point out the essentially different part in the proof.

Without loss of generality, we may suppose that

$$0 < \mu \le \frac{3}{6\alpha + 7}. \tag{3.7.10}$$

Noting (3.3.25) and using Taylor's formula, in the normalized coordinates we have

$$b_i(u) = \sum_{j=1}^{n} b_i^j(u)\,(u_j)^p + b_i^*(u) \quad (i = 1, \cdots, n), \tag{3.7.11}$$

where $b_i^j(u)$ and $b_i^*(u)$ $(i, j = 1, \cdots, n)$ are C^1 functions which are given by Taylor's expansion, and $b_i^*(u)$ $(i = 1, \cdots, n)$ satisfy the matching condition. It follows from (3.7.11) that

$$(b_i(u))_x = \sum_{j,k=1}^{n} \left[b_{ik}^j(u)\,(u_j)^p\, w_k + \tilde{b}_{ik}^j(u)\,(u_j)^{p-1}\, w_k \right] + (b_i^*(u))_x \quad (i = 1, \cdots, n), \tag{3.7.12}$$

where

$$b_{ik}^j(u) = \nabla b_i^j(u) r_k(u) \tag{3.7.13}$$

and

$$\tilde{b}_{ik}^j(u) = p b_i^j(u) r_{kj}(u). \tag{3.7.14}$$

Then, in the present situation, instead of (3.6.14) we have

$$\begin{aligned}
&\int_{t_0}^{t_2} |v_i\,(t, x_j(t))\,|\; |\lambda_j(u(t, x_j(t))) - \lambda_i(u(t, x_j(t)))| dt \\
&\le \int_0^{\frac{y_2}{\lambda_n(0)+\delta_0}} |v_i(t, (\lambda_n(0) + \delta_0)t)| (\lambda_n(0) + \delta_0 - \lambda_i(t, (\lambda_n(0) + \delta_0)t)) dt \\
&\quad + \iint_{P_0OA_2P_2} \left| \sum_{j\neq k} \tilde{\beta}_{ijk}(u) v_j w_k \right| dtdx \\
&\quad + \iint_{P_0OA_2P_2} \left| \sum_{j=1}^{n} \left(\tilde{\beta}_{ijj}(u) - \tilde{\beta}_{ijj}(u_j e_j) \right) v_j w_j \right| dtdx \\
&\quad + \iint_{P_0OA_2P_2} \left| \tilde{\beta}_{iii}(u_i e_i) v_i w_i \right| dtdx \\
&\quad + \iint_{P_0OA_2P_2} \left| \left[\sum_{j,k=1}^{n} \nu_{ijk}(u) v_j b_k^*(u) + b_i^*(u) \right] \right| dtdx \\
&\quad + \iint_{P_0OA_2P_2} \left| \left[\sum_{j,k,l=1}^{n} \nu_{ijk}(u) b_k^l(u) v_j (u_l)^p + \sum_{j=1}^{n} b_i^j(u)(u_j)^p \right] \right| dtdx.
\end{aligned} \tag{3.7.15}$$

Since $b_i^*(u)$ $(i = 1, \cdots, n)$ satisfy the matching condition, we only need to estimate the last term of the right-hand side of (3.7.15). Noting (3.4.26), we get

$$\begin{aligned}&\iint_{P_0OA_2P_2} \left|\left[\sum_{l,j,k=1}^{n} \nu_{ijk}(u)b_k^l(u)v_j(u_l)^p + \sum_{j=1}^{n} b_i^j(u)(u_j)^p\right]\right| dtdx \\ &\quad \le C_1\left[(1+V_\infty(T))\left((U_\infty^c(T))^p + (V_\infty(T))^{p-1}U_1(T)T\right)\right],\end{aligned} \tag{3.7.16}$$

henceforth C_j $(j = 1, 2, \cdots)$ will denote positive constants independent of ε and T. Thus, noting (3.4.4) and (3.4.74), similar to (3.6.18), from (3.7.15)-(3.7.16) we obtain

$$\begin{aligned}\tilde{V}_1(T) \le\ & C_2\{\varepsilon + V_\infty^c(T)W_\infty^c(T) + V_1(T)W_\infty^c(T) \\ & + V_\infty^c(T)W_1(T) + U_\infty^c(T)W_1(T) + U_\infty^c(T)W_\infty^c(T) \\ & + (U_\infty^c(T))^2 + U_\infty^c(T)V_1(T) + U_\infty^c(T)V_\infty^c(T) \\ & + (V_\infty(T))^{1+\alpha}(W_\infty^c(T) + W_1(T))T + (U_\infty^c(T))^p \\ & + (V_\infty(T))^{p-1}V_1(T)T + (V_\infty(T))^{p-1}V_\infty^c(T)T\Big\}.\end{aligned} \tag{3.7.17}$$

Similarly, instead of (3.6.19) we have

$$\begin{aligned}V_1(T) \le\ & C_3\{\varepsilon + V_\infty^c(T)W_\infty^c(T) + V_1(T)W_\infty^c(T) \\ & + V_\infty^c(T)W_1(T) + U_\infty^c(T)W_1(T) + U_\infty^c(T)W_\infty^c(T) \\ & + (U_\infty^c(T))^2 + U_\infty^c(T)V_1(T) + U_\infty^c(T)V_\infty^c(T) \\ & + (V_\infty(T))^{1+\alpha}(W_\infty^c(T) + W_1(T))T + (U_\infty^c(T))^p \\ & + (V_\infty(T))^{p-1}V_1(T)T + (V_\infty(T))^{p-1}V_\infty^c(T)T\Big\}.\end{aligned} \tag{3.7.18}$$

Clearly, (3.4.73) is still valid, namely, we have

$$U_\infty^c(T) \le C_4V_\infty^c(T) + C_5V_\infty(T)U_\infty^c(T). \tag{3.7.19}$$

Moreover, noting (3.7.11)-(3.7.12), similar to (3.6.21)-(3.6.25), we have

$$\begin{aligned}\tilde{W}_1(T) \le\ & C_6\Big\{\varepsilon + (W_\infty^c(T))^2 + W_\infty^c(T)W_1(T) + U_\infty^c(T)W_\infty^c(T) \\ & + W_\infty^c(T)V_\infty^c(T) + W_\infty^c(T)V_1(T) + U_\infty^c(T)W_1(T) \\ & + W_\infty^c(T)(U_\infty^c(T))^{p-1} + (V_\infty(T))^{p-1}W_1(T)T\Big\},\end{aligned} \tag{3.7.20}$$

$$\begin{aligned} W_\infty^c(T) \;\le\; & C_8\Big\{\varepsilon + (W_\infty^c(T))^2 + W_\infty^c(T)\tilde{W}_1(T) + U_\infty^c(T)W_\infty^c(T) \\ & +\tilde{V}_1(T)W_\infty^c(T) + V_\infty^c(T)W_\infty^c(T) + U_\infty^c(T)\tilde{W}_1(T) \\ & +W_\infty^c(T)\,(U_\infty^c(T))^{p-1} + (V_\infty(T))^{p-1}\,\tilde{W}_1(T)T^{1+\mu}\Big\}, \end{aligned} \tag{3.7.21}$$

$$\begin{aligned} V_\infty^c(T) \;\le\; & C_9\Big\{\varepsilon + V_\infty^c(T)W_\infty^c(T) + V_\infty^c(T)\tilde{W}_1(T) + \tilde{V}_1(T)W_\infty^c(T) \\ & +U_\infty^c(T)W_\infty^c(T) + U_\infty^c(T)\tilde{W}_1(T) + \tilde{V}_1(T)W_\infty^c(T) \\ & +(U_\infty^c(T))^2 + U_\infty^c(T)\tilde{V}_1(T) + U_\infty^c(T)V_\infty^c(T) \\ & +(U_\infty^c(T))^p + (V_\infty(T))^{p-1}\,\tilde{V}_1(T)T^{1+\mu}\Big\}, \end{aligned} \tag{3.7.22}$$

$$\begin{aligned} W_1(T) \;\le\; & C_7\Big\{\varepsilon + (W_\infty^c(T))^2 + W_\infty^c(T)W_1(T) + U_\infty^c(T)W_\infty^c(T) \\ & +W_\infty^c(T)V_\infty^c(T) + W_\infty^c(T)V_1(T) + U_\infty^c(T)W_1(T) \\ & +W_\infty^c(T)\,(U_\infty^c(T))^{p-1} + (V_\infty(T))^{p-1}\,W_1(T)T\Big\}, \end{aligned} \tag{3.7.23}$$

$$V_\infty(T) \;\le\; C_{10}\left\{\varepsilon + W_\infty^c(T) + W_1(T)\right\}. \tag{3.7.24}$$

Noting (3.7.10) and the fact that $p > 2+\alpha$, we have

$$\varepsilon^{p-1}\cdot\varepsilon^{2+\alpha}\varepsilon^{-(\alpha+\frac{7}{6})}\cdot\varepsilon^{-(\alpha+\frac{7}{6})(1+\mu)} \le \varepsilon^{\frac{7}{6}}. \tag{3.7.25}$$

In a manner similar to the proof of (3.6.5)-(3.6.10), noting (3.7.25) and using (3.7.17)-(3.7.24), we can easily prove (3.7.1)-(3.7.6) under the condition (3.7.7).

The proof of (3.7.8) is almost the same as that of (3.6.12) and can be handled similarly. For brevity, we omit the details. Thus the proof of Lemma 3.12 is finished. Q.E.D.

Remark 3.12. By (3.7.6) and (3.7.8), when $\varepsilon_0 > 0$ is suitably small, the Cauchy problem (3.6.1)-(3.6.2) admits a unique C^1 solution $u = u(t,x)$ on $0 \le t \le T$, where T satisfies (3.7.9). Hence, we get the following lower bound on the life span $\tilde{T}(\varepsilon)$ of the C^1 solution

$$\tilde{T}(\varepsilon) \ge K_\star\varepsilon^{-(1+\alpha)}, \tag{3.7.26}$$

where $K_\star\,(= k_{10})$ is a positive constant independent of ε. □

Proof of Theorem 3.3. In a manner similar to the proof of Theorem 3.2, using Lemma 3.8 and Lemma 3.12, we can easily prove Theorem 3.3. For brevity, the details are omitted. Q.E.D.

§3.8. Formation of envelope of characteristics of the same family — Proof of Theorem 3.4

We prove Theorem 3.4 only under the hypotheses of Theorem 3.2. Under the assumptions of Theorem 3.3, the proof of Theorem 3.4 is similar.

We still adopt the normalized coordinates $\tilde{u}$ mentioned in Remark 3.3, and still simply denote it by u. Let (t^*, x^*) be a starting point of singularity of the C^1 solution to the Cauchy problem (3.3.1)-(3.3.2). By (3.6.58), we have

$$\frac{1}{2}M_0\varepsilon^{-(1+\alpha)} < t^* < 2M_0\varepsilon^{-(1+\alpha)}, \tag{3.8.1}$$

where M_0 is given by (3.6.59). In the domain

$$R(t^*) \triangleq \{(t, x) | t \in [0, t^*),\ x \in R\} \tag{3.8.2}$$

the Cauchy problem (3.3.1)-(3.3.2) admits a unique C^1 solution $u = u(t, x)$, and by Lemma 3.9 we have

$$||u(t, x)||_{C^0(R(t^*))} \le K_1\varepsilon, \tag{3.8.3}$$

where K_1 is a positive constant independent of ε.

Let $\xi = x_i(s, y_i)$ be the i-th characteristic passing through any given point (t, x) in the domain $R(t^*)$, where (s, ξ) denote the coordinates of variable point on this characteristic and y_i stands for the x-coordinate of the intersection point of this characteristic with the x-axis. We have

$$\frac{dx_i(s, y_i)}{ds} = \lambda_i(u(s, x_i(s, y_i))), \tag{3.8.4}$$

$$x_i(0, y_i) = y_i \quad \text{and} \quad x_i(t, y_i) = x. \tag{3.8.5}$$

Lemma 3.13. For each $i = 1, 2, \cdots, n$ and for any given point (t, x), in the domain $R(t^*)$ we have

$$\left| w_i(t, x)\frac{\partial x_i(t, y_i)}{\partial y_i} \right| \le K_2\varepsilon, \tag{3.8.6}$$

provided that $\varepsilon > 0$ is suitably small, where K_2 is a positive constant independent of i, t, y_i (or x) and ε. □

Lemma 3.13 shows that the singularity is produced by the envelope of characteristics of the same family, and this solution is a "blow-up solution of cusp type" according to the terminology of [A1]. In particular, if the i-th characteristics is genuinely nonlinear, then our result goes back to the corresponding result in [A1] and [Ma].

Proof of Lemma 3.13. As in the proof of Theorem 3.2, the initial condition is given by (3.6.60), and (3.6.61) holds. Let

$$\tilde{\psi}_i(x) = l_i(0)\psi(x) \quad (i = 1, \cdots, n). \tag{3.8.7}$$

It follows from (3.6.61) that

$$v_i(0,x) = \varepsilon\tilde{\psi}_i(x) + O(\varepsilon^2) \quad (i = 1, \cdots, n) \tag{3.8.8}$$

and

$$w_i(0,x) = \varepsilon\tilde{\psi}_i'(x) + O(\varepsilon^2) \quad (i = 1, \cdots, n). \tag{3.8.9}$$

For each $i = 1, 2, \cdots, n$ and for any given point (t, x), in the domain $R(t^*)$, noting the definition of M_0 (see (3.6.59)) we see that

$$\max_{i=1,\cdots,n} \left\{ -\frac{1}{\alpha!}\frac{\partial^{1+\alpha}\lambda_i}{\partial u_i^{1+\alpha}}(0)\left(\tilde{\psi}_i(y_i)\right)^\alpha \tilde{\psi}_i'(y_i)\right\} \le \frac{1}{M_0}. \tag{3.8.10}$$

Hence, for any fixed $y_i \in \mathbf{R}$, there are two and only two possibilities:

Case I:

$$a_i\left(\tilde{\psi}_i(y_i)\right)^\alpha \tilde{\psi}_i'(y_i) < \frac{1}{4M_0} \tag{3.8.11}$$

and

Case II:

$$a_i\left(\tilde{\psi}_i(y_i)\right)^\alpha \tilde{\psi}_i'(y_i) \in \left[\frac{1}{4M_0}, \frac{1}{M_0}\right], \tag{3.8.12}$$

where

$$a_i = -\frac{1}{\alpha!}\frac{\partial^{1+\alpha}\lambda_i}{\partial u_i^{1+\alpha}}(0). \tag{3.8.13}$$

We first consider Case I.

Noting (3.8.11), by Remark 3.11 (in which we take $B_0 = 4M_0$ and $B = 2M_0$) we get

$$|w_i(t, x_i(t, y_i))| \leq C_1\varepsilon, \quad \forall\, t \in [0, t^*), \tag{3.8.14}$$

henceforth C_j $(j = 1, 2, \cdots,)$ denote positive constants independent of i, t, y_i and ε.

Differentiating (3.8.4) with respect to y_i gives

$$\frac{d}{ds}\left(\frac{\partial x_i(s, y_i)}{\partial y_i}\right) = \nabla\lambda_i(u)\frac{\partial u}{\partial x}\frac{\partial x_i(s, y_i)}{\partial y_i}, \tag{3.8.15}$$

in which

$$u = u(s, x_i(s, y_i)). \tag{3.8.16}$$

Noting (2.2.6) and (2.2.22), from (3.8.15) we obtain

$$\frac{d}{ds}\left(\frac{\partial x_i(s, y_i)}{\partial y_i}\right) = \sum_{j\neq i}\gamma_{ij}(u)w_j\frac{\partial x_i(s, y_i)}{\partial y_i} - \gamma_{iii}(u)w_i\frac{\partial x_i(s, y_i)}{\partial y_i}, \tag{3.8.17}$$

where

$$\gamma_{ij}(u) = \nabla\lambda_i(u)r_j(u). \tag{3.8.18}$$

Making use of Hadamard's formula, we have

$$\begin{aligned} \gamma_{iii}(u) - \gamma_{iii}(u_i e_i) &= \int_0^1 \sum_{j\neq i}\tfrac{\partial\gamma_{iii}}{\partial u_j}(\tau u_1, \cdots, \tau u_{i-1}, u_i, \tau u_{i+1}, \cdots, \tau u_n)u_j d\tau \\ &\triangleq \sum_{j\neq i}\gamma_{iii}^j(u)u_j, \end{aligned} \tag{3.8.19}$$

then (3.8.17) can be rewritten as

$$\frac{d}{ds}\left(\frac{\partial x_i(s, y_i)}{\partial y_i}\right) = \sum_{j\neq i}\left(\gamma_{ij}(u)w_j - \gamma_{iii}^j(u)u_j w_i\right)\frac{\partial x_i(s, y_i)}{\partial y_i} - \gamma_{iii}(u_i e_i)w_i\frac{\partial x_i(s, y_i)}{\partial y_i}. \tag{3.8.20}$$

On the other hand, it follows from the first equality of (3.8.5) that

$$\frac{\partial x_i(0, y_i)}{\partial y_i} = 1. \tag{3.8.21}$$

Solving the initial value problem (3.8.20)-(3.8.21) along the characteristic $\xi = x_i(s, y_i)$ yields

$$\frac{\partial x_i(s, y_i)}{\partial y_i} = \exp\left[\int_0^s (P_1(\tau) + P_2(\tau) + P_3(\tau))\,d\tau\right], \quad \forall\, s \in (0, t^*], \tag{3.8.22}$$

where

$$P_1(\tau) = \sum_{j\neq i} (\gamma_{ij}(u)w_j)(\tau, x_i(\tau, y_i)), \tag{3.8.23}$$

$$P_2(\tau) = -\sum_{j\neq i} \left(\gamma_{iii}^j(u)u_j w_j\right)(\tau, x_i(\tau, y_i)) \tag{3.8.24}$$

and

$$P_3(\tau) = -(\gamma_{iii}(u_i e_i)w_i)(\tau, x_i(\tau, y_i)). \tag{3.8.25}$$

Then it follows from (3.8.22) that

$$\frac{\partial x_i(s, y_i)}{\partial y_i} > 0, \quad \forall\, s \in [0, t^*) \tag{3.8.26}$$

and

$$\frac{\partial x_i(s, y_i)}{\partial y_i} \le \exp\left[\int_0^s (|P_1(\tau)| + |P_2(\tau)| + |P_3(\tau)|)\, d\tau\right], \quad \forall\, s \in [0, t^*). \tag{3.8.27}$$

We first estimate $\int_0^s |P_3(\tau)| d\tau$.

Noting (3.6.53) and using (3.8.3) and (3.8.14), from (3.8.25) we get

$$|P_3(\tau)| \le C_2 |u_i|^\alpha |w_i| \le C_3 \varepsilon^{1+\alpha}, \quad \forall\, \tau \in [0, t^*). \tag{3.8.28}$$

Then, noting (3.8.1) we obtain

$$\int_0^s |P_3(\tau)| d\tau \le C_3 \varepsilon^{1+\alpha} t^* \le 2C_3 M_0, \quad \forall\, s \in [0, t^*). \tag{3.8.29}$$

We now estimate $\int_0^s |P_1(\tau)| d\tau$.

Noting (3.8.3), by (3.8.23) we have

$$\begin{aligned}
\int_0^s |P_1(\tau)| d\tau &\le C_4 \sum_{j\neq i} \int_0^s |w_j(\tau, x_i(\tau, y_i))| d\tau \\
&\le C_5 \left\{\tilde{W}_1(s) + \left(W(D_-^s) + W(D_0^s) + W(D_+^s) + W_\infty^c(s)\right) I(s)\right\} \\
&\le C_6 \left\{\tilde{W}_1(s) + W(D_-^s) + W(D_0^s) + W(D_+^s) + W_\infty^c(s)\right\}, \\
&\qquad \forall\, s \in [0, t^*),
\end{aligned} \tag{3.8.30}$$

where $I(s) = \int_0^s (1+\tau)^{-(1+\mu)} d\tau$. Then, by (3.6.4), (3.6.7) and (3.6.39) we get

$$\int_0^s |P_1(\tau)| d\tau \le C_7 \varepsilon, \quad \forall\, s \in [0, t^*). \tag{3.8.31}$$

Finally, we estimate $\int_0^s |P_2(\tau)|d\tau$.

Making use of (3.8.3) and (3.8.14), from (3.8.24) we obtain

$$\begin{aligned}\int_0^s |P_2(\tau)|d\tau &\le C_8\varepsilon\sum_{j\neq i}\int_0^s |u_j(\tau, x_i(\tau, y_i))|d\tau \\ &\le C_9\varepsilon\left\{\tilde{U}_1(s) + \left(V(D_-^s) + V(D_0^s) + V(D_+^s) + U_\infty^c(s)\right) I(s)\right\} \\ &\le C_{10}\varepsilon\left\{\tilde{U}_1(s) + V(D_-^s) + V(D_0^s) + V(D_+^s) + U_\infty^c(s)\right\}, \\ &\qquad \forall\, s \in [0, t^*),\end{aligned} \tag{3.8.32}$$

where, as before, $I(s) = \int_0^s (1+\tau)^{-(1+\mu)}d\tau$. Noting (3.4.101), (3.6.4)-(3.6.6), (3.6.40) and (3.8.1), from (3.8.32) we have

$$\begin{aligned}\int_0^s |P_2(\tau)|d\tau &\le C_{11}\varepsilon\left\{\tilde{V}_1(s) + V_\infty^c(s) + V(D_-^s) + V(D_0^s) + V(D_+^s) + U_\infty^c(s)\right\} \\ &\le C_{12}\varepsilon^2, \quad \forall\, s \in [0, t^*).\end{aligned} \tag{3.8.33}$$

Hence, noting (3.8.26)-(3.8.27) and combining (3.8.29), (3.8.31) and (3.8.33), we have proved that, for any fixed y_i belonging to Case I, there exists a positive constant K_3 independent of i, t, y_i and ε such that

$$0 < \frac{\partial x_i(t, y_i)}{\partial y_i} \le K_3, \quad \forall\, t \in [0, t^*). \tag{3.8.34}$$

Thus, by (3.8.14) and (3.8.34), (3.8.6) is proved in Case I.

We next consider Case II.

By (3.3.12), there exist two positive constants b_1 and b_2 independent of ε, such that for all y_i satisfying (3.8.12) we have

$$y_i \in [b_1, b_2]. \tag{3.8.35}$$

Then it is easy to show that there exists a positive number t_0 independent of ε, such that for each $i = 1, \cdots, n$ the i-th characteristic $x = x_i(t, y)$ passing through any given point $(0, y)$ on the interval $[b_1, b_2]$ on the x-axis must stay in the domain $D_i^{t^*}$ for $t_0 \le t < t^*$.

Consider any fixed y_i satisfying (3.8.12). Without loss of generality, we may assume that

$$\tilde{\psi}_i'(y_i) > 0 \quad \text{and} \quad a_i \left(\tilde{\psi}_i(y_i)\right)^\alpha. \tag{3.8.36}$$

Noting (3.6.68), we have

$$|w_j(t,x)| \le C_{13}\varepsilon, \quad \forall\,(t,x) \in [0,T_*] \times R, \quad \forall\, j = 1,\cdots,n, \tag{3.8.37}$$

where $T_* = K_*\varepsilon^{-(1+\alpha)} \ge t_0$, in which K_* is a positive constant independent of ε (see Remark 3.10 or Lemma 3.9).

Furthermore, by (3.6.94) there exists a constant $\tilde{\delta} > 0$ suitably small and independent of ε such that for $\varepsilon > 0$ suitably small we have

$$w_i(t, x_i(t,y_i)) \ge \frac{1}{2}\varepsilon\tilde{\psi}_i'(y_i) > 0, \quad \forall\, t \in [0, T_{\tilde{\delta}}], \tag{3.8.38}$$

where $T_{\tilde{\delta}} = \tilde{\delta}\varepsilon^{-(1+\alpha)}$ with

$$t_0 \le T_{\tilde{\delta}} \le T_*. \tag{3.8.39}$$

We now prove that $w_i(t, x_i(t,y_i))$ is a strictly increasing function of t for $t \ge T_{\tilde{\delta}}$, and then

$$w_i(t, x_i(t,y_i)) \ge \frac{1}{2}\varepsilon\tilde{\psi}_i'(y_i) > 0, \quad \forall\, t \in [T_{\tilde{\delta}}, t^*]. \tag{3.8.40}$$

In fact, in the present situation, (3.6.96) reduces to

$$\frac{dw_i}{d_i t} = a_0(t;i,y_i)w_i^2 + a_1(t;i,y_i)w_i + a_2(t;i,y_i), \tag{3.8.41}$$

where $a_0(t;i,y_i)$, $a_1(t;i,y_i)$, $a_2(t;i,y_i)$ are defined by (3.6.97)-(3.6.99), in which $u = u(t, x_i(t,y_i))$ and $w_j = w_j(t, x_i(t,y_i))$ $(j = 1,\cdots,n)$, respectively.

Similar to (3.6.117), we have

$$\gamma_{iii}(u_i(t, x_i(t,y_i))e_i) = b_i\varepsilon^\alpha + O\left(\varepsilon^{1+\alpha}\right), \quad \forall\, t \in [0, t^*), \tag{3.8.42}$$

where

$$b_i = a_i\left(\tilde{\psi}_i(y_i)\right)^\alpha > 0, \tag{3.8.43}$$

in which a_i is defined by (3.8.13). Furthermore, similar to (3.6.118), we have

$$\begin{aligned} |\gamma_{iii}(u(t,x_i(t,y_i))) - \gamma_{iii}(u_i(t,x_i(t,y_i))e_i)| &\le C_{14}(1+t)^{-(1+\mu)}U^c_\infty(t) \\ &\le C_{15}\varepsilon(1+t)^{-(1+\mu)} \\ &\le C_{16}\varepsilon(1+T_{\tilde{\delta}})^{-(1+\mu)} \\ &\le C_{17}\varepsilon^{1+\alpha}, \quad \forall\, t \in [T_{\tilde{\delta}}, t^*). \end{aligned} \tag{3.8.44}$$

Then we obtain

$$a_0(t;i,y_i) = b_i\varepsilon^\alpha + O\left(\varepsilon^{1+\alpha}\right), \quad \forall\, t \in [T_{\tilde{\delta}}, t^*). \tag{3.8.45}$$

On the other hand, using Lemma 3.9 and noting the fact that the characteristic $x = x_i(t, y_i)$ must stay in $D_i^{t^*}$ for $t \ge T_{\tilde{\delta}}$, from (3.6.98) we obtain

$$\begin{aligned} |a_1(t;i,y_i)| &\le C_{18}\left\{\sum_{j\neq i}(|w_j| + |u_j|)(t, x_i(t,y_i))\right\} \\ &\le C_{19}\left\{\left(W^c_\infty(t) + U^c_\infty(t)\right)(1+T_{\tilde{\delta}})^{-(1+\mu)}\right\} \\ &\le C_{20}\varepsilon^{2+\alpha}, \quad \forall\, t \in [T_{\tilde{\delta}}, t^*), \end{aligned} \tag{3.8.46}$$

where we have made use of Hadamard's formula for $\tilde{b}_{ii}(u) - \tilde{b}_{ii}(u_ie_i)$:

$$\tilde{b}_{ii}(u) - \tilde{b}_{ii}(u_ie_i) = \sum_{j\neq i}\int_0^1 \frac{\partial \tilde{b}_{ii}}{\partial u_j}(\tau u_1, \cdots, \tau u_{i-1}, u_i, \tau u_{i+1}, \cdots, \tau u_n)u_j d\tau. \tag{3.8.47}$$

Similarly, we can show that

$$|a_2(t;i,y_i)| \le C_{21}\varepsilon^{3+\alpha}, \quad \forall\, t \in [T_{\tilde{\delta}}, t^*). \tag{3.8.48}$$

Noting (3.8.45)-(3.8.46) and (3.8.48), from (3.8.41) we get

$$\frac{dw_i}{d_it} \ge \frac{1}{2}b_i\varepsilon^\alpha w_i^2 - C_{22}\varepsilon^{3+\alpha}, \quad \forall\, t \in [T_{\tilde{\delta}}, t^*). \tag{3.8.49}$$

At $t = T_{\tilde{\delta}}$, by (3.8.38) we observe that the right-hand side of (3.8.49) is positive. Hence $w_i(t, x_i(t, y_i))$ is a strictly increasing function of t at lease in a neighbourhood of $t = T_{\tilde{\delta}}$, then noting (3.8.49) again, $w_i(t, x_i(t, y_i))$ is always a strictly increasing function of t for $t \in [T_{\tilde{\delta}}, t^*)$. Thus, (3.8.40) holds.

For any $j \neq i$, using (3.6.7) we have

$$\begin{aligned} |w_j(t, x_i(t, y_i))| &\leq C_{23}(1+t)^{-(1+\mu)} W_\infty^c(t) \\ &\leq C_{24}(1+T_{\bar{\delta}})^{-(1+\mu)}\varepsilon \\ &\leq C_{25}\varepsilon^2, \quad \forall\, t \in [T_{\bar{\delta}}, t^*), \end{aligned} \tag{3.8.50}$$

provided that $\varepsilon > 0$ is suitable small.

Since the C^1 norm of $\psi(x)$ is bounded, using the first inequality of (3.8.36) and noting (3.8.12), for any $j \neq i$ we have

$$|w_j(t, x_i(t, y_i))| < w_i(t, x_i(t, y_i)), \quad \forall\, t \in [T_{\bar{\delta}}, t^*). \tag{3.8.51}$$

Noting (3.8.37) and (3.8.39) and making use of (3.8.23)-(3.8.25), similar to Case I, we can prove easily

$$0 < \frac{\partial x_i(t, y_i)}{\partial y_i} \leq K_4, \quad \forall\, t \in [0, T_{\bar{\delta}}], \tag{3.8.52}$$

where K_4 is a positive constant independent of i, t, y_i and ε. Moreover, (3.8.26) still holds.

Noting (2.2.19), (2.2.6) and (3.8.15), we have

$$\frac{d}{d_i t}\left(w_i(t, x_i(t, y_i))\frac{\partial x_i(t, y_i)}{\partial y_i}\right) = \sum_{j,k=1}^{n} \tilde{\gamma}_{ijk}(u) w_j w_k \frac{\partial x_i(t, y_i)}{\partial y_i} + (b_i(u))_x \frac{\partial x_i(t, y_i)}{\partial y_i}, \tag{3.8.53}$$

where $\tilde{\gamma}_{ijk}(u)$ is defined by (2.2.26).

Noting (3.8.3), (3.8.26), (3.8.51), (2.2.27), (3.2.7) and (3.2.12), from (3.8.53) we obtain

$$\begin{aligned} |w_i(t, x_i(t, y_i))|\tfrac{\partial x_i(t, y_i)}{\partial y_i} \leq\ & |w_i(T_{\bar{\delta}}, x_i(T_{\bar{\delta}}, y_i))|\tfrac{\partial x_i(T_{\bar{\delta}}, y_i)}{\partial y_i} + \\ & C_{26}\left\{ W_\infty^c(t)\int_{T_{\bar{\delta}}}^{t} (1+\tau)^{-(1+\mu)} |w_i(\tau, x_i(\tau, y))| \tfrac{\partial x_i(\tau, y_i)}{\partial y_i} d\tau + \right. \\ & \left. \sum_{k=1}^{n} \int_{T_{\bar{\delta}}}^{t} \left|\tilde{b}_{ik}(u) - \tilde{b}_{ik}(u_k e_k)\right| |w_k| \tfrac{\partial x_i(\tau, y_i)}{\partial y_i} d\tau \right\}, \quad \forall\, t \in [T_{\bar{\delta}}, t^*). \end{aligned} \tag{3.8.54}$$

By Hadamard's formula, we have

$$\tilde{b}_{ik}(u) - \tilde{b}_{ik}(u_k e_k) = \sum_{j \neq k} \int_0^1 \frac{\partial \tilde{b}_{ik}}{\partial u_j}(\tau u_1, \cdots, \tau u_{k-1}, u_k, \tau u_{k+1}, \cdots, \tau u_n) u_j d\tau. \tag{3.8.55}$$

Noting (3.8.3) and (3.8.40), we get

$$|u_j(t, x_i(t, y_i))| \leq C_{27} w_i(t, x_i(t, y_i)), \quad \forall\, t \in [T_{\bar{\delta}}, t^*), \quad \forall\, j = 1, \cdots, n. \tag{3.8.56}$$

Then, using (3.8.55)-(3.8.56) and noting the definitions of $U^c_\infty(t)$ and $W^c_\infty(t)$, from (3.8.54) we obtain

$$\begin{aligned} |w_i(t, x_i(t, y_i))| \tfrac{\partial x_i(t, y_i)}{\partial y_i} \leq \left| w_i(T_{\bar{\delta}}, x_i(T_{\bar{\delta}}, y_i)) \right| \tfrac{\partial x_i(T_{\bar{\delta}}, y_i)}{\partial y_i} + C_{28} \left(W^c_\infty(t) + U^c_\infty(t) \right) \times \\ \int_{T_{\bar{\delta}}}^t (1+\tau)^{-(1+\mu)} |w_i(\tau, x_i(\tau, y_i))| \tfrac{\partial x_i(\tau, y_i)}{\partial y_i} d\tau, \quad \forall\, t \in [T_{\bar{\delta}}, t^*), \end{aligned} \tag{3.8.57}$$

where we are aware of the fact that the term $|u_j(\tau, x_i(\tau, y_i))| w_i(\tau, x_i(\tau, y_i))$ $(j \neq i)$ can be controled by $(1+\tau)^{-(1+\mu)} U^c_\infty(t) w_i(\tau, x_i(\tau, y_i))$, and the fact that the term $|u_j(\tau, x_i(\tau, y_i)) w_k(\tau, x_i(\tau, y_i))|$ $(j = 1, \cdots, n;\ k \neq i)$ can be bounded by $C_{27} |w_k(\tau, x_i(\tau, y_i))| w_i(\tau, x_i(\tau, y_i))$, then by $C_{27}(1+\tau)^{-(1+\mu)} W^c_\infty(t) w_i(\tau, x_i(\tau, y_i))$, in which τ satisfies that $T_{\bar{\delta}} \leq \tau \leq t < t^*$. Hence, noting Lemma 3.9 and using (3.8.57), we can easily prove

$$|w_i(t, x_i(t, y_i))| \frac{\partial x_i(t, y_i)}{\partial y_i} \leq C_{29} \left| w_i(T_{\bar{\delta}}, x_i(T_{\bar{\delta}}, y_i)) \right| \frac{\partial x_i(T_{\bar{\delta}}, y_i)}{\partial y_i}, \quad \forall\, t \in [T_{\bar{\delta}}, t^*), \tag{3.8.58}$$

provided that $\varepsilon > 0$ is suitably small. Thus, noting (3.8.39) and using (3.8.37) and (3.8.52), we get easily

$$|w_i(t, x_i(t, y_i))| \frac{\partial x_i(t, y_i)}{\partial y_i} \leq C_{30} \varepsilon, \quad \forall\, t \in [0, t^*). \tag{3.8.59}$$

This proves (3.8.6) in Case II.

Therefore the proof of Lemma 3.13 is completed. Q.E.D.

Proof of Theorem 3.4. By Lemma 3.9, on the existence domain of the C^1 solution $u = u(t, x)$ to the Cauchy problem (3.3.1)-(3.3.2), the C^0 norm of $u = u(t, x)$ is uniformly bounded, then the first order derivatives of $u = u(t, x)$ must tend to the infinity at the starting point of singularity. By Lemma 3.13, if there exist $y_0 \in R$ and $i \in \{1, \cdots, n\}$ such that

$$|w_i(t, x_i(t, y_0))| \longrightarrow +\infty \quad \text{as} \quad t \nearrow t^*, \tag{3.8.60}$$

then

$$\frac{\partial x_i(t, y_0)}{\partial y_0} \longrightarrow 0 \quad \text{as} \quad t \nearrow t^*. \tag{3.8.61}$$

In particular, for any given $i \notin J_1$, a_i (see (3.8.13)) is equal to zero, thus for any given $y_i \in R$, (3.8.11) holds and then (3.8.14) is valid. That is to say, the corresponding value of w_i remains bounded, hence it is impossible to have (3.8.60) and (3.8.61)[4]. Therefore, the characteristics of the i-th family never form an envelope on the domain $0 \le t \le \tilde{T}(\varepsilon)$. This proves Theorem 3.4. Q.E.D.

§3.9. Remarks on the critical case

Similar to Remark 3.10, we can easily prove the following proposition.

Proposition 3.2. Under the assumptions mentioned at the beginning of §3.3, suppose that $A(u) \in C^\infty$ and $B(u) \in C^2$ in a neighbourhood of $u = 0$. Suppose furthermore that $\varphi(x) = \varepsilon\psi(x)$, where $\varepsilon > 0$ is a small parameter and $\psi(x)$ is a C^1 vector function satisfying (3.3.12). Suppose finally that each characteristic of system (3.3.1) is either critical or weakly linearly degenerate, and $B(u)$ satisfies the matching condition. Then, for any given integer $N \ge 1$, there exists $\varepsilon_0 = \varepsilon_0(N) > 0$ so small that for any fixed $\varepsilon \in (0, \varepsilon_0]$, the life span $\tilde{T}(\varepsilon)$ of the C^1 solution $u = u(t, x)$ to the Cauchy problem (3.3.1)-(3.3.2) satisfies

$$\tilde{T}(\varepsilon) \ge C_N \varepsilon^{-N}, \tag{3.9.1}$$

where C_N is a positive constant independent of ε. □

When $B(u)$ does not satisfy the matching condition, we have

Proposition 3.3. Under the assumptions mentioned at the beginning of §3.3, suppose that $A(u)$ and $B(u)$ are suitably smooth in a neighbourhood of $u = 0$.

[4]In the present situation, (3.8.14) is valid. Then it follows from (3.8.14) directly that (3.8.60) is impossible. Noting (3.8.29), (3.8.31) and (3.8.33), from (3.8.22) we observe that (3.8.61) is also impossible.

Suppose furthermore that $\varphi(x) = \varepsilon\psi(x)$, where $\varepsilon > 0$ is a small parameter and $\psi(x)$ is a C^1 vector function satisfying (3.3.12). Suppose finally that each characteristic of system (3.3.1) is either critical or weakly linearly degenerate, and $B(u)$ satisfies

$$B(u) = O\left(|u|^p\right) \tag{3.9.2}$$

in a neighbourhood of $u = 0$, where p is an integer ≥ 2. Then there exists $\varepsilon_0 = \varepsilon_0(p) > 0$ so small that for any fixed $\varepsilon \in (0, \varepsilon_0]$, the life span $\tilde{T}(\varepsilon)$ of the C^1 solution $u = u(t, x)$ to the Cauchy problem (3.3.1)-(3.3.2) satisfies

$$\tilde{T}(\varepsilon) \geq C_p \varepsilon^{-(p-1-\tilde{\mu})}, \tag{3.9.3}$$

where $\tilde{\mu} \in (0, 1)$ is an arbitrary fixed real number and C_p is a positive constant independent of ε. □

In the present situation, Lemma 3.8 is still valid. In order to prove Proposition 3.3 it suffices to show the following.

Lemma 3.14. Under the assumptions of Proposition 3.3, in the normalized coordinates there exists $\varepsilon_0 > 0$ so small that for any fixed $\varepsilon \in (0, \varepsilon_0]$, on any given existence domain $0 \leq t \leq T$ of the C^1 solution $u = u(t, x)$ to the Cauchy problem (3.6.1)-(3.6.2) there exist positive constants κ_i $(i = 1, 2, \cdots, 8)$ independent of ε and T, such that the following uniform *a priori* estimates hold:

$$U_\infty^c(T) \leq \kappa_1 \varepsilon, \tag{3.9.4}$$

$$V_\infty^c(T) \leq \kappa_2 \varepsilon, \tag{3.9.5}$$

$$W_\infty^c(T) \leq \kappa_3 \varepsilon, \tag{3.9.6}$$

$$W_1(T) \leq \kappa_4 \varepsilon, \tag{3.9.7}$$

$$V_1(T) \leq \kappa_5 \varepsilon, \tag{3.9.8}$$

$$V_\infty(T) \leq \kappa_6 \varepsilon \tag{3.9.9}$$

and

$$W_\infty(T) \leq \kappa_7 \varepsilon, \tag{3.9.10}$$

where

$$T\varepsilon^{p-1-\tilde{\mu}} \le \kappa_8, \tag{3.9.11}$$

and $\tilde{\mu} \in (0,1)$ is an arbitrary fixed real number. □

Proof. The proof of this Lemma is almost the same as that of Lemma 3.12 and can be handled similarly.

Without loss of generality, we may suppose

$$0 < \mu \le \frac{\tilde{\mu}}{p-1-\tilde{\mu}}. \tag{3.9.12}$$

(3.9.12) implies that

$$\varepsilon^p \cdot \varepsilon^{-(1+\mu)(p-1-\tilde{\mu})} \le \varepsilon. \tag{3.9.13}$$

In a way similar to the proof of (3.7.1)-(3.7.6) and (3.7.8), noting (3.9.13) and using (3.7.17)-(3.7.24) (in which we may assume that $\alpha \ge p$ since each characteristic of system (3.6.1) is either weakly linearly degenerate or critical), we can prove that there exist positive constants κ_i $(i = 1, \cdots, 7)$ and a small $\kappa_8 > 0$ such that, when T satisfies (3.9.11), the estimates (3.9.4)-(3.9.10) hold. For brevity, the details are omitted. Q.E.D.

In the critical case, however, the precise estimate on the life span of the C^1 solution might be very complicated. A discussion for a single equation can be found in §5 of [LZK2]. Recently, A.Hoshiga [H] generalizes the result given in §5 of [LZK2] to the case that system may be 2×2, under some assumptions he gives a similar result. A discussion on general critical quasilinear hyperbolic systems in diagonal form is carried out in Appendix 2, and a sharp estimate on life span of the classical solutions is given.